TWO KINDS OF STRONG PSEUDOPRIMES UP TO $10^{36}$

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Dedicated to the memory of Kencheng Zeng (1927–2004)

Abstract. Let $n > 1$ be an odd composite integer. Write $n - 1 = 2^sd$ with $d$ odd. If either $b^d \equiv 1 \pmod{n}$ or $b^{2^rd} \equiv -1 \pmod{n}$ for some $r = 0, 1, \ldots, s - 1$, then we say that $n$ is a strong pseudoprime to base $b$, or spsp($b$) for short. Define $\psi_t$ to be the smallest strong pseudoprime to all the first $t$ prime bases. If we know the exact value of $\psi_t$, we will have, for integers $n < \psi_t$, a deterministic efficient primality testing algorithm which is easy to implement. Thanks to Pomerance et al. and Jaeschke, the $\psi_t$ are known for $1 \leq t \leq 8$. Conjectured values of $\psi_9, \ldots, \psi_{12}$ were given by us in our previous papers (Math. Comp. 72 (2003), 2085–2097; 74 (2005), 1009–1024).

The main purpose of this paper is to give exact values of $\psi'_t$ for $13 \leq t \leq 19$; to give a lower bound of $\psi'_20$: $\psi'_20 > 10^{36}$; and to give reasons and numerical evidence of $K2$- and $C3$-spsp's < $10^{36}$ to support the following conjecture: $\psi_t = \psi'_t < \psi''_t$ for any $t \geq 12$, where $\psi'_t$ (resp. $\psi''_t$) is the smallest $K2$- (resp. $C3$-) strong pseudoprime to all the first $t$ prime bases. For this purpose we describe procedures for computing and enumerating the two kinds of spsp's < $10^{36}$ to the first 9 prime bases. The entire calculation took about 4000 hours on a PC Pentium IV/1.8GHz. (Recall that a $K2$-spsp is an spsp of the form: $n = pq$ with $p, q$ primes and $q - 1 = 2(p - 1)$; and that a $C3$-spsp is an spsp and a Carmichael number of the form: $n = q_1q_2q_3$ with each prime factor $q_i \equiv 3 \pmod{4}$.)

1. Introduction

Let $n > 1$ be an odd integer. Write $n - 1 = 2^sd$ with $d$ odd. We say that $n$ passes the Miller (strong probable prime) test [5] to base $b$, or that $n$ is an sprp($b$) for short, if

(1.1) either $b^d \equiv 1 \pmod{n}$ or $b^{2^rd} \equiv -1 \pmod{n}$ for some $r = 0, 1, \ldots, s - 1$.

(The original test of Miller [5] was somewhat more complicated and was a deterministic, ERH-based test; see [1, Section 3.4].) If $n$ is composite and (1.1) holds, then we say that $n$ is a strong pseudoprime to base $b$, or spsp($b$) for short. An spsp($b_1, \ldots, b_t$) is an spsp to all the $t$ bases. Define

(1.2) $SB(n) = \#\{b \in \mathbb{Z} : 1 \leq b \leq n - 1, n$ is an spsp($b$) $\}$ and $P_R(n) = \frac{SB(n)}{\phi(n)}$.
where \( \varphi \) is Euler's function. Monier \[6\] and Rabin \[8\] proved that if \( n \) is an odd composite positive integer, then \( SB(n) \leq (n-1)/4 \). In fact, as pointed out by Damgård, Landrock and Pomerance \[2\], if \( n \neq 9 \) is odd and composite, then \( SB(n) \leq \varphi(n)/4 \), i.e., \( P_{BL}(n) \leq 1/4 \). These facts lead to the Rabin-Miller test: given a positive integer \( n \), pick \( k \) different positive integers less than \( n \) and perform the Miller test on \( n \) for each of these bases; if \( n \) is composite, the probability that \( n \) passes all \( k \) tests is less than \( 1/4^k \).

Define \( \psi_t \) to be the smallest strong pseudoprime to all the first \( t \) prime bases. If \( n < \psi_t \), then only \( t \) Miller tests are needed to find out whether \( n \) is prime or not. This means that if we know the exact value of \( \psi_t \), then for integers \( n < \psi_t \) we will have a deterministic primality testing algorithm which is not only easier to implement but also faster than existing deterministic primality testing algorithms. From Pomerance et al. \[7\] and Jaeschke \[4\] we know the exact values of \( \psi_t \) for \( 1 \leq t \leq 8 \) and upper bounds for \( \psi_9 \), \( \psi_{10} \) and \( \psi_{11} \):

\[
\psi_9 \leq 41234 \ 31613 \ 57056 \ 89041 \ \text{(20 digits)}
= 4540612081 \cdot 9081224161,
\psi_{10} \leq 155 \ 33605 \ 66073 \ 14320 \ 55410 \ 02401 \ \text{(28 digits)}
= 22754930352733 \cdot 68264791058197,
\psi_{11} \leq 5689 \ 71935 \ 26942 \ 02437 \ 03269 \ 72321 \ \text{(29 digits)}
= 137716125329053 \cdot 413148375987157.
\]

Jaeschke \[4\] tabulated all strong pseudoprimes \( < 10^{12} \) to the bases 2, 3, and 5. There are in total 101 of them. Among these 101 numbers there are 95 numbers \( n \) having the form

\begin{equation}
(1.3) \quad n = pq \quad \text{with } p, q \text{ odd primes and } q - 1 = k(p - 1),
\end{equation}

with \( k \in \{2, 3, 4, 5, 6, 7, 13, 4/3, 5/2\} \); the other six numbers are Carmichael numbers with three prime factors in the sense that:

\begin{equation}
(1.4) \quad n = q_1q_2q_3 \quad \text{with } q_1 < q_2 < q_3 \text{ odd primes and each } q_i - 1 \mid n - 1.
\end{equation}

For short we call numbers (strong pseudoprimes) having the form (1.3) \( Kk \)-numbers (spsp’s), say, \( K2 \)-spsp’s if \( k = 2 \).

