NEW EXPANSIONS OF NUMERICAL EIGENVALUES FOR
\(-\Delta u = \lambda \rho u\) BY NONCONFORMING ELEMENTS

QUN LIN, HUNG-TSAI HUANG, AND ZI-CAI LI

ABSTRACT. The paper explores new expansions of the eigenvalues for
\(-\Delta u = \lambda \rho u\) in \(S\) with Dirichlet boundary conditions by the bilinear element (denoted \(Q_1\)) and three nonconforming elements: the rotated bilinear element (denoted \(Q_1^{rot}\)), the extension of \(Q_1^{rot}\) (denoted \(EQ_1^{rot}\)) and Wilson’s elements. The expansions indicate that \(Q_1\) and \(Q_1^{rot}\) provide upper bounds of the eigenvalues, and that \(EQ_1^{rot}\) and Wilson’s elements provide lower bounds of the eigenvalues. By extrapolation, the \(O(h^4)\) convergence rate can be obtained, where \(h\) is the maximal boundary length of uniform rectangles. Numerical experiments are carried out to verify the theoretical analysis made.

1. INTRODUCTION

In this paper, we consider the eigenvalue problem
\[
-\Delta u = \lambda \rho u \quad \text{in } S, \\
\tag{1.1}
\]
\[
u = 0 \quad \text{in } \partial S, \\
\tag{1.2}
\]
where \(S = [0,1]^2\), the function \(\rho = \rho(x,y) > 0\) and \(\rho \in C^2(S)\). Then Eqs. (1.1) and (1.2) can be written in a weak form: To seek \((\lambda, u) \in R \times H_0^1(S)\) with \(u \neq 0\) such that
\[
a(u,v) = \lambda(u,v), \quad \forall v \in H_0^1(S),
\tag{1.3}
\]
where \(H_0^1(S) = \{ v \mid v \in H^1(S), v|_{\partial S} = 0 \}\), and
\[
a(u,v) = \iint_S \nabla u \nabla v, \\
\tag{1.4}
\]
\[
(u,v) = \iint_S \rho uv. \\
\tag{1.5}
\]

We choose one conforming element, the bilinear element \(Q_1\), and three nonconforming elements: the rotated \(Q_1\) (denoted \(Q_1^{rot}\)), the extension of \(Q_1^{rot}\) (denoted \(EQ_1^{rot}\)) and Wilson’s element. All the above elements are defined on rectangles \(\square_{ij}\) (see Figure 1), and their admissible functions are defined as follows.

(1) Bilinear element \(Q_1\). The piecewise interpolation functions \(u_I \in Q_1 = \text{span}\{1, x, y, xy\}\) are formulated as
\[
u(Z_i) = u_I(Z_i), \quad i = 1, 2, 3, 4,
\tag{1.6}
\]
where $Z_i$ are the four corners of $\square_{ij}$, and $\square_{ij} = \{(x, y) | x_i - h_i \leq x \leq x_i + h_i, y_j - k_j \leq y \leq y_j + k_j\}$.

(2) Rotated $Q_1$ element ($Q_1^{\text{rot}}$). The piecewise interpolation functions $u_I \in \text{span}\{1, x, y, x^2 - y^2\}$ are formulated by

$$\int_{\ell_k} u = \int_{\ell_k} u_I, \quad k = 1, 2, 3, 4,$$

where $\ell_k$ are the edges of $\square_{ij}$.

(3) Extension of $Q_1^{\text{rot}}$ ($EQ_1^{\text{rot}}$). The piecewise interpolation functions $u_I \in \text{span}\{1, x, y, x^2, y^2\}$ are formulated by

$$\int_{\ell_k} u = \int_{\ell_k} u_I, \quad k = 1, 2, 3, 4,$$

$$\int_{\square_{ij}} u = \int_{\square_{ij}} u_I.$$

(4) Wilson’s element. The piecewise interpolation functions $u_I \in P_2 = \text{span}\{1, x, y, xy, x^2, y^2\}$ are formulated by

$$u(Z_i) = u_I(Z_i), \quad i = 1, 2, 3, 4,$$

$$u_{xx}(O) = (u_I)_{xx}(O), \quad u_{yy}(O) = (u_I)_{yy}(O),$$

where $O$ is the center of $\square_{ij}$.

Let $S = \bigcup_{ij} \square_{ij}$, where $\square_{ij}$ are quasi-uniform. Denote by $V_h^0 \in L^2(S)$ the finite-dimensional collection of the admissible functions defined in $Q_1$, $Q_1^{\text{rot}}$, $EQ_1^{\text{rot}}$ and Wilson’s elements. The conforming $Q_1$ element is used to seek the solution $(\lambda_h, u_h) \in R \times V_h^0$ ($V_h^0 \subset H_0^1(S)$) such that

$$a(u_h, v) = \lambda_h(u_h, v), \quad \forall v \in V_h^0,$$

and the nonconforming elements, such as $Q_1^{\text{rot}}$, $EQ_1^{\text{rot}}$ and Wilson’s elements, are used to seek $(\lambda_h, u_h) \in R \times V_h^0$ such that

$$a_h(u_h, v) = \lambda_h(u_h, v), \quad \forall v \in V_h^0,$$

where $V_h^0$ is not a subset of $H_0^1(S)$.
where

\begin{equation}
\alpha_h(u,v) = \sum_{ij} \int_{\Delta_{ij}} \nabla u \cdot \nabla v.
\end{equation}

In \(Q_1\) and Wilson’s elements, the nodal variables are used, but in \(Q_1^{rot}\) and \(EQ_1^{rot}\), the line and the area variables are also chosen, which can be interpreted as the average values on the edges \(\partial \Delta_{ij}\) and those in the area \(\Delta_{ij}\). The line-area interpolation in \(Q_1^{rot}\) and \(EQ_1^{rot}\) is, rather than the nodal interpolation, advantageous in global superconvergence.

In this paper, we explore the expansions of the eigenvalues \(\lambda_h\). When \(\Delta_{ij}\) are uniform squares with the boundary length \(h\), we obtain the following formulas:

\begin{equation}
\lambda_h - \lambda = \begin{cases} 
\frac{h^2}{3} \int_{\Delta} (u_{xx}^2 + u_{yy}^2) + O(h^4), & \text{for } Q_1 \text{ element,} \\
\frac{h^2}{6} \int_{\Delta} (u_{xx} - u_{yy})^2 + O(h^4), & \text{for } Q_1^{rot} \text{ element,} \\
-\frac{2h^2}{3} \int_{\Delta} u_{xy}^2 + O(h^4), & \text{for } EQ_1^{rot} \text{ element,} \\
-\frac{2h^2}{3} \int_{\Delta} u_{xx}u_{yy} - \frac{h^2}{3} \int_{\Delta} [u_{xx}(u_h)_{yy} + u_{yy}(u_h)_{xx}] + O(h^4), & \text{for Wilson’s element.}
\end{cases}
\end{equation}

The detailed proof for \(Q_1^{rot}\) and \(EQ_1^{rot}\) elements is deferred to Section 3, and the proof for \(Q_1\) and Wilson’s elements will appear elsewhere. From the expansions of \(\lambda_h\) in (1.15), we may draw a few important conclusions:

1. Both \(Q_1\) and \(Q_1^{rot}\) provide an upper bound of \(\lambda\), but in contrast, \(EQ_1^{rot}\) and Wilson’s elements provide a lower bound of \(\lambda\). The lower estimation of \(\lambda\) is particularly interesting, because all conforming FEMs can only provide an upper estimation on \(\lambda\).

2. Suppose that \(\rho(x,y)\) is symmetric with respect to \(x\) and \(y\). For the minimal eigenvalue \(\lambda_{\text{min}} = \lambda_1\), since the corresponding eigenfunction satisfies \(u_{xx} = u_{yy}\), the \(Q_1^{rot}\) element yields the high \(O(h^4)\) convergence rate. Such an ultraconvergence of \(Q_1^{rot}\) is retained for any eigenvalue whose corresponding eigenfunction is symmetric with respect to \(x\) and \(y\).

3. The errors of \(\lambda\) by \(Q_1\), \(Q_1^{rot}\) and \(EQ_1^{rot}\) have the following relation:

\begin{equation}
E_{|Q_1^{rot}|} - \frac{1}{2}(E_{|Q_1|} + E_{|EQ_1^{rot}|}) = O(h^4),
\end{equation}

where \(E = \lambda_h - \lambda\).

4. By the extrapolation we may reach the high \(O(h^4)\) convergence rates for \(Q_1\), \(Q_1^{rot}\), \(EQ_1^{rot}\), and Wilson’s elements.

In our numerical experiments, the \(O(h^4)\) convergence rate has been confirmed by the extrapolation for all four elements, and the further extrapolation can be carried out for the \(Q_1\) element to reach the \(O(h^{2k})\) \((k \geq 2)\) convergence rates.

Let us mention the references related to this paper. Numerical eigenvalues are discussed in Babuska and Osborn [12, 3], Chatelin [6], Koluta [10], Mercier et al.
The nonconforming elements, such as the rotated bilinear element (i.e., $Q_{rot}^1$) and Wilson’s element, are studied in Chen and Li [7], Hu et al. [9], Lua and Lin [16], and Lin and Lin [13], and the extrapolations for eigenvalues are explored in Blum et al. [4], Lin [12], Lin and Zhu [14], and Lü et al. [15].

It is worth pointing out that asymptotic lower bounds for eigenvalues have been obtained by the finite difference method (FDM) in Forsythe [8] and Weinberger [21]. In [8], for a convex $S$, the numerical eigenvalues by the standard five-node finite difference equations have lower bounds, and upper and lower bounds of numerical eigenvalues by FDM are also discussed in [21]. Since the FDM can be regarded as a special kind of FEM involving different integration rules in Li [11], the variational crimes, the terminology used in [20] for FEM with nonconforming elements and numerical integration, may produce the lower bounds of approximate eigenvalues.

