ON THE POLYNOMIAL REPRESENTATION FOR THE NUMBER OF PARTITIONS WITH FIXED LENGTH

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Abstract. In this paper, it is shown that the number \( M(n, k) \) of partitions of a nonnegative integer \( n \) with \( k \) parts can be described by a set of \( \tilde{k} \) polynomials of degree \( k - 1 \) in \( Q_k \), where \( \tilde{k} \) denotes the least common multiple of the \( k \) integers \( 1, 2, \ldots, k \) and \( Q_k \) denotes the quotient of \( n \) when divided by \( \tilde{k} \). In addition, the sets of the \( \tilde{k} \) polynomials are obtained and shown explicitly for \( k = 3, 4, 5, \) and 6.

1. Preliminaries

Let \( N(n) \) and \( M(n, k) \) denote the number of partitions of \( n \) and that with length \( k \) (or with \( k \) parts), respectively, where \( n \) and \( k \) are nonnegative integers. Obviously,
\[
M(n, k) = 0, \quad \text{if } k > n. \tag{1.1}
\]

Defining
\[
N(0) = 1 \tag{1.2}
\]
and
\[
M(0, 0) = 1 \tag{1.3}
\]
for convenience, we have
\[
N(n) = \sum_{k=0}^{n} M(n, k). \tag{1.4}
\]

While (1.4) allows us to express the number \( N(n) \) in terms of \( \{M(n, k)\}_{k=0}^{n} \), the number \( M(n, k) \) for \( k \geq \left\lfloor \frac{n}{3} \right\rfloor \) can be described in terms of \( \{N(n)\}_{n=0}^{n-k} \) by
\[
M(n, k) = \begin{cases} 
N(n - k), & 2k \geq n \\
N(n - k) - \sum_{j=0}^{n-2k-1} N(j), & 2k < n \leq 3k + 2
\end{cases} \tag{1.5}
\]

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as shown in the Appendix, where \([x]\) and \([x]\) denote the smallest integer no smaller than \(x\) and the largest integer no larger than \(x\), respectively. More generally, it is well known \([2, 3, 4, 5, 7]\) that \(M(n, k)\) satisfies the recursion

\[
M(n, k) = M(n - 1, k - 1) + M(n - k, k).
\]

(1.6)

Clearly, once \(n\) and \(k\) are given, we can evaluate \(M(n, k)\) recursively with (1.6) using \(M(n, 0) = 0\) for \(n \geq 1\) and \(M(0, 0) = 1\). In this paper we are interested in finding a polynomial representation for the evaluation of \(M(n, k)\).

2. Polynomial representations

2.1. A nonrecursive formula. Let us denote by \(Q_{n,k}\) and \(R_{n,k}\) the quotient and remainder, respectively, of \(n\) when divided by \(k\). Unless it is ambiguous, we use for brevity the notations \(Q_k\) and \(R_k\) for \(Q_{n,k}\) and \(R_{n,k}\), respectively. Now, from (1.6), we have

\[
M(n, k) - M(n - k, k) = M(n - 1, k - 1),
\]

\[
M(n - k, k) - M(n - 2k, k) = M(n - k - 1, k - 1),
\]

\[
\vdots
\]

\[
M(k + R_k, k) - M(R_k, k) = M(k + R_k - 1, k - 1),
\]

which can be added to produce

\[
M(n, k) = \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} M(n + k - ki - 1, k - 1), \quad k \geq 1
\]

(2.1)

since \(R_k = 0, 1, \cdots, k - 1\), and consequently, \(M(R_k, k) = 0\) from (1.1). Based on (2.1), we will show next that \(M(n, k)\) can be described by a set of polynomials.

2.2. Simple examples. Let us first obtain polynomial representations for \(M(n, k)\) when \(k = 1, 2, 3\) using (2.1). Clearly, when \(k = 1\), we have

\[
M(n, 1) = \sum_{i=1}^{n} M(n - i, 0)
\]

(2.2)

\[
= 1, \quad n \geq 1
\]

from (2.1) using \(M(0, 0) = 1\). If we write \(n = 2Q_2 + R_2\) when \(k = 2\), we have \(\lfloor \frac{n}{2} \rfloor = Q_2\). Thus, we get

\[
M(n, 2) = \sum_{i=1}^{Q_2} M(n - 2i + 1, 1)
\]

(2.3)

\[
= Q_2
\]

from (2.1) using (2.2). Next, writing \(n = 6Q_6 + R_6\) when \(k = 3\), we have \(\lfloor \frac{n}{3} \rfloor = 2Q_6 + Q_{R_6,3}\). Then, remembering \(M(R_6 - 1, 2) = 0\) for \(R_6 = 0, 1, 2\) from (1.1), we
obtain

\[
M(n, 3) = \sum_{i=1}^{\lfloor n/3 \rfloor} M(n - 3i + 2, 2)
\]

\[
= \begin{cases} 
M(n - 1, 2) + M(n - 4, 2) + \cdots + M(R_6 + 2, 2), & R_6 = 0, 1, 2, \ (2Q_6 \text{ terms added}) \\
M(n - 1, 2) + M(n - 4, 2) + \cdots + M(R_6 + 2, 2) + M(R_6 - 1, 2), & R_6 = 3, 4, 5, \ (2Q_6 + 1 \text{ terms added})
\end{cases}
\]

\[
= \sum_{i=1}^{Q_6} \{ M(6(i - 1) + R_6 + 2, 2) + M(6(i - 1) + R_6 + 5, 2) \}
+ M(R_6 - 1, 2)
\]

\[
= \sum_{i=1}^{Q_6} \sum_{j=1}^{2} M(2\{3(i - 1)\} + R_6 + 2 + 3(j - 1), 2) + M(R_6 - 1, 2)
\]

