MAXIMUM OF THE MODULUS OF KERNELS IN GAUSS-TURÁN QUADRATURES

GRADIMIR V. MILOVANOVIĆ, MIODRA M. SPALEVIĆ, AND MIROSLAV S. PRANIĆ

Abstract. We study the kernels $K_{n,s}(z)$ in the remainder terms $R_{n,s}(f)$ of the Gauss-Turán quadrature formulae for analytic functions on elliptical contours with foci at ±1, when the weight $ω$ is a generalized Chebyshev weight function. For the generalized Chebyshev weight of the first (third) kind, it is shown that the modulus of the kernel $|K_{n,s}(z)|$ attains its maximum on the real axis (positive real semi-axis) for each $n \geq n_0$, $n_0 = n_0(\rho, s)$. It was stated as a conjecture in [Math. Comp. 72 (2003), 1855–1872]. For the generalized Chebyshev weight of the second kind, in the case when the number of the nodes $n$ in the corresponding Gauss-Turán quadrature formula is even, it is shown that the modulus of the kernel attains its maximum on the imaginary axis for each $n \geq n_0$, $n_0 = n_0(\rho, s)$. Numerical examples are included.

1. Introduction

We consider the Gauss-Turán quadrature formula with multiple nodes

$$\int_{-1}^{1} f(t) \omega(t) dt = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_{\nu}) + R_{n,s}(f) \quad (n \in \mathbb{N}; \ s \in \mathbb{N}_0)$$

where $\omega$ is a nonnegative and integrable function on the interval $(-1, 1)$, which is exact for all algebraic polynomials of degree at most $(2s + 1)n - 1$. The nodes $\tau_{\nu}$ in (1.1) must be zeros of the $s$-orthogonal polynomials with respect to the weight function $\omega(t)$. The $s$-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $\omega(t)$ are polynomials which satisfy the following orthogonality conditions:

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k \omega(t) dt = 0, \quad k = 0, 1, \ldots, n - 1.$$

Numerically stable methods for constructing nodes $\tau_{\nu}$ and coefficients $A_{i,\nu}$ can be found in [8, 10, 13]. For more details on quadrature formulae with multiple nodes see [7] and [9].

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Let $\Gamma$ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let $D$ be its interior. If the integrand $f$ is analytic on $D$ and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation (see [14, 11])

\[
R_{n,s}(f) = \frac{1}{2\pi i} \oint_{\Gamma} K_{n,s}(z)f(z)dz.
\]

The kernel is given by $K_{n,s}(z) = \rho_{n,s}(z)/[\pi_{n,s}(z)]^{2s+1}$, $z \notin [-1, 1]$, where

\[
\rho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t} \omega(t)dt.
\]

The modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(\pm z)| = |K_{n,s}(z)|$. If the weight function in (1.1) is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\overline{z})| = |K_{n,s}(z)|$ (see [11] Lemma 2.1).

A particularly interesting case is the Chebyshev weight $\omega_1(t) = (1 - t^2)^{-1/2}$. In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $T_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

\[
\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}} dt \quad (k \geq 0).
\]

This means that the Chebyshev polynomials $T_n$ are $s$-orthogonal on $(-1, 1)$ for each $s \geq 0$. Ossicini and Rosati [14] found three other weights $\omega_k(t)$ ($k = 2, 3, 4$) for which the $s$-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third and fourth kind: $U_n$, $V_n$, and $W_n$, which are defined by

\[
U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad V_n(\cos \theta) = \frac{\cos(n + \frac{1}{2})\theta}{\cos \frac{1}{2}\theta}, \quad W_n(\cos \theta) = \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta},
\]

respectively (cf. Gautschi and Notaris [4]). However, these weights depend on $s$,

\[
\omega_2(t) = (1 - t^2)^{1/2+s}, \quad \omega_3(t) = \frac{(1 + t)^{1/2+s}}{(1 - t)^{1/2}}, \quad \omega_4(t) = \frac{(1 - t)^{1/2+s}}{(1 + t)^{1/2}}.
\]

It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that in the investigation it is sufficient to study only the first three generalized Chebyshev weights $\omega_k(t)$, $k = 1, 2, 3$.