2. Basic theorems

We rewrite (1.3) as:

$$a(u, v) = (f, v), \forall v \in H^1_0(S),$$

where $f = \lambda u$. Define the finite element projection $R_h$ by

$$a_h(R_h u, v) = (f, v), \forall v \in V_h^0.$$  

For simplicity, we assume the simple eigenvalues, and consider only a few leading eigenvalues

$$\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k,$$

where $k$ is a small integer. Note that the minimal eigenvalue $\lambda_1 = \lambda_{\text{min}}$ is of great interest in practical applications.

For the above elements, we cite the known results in [23, 24] as a lemma.

Lemma 2.1. For the quasi-uniform $\boxdim_{ij}$ with the maximal boundary length $h$, there exists the following bound for leading eigenvalues $\lambda$ and their corresponding eigenfunctions $u$:

$$|\lambda - \lambda_h| + \|u - u_h\|_{0, S} + \|u - R_h u\|_{0, S} \leq Ch^2,$$

where $C$ is a constant independent of $h$, and $(\lambda_h, u_h)$ are the FEM solutions by $Q_1$, $Q_{rot}^1$, EQ_{rot}^1 and Wilson’s elements.

Below we give a new theorem.

Theorem 2.1 (Nonconforming). Let $\boxdim_{ij}$ be quasi-uniform with the maximal boundary length $h$. For the nonconforming elements, there exists the error formula

$$\lambda_h - \lambda = \lambda(u - u_I, u_h) - a_h(u - u_I, u_h) + a_h(u - R_h u, u_h) + O(h^4),$$

where $u$ and $u_I$ are the true solution (i.e., eigenfunction) and the FEM interpolation of $u$, respectively, and $u_h$ and $R_h u$ are the FEM solution of (1.13) and the FEM projection in (2.2), respectively.

Proof. For the eigenfunctions,

$$\langle u, u \rangle = 1, \quad \langle u_h, u_h \rangle = 1.$$
We choose a different scale of $u_h$ by $\bar{u}_h = \frac{u_h}{(u, u_h)}$. Then we have $(u, \bar{u}_h) = 1$, which yields
\begin{equation}
\lambda_h = \lambda_h(u, \bar{u}_h) = \lambda_h(R_h u, \bar{u}_h) + \lambda_h(u - R_h u, \bar{u}_h).
\end{equation}

Moreover, from (2.13) and (2.2), we obtain
\begin{equation}
\lambda_h(R_h u, \bar{u}_h) = a_h(R_h u, \bar{u}_h) = \lambda(u, \bar{u}_h) = \lambda.
\end{equation}

Since $\bar{u}_h$ has a small difference from $u_h$, we obtain from Lemma 2.1
\begin{equation}
\|\bar{u}_h - u_h\|_{0,S} = \|(u, u_h)\|_{0,S} \leq Ch^2.
\end{equation}

Hence by means of Lemma 2.1 again, a primary expansion from (2.7) to (2.9) is given by
\begin{equation}
\lambda_h = \lambda + \lambda_h(u - R_h u, \bar{u}_h) = \lambda + \lambda_h(u - R_h u, u_h) + O(h^4).
\end{equation}

Finally, a further expansion can be obtained:
\begin{equation}
\lambda_h = \lambda + \lambda_h(u - u_I, u_h) + \lambda_h(u_I - R_h u, u_h) + O(h^4)
= \lambda + \lambda_h(u - u_I, u_h) + a_h(u_I - R_h u, u_h) + O(h^4)
= \lambda + \lambda(u - u_I, u_h) + a_h(u_I - u, u_h) + a_h(u - R_h u, u_h) + O(h^4),
\end{equation}

where we have replaced $\lambda_h$ by $\lambda$ from Lemma 2.1. This is the desired result (2.10), and completes the proof of Theorem 2.1.

In Theorem 2.1 in order to derive the errors $\lambda_h - \lambda$, we need to evaluate the following interpolation errors:
\begin{equation}
(u - u_I, v), \ a_h(u - u_I, v), \ \forall v \in V_h^0,
\end{equation}

and the projection error
\begin{equation}
a_h(u - R_h u, v), \ \forall v \in V_h^0.
\end{equation}

Note that the projection error (2.13) is null for the conforming element, and that the estimation of (2.12) is similar to that for Poisson’s equation. Hence the key analysis of the nonconforming elements is to derive the expansions of (2.13). In this paper, the detailed proof is provided only for $Q_1^\text{rot}$ and $EQ_1^\text{rot}$ (see the next section), and the proof for the $Q_1$ and Wilson’s elements in (1.15) appears elsewhere.

In error estimates, we often use the Bramble-Hilbert lemma [5]: Denote by $B(u)$ a bounded linear function from $H^k(S)$ to $R$, if for a polynomial $P_k$ of degree $k$, $B(P_k) = 0$, then there exists a constant $C$ independent of $u$ such that
\begin{equation}
|B(u)| \leq C|u|_{k+1,S}.
\end{equation}

In this paper, we need more expansions of higher terms of degree $k + 1$. We solicit the generalized Bramble-Hilbert Lemma. Let
\begin{equation}
B(u) = \sum_{|\alpha| = k+1} \frac{B(x^\alpha)}{\alpha!|S|} \int_S D^\alpha u + H(u),
\end{equation}\footnote{For the conforming $Q_1$ element, the expansions of (2.12) will lead to those in (1.15), by using the same proof techniques in this paper.}
where \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \), \( \alpha_1 + \alpha_2 = \alpha \), and \( \alpha! = \alpha_1! \alpha_2! \). \( H(u) \) in (2.15) is also a bounded linear function from \( H^{k+1}(S) \) to \( R \). We write the following lemma without proof, whose proof is given in Lin and Lin [13].

**Lemma 2.2** (Generalized Bramble and Hilbert Lemma). Let \( u \in H^{k+2}(S) \) and \( B(P_k) = 0 \). Suppose that \( H(P_{k+1}) = 0 \) in (2.15). There exists a bound,

\[
|H(u)| \leq C|u|_{k+2,S},
\]

where \( C \) is a constant independent of \( u \).

### 3. \( Q_1^{rot} \) and \( EQ_1^{rot} \) Elements

In this paper, we will derive the expansions in (1.15) for \( Q_1^{rot} \) and \( EQ_1^{rot} \). We merge their proofs together, because the main proof for both nonconforming elements has many features in common. Based on Theorem 2.1, the three terms in (2.12) and (2.13) need to be evaluated. For both \( Q_1^{rot} \) and \( EQ_1^{rot} \), from their definition of \( u_I \) and by integration by parts, we can show the following equality easily:

\[
a_h(u - u_I, v) = \int\int_S \nabla (u - u_I) \nabla v = 0, \quad \forall v \in Q_1^{rot} \text{ or } EQ_1^{rot}.
\]

\[3.1\]

**Figure 2.** (1) \( \widehat{e} = [-1,1] \times [-1,1] \). (2) \( \Box_{ij} = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e] \).

To obtain the expansions of the other two terms in (2.12) and (2.13), we need the following lemmas.

**Lemma 3.1.** For \( v \in EQ_1^{rot} \) or \( Q_1^{rot} \), there exists the equality

\[
a_h(u - R_h u, v) = \sum_e \left[ \int\int_e u_{xy} v_y - \frac{4}{45} \int\int_e u_{xxy} v_{yy} + \frac{\varepsilon}{3} \int\int_e u_{xxy} v_{xy} + \frac{4}{45} \int\int_e u_{yyx} v_{xx} \right] + O(h^5)|u|_5|v|_{2,h},
\]

where \( |v|_{m,h} = \sqrt{\sum_m |v|_{m,e}^2} \) (\( m = 1,2 \)), and \( e = \Box_{ij} = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e] \) (see Figure 2). Moreover, for uniform rectangles \( \Box_{ij} \) with \( h_e = h \) and...
k_e = k, there exists the following equality for \( v \in EQ^1_{\text{rot}} \) or \( Q^1_{\text{rot}} \):

\[
(3.3) \quad a_h(u - R_h u, v) = -\frac{h^2 + k^2}{3} \int_S u_{xyy}v + O(h^4)\|u\|_S\|v\|_{1,h}.
\]

**Lemma 3.2.** For \( v \in EQ^1_{\text{rot}}(e) \),

\[
(3.4) \quad \int_e (u - u_I)v = -\frac{h^4}{45} \int_e u_{xx}v_x - \frac{k^4}{45} \int_e u_{yy}v_y + O(h^5)\|u\|_{4,e}\|v\|_{1,e}.
\]

**Lemma 3.3.** For \( v \in Q^1_{\text{rot}}(e) \),

\[
(3.5) \quad \int_e (u - u_I)v = -\frac{h^2}{6} \int_e (u_{xx} + u_{yy})v + \frac{h^4}{30} \int_e (u_{xxx}v_x + u_{yyy}v_y) + \frac{h^4}{18} \int_e (u_{xx}v_x + u_{yy}v_y) + O(h^5)\|u\|_{4,e}\|v\|_{1,e}.
\]

**Lemma 3.4.** For \( v \in EQ^1_{\text{rot}}(e) \) or \( Q^1_{\text{rot}}(e) \), there exists the integral equality

\[
(3.6) \quad \int_e u_{xx}(y - y_e)v(x, y_e) + ((y - y_e)^2 - \frac{k^2}{3})v_{yy}(x, y_e) = \frac{k^2}{3} \int_e u_{xyy}v_y - \frac{4k^4}{45} \int_e u_{xyy}v_{yy} + O(h^5)\|u\|_{5,e}\|v\|_{2,e}.
\]

The proof of Lemmas 3.2-3.4 is deferred to Sections 3.1-3.4. For \( EQ^1_{\text{rot}} \), we have the following theorem.