\[
\begin{cases} 
\sum_{i=1}^{Q_6} \{ (3i - 2) + (3i - 1) \} = 3Q_6^2, & R_6 = 0, \\
\sum_{i=1}^{Q_6} \{ (3i - 2) + 3i \} = 3Q_6^2 + Q_6, & R_6 = 1, \\
\sum_{i=1}^{Q_6} \{ (3i - 1) + 3i \} = 3Q_6^2 + 2Q_6, & R_6 = 2, \\
1 + \sum_{i=1}^{Q_6} \{ (3i - 1) + (3i + 1) \} = 3Q_6^2 + 3Q_6 + 1, & R_6 = 3, \\
1 + \sum_{i=1}^{Q_6} \{ 3i + (3i + 1) \} = 3Q_6^2 + 4Q_6 + 1, & R_6 = 4, \\
2 + \sum_{i=1}^{Q_6} \{ 3i + (3i + 2) \} = 3Q_6^2 + 5Q_6 + 2, & R_6 = 5.
\end{cases}
\]

(2.4)

\[
= 3Q_6^2 + R_6Q_6 + c_{0,3}(R_6)
\]

using (2.3) in (2.1), where \( \{c_{0,3}(r)\}_{r=0}^{5} = \{0, 0, 0, 1, 1, 2\} \). It is interesting to note that \( M(n, 3) \) can also be expressed as (41 p. 451)

\[
M(n, 3) = \left\lfloor \frac{n^2 + 6}{12} \right\rfloor.
\]

(2.6)

2.3. The main result. By extending the results (2.2)-(2.3) for \( M(n, 1), M(n, 2), \) and \( M(n, 3) \), we obtain the main result in the form of a theorem below.

**Theorem 2.1.** When \( k \geq 1 \), let us write the natural number \( n \) as \( n = kQ_k + R_k \), where \( k \) denotes the least common multiple of the \( k \) integers \( 1, 2, 3, \ldots, k \). Then the number \( M(n, k) = M(kQ_k + R_k, k) \) of partitions of \( n \) with \( k \) parts can be specified with the \( \bar{k} \) polynomials

\[
M(kQ_k + R_k, k) = \sum_{i=0}^{k-1} c_{i,k}(R_k)Q_k^i, \quad R_k = 0, 1, \ldots, \bar{k} - 1
\]

of degree \( k - 1 \) in \( Q_k \). In (2.7), the constant terms \( c_{0,k}(r) = M(r, k) \) for \( r = 0, 1, \ldots, \bar{k} - 1 \), are obtained as

\[
c_{0,k}(r) = \sum_{j=1}^{Q_{r,k}} M(r + k - 1 - jk, k - 1),
\]

the coefficients

\[
c_{k-1,k}(r) = \frac{\bar{k}^{k-1}}{k!(k-1)!}, \quad k \geq 1
\]
for $r = 0, 1, \cdots, \tilde{k} - 1$, of the highest degree term are dependent on $k$ but not on $r$, the coefficients

\begin{align}
(2.10) \quad c_{k-2,k}(r) &= \frac{\tilde{k}^{-2}}{4k!(k-2)!} \{4r + k(k-3)\}, \quad k \geq 3
\end{align}

for $r = 0, 1, \cdots, \tilde{k} - 1$, of the second-highest degree term are linear functions of $r$, and the coefficients

\begin{align}
(2.11) \quad c_{1,4}(r) &= \begin{cases} (r^2 + 2r)/4, & R_{r,2} = 0, \\ (r^2 + 2r - 3)/4, & R_{r,2} = 1, \end{cases}
\end{align}

and

\begin{align}
(2.12) \quad c_{k-3,k}(r) &= \frac{\tilde{k}^{-3}}{288k!(k-3)!} \\
&\times \{144r^2 + 72k(k-3)r + k(9k^3 - 58k^2 + 75k - 2)\}, \quad k \geq 5
\end{align}

for $r = 0, 1, \cdots, \tilde{k} - 1$, of the third-highest degree term are quadratic functions of $r$.

**Proof.** Let us prove the theorem using mathematical induction. Obviously, (2.2)–(2.3) imply that (2.7) is true when $k = 1, 2, 3$. Assuming (2.7) is true, let us now show that $M(n, k + 1)$ is described by $k + 1$ polynomials of degree $k$ in $Q_{k+1}$.

Putting $n$ as $n = k + 1Q_{k+1} + R_{k+1}$, we have $n = k + 1Q_{k+1} + Q_{R_{k+1},k+1}$. Thus, denoting $v = R_{k+1} - 1 + l(k + 1)$ for convenience, we get

\begin{align}
M(n, k + 1) &= \sum_{i=1}^{\lfloor n/v \rfloor} M(n + k - (k + 1)i, k) \\
&= \begin{cases} M(n - 1, k) + M(n - 2 - k, k) + \cdots + M(R_{k+1}, k), & \text{when } 0 \leq R_{k+1} \leq k \\
&\quad (\frac{k+1}{k+1}Q_{k+1} \text{ terms added}) \\
M(n - 1, k) + M(n - 2 - k, k) + \cdots + M(R_{k+1}, k) + M(R_{k+1} - 1, k), & \text{when } k + 1 \leq R_{k+1} \leq 2k + 1 \\
&\quad (\frac{k+1}{k+1}Q_{k+1} + 1 \text{ terms added}) \\
&\vdots \\
M(n - 1, k) + M(n - 2 - k, k) + \cdots + M(R_{k+1}, k) + M(R_{k+1} - 1, k), & \text{when } R_{k+1} \leq k + 1 - k - 1 \leq R_{k+1} \leq k + 1 - 1 \\
&\quad (\frac{k+1}{k+1}Q_{k+1} + \frac{k+1}{k+1} \text{ terms added}) \end{cases}
\end{align}
which can be rewritten as

