THE EVALUATION OF \( \kappa_3 \)

J. M. CHICK AND G. H. DAVIES

Abstract. Numerical evidence relevant to the evaluation of the constant \( \kappa_3 \) in the conjectural distribution of three-prime Carmichael numbers of Granville and Pomerance (2001) is summarised.

1. Introduction

Let \( C_3(X) \) be the number of three-prime Carmichael numbers up to \( X \). In §8 of [2] Granville and Pomerance conjecture that

\[
C_3(X) \sim \tau_3 \frac{X^\frac{4}{3}}{(\log X)^3} \sim \frac{\tau_3}{27} \int_2^X \frac{dt}{(\log t)^3},
\]

where \( \tau_3 := \kappa_3 \lambda \), with \( \lambda := \frac{243}{2} \prod_{p > 3} \left(1 - \frac{1}{p^3}\right) \approx 77.1727 \) and

\[
\kappa_3 := \sum_{n \geq 1} \frac{(n,6)}{n^\frac{4}{3}} \prod_{p \mid n, p > 3} \frac{\rho}{p - 3} \sum_{a < b < c, n = abc, a, b, c \text{ pairwise coprime}} \delta_3(a, b, c) \prod_{p > 3, \text{p prime}} \frac{p - \omega_{a,b,c}(p)}{p - 3}
\]

where \( \delta_3(a, b, c) = 2 \) if \( a + b + c \equiv 0 \pmod{3} \) and 1 otherwise and \( \omega_{a,b,c}(p) \) is the number of distinct residues modulo \( p \) represented by \( a, b, c \).

The infinite series for \( \kappa_3 \) converges exceedingly slowly, and Carl Pomerance invited the authors to attempt a better evaluation than that in the first preprint of [2]. This paper is a brief account of our computational work on this problem.

2. Algorithm and Implementation

Let \( u_n \) be the general term of the above series for \( \kappa_3 \), write \( \kappa := \kappa_3 = \sum_{n=1}^{\infty} u_n \), \( \kappa(N) := \sum_{n=1}^{N} u_n \), and suppose \( n = \prod_{j=1}^{k} p_j^{a_j} \) is the prime factorisation of \( n \), and \( q_j := p_j^{a_j} \). Clearly, if \( k = 1 \), then \( u_n = 0 \), and it is easy to show that the number of terms in the summation for \( u_n \) over \( (a, b, c) \) triples is \( t_k := \frac{3^{k-1} - 1}{2} \). For each \( n \) to find the set \( S_k \) of all possible coprime triples \( (a, b, c) \) we used the following recursion: start with \( S_2 := \{(1, q_1, q_2)\} \), and if \( S_j := \bigcup_{i=1}^{j} \{(a_{ij}, b_{ij}, c_{ij})\}, \) then \( S_{j+1} = \left( \bigcup_{i=1}^{j} \{(a_{ij}q_{j+1}, b_{ij}, c_{ij}), (a_{ij}, b_{ij}q_{j+1}, c_{ij}), (a_{ij}, b_{ij}, c_{ij}q_{j+1})\} \right) \cup \{(1, q_{j+1}, \prod_{i=1}^{j} q_i)\} \)

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until \( S_k \) is formed. (In general \( a < b < c \) will not hold, but this is irrelevant for evaluation.)

For each \((a, b, c)\), to evaluate the final infinite product in the expression for \( u_n \) we observe that when \( \omega_{a,b,c}(p) = 3 \) the corresponding factor in the product is 1, and that those \( p \) for which \( \omega_{a,b,c}(p) = 1 \) or 2 are the prime factors of \(|a - b|, |b - c|\) or \(|c - a|\), so the product is essentially finite; thus we found such \( p \) and \( \omega_{a,b,c}(p) \), checking that \( p \nmid n \) and \( p > 3 \).

To minimise the effect of rounding errors when adding large numbers of small terms we used an array to retain all decimal digits of the computation (so for \( \kappa(10^7) \), for example, we had 18 decimal places of which the first 7 or 8 might reasonably be expected to be correct).

Accordingly a program was devised in BASIC V. Using a RISC PC, running RISC OS 3.7 with 16 MB of RAM, \( \kappa(10^7) \) was found and, discrepancies having been resolved, Granville and Pomerance encouraged us to advance. After several hundred hours of computing \( \kappa(10^8) \) was reached: towards the end, each million increase in \( N \) took about 16 hours.

3. Statistics and evaluation

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \kappa(N) )</th>
<th>( \kappa(N) )</th>
<th>( \kappa(N) )</th>
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<tr>
<td>10</td>
<td>0.782401</td>
<td>7 \times 10^7</td>
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</tr>
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<td>( 10^2 )</td>
<td>5.354251</td>
<td>10^7</td>
<td>25.666883</td>
</tr>
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<td>16.736742</td>
<td>3 \times 10^7</td>
<td>25.709247</td>
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<td>( 10^5 )</td>
<td>20.593777</td>
<td>4 \times 10^7</td>
<td>25.728184</td>
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<tr>
<td>( 10^6 )</td>
<td>23.169099</td>
<td>5 \times 10^7</td>
<td>25.745867</td>
</tr>
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<td>2.5 \times 10^6</td>
<td>23.908847</td>
<td>6 \times 10^7</td>
<td>25.762436</td>
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</table>