**Theorem 3.1.** Let \( \Box_{ij} \) be quasi-uniform. For \( EQ^1_{\text{rot}} \), there exists the eigenvalue error

\[
(3.7) \quad \lambda_h - \lambda = \frac{1}{3} \sum_e \left[ \int_e k_e^2 u_{xyy}u_y + \frac{h^2}{3} \int_e u_{xyy}u_x \right] + O(h^4).
\]

Moreover for uniform \( \Box_{ij} \),

\[
(3.8) \quad \lambda_h - \lambda = -\frac{h^2 + k^2}{3} \int_S u_{xy} + O(h^4).
\]

**Proof.** From Lemma 3.2,

\[
(3.9) \quad \lambda(u - u_I, u_h) = \lambda \int_S (u - u_I)\rho u_h
\]

\[
= \lambda \sum_e \left[ \int_e (u - u_I)(\rho u_h)_I - \int_e (u - u_I)(\rho u_h - (\rho u_h)_I) \right]
\]

\[
= -\lambda \sum_e \left[ \frac{h^4}{45} \int_e u_{xx}((\rho u_h)_I)_x - \frac{k^4}{45} \int_e u_{yy}((\rho u_h)_I)_y \right] + O(h^4) = O(h^4),
\]

where we have used

\[
(3.10) \quad \int_e (u - u_I)(\rho u_h - (\rho u_h)_I) = \int_e (u - u_I)u_h(\rho - \rho_I) = O(h^4).
\]

Also from Lemma 3.1

\[
(3.11) \quad a_h(u - R_h u, u_h) = \sum_e \frac{k^2}{3} \int_e u_{xyy}(u_h)_y + \sum_e \frac{h^2}{3} \int_e u_{xyy}(u_h)_x + O(h^4)
\]

\[
= \sum_e \frac{k^2}{3} \int_e u_{xyy}u_y + \sum_e \frac{h^2}{3} \int_e u_{xyy}u_x + O(h^3).
\]
Based on Theorem 2.1 combining (3.1), (3.9) and (3.11) yields the first desired result (3.7).

Next, we prove (3.8) for the uniform rectangles \( \Box_{ij} \). From Lemmas 3.1 and 2.1 and by integration by parts,

\[
a_h(u - R_h u, u_h) = \frac{h^2}{3} \sum_e \int_{e} u_{xyy}(u_h)_x \\
+ \frac{k^2}{3} \sum_e \int_{e} u_{xy}(u_h)_y + O(h^4) |u|_3 |u|_{2,h}
\]

(3.12)

where we have used the integration by parts again,

\[
\int_S u_{xyy}u = - \int_S u_{xxy}u_x + \int_S u_{xyy}^2,
\]

and

\[
|u_h|_{2,h} \leq |u_h - u_I|_{2,h} + |u_I - u|_{2,h} + |u|_2
\]

\[
\leq Ch^{-1}|u_h - u_I|_{1,h} + C|u|_2 \leq C|u|_2.
\]

Based on Theorem 2.1, combining (3.1), (3.9) and (3.11) yield the second desired result (3.8) (i.e., (1.15) for \( EQ_1^{\text{rot}} \) with \( k_e = h_e = h \)). This completes the proof of Theorem 3.1.

Below, for \( Q_1^{\text{rot}} \), we have the following theorem.

**Theorem 3.2.** Let \( \Box_{ij} \) be uniform squares. For \( Q_1^{\text{rot}} \) there exists the eigenvalue error

\[
\lambda_h - \lambda = \frac{h^2}{6} \int_S (u_{xx} - u_{yy})^2 + O(h^4).
\]

**Proof.** For \( Q_1^{\text{rot}} \) on uniform square \( \Box_{ij} \) with \( h = k \), we have from Lemmas 3.3 and 2.1

\[
\lambda(u - u_I, u_h) = \lambda \int_S (u - u_I)(\rho u_h)_I + O(h^4)
\]

\[
= -\lambda \frac{h^2}{6} \int_S (u_{xx} + u_{yy})(\rho u_h)_I + O(h^4)
\]

\[
= -\lambda \frac{h^2}{6} \int_S (u_{xx} + u_{yy})\rho u + O(h^4)
\]

(3.16)

where we have used (1.1). From integration by parts, there exists the equality

\[
\int_S u_{xx}u_{yy} = \int_S u_{xy}^2.
\]

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\[ \ell_2 \]
\[ \ell_3 \]
\[ e \]
\[ \ell_1 \]
\[ \ell_4 \]

Figure 3. The rectangle.

For uniform squares $\square_{ij}$ with $h = k$, based on Theorem 2.1 combining (3.1), (3.12), (3.15) and (3.16) yields the desired result (3.15) (i.e., (1.15) for $Q^1_{\text{rot}}$). This completes the proof of Theorem 3.2.

Theorems 3.1 and 3.2 provide the desired expansions in (1.15) for $EQ^1_{\text{rot}}$ and $Q^1_{\text{rot}}$ elements. It is interesting to note that $Q^1_{\text{rot}}$ and $EQ^1_{\text{rot}}$ give the upper and the lower bounds of the leading eigenvalues, respectively.

3.1. Proof of Lemma 3.1 For the nonconforming errors of $Q^1_{\text{rot}}$ and $EQ^1_{\text{rot}}$, we have from (2.1), (2.2) and the Green formula,

\[ a_h(u - R_h u, v) = \sum_e \int_{\partial e} \frac{\partial u}{\partial n} v \, ds \]

\[ = \sum_e (\int_{\ell_1} - \int_{\ell_3}) u_x v \, dy + \sum_e (\int_{\ell_2} - \int_{\ell_4}) u_y v \, dx, \]

where $\ell_i$ are the edges in Figure 3. Since the average on $\ell_k$ is continuous based on the definitions in (1.7) and (1.8), we have

\[ \sum_e (\int_{\ell_1} - \int_{\ell_3}) u_x \bar{v} \, dy = 0, \]

where $\bar{v} = \int_{\ell_i} v \, ds/|\ell_i|$ is constant on $\ell_i$. Hence we obtain

\[ \sum_e (\int_{\ell_1} - \int_{\ell_3}) u_x v \, dy = \sum_e (\int_{\ell_2} - \int_{\ell_4}) u_x (v - \bar{v}) \, dy. \]

Also since $v|_{\ell_1 \cup \ell_3} = \text{span}\{1, y - y_e, (y - y_e)^2\}$, then $\bar{v}|_{\ell_1 \cup \ell_3} = \text{span}\{1, 0, \frac{k^2}{3}\}$. Moreover, from Taylor’s formula in the $y$ variable for each $x$, we have

\[ (v - \bar{v})|_{\ell_i} = (y - y_e) v_y(x, y_e) + \left((y - y_e)^2 - \frac{k^2}{3}\right) \frac{v_{yy}(x, y_e)}{2}, \quad i = 1, 3. \]
Table 1. The integration, \( \iint_{\Omega}(u - u_I)v \) for \( u \in P_3 \setminus EQ_1^{rot}, v \in EQ_1^{rot} = \text{span}\{1, x, y, x^2, y^2\} \), where \( \hat{c} = [-1, 1]^2 \) and the sign \( 0^+ \) denotes that the computed integrals are zero.

<table>
<thead>
<tr>
<th>( u )</th>
<th>( xy )</th>
<th>( x^3 )</th>
<th>( x^2y )</th>
<th>( xy^2 )</th>
<th>( y^3 )</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_I )</td>
<td>0</td>
<td>( x )</td>
<td>( \frac{1}{2}y )</td>
<td>( \frac{1}{4}x )</td>
<td>( y )</td>
<td>/</td>
</tr>
<tr>
<td>( u - u_I )</td>
<td>( xy )</td>
<td>( x^3 - x )</td>
<td>( x^2y - \frac{1}{2}y )</td>
<td>( xy^2 - \frac{1}{4}x )</td>
<td>( y^3 - y )</td>
<td>/</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\int_{\Omega}(u - u_I) & = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad v = 1 \\
\int_{\Omega}(u - u_I)x & = 0 \quad -\frac{8}{15} \quad 0 \quad 0^+ \quad 0 \quad v = x \\
\int_{\Omega}(u - u_I)y & = 0 \quad 0 \quad 0 \quad 0 \quad -\frac{8}{15} \quad v = y \\
\int_{\Omega}(u - u_I)x^2 & = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad v = x^2 \\
\int_{\Omega}(u - u_I)y^2 & = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad v = y^2
\end{align*}
\]

Then

\[
\left( \int_{\Omega} \int_{\Omega} u_x(v - \bar{v}) \right) dy
\]

\[
= \left( \int_{\Omega} \int_{\Omega} u_x \left( (y - y_e)v_y(x, y_e) + \left( (y - y_e)^2 - \frac{k_e^2}{3} \frac{v_y(y, y_e)}{2} \right) \right) \right) dy
\]

\[
\int_{\Omega} \int_{\Omega} u_{xx} \left( (y - y_e)v_y(x, y_e) + \left( (y - y_e)^2 - \frac{k_e^2}{3} \frac{v_y(x, y_e)}{2} \right) \right) dxdy,
\]

where we have used that

\[
\left( (y - y_e)v_y(x, y_e) + \left( (y - y_e)^2 - \frac{k_e^2}{3} \frac{v_y(x, y_e)}{2} \right) \right)_x = 0,
\]

based on \( v_{xy} = v_{xxy} = 0 \) for \( Q_1^{rot} \) and \( EQ_1^{rot} \) elements.