\[ M(n, k + 1) = \sum_{i=1}^{\ell+1} \sum_{l=1}^{n} M \left( \hat{k} \left\{ \frac{k+1}{k} (i - 1) \right\} + R_{k+1}^{-1} - 1 + l(k + 1), k \right) \]

\[ + D(R_{k+1}^{-1}) \]

\[ = \sum_{i=1}^{\ell+1} \sum_{l=1}^{n} M \left( \hat{k} \left\{ \frac{k+1}{k} (i - 1) + Q_{v_i, \hat{k}} \right\} + \left( v_l - \bar{k}Q_{v_i, \hat{k}} \right), k \right) \]

\[ + D(R_{k+1}^{-1}) \]

\[ = \sum_{i=1}^{\ell+1} \sum_{l=1}^{n} \left[ c_{k-1, \hat{k}} (v_l - \bar{k}Q_{v_i, \hat{k}}) \left\{ \frac{k+1}{k} (i - 1) + Q_{v_i, \hat{k}} \right\}^{k-1} \right. \]

\[ + c_{k-2, \hat{k}} (v_l - \bar{k}Q_{v_i, \hat{k}}) \left\{ \frac{k+1}{k} (i - 1) + Q_{v_i, \hat{k}} \right\}^{k-2} \]

\[ + c_{k-3, \hat{k}} (v_l - \bar{k}Q_{v_i, \hat{k}}) \left\{ \frac{k+1}{k} (i - 1) + Q_{v_i, \hat{k}} \right\}^{k-3} \]

\[ + \cdots + c_{0, \hat{k}} (v_l - \bar{k}Q_{v_i, \hat{k}}) \right] + D(R_{k+1}^{-1}) \]

\[ = \sum_{i=1}^{\ell+1} \left\{ t_{k-1, \hat{k}} (R_{k+1}^{-1})(i - 1)^{k-1} + t_{k-2, \hat{k}} (R_{k+1}^{-1})(i - 1)^{k-2} \right. \]

\[ + t_{k-3, \hat{k}} (R_{k+1}^{-1})(i - 1)^{k-3} + \cdots + t_{0, \hat{k}} (R_{k+1}^{-1}) \left\} + D(R_{k+1}^{-1}) \right\}

(2.14) from (2.1) using (2.7) since \( 0 \leq v_l - \bar{k}Q_{v_i, \hat{k}} \leq \hat{k} - 1 \). Here,

\[ D(R_{k+1}^{-1}) = \sum_{j=1}^{Q_{R_{k+1}^{-1}, k+1}} M(R_{k+1}^{-1} + k - j(k + 1), k) \]

(2.15) is zero unless \( R_{k+1}^{-1} \geq k + 1 \) since \( Q_{R_{k+1}^{-1}, k+1} = 0 \) when \( R_{k+1}^{-1} \leq k \),

(2.16) \[ t_{m, \hat{k}} (R_{k+1}^{-1}) = \left\{ \frac{k+1}{k} \right\} \sum_{l=1}^{m \bar{c}_{m, \hat{k}} (v_l), \ m = 0, 1, \cdots, k - 1,} \]

and

(2.17) \[ \bar{c}_{m, \hat{k}} (r) = \sum_{l=m}^{k-1} \left( \frac{t}{t - m} \right) c_{l, \hat{k}} (r - \bar{k}Q_{r, \hat{k}}) Q_{r, \hat{k}}^{t-m}. \]

Recollecting \( R_{k+1}^{-1} = 0, 1, \cdots, \hat{k} + 1 - 1 \), it is clear from (2.14) that \( M(n, k + 1) \) is described by \( k + 1 \) polynomials of degree \( k \) in \( Q_{k+1}^{-1} \), proving that (2.7) holds when
is increased by 1. In addition, we have $c_{0,k+1}(R_{k+1}^{-1}) = D(R_{k+1}^{-1})$ which is the same as (2.3) when $k + 1$ is replaced with $k$. Let us show next that the coefficients of $Q_{k+1}^{k-1} R_{k+1}^{-1}$ and $Q_{k+1}^{k-2} R_{k+1}^{-1}$ in the polynomial representation of $M(n,k+1)$ are those obtained from (2.19), (2.10), and (2.17), respectively, by replacing $k+1$ for $k$. Concentrating on the terms of $t^{k-1}$, $t^{k-2}$, and $t^{k-3}$ in (2.14), we have

$$M(n,k+1) = \sum_{i=1}^{Q_{k+1}^{k-1}} t_{k-1,k}(R_{k+1}^{-1})t^{k-1}$$

$$+ \sum_{i=1}^{Q_{k+1}^{k-1}} \left\{ (-1)^{\frac{t}{2}} \binom{k-2}{0} t_{k-2,k}(R_{k+1}^{-1}) + \binom{k-1}{1} t_{k-1,k+1}(R_{k+1}^{-1}) \right\} t^{k-2}$$

$$+ \sum_{i=1}^{Q_{k+1}^{k-1}} \left\{ (-1)^{0} \binom{k-3}{0} t_{k-3,k}(R_{k+1}^{-1}) + \binom{k-2}{1} t_{k-2,k}(R_{k+1}^{-1}) \right\} t^{k-3} + O(t^{k-3})$$

$$= t_{k-1,k}(R_{k+1}^{-1}) \left( H_{1,k-1} Q_{k+1}^{k-1} + H_{2,k-1} Q_{k+1}^{k-2} + H_{3,k-1} Q_{k+1}^{k-3} + \cdots \right)$$