The integral representation (1.2) leads directly to the error estimate

\[
|R_{n,s}| \leq \frac{l(\Gamma)}{2\pi} \left( \max_{z \in \Gamma} |K_{n,s}(z)| \right) \left( \max_{z \in \Gamma} |f(z)| \right),
\]

where $l(\Gamma)$ denotes the length of the contour $\Gamma$. First maximum depends only on the quadrature rule (i.e., on $\omega$) and not on $f$. The first unified approach described above was taken by Donaldson and Elliot [2]. They applied it to several kinds of interpolatory and non-interpolatory quadrature rules. Error bounds for Gaussian quadratures of analytic functions were studied by Gautschi and Varga [5] (see also [6]), and later by Schira [15, 16], Hunter and Nikolov [8].

As a contour $\Gamma$ we take an ellipse $E_{\rho}$ with foci at points $\pm 1$ and a sum of semi-axes $\rho > 1$,

\[
E_{\rho} = \left\{ z \in \mathbb{C} : \; z = \frac{1}{2} (u + u^{-1}), \quad 0 \leq \theta \leq 2\pi \right\}, \quad u = \rho e^{i\theta}.
\]
When \( \rho \to 1 \), the ellipse shrinks to the interval \([-1, 1]\), while with increasing \( \rho \) it becomes more and more circle-like.

When \( \omega \) is the generalized Chebyshev weight of the first (third) kind, it is conjectured, on the basis of numerical experiments (see [11]), that the modulus of the kernel attains its maximum on the real axis (positive real semi-axis) for each \( n \geq n_0, \; n_0 = n_0(\rho, s) \).

In this paper we prove those conjectures. Moreover, for the generalized Chebyshev weight of the second kind, in the case when the number of the nodes \( n \) in the corresponding Gauss-Turán quadrature formula is even, we show that the modulus of the kernel attains its maximum on the imaginary axis for each \( n \geq n_0, \; n_0 = n_0(\rho, s) \). Numerical examples are included.

2. The maximum modulus of the kernel on confocal ellipses

We study the magnitude of \(|K_{n,s}(z)|\) on the contour \( E_\rho \) for the generalized Chebyshev weight functions of the first, second and third kind, respectively. The particular case \(|K_{n,0}(z)|\) was analyzed in details by Gautschi et al. [5, 6].

2.1. The weight function \( \omega_1(t) = (1 - t^2)^{-1/2} \). An explicit representation of the kernel \( K_{n,s}^{(1)}(z) \) on the ellipse \( E_\rho \) for the weight function \( \omega_1(t) \) was given by Milovanović and Spalević in [11], as well as

\[
|K_{n,s}^{(1)}(z)| = \frac{2^{1-s} \pi}{\rho^n} \frac{|Z_{n,s}^{(1)}(\rho \e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}}, \quad z \in E_\rho,
\]

where

\[
a_j = a_j(\rho) = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j \in \mathbb{N},
\]

and

\[
Z_{n,s}^{(1)}(u) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} u^{-2nk} = \sum_{k=0}^{s} \frac{(2s+1)_{s}}{k} u^{-2n(s-k)}.
\]

The weight function \( \omega_1(t) \) is even, so we can take \( \theta \in [0, \pi/2] \).

The following result was conjectured in [11]:

**Theorem 2.1.** For each fixed \( \rho > 1 \) and \( s \in \mathbb{N}_0 \) there exists \( n_0 = n_0(\rho, s) \) such that

\[
\max_{z \in E_\rho} |K_{n,s}^{(1)}(z)| = K_{n,s}^{(1)} \left( \frac{1}{2}(\rho + \rho^{-1}) \right)
\]

for each \( n \geq n_0 \).

