MAXIMUM OF THE MODULUS OF KERNELS IN GAUSS-TURÁN QUADRATURES

GRADIMIR V. MILOVANOVIC, MIODRAG M. SPALEVIĆ, AND MIROSLAV S. PRANIĆ

ABSTRACT. We study the kernels $K_{n,s}(z)$ in the remainder terms $R_{n,s}(f)$ of the Gauss-Turán quadrature formulae for analytic functions on elliptical contours with foci at $\pm 1$, when the weight $\omega$ is a generalized Chebyshev weight function. For the generalized Chebyshev weight of the first (third) kind, it is shown that the modulus of the kernel $|K_{n,s}(z)|$ attains its maximum on the real axis (positive real semi-axis) for each $n \geq n_0$, $n_0 = n_0(\rho, s)$. It was stated as a conjecture in [Math. Comp. 72 (2003), 1855–1872]. For the generalized Chebyshev weight of the second kind, in the case when the number of the nodes $n$ in the corresponding Gauss-Turán quadrature formula is even, it is shown that the modulus of the kernel attains its maximum on the imaginary axis for each $n \geq n_0$, $n_0 = n_0(\rho, s)$. Numerical examples are included.

1. Introduction

We consider the Gauss-Turán quadrature formula with multiple nodes

$$
\int_{-1}^{1} f(t)\omega(t)dt = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}f^{(i)}(\tau_{\nu}) + R_{n,s}(f) \quad (n \in \mathbb{N}; \ s \in \mathbb{N}_0)
$$

where $\omega$ is a nonnegative and integrable function on the interval $(-1,1)$, which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$. The nodes $\tau_{\nu}$ in (1.1) must be zeros of the $s$-orthogonal polynomials with respect to the weight function $\omega(t)$. The $s$-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $\omega(t)$ are polynomials which satisfy the following orthogonality conditions:

$$
\int_{-1}^{1} \pi_n(t)^2 s+1 t^k \omega(t)dt = 0, \quad k = 0, 1, \ldots, n - 1.
$$

Numerically stable methods for constructing nodes $\tau_{\nu}$ and coefficients $A_{i,\nu}$ can be found in [8] [10] [13]. For more details on quadrature formulae with multiple nodes see [7] and [9].

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Let $\Gamma$ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let $D$ be its interior. If the integrand $f$ is analytic on $D$ and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation (see [14, 11])

$$
R_{n,s}(f) = \frac{1}{2\pi i} \oint_{\Gamma} K_{n,s}(z)f(z)dz.
$$

The kernel is given by $K_{n,s}(z) = \rho_{n,s}(z)/[\pi_{n,s}(z)]^{2s+1}$, $z \notin [-1, 1]$, where

$$
\rho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t} \omega(t)dt.
$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$. If the weight function in (1.1) is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$ (see [11, Lemma 2.1]).

A particularly interesting case is the Chebyshev weight $\omega_1(t) = (1-t^2)^{-1/2}$. In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^n$—minimizes all integrals of the form

$$
\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}} dt \quad (k \geq 0).
$$

This means that the Chebyshev polynomials $T_n$ are $s$-orthogonal on $(-1, 1)$ for each $s \geq 0$. Oscicini and Rosati [14] found three other weights $\omega_k(t)$ ($k = 2, 3, 4$) for which the $s$-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third and fourth kind: $U_n$, $V_n$, and $W_n$, which are defined by

$$
U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad V_n(\cos \theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos \frac{1}{2}\theta}, \quad W_n(\cos \theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta},
$$

respectively (cf. Gautschi and Notaris [4]). However, these weights depend on $s$,

$$
\omega_2(t) = (1-t^2)^{1/2+s}, \quad \omega_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}, \quad \omega_4(t) = \frac{(1-t)^{1/2+s}}{(1+t)^{1/2}}.
$$

It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that in the investigation it is sufficient to study only the first three generalized Chebyshev weights $\omega_k(t)$, $k = 1, 2, 3$.

The integral representation (1.2) leads directly to the error estimate

$$
|R_{n,s}| \leq \frac{l(\Gamma)}{2\pi} \left( \max_{z \in \Gamma} |K_{n,s}(z)| \right) \left( \max_{z \in \Gamma} |f(z)| \right),
$$

where $l(\Gamma)$ denotes the length of the contour $\Gamma$. First maximum depends only on the quadrature rule (i.e., on $\omega$) and not on $f$. The first unified approach described above was taken by Donaldson and Elliot [2]. They applied it to several kinds of interpolatory and non-interpolatory quadrature rules. Error bounds for Gaussian quadratures of analytic functions were studied by Gautschi and Varga [5] (see also [6]), and later by Schira [15, 16], Hunter and Nikolov [8].

