STRUCTURE OF GRÖBNER BASES WITH RESPECT TO BLOCK ORDERS

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Abstract. In this paper we study the structure of Gröbner bases with respect to block orders. We extend Lazard’s theorem and the Gianni-Kalkbrenner theorem to the case of a zero-dimensional ideal whose trace in the ring generated by the first block of variables is radical. We then show that they do not hold for general zero-dimensional ideals.

1. Introduction

The concept of Gröbner bases, introduced by Buchberger [6] in 1965, is nowadays one of the main tools for studying algebraic systems and various related problems in computational algebra; see [7] and the standard reference books [5, 2, 8, 15] for basic facts and applications of such a concept.

An important question is to understand the structure of Gröbner bases. In the case of two variables, Lazard [13] gave a complete structural understanding of lexicographic Gröbner bases. This result is extended to the case of univariate polynomial rings over Dedekind domains in [3]. In the case of more than two variables an extension of Lazard’s theorem is given by Marinari and Mora [14] in the case of radical zero-dimensional ideals.

Another important result is proved by Gianni [10] and Kalkbrenner [12] for lexicographic Gröbner bases of zero-dimensional ideals. More precisely, the result states that if $G$ is a Gröbner basis of a zero-dimensional ideal $\mathcal{I}$ in the polynomial ring $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ and if $(\mu_1, \ldots, \mu_{n-1})$ is a zero of $\mathcal{I} \cap \mathbb{K}[x_1, \ldots, x_{n-1}]$, then the list of polynomials obtained from $G$ after substituting the $\mu_i$’s to the $x_i$’s already contains a gcd of the nonzero polynomials in it. This result is extended by Becker in [4], for the radical zero-dimensional case, in the sense that for any $i \leq n - 1$ and any zero $(\mu_1, \ldots, \mu_i)$ of $\mathcal{I} \cap \mathbb{K}[x_1, \ldots, x_i]$ the image of $\mathcal{I}$ under the specialization homomorphism $\phi : x_i \mapsto \mu_i$ has $\phi(G)$ as a Gröbner basis.

We are concerned in this paper with the question of understanding the structure of Gröbner bases with respect to block orders (see section 2 for the definition of block orders). We give extensions of Lazard’s theorem and Becker’s extension of the Gianni-Kalkbrenner theorem. More precisely, let $J = x_1, \ldots, x_j$ be a sub-list of $x$ and $\prec$ be a monomial block order built out of monomial orders on $J$ and its complement $J'$. Given a zero-dimensional ideal $\mathcal{I}$ of $\mathbb{K}[x]$ such that $\mathcal{I} \cap \mathbb{K}[J]$ is radical, we show that $\mathcal{I}$ has a Gröbner basis which exhibits a factorized form
similar to the one in Lazard’s theorem. We also show that Becker’s extension of
the Gianni-Kalkbrenner theorem holds in this case. We finally show that these
results no longer hold in general if \( \mathcal{I} \cap \mathbb{K}[J] \) is not radical (even for lexicographic
orders).

2. Notation and basic facts

Throughout this paper we will denote by \( \mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[x] \) the ring of polynomials
in \( n \) indeterminates with coefficients in a commutative field \( \mathbb{K} \). The multiplicative
semigroup generated by \( x = x_1, \ldots, x_n \) is denoted by \( \mathbb{M} \), and its divisibility
partial order by \( \preceq \).

For any sub-list \( J \) of \( x \), with \( |J| \) as the number of elements and \( J^c \) as the
complement, we let \( \mathbb{M}[J] \) be the sub-semigroup of \( \mathbb{M} \) generated by \( J \). Identified to
\( \mathbb{K}[J][J^c] \), the ring \( \mathbb{K}[x] \) has a natural structure of the free \( \mathbb{K}[J] \)-module with \( \mathbb{M}[J^c] \)
as the canonical basis. Thus, any polynomial \( f \in \mathbb{K}[x] \) is uniquely written as
\[
\sum a_\alpha x^\alpha,
\]
where the sum ranges over a finite subset of \( \mathbb{M}[J^c] \) and the \( a_\alpha \)'s are elements of
\( \mathbb{K}[J] \).