**Proof.** The inequality \(|Z_{n,s}^{(1)}(\rho \e^{i\theta})| \leq Z_{n,s}^{(1)}(\rho)\) immediately follows from (2.3). Because of that and (2.4), it is sufficient to prove

\[
\frac{1}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}} \leq \frac{1}{(a_2 - 1)^{1/2}(a_{2n} + 1)^{1/2+s}}
\]

for a sufficiently large \( n \) (\( n \geq n_0(\rho, s) \)) and \( \theta \in (0, \pi/2] \), where \( a_j \) are given by (2.2).

By squaring (2.4) it is reduced to

\[
(a_2 - 1)(a_{2n} + 1)^{2s+1} \leq (a_2 - \cos 2\theta)(a_{2n} + \cos 2n\theta)^{2s+1}.
\]

The following transformation will be used

\[
a_2 - \cos 2\theta = (a_2 - 1) + 2 \sin^2 \theta.
\]
Further, we will use
\[(a_{2n} + \cos 2n\theta)^{2s+1} = (a_{2n} + 1 - 2\sin^2 n\theta)^{2s+1},\]
\[= (a_{2n} + 1)^{2s+1} + \sum_{k=1}^{2s+1} (-2)^k \binom{2s+1}{k} (a_{2n} + 1)^{2s+1-k} \sin^{2k} n\theta,\]
i.e.,
\[(2.7) \quad (a_{2n} + \cos 2n\theta)^{2s+1} = (a_{2n} + 1)^{2s+1} - 2(\sin^2 n\theta)E_{\rho,s}(n, \theta),\]
where
\[E_{\rho,s}(n, \theta) = \sum_{k=1}^{2s+1} (-2)^{k-1} \binom{2s+1}{k} (a_{2n} + 1)^{2s+1-k} \sin^{2k} n\theta \quad (\geq 0).\]

It is easy to see that \(E_{\rho,s}(n, \theta)\) can be represented in the form
\[E_{\rho,s}(n, \theta) = (2s+1)(a_{2n} + 1)^{2s} + \sum_{k=2}^{2s+1} (-2)^{k-1} \binom{2s+1}{k} (a_{2n} + 1)^{2s-k} \sin^{2k} n\theta,\]
i.e.,
\[(2.8) \quad E_{\rho,s}(n, \theta) = (2s+1)(a_{2n} + 1)^{2s} - \sum_{k=1}^{s} 2^{2k-1} \binom{2s+1}{2k} (a_{2n} + 1)^{2s-2k} \sin^{4k} n\theta + \sum_{k=1}^{s} 2^{2k} \binom{2s+1}{2k+1} (a_{2n} + 1)^{2s-2k} \sin^{4k} n\theta.\]

Using (2.6) and (2.7), the inequality (2.5) is reduced to
\[(a_{2n} + 1 - 1)(a_{2n} + 1)^{2s+1} \leq \left[ (a_{2n} + 1 - 2\sin^2 \theta) \left[ (a_{2n} + 1)^{2s+1} - 2(\sin^2 n\theta)E_{\rho,s}(n, \theta) \right] \right],\]
i.e.,
\[2\sin^2 \theta (a_{2n} + 1)^{2s+1} - 2\sin^2 n\theta \left[ (a_{2n} + 1 - 2\sin^2 \theta)E_{\rho,s}(n, \theta) \right] \geq 0.\]

Dividing this inequality by \(2\sin^2 \theta\), it becomes
\[(2.9) \quad (a_{2n} + 1)^{2s+1} - \frac{\sin^2 n\theta}{\sin^2 \theta} \left[ (a_{2n} + 1 - 2\sin^2 \theta)E_{\rho,s}(n, \theta) \right] \geq 0.\]

By using the well-known fact \(|\sin n\theta| / \sin \theta| \leq n\), it is easy to see that
\[(2.10) \quad \frac{\sin^2 n\theta}{\sin^2 \theta} \left[ (a_{2n} + 1 + 2\sin^2 \theta)E_{\rho,s}(n, \theta) \right] \leq 2\sin^2 n\theta \leq (a_{2n} - 1)n^2 + 2.\]