As a contour $\Gamma$ we take an ellipse $E_\rho$ with foci at points $\pm 1$ and a sum of semi-axes $\rho > 1$,

$$
E_\rho = \left\{ z \in \mathbb{C} : z = \frac{1}{2}(u + u^{-1}), \quad 0 \leq \theta \leq 2\pi \right\}, \quad u = \rho e^{i\theta}.
$$
When \( \rho \to 1 \), the ellipse shrinks to the interval \([-1, 1]\), while with increasing \( \rho \) it becomes more and more circle-like.

When \( \omega \) is the generalized Chebyshev weight of the first (third) kind, it is conjectured, on the basis of numerical experiments (see [11]), that the modulus of the kernel attains its maximum on the real axis (positive real semi-axis) for each \( n \geq n_0, \ n_0 = n_0(\rho, s) \).

In this paper we prove those conjectures. Moreover, for the generalized Chebyshev weight of the second kind, in the case when the number of the nodes \( n \) in the corresponding Gauss-Turán quadrature formula is even, we show that the modulus of the kernel attains its maximum on the imaginary axis for each \( n \geq n_0, \ n_0 = n_0(\rho, s) \). Numerical examples are included.

2. The maximum modulus of the kernel on confocal ellipses

We study the magnitude of \(|K_{n,s}(z)|\) on the contour \( \mathcal{E}_\rho \) for the generalized Chebyshev weight functions of the first, second and third kind, respectively. The particular case \(|K_{n,0}(z)|\) was analyzed in details by Gautschi et al. [5 6].

2.1. The weight function \( \omega_1(t) = (1 - t^2)^{-1/2} \). An explicit representation of the kernel \( K_{n,s}^{(1)}(z) \) on the ellipse \( \mathcal{E}_\rho \) for the weight function \( \omega_1(t) \) was given by Milovanović and Spalević in [11], as well as

\[
K_{n,s}^{(1)}(z) = \frac{2^{1-s} \pi}{\rho^n} \frac{|\rho_{n,0}(\rho e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}}, \quad z \in \mathcal{E}_\rho,
\]

where

\[
a_j = a_j(\rho) = \frac{1}{2} \left( \rho^j + \rho^{-j} \right), \quad j \in \mathbb{N},
\]

and

\[
Z_{n,s}^{(1)}(u) = \sum_{k=0}^{s} \binom{2s + 1}{s + k + 1} u^{-2nk} = \sum_{k=0}^{s} \binom{2s + 1}{k} u^{-2(n-s-k)}.
\]

The weight function \( \omega_1(t) \) is even, so we can take \( \theta \in [0, \pi/2] \).

The following result was conjectured in [11]:

**Theorem 2.1.** For each fixed \( \rho > 1 \) and \( s \in \mathbb{N}_0 \) there exists \( n_0 = n_0(\rho, s) \) such that

\[
\max_{z \in \mathcal{E}_\rho} |K_{n,s}^{(1)}(z)| = K_{n,s}^{(1)}(\frac{1}{2}(\rho + \rho^{-1}))
\]

for each \( n \geq n_0 \).

**Proof.** The inequality \(|Z_{n,s}^{(1)}(\rho e^{i\theta})| \leq Z_{n,s}^{(1)}(\rho)\) immediately follows from (2.3). Because of that and (2.4), it is sufficient to prove

\[
\frac{1}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}} \leq \frac{1}{(a_2 - 1)^{1/2}(a_{2n} + 1)^{1/2+s}}
\]

for a sufficiently large \( n \ (n \geq n_0(\rho, s)) \) and \( \theta \in (0, \pi/2) \), where \( a_j \) are given by (2.2).