$$+ \left\{ t_{k-2,k}(R_{k+1}^{-1}) - (k-1)t_{k-1,k+1}(R_{k+1}^{-1}) \right\}$$

$$+ \left\{ H_{1,k-2} Q_{k+1}^{k-1} + H_{2,k-2} Q_{k+1}^{k-2} + H_{3,k-2} Q_{k+1}^{k-3} + \cdots \right\}$$

$$+ \left\{ t_{k-3,k}(R_{k+1}^{-1}) - (k-2)t_{k-2,k}(R_{k+1}^{-1}) + \frac{(k-1)(k-2)}{2} t_{k-1,k+1}(R_{k+1}^{-1}) \right\}$$

$$+ \left\{ H_{1,k-3} Q_{k+1}^{k-1} + H_{2,k-3} Q_{k+1}^{k-2} + H_{3,k-3} Q_{k+1}^{k-3} + \cdots \right\} + O(Q_{k+1}^{k-2})$$

$$(2.18)$$

Here, $H_{j,k}$ denotes the coefficient of (the $j$-th highest degree term) $n^{k-j}$ in $\sum_{i=1}^{n} n^k$, and can straightforwardly be shown to be $H_{1,k} = \frac{1}{k+1}$ for $k \geq 0$, $H_{2,k} = \frac{1}{2}$ for $k \geq 1$, and $H_{3,k} = \frac{k}{12}$ for $k \geq 2$. Now using (2.19), (2.10), and (2.17), we get
\[ t_{k-1,k}(R_{k+1}) = \begin{cases} \left(\frac{k+1}{k}\right)^{k-1} & k - 1 = 0, \\ \left(\frac{k+1}{k}\right)^{k-1} - \frac{k-1}{k+1} & k - 1 \geq 1, \end{cases} \]

(2.19)

\[ = \frac{\tilde{k}+1}{(k+1)!}(k-1)! \quad k \geq 1 \]

since \( c_{0,1}(r) = 1 \) from (2.2). The result (2.19) can in turn be used in the coefficient of \( Q_k^{R_{k+1}} \) in (2.18) to produce

\[ c_{k,k+1}(R_{k+1}) = H_{1,k-1}t_{k-1,k}(R_{k+1}) \]

\[ = \frac{1}{k+1} \frac{\tilde{k}+1}{k+1)!}((k-1)! \]

(2.20)

After some calculations from (2.16) when \( k = 2 \) since \( c_{0,2}(r) = 0 \) from (2.3) and \( c_{1,2}(r) = 1 \) from (2.20), and

\[ t_{0,2}(R_6) = 2 \sum_{l=1}^2 \{c_{0,2}(v_l - 2Q_{v_1,2}) + c_{1,2}(v_l - 2Q_{v_1,2})Q_{v_1,2}\} \]

\[ = Q_{v_0+2,2} + Q_{v_0+5,2} \]

(2.21)

Next, we have

\[ t_{k-2,k}(R_{k+1}) = \left(\frac{k+1}{k}\right)^{k-2} \sum_{l=1}^{k-2} \left\{ c_{k-2,k}(v_l - \tilde{k}Q_{v_1,k}) \right\} \]

\[ + (k-1)c_{k-1,k}(v_l - \tilde{k}Q_{v_1,k})Q_{v_1,k} \]

\[ = \left(\frac{k+1}{k}\right)^{k-2} \sum_{l=1}^{k-2} \left\{ \frac{\tilde{k}^{k-2}}{4k!(k-2)!} \{4(v_l - \tilde{k}Q_{v_1,k}) + k(k-3)\} \right\} \]

\[ + (k-1)\frac{\tilde{k}^{k-1}}{k!(k-1)!}Q_{v_1,k} \]

\[ = \left(\frac{k+1}{k}\right)^{k-2} \frac{\tilde{k}^{k-2}}{4k!(k-2)!} \sum_{l=1}^{k-2} \{4v_l + k(k-3)\} \]

\[ = \frac{k+1}{4k!(k-2)!} \]
\[
\times \left[ \frac{2}{k+1} \left\{ \tilde{k} + 1^2 + \tilde{k} + 1 (2R_{k+1} - k - 1) \right\} + \frac{\tilde{k} + 1}{k+1} k(k-3) \right]
\]

\begin{equation}
(2.22)
\end{equation}

\[
= \frac{k+1}{4(k+1)!(k-2)!} \left\{ 2\tilde{k} + 1 + 4R_{k+1} + (k+1)(k-2) \right\}, \quad k \geq 3
\]

from (2.16) using (2.10) since

\[
\sum_{l=1}^{4} v_{l} = 4 \left\{ (R_{k+1} - 1) \left( \frac{\tilde{k} + 1}{k+1} + \frac{\tilde{k} + 1}{k+1} \left( \frac{\tilde{k} + 1}{k+1} + 1 \right) \right) \right\}
\]

\begin{equation}
(2.23)
\end{equation}

As the result \(R_6 + 3\) obtained in (2.21) can also be obtained from (2.22) when we put \(k = 2\), we note that (2.22) holds for \(k \geq 2\). Using (2.19) and (2.22), we have the coefficient of \(Q^{k-1}_{k+1}\) in (2.18) as

\[
c_{k-1,k+1}(R_{k+1}) = H_{2,k-1} t_{k-1,k}(R_{k+1})
\]

\[
+ H_{1,k-2} \left\{ t_{k-2,k}(R_{k+1}) - (k-1) t_{k-1,k}(R_{k+1}) \right\}
\]

\[
= -\frac{1}{2} t_{k-1,k}(R_{k+1}) + \frac{1}{k-1} t_{k-2,k}(R_{k+1})
\]