Let \( \mathcal{I} \) be an ideal of \( \mathbb{K}[x] \), \( J \) be a sub-list of \( x \) and \( \prec \) be an admissible order on
the monomials. Given \( x^\alpha \in \mathbb{M}[J^c] \) we let \( \mathcal{J}(\mathcal{I}, J, \prec, x^\alpha) \) be the set
\[
\{a \in \mathbb{K}[J]: a = 0 \text{ or } \exists \mathcal{I} \text{ in } \mathcal{I} \text{ Lm}(f, J, \prec) = x^\alpha, \text{ Lc}(f, J, \prec) = a \}.
\]
This is obviously an ideal of \( \mathbb{K}[J] \) and the sequence \( (\mathcal{J}(\mathcal{I}, J, \prec, x^\alpha))_\alpha \) is non-decreasing,
\( i.e. \mathcal{J}(\mathcal{I}, J, \prec, x^\alpha) \subseteq \mathcal{J}(\mathcal{I}, J, \prec, x^\beta) \) whenever \( x^\alpha \prec x^\beta \).

Given any sub-list \( J \) of \( x \) and a finite subset \( G \) of \( \mathbb{K}[x] \) we will write \( G = \{B_{1,i}, \ldots, B_{s,i}: i = 0, \ldots, s\} \), where \( \operatorname{Lm}(B_{j,i}, J, \prec) = x^{\alpha(1)}_j \) and \( x^{\alpha(0)} \prec x^{\alpha(1)} \prec \cdots \prec x^{\alpha(s)} \).

The following definition of block orders and their main properties can be found in [8].

**Definition 2.1.** Given an admissible order \( \prec \) on \( \mathbb{M} \) and a sub-list \( J \) of \( x \), we will
say that \( \prec \) is a block order with respect to \( J \) if for any \( a, a' \in \mathbb{M}[J] \) and \( b, b' \in \mathbb{M}[J^c] \) we have:
\[
ab \prec a'b' \iff b \prec b' \text{ or } (b = b' \text{ and } a \prec a').
\]

A typical example of block order is given by the lexicographic order \( x_1 \prec_{\text{lex}} x_2 \prec_{\text{lex}} \cdots \prec_{\text{lex}} x_n \) and \( J = x_1, \ldots, x_i \), where \( i \leq n - 1 \). In general, let \( J \) be a sub-list of \( x \) and \( \prec_J \) (resp. \( \prec_{J^c} \)) be an admissible order on \( \mathbb{M}[J] \) (resp. \( \mathbb{M}[J^c] \)). If \( \prec \) is the lexicographic order on \( \mathbb{M} = \mathbb{M}[J] \times \mathbb{M}[J^c] \) built with (\( \mathbb{M}[J], \prec_J \)) and
(\( \mathbb{M}[J^c], \prec_{J^c} \)), then \( \prec \) is a block order with respect to \( J \).

If \( \prec \) is a block order with respect to \( J \), then for any polynomial \( f \in \mathbb{K}[x] \) we have:
\[
(2.1) \quad \operatorname{Lm}(f, \prec) = \operatorname{Lm}(\operatorname{Lc}(f, J, \prec), \prec) \operatorname{Lm}(f, J, \prec).
\]
We will also need the following classical result; see [1] for a more general result.

**Theorem 2.2.** Let $I$ be an ideal of $K[x]$ and $J$ be a sub-list of $x$. Let $\prec$ be a block order with respect to $J$. Then, for any finite set $G$ of $I$ the following assertions are equivalent:

i) $G$ is a Gröbner basis of the ideal $I$ with respect to $\prec$,

ii) for any $x^\alpha \in M[J']$ the set

$$\{Lc(f, J, \prec) : f \in G, \text{Lm}(f, J, \prec) \mid x^\alpha\}$$

is a Gröbner basis of the ideal $J(I, J, \prec, x^\alpha)$ with respect to $\prec$.