In \[9\], we used biquadratic residue characters and cubic residue characters as main tools to find all \( K2 \)-, \( K3 \)-, \( K4 \)-strong pseudoprimes \( < 10^{24} \) to the first nine or ten prime bases. As a result the upper bounds for \( \psi_{10} \) and \( \psi_{11} \) were considerably lowered:

\[
\psi_{10} \leq N_{10} = 19 \ 55097 \ 53037 \ 45565 \ 03981 \ \text{(22 digits)}
= 31265776261 \cdot 62531552521,
\psi_{11} \leq N_{11} = 73 \ 95010 \ 24079 \ 41207 \ 09381 \ \text{(22 digits)}
= 60807114061 \cdot 121614228121,
\]

and a 24-digit upper bound for \( \psi_{12} \) was obtained:

\[
\psi_{12} \leq N_{12} = 3186 \ 65857 \ 83403 \ 11511 \ 67461 \ \text{(24 digits)}
= 399165290221 \cdot 798330580441.
\]

In \[11\], we first followed our previous work \[9\] to find all \( K4/3 \)-, \( K5/2 \)-, \( K3/2 \)-, \( K6 \)-spsp’s \( < 10^{24} \) to the first several prime bases. No spsp’s of such forms to the first 8 prime bases are found. Note that the three bounds \( N_{10}, N_{11} \) and \( N_{12} \) are all
K2-spss’s with $P_R(n) = 3/16$. These facts give us a hint that to lower these upper bounds, we should find those numbers $n$ with $P_R(n)$ equal to or close to $1/4$.

For short, we call a Carmichael number $n = q_1q_2q_3$ with each prime factor $q_i \equiv 3 \mod 4$ a $K_2$-number. If $n$ is a $K_2$-number and an spsp($b_1, b_2, \ldots, b_t$), we call $n$ a $C_3$-spsp($b_1, b_2, \ldots, b_t$). It is easy to prove that (see [11 §5])

\begin{equation}
P_R(n) = 1/4 \iff \begin{array}{l}
either n = pq is a K_2-number with p \equiv 3 \mod 4 or n is a C_3-number; \\
\end{array}
\end{equation}

and that

\begin{equation}
if n is an spsp(2), then P_R(n) = 1/4 \iff n is a C_3-number.
\end{equation}

We [11] then focused our attention to develop a method for finding all $C_3$-spsp($2, 3, 5, 7, 11$)'s $< 10^{36}$. As a result the upper bounds for $\psi_9, \psi_{10}$ and $\psi_{11}$ are lowered from 20- and 22-decimal-digit numbers to a 19-decimal-digit number:

$\psi_9 \leq \psi_{10} \leq \psi_{11} \leq Q_{11} = 3825123056546413051 \ (19 \ digits) = 149491 \cdot 747451 \cdot 34233211$.

We [11] at last gave reasons to support the following Conjecture 1 (see also [3, Problem A12]).

**Conjecture 1.** We have

$\psi_9 = \psi_{10} = \psi_{11} = 3825123056546413051 \ (19 \ digits)$.

Let $q_1 < q_2 < q_3$ be odd primes and $N = q_1q_2q_3$. Put

$d = \gcd(q_1 - 1, q_2 - 1, q_3 - 1)$ and $h_i = \frac{2q_i - 1}{d}, \ i = 1, 2, 3$.

Then we call $d$ the kernel, the triple $(h_1, h_2, h_3)$ the signature, and $H = h_1h_2h_3$ the height of $N$, respectively. In [10, Section 2], we described a procedure for finding $C_3$-spsp($2$)'s, to a given limit, with heights bounded. There are in total 21978 $C_3$-spsp($2$)'s $< 10^{24}$ with heights $< 10^{9}$, only three of which are spsp’s to the first 11 prime bases up to 31. No $C_3$-spsp’s $< 10^{24}$ to the first 12 prime bases with heights $< 10^{9}$ were found.

Denote by $\psi'_t$ (resp. $\psi''_t$) the smallest K2- (resp. $C_3$-) spsp to all the first $t$ prime bases. In [10, §5], we gave reasons to support the following Conjecture 2.

**Conjecture 2.** We have

\begin{equation}
\psi_12 = \psi'_12 < 10^{24} < \psi''_12
\end{equation}

where

$\psi'_12 = N_{12} = 318665857834031151167461 \ (24 \ digits) = 399165290221 \cdot 798330580441$.

was found in [9] (see (1.5)).

The main purpose of this paper is to give reasons and numerical evidence of K2- and $C_3$- strong pseudoprimes $< 10^{36}$ to support the following Conjecture 3, where the exact values of $\psi'_t$ for $13 \leq t \leq 19$ and an upper bound of $\psi''_{20}$ are given in the following Proposition 1.1.

**Conjecture 3.** We have

$\psi_t = \psi'_t < \psi''_t$

for any $t \geq 12$. 

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Remark 1.1. Conjecture 3 covers Conjecture 2, which is the case \( t = 12 \).

**Proposition 1.1.** We have

\[
\psi'_{13} = 33170 44064 67988 73859 61981 \quad (25 \text{ digits})
\]

\[
= 1287836182261 \cdot 2575672364521;
\]

\[
\psi'_{14} = 600 39042 89670 10580 03125 96501 \quad (28 \text{ digits})
\]

\[
= 54786377365501 \cdot 1062198595387801;
\]

\[
\psi'_{15} = 5927 63610 75595 57326 34463 30101 \quad (29 \text{ digits})
\]

\[
= 172157429516701 \cdot 344314859033401;
\]

\[
\psi'_{16} = \psi'_{17} = 56413 29280 21909 22101 40875 01701 \quad (30 \text{ digits})
\]

\[
= 531099297693901 \cdot 1062198595387801;
\]

\[
\psi'_{18} = \psi'_{19} = 1543 26786 44434 20616 87767 76407 51301 \quad (34 \text{ digits})
\]

\[
= 27778299663977101 \cdot 55556599327954201;
\]

\[
\psi'_{20} > 10^{36}.
\]

In Section 2 we describe a procedure for finding all K2-spss \( < L = 10^{36} \) to the first \( t = 9 \) prime bases. There are in total 90002828 numbers, 100920 of which are spss to the first 13 prime bases. We tabulate 24 of them, which are spss to the first 18 prime bases up to 61, 4 of which are spss to the first 19 prime bases up to 67. No K2-spss \( < 10^{36} \) to the first 20 prime bases are found. Thus the 100920 numbers prove Proposition 1.1. In Section 3 we describe procedures for finding all \( C_3 \)-spss \( < 10^{36} \), to the first \( t = 9 \) prime bases, with heights \( < 10^{12} \). There are in total 43278 numbers. We tabulate 20 of them, which are spss to the first 15 prime bases up to 47, 2 numbers are spss to the first 16 prime bases up to 53. No \( C_3 \)-spss \( < 10^{36} \) to the first 17 prime bases with heights \( < 10^{12} \) are found. Moreover, no \( C_3 \)-spss \( < \psi'_t \) to the first \( t \) prime bases with heights \( < 10^{12} \) are found for \( t \geq 12 \). In Section 4 we reasonably predict that \( \psi'_t < \psi''_t \) for any \( t \geq 12 \). Since K2-spss’s and \( C_3 \)-spss’s have \( P_R(n) \) close to or equal to 1/4 (the upper bound of the probability of error for the Rabin-Miller test), Conjecture 3 would be most likely correct. The entire calculation for computing the two kinds of spss’s \( < 10^{36} \) took about 4000 hours on a PC Pentium IV/1.8GHz.