In §8 of [2] Granville and Pomerance give a heuristic argument to justify \( \kappa = \kappa(N) + (\alpha + o(1)) \int_N^\infty \frac{(\log t)^2}{t^\frac{3}{2}} \, dt \), where \( \alpha \) is a constant. We find

\[
\int_N^\infty \frac{(\log t)^2}{t^\frac{3}{2}} \, dt = f(N) := \frac{3}{\sqrt{N}} \{ (\log N)^2 + 6 \log N + 18 \},
\]

so for given \( N \) we may regard this relationship as an equation in \( \kappa \) and \( \alpha \) with known coefficients, but for which the constant \( \kappa(N) + o(1)f(N) \) is not known precisely. If now we suppose that, for largish \( N \), the \( o(1) \) in the formula is negligible and may be discarded, then from two values \( N_i, N_j \) of \( N \) we can solve simultaneously to obtain a heuristic estimate for \( \kappa \). Writing \( \kappa(i, j) \) for the estimate thus obtained from \( N_i = 10^i \) and \( N_j = 10^j \) we get

\[
\kappa(i, j) = \frac{\kappa(N_i)\kappa(N_j) - \kappa(N_i)}{f(N_j) - f(N_i)}.
\]

Table 2 shows some values of \( \kappa(i, j) \).
Ultra cautious extrapolation from Tables 1 and 2 would seem to justify $26 < \kappa < 27.09$, and various empirical approaches with no theoretical justification suggest a value near the upper end of this interval. For example, if we write $\kappa := \kappa(2\mu - 3, 2\mu - 2)$, $\delta_\mu := k_\mu - k_{\mu+1}$ and $\rho_\mu := \delta_{\mu+1}/\delta_\mu$, Table 1 enables us to calculate $k_1 = \kappa(0.25, 0.5) = 27.16985$, $k_2 = 27.14451$, $k_3 = 27.12610$, $k_4 = 27.11024$, $k_5 = 27.09862$; $\delta_1 = 0.02434$, $\delta_2 = 0.01941$, $\delta_3 = 0.01527$, $\delta_4 = 0.01221$; and $\rho_1 = 0.7975$, $\rho_2 = 0.7867$. We also pursued an empirical method which utilised our categorisation of terms $u_n$ according to the number of consecutive zeros after the decimal point before the first non-zero digit, which we needed to retain all decimal digits as mentioned above. Thus we found $s_i := \sum_{10^{-i-1} < u_i \leq 10^{-i}} u_n$ for $0 < i < 6$; for $i = 4$, $s_4$ was complete with $n < 10^8$, but to find $s_5$ and $s_6$ we had to implement programs to find $u_n$, with $n > 10^8$ and $6 \leq k = \omega(n) \leq 9$ up to various upper bounds for $n$, depending on $k$. We note *en passant* that forming the sum of $\kappa(10^k)$ and the extra terms thus found with $n > 10^8$ belonging to $s_5$, $s_6$ and $s_7$ gave a total of 25.817473, so certainly $\kappa$ is greater than this. Then with $\alpha_i := \frac{s_i}{s_{i-1}}$ we get $s_0 = 4.952693$, $s_1 = 5.992839$, $s_2 = 4.965006$, $s_3 = 3.727265$, $s_4 = 2.586746$, $s_5 = 1.737140$, $s_6 = 1.138759$ and $\alpha_3 = 0.828498$, $\alpha_4 = 0.750707$, $\alpha_5 = 0.694006$, $\alpha_6 = 0.655536$. If now we *arbitrarily* assume that $\alpha_i$ continues to decrease as $i$ increases, comparison with the appropriate GP’s gives $\kappa < K_i := \sum_{j=0}^{i-2} s_j + \frac{s_{i-1}}{1 - \alpha_i}$, $K_3 = 30.8619$, $K_4 = 28.0914$, $K_5 = 27.5135$, $K_6 = 27.2676$ and $K_7$ is a decreasing sequence with $K_\infty = \kappa$.

A similar empirical approach based on $S_i := \kappa(10^i)$, $\Delta_i := S_{i+1} - S_i$ and $r_i := \Delta_{i+1}/\Delta_i$ yields (assuming decreasing $r_i$) another decreasing sequence $K^*_i := S_i + \frac{\Delta_i}{1-r_i}$ with $K^*_3 = 31.1398$, $K^*_4 = 28.3437$, $K^*_5 = 27.5245$, $K^*_6 = 27.2397$ and $K^*_\infty = \kappa$.

These numerical discussions suggest sequences of approximations decreasing to $\kappa$. Even with $\kappa = \kappa_3$ at the top end of our above ultra cautious extrapolation range, we get $\gamma_3 = 2090$ rather than 2100, which is the 2-significant figure value taken by Granville and Pomerance in [2] based on the numerical evidence then available. Our best provisional estimate, $\kappa = 27.05$ (see above), gives $\gamma_3 = 2087.5$. In [2] they define $\gamma := \gamma(X) = C_3(X) / \int_2^X \frac{dt}{(\log t)^3}$ and state that $\gamma \rightarrow \tau_3$ from below as $X \rightarrow \infty$. Table 3 below (from Table 1 of [3]) shows $\gamma$ approaching 2087.5 rather rapidly as $N$ increases, where $X = N^{10^4}$.
Table 3.

<table>
<thead>
<tr>
<th>( N )</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
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</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>1839</td>
<td>1899</td>
<td>1947</td>
<td>1984</td>
<td>2019</td>
<td>2047</td>
<td>2067</td>
<td>2081</td>
</tr>
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</table>

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