\[
= -\frac{1}{2} \left( \frac{k+1}{(k+1)!(k-1)!} \right) + \frac{k+1}{4(k+1)!(k-1)!} \left\{ 2\tilde{k} + 1 + 4R_{k+1} + (k+1)(k-2) \right\}
\]

\begin{equation}
(2.24)
\end{equation}

For convenience, we have

\[
\alpha_k = \frac{k+3}{2k(k-3)!}, \quad \beta_k = \frac{k+3}{4(k-1)(k-4)!}, \quad \text{and} \quad \gamma_k = \frac{k+3}{288(k-1)(k-3)!} (9k^3 - 58k^2 + 75k - 2)
\]
\[ t_{k-3,k}(R_{\widetilde{k+1}}) = \left\{ \frac{k+1}{k} \right\}^{k-3} \sum_{i=1}^{\frac{k+1}{k+1}} \left\{ c_{k-3,k}(v_l - \tilde{k}Q_{v_l,k}) + (k-2)c_{k-2,k}(v_l - \tilde{k}Q_{v_l,k})Q_{v_l,k} + \frac{(k-1)(k-2)}{2} c_{k-1,k}(v_l - \tilde{k}Q_{v_l,k})Q_{v_l,k}^2 \right\} \]

\[ = \left\{ \frac{k+1}{k} \right\}^{k-3} \sum_{i=1}^{\frac{k+1}{k+1}} \left\{ (v_l - \tilde{k}Q_{v_l,k})^2 \alpha_k + (v_l - \tilde{k}Q_{v_l,k}) \beta_k + \gamma_k \right\} \]

which can further be shown to be

\[ t_{0,3}(R_{12}) = \sum_{i=1}^{3} \sum_{j=0}^{2} c_{j,3}(v_l - 6Q_{v_l,6})Q_{v_l,6}^j \]

\[ = \left\{ \begin{array}{ll}
(R_{12}^2 + 14R_{12} + 60)/4, & R_{r,2} = 0, \\
(R_{12}^2 + 14R_{12} + 57)/4, & R_{r,2} = 1,
\end{array} \right. \]

(2.26)

and

\[ t_{k-3,k}(R_{\widetilde{k+1}}) = \left\{ \frac{k+1}{k} \right\}^{k-3} \frac{k+1}{6(k+1)} \left[ \alpha_k(\tilde{k} + 1 + k + 1)(2\tilde{k} + 1 + k + 1) \right. \]

\[ + 3\left\{ 2\alpha_k(R_{\widetilde{k+1}} - 1) + \beta_k \right\}(\tilde{k} + 1 + k + 1) \]

\[ + 6\left\{ \alpha_k(R_{\widetilde{k+1}} - 1)^2 + \beta_k(R_{\widetilde{k+1}} - 1) + \gamma_k \right\} \]

\[ = \left\{ \frac{k+1}{k} \right\}^{k-3} \frac{k+1}{6(k+1)} \left[ 6\alpha_k R_{\widetilde{k+1}}^2 \right. \]

\[ + \left\{ 6\alpha_k(\tilde{k} + 1 + k + 1) - 12\alpha_k + 6\beta_k \right\} R_{\widetilde{k+1}} \]

\[ + \alpha_k(\tilde{k} + 1 + k + 1)(2\tilde{k} + 1 + k + 1) \]

\[ + 3(\tilde{k} + 1 + k + 1)(\beta_k - 2\alpha_k) + 6(\alpha_k - \beta_k + \gamma_k) \]

\[ = \left\{ \frac{k+1}{k} \right\}^{k-3} \frac{k+1}{(k+1)} \left[ \frac{\tilde{k} k-3}{2k!(k-3)!} R_{\widetilde{k+1}}^2 \right. \]
since \( \{c_0,r\}_{r=0}^5 = \{0, 0, 0, 1, 1, 2\} \), \( c_{1,3}(r) = r \), and \( c_{2,3}(r) = 3 \) from (2.5). The results (2.26) and (2.27), together with (2.19) and (2.22), can now be used in the coefficient of \( Q_{k+1}^{k-2} \) in (2.18) to produce

\[
c_{k-2,k+1}(R_{k+1}) = \frac{k-1}{12} t_{k-1,k}(R_{k+1}) + \frac{1}{2} \left\{ t_{k-2,k}(R_{k+1}) - (k-1)t_{k-1,k}(R_{k+1}) \right\}
+ \frac{1}{k-2} \left\{ t_{k-3,k}(R_{k+1}) - (k-2)t_{k-2,k}(R_{k+1}) + \frac{(k-1)(k-2)}{2} t_{k-1,k}(R_{k+1}) \right\}
= \frac{k-1}{12} t_{k-1,k}(R_{k+1}) - \frac{1}{2} t_{k-2,k}(R_{k+1}) + \frac{1}{k-2} t_{k-3,k}(R_{k+1})
= \frac{k+1}{12(k+1)!(k-2)!} \kappa^{k-1}
+ \frac{k+1}{8(k+1)!(k-2)!} \left\{ 2\kappa^{k+1} + 4R_{k+1} + (k+1)(k-2) \right\}
+ \frac{k+1}{6(k+1)!(k-2)!} \left[ 3R_{k+1}^2 + \frac{3}{2} \left( 2\kappa + (k+1)(k-2) \right) R_{k+1} \right]
\]