2. **K2-strong pseudoprimes up to \( 10^{36} \)**

Let \( \pi \) be a primary irreducible of the ring \( \mathbb{Z}[i] \) of Gaussian integers such that \( q = \pi \overline{\pi} \equiv 1 \pmod{4} \) and \( p = (q + 1)/2 \) are two primes determined by \( \pi \). Denote by \( \left( \frac{\alpha}{\pi} \right)_4 \) the biquadratic residue character symbol of \( \alpha \) modulo \( \pi \). Put \( p_\alpha = (\alpha \overline{\alpha} + 1)/2 \) for \( \alpha \in \mathbb{Z}[i] \). Let

\[
R_2 = \{ \text{primary } \alpha = x + yi : 0 \leq x, y < 8, i \overline{x} = (-1)^{\frac{x^2 - 1}{8}} \} = \{ 1, 5 + 4i \}.
\]

For a prime \( b \equiv 3 \pmod{4} \), let

\[
R_b = \left\{ \alpha = x + yi : 0 \leq x, y < 4b, \alpha \equiv 1 \pmod{4}, \left( \frac{\alpha}{b} \right)_4 = \left( \frac{p_\alpha}{b} \right) \right\};
\]

and for a prime \( b \equiv 1 \pmod{4} \), let

\[
R_b = \left\{ \alpha = x + yi : 0 \leq x, y < 4b, \alpha \equiv 1 \pmod{4}, \left( \frac{\alpha \overline{\alpha} - 1}{\beta} \right)_4 = \left( \frac{p_\alpha}{b} \right) \right\},
\]
where \( \left( \frac{a}{b} \right) \) is the Jacobi symbol. Let \( M^{(t)} = \prod_{j=1}^{t} b_j \), where \( b_j \) is the \( j \)th prime. Applying the Chinese Remainder Theorem, it is easy to compute the set
\[
R^{(t)} = \{ x + yi : 0 \leq x, y < M^{(t)}, x + yi \pmod{4b} \in R_b \text{ for all the first } t \text{ prime bases } b \}.
\]
In [9], we described a procedure to compute all K2-numbers \( n = pq \), below a given limit \( L \) (say \( 10^{24} \)), which are strong pseudoprimes to the first \( t \geq 6 \) prime bases. The procedure is based on the following proposition.

**Proposition 2.1 ([9] Proposition 3.2).** If \( n = pq \) is an spsp to the first \( t \) prime bases, then there exists \( \alpha \in R^{(t)} \) such that \( \pi \equiv \alpha \mod M^{(t)} \).

Since \( M^{(6)} = 4 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 120120 \) and \( \#R^{(6)} = 2 \cdot 2 \cdot 12 \cdot 30 \cdot 30 = 86400 \) are of suitable size for programming on a PC486 with Turbo Pascal 6.0, we successfully found all K2-strong pseudoprimes \(< 10^{24}\) to the first six prime bases.

Now our objective is to compute all K2-numbers \( n = pq \leq 10^{36} \) on a PC Pentium IV/1.8 GHz with Delphi 6.0, which are strong pseudoprimes to the first 9 prime bases. To speed things up, we should use a larger database \( R^{(t)} \). However, since
\[
\#R^{(9)} = \#R^{(6)} \cdot 56 \cdot 90 \cdot 132 = 4838400 \cdot 11880 = 57480192000,
\]
the set \( R^{(9)} \) is too large to fit in either memory or in a disk file. Considering the storage requirements and the efficiency of the algorithm, we pre-compute the set \( R^{(7)} \) and the set
\[
S = \{ x + yi : 0 \leq x, y < 4 \cdot 19 \cdot 23, x + yi \pmod{4b} \in R_b \text{ for } b = 19 \text{ and } 23 \}
\]
with
\[
M^{(7)} = M^{(6)} \cdot 17 = 2042040, \quad \#R^{(7)} = \#R^{(6)} \cdot 56 = 4838400
\]
and \( \#S = 90 \cdot 132 = 11880 \).

Now we are ready to describe a procedure to compute all K2-spsp’s \(< L = 10^{36}\), to the first \( t \geq 9 \) prime bases, with \( M^{(9)} = M^{(7)} \cdot 19 \cdot 23 = 892371480 \).

**PROCEDURE 2.1.** Finding K2-spsp to the first \( t \geq 9 \) prime bases;

**BEGIN**

For every \( x_1 + y_1i \in R^{(7)} \) Do

For every \( x_2 + y_2i \in S \) Do

begin Using the CRT, compute \( x \) and \( y \) such that
\[
0 \leq x, y < M^{(9)}, \quad x + yi \pmod{M^{(7)}} \in R^{(7)}
\]

and \( x + yi \pmod{4 \cdot 19 \cdot 23} \in S \);

For \( u \geq 0, v \geq 0, u + v \leq \frac{\sqrt{M^{(9)}}}{M^{(7)}} + 1 \) Do

Begin

\[
q \leftarrow (x + uM^{(9)})^2 + (y + vM^{(9)})^2; \quad p \leftarrow (q + 1)/2; \quad n \leftarrow p \cdot q;
\]

If \( n \) is an spsp to the first \( t \) prime bases Then output \( n, p \) and \( q \);

\[
q \leftarrow (x - uM^{(9)})^2 + (y + vM^{(9)})^2; \quad p \leftarrow (q + 1)/2; \quad n \leftarrow p \cdot q;
\]

If \( n \) is an spsp to the first \( t \) prime bases Then output \( n, p \) and \( q \)

End

end

**END.**

The Delphi-Pascal program (with multi-precision package partially written in Assembly language) ran about 2400 hours on a PC Pentium IV/1.8GHz (in fact we
used 10 PCs with each running 10 days) to get all K2-spsp’s < $10^{36}$ to the first nine prime bases 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are in total 90002828 numbers, 100920 of which are spsp’s to the first 13 prime bases, 24 numbers are spsp’s to the first 18 prime bases up to 61 (listed in Table 1), 4 numbers are spsp’s to the first 19 prime bases up to 67. No K2-spsp’s < $10^{36}$ to the first 20 prime bases are found. Thus, the 100920 numbers prove Proposition 1.1.