Similarly, we have

\[
\left( \int_{\Omega} \int_{\Omega} u_y(v - \bar{v}) \right) dx
\]

\[
= \int_{\Omega} \int_{\Omega} u_{yy} \left( (x - x_e)v_x(x_e, y) + \left( (x - x_e)^2 - \frac{k_e^2}{3} \frac{v_x(x_e, y)}{2} \right) \right) dxdy,
\]

Hence for both \( Q_1^{rot} \) and \( EQ_1^{rot} \), we obtain from \( 3.16 \), \( 3.22 \) and \( 3.24 \),

\[
a_h(u - R_hu, v) = \sum_e \int_{\Omega} \int_{\Omega} \left( (y - y_e)v_y(x, y_e) + \left( (y - y_e)^2 - \frac{k_e^2}{3} \frac{v_y(x, y_e)}{2} \right) \right)
\]

\[
+ \sum_e \int_{\Omega} \int_{\Omega} \left( (x - x_e)v_x(x, y_e) + \left( (x - x_e)^2 - \frac{k_e^2}{3} \frac{v_x(x_e, y)}{2} \right) \right).
\]

The desired result (3.22) in Lemma 3.1 follows from Lemma 3.3. This completes the proof of Lemma 3.1 \( \square \)

3.2. Proof of Lemma 3.2 Denote \( B(u, v) = \iint_{\Omega}(u - u_I)v \), where \( \hat{c} = [-1, 1]^2 \) in Figure 2. For \( u \in EQ_1^{rot} \), we have \( \iint_{\Omega}(u - u_I)v = 0 \). For \( u \in P_3 \setminus EQ_1^{rot} \), the integration terms needed are given in Table I. In Table I and other tables given below, the zero values can be easily seen by checking odd polynomials with respect
to $x$ or $y$, and the zero values with “$+$” in the tables are confirmed by real integral evaluation. Hence, we only examine those zeros with “$+$” and the nontrivial terms. First, take $u = x^2y$ and $v = y$ for example. We have

$$B(x^2y, y) = \iint_\mathcal{E} (x^2y - \frac{y}{3})y = 0. \quad (3.26)$$

Similarly, for $u = xy^2$ and $v = x$,

$$B(xy^2, x) = 0. \quad (3.27)$$

Next, we examine the nontrivial terms in Table 1. When $u = x^3$ and $v = x$,

$$B(x^3, x) = \iint_\mathcal{E} (x^3 - x)x = \iint_\mathcal{E} (x^4 - x^2) = -\frac{8}{15} = -\frac{1}{45} \iint_\mathcal{E} u_{xxx}v_x. \quad (3.28)$$

Similarly, when $u = y^3$ and $v = y$,

$$B(y^3, y) = \iint_\mathcal{E} (y^3 - y)y = \iint_\mathcal{E} (y^4 - y^2) = -\frac{8}{15} = -\frac{1}{45} \iint_\mathcal{E} u_{yyy}v_y. \quad (3.29)$$

Define the new function

$$H(u, v) = B(u, v) + \frac{1}{45} \iint_\mathcal{E} u_{xxx}v_x + \frac{1}{45} \iint_\mathcal{E} u_{yyy}v_y. \quad (3.30)$$

Hence for $u \in P_3$, $H(u, v) = 0$, and then from Lemma 2.2,

$$|H(u, v)| \leq C|u|_{4, \mathcal{E}}|v|_{1, \mathcal{E}}. \quad (3.31)$$

Denote $e = \square_{ij} = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$ with the boundary lengths $2h_e$ and $2k_e$ (see Figure 2), where $h_e = O(h)$, $k_e = O(h)$ max$\left\{\frac{h_e}{k_e}, \frac{k_e}{h_e}\right\} \leq C_0$, and $C_0$ is a constant independent of $h$. Define an affine transformation $T : (x, y) \rightarrow (\hat{x}, \hat{y})$ with

$$\hat{x} = \frac{x - x_e}{h_e}, \quad \hat{y} = \frac{y - y_e}{k_e}. \quad (3.32)$$

Then, under $T$, we have that $e \rightarrow \hat{e} = [-1, 1]^2$ and the following equations:

$$\hat{u}(\hat{x}, \hat{y}) = u(x, y), \quad \hat{u}_I(\hat{x}, \hat{y}) = u_I(x, y),$$

$$\frac{d\hat{x}}{h_e}, \quad \frac{d\hat{y}}{k_e},$$

$$\hat{u}_{\hat{x}} = h_e u_x, \quad \hat{u}_{\hat{y}} = k_e u_y.$$  

By the affine transformation $T$ in (3.32) we have

$$\int_e (u - u_I)v = h_e k_e \int_{\hat{e}} (\hat{u} - \hat{u}_I)v = \frac{h_e k_e}{45} \int_{\hat{e}} u_{xxx}v_x - \frac{1}{45} \int_{\hat{e}} u_{yyy}v_y + O(1)|u|_{4, \mathcal{E}}|v|_{1, \mathcal{E}} \quad (3.33)$$

This is the desired result (3.31) and completes the proof of Lemma 3.2.  

\footnote{For simplicity, we omit the hat notation on the top in the integral of $\hat{e}$. For instance, the integration $\int_{\hat{e}} \hat{u}_{\hat{x}} \hat{v}_x$ is simplified as $\iint_{\hat{e}} u_{xxx}v_x$ in (3.33).}
3.3. Proof of Lemma 3.3. Denote $B(u, v) = \iint_{\mathcal{E}} (u - u_I)v$. For $u \in P_3 \setminus Q_1^{\text{rot}}$, the integration is given in Table 2. Let us check the terms with $0^+$ and the nontrivial terms in Table 2. First for $u = x^2$ and $v = x^2 - y^2$, we have

\begin{equation}
\int_{\mathcal{E}} (u - u_I)v = \int_{\mathcal{E}} \left[ \frac{1}{2}(x^2 + y^2) - \frac{2}{3}(x^2 - y^2) \right] = 0,
\end{equation}

where we have used the symmetry: $\iint_{\mathcal{E}} x^2 = \iint_{\mathcal{E}} y^2$ and $\iint_{\mathcal{E}} x^4 = \iint_{\mathcal{E}} y^4$. Similarly, for $u = y^2$ and $v = x^2 - y^2$,

\begin{equation}
\int_{\mathcal{E}} (u - u_I)v = 0.
\end{equation}

Next, we examine the nontrivial terms. When $u = x^2$ and $v = 1$,

\begin{equation}
\int_{\mathcal{E}} (u - u_I)v = \int_{\mathcal{E}} \left[ \frac{1}{2}(x^2 + y^2) - \frac{2}{3} \right] = -\frac{4}{3} = -\frac{4}{3} \iint_{\mathcal{E}} u_{xx}v.
\end{equation}

Similarly, when $u = y^2$ and $v = 1$,

\begin{equation}
\int_{\mathcal{E}} (u - u_I)v = -\frac{4}{3} = -\frac{4}{3} \iint_{\mathcal{E}} u_{yy}v.
\end{equation}

Define a functional

\begin{equation}
H(u, v) = B(u, v) + \frac{1}{6} \iint_{\mathcal{E}} u_{xx}v + \frac{1}{6} \iint_{\mathcal{E}} u_{yy}v.
\end{equation}

Hence for $u \in P_2$, $H(u, v) = 0$, $\forall v \in Q_1^{\text{rot}}$, and then from Lemma 2.2,

\begin{equation}
\forall \eta \in [3, \varepsilon], |H(u, v)| \leq C|u|_{[3, \varepsilon]}|v|_{0, \varepsilon}.
\end{equation}
Then we have
\begin{equation}
B(u, v) = -\frac{1}{6} \int \int u_{xx}v - \frac{1}{6} \int \int u_{yy}v + O(1)|u|_{4, \tilde{\varepsilon}}|v|_{0, \tilde{\varepsilon}}.
\end{equation}

Below, we consider the additional terms in \( P_3 \setminus P_2 \), whose results are also listed in Table 2. First, when \( u = x^2y \) and \( v = y \), we have
\begin{equation}
\int \int (u - u_I)v = \int \int (x^2y - \frac{1}{3}y)y = 0,
\end{equation}
and when \( u = xy^2 \) and \( v = x \),
\begin{equation}
\int \int (u - u_I)v = \int \int (xy^2 - \frac{1}{3}x)x = 0.
\end{equation}
Next, when \( u = x^3 \) and \( v = x \),
\begin{equation}
\int \int (u - u_I)v = \int \int (x^3 - x)x = \int \int (x^4 - x^2) = -\frac{8}{15},
\end{equation}
and when \( u = y^3 \) and \( v = y \), similarly
\begin{equation}
\int \int (u - u_I)v = -\frac{8}{15}.
\end{equation}
Now we have to recount \( H(u, v) \) for those extra nontrivial terms of \( P_3 \setminus P_2 \), and obtain from (3.38):
\begin{enumerate}
\item When \( u = x^2y \) and \( v = y \),
\[ H(u, v) = B(x^2y, y) + \frac{1}{6} \int \int u_{xx}v + \frac{1}{6} \int \int u_{yy}v = 0 + \frac{4}{9} + 0 = \frac{1}{18} \int \int u_{xxyy}. \]
\item When \( u = xy^2 \) and \( v = x \), similarly
\[ H(u, v) = \frac{4}{9} = \frac{1}{18} \int \int u_{xxyy}. \]
\item When \( u = x^3 \) and \( v = x \),
\[ H(u, v) = B(x^3, x) + \frac{1}{6} \int \int u_{xx}v = -\frac{8}{15} + \frac{1}{6} \int \int 6x^2 \]
\[ = -\frac{8}{15} + \frac{4}{3} = \frac{1}{30} \int \int u_{xxx}v. \]
\item When \( u = y^3 \) and \( v = y \), similarly
\[ H(u, v) = \frac{4}{5} = \frac{1}{30} \int \int u_{yyy}v. \]
\end{enumerate}

Hence we define a new functional
\begin{equation}
X(u, v) = H(u, v) - \frac{1}{30} \int \int u_{xxx}v - \frac{1}{30} \int \int u_{yyy}v
\end{equation}
\[ - \frac{1}{18} \int \int u_{xxyy} - \frac{1}{18} \int \int u_{xyy}v. \]

Obviously, for \( u \in P_3 \), \( H(u, v) = 0 \), \( v \in Q^1 \), and then from Lemma 2.2
\[ X(u, v) \leq C|u|_{4, \tilde{\varepsilon}}|v|_{1, \tilde{\varepsilon}}. \]
Then, we conclude that
\[
B(u, v) = -\frac{1}{6} \int_{\bar{\varepsilon}} (u_{xx} + u_{yy})v + \frac{1}{30} \int_{\bar{\varepsilon}} (u_{xxx} + u_{yyy})v + \frac{1}{18} \int_{\bar{\varepsilon}} (u_{xxy} + u_{yyx})v_x + O(1)|u|^3|v|_{1, \bar{\varepsilon}}.
\]

The desired result (3.5) follows by the proof techniques via the affine transformation $T$ in (3.32). This completes the proof of Lemma 3.3.