By squaring (2.4) it is reduced to

\[
(a_2 - 1)(a_{2n} + 1)^{2s+1} \leq (a_2 - \cos 2\theta)(a_{2n} + \cos 2n\theta)^{2s+1}.
\]

The following transformation will be used

\[
a_2 - \cos 2\theta = (a_2 - 1) + 2\sin^2 \theta.
\]
Further, we will use
\( (a_{2n} + \cos 2n\theta)2s+1 = ((a_{2n} + 1) - 2\sin^2 n\theta)2s+1 \)
\[= (a_{2n} + 1)^{2s+1} + \sum_{k=1}^{2s+1} (-2)^k \binom{2s+1}{k} (a_{2n} + 1)^{2s+1-k} \sin^{2k} n\theta, \]
i.e.,
\[ (2.7) \quad (a_{2n} + \cos 2n\theta)2s+1 = (a_{2n} + 1)^{2s+1} - 2(\sin^2 n\theta)E_{\rho,s}(n, \theta), \]
where
\[ E_{\rho,s}(n, \theta) = \sum_{k=1}^{2s+1} (-2)^{k-1} \binom{2s+1}{k} (a_{2n} + 1)^{2s+1-k} \sin^{2k-2} n\theta \quad (\geq 0). \]
It is easy to see that \( E_{\rho,s}(n, \theta) \) can be represented in the form
\[ E_{\rho,s}(n, \theta) = (2s+1)(a_{2n} + 1)^{2s+1} + \sum_{k=2}^{2s+1} (-2)^{k-1} \binom{2s+1}{k} (a_{2n} + 1)^{2s+1-k} \sin^{2k-2} n\theta, \]
i.e.,
\[ (2.8) \quad E_{\rho,s}(n, \theta) = (2s+1)(a_{2n} + 1)^{2s+1} - \sum_{k=1}^{s} 2^{2k-1} \binom{2s+1}{2k} (a_{2n} + 1)^{2s-2k+1} \sin^{4k-2} n\theta \]
\[+ \sum_{k=1}^{s} 2^{2k} \binom{2s+1}{2k+1} (a_{2n} + 1)^{2s-2k} \sin^{4k} n\theta. \]
Using (2.6) and (2.7), the inequality (2.5) is reduced to
\[ (a_{2} - 1) (a_{2n} + 1)^{2s+1} \]
\[ \leq \left[ (a_{2} - 1) + 2\sin^2 \theta \right] \left[ (a_{2n} + 1)^{2s+1} - 2(\sin^2 n\theta)E_{\rho,s}(n, \theta) \right], \]
i.e.,
\[ 2\sin^2 \theta (a_{2n} + 1)^{2s+1} - 2\sin^2 n\theta \left[ (a_{2} - 1) + 2\sin^2 \theta \right] E_{\rho,s}(n, \theta) \geq 0. \]
Dividing this inequality by \( 2\sin^2 \theta \), it becomes
\[ (a_{2n} + 1)^{2s+1} - \frac{\sin^2 n\theta}{\sin^2 \theta} \left[ (a_{2} - 1) + 2\sin^2 \theta \right] E_{\rho,s}(n, \theta) \geq 0. \]
By using the well-known fact \( |\sin n\theta/\sin \theta| \leq n \), it is easy to see that
\[ (2.10) \quad \frac{\sin^2 n\theta}{\sin^2 \theta} \left[ (a_{2} - 1) + 2\sin^2 \theta \right] E_{\rho,s}(n, \theta) \geq 0. \]
According to (2.8), we conclude that
\[ E_{\rho,s}(n, \theta) - (2s+1)(a_{2n} + 1)^{2s} = \sum_{k=1}^{s} \frac{4^k(2s+1)!}{(2k)!(2s-2k)!} (a_{2n} + 1)^{2s-2k} \]
\[\times \left( \frac{\sin^2 n\theta}{2k+1} - \frac{a_{2n} + 1}{2(2s-2k+1)} \right) \sin^{4k-2} n\theta. \]
Since \( \sin^{4k-2} n\theta \leq 1 \) and
\[ \frac{\sin^2 n\theta}{2k+1} - \frac{a_{2n} + 1}{2(2s-2k+1)} \leq \frac{1}{2k+1} - \frac{a_{2n} + 1}{2(2s-2k+1)}, \]
from the previous equality we obtain
\[ E_{\rho,s}(n, \theta) - (2s+1)(a_{2n+1})^{2s} \leq \sum_{k=1}^{s} 4^k (a_{2n+1})^{2s-2k} \left( \frac{2s+1}{2k+1} \right)^{2s-2k} - \frac{1}{2} \sum_{k=1}^{s} 4^k (a_{2n+1})^{2s-2k+1}. \]