Table 1. List of all K2-spsp’s $n = p(2p - 1) < 10^{36}$ to the first 18 prime bases

<table>
<thead>
<tr>
<th>$n = p(2p - 1)$</th>
<th>$p$</th>
<th>spsp-base</th>
</tr>
</thead>
<tbody>
<tr>
<td>1543 26786 44344 20616 78767 76407 51301</td>
<td>27778299663977101</td>
<td>67 71 73</td>
</tr>
<tr>
<td>3573 96616 43156 88081 50998 90376 80081</td>
<td>422722672638961</td>
<td>0 0 1</td>
</tr>
<tr>
<td>3957 57033 03519 44936 54580 63660 50221</td>
<td>448337968286341</td>
<td>0 0 0</td>
</tr>
<tr>
<td>7134 11233 75303 11731 29413 76567 93141</td>
<td>60946766494438781</td>
<td>0 0 0</td>
</tr>
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<td>11068 90507 54608 90469 45172 83403 23501</td>
<td>74393990123724501</td>
<td>0 0 0</td>
</tr>
<tr>
<td>15837 76949 08904 27362 08053 96976 19981</td>
<td>8898811573988261</td>
<td>0 0 0</td>
</tr>
<tr>
<td>28474 73406 99486 58439 94849 77173 56181</td>
<td>11932043846288161</td>
<td>0 0 0</td>
</tr>
<tr>
<td>40367 25471 42188 26380 27978 16596 56101</td>
<td>142069093602758701</td>
<td>0 0 0</td>
</tr>
<tr>
<td>71361 96942 79138 04551 29896 91981 34381</td>
<td>188894109791589061</td>
<td>0 0 0</td>
</tr>
<tr>
<td>1 55559 41557 67059 71544 44226 02410 85421</td>
<td>278890135695676741</td>
<td>0 0 0</td>
</tr>
<tr>
<td>1 79574 77020 19886 65981 89846 22792 23261</td>
<td>2996453204572821</td>
<td>0 0 1</td>
</tr>
<tr>
<td>1 94375 57764 05156 30021 54227 05343 35741</td>
<td>311749561058644981</td>
<td>0 0 0</td>
</tr>
<tr>
<td>2 19337 01033 96078 99908 19416 68785 91181</td>
<td>331162354699026661</td>
<td>0 0 0</td>
</tr>
<tr>
<td>2 52100 58000 17252 74015 37676 90691 76741</td>
<td>35503617932711981</td>
<td>0 0 0</td>
</tr>
<tr>
<td>2 55173 55898 34500 77412 04551 29896 91981</td>
<td>357192916351549501</td>
<td>0 0 0</td>
</tr>
<tr>
<td>4 07387 35132 62980 70201 48352 55040 09841</td>
<td>45132437489321681</td>
<td>0 0 0</td>
</tr>
<tr>
<td>5 27946 43421 43471 71361 92967 10213 53301</td>
<td>51378329418311101</td>
<td>0 0 0</td>
</tr>
<tr>
<td>5 34925 55589 34500 77412 12805 09551 48501</td>
<td>537192916351549501</td>
<td>0 0 0</td>
</tr>
<tr>
<td>8 08460 12435 54390 22735 02512 76681 37421</td>
<td>635790895010080741</td>
<td>0 0 0</td>
</tr>
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<td>8 12968 03943 72552 74477 38266 69713 53901</td>
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</tr>
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<td>8 16345 93783 72388 10402 16654 72644 26261</td>
<td>63888159232813821</td>
<td>0 0 0</td>
</tr>
<tr>
<td>9 64006 87022 43616 26772 79703 83092 14301</td>
<td>694264672233998101</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>

Define the function

$$F(t, L) = \# \{ N : N \text{ is a K2-spsp } < L \text{ to the first } t \text{ prime bases} \}.$$ 

In Table 2 we give $F(t, L)$ for $9 \leq t \leq 19$ and $L = 10^{24}, \ldots, 10^{36}$.

Table 2. The function $F(t, L)$

<table>
<thead>
<tr>
<th>$t$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 10^{24}$</td>
<td>214</td>
<td>41</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L = 10^{28}$</td>
<td>15099</td>
<td>2680</td>
<td>551</td>
<td>105</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L = 10^{32}$</td>
<td>1146700</td>
<td>199736</td>
<td>38915</td>
<td>6913</td>
<td>1290</td>
<td>224</td>
<td>49</td>
<td>11</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L = 10^{36}$</td>
<td>90062828</td>
<td>15684487</td>
<td>3087051</td>
<td>546925</td>
<td>100920</td>
<td>18924</td>
<td>3778</td>
<td>664</td>
<td>128</td>
<td>24</td>
<td>4</td>
</tr>
</tbody>
</table>
3. \textsc{C}_3\text{-strong pseudoprimes} $< 10^{36}$ with heights $< 10^{12}$

The triple \((h_1, h_2, h_3)\) is called \(\text{C}_3\text{-spsp}(2)\)-\textit{acceptable} if the \(h_i\) are all odd positive integers, pairwise relatively prime, and \(h_1 \equiv h_2 \equiv h_3 \mod 4\). Let \(q_1 < q_2 < q_3\) be odd primes and \(N = q_1q_2q_3\) with kernel \(d\), signature \((h_1, h_2, h_3)\), and height \(H = h_1h_2h_3\). The kernel \(d\) is called \(\text{C}_3\text{-spsp}(2)\)-\textit{acceptable} if \((h_1, h_2, h_3)\) is \(\text{C}_3\text{-spsp}(2)\)-acceptable and

\[
d \equiv \overline{x_0} \mod 4H,
\]

where \(\overline{x_0} = x_0 + j_0H \equiv 2 \mod 4\), \(j_0 = (2 - x_0)H \mod 4\), \(0 \leq j_0 \leq 3\), \(x_0\) is the unique integer with \(0 \leq x_0 < H\) such that

\[
x_0 = \begin{cases} 
-h_{2,1} - h_{3,1} \mod h_1, \\
-h_{1,2} - h_{3,2} \mod h_2, \\
-h_{1,3} - h_{2,3} \mod h_3,
\end{cases}
\]

and \(h_{i,j} = h_{i}^{-1} \mod h_j\) for \(1 \leq i \neq j \leq 3\).

In [10] Section 2, we described a procedure for finding \(\text{C}_3\text{-spsp}(2)\)'s, to a given limit, with heights bounded. The method is based on the following Lemma 3.1.