Table 3. The integration, $\int_{\bar{\varepsilon}} u_{xxx} D(v)$ for $u \in P_4 \setminus EQ_1^{rot}$, $v \in EQ_1^{rot} = \text{span}\{1, x, y, x^2, y^2\}$ and $D(v) \in \text{span}\{1, 0, 1, 0, y^2 - \frac{1}{3}\}$, where $\bar{\varepsilon} = [-1, 1]^2$ and the sign “/” denotes the zero of integrals due to $u_{xx} = 0$.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$1$</th>
<th>$x$</th>
<th>$y$</th>
<th>$x^2$</th>
<th>$y^2$</th>
<th>$xy$</th>
<th>$x^3$</th>
<th>$x^2y$</th>
<th>$xy^2$</th>
<th>$y^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{xx}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>6$x$</td>
<td>2$y$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\int_{\bar{\varepsilon}} u_{xx} y$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>0</td>
<td>/</td>
<td>/</td>
<td>0</td>
<td>$\frac{8}{3}$</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>$\int_{\bar{\varepsilon}} u_{xx} (y^2 - \frac{1}{2})$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>0$^*$</td>
<td>/</td>
<td>/</td>
<td>0</td>
<td>0</td>
<td>/</td>
<td>/</td>
</tr>
</tbody>
</table>

$^*$ Denote $D(v) = y v_y (x, 0) + \left(y^2 - \frac{1}{3}\right) v_{yy}(x, 0)$ on $\bar{\varepsilon}$ and

(3.46) \[
B(u, v) = \int_{\bar{\varepsilon}} u_{xx} \left[y v_y (x, 0) + \left(y^2 - \frac{1}{3}\right) v_{yy}(x, 0)\right] = \int_{\bar{\varepsilon}} u_{xx} D(v),
\]

where $v \in \text{span}\{1, x, y, x^2, y^2\}$ and $D(v) = \text{span}\{0, 0, y, 0, y^2 - \frac{1}{3}\}$. We list in Table 3 the integration $\int_{\bar{\varepsilon}} u_{xx} D(v)$ for $u \in P_4$ and $v \in EQ_1^{rot}$. Let us check the terms with $0^*$ and the nontrivial terms in Table 3. First, when $u = x^2$, $v = y^2$ and $D(v) = y^2 - \frac{1}{3}$, the integral is zero:

\[
\int_{\bar{\varepsilon}} u_{xx} D(v) = 2 \int_{\bar{\varepsilon}} \left(y^2 - \frac{1}{3}\right) = 0.
\]

Hence for $u \in P_2$, $B(u, v) = 0$, $v \in EQ_1^{rot}$, and then from Lemma 2.2

$|B(u, v)| \leq C|u|^3|v|_{1, \bar{\varepsilon}}$.

Next consider $u \in P_3/P_2$. When $u = x^2y$ and $v = y$,

(3.47) \[
\int_{\bar{\varepsilon}} u_{xx} D(v) = \int_{\bar{\varepsilon}} 2y^2 = \frac{8}{3} = \frac{1}{3} \int_{\bar{\varepsilon}} u_{xxy} v_y.
\]

Also $B(u, v) = 0$ for $u = x^3, x^2 y, y^3$ and $v \in EQ_1^{rot}$ (see Table 3). Define a functional

$H(u, v) = B(u, v) - \frac{1}{3} \int_{\bar{\varepsilon}} u_{xxy} v_y$. 
For \( u \in P_3 \), \( H(u, v) = 0 \). From Lemma 2.2
\[
|H(u, v)| \leq C|u|_{4, \varepsilon}|v|_{1, \varepsilon},
\]
which yields
\[
B(u, v) = \frac{1}{3} \int \int_{\varepsilon} u_{xxy} v_y + O(1)|u|_{4, \varepsilon}|v|_{1, \varepsilon}.
\]

By the affine transformation \( (3.32) \), we have
\[
\int \int_{\varepsilon} u_{xxy} \left[ (y - y_c)v_y(x, y_c) + \left( (y - y_c)^2 - \frac{k_v^2}{3} \right) \frac{v_y}{2} \right] = \frac{k_v}{h_v} B(\tilde{u}, \tilde{v}) = \frac{k_v}{h_v} \left[ \frac{1}{3} \int \int_{\varepsilon} \tilde{u}_{xxy} \tilde{v}_y + O(1)|\tilde{u}|_{4, \varepsilon}|\tilde{v}|_{1, \varepsilon} \right] = \frac{k_v^2}{3} \int \int_{\varepsilon} u_{xxy} v_y + O(h^3)|u|_{4, \varepsilon}|v|_{1, \varepsilon}.
\]

To discover the higher remainders of \( O(h^4) \), we should also consider \( v \in P_4 \setminus P_3 \); the additional integrations are listed in Table 3. Below we consider the nontrivial terms only. For \( u = x^2y^2, v = y^2 \) and \( D(v) = y^2 - \frac{1}{3} \), we have
\[
\int \int_{\varepsilon} u_{xxy} D(v) = \int \int_{\varepsilon} 2y^2 \left( y^2 - \frac{1}{3} \right) = \frac{32}{45},
\]
which gives
\[
H(u, v) = H(x^2y^2, y^2) = B(x^2y^2, y^2) - \frac{1}{3} \int \int_{\varepsilon} u_{xxy} v_y = \frac{32}{45} - \frac{1}{3} \int \int_{\varepsilon} 4y \cdot 2y = \frac{32}{45} - \frac{32}{9} = -\frac{128}{45} = \frac{4}{45} \int \int_{\varepsilon} u_{xxy} v_{yy}.
\]

Now we define a new functional
\[
X(u, v) = H(u, v) + \frac{4}{45} \int \int_{\varepsilon} u_{xxy} v_{yy}.
\]

Hence for \( u \in P_4 \), \( X(u, v) = 0 \), and then from Lemma 2.2
\[
|X(u, v)| \leq C|u|_{5, \varepsilon}|v|_{2, \varepsilon}.
\]

This yields
\[
B(u, v) = \frac{1}{3} \int \int_{\varepsilon} u_{xxy} v_y + \frac{4}{45} \int \int_{\varepsilon} u_{xxy} v_{yy} + O(1)|u|_{5, \varepsilon}|v|_{2, \varepsilon}.
\]

The desired result \( (3.6) \) in Lemma 3.4 for \( EQ_{1}^{rot} \) follows from the affine transformation \( T \) in \( (3.32) \).

Next for \( Q_{1}^{rot} \), we have from Table 3
\[
|B(u, v)| \leq C|u|_{3, \varepsilon}|v|_{1, \varepsilon}.
\]

The rest of the proof is exactly the same as that for \( EQ_{1}^{rot} \). This completes the proof of Lemma 3.4. \( \square \)
4. Numerical experiments

In this section, we provide two numerical experiments of the four elements, $Q_1, Q_1^{rot}, EQ_1^{rot}$ and Wilson’s element for solving (1.1) and (1.2).

4.1. Function $\rho = 1$. Consider the eigenvalue problem of Laplace’s operator with $\rho = 1$,

$$- \triangle u = - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \lambda u \text{ in } S,$$
$$u = 0 \text{ on } \Gamma = \partial S,$$

where $S = \{(x, y), 0 \leq x, y \leq 1\}$. Then we have the exact eigenfunctions and eigenvalues

$$u_{k, \ell} = 2 \sin(k \pi x) \sin(\ell \pi y), \quad \lambda_{k, \ell} = (k^2 + \ell^2)\pi^2, \quad 1 \leq k, \ell \leq N - 1. \quad (4.1)$$

Since the minimal and the next minimal eigenvalues, denoted by $\lambda_1$ and $\lambda_2$, are the most interesting, we only provide their computed results. In Tables 4 and 5, we list the numerical eigenvalues, their errors and the ratios $\frac{|\lambda_{k, \ell}|}{\lambda_{k, \ell}}$ for all four elements, where $\varepsilon_k = \lambda_k - \lambda$, and $\lambda_h$ and $\lambda$ are the approximate and the true eigenvalues, respectively. Denote $h = 1/(2N)$ from Figure 2, and $N = 2^m$, $m = 1, 2, \ldots$. When $|\lambda_{k, \ell}| \approx 2^p$, we may conclude the empirical convergence rates $O(h^p)$.

For the $Q_1$, the $EQ_1^{rot}$ and the Wilson’s element, we can see from Tables 4 and 5 that

$$\lambda_{1, h} - \lambda = O(h^2), \quad (4.2)$$

where $\lambda_{1, h}$ denotes the computed $\lambda_{\ell}$ ($\ell = 1, 2$) at the mesh size $h$. However, for the $Q_1^{rot}$,

$$\lambda_{1, h} - \lambda_1 = O(h^4), \quad (4.3)$$
$$\lambda_{2, h} - \lambda_2 = O(h^2). \quad (4.4)$$

Equations (4.2)–(4.4) agree with those in (1.15) perfectly. The high convergence rate $O(h^4)$ in (4.3) results from the symmetry of $u_{xx} = u_{yy}$ for the eigenfunction $u(x, y)$ corresponding to $\lambda_1$.