**Proof.** “i) $\Rightarrow$ ii)” Let $c \in J(I, J, \prec, x^\alpha)$ and $f \in I$ be such that $\text{Lm}(f, J, \prec) = x^\alpha$ and $Lc(f, J, \prec) = c$. Since $G$ is a Gröbner basis of $I$ with respect to $\prec$ there exists $g \in G$ such that $\text{Lm}(g, \prec) \mid \text{Lm}(f, \prec)$.

According to the relation (2.1) we have

$$\text{Lm}(g, J, \prec) \mid \text{Lm}(f, J, \prec) = x^\alpha,$$

$$\text{Lm}(Lc(g, J, \prec), \prec) \mid \text{Lm}(Lc(f, J, \prec), \prec) = \text{Lm}(c, \prec).$$

Therefore, the set $\{Lc(g, J, \prec) : g \in G \text{ and } \text{Lm}(g, J, \prec) \mid x^\alpha\}$ is a Gröbner basis of the ideal $J(I, J, \prec, x^\alpha)$ with respect to $\prec$.

“i) $\Rightarrow$ ii)” Let $f \in I$ and write $\text{Lm}(f, J, \prec) = x^\alpha$. Since the set $\{Lc(g, J, \prec) : g \in G \text{ and } \text{Lm}(g, J, \prec) \mid x^\alpha\}$ is a Gröbner basis of the ideal $J(I, J, \prec, x^\alpha)$ with respect to $\prec$ there exists $g \in G$ such that

$$\text{Lm}(g, J, \prec) \mid \text{Lm}(f, J, \prec) = x^\alpha,$$

$$\text{Lm}(Lc(g, J, \prec), \prec) \mid \text{Lm}(Lc(f, J, \prec), \prec).$$

Therefore, $\text{Lm}(g, \prec) \mid \text{Lm}(f, \prec)$ according to the relation (2.1). This shows that $G$ is a Gröbner basis of $I$ with respect to $\prec$. \hfill $\Box$

3. **The case of zero-dimensional ideals with radical trace on the first block**

Let $J$ be a sub-list of $x$ and $\prec$ be a block order with respect to $J$. In this section we study the structure of a zero-dimensional ideal $I$ of $K[x]$ as a $K[J]$-submodule of $K[J][J']$. Under the additional assumption that $I \cap K[J]$ is radical, we give an extension of Lazard’s theorem and we show that Becker’s extension of the Gianni-Kalkbrenner theorem also holds in this case.

**Lemma 3.1.** Let $I$ be a zero-dimensional ideal of $K[x]$, $J$ be a sub-list of $x$ and $\prec$ be a block order with respect to $J$. Assume that the ideal $I \cap K[J]$ is radical. Then for any $x^\alpha \in M[J']$ there exists a polynomial $H_\alpha \in K[x]$ such that:

i) $\text{Lm}(H_\alpha, J, \prec) = x^\alpha$ and $\text{Lc}(H_\alpha, J, \prec) = 1$,

ii) for any $c \in J(I, J, \prec, x^\alpha)$ we have $cH_\alpha \in I$,

iii) for any $f \in I$ there exists a unique sequence $(c_\alpha)_{x^\alpha \in M[J']}$, having a finite support, such that $f = \sum c_\alpha H_\alpha$ and $c_\alpha \in J(I, J, \prec, x^\alpha)$.

**Proof.** To simplify let $J_\alpha = J(I, J, \prec, x^\alpha)$, and notice that $J_0 = I \cap K[J]$ is radical zero-dimensional and $J_0 \subset J_\alpha$, $J_0 \subset J_\alpha : J_\alpha$ for any $\alpha$. This implies in particular that $J_\alpha$, as well as $J_0 : J_\alpha$, is either radical zero-dimensional or equal to $K[J]$. Moreover we have $(J_0 : J_\alpha) + J_\alpha = K[J]$.