Therefore,
\[ E_{\rho,s}(n, \theta) \leq \sum_{k=0}^{s} 4^k \left( \frac{2s+1}{2k+1} \right) (a_{2n+1})^{2s-2k} - \frac{1}{2} \sum_{k=1}^{s} 4^k \left( \frac{2s+1}{2k} \right) (a_{2n+1})^{2s-2k+1}. \]

Using the last inequality and (2.10), we conclude that the left-hand side of (2.9) is greater than or equal to \( F(\rho,s) \), where
\[ F_{\rho,s}(n) := (a_{2n+1})^{2s+1} - \left[ (a_2 - 1)n^2 + 2 \right] \times \left[ \sum_{k=0}^{s} 4^k \left( \frac{2s+1}{2k+1} \right) (a_{2n+1})^{2s-2k} - \frac{1}{2} \sum_{k=1}^{s} 4^k \left( \frac{2s+1}{2k} \right) (a_{2n+1})^{2s-2k+1} \right]. \]

Since \( F_{\rho,s}(n) \) (\( \rho, s \) are fixed) is continuous on \( \mathbb{R} \) and \( \lim_{n \to +\infty} F_{\rho,s}(n) = +\infty \), it follows that \( F_{\rho,s}(n) > 0 \), for each \( n > t \), where \( t \) is the largest zero of \( F_{\rho,s}(n) \). For \( n_0 \) we can take \( [t] + 1 \).

**Table 1.** The smallest possible (s.p.) values of \( n_0 \) and their approximations \( [t] + 1 \) (\( t \) is the largest zero of \( F \))

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<tr>
<th>( n )</th>
<th>( [t] + 1 ) the s.p. ( n_0 )</th>
<th>( [t] + 1 ) the s.p. ( n_0 )</th>
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<td>1</td>
<td>41</td>
<td>34</td>
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<td>2</td>
<td>50</td>
<td>46</td>
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<td>3</td>
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<td>4</td>
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<td>9</td>
<td>70</td>
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The proof of Theorem 2.1 is not only of theoretical, but also of practical importance. We can use the function \( F(n) \) from the proof to estimate \( n_0 \). Numerical values of \( [t] + 1 \) (\( t \) is the largest zero of \( F \)) for some values of \( \rho \) and \( s \) are presented in Tables 1 and 2. The smallest possible (s.p.) values of \( n_0 \) are also presented. We can see that the smallest possible \( n_0 \) is estimated by \( [t] + 1 \) very well.

A typical graph illustrating the relationship between \( n \) and \( F(n) \) is given in Figure 1. Here, \( \rho = 1.05, s = 1; n \in [1,42] \).

2.2. The weight function \( \omega_2(t) = (1 - t^2)^{s+1/2}, s \in N_0 \). An explicit representation of the kernel \( K_{n,s}^{(2)}(z) \) on the ellipse \( \mathcal{E}_\rho \) for the weight function \( \omega_2(t) \) was given in [11], as well as
\[ |K_{n,s}^{(2)}(z)| = \frac{\pi}{4s \rho^{n+1}} \left| \frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos (2n+2)\theta} \right|^{s+1/2} |Z_{n,s}^{(2)}(\rho e^{i\theta})|. \]
Table 2. The smallest possible (s.p.) values of $n_0$ and their approximations $|t| + 1$ ($t$ is the largest zero of $F$)

<table>
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<tr>
<th>$s$</th>
<th>$n_0 = 1$</th>
<th>$n_0 = 5$</th>
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<tr>
<td>$s = 1$</td>
<td>$</td>
<td>t</td>
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<tr>
<td>$s = 5$</td>
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<td>t</td>
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</table>

Figure 1. The typical graph of $F(n)$.

where

$$Z_{n,s}^{(2)}(pe^{i\theta}) = \sum_{k=0}^{s} (-1)^k \left( \frac{2s + 1}{s + k + 1} \right) (pe^{i\theta})^{-2(n+1)k}.$$  

There we proved the following statement:

**Theorem 2.2.** If $\omega_2(t) = (1 - t^2)^{s+1/2}$ on $(-1, 1)$, $s \in \mathbb{N}_0$, and $n$ is odd, then

$$\max_{z \in E_{\rho}} |K_{n,s}^{(2)}(z)| = \left| K_{n,s}^{(2)} \left( \frac{i}{2}(\rho - \rho^{-1}) \right) \right| .$$

In this section we consider the case when $n$ is even.
Theorem 2.3. For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists even $n_0 = n_0(\rho, s)$ such that

$$\max_{z \in \mathbb{C}_+} |K_{n,s}^{(2)}(z)| = \left| K_{n,s}^{(2)} \left( \frac{i}{2}(\rho - \rho^{-1}) \right) \right|$$

for each even $n \geq n_0$.