\textbf{Lemma 3.1 (10, Theorem 2.1).} \(N = q_1q_2q_3\) be a product of three different odd primes. Then we have

\(N\) is a \(\text{C}_3\text{-spsp}(2)\) \iff its kernel \(d\) is \(\text{C}_3\text{-spsp}(2)\)-acceptable.

Let \(b_i\) be the \(i\)th prime, \(t \geq 2\) and \(M_t = 4b_2 \cdots b_t\), and suppose that \((h_1, h_2, h_3)\) is \(\text{C}_3\text{-spsp}(2)\)-acceptable. For an odd prime \(b\), define the set

\[
S_{\text{h}_i}^{(h_1,h_2,h_3)} = \left\{ u : u = 2 + 4k, 0 \leq k < b, \left(\frac{b}{uh_1+1}\right) = \left(\frac{b}{uh_2+1}\right) = \left(\frac{b}{uh_3+1}\right) \right\}.
\]

Suppose

\[
S_{\text{h}_i}^{(h_1,h_2,h_3)} \neq \emptyset
\]

for \(2 \leq i \leq t\). Define the set

\[
R_{\text{h}_i}^{(h_1,h_2,h_3)} = \left\{ r : 0 \leq r < M_t, r \equiv u_i \mod 4b_i \text{ for some } u_i \in S_{\text{h}_i}^{(h_1,h_2,h_3)}, 2 \leq i \leq t\right\}.
\]

The triple \((h_1, h_2, h_3)\) is called \(\text{C}_3\text{-spsp}(b_1, b_2, \ldots, b_t)\)-\textit{acceptable} if the system of linear congruences

\[
\begin{cases} 
x \equiv \overline{x_0} \mod 4H, \\
x \equiv u_i \mod 4b_i \text{ for some } u_i \in S_{\text{h}_i}^{(h_1,h_2,h_3)}, 2 \leq i \leq t,
\end{cases}
\]

has solutions, or in other words, the system

\[
\begin{cases} 
x \equiv \overline{x_0} \mod 4H, \\
x \equiv r \mod M_t \text{ for some } r \in R_{\text{h}_i}^{(h_1,h_2,h_3)}
\end{cases}
\]

has solutions. The kernel \(d\) is called \(\text{C}_3\text{-spsp}(b_1, b_2, \ldots, b_t)\)-\textit{acceptable} if \((h_1, h_2, h_3)\) is \(\text{C}_3\text{-spsp}(b_1, b_2, \ldots, b_t)\)-acceptable and \((3.3)\) holds with \(x\) replaced by \(d\).

In [10] Section 4, we speeded up the procedure described in [10] Section 2 so that we can find all \(\text{C}_3\text{-spsp}\)'s less than a larger limit \(L = 10^{50}\), with the same signature \((1, 37, 41)\), to the first \(t \geq 9\) prime bases. The accelerated procedure is based on the following Lemma 3.2.
Lemma 3.2 ([10 Corollary 4.1]). Let $N = q_1 q_2 q_3$ be a product of three different odd primes and let $b_i$ be the $i$th prime and $t \geq 2$; and suppose $(h_1, h_2, h_3)$ is $C_3$-spsp\(^{2}\)-acceptable. Then $N$ is a $C_3$-spsp$(b_1, b_2, \ldots, b_t)$ if and only if its kernel $d$ is $C_3$-spsp$(b_1, b_2, \ldots, h_1)$-acceptable.

Now our objective is to find all $N = q_1 q_2 q_3 \leq L = 10^{36}$ which are $C_3$-spsp’s to the first $t = 9$ prime bases with heights $H < 10^{12}$. For this purpose, we use Procedure 3.1 (based on Lemma 3.1) for finding all $N$ with heights $10^9 < H < 10^{12}$; and use Procedure 3.2 (based on Lemma 3.2) for finding all $N$ with heights $H < 10^9$.

**PROCEDURE 3.1.** Finding $C_3$-spsp$(b_1, \ldots, b_t)$’s $< L$ with $10^9 < H < 10^{12}$;

BEGIN For each $C_3$-spsp\(^{2}\)$-acceptable triple $(h_1, h_2, h_3)$ with $10^9 < H = h_1 h_2 h_3 < 10^{12}$ Do
begin Using the Euclidean Algorithm and the Chinese Remainder Theorem, compute the seed $x_0$ of the triple $(h_1, h_2, h_3)$:
\[ x_0 \leftarrow x_0; \quad j_0 \leftarrow (6 - x_0 \mod 4) H \mod 4; \quad \text{If} \quad j_0 > 0 \quad \text{Then} \quad x_0 \leftarrow x_0 + j_0 H; \]

For $i := 1$ To 3 Do $q_i \leftarrow x_0 h_i + 1$; $q_1 q_2 \leftarrow q_1 \cdot q_2$; $N \leftarrow q_1 q_2 \cdot q_3$;

If $N < L$ Then
begin If $2^N \equiv 2 \mod q_1 q_2$ Then
begin If $(q_1, q_2$ and $q_3$ are all spsp’s to the first several prime bases) And $(N$ is an spsp$(b_1, \ldots, b_t))$ Then
output($N, q_1, q_2, q_3, h_1, h_2, h_3, \ldots$)
end;
End
end

END.

Remark 3.1. Procedure 3.1 is in fact the same but simpler than the original procedure described in [10 Section 2] for finding all $C_3$-spsp\(^{2}\)$’s, but we save only those $C_3$-spsp’s to the first nine prime bases. Note that, “simpler” means that in the sentence “If $N < L$ Then Begin ... End” there is no loop “repeat ... until $N > L$”, since $H > 10^9$ and $N < 10^{36}$.

**Remark 3.2.** Using Procedure 3.1 for finding all $N$ with heights $10^9 < H < 10^{12}$, we use 10 PCs. We divide the computations into ten parts with each on one PC: $10^9 < H < 10^{11}$, $j \cdot 10^{11} < H < (j + 1) \cdot 10^{11}$, $1 \leq j \leq 9$. For each part $H_0 \leq H = h_1 h_2 h_3 \leq H_1$, we loop on $C_3$-spsp\(^{2}\)$-acceptable triples $(h_1, h_2, h_3)$ with
\[
1 \leq h_1 \leq \left\lfloor \frac{\ln (H_1)}{\ln (h_1)} \right\rfloor, \quad h_1 + 4 \leq h_2 \leq \left\lfloor \frac{H_1}{h_1} \right\rfloor,
\]
and
\[
\max \left\{ h_2, \frac{H_0}{h_1 h_2} \right\} < h_3 \leq \min \left\{ \frac{H_1}{h_1 h_2}, \frac{1}{8} \left( h_1 + h_2 + \sqrt{(h_1 + h_2)^2 + 8 h_1 h_2 \sqrt{L}} \right) \right\}.
\]

Remark 3.3. Procedure 3.2 is an extended version of the accelerated procedure which was mentioned (but not explicitly written) and which ran only for the triple $(1, 37, 41)$ in [10 Section 4], whereas Procedure 3.2 loops on all $C_3$-acceptable triples $(h_1, h_2, h_3)$ with $h_1 h_2 h_3 < 10^9$.