(2.27)
The results (2.20), (2.24), (2.28), and (2.29) complete the proof of the theorem. □

\[ \text{(2.29) } \]

and

\[ M(n, k) = \sum_{i=0}^{k-1} c_{i,k}(r - \overline{k}Q_{r,k})(q + Q_{r,k})^i \]

\[ = \sum_{i=0}^{k-1} \left\{ \binom{i+1}{1} c_{i+1,k}(r - \overline{k}Q_{r,k})Q_{r,k} + \binom{i+2}{2} c_{i+2,k}(r - \overline{k}Q_{r,k})Q_{r,k}^2 \right. \]

\[ + \cdots + \left. \binom{k-i-1}{k-i-1} c_{k-i-1,k}(r - \overline{k}Q_{r,k})Q_{r,k}^{k-i-1} \right\} q^i \]

\[ = \sum_{i=0}^{k-1} \tilde{c}_{i,k}(r)q^i, \]

from (2.27) since \( Q_{r,k} = q + Q_{r,k} \) and \( R_{r,k} = r - \overline{k}Q_{r,k} \). The result (2.30) implies that, to compute \( M(n, k) \), the \( \overline{k} \) polynomials in (2.27) can be extended by expressing the number \( n \) more generally as the sum \( n = \overline{k}q + r \) of a multiple of \( \overline{k} \) and a nonnegative integer, where \( q \) does not need to be \( Q_{r,k} \). For example, when \( n = 42 \), \( M(42, 3) = 147 \) can be evaluated as \( M(6 \cdot 7 + 0, 3) = 3 \cdot 7^2 = 147 \), \( M(6 \cdot 6 + 6, 3) = 3 \cdot 6^2 + 6 \cdot 6 + 3 = 147 \), or \( M(6 \cdot 5 + 12, 3) = 3 \cdot 5^2 + 12 \cdot 5 + 12 = 147 \). \( \cdots \) from the polynomials \( 3m^2 \) for \( r = 0, 3m^2 + 6m + 3 \) for \( r = 6 \), and \( 3m^2 + 12m + 12 \) for \( r = 12 \) (shown in (2.24) and those extended from (2.25) as in (2.30) \( \cdots \).
2.4. More examples. Let us now obtain the $\tilde{k}$ polynomials explicitly for $k = 4, 5, 6$. Writing $n = 12Q_{12} + R_{12}$ when $k = 4$, we have

\begin{equation}
M(n, 4) = 12Q_{12}^3 + 3(R_{12} + 1)Q_{12}^2 + c_{1.4}(R_{12})Q_{12} + c_{0.4}(R_{12})
\end{equation}

after some manipulations, where $\{c_{0.4}(r)\}_{r=0}^{11} = \{0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 11\}$. For example, $M(577, 4) = M(48 \cdot 12 + 1, 4) = 12 \cdot 48^3 + 6 \cdot 48^2 = 1340928$. When $k = 5$, we similarly have

\begin{equation}
M(n, 5) = 4500Q_{60}^4 + (300R_{60} + 750)Q_{60}^3 + \frac{5}{2}(3R_{60}^2 + 15R_{60} + 5)Q_{60}^2
\end{equation}

\begin{equation}
+ c_{1.5}(R_{60})Q_{60} + c_{0.5}(R_{60}),
\end{equation}

where

\begin{equation}
24 c_{1.5}(r) = \begin{cases}
2r^3 + 15r^2 + 10r - 60, & R_{r.2} = 0, \\
2r^3 + 15r^2 + 10r - 15, & R_{r.2} = 1,
\end{cases}
\end{equation}

and $\{c_{0.5}(r)\}_{r=0}^{59} = \{0, 0, 0, 0, 1, 1, 2, 3, 5, 7, 10, \ldots, 4932\}$. For example, we have $M(577, 5) = M(9 \cdot 60 + 37, 5) = 4500 \cdot 9^4 + 11850 \cdot 9^3 + \frac{23335}{2} \cdot 9^2 + \frac{10183}{2} \cdot 9 + 831 = 39154872$. Finally, when $k = 6$, we can obtain

\begin{equation}
M(n, 6) = 9000Q_{60}^5 + (750R_{60} + 3375)Q_{60}^4 + \frac{25}{3}(3R_{60}^2 + 27R_{60} + 38)Q_{60}^3
\end{equation}

\begin{equation}
+ c_{2.6}(R_{60})Q_{60}^2 + c_{1.6}(R_{60})Q_{60} + c_{0.6}(R_{60}),
\end{equation}

where

\begin{equation}
\frac{24}{5} c_{2.6}(r) = \begin{cases}
2r^3 + 27r^2 + 76r, & R_{r.2} = 0, \\
2r^3 + 27r^2 + 76r - 45, & R_{r.2} = 1,
\end{cases}
\end{equation}

\begin{equation}
288 c_{1.6}(r) = \begin{cases}
r^4 + 18r^3 + 76r^2 + 96, & R_{r.6} = 0, \\
r^4 + 18r^3 + 76r^2 - 90r - 629, & R_{r.6} = 1.5, \\
r^4 + 18r^3 + 76r^2 - 224, & R_{r.6} = 2.4, \\
r^4 + 18r^3 + 76r^2 - 90r - 309, & R_{r.6} = 3,
\end{cases}
\end{equation}

and $\{c_{0.6}(r)\}_{r=0}^{59} = \{0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 7, 11, 14, \ldots, 11720\}$. For example, $M(577, 6) = M(9 \cdot 60 + 37, 6) = 9000 \cdot 9^5 + 31125 \cdot 9^4 + \frac{129600}{3} \cdot 9^3 + \frac{58765}{2} \cdot 9^2 + \frac{60125}{6} \cdot 9 + 1360 = 769373455$.