Proof. First we prove the inequality

$$(2.13) \quad |Z_{n,s}^{(2)}(\rho e^{i\theta})| \leq Z_{n,s}^{(2)}(i\rho), \quad \theta \in [0, \pi/2), \text{ } n \text{ is even.}$$

We note that (see [11, Eq. (3.13)])

$$Z_{n,s}^{(2)}(u) = \sum_{\nu=0}^{[(s-1)/2]} \left( \sum_{k=2\nu}^{2\nu+1} (-1)^k \frac{2s+1}{s+k+1} u^{-2(n+1)k} \right) + \zeta_{n,s}(u)$$

introducing $q = \alpha \rho^{-2(n+1)}$, now we get

$$|Z_{n,s}^{(2)}(u)| \leq \sum_{\nu=0}^{[(s-1)/2]} \left( \frac{2s+1}{s+2\nu+1} \right) \rho^{-4\nu(n+1)} \sqrt{1 - 2q \cos(2n+2)\theta + q^2} + \zeta_{n,s}(i\rho)$$

Therefore, in order to prove the statement, on the basis of (2.11) and (2.13), it is sufficient to prove

$$\frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos (2n+2)\theta} \leq \frac{a_2 + 1}{a_{2n+2} + 1}, \quad \theta \in [0, \pi/2), \text{ } n \text{ is even,}$$

for sufficiently large $n$ ($n \geq n_0$; $n_0 = n_0(\rho)$ even). This is equivalent to

$$a_{2n+2} + a_{2n+2} \cos 2\theta - a_2 - a_2 \cos (2n+2)\theta + \cos 2\theta - \cos (2n+2)\theta \geq 0,$$

and furthermore to

$$a_{2n+2}(1 + \cos 2\theta) - a_2(1 + \cos (2n+1)\theta) + (1 + \cos 2\theta) - (1 + \cos (2n+1)\theta) \geq 0,$$
introducing half-angles, to \((a_{2n+2} + 1) \cos^2 \theta - (a_2 + 1) \cos^2 (n+1) \theta \geq 0\), and to
\begin{equation}
(a_{2n+2} + 1) - \frac{\cos^2 (n+1) \theta}{\cos^2 \theta} (a_2 + 1) \geq 0.
\end{equation}

Since \(|\cos(n+1)\theta/\cos \theta| \leq n+1\) for even \(n\), we have
\begin{equation}
(a_{2n+2} + 1) - \frac{\cos^2 (n+1) \theta}{\cos^2 \theta} (a_2 + 1) \geq (a_{2n+2} + 1) - (n + 1)^2(a_2 + 1),
\end{equation}
which means that (2.14) holds if \((a_{2n+2} + 1) - (n + 1)^2(a_2 + 1) \geq 0\).

Since \(a_2 + 1 = 2a_1^2\) and \(a_{2n+2} + 1 = 2a_{n+1}^2\), the last inequality is equivalent to
\(a_{n+1}^2 - [(n + 1)a_1]^2 \geq 0\) or to \(a_{n+1} - (n + 1)a_1 \geq 0\). Substituting \(a_1, a_{n+1}\) by (2.2),
this inequality becomes
\[G_\rho(n) \equiv G(n) := \rho^{n+1} - (n + 1)\rho - (n + 1)\rho^{-1} + \rho^{-(n+1)} \geq 0.\]

Since \(G_\rho(n)\) (\(\rho\) is fixed) is continuous on \(\mathbb{R}\) and \(\lim_{n \to +\infty} G_\rho(n) = +\infty\),
it follows that \(G_\rho(n) > 0\), for each \(n > t\), where \(t\) is the largest zero of \(G_\rho(n)\). For \(n_0\)
we can take the smallest even integer which is greater than or equal to \(t\).

