ZEROS OF THE DAVENPORT-HEILBRONN COUNTEREXAMPLE

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Abstract. We compute zeros off the critical line of a Dirichlet series considered by H. Davenport and H. Heilbronn. This computation is accomplished by deforming a Dirichlet series with a set of known zeros into the Davenport-Heilbronn series.

1. Introduction

Let \( \xi = (\sqrt{10} - 2\sqrt{5} - 2)/(\sqrt{5} - 1) \). For \( s = \sigma + it \) with \( \sigma > 1 \), let

\[
 f_1(s) = 1 + \frac{\xi}{2^s} - \frac{\xi}{3^s} - \frac{1}{4^s} + \frac{0}{5^s} + \cdots
\]

be a Dirichlet series with periodic coefficients of period 5. Then \( f_1(s) \) defines an entire function satisfying the following functional equation

\[
 f(s) = T^{-s+\frac{1}{2}} \chi_2(s) f(1-s)
\]

with \( T = 5 \) and

\[
 \chi_2(s) = 2(2\pi)^{s-1} \Gamma(1-s) \cos\left(\frac{\pi s}{2}\right).
\]

In 1936, H. Davenport and H. Heilbronn (see [2]) showed that \( f_1(s) \), as defined in (1), has zeros off the critical line \( \sigma = 1/2 \). In 1994, R. Spira (see [3]) computed the following zeros of the Davenport-Heilbronn example:

\[
 \begin{align*}
 .808517 + 85.699348i, & \quad .650830 + 114.163343i, \\
 .574356 + 166.479306i, & \quad .724258 + 176.702461i.
\end{align*}
\]

In this note we present a scheme for computing additional zeros of the Davenport-Heilbronn Dirichlet series.

2. Continuity of zeros

In order to compute zeros of \( f_1 \), let us consider

\[
 f_0(s) = \left(1 + \frac{\sqrt{5}}{5^s}\right) \zeta(s),
\]

where \( \zeta(s) \) is the Riemann zeta function. A good number of zeros of \( \zeta(s) \) have been computed and are readily available. On the other hand

\[
 1 + \frac{\sqrt{5}}{5^s} = 0 \quad \text{if and only if} \quad s = \frac{1}{2} + \frac{2k+1}{\log 5} \pi i \quad \text{with} \quad k \in \mathbb{Z}.
\]

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For each $\tau \in [0, 1]$, let
\begin{equation}
\label{eq:5}
f_{\tau} = f_0 \cdot (1 - \tau) + f_1 \cdot \tau.
\end{equation}
The next theorem shows that if $\rho_0$ is a zero of $f_0$, and $\tau > 0$ is small, then $f_{\tau}$ has a zero $\rho_{\tau}$ in a small neighborhood of $\rho_0$. Beginning with $\rho_0$ as initial data, it is easy (provided $\tau > 0$ is small) to numerically compute a zero $\rho_{\tau}$ of $f_{\tau}$ in a small neighborhood of $\rho_0$. Repeating this process a number of times, we end up with a zero $\rho_1$ of the Davenport-Heilbronn series $f_1$.

**Theorem 1.** Let $q$ be a fixed positive integer. Let $a_{\tau}(j): \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{C}$ be a sequence of continuous functions such that $a_{\tau}(j + q) = a_{\tau}(j)$ for all $\tau \in \mathbb{R}$ and all $j \in \mathbb{N}$. Let $f_{\tau}(s)$ be the meromorphic function, defined initially for $\sigma > 1$ by
\begin{equation}
\label{eq:6}
f_{\tau}(s) = \sum_{j=1}^{\infty} \frac{a_{\tau}(j)}{j^s},
\end{equation}
and extended to the whole complex plane by analytic continuation. Let $\rho$ be such that $0 < \Re(\rho) < 1$ and $f_0(\rho) = 0$. If $\delta > 0$ and $\tau \in \mathbb{R}$ are sufficiently small, then there exists $s$ such that $f_{\tau}(s) = 0$ and $|s - \rho| < \delta$.