According to (2.8), we conclude that
\[E_{\rho,s}(n, \theta) - (2s+1)(a_{2n} + 1)^{2s} = \sum_{k=1}^{s} \frac{4^k(2s+1)!}{(2k)!(2s-2k)!} (a_{2n} + 1)^{2s-2k} \times \frac{\sin^2 n\theta}{2k+1} - \frac{a_{2n} + 1}{2(2s - 2k + 1)} \sin^{4k-2} n\theta.\]

Since \(\sin^{4k-2} n\theta \leq 1\) and
\[
\frac{\sin^2 n\theta}{2k+1} - \frac{a_{2n} + 1}{2(2s - 2k + 1)} \leq \frac{1}{2k+1} - \frac{a_{2n} + 1}{2(2s - 2k + 1)},
\]
from the previous equality we obtain

\[ E_{\rho,s}(n,\theta) - (2s+1)(a_{2n}+1)^{2s} \leq \sum_{k=0}^{s} 4^k(a_{2n}+1)^{2s-2k} \left( \frac{2s+1}{2k+1} - \frac{a_{2n}+1}{2} \left( \frac{2s+1}{2k} \right) \right). \]

Therefore,

\[ E_{\rho,s}(n,\theta) \leq \sum_{k=0}^{s} 4^k \left( \frac{2s+1}{2k+1} \right) (a_{2n}+1)^{2s-2k} - \frac{1}{2} \sum_{k=1}^{s} 4^k \left( \frac{2s+1}{2k} \right) (a_{2n}+1)^{2s-2k} + 1. \]

Using the last inequality and (2.11), we conclude that the left-hand side of (2.9) is greater than or equal to \( F(n) \equiv F_{\rho,s}(n) \), where

\[ F_{\rho,s}(n) := (a_{2n}+1)^{2s+1} - [(a_{2} - 1)n^2 + 2] \times \left[ \sum_{k=0}^{s} 4^k \left( \frac{2s+1}{2k+1} \right) (a_{2n}+1)^{2s-2k} - \frac{1}{2} \sum_{k=1}^{s} 4^k \left( \frac{2s+1}{2k} \right) (a_{2n}+1)^{2s-2k} + 1 \right]. \]

Since \( F_{\rho,s}(n) \) is continuous on \( \mathbb{R} \) and \( \lim_{n \to +\infty} F_{\rho,s}(n) = +\infty \), it follows that \( F_{\rho,s}(n) > 0 \), for each \( n > t \), where \( t \) is the largest zero of \( F_{\rho,s}(n) \). For \( n_0 \) we can take \( \lfloor t \rfloor + 1 \).

Theorem 2.1 is not only of theoretical, but also of practical importance. We can use the function \( F(n) \) from the proof to estimate \( n_0 \). Numerical values of \( \lfloor t \rfloor + 1 \) (\( t \) is the largest zero of \( F \)) for some values of \( \rho \) and \( s \) are presented in Tables 1 and 2. The smallest possible (s.p.) values of \( n_0 \) are also presented. We can see that the smallest possible \( n_0 \) is estimated by \( \lfloor t \rfloor + 1 \) very well.

A typical graph illustrating the relationship between \( n \) and \( F(n) \) is given in Figure 1. Here, \( \rho = 1.05 \), \( s = 1 \); \( n \in [1,42] \).

### Table 1. The smallest possible (s.p.) values of \( n_0 \) and their approximations \( \lfloor t \rfloor + 1 \) (\( t \) is the largest zero of \( F \))

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \rho = 1.05 )</th>
<th>( \rho = 1.3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lfloor t \rfloor + 1 )</td>
<td>the s.p. ( n_0 )</td>
<td>( \lfloor t \rfloor + 1 )</td>
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<tr>
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<tr>
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<td>50</td>
<td>46</td>
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<td>3</td>
<td>56</td>
<td>53</td>
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<tr>
<td>4</td>
<td>59</td>
<td>57</td>
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<td>9</td>
<td>70</td>
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</tr>
</tbody>
</table>

The proof of Theorem 2.1 is not only of theoretical, but also of practical importance. We can use the function \( F(n) \) from the proof to estimate \( n_0 \). Numerical values of \( \lfloor t \rfloor + 1 \) (\( t \) is the largest zero of \( F \)) for some values of \( \rho \) and \( s \) are presented in Tables 1 and 2. The smallest possible (s.p.) values of \( n_0 \) are also presented. We can see that the smallest possible \( n_0 \) is estimated by \( \lfloor t \rfloor + 1 \) very well.