Let \(\overline{t}\) be the smallest even integer \(\geq t\). If \(t\) is an even integer, we have \(\overline{t} = t\),
otherwise
\[\overline{t} := \begin{cases} \lceil t \rceil + 1 & \text{if } \lceil t \rceil \text{ is odd,} \\ \lceil t \rceil + 2 & \text{if } \lceil t \rceil \text{ is even.} \end{cases} \]

We can use the function \(G(n)\) from the proof to estimate \(n_0\). Numerical values of \(\overline{t}\) for some values of \(\rho\)
are presented in Table [11]. The smallest possible (s.p.) values of \(n_0\), for \(s = 1, \ldots, 10\),
are also presented in the same table. We can see that the smallest possible \(n_0\) (which is even)
was estimated very well, independently of \(s\).

Finally, observe that the function \(G_\rho(n) \equiv G(n)\) in this case has rather simple form.
Because of \(G(0) = 0\), and \(G'(n) = 2a_{n+1} \log^2 \rho > 0\), for \(n \in [0, +\infty)\),
we conclude that \(G(n)\) has at least one zero \(t\) in the interval \((0, +\infty)\).

2.3. The weight function \(\omega_3(t) = (1 + t)^{1/2 + s}(1 - t)^{-1/2}, s \in \mathbb{N}_0\). An explicit
representation of the kernel \(K_{n,s}^{(3)}(z)\) on the ellipse \(E_\rho\) for the generalized Chebyshev
weight function of the third kind \(\omega_3(t)\) was given in [11], as well as
\begin{equation}
|K_{n,s}^{(3)}(z)| = \frac{2^{1-s} \pi}{\rho^{n+1/2}(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} + \cos (2n + 1)\theta)^{1/2+s}}
\left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|.
\end{equation}

where
\[Z_{n,s}^{(3)}(u) = \sum_{k=0}^{s} \frac{2s + 1}{s + k + 1} u^{-(2n+1)k}.\]

The following result was conjectured in [11]:

**Theorem 2.4.** For each fixed \(\rho > 1\) and \(s \in \mathbb{N}_0\) there exists \(n_0 = n_0(\rho, s)\) such that
\[\max_{z \in E_\rho} |K_{n,s}^{(3)}(z)| = K_{n,s}^{(3)}\left(\frac{1}{2}(\rho + \rho^{-1})\right)\]
for each \(n \geq n_0\).
Table 3. The smallest possible (s.p.) values of \( n_0 \) and their approximations \( \overline{t} \) (\( t \) is the largest zero of \( G \))

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**Proof.** Because of (2.15), it is sufficient to prove

\[
\frac{(a_1 + \cos \theta)^{s+1} \left| Z_{n,s}^{(3)}(\rho e^{i\theta}) \right|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} + \cos (2n + 1)\theta)^{1/2+s}} \leq \frac{(a_1 + 1)^{s+1} Z_{n,s}^{(3)}(\rho)}{(a_2 - 1)^{1/2}(a_{2n+1} + 1)^{1/2+s}}
\]

for sufficiently large \( n \) (\( n \geq n_0(\rho, s) \)) and \( \theta \in (0, \pi] \), where \( a_j \) are given by (2.2).

It is obvious that for each \( n \geq 1 \), we have \( (a_1 + \cos \theta)^{s+1} \leq (a_1 + 1)^{s+1} \). On the basis of the results from Subsection 2.1 we obtain

\[
\frac{Z_{n,s}^{(3)}(\rho e^{i\theta})}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} + \cos (2n + 1)\theta)^{1/2+s}}
\]

\[
= \frac{Z_{n+1/2,s}^{(1)}(\rho e^{i\theta})}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1/2} + \cos (2n + 1/2)\theta)^{1/2+s}}
\]

\[
\leq \frac{Z_{n+1/2,s}^{(1)}(\rho)}{(a_2 - 1)^{1/2}(a_{2n+1/2} + 1)^{1/2+s}} = \frac{Z_{n,s}^{(3)}(\rho)}{(a_2 - 1)^{1/2}(a_{2n+1} + 1)^{1/2+s}}.
\]

for each \( n \geq n_0 \) (\( n_0 = n_0(\rho, s) \)). Therefore, we conclude that

\[
\left| K_{n,s}^{(3)} \left( \frac{1}{2} (\rho e^{i\theta} + \rho^{-1} e^{-i\theta}) \right) \right| \leq K_{n,s}^{(3)} \left( \frac{1}{2} (\rho + \rho^{-1}) \right),
\]

for each \( n \geq n_0 \) (\( n_0 = n_0(\rho, s) \)).

If \( t \) is the largest zero of \( F \), for \( n_0 \) we can take \( [(2t - 1)/2] + 1 \). \( \square \)
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REFERENCES


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