Let $a \in J_0 : J_\alpha$ and $b \in J_\alpha$ be such that $a + b = 1$, and let $f \in I$ be such that $\text{Lm}(f, J, \prec) = x^\alpha$ and $\text{Lc}(f, J, \prec) = b$. If we let $H_\alpha = f + ax^\alpha$, then we have
\( \text{Lm}(H_\alpha, J, \prec) = x^\alpha \) and \( \text{Lc}(H_\alpha, J, \prec) = a + b = 1 \). On the other hand, given any \( c \in \mathcal{J}_0 \), we have \( ca \in \mathcal{J}_0 \subset \mathcal{I} \) and hence \( ch_\alpha = cf + ca x^\alpha \) belongs to the ideal \( \mathcal{I} \).

Let \( f \in \mathcal{I} \setminus \{0\} \), \( x^{\alpha(0)} = \text{Lm}(f, J, \prec) \) and \( c^{\alpha(0)} = \text{Lc}(f, J, \prec) \). Then \( f_1 = f - c^{\alpha(0)}H_\alpha(0) \) belongs to \( \mathcal{I} \) and satisfies \( x^{\alpha(1)} = \text{Lm}(f_1, J, \prec) \not\sim \text{Lm}(f, J, \prec) \). If \( f_1 \neq 0 \), then we repeat the process to construct another polynomial

\[
 f_2 = f_1 - \text{Lc}(f_1, J, \prec)H_\alpha(1)
\]

which satisfies the relation \( x^{\alpha(2)} = \text{Lm}(f_2, J, \prec) \not\sim \text{Lm}(f_1, J, \prec) \). Continuing this way we construct a sequence \( f_0 = f, f_1, \ldots, f_t \ldots \in \mathcal{I} \) such that

\[
 \text{Lm}(f_0, J, \prec) \not\sim \text{Lm}(f_1, J, \prec) \not\sim \cdots \not\sim \text{Lm}(f_t, J, \prec) \not\sim \cdots.
\]

According to the Artinian nature of the order \( \prec \), the constructed sequence \( (f_t)_{t \geq 0} \) should stop at some \( t \). If \( f_t \neq 0 \), then we can construct another polynomial \( f_{t+1} = f_t - \text{Lc}(f_t, J, \prec)H_\alpha(t) \) with \( \text{Lm}(f_{t+1}, J, \prec) \not\sim \text{Lm}(f_t, J, \prec) \), and this contradicts the fact that the sequence stops at \( t \). Thus we have \( f = \sum_{i=0}^{t-1} c^{\alpha(i)}H_\alpha(i) \). To prove the uniqueness it suffices to show that the \( H_\alpha \)'s are linearly independent over \( \mathbb{K}[J] \). But this obviously follows from the fact that the polynomials \( H_\alpha \) have pair-wise distinct leading monomials with respect to \( J \) and \( \prec \).

As a consequence of Lemma 3.1 we get the following extension of Lazard’s structure theorem [13]. A similar result is proven in [9, 14] for radical zero-dimensional ideals and \( J = x_1, \ldots, x_{n-1} \).

**Theorem 3.2.** Let \( \mathcal{I} \) be a zero-dimensional ideal of \( \mathbb{K}[x] \), \( J \) be a sub-list of \( x \) and \( \prec \) be a block order with respect to \( J \). Assume that \( \mathcal{I} \cap \mathbb{K}[J] \) is radical and let \( G = \{ B_{j,i} : i = 0, \ldots, s, j = 1, \ldots, r_i \} \) be a Gröbner basis of \( \mathcal{I} \) with respect to \( \prec \), with \( \text{Lm}(B_{j,i}, J, \prec) = x^{\alpha(i)} \). Then one can construct polynomials \( H_0, \ldots, H_s \) such that:

i) \( \text{Lm}(H_i, J, \prec) = x^{\alpha(i)} \) and \( \text{Lc}(H_i, J, \prec) = 1 \),

ii) for any \( c \in \mathcal{J}(\mathcal{I}, J, \prec, x^{\alpha(i)}) \) we have \( cH_i \in \mathcal{I} \),

iii) the set \( G_1 = \{ \text{Lc}(B_{j,i}, J, \prec)H_i : i = 0, \ldots, s, j = 1, \ldots, r_i \} \) is a Gröbner basis of the ideal \( \mathcal{I} \) with respect to \( \prec \).