Some of the results are summarized in Tables 1. As shown in Table 1, the leading coefficients $c_{k-1,k}(r)$ form a new integer sequence of infinite length.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{k-1,k}(r)$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>12</td>
<td>4500</td>
<td>9000</td>
<td>1512630000</td>
<td>1452124800000</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>
The coefficients \( c_{j,4}(r) \) for \( M(n, 4) = c_{3,4}(R_{12})Q_{12}^3 + c_{2,4}(R_{12})Q_{12}^2 + c_{1,4}(R_{12})Q_{12} + c_{0,4}(R_{12}) \) are given in Table 2.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( (c_{3,4}(r), c_{2,4}(r), c_{1,4}(r), c_{0,4}(r)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(12, 3, 0, 0)</td>
</tr>
<tr>
<td>2</td>
<td>(12, 9, 2, 0)</td>
</tr>
<tr>
<td>4</td>
<td>(12, 15, 6, 1)</td>
</tr>
<tr>
<td>6</td>
<td>(12, 21, 12, 2)</td>
</tr>
<tr>
<td>8</td>
<td>(12, 27, 20, 5)</td>
</tr>
<tr>
<td>10</td>
<td>(12, 33, 30, 9)</td>
</tr>
</tbody>
</table>

The coefficients \( c_{j,5}(r) \) for \( M(n, 5) = c_{4,5}(R_{60})Q_{60}^4 + c_{3,5}(R_{60})Q_{60}^3 + c_{2,5}(R_{60})Q_{60}^2 + c_{1,5}(R_{60})Q_{60} + c_{0,5}(R_{60}) \) are given in Table 3.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( (c_{4,5}(r), c_{3,5}(r), c_{2,5}(r), c_{1,5}(r), c_{0,5}(r)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(4500, 750, 25/2, -5/2, 0)</td>
</tr>
<tr>
<td>2</td>
<td>(4500, 1350, 235/2, 3/2, 0)</td>
</tr>
<tr>
<td>4</td>
<td>(4500, 1950, 565/2, 29/2, 0)</td>
</tr>
<tr>
<td>6</td>
<td>(4500, 2550, 1015/2, 81/2, 1)</td>
</tr>
<tr>
<td>8</td>
<td>(4500, 3150, 1585/2, 167/2, 3)</td>
</tr>
<tr>
<td>10</td>
<td>(4500, 3750, 2275/2, 295/2, 7)</td>
</tr>
<tr>
<td>12</td>
<td>(4500, 4350, 3085/2, 473/2, 13)</td>
</tr>
<tr>
<td>14</td>
<td>(4500, 4950, 4015/2, 709/2, 23)</td>
</tr>
<tr>
<td>16</td>
<td>(4500, 5550, 5065/2, 1011/2, 37)</td>
</tr>
<tr>
<td>18</td>
<td>(4500, 6150, 6235/2, 1387/2, 57)</td>
</tr>
<tr>
<td>20</td>
<td>(4500, 6750, 7525/2, 1845/2, 84)</td>
</tr>
<tr>
<td>22</td>
<td>(4500, 7350, 8935/2, 2393/2, 119)</td>
</tr>
<tr>
<td>24</td>
<td>(4500, 7950, 10465/2, 3039/2, 164)</td>
</tr>
<tr>
<td>26</td>
<td>(4500, 8550, 12115/2, 3791/2, 221)</td>
</tr>
<tr>
<td>28</td>
<td>(4500, 9150, 13885/2, 4657/2, 291)</td>
</tr>
<tr>
<td>30</td>
<td>(4500, 9750, 15775/2, 5645/2, 377)</td>
</tr>
<tr>
<td>32</td>
<td>(4500, 10350, 17785/2, 6763/2, 480)</td>
</tr>
<tr>
<td>34</td>
<td>(4500, 10950, 19915/2, 8019/2, 603)</td>
</tr>
<tr>
<td>36</td>
<td>(4500, 11550, 22165/2, 9421/2, 748)</td>
</tr>
<tr>
<td>38</td>
<td>(4500, 12150, 24535/2, 10977/2, 918)</td>
</tr>
<tr>
<td>40</td>
<td>(4500, 12750, 27025/2, 12695/2, 1115)</td>
</tr>
<tr>
<td>42</td>
<td>(4500, 13350, 29635/2, 14583/2, 1342)</td>
</tr>
<tr>
<td>44</td>
<td>(4500, 13950, 32365/2, 16649/2, 1602)</td>
</tr>
<tr>
<td>46</td>
<td>(4500, 14550, 35215/2, 18901/2, 1898)</td>
</tr>
<tr>
<td>48</td>
<td>(4500, 15150, 38185/2, 21347/2, 2233)</td>
</tr>
<tr>
<td>50</td>
<td>(4500, 15750, 41275/2, 23995/2, 2611)</td>
</tr>
<tr>
<td>52</td>
<td>(4500, 16350, 44485/2, 26853/2, 3034)</td>
</tr>
<tr>
<td>54</td>
<td>(4500, 16950, 47815/2, 29929/2, 3507)</td>
</tr>
<tr>
<td>56</td>
<td>(4500, 17550, 51265/2, 32321/2, 4033)</td>
</tr>
<tr>
<td>58</td>
<td>(4500, 18150, 54835/2, 36767/2, 4616)</td>
</tr>
</tbody>
</table>
2.5. Remarks. In passing, we would like to make some remarks on several interesting issues.