### 2.2. The weight function \( \omega_2(t) = (1 - t^2)^{s+1/2} \), \( s \in N_0 \)

An explicit representation of the kernel \( K_{n,s}^{(2)}(z) \) on the ellipse \( \mathcal{E}_\rho \) for the weight function \( \omega_2(t) \) was given in (11), as well as

\[ |K_{n,s}^{(2)}(z)| = \frac{\pi}{4^s \rho^{n+1}} \left[ \frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos (2n+2)\theta} \right]^{s+1/2} |Z_{n,s}^{(2)}(pe^{i\theta})|, \]
Table 2. The smallest possible (s.p.) values of $n_0$ and their approximations $|t| + 1$ ($t$ is the largest zero of $F$)

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$s = 1$</th>
<th>$s = 5$</th>
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</tr>
<tr>
<td>2.0</td>
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<td>3</td>
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</table>

Figure 1. The typical graph of $F(n)$.

where

$$Z_{n,s}(\rho e^{i\theta}) = \sum_{k=0}^{s} (-1)^{k} \binom{2s + 1}{s + k + 1} (\rho e^{i\theta})^{-2(n+1)k}. \tag{2.12}$$

There we proved the following statement:

**Theorem 2.2.** If $\omega_2(t) = (1 - t^2)^{s+1/2}$ on $(-1, 1)$, $s \in \mathbb{N}_0$, and $n$ is odd, then

$$\max_{z \in E_{\rho}} |K_{n,s}^{(2)}(z)| = \left| K_{n,s}^{(2)} \left( \frac{t}{2} (\rho - \rho^{-1}) \right) \right|. \tag{2.13}$$

In this section we consider the case when $n$ is even.
Theorem 2.3. For each fixed \( \rho > 1 \) and \( s \in \mathbb{N}_0 \) there exists even \( n_0 = n_0(\rho, s) \) such that

\[
\max_{z \in C_\rho} |K_{n,s}^{(2)}(z)| = |K_{n,s}^{(2)}(\frac{i}{2}(\rho - \rho^{-1}))|
\]

for each even \( n \geq n_0 \).

Proof. First we prove the inequality

\[
|Z_{n,s}^{(2)}(\rho e^{i\theta})| \leq Z_{n,s}^{(2)}(i\rho), \quad \theta \in [0, \pi/2), \ n \text{ even}.
\]

We note that (see [11, Eq. (3.13)])

\[
Z_{n,s}^{(2)}(u) = \sum_{\nu=0}^{[s-1]/2} \left( \sum_{k=2\nu}^{2\nu+1} (-1)^k \left( \frac{2s+1}{s+k+1} \right) u^{-(n+1)k} \right) + \zeta_{n,s}(u)
\]

where \( u = \rho e^{i\theta} \), \( \alpha = (s - 2\nu)/(s + 2\nu + 2) \), \( 0 < \alpha < 1 \), and

\[
\zeta_{n,s}(u) = \zeta_{n,s}(\rho e^{i\theta}) := \begin{cases} 
0 & \text{if } s \text{ is odd}, \\
(\rho e^{i\theta})^{-2(n+1)} & \text{if } s \text{ is even}, 
\end{cases}
\]

as well as \( |\zeta_{n,s}(\rho e^{i\theta})| = \zeta_{n,s}(i\rho) \).