**PROCEDURE 3.2.** Finding $C_3$-spsp$(b_1, \ldots, b_t)$’s $< L$ with heights $H < 10^9$;

BEGIN For each $C_3$-spsp\(^{2}\)$-acceptable triple $(h_1, h_2, h_3)$
with \( H = h_1h_2h_3 < 10^9 \)

**Do**

**begin** Using the Euclidean Algorithm and the Chinese Remainder Theorem, compute the seed \( x_0 \) of the triple \((h_1, h_2, h_3)\):

\[
\overline{x_0} \leftarrow x_0; \quad j_0 \leftarrow (6 - x_0 \mod 4)H \mod 4; \quad \text{If } j_0 > 0 \text{ Then } \overline{x_0} \leftarrow x_0 + j_0H;
\]

Compute the sets \( S_{b_i}^{(h_1, h_2, h_3)} \) for \( 2 \leq i \leq t \);

Using the Chinese Remainder Theorem, compute the set \( R_t^{(h_1, h_2, h_3)} \);

**For** each element \( r \) of the set \( R_t^{(h_1, h_2, h_3)} \)

**Begin** Using the Chinese Remainder Theorem, find a solution \( x < L = \text{lcm}[4H, M_t] \) to the system of congruence

\[
\begin{align*}
x & \equiv \overline{x_0} \mod 4H, \\
x & \equiv r \mod M_t;
\end{align*}
\]

**For** \( i := 1 \text{ To } 3 \)

\[
q_i \leftarrow xh_i + 1; \quad q_1q_2 \leftarrow q_1 \cdot q_2; \quad N \leftarrow q_1q_2 \cdot q_3;
\]

**If** \( N < L \text{ Then}

**repeat** If \( 2^N \equiv 2 \mod q_1q_2 \) **Then**

**begin** If \((q_1, q_2)\) are spsp’s to the first several prime bases and \((N)\text{ is an spsp}(b_1, \ldots, b_t)\) **Then**

output \((N, q_1, q_2, q_3, h_1, h_2, h_3, \ldots)\)

**end:**

**For** \( i := 1 \text{ To } 3 \)

\[
q_i \leftarrow q_i + h_iL; \\
q_1q_2 \leftarrow q_1 \cdot q_2; \quad N \leftarrow q_1q_2 \cdot q_3
\]

**until** \( N > L \)

**End**

**End.**

The two Delphi-Pascal programs with multi-precision package partially written in Assembly language ran about 1600 hours in total on a PC Pentium IV/1.8GHz to get all \( C_3\)-spsp’s \(< 10^{36} \) to the first nine prime bases 2, 3, 5, 7, 11, 13, 17, 19, and 23, with heights \( H < 10^{12} \). There are in total 43278 numbers, among which 20 numbers are spsp’s to the first 15 prime bases up to 47 (listed in Table 3), 2 numbers are spsp’s to the first 16 prime bases up to 53. No \( C_3\)-spsp’s \(< 10^{36} \) to the first 17 prime bases with heights \(< 10^{12} \) are found. Define the function

\[
(3.4) \quad f(t, L, \mathcal{H}) = \# \{ N : N \text{ is a } C_3\text{-spsp}(b_1, b_2, \ldots, b_t) < L \text{ with height } < \mathcal{H} \}.
\]

In Table 4 we give \( f(t, L, 10^{12}) \) for \( 9 \leq t \leq 16 \) and \( L = 10^{24}, \ldots, 10^{36} \). In Table 5 we give \( f(t, 10^{36}, \mathcal{H}) \) for \( 9 \leq t \leq 16 \) and \( \mathcal{H} = 10^2, 10^3, \ldots, 10^{12} \). Note that \( f(t, 10^{36}, 10^{12}) = 0 \) for \( t > 16 \).

**Remark 3.4.** Procedure 3.2 ran only 11 hours on a PC Pentium IV/1.8GHz for computing all \( C_3\)-spsp\((b_1, b_2, \ldots, b_t)\)’s \(< 10^{36} \) with heights \( H < 10^9 \). Note that, since

\[
\max \left\{ \#R_9^{(h_1, h_2, h_3)} : (h_1, h_2, h_3) \text{ is } C_3\text{-spsp}(b_1, b_2, \ldots, b_9)\text{-acceptable} \right. \\
\left. \quad \text{with } H = h_1h_2h_3 < 10^9 \right\} = 129600,
\]

each \( R_9^{(h_1, h_2, h_3)} \) is of suitable size to fit in the memory of a PC Pentium IV/1.8GHz.
Table 3. List of all $C_3$-spsp’s $n = q_1q_2q_3 < 10^{36}$ to the first 15 prime bases with signature $(h_1, h_2, h_3)$ and height $H = h_1h_2h_3 < 10^{12}$