**Proof.** For any \( i = 0, 1, \ldots, s \) let \( H_i \) be a polynomial satisfying the properties of lemma 3.1 with \( x^\alpha = x^{\alpha(i)} \). On the other hand, following the construction of \( G_1 \) we have

\[
\{ \text{Lc}(f, J, \prec) : f \in G_1, \text{Lm}(f, J, \prec) \mid x^\alpha \} = \{ \text{Lc}(B_{j,i}, J, \prec) : x^{\alpha(i)} \mid x^\alpha, j = 1, \ldots, r_i \}
\]

for any monomial \( x^\alpha \), and therefore this system is a Gröbner basis of \( \mathcal{J}(\mathcal{I}, J, \prec, x^\alpha) \) according to Theorem 2.2 and to the fact that \( G \) is a Gröbner basis of \( \mathcal{I} \) with respect to \( \prec \). By applying once again Theorem 2.2 we deduce that \( G_1 \) is a Gröbner basis of \( \mathcal{I} \) with respect to \( \prec \). □

Another consequence of Lemma 3.1 is the following extension of Becker’s result [4].

**Theorem 3.3.** Let \( \mathcal{I} \) be a zero-dimensional ideal of \( \mathbb{K}[x] \), \( J \) be a sub-list of \( x \) and \( \prec \) be a block order with respect to \( J \). Assume that \( \mathcal{I} \cap \mathbb{K}[J] \) is radical. Let \( G \) be a Gröbner basis of \( \mathcal{I} \) with respect to \( \prec \) and let \( \phi : \mathbb{K}[J] \rightarrow F \) be a ring homomorphism, where \( F \) is a field extension of \( \mathbb{K} \). Then \( \phi(G) \) is a Gröbner basis, with respect to \( \prec_J \), of the ideal generated by \( \phi(\mathcal{I}) \) in \( F[J'] \).
Let holds. Consider the following system of polynomials:

\[ p = u_1 B_1 + \cdots + u_t B_t + R, \]

where \( B_i \in G \), \( \text{Lm}(B_i, J, \prec) \). \( \text{Lm}(u_j, J, \prec) = x^\alpha \) and \( \text{Lm}(R, J, \prec) \prec x^\alpha \). In particular, if \( \phi(c) \neq 0 \), then there exists \( i \) such that \( \text{Lm}(\phi(B_i), \prec) = x^\alpha \), i.e.

\[ \phi(\text{Lc}(B_i, J, \prec)) \neq 0. \]

Now let \( (H_\alpha)_{x^\alpha \in \mathbb{M}^{\mathbb{J}}_\prec} \) be a sequence of polynomials satisfying the properties of lemma \ref{lem:example}. Let \( p \in \mathcal{I} \) and write \( p = \sum_\alpha c_\alpha H_\alpha \), with \( c_\alpha \in \mathcal{J}(\mathcal{I}, J, \prec, x^\alpha) \). Since the \( H_\alpha \) have pairwise distinct leading coefficients we have \( \text{Lm}(\phi(p), \prec_J) = x^\beta \), where \( x^\beta \) is the highest monomial with \( \phi(c_\beta) \neq 0 \).