Since

\[
|Z_{n,s}^{(2)}(u)| \leq \sum_{\nu=0}^{[s-1]/2} \left( \frac{2s+1}{s+2\nu+1} \right) |u^{-(n+1)}(1 - \alpha u^{-(n+1)})| + |\zeta_{n,s}(u)|,
\]

introducing \( q = \alpha \rho^{-2(n+1)} \), now we get

\[
|Z_{n,s}^{(2)}(u)| \leq \sum_{\nu=0}^{[s-1]/2} \left( \frac{2s+1}{s+2\nu+1} \right) \rho^{-4\nu(n+1)} \sqrt{1 - 2q \cos(2n+2)\theta + q^2} + \zeta_{n,s}(i\rho)
\]

\[
\leq \sum_{\nu=0}^{[s-1]/2} \left( \frac{2s+1}{s+2\nu+1} \right) \rho^{-4\nu(n+1)}(1 + q) + \zeta_{n,s}(i\rho)
\]

\[
= \sum_{\nu=0}^{[s-1]/2} \left( \frac{2s+1}{s+2\nu+1} \right) (i\rho)^{-4\nu(n+1)}(1 - \alpha(i\rho)^{-2(n+1)}) + \zeta_{n,s}(i\rho)
\]

\[
= Z_{n,s}^{(2)}(i\rho).
\]

Therefore, in order to prove the statement, on the basis of (2.11) and (2.13), it is sufficient to prove

\[
\frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos(2n+2)\theta} \leq \frac{a_2 + 1}{a_{2n+2} + 1}, \quad \theta \in [0, \pi/2), \ n \text{ even},
\]

for sufficiently large \( n \) (\( n \geq n_0; \ n_0 = n_0(\rho) \) - even). This is equivalent to

\[
a_{2n+2} + a_{2n+2} \cos 2\theta - a_2 - a_2 \cos(2n+2)\theta + \cos 2\theta - \cos(2n+2)\theta \geq 0,
\]

and furthermore to

\[
a_{2n+2}(1 + \cos 2\theta) - a_2(1 + \cos(2n+1)\theta) + (1 + \cos 2\theta) - (1 + \cos(2n+1)\theta) \geq 0,
\]
introducing half-angles, to \((a_{2n+2} + 1) \cos^2 \theta - (a_2 + 1) \cos^2 (n+1)\theta \geq 0\), and to
\begin{equation}
(a_{2n+2} + 1) - \frac{\cos^2 (n+1)\theta}{\cos^2 \theta} (a_2 + 1) \geq 0.
\end{equation}

Since \(|\cos(n+1)\theta/\cos \theta| \leq n+1\) for even \(n\), we have
\begin{equation}
(a_{2n+2} + 1) - \frac{\cos^2 (n+1)\theta}{\cos^2 \theta} (a_2 + 1) \geq (a_{2n+2} + 1) - (n+1)^2 (a_2 + 1),
\end{equation}
which means that (2.14) holds if \((a_{2n+2} + 1) - (n+1)^2 (a_2 + 1) \geq 0\).

Since \(a_2 + 1 = 2a_1^2\) and \(a_{2n+2} + 1 = 2a_{n+1}^2\), the last inequality is equivalent to
\(a_{n+1}^2 - [(n+1)a_1]^2 \geq 0\) or to \(a_{n+1} - (n+1)a_1 \geq 0\). Substituting \(a_1, a_{n+1}\) by (2.2),
this inequality becomes
\[G_{\rho}(n) \equiv G(n) := \rho^{n+1} - (n+1)\rho - (n+1)\rho^{-1} + \rho^{-(n+1)} \geq 0.\]

Since \(G_{\rho}(n) \ (\rho \text{ is fixed})\) is continuous on \(\mathbb{R}\) and \(\lim_{n \to +\infty} G_{\rho}(n) = +\infty\), it
follows that \(G_{\rho}(n) > 0\), for each \(n > t\), where \(t\) is the largest zero of \(G_{\rho}(n)\). For \(n_0\)
we can take the smallest even integer which is greater than or equal to \(t\).