Thus, our scheme of computation of zeros of $f_1$ is to keep track of zeros of $f_0$ while performing a 'deformation' of $f_0$ into $f_1$. By keeping track of the first known zeros of $f_0$ as defined in (4), we found the following additional zeros of the Davenport-Heilbronn Dirichlet series $f_1$ defined in (1):

\begin{align*}
.86953 + 240.4046i, & \quad .81955 + 320.8764i, & \quad .76822 + 331.0502i, \\
.62850 + 366.6409i, & \quad .81587 + 411.7967i, & \quad .70882 + 440.4845i, \\
.51591 + 520.9438i, & \quad .84695 + 531.2797i, & \quad .72953 + 548.9067i, \\
.78655 + 566.5097i, & \quad .58285 + 595.0233i, & \quad .62825 + 611.7750i, \\
.61076 + 646.9868i, & \quad .76059 + 657.1083i, & \quad .78870 + 692.8924i, \\
.77736 + 737.7669i, & \quad .85300 + 783.6530i, & \quad .66855 + 811.7657i, \\
.56194 + 847.4657i, & \quad .85610 + 857.2958i, & \quad .68089 + 864.1180i, \\
.68843 + 892.1490i, & \quad .75935 + 921.1726i, & \quad .76249 + 983.7521i, \\
.69140 + 1012.019i, & \quad .69809 + 1018.795i, & \quad .58613 + 1029.004i, \\
.61106 + 1078.490i, & \quad .85462 + 1092.454i, & \quad .60577 + 1109.548i.
\end{align*}

3. Two hypotheses

In this section, we consider the case in which $f_0$ and $f_1$ are two linearly independent Dirichlet series satisfying a given functional equation. For example we might take $f_0$ to be as in equation (4) above, or we might take $f_0$ to be
\begin{equation}
\label{eq:7}
L(s, \chi_2^{(5)}) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{0}{5^s} + \cdots.
\end{equation}
Notice that $f_0$ as given in (4) and $L(s, \chi_2^{(5)})$ both satisfy the functional equation
\begin{equation}
\label{eq:8}
f(s) = T^{-s+\frac{1}{2}} \chi_1(s) f(1-s)
\end{equation}
with $T = 5$ and
\begin{equation}
\label{eq:9}
\chi_1(s) = 2(2\pi)^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).
\end{equation}
Thus we have two linearly independent Dirichlet series satisfying a given functional equation. Hence, by taking appropriate linear combinations of these, we can produce a Dirichlet series $f_1$ satisfying the above functional equation and having a zero off the critical line, in fact, at any preassigned place in the complex plane. With these $f_0$ and $f_1$, we let $f_\tau$ be as in (5). Then $f_\tau$ satisfies the functional equation (6) for all $\tau \in [0, 1]$.

Since $f_\tau$ satisfies (6), then its nonreal zeros lie symmetrically about the critical line $\sigma = 1/2$. Hence, by Theorem 1 in section §2, a simple zero must move along the critical line. If we assume that all zeros of $f_0$ are simple, how then might we obtain any zero of $f_1$ lying off the critical line? It is easy to see that there must exist $0 \leq \tau^* < 1$ such that $f_{\tau^*}$ has a zero in the critical line with an even multiplicity.

Loosely speaking, we might say that zeros of multiplicity greater than one must exist before the Riemann hypothesis fails.

4. Other periodic Dirichlet series

The Dirichlet series considered by Davenport and Heilbronn satisfies a functional equation akin to the functional equation satisfied by the Riemann zeta function. Moreover, this Dirichlet series of Davenport and Heilbronn is the unique solution of its functional equation. It is natural to consider all those periodic Dirichlet series arising as the unique solution to a fixed functional equation of the type satisfied by the Riemann zeta function. The following result will help us determine all such Dirichlet series; see [1].