Remark 1: Applying mathematical induction in (2.1) with the initial result

\[ M(n, k - 1) = \frac{n^{k-2}}{(k-1)!(k-2)!} + o(n^{k-3}) \]

deduced from (2.2) and (2.3), we can show that

\[
M(n, k) = \sum_{i=1}^{\lfloor k/2 \rfloor} \left\{ \frac{(n+k-i-1)^{k-2}}{(k-i-1)!(k-2)!} + o(n^{k-3}) \right\}
\]

\[
= \frac{1}{(k-1)!(k-2)!} \sum_{i=1}^{\lfloor k/2 \rfloor} \left\{ \sum_{j=0}^{k-2} \binom{k-2}{j} (-k)^{k-2-j} \right\} + o(n^{k-2})
\]

\[
= \frac{1}{(k-1)!(k-2)!} \left\{ \sum_{j=0}^{k-2} \binom{k-2}{j} (-k)^{j} \sum_{i=1}^{\lfloor k/2 \rfloor} i^j n^{k-2-j} \right\} + o(n^{k-2})
\]

\[
= \frac{1}{(k-1)!(k-2)!} \left\{ \sum_{j=0}^{k-2} \binom{k-2}{j} (-1)^{j} \left( \frac{k-1}{j+1} \right) n^{k-1-j} + o(n^{k-2}) \right\}
\]

\[ = \frac{n^{k-1}}{k!(k-1)!} + o(n^{k-2}) \]
after some calculations using $\sum_{i=1}^{k-1} i^j = \frac{n^{i+1}}{(i+1)k} + o(n^j)$ and $\sum_{j=0}^{k-2} \binom{k-1}{j+1}(-1)^j = \binom{k-1}{0} = 1$. This implies that the sum of degrees in $Q_{\tilde{k}}$ and $R_{\tilde{k}}$ for any term on the right-hand side of (2.7) is $k-1$ since $n = \tilde{k}Q_{\tilde{k}} + R_{\tilde{k}}$. Consequently, the coefficient $c_{i,k}(R_{\tilde{k}})$ in (2.7) has degree $k - 1 - i$ in $R_{\tilde{k}}$ for $i = 0, 1, \ldots, k - 1$.

Remark 2: By replacing $Q_{\tilde{k}}$ with $(n - R_{\tilde{k}})/\tilde{k}$ in (2.7), the polynomial in $n$ of (2.38) for the numbers $M(n, k)$ can be obtained explicitly. Specifically, we have

\begin{equation}
M(n, k) = \frac{n^{k-1}}{k!(k-1)!} + \frac{(k-3)n^{k-2}}{4(k-1)! (k-2)!} + o(n^{k-3})
\end{equation}

from (2.7) using (2.9) and (2.10). For example, after some straightforward manipulations in (2.4) and (2.31), we have

\begin{equation}
12 M(n, 3) = \left\{ \begin{array}{ll}
n^2, & R_6 = 0, \\
n^2 - 1, & R_6 = 1, \quad 5, \\
n^2 - 4, & R_6 = 2, \quad 4, \\
n^2 + 3, & R_6 = 3,
\end{array} \right.
\end{equation}

and

\begin{equation}
144 M(n, 4) = \left\{ \begin{array}{ll}
n^3 + 3n^2, & R_{12} = 0, \\
n^3 + 3n^2 - 20, & R_{12} = 2, \\
n^3 + 3n^2 + 32, & R_{12} = 4, \\
n^3 + 3n^2 - 36, & R_{12} = 6, \\
n^3 + 3n^2 + 16, & R_{12} = 8, \\
n^3 + 3n^2 - 4, & R_{12} = 10, \\
n^3 + 3n^2 - 9n + 5, & R_{12} = 1, \quad 7, \\
n^3 + 3n^2 - 9n - 27, & R_{12} = 3, \quad 9, \\
n^3 + 3n^2 - 9n - 11, & R_{12} = 5, \quad 11.
\end{array} \right.
\end{equation}

Remark 3: As a final remark, let us mention that we have not made use of generating functions in this paper. Recall that, for fixed $k$, the numbers $\{M(n, k)\}_{n=1}^{\infty}$ have the generating function

\begin{equation}
\sum_{n=1}^{\infty} M(n, k)x^{n-k} = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}
\end{equation}

\begin{equation}
= \frac{1}{(1-x)^k} \prod_{j=1}^{k} \frac{1 - x^j}{1 - x^j}.
\end{equation}

It would be an interesting topic to use this generating function for a possible alternative point of view of the arguments of this paper.

**Appendix**

First, decreasing $n$ and $k$ by 1 until $k = 1$ in (1.6) and adding all of the resultant equations, we can obtain

\begin{equation}
M(n, k) = \sum_{j=0}^{k} M(n - k, j).
\end{equation}
Since $M(n - k, j) = 0$ for $j \geq n - k + 1$ from (1.1), we have

$$M(n, k) = \sum_{j=0}^{n-k} M(n - k, j) + \sum_{j=n-k+1}^{k} M(n - k, j)$$

$$= \sum_{j=0}^{n-k} M(n - k, j)$$

(A.2) $$= N(n - k), \quad 2k \geq n$$

from (1.4) and (A.1) when $k \geq n - k$. Next, when $k < n - k$, we have

$$M(n, k) = \sum_{j=0}^{n-k} M(n - k, j) - \sum_{j=k+1}^{n-k} M(n - k, j)$$

$$= N(n - k) - \sum_{j=k+1}^{n-k} M(n - k, j)$$

(A.3) $$= N(n - k) - \sum_{q=0}^{n-2k-1} M(n - k, q + k + 1)$$

from (1.4) and (A.1). Now, if $k + 1 \geq (n-k) - (k+1)$, we have $q + k + 1 \geq (n-k) - (q + k + 1)$ when $q = 0, 1, \ldots, n - 2k - 1$, and consequently, $M(n - k, q + k + 1) = N(n - 2k - 1 - q)$ for $q = 0, 1, \ldots, n - 2k - 1$ from (A.2). In other words, we can rewrite (A.3) as

$$M(n, k) = N(n - k) - \sum_{q=0}^{n-2k-1} N(n - 2k - 1 - q)$$

(A.4) $$= N(n - k) - \sum_{q=0}^{n-2k-1} N(q), \quad 2k < n \leq 3k + 2.$$
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