From Table 4 we can find the following relative errors of $\lambda_1$ at $N = 32$:

$$\frac{\lambda_{1, h} - \lambda_1}{\lambda_1} = 0.803(-3), -0.387(-6), -0.802(-3), -0.240(-2), \quad (4.5)$$

for $Q_1, Q_1^{rot}, EQ_1^{rot}$ and Wilson’s elements, respectively. From (4.5) we can see that $Q_1$ provides an upper bound due to a positive relative error, and $EQ_1^{rot}$ and Wilson’s elements provide lower bounds due to negative relative errors. From Tables 4 and 5, the $Q_1^{rot}$ provide the lower and the upper bounds for $\lambda_1$ and $\lambda_2$, respectively.

To verify (1.16) we have computed $E_h = E_h|_{EQ_1^{rot}} - \frac{1}{2}(E_h|_{Q_1} + E_h|_{EQ_1^{rot}})$, where $E_h = \lambda_{1, h} - \lambda_1$. Table 6 lists the results to display the $O(h^4)$ convergence rate perfectly.

More importantly, the expansions of eigenvalues can be applied to raise the accuracy by the extrapolation techniques. Based on the computed eigenvalues in

5The constant 2 of the eigenfunctions in (4.1) is used for $(u, u) = 1$. 
Tables 4 and 5 we may use the following extrapolation formulas for \( \lambda_{1,h} \):

\[
\lambda_h^{(k)} = \frac{2^{2k} \lambda_h^{(k-1)} - \lambda_{2h}^{(k-1)}}{2^{2k} - 1}, \quad k = 1, 2, 3, 4,
\]

for \( Q_1 \), \( EQ_1^{rot} \) and Wilson's elements, where \( \lambda_0^h = \lambda_h \). Eq. (4.6) is also used for \( \lambda_{2,h} \) by the \( Q_1^{rot} \). Since the \( \lambda_{1,h} \) by the \( Q_1^{rot} \) has the higher convergence rate, the following extrapolation formulas should be used:

\[
\lambda_h^{(k)} = \frac{2^{2k+2} \lambda_h^{(k-1)} - \lambda_{2h}^{(k-1)}}{2^{2k+2} - 1}, \quad k = 1, 2, 3, 4.
\]

Note that in (4.6) and (4.7), \( \lambda_h^{(1)} \) denotes the first level of extrapolation. In computation, we have computed from the first to the fourth levels of extrapolation. Such a procedure is like that in the Romberg integration. All the extrapolation results are listed in Tables 7 and 8 for \( \lambda_{1,h} \) by \( Q_1^{rot} \) and \( EQ_1^{rot} \). From Tables 7 and 8 we can see

\[
\lambda_{1,h}^{(1)} - \lambda = O(h^4) \text{ for } EQ_1^{rot},
\]

\[
\lambda_{1,h}^{(1)} - \lambda = O(h^6) \text{ for } Q_1^{rot},
\]

where \( \lambda_{1,h}^{(1)} \) is the better approximation of \( \lambda_{1,h} \) at the first level of extrapolation. Below, we list the following eigenvalues at the first and fourth levels of extrapolation:

\[
\lambda_{1,h}^{(1)} - \lambda_1 = 0.472(-9), \quad -0.512(-5),
\]

\[
\lambda_{1,h}^{(4)} - \lambda_1 = -0.135(-13), \quad -0.454(-8),
\]

for \( Q_1^{rot} \) and \( EQ_1^{rot} \) at \( N = 32 \) respectively. Evidently, the errors in (4.10) and (4.11) are much smaller than those in (4.5). Interestingly, the \( \lambda_{1,h}^{(4)} = 19.73920880217845 \) by the \( Q_1^{rot} \) has 14 significant digits, which is the most accurate value in our computation.

Suppose that we only carry out the computation for \( N = 2, 4, 8 \), but not for \( N = 16 \) and \( N = 32 \) due to some reasons (e.g., the limitation of computer memory or the CPU time). Based on those results, we may use (4.6) and (4.7) until the second level of extrapolation only. The corresponding results are found from Tables 7 and 8 at \( N = 8 \):

\[
\left| \frac{\lambda_{1,h}^{(2)} - \lambda_1}{\lambda_1} \right| = 0.422(-6), \quad 0.347(-3),
\]

for \( Q_1^{rot} \) and \( EQ_1^{rot} \) respectively. The relative errors in (4.12) are close to those in (4.5), but their signs may be changed. This fact displays a significance of the extrapolation, based on the expansions of eigenvalue solutions given in this paper.

The above examination is for the convergence rate; it is crucial to scrutinize numerically the principal terms of the error expansions in (4.10). First, take \( EQ_1^{rot} \) for \( \lambda_1 \) for example. Since the corresponding eigenfunction \( u_{1,1} = 2 \sin(\pi x) \sin(\pi y) \) from (4.3), we have the principal term from (4.15),

\[
E_1 = \frac{-2h^2}{3} \int \int_S u_{xy}^2 = \frac{-2h^2 \pi^4}{3} = -\frac{\pi^4}{6N^2},
\]
where we have used $h = \frac{1}{2N}$. Then the relative value is given by

\[
\bar{\epsilon}_1 = \frac{E_1}{\lambda_1} = -\frac{\pi^4}{6N^2(2\pi^2)} = -\frac{\pi^2}{12N^2}.
\]

Based on (4.14), for $N = 2, 4, 8, 16, 32$, we obtain respectively

\[
(4.15) \bar{\epsilon}_1 = -0.206, -0.514(-1), -0.129(-1), -0.321(-2), -0.803(-3).
\]

Eq. (4.15) coincides with the numerical data in Table 4 for $\rho$ for Eq. (4.18) also coincides with the numerical data in Table 5 for $\lambda$ such as those in Tables 10 and 11 can also be computed.


\[
\text{(4.16)} \quad E_1 = \frac{h^2}{6} \int \int_S (u_{xx} - u_{yy})^2 = \frac{3h^2\pi^4}{2} = \frac{3\pi^4}{8N^2},
\]

which gives

\[
\bar{\epsilon}_2 = \frac{E_2}{\lambda_2} = \frac{3\pi^4}{8N^2(5\pi^2)} = \frac{3\pi^2}{40N^2}.
\]

Based on (4.17), for $N = 2, 4, 8, 16, 32$, we obtain respectively

\[
(4.18) \quad \bar{\epsilon}_2 = 0.185, 0.463(-1), 0.116(-1), 0.289(-2), 0.723(-3).
\]

Eq. (4.16) also coincides with the numerical data in Table 5 for $Q_1^{rot}$, which verifies the principal term in (4.14).

4.2. Function $\rho \neq 1$. Since the error analysis is valid for the function $\rho = \rho(x, y) \geq \rho_0 > 0$, to verify the analysis made, we also carry out the numerical experiments for $\rho \neq 1$. Choose

\[
\rho = \rho(x, y) = 1 + (x - \frac{1}{2})(y - \frac{1}{2}),
\]

which is symmetric with respect to $x$ and $y$. We have

\[
- \Delta u = - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \lambda \rho u \text{ in } S,
\]

\[
u = 0 \text{ on } \Gamma = \partial S,
\]

where $S$ is also the unit square. For the $\rho$ in (4.19), we may evaluate $\int \int_S \rho uv$ in (4.2) exactly. The FEM as (1.3) can be easily performed. We provide the results for $\lambda_1$ by $Q_1^{rot}$ and $EQ_1^{rot}$ only, and list them in Tables 10 and 11. Since for $\rho$ in (4.19), the true solution of $\lambda_1$ is unknown, we may compute the ratios of sequential errors to display the empirical convergence rates. The numerical solutions, the sequential errors and their ratios are listed in Tables 10 and 11 for $Q_1^{rot}$ and $EQ_1^{rot}$. Since only the sign of $\varepsilon^{(0)}$ is significant, it is listed in Tables 10 and 11. From Table 11 we can see the sequential errors.

\footnote{An a posteriori error may be evaluated as follows. Since $Q_1^{rot}$ may provide the most accurate solution, we may choose $\lambda_1^{(0)} = 19.73225552487$ in Table 10 as the true solution. Then the errors such as those in Tables 10 and 11 can also be computed.}
NEW EXPANSIONS OF NUMERICAL EIGENVALUES FOR $-\Delta u = \lambda \rho u$

(4.20) \[
\frac{\lambda_{1,2h} - \lambda_{1,4h}}{\lambda_{1,h} - \lambda_{1,2h}} = O(h^2),
\]
for $EQ_1^{rot}$ elements. However, from Table [10]

(4.21) \[
\frac{\lambda_{1,2h} - \lambda_{1,4h}}{\lambda_{1,h} - \lambda_{1,2h}} = O(h^4),
\]
for the $Q_1^{rot}$ element. The empirical convergence rates of $\lambda_1$ are exactly the same as those in Section 4.1 for $\rho = 1$.

4.3. Numerical conclusions. Based on the numerical results, we may draw a few important conclusions:

1. The $Q_1$ and the $EQ_1^{rot}$ provide the upper and the lower bounds respectively. The $Q_1^{rot}$ provides the lower bound for $\lambda_2$ and other $\lambda$ whose corresponding function $u$ satisfies $u_{xx} \neq u_{yy}$.

2. For the minimal eigenvalue $\lambda_{\min} = \lambda_1$, the corresponding eigenfunctions satisfy $u_{xx} = u_{yy}$, and the $Q_1^{rot}$ element yields the high $O(h^4)$ convergence rates. Such an ultraconvergence of $Q_1^{rot}$ holds for any eigenvalues whose eigenfunctions are symmetric with respect to $x$ and $y$.