<table>
<thead>
<tr>
<th>$n = q_1q_2q_3$</th>
<th>$q_1$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>spsp-base</th>
</tr>
</thead>
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<tr>
<td>168 79087 75236 76911 80019 24541 71451</td>
<td>168</td>
<td>1</td>
<td>29</td>
<td>57</td>
<td>0 1 0</td>
</tr>
<tr>
<td>404 00641 91965 17352 34018 69881 7634</td>
<td>404</td>
<td>17</td>
<td>21</td>
<td>9183</td>
<td>0 0 1</td>
</tr>
<tr>
<td>638 44429 45452 74988 12832 64086 78791</td>
<td>638</td>
<td>1</td>
<td>29</td>
<td>31713</td>
<td>0 0 1</td>
</tr>
<tr>
<td>7386 49195 93228 96977 41258 18632 67131</td>
<td>7386</td>
<td>1</td>
<td>13</td>
<td>2233</td>
<td>1 0 0</td>
</tr>
<tr>
<td>14987 06615 24763 89521 37359 79490 57667</td>
<td>14987</td>
<td>1</td>
<td>17</td>
<td>21</td>
<td>0 0 0</td>
</tr>
<tr>
<td>22072 88440 42828 12126 78317 84805 82551</td>
<td>22072</td>
<td>1</td>
<td>5</td>
<td>29</td>
<td>0 0 1</td>
</tr>
<tr>
<td>22237 51676 18766 76325 87593 37145 05851</td>
<td>22237</td>
<td>5</td>
<td>13</td>
<td>57</td>
<td>0 1 0</td>
</tr>
<tr>
<td>22535 13758 55715 82393 57550 88590 41539</td>
<td>22535</td>
<td>1</td>
<td>341</td>
<td>4641</td>
<td>1 0 0</td>
</tr>
<tr>
<td>33197 80568 91606 49280 97700 67802 95751</td>
<td>33197</td>
<td>1</td>
<td>5</td>
<td>89</td>
<td>0 0 0</td>
</tr>
<tr>
<td>73956 31848 86374 65060 47922 10797 09511</td>
<td>73956</td>
<td>5</td>
<td>261</td>
<td>61621</td>
<td>0 0 1</td>
</tr>
<tr>
<td>84699 94086 53768 76461 36135 59745 27971</td>
<td>84699</td>
<td>3</td>
<td>23</td>
<td>247</td>
<td>0 1 0</td>
</tr>
<tr>
<td>86114 94622 43439 91256 73057 32901 32851</td>
<td>86114</td>
<td>41</td>
<td>69</td>
<td>145</td>
<td>0 1 0</td>
</tr>
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<td>1 57909 67938 52590 51550 19158 97075 64171</td>
<td>1 57909</td>
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<td>165</td>
<td>281</td>
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</tr>
<tr>
<td>1 64290 81802 11124 38861 65231 15234 05671</td>
<td>1 64290</td>
<td>3</td>
<td>11</td>
<td>19</td>
<td>0 0 1</td>
</tr>
<tr>
<td>3 29688 82238 37447 72420 05277 30066 75151</td>
<td>3 29688</td>
<td>1</td>
<td>5</td>
<td>472269</td>
<td>0 0 1</td>
</tr>
<tr>
<td>3 81295 37619 60634 49827 51838 84361 05491</td>
<td>3 81295</td>
<td>1</td>
<td>5</td>
<td>41</td>
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<tr>
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<td>4 88964</td>
<td>1</td>
<td>5</td>
<td>69</td>
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<td>31</td>
<td>391</td>
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</tr>
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<td>6 78033</td>
<td>3</td>
<td>23</td>
<td>175</td>
<td>0 0 1</td>
</tr>
</tbody>
</table>

Table 4. The function $f(t, L, 10^{12})$

<table>
<thead>
<tr>
<th>$t$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 10^{24}$</td>
<td>21</td>
<td>8</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L = 10^{28}$</td>
<td>217</td>
<td>70</td>
<td>20</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L = 10^{32}$</td>
<td>2808</td>
<td>821</td>
<td>249</td>
<td>70</td>
<td>17</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L = 10^{36}$</td>
<td>43278</td>
<td>12623</td>
<td>3655</td>
<td>1019</td>
<td>271</td>
<td>80</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 5. The function $f(t, 10^{36}, \mathcal{H})$

<table>
<thead>
<tr>
<th>$\mathcal{H}$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H} = 10^2$</td>
<td>894</td>
<td>211</td>
<td>51</td>
<td>16</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^3$</td>
<td>11592</td>
<td>3225</td>
<td>814</td>
<td>221</td>
<td>56</td>
<td>15</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^4$</td>
<td>22249</td>
<td>6367</td>
<td>1717</td>
<td>482</td>
<td>125</td>
<td>30</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^5$</td>
<td>30385</td>
<td>8811</td>
<td>2531</td>
<td>710</td>
<td>176</td>
<td>42</td>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^6$</td>
<td>35526</td>
<td>10377</td>
<td>2995</td>
<td>846</td>
<td>216</td>
<td>57</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^7$</td>
<td>38811</td>
<td>11344</td>
<td>3287</td>
<td>922</td>
<td>238</td>
<td>69</td>
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<td>2</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^8$</td>
<td>40872</td>
<td>11915</td>
<td>3454</td>
<td>966</td>
<td>253</td>
<td>76</td>
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<td>2</td>
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<td>$\mathcal{H} = 10^9$</td>
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<td>12270</td>
<td>3548</td>
<td>989</td>
<td>264</td>
<td>79</td>
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<td>2</td>
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<tr>
<td>$\mathcal{H} = 10^{10}$</td>
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<td>1003</td>
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<td>2</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{11}$</td>
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<td>12567</td>
<td>3637</td>
<td>1011</td>
<td>270</td>
<td>80</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{12}$</td>
<td>43278</td>
<td>12623</td>
<td>3655</td>
<td>1019</td>
<td>271</td>
<td>80</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

4. Discussion

In Tables 6, 7, 8, 9, and 10 we give $f(t, \psi_u^t, \mathcal{H})$ for $u = 13, 14, 15, 16$ and 18; and for $t \geq 9, \mathcal{H} \leq 10^{12}$. In these tables, if $f(t, \psi_u^t, \mathcal{H}) > 0$ for $\mathcal{H} = 10^{12}$ and for $t \geq t_0$, then the columns for $t \geq t_0$ are all deleted since all entries are 0.
Table 6. The function $f(t, \psi'_1, \mathcal{H})$

<table>
<thead>
<tr>
<th>$t$</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H} = 10^1$</td>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^2$</td>
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<td>4</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^3$</td>
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<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^4$</td>
<td>25</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^5$</td>
<td>26</td>
<td>12</td>
<td>5</td>
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<tr>
<td>$\mathcal{H} = 10^6$</td>
<td>27</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^7$</td>
<td>28</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^8$</td>
<td>28</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^9$</td>
<td>28</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{10}$</td>
<td>28</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{11}$</td>
<td>28</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{12}$</td>
<td>28</td>
<td>13</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 7. The function $f(t, \psi'_1, \mathcal{H})$

<table>
<thead>
<tr>
<th>$t$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H} = 10^2$</td>
<td>4</td>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^3$</td>
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<td>15</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^4$</td>
<td>95</td>
<td>37</td>
<td>9</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^5$</td>
<td>129</td>
<td>44</td>
<td>12</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^6$</td>
<td>150</td>
<td>52</td>
<td>13</td>
<td>2</td>
<td>0</td>
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<tr>
<td>$\mathcal{H} = 10^7$</td>
<td>161</td>
<td>56</td>
<td>14</td>
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<td>0</td>
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<tr>
<td>$\mathcal{H} = 10^8$</td>
<td>172</td>
<td>58</td>
<td>14</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^9$</td>
<td>179</td>
<td>60</td>
<td>15</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{10}$</td>
<td>180</td>
<td>60</td>
<td>15</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{11}$</td>
<td>180</td>
<td>60</td>
<td>15</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{12}$</td>
<td>182</td>
<td>61</td>
<td>15</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8. The function $f(t, \psi'_1, \mathcal{H})$