**Theorem 2.** Let $f(s)$ be a $T$-periodic Dirichlet series. Let $\chi_1(s)$ and $\chi_2(s)$ be as in (7) and (3) respectively. Let

$$V_{\alpha,\beta} = \left\{ f(s) : f(s) = (-1)^\alpha T^{s+\frac{1}{2}} \chi_\beta(s) f(1-s) \right\}.$$ 

Let $d_1 = \dim V_{0,1}$, $d_2 = \dim V_{1,1}$, $d_3 = \dim V_{0,2}$, and $d_4 = \dim V_{1,2}$, where $\dim V_{\alpha,\beta}$ is the dimension of $V_{\alpha,\beta}$ as a vector space. Then $d_j$ is given by the following table:

<table>
<thead>
<tr>
<th>$T$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4m$</td>
<td>$m+1$</td>
<td>$m$</td>
<td>$m$</td>
<td>$m-1$</td>
</tr>
<tr>
<td>$4m+1$</td>
<td>$m+1$</td>
<td>$m$</td>
<td>$m$</td>
<td></td>
</tr>
<tr>
<td>$4m+2$</td>
<td>$m+1$</td>
<td>$m+1$</td>
<td>$m$</td>
<td></td>
</tr>
<tr>
<td>$4m+3$</td>
<td>$m+1$</td>
<td>$m+1$</td>
<td>$m+1$</td>
<td>$m$</td>
</tr>
</tbody>
</table>

Thus, for $T = 2$,

$$\zeta(s) + \frac{\sqrt{2}}{2\pi} \zeta(s), \quad \zeta(s) - \frac{\sqrt{2}}{2\pi} \zeta(s)$$

is the list of all 2-periodic Dirichlet series which are the unique solution of a functional equation.

For $T = 3$,

$$L(s, \chi_1^{(3)}), \quad \frac{\sqrt{3}}{3\pi} \zeta(s), \quad \frac{\sqrt{3}}{3\pi} \zeta(s)$$

is the list of all 3-periodic Dirichlet series which are the unique solution of a functional equation. Here $\chi_1^{(3)}$ is the nonprincipal character modulo 3.
For $T = 4$,
\[ L(s, \chi_4^{(4)}), \quad (1 - \frac{1}{4^s})\zeta(s) \]

is the list of all 4-periodic Dirichlet series which are the unique solution of a functional equation. Here $\chi_4^{(4)}$ is the nonprincipal character modulo 4.

For $T = 5$,
\[ (1 - \frac{\sqrt{5}}{\xi^s})\zeta(s), \quad f_1(s), \quad f_2(s) = 1 - \frac{1/\xi}{2^s} + \frac{1/\xi}{3^s} - \frac{1}{4^s} + \frac{0}{5^s} + \cdots \]

is the list of all 5-periodic Dirichlet series which are the unique solution of a functional equation. Here $f_1$ is the Dirichlet series of Davenport and Heilbronn, and the constant $\xi$ in the definition of $f_2$ is defined in the Introduction, §1.

For $T = 6$,
\[ (1 - \frac{1 - \sqrt{2}}{1 + 2^s})L(s, \chi_6^{(6)}), \quad (1 - \frac{1 + \sqrt{2}}{1 + 2^s})L(s, \chi_6^{(6)}) \]

is the list of all 6-periodic Dirichlet series which are the unique solution of a functional equation. Here $\chi_6^{(6)}$ is the nonprincipal character modulo 6.

For $T = 7$,
\[ f_3(s) = 1 - \frac{1 + \alpha}{2^s} - \frac{\alpha}{3^s} + \frac{\alpha}{4^s} + \frac{1 + \alpha}{5^s} - \frac{1}{6^s} + \frac{0}{7^s} + \cdots \]

is the only 7-periodic Dirichlet series which is the unique solution of a functional equation. Here $\alpha = 0.80194 \cdots$.