3. We list in Table [6] the computed results, to show the validation of (1.16).

4. By the first level of extrapolation, the superconvergence $O(h^4)$ can be obtained by all four FEMs.

5. For $Q_1^{rot}$, the ultraconvergence for $\lambda_1$ as

$$\lambda^{(i)}_{1,h} - \lambda_1 = O(h^{2i+4}), \quad i = 0, 1, 2, 3,$$

can be achieved numerically by multiple levels of extrapolation; see Table [8].

6. The principal terms of the eigenvalue errors for $Q_1^{rot}$ and $EQ_1^{rot}$ have been verified by our numerical experiments.

Concluding remarks. The new expansions of numerical eigenvalues by four FEMs are summarized in [11,12], whose proof for the two nonconforming elements $Q_1^{rot}$ and $EQ_1^{rot}$ is provided in this paper. Not only can [11,12] display an upper or a lower bound of the FEM solution of leading eigenvalues, but it can also lead to higher superconvergence rates by the extrapolation techniques. All the theoretical analyses have been verified by the numerical experiments in Section 4. Moreover, the best convergence rates have been obtained numerically by multiple levels of extrapolation for both $Q_1^{rot}$ and $EQ_1^{rot}$ elements.

\[\text{7Numerically, the } Q_1^{rot} \text{ also provides the lower bound of } \lambda_1, \text{ based on Table [4] for } \rho = 1, \text{ and on Table [10] for } \rho \neq 1.\]
Table 4. The first eigenvalue solutions $\lambda_{1,h}$ for $-\Delta u = \lambda u$ by the four FEMs, where the true $\lambda_1 = \frac{2\pi^2}{d} = 19.73920880217872$, $\varepsilon_h = \frac{\lambda_{1,h} - \lambda_1}{\lambda_1}$, Ratio $= \left|\frac{\varepsilon_h}{\varepsilon_h}\right|$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,h}$ by $Q_1$</td>
<td>24.00000</td>
<td>20.75328</td>
<td>19.99416</td>
<td>19.802707</td>
<td>19.755068</td>
</tr>
<tr>
<td>$\lambda_{1,h}$ by $Q_1^{rot}$</td>
<td>19.29000</td>
<td>19.707183</td>
<td>19.737241</td>
<td>19.739086</td>
<td>19.739201</td>
</tr>
<tr>
<td>$\lambda_{1,h}$ by $EQ_1^{rot}$</td>
<td>16.848019</td>
<td>18.818638</td>
<td>19.491886</td>
<td>19.676196</td>
<td>19.723380</td>
</tr>
<tr>
<td>$\lambda_{1,h}$ by Wilson’s</td>
<td>13.321013</td>
<td>17.296011</td>
<td>19.023223</td>
<td>19.551919</td>
<td>19.691833</td>
</tr>
<tr>
<td>$\varepsilon_h$ by $Q_1$</td>
<td>-0.216</td>
<td>-0.524(-1)</td>
<td>-0.129(-1)</td>
<td>-0.322(-2)</td>
<td>-0.803(-5)</td>
</tr>
<tr>
<td>$\varepsilon_h$ by $Q_1^{rot}$</td>
<td>-0.273(-1)</td>
<td>-0.162(-2)</td>
<td>-0.997(-4)</td>
<td>-0.620(-5)</td>
<td>-0.387(-6)</td>
</tr>
<tr>
<td>$\varepsilon_h$ by $EQ_1^{rot}$</td>
<td>-0.146</td>
<td>-0.466(-1)</td>
<td>-0.125(-1)</td>
<td>-0.319(-2)</td>
<td>-0.802(-3)</td>
</tr>
<tr>
<td>$\varepsilon_h$ by Wilson’s</td>
<td>-0.325</td>
<td>-0.124</td>
<td>-0.363(-1)</td>
<td>-0.949(-2)</td>
<td>-0.240(-2)</td>
</tr>
<tr>
<td>Ratio by $Q_1$</td>
<td>/</td>
<td>4.12</td>
<td>4.06</td>
<td>4.02</td>
<td>4.00</td>
</tr>
<tr>
<td>Ratio by $Q_1^{rot}$</td>
<td>/</td>
<td>16.8</td>
<td>16.3</td>
<td>16.1</td>
<td>16.0</td>
</tr>
<tr>
<td>Ratio by $EQ_1^{rot}$</td>
<td>/</td>
<td>3.14</td>
<td>3.72</td>
<td>3.92</td>
<td>3.98</td>
</tr>
<tr>
<td>Ratio by Wilson’s</td>
<td>/</td>
<td>2.63</td>
<td>3.41</td>
<td>3.82</td>
<td>3.95</td>
</tr>
</tbody>
</table>

Table 5. The second eigenvalue solutions $\lambda_{2,h}$ for $-\Delta u = \lambda u$ by the four FEMs, where the true $\lambda_2 = \frac{5\pi^2}{d} = 49.3480220054$, $\varepsilon_h = \frac{\lambda_{2,h} - \lambda_2}{\lambda_2}$, Ratio $= \left|\frac{\varepsilon_h}{\varepsilon_h}\right|$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{2,h}$ by $Q_1$</td>
<td>/</td>
<td>58.3866</td>
<td>51.3436</td>
<td>49.8897</td>
<td>49.4829</td>
</tr>
<tr>
<td>$\lambda_{2,h}$ by $Q_1^{rot}$</td>
<td>58.5306</td>
<td>51.3290</td>
<td>49.9022</td>
<td>49.4897</td>
<td>49.3836</td>
</tr>
<tr>
<td>$\lambda_{2,h}$ by $EQ_1^{rot}$</td>
<td>31.6696</td>
<td>46.3304</td>
<td>48.3088</td>
<td>49.0993</td>
<td>48.3394</td>
</tr>
<tr>
<td>$\lambda_{2,h}$ by Wilson’s</td>
<td>23.8182</td>
<td>38.6181</td>
<td>45.7010</td>
<td>48.3394</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_h$ by $Q_1$</td>
<td>/</td>
<td>0.183</td>
<td>0.445(-1)</td>
<td>0.110(-1)</td>
<td>0.273(-2)</td>
</tr>
<tr>
<td>$\varepsilon_h$ by $Q_1^{rot}$</td>
<td>0.180</td>
<td>0.401(-1)</td>
<td>0.112(-1)</td>
<td>0.287(-2)</td>
<td>0.722(-3)</td>
</tr>
<tr>
<td>$\varepsilon_h$ by $EQ_1^{rot}$</td>
<td>-0.156</td>
<td>-0.611(-1)</td>
<td>-0.190(-1)</td>
<td>-0.504(-2)</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_h$ by Wilson’s</td>
<td>-0.558</td>
<td>-0.217</td>
<td>-0.739(-1)</td>
<td>-0.204(-1)</td>
<td></td>
</tr>
<tr>
<td>Ratio by $Q_1$</td>
<td>/</td>
<td>/</td>
<td>4.12</td>
<td>4.05</td>
<td>4.01</td>
</tr>
<tr>
<td>Ratio by $Q_1^{rot}$</td>
<td>/</td>
<td>4.64</td>
<td>3.57</td>
<td>3.91</td>
<td>3.98</td>
</tr>
<tr>
<td>Ratio by $EQ_1^{rot}$</td>
<td>/</td>
<td>2.55</td>
<td>3.21</td>
<td>3.78</td>
<td></td>
</tr>
<tr>
<td>Ratio by Wilson’s</td>
<td>/</td>
<td>2.57</td>
<td>2.94</td>
<td>3.62</td>
<td></td>
</tr>
</tbody>
</table>

Table 6. The errors $\lambda_{1,h} - \lambda_1$ for $Q_1$, $Q_1^{rot}$ and $EQ_1^{rot}$, where $E = \frac{\lambda_{1,h} - \lambda_1}{\lambda_1}$, $\tilde{E}_h = E_h|Q_1^{rot} - \frac{1}{2}(E_h|Q_1 + E_h|EQ_1^{rot})$, Ratio $= \tilde{E}_{2h}/\tilde{E}_h$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,h}$ by $Q_1$</td>
<td>24.00000000000</td>
<td>20.7532840104</td>
<td>19.994161312</td>
<td>19.8027075368</td>
<td>19.7550682451</td>
</tr>
<tr>
<td>$\lambda_{1,h}$ by $Q_1^{rot}$</td>
<td>19.29000000000</td>
<td>19.7071838998</td>
<td>19.7372410126</td>
<td>19.7390863687</td>
<td>19.7392011588</td>
</tr>
<tr>
<td>$\lambda_{1,h}$ by $EQ_1^{rot}$</td>
<td>16.84801900000</td>
<td>18.8186378768</td>
<td>19.4918862529</td>
<td>19.6761956158</td>
<td>19.7237988271</td>
</tr>
<tr>
<td>$E_h$ by $Q_1$</td>
<td>-4.26079</td>
<td>-1.03408</td>
<td>-0.254953</td>
<td>-0.635686(-1)</td>
<td>-0.158594(-1)</td>
</tr>
<tr>
<td>$E_h$ by $Q_1^{rot}$</td>
<td>-0.539099</td>
<td>-0.320261(-1)</td>
<td>-0.196779(-2)</td>
<td>-0.122343(-3)</td>
<td>-0.764336(-5)</td>
</tr>
<tr>
<td>$E_h$ by $EQ_1^{rot}$</td>
<td>-2.89119</td>
<td>-0.920571</td>
<td>-0.247323</td>
<td>-0.630126(-1)</td>
<td>-0.158299(-1)</td>
</tr>
<tr>
<td>Ratio</td>
<td>/</td>
<td>13.79</td>
<td>15.35</td>
<td>15.83</td>
<td>15.96</td>
</tr>
</tbody>
</table>
Table 7. The first $\lambda_{1,h}$ by extrapolation from the $Q_1^{rot}$ solutions, where the true $\lambda_1 = 2\pi^2 \approx 19.73920880217872$, $\lambda_{1,h}^{(k)} = \frac{\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{k+1} - 1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{1,h}^{(k)} - \lambda_1}{\lambda_1}$, Ratio$(k) = |\varepsilon_h^{(k)} / \varepsilon_h^{(0)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda_{1,h}$</th>
<th>$\lambda_{1,h}^{(1)}$</th>
<th>$\lambda_{1,h}^{(2)}$</th>
<th>$\lambda_{1,h}^{(3)}$</th>
<th>$\lambda_{1,h}^{(4)}$</th>
</tr>
</thead>
</table>

| $\varepsilon_h^{(0)}$ | -0.273(-1) | -0.162(-2) | -0.997(-4) | -0.629(-5) | -0.387(-6) |
| $\varepsilon_h^{(1)}$ | -0.905(-4) | 0.183(-5) | 0.299(-7) | 0.472(-9) | -0.472(-9) |
| $\varepsilon_h^{(2)}$ | 0.422(-6) | 0.135(-8) | -0.297(-9) | -0.304(-12) | -0.304(-12) |
| $\varepsilon_h^{(3)}$ | -0.135(-13) | -0.135(-13) | -0.135(-13) | -0.135(-13) | -0.135(-13) |