<table>
<thead>
<tr>
<th>$t$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H} = 10^2$</td>
<td>9</td>
<td>4</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>$\mathcal{H} = 10^3$</td>
<td>91</td>
<td>26</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^4$</td>
<td>180</td>
<td>60</td>
<td>15</td>
<td>2</td>
<td>0</td>
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<tr>
<td>$\mathcal{H} = 10^5$</td>
<td>245</td>
<td>71</td>
<td>21</td>
<td>5</td>
<td>0</td>
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<tr>
<td>$\mathcal{H} = 10^6$</td>
<td>283</td>
<td>83</td>
<td>25</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^7$</td>
<td>311</td>
<td>93</td>
<td>26</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^8$</td>
<td>336</td>
<td>97</td>
<td>27</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^9$</td>
<td>344</td>
<td>99</td>
<td>28</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{10}$</td>
<td>348</td>
<td>101</td>
<td>28</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{11}$</td>
<td>348</td>
<td>101</td>
<td>28</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{H} = 10^{12}$</td>
<td>350</td>
<td>102</td>
<td>28</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
From Table 6 we may predict that $f(13, \psi'_{13}, \mathcal{H}) = 0$ for $\mathcal{H} > 10^{12}$ and so that

\begin{equation}
\psi'_{13} < \psi''_{13}.
\end{equation}

From Table 7 we may predict that $f(14, \psi'_{14}, \mathcal{H}) = 0$ for $\mathcal{H} > 10^{12}$ and so that

\begin{equation}
\psi'_{14} < \psi''_{14}.
\end{equation}

From Table 8 we may predict that $f(15, \psi'_{15}, \mathcal{H}) = 0$ for $\mathcal{H} > 10^{12}$ and so that

\begin{equation}
\psi'_{15} < \psi''_{15}.
\end{equation}

From Table 9 we may predict that $f(16, \psi'_{16}, \mathcal{H}) = 0$ for $\mathcal{H} > 10^{12}$ and so that

\begin{equation}
\psi'_{16} = \psi'_{17} < \psi''_{16} \leq \psi''_{17}.
\end{equation}

From Table 10 we may predict that $f(18, \psi'_{18}, \mathcal{H}) = 0$ for $\mathcal{H} > 10^{12}$ and so that

\begin{equation}
\psi'_{18} < \psi''_{18} \leq \psi''_{19}.
\end{equation}

\begin{table}[h]
\centering
\caption{The function $f(t, \psi'_{16}, \mathcal{H})$}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{\mathcal{H}} & 9 & 10 & 11 & 12 & 13 \\
\hline
\hline
$\mathcal{H} = 10^2$ & 11 & 4 & 0 & 0 & 0 & 0 \\
\hline
$\mathcal{H} = 10^3$ & 165 & 50 & 13 & 4 & 1 & 0 \\
\hline
$\mathcal{H} = 10^4$ & 332 & 106 & 30 & 7 & 2 & 0 \\
\hline
$\mathcal{H} = 10^5$ & 452 & 128 & 42 & 12 & 2 & 0 \\
\hline
$\mathcal{H} = 10^6$ & 519 & 146 & 46 & 12 & 2 & 0 \\
\hline
$\mathcal{H} = 10^7$ & 570 & 164 & 50 & 14 & 2 & 0 \\
\hline
$\mathcal{H} = 10^8$ & 605 & 169 & 51 & 14 & 2 & 0 \\
\hline
$\mathcal{H} = 10^9$ & 618 & 175 & 52 & 15 & 3 & 0 \\
\hline
$\mathcal{H} = 10^{10}$ & 622 & 177 & 52 & 15 & 3 & 0 \\
\hline
$\mathcal{H} = 10^{11}$ & 625 & 177 & 52 & 15 & 3 & 0 \\
\hline
$\mathcal{H} = 10^{12}$ & 630 & 179 & 53 & 15 & 3 & 0 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{The function $f(t, \psi'_{18}, \mathcal{H})$}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{\mathcal{H}} & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
\hline
$\mathcal{H} = 10^2$ & 133 & 31 & 8 & 3 & 0 & 0 & 0 & 0 & 0 \\
\hline
$\mathcal{H} = 10^3$ & 1641 & 438 & 104 & 29 & 6 & 0 & 0 & 0 & 0 \\
\hline
$\mathcal{H} = 10^4$ & 3212 & 900 & 251 & 75 & 19 & 4 & 1 & 0 & 0 \\
\hline
$\mathcal{H} = 10^5$ & 4424 & 1242 & 370 & 110 & 26 & 6 & 1 & 0 & 0 \\
\hline
$\mathcal{H} = 10^6$ & 5207 & 1505 & 449 & 133 & 30 & 9 & 2 & 0 & 0 \\
\hline
$\mathcal{H} = 10^7$ & 5714 & 1658 & 491 & 142 & 33 & 11 & 3 & 0 & 0 \\
\hline
$\mathcal{H} = 10^8$ & 6036 & 1745 & 519 & 150 & 35 & 13 & 3 & 0 & 0 \\
\hline
$\mathcal{H} = 10^9$ & 6211 & 1797 & 527 & 153 & 38 & 14 & 3 & 0 & 0 \\
\hline
$\mathcal{H} = 10^{10}$ & 6309 & 1832 & 544 & 157 & 39 & 15 & 3 & 0 & 0 \\
\hline
$\mathcal{H} = 10^{11}$ & 6352 & 1842 & 547 & 158 & 39 & 15 & 3 & 0 & 0 \\
\hline
$\mathcal{H} = 10^{12}$ & 6384 & 1853 & 549 & 159 & 39 & 15 & 3 & 0 & 0 \\
\hline
\end{tabular}
\end{table}
Since the conditions for a number to be a \( C_3 \)-spsp are more stringent than those for it to be a K2-spsp, K2-spsp’s are much more numerous than \( C_3 \)-spsp’s as can be seen from Tables 2 and 4. Thus, combining (4.1)-(4.5) and (1.8), it is reasonable to predict that

\[ \psi'_t < \psi''_t \]

for any \( t \geq 12 \).

ACKNOWLEDGMENT

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REFERENCES