For $T = 8$,
\[ (1 - \frac{\sqrt{2}}{2^s})L(s, \chi_8^{(8)}) \]

is the only 8-periodic Dirichlet series which is the unique solution of a functional equation. Here $\chi_8^{(8)}(3) = -1, \chi_8^{(8)}(5) = 1, \chi_8^{(8)}(7) = -1$ is a Dirichlet character.

Of all these periodic Dirichlet series arising as the unique solution of a functional equation, only three are not Euler products. These three series are $f_1$ as given in (1), $f_2$ as given in (8) and $f_3$ as given in (9).

Now we list a few zeros off the critical line of $f_2$ as given in (8). Notice that most of these zeros have real part greater than 1:

\[
\begin{align*}
2.30862 + 8.91836i, & \quad 1.94374 + 18.8994i, & \quad 2.09106 + 26.5450i, \\
2.15626 + 36.5556i, & \quad 1.50497 + 44.8057i, & \quad 2.33262 + 54.4201i, \\
1.78509 + 64.3711i, & \quad 2.17279 + 72.0637i, & \quad 0.69340 + 77.3469i, \\
2.05503 + 82.0598i, & \quad 1.83279 + 89.9631i, & \quad 2.34551 + 99.8614i, \\
1.18952 + 107.106i, & \quad 1.33795 + 109.439i, & \quad 2.22293 + 117.572i, \\
\end{align*}
\]

Finally, we list a few zeros off the critical line of $f_3$ as given in (9). Notice that most of these zeros have real part greater than 1:

\[
\begin{align*}
1.34746 + 17.5286i, & \quad 1.06162 + 28.4426i, & \quad 1.30492 + 45.5320i, \\
1.01460 + 56.2793i, & \quad 0.91718 + 63.7111i, & \quad 1.33196 + 80.3522i, \\
1.22180 + 91.1756i, & \quad 1.22009 + 108.402i, & \quad 0.92165 + 119.323i, \\
1.28500 + 126.482i, & \quad 1.08964 + 137.285i, & \quad 0.91608 + 143.175i, \\
0.78002 + 146.163i, & \quad 1.28909 + 154.268i, & \quad 0.65384 + 161.521i, \\
\end{align*}
\]
5. Proof of Theorem 1

In order to prove the theorem in §2, we let

\[ A_\tau = \frac{1}{q} \sum_{j=1}^{q} a_\tau(j) \quad \text{and} \quad A_\tau(x) = \sum_{j \leq x} a_\tau(j). \]

For \( \sigma > 0 \) we have

\[ f_{\tau}(s) = \frac{A_\tau s}{s-1} + s \int_{1}^{\infty} \frac{A_\tau(x) - A_\tau x}{x^{s+1}} dx. \]

Assume \( \rho \in \mathbb{C} \) is such that \( 0 < \Re(\rho) < 1 \) and \( f_{0}(\rho) = 0 \). There exist \( \delta > 0 \) such that \( f_{0}(s) \) does not vanish for \( 0 < |s - \rho| \leq \delta \). Let

\[ \epsilon = \min \{ |f_{0}(s)| : |s - \rho| = \delta \}. \]

Since

\[ |A_\tau(x) - A_\tau x - A_0(x) + A_0 x| \leq 2 \sum_{j=1}^{q} |a_\tau(j) - a_0(j)| \]

and \( a_\tau(j): \mathbb{R} \times \mathbb{N} \to \mathbb{C} \) are continuous functions of \( \tau \), then it follows that

\[ |f_{\tau}(s) - f_{0}(s)| \leq \left( \frac{1}{q} \left| \frac{s}{s-1} \right| + 2 \left| \frac{s}{\sigma} \right| \right) \sum_{j=1}^{q} |a_\tau(j) - a_0(j)| < \epsilon \]

provided \( |\tau| \) and \( \delta \) are sufficiently small and \( |s - \rho| = \delta \). By Rouche’s theorem

\[ f_{\tau}(s) = f_{0}(s) + \left\{ f_{\tau}(s) - f_{0}(s) \right\} \]

vanishes for some \( s \) such that \( |s - \rho| < \delta \). \( \square \)

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References


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