Table 8. The first $\lambda_{1,h}$ by extrapolation from the $E_{Q_1^{rot}}$ solutions, where the true $\lambda_1 = 2\pi^2 \approx 19.73920880217872$, $\lambda_{1,h}^{(k)} = \frac{\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{k+1} - 1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{1,h}^{(k)} - \lambda_1}{\lambda_1}$, Ratio$(k) = |\varepsilon_h^{(k)} / \varepsilon_h^{(0)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda_{1,h}$</th>
<th>$\lambda_{1,h}^{(1)}$</th>
<th>$\lambda_{1,h}^{(2)}$</th>
<th>$\lambda_{1,h}^{(3)}$</th>
<th>$\lambda_{1,h}^{(4)}$</th>
</tr>
</thead>
</table>

| $\varepsilon_h^{(0)}$ | -0.146 | -0.466(-1) | -0.125(-1) | -0.319(-2) | -0.802(-3) |
| $\varepsilon_h^{(1)}$ | -0.134(-1) | -0.116(-2) | -0.798(-4) | -0.512(-5) | -0.512(-5) |
| $\varepsilon_h^{(2)}$ | -0.347(-3) | -0.241(-5) | -0.241(-5) | -0.241(-5) | -0.241(-5) |
| $\varepsilon_h^{(3)}$ | -0.454(-8) | -0.454(-8) | -0.454(-8) | -0.454(-8) | -0.454(-8) |

| Ratio$(0)$ | 3.14 | 3.72 | 3.92 | 3.98 |
| Ratio$(1)$ | 11.5 | 14.5 | 15.6 |
| Ratio$(2)$ | 44.5 | 57.5 |
| Ratio$(3)$ | 17.3 |
Table 9. The second $\lambda_{2,h}$ by extrapolation from the $Q^{rot}_{1}$ solutions, where the true $\lambda_{2} = 5\pi^{2} \approx 49.34802200544679$, $\lambda_{2,h}^{(k)} = \frac{2^{k} \lambda_{2}^{(k-1)} - \lambda_{2,h}^{(k-1)}}{2^{k} - 1}$, $\varepsilon_{h}^{(k)} = \frac{\lambda_{2}^{(k)} - \lambda_{2,h}^{(k)}}{\lambda_{2}}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_{h}^{(k)}|$, $\lambda_{2,h}^{(0)} = \lambda_{2,h}$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>N</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{2,h}$</td>
<td>58.5365853659</td>
<td>51.3289965050</td>
<td>49.9021574073</td>
<td>49.4897065300</td>
<td>49.3836320249</td>
</tr>
<tr>
<td>$\lambda_{2,h}^{(1)}$</td>
<td>/</td>
<td>48.9264668847</td>
<td>49.4265443748</td>
<td>49.352229042</td>
<td>49.3482738566</td>
</tr>
<tr>
<td>$\lambda_{2,h}^{(2)}$</td>
<td>/</td>
<td>/</td>
<td>49.4598828741</td>
<td>49.347681395</td>
<td>49.3480105867</td>
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<tr>
<td>$\lambda_{2,h}^{(3)}$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>49.345360400</td>
<td>49.348023716</td>
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<tr>
<td>$\lambda_{2,h}^{(4)}$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>49.348032393</td>
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<tr>
<td>$\varepsilon_{h}^{(0)}$</td>
<td>0.186</td>
<td>0.401(-1)</td>
<td>0.112(-1)</td>
<td>0.287(-2)</td>
<td>0.722(-3)</td>
</tr>
<tr>
<td>$\varepsilon_{h}^{(1)}$</td>
<td>/</td>
<td>-0.854(-2)</td>
<td>0.159(-2)</td>
<td>0.851(-4)</td>
<td>0.510(-5)</td>
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<tr>
<td>$\varepsilon_{h}^{(2)}$</td>
<td>/</td>
<td>/</td>
<td>0.227(-2)</td>
<td>-0.153(-4)</td>
<td>-0.23(-6)</td>
</tr>
<tr>
<td>$\varepsilon_{h}^{(3)}$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>-0.515(-4)</td>
<td>0.742(-8)</td>
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<tr>
<td>$\varepsilon_{h}^{(4)}$</td>
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<td>/</td>
<td>/</td>
<td>/</td>
<td>0.209(-6)</td>
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<td>Ratio(0)</td>
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<td>4.64</td>
<td>3.57</td>
<td>3.91</td>
<td>3.98</td>
</tr>
<tr>
<td>Ratio(1)</td>
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<td>/</td>
<td>5.37</td>
<td>18.7</td>
<td>16.7</td>
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<tr>
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<td>/</td>
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<td>66.0</td>
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<tr>
<td>Ratio(3)</td>
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<td>/</td>
<td>/</td>
<td>6940</td>
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</tbody>
</table>

Table 10. The first $\lambda_{1,h}$ for $-\Delta u = \lambda p u$ by $Q^{rot}_{1}$, where $\lambda_{1,h}^{(k)} = \frac{2^{k+1} \lambda_{1}^{(k-1)} - \lambda_{1,h}^{(k-1)}}{2^{k+1} - 1}$, $\varepsilon_{h}^{(k)} = \lambda_{1}^{(k)} - \lambda_{1,h}^{(k)}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_{h}^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>N</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,h}$</td>
<td>19.2000000000</td>
<td>19.703183280</td>
<td>19.7309925515</td>
<td>19.7352149854</td>
<td>19.732281452</td>
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<tr>
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<td>19.7345933897</td>
<td>19.7323399681</td>
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<td>19.7322532892</td>
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<td>$\lambda_{1,h}^{(2)}$</td>
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<td>/</td>
<td>19.7323041995</td>
<td>19.7322562394</td>
<td>19.7322552534</td>
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<tr>
<td>$\lambda_{1,h}^{(3)}$</td>
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<td>/</td>
<td>/</td>
<td>19.7322560513</td>
<td>19.7322552495</td>
</tr>
<tr>
<td>$\lambda_{1,h}^{(4)}$</td>
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<td>/</td>
<td>/</td>
<td>/</td>
<td>19.7322552487</td>
</tr>
<tr>
<td>$\varepsilon_{h}^{(0)}$</td>
<td>/</td>
<td>-0.501</td>
<td>-0.292(-1)</td>
<td>-0.175(-2)</td>
<td>-0.107(-3)</td>
</tr>
<tr>
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<td>/</td>
<td>0.225(-2)</td>
<td>0.824(-4)</td>
<td>0.226(-5)</td>
</tr>
<tr>
<td>$\varepsilon_{h}^{(2)}$</td>
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<td>/</td>
<td>/</td>
<td>-0.480(-4)</td>
<td>0.986(-6)</td>
</tr>
<tr>
<td>$\varepsilon_{h}^{(3)}$</td>
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<td>/</td>
<td>/</td>
<td>/</td>
<td>0.802(-6)</td>
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<tr>
<td>Ratio(0)</td>
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<td>/</td>
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<td>16.71</td>
<td>16.32</td>
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<td>48.64</td>
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NEW EXPANSIONS OF NUMERICAL EIGENVALUES FOR $-\Delta u = \lambda \rho u$  2083

Table 11. The first $\lambda_{1,h}$ for $-\Delta u = \lambda \rho u$ by $EQ^{rot}_{1}$, where $\lambda^{(k)}_1 = \frac{2^{2k} \lambda^{(k-1)}_1 - \lambda^{(k-1)}_{2h}}{2^{2k-1}}$, $\varepsilon^{(k)}(k) = \lambda^{(k)}_{1,h} - \lambda^{(k)}_{1,2h}$, Ratio(k) = $|\varepsilon^{(k)}(k)/\varepsilon^{(k)}(h)|$, $\lambda^{(0)}_{1,h} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
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<td>$\lambda_{1,h}$</td>
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<td>19.5800759724</td>
<td>19.6984491727</td>
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<td>/</td>
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<tr>
<td>$</td>
<td>\varepsilon^{(2)}_h</td>
<td>$</td>
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<tr>
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<td>\varepsilon^{(3)}_h</td>
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</table>

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REFERENCES


Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, China

E-mail address: qlin@lsec.cc.ac.cn

Department of Applied Mathematics, I-Shou University, Taiwan 840

E-mail address: huanght@isu.edu.tw

(Responding author) Department of Applied Mathematics, and Department of Computer Science and Engineering, National Sun Yat-sen University, Kaohsiung, Taiwan 80424

E-mail address: zcli@math.nsysu.edu.tw