The fact that \( c_\alpha H_\beta \in \mathcal{I} \) and \( \phi(c_\beta) \neq 0 \) implies the existence of \( B \in G \) such that \( \phi(\text{Lc}(B, J, \prec)) \neq 0 \) and \( \text{Lm}(B, J, \prec) \prec x^\beta \). This proves that \( \text{Lm}(\phi(B), \prec_J) \) can be reduced by \( \phi(G) \). This is enough to show that \( \phi(G) \) is a Gröbner basis of the ideal \( \phi(\mathcal{I}) \) with respect to \( \prec_J \).

\[ \square \]

4. Counterexamples to the general case

In this section we give examples of zero-dimensional nonradical ideals for which the results of section 3 do not hold. Symbolic computations of this section are performed using Singular \cite{ Singular } and Maple 9.

Consider the ideal \( \mathcal{I} \) generated by the following list of polynomials:

\[ p_1 = t^3 + zy - z^2, \quad p_2 = xt^3 - y^3, \]
\[ p_3 = yt^2 - x^2 - y^2, \quad p_4 = zt + y + z^2. \]

The reduced Gröbner basis \( G \) of \( \mathcal{I} \) with respect to the lexicographic order \( x \prec y \prec z \prec t \) contains 16 polynomials.

After substituting 0 to \( x \) in the \( p_i \)'s we obtain an ideal \( \mathcal{I}_0 \) whose reduced Gröbner basis is

\[ G_0 = y^3, zy^2, yz^2, z^4 + z^3 + 2y^2, y^2t, zt + y + z^2, yt^2 - y^2, t^3 + zy - z^2. \]

On the other hand, if we substitute 0 to \( x \) in the elements of \( G \), remove the contents and reduce, we get the following system:

\[ G_1 = y^3, y^2z^2, z^3y, z^4 + z^3 + yz^2 + 2y^2, -yz^2 + y^2, y^2zt, zt + z^2 + y, \]
\[ yt^2 - y^2, t^3 - z^2 + zy. \]

Clearly, the polynomial \( zy^2 \in G_0 \) cannot be reduced to 0 by using \( G_1 \), which proves that \( G_1 \) is not a Gröbner basis of \( \mathcal{I}_0 \).

The above example shows that Theorem \ref{thm:counterexample} does not hold in the general case. Also, the decomposition of Lemma \ref{lem:decomposition} does not hold since Theorem \ref{thm:decomposition} is one of its consequences.

Next we give an example for which Theorem \ref{thm:counterexample} does not hold but Theorem \ref{thm:counterexample} holds. Consider the following system of polynomials:

\[ f = z^3 - x^2, \quad g = z^3 + xy^2 + x^2, \quad h = z^3 + xyz - y^3. \]
A Gröbner basis $G$ of $\mathcal{I} = \mathcal{I}(f, g, h)$ with respect to the lexicographic order $x < y < z$ consists of the following elements:

$$
\begin{align*}
B_1 &= 8x^7 - 47x^6 + 152x^5 + 192x^4 + 512x^3, \\
B_2 &= 8192yx^2 + 168x^6 - 1051x^5 + 2032x^4 + 7744x^3, \\
B_3 &= y^2x + 2x^2, \\
B_4 &= 2048y^5 + 168x^6 - 1051x^5 + 2032x^4 + 7744x^3, \\
B_5 &= 16384zx^2 + 8192y^4 + 168x^6 - 1051x^5 + 2032x^4 + 7744x^3, \\
B_6 &= zyx - y^3 + x^2, \\
B_7 &= 8192z^2y^3 + 4096zy^4 - 232x^6 - 109x^5 + 1680x^4 - 14400x^3, \\
B_8 &= x^3 - x^2.
\end{align*}
$$

If we let $J = x$, then we have $\mathcal{J}(\mathcal{I}, J, <, z) = \mathcal{I}(x^2)$. Now if there exists a polynomial $H_z = z + q(x, y)$ such that $x^3H_z \in \mathcal{I}$, then we have $p = 16384x^2H_z - B_5 \in \mathcal{I} \cap \mathbb{K}[x, y]$, and so $p = \sum u_iB_i$. By substituting 0 to $x$ in this equality we get $y^4 = \alpha(y_0)^2$, which is impossible. On the other hand, a direct computation shows that Theorem 13.3 holds for this example.

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