Let \(\bar{t}\) be the smallest even integer \(\geq t\). If \(t\) is an even integer, we have \(\bar{t} = t\),
otherwise
\[\bar{t} := \begin{cases} \lfloor t \rfloor + 1 & \text{if } \lfloor t \rfloor \text{ is odd,} \\ \lfloor t \rfloor + 2 & \text{if } \lfloor t \rfloor \text{ is even.} \end{cases}\]

We can use the function \(G(n)\) from the proof to estimate \(n_0\). Numerical values of \(G(n)\) for some values of \(\rho\) are presented in Table 3. The smallest possible (s.p.) values of \(n_0\), for \(s = 1, \ldots, 10\), are also presented in the same table. We can see that the smallest possible \(n_0\) (which is even) is estimated very well, independently of \(s\).

Finally, observe that the function \(G_{\rho}(n) \equiv G(n)\) in this case has rather simple
form. Because of \(G(0) = 0\), and \(G''(n) = 2a_{n+1} \log^2 \rho > 0\), for \(n \in [0, +\infty)\), we conclude that \(G(n)\) has at most one zero \(t\) in the interval \((0, +\infty)\).

2.3. The weight function \(\omega_3(t) = (1 + t)^{1/2+s}(1 - t)^{-1/2}, \ s \in \mathbb{N}_0\). An explicit
representation of the kernel \(K_{n,s}^{(3)}(z)\) on the ellipse \(E_{\rho}\) for the generalized Chebyshev
weight function of the third kind \(\omega_3(t)\) was given in [11], as well as
\begin{equation}
|K_{n,s}^{(3)}(z)| = \frac{2^{1-s} \pi}{\rho^{n+1/2} (a_2 - \cos 2\theta)^{1/2} (a_{2n+1} + \cos (2n+1)\theta)^{1/2+s}} \left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|
\end{equation}
where
\[Z_{n,s}^{(3)}(u) = \sum_{k=0}^{s} \left(\frac{2s+1}{s+k+1}\right) u^{-(2n+1)k}.
\]

The following result was conjectured in [11]:

**Theorem 2.4.** For each fixed \(\rho > 1\) and \(s \in \mathbb{N}_0\) there exists \(n_0 = n_0(\rho, s)\) such that
\[\max_{z \in E_{\rho}} |K_{n,s}^{(3)}(z)| = K_{n,s}^{(3)} \left(\frac{1}{2}(\rho + \rho^{-1})\right)\]
for each \(n \geq n_0\).
Table 3. The smallest possible (s.p.) values of $n_0$ and their approximations $\bar{t}$ (t is the largest zero of $G$)

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Proof. Because of (2.15), it is sufficient to prove

\[
\left( a_1 + \cos \theta \right)^{s+1} \left| Z_{n,s}^{(3)} (\rho e^{i\theta}) \right| \leq \left( a_1 + 1 \right)^{s+1} Z_{n,s}^{(3)} (\rho) \leq \left( a_2 - 1 \right)^{s+1} \left( a_{2n+1} + 1 \right)^{1/2+s}
\]

for sufficiently large $n$ ($n \geq n_0(\rho, s)$) and $\theta \in [0, \pi]$, where $a_j$ are given by (2.2).

It is obvious that for each $n \geq 1$, we have \((a_1 + \cos \theta)^{s+1} \leq (a_1 + 1)^{s+1}\). On the basis of the results from Subsection 2.1, we obtain

\[
\left( a_2 - \cos 2\theta \right)^{1/2} \left( a_{2n+1} + \cos (2n + 1)\theta \right)^{1/2+s}
\]

for each $n \geq n_0 \ (n_0 = n_0(\rho, s))$. Therefore, we conclude that

\[
\left| K_{n,s}^{(3)} \left( \frac{1}{2} (\rho e^{i\theta} + \rho^{-1} e^{-i\theta}) \right) \right| \leq K_{n,s}^{(3)} \left( \frac{1}{2} (\rho + \rho^{-1}) \right),
\]

for each $n \geq n_0 \ (n_0 = n_0(\rho, s))$.

If $t$ is the largest zero of $F$, for $n_0$ we can take $\lfloor (2t - 1) / 2 \rfloor + 1$. \qed
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REFERENCES


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