POSITIVE QUADRATURE FORMULAS III: ASYMPTOTICS OF WEIGHTS

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ABSTRACT. First we discuss briefly our former characterization theorem for positive interpolation quadrature formulas (abbreviated qf), provide an equivalent characterization in terms of Jacobi matrices, and give links and applications to other qf, in particular to Gauss-Kronrod quadratures and recent rediscoveries. Then for any polynomial t_n which generates a positive qf, a weight function (depending on n) is given with respect to which t_n is orthogonal to $\mathbb{P}_{n-1}$. With the help of this result an asymptotic representation of the quadrature weights is derived. In general the asymptotic behaviour is different from that of the Gaussian weights. Only under additional conditions do the quadrature weights satisfy the so-called circle law. Corresponding results are obtained for positive qf of Radau and Lobatto type.

1. INTRODUCTION

Let σ be a positive measure on $[-1,+1]$ normed by $\int_{-1}^{+1} d\sigma = 1$ and such that the support of $d\sigma$ contains an infinite set of points. We call an interpolatory quadrature formula (abbreviated qf) of the form

$$(1.1) \qquad \int_{-1}^{+1} f(x)d\sigma = \sum_{j=1}^{n} \lambda_{j,n}(d\sigma)f(x_{j,n}) + R_n(f),$$

where $-1 < x_1 < x_2 < \cdots < x_n < 1$ and $R_n(f) = 0$ for $f \in \mathbb{P}_{2n-1-m}$ ($\mathbb{P}_n$ denotes as usual the set of polynomials of degree at most n), $0 \leq m \leq n$, a $(2n-1-m, n, d\sigma)$ qf; if $d\sigma = w(x)dx$, then we write $\lambda_{j,n}(w)$, $(2n-1-m, n, w)$qf, ..., and if there is no confusion possible $\lambda_j, x_j, \ldots; \lambda_j^G$ denotes the Gaussian weights. Furthermore, we say that a polynomial $t_n \in \mathbb{P}_n$ generates a $(2n-1-m, n, d\sigma)$ qf if t_n has n simple zeros $x_1 < x_2 < \cdots < x_n$ in $(-1,1)$ and if the interpolatory qf based on the nodes x_j, $j = 1, \ldots, n$, is a $(2n-1-m, n, d\sigma)$ qf. For $q_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ we denote by $q_n^{[1]}$ the polynomial of second kind of q_n with respect to $d\sigma$, i.e.,

$$(1.2) \qquad q_n^{[1]}(y) = \int_{-1}^{+1} \frac{q_n(y) - q_n(x)}{y - x} d\sigma(x).$$

Note that $q_n^{[1]} \in \mathbb{P}_{n-1}$. With the help of (1.2) we obtain for the weights λ_j of the interpolatory qf (1.1) the known formula

(1.3) $\lambda_j = \int_{-1}^{+1} \frac{t_n(x)}{(x-x_j)t_n'(x_j)}d\sigma(x) = \frac{t_n^{[1]}(x_j)}{t_n'(x_j)}$ for $j = 1,\ldots,n$,

where

(1.4) $$t_n(x) = \prod_{j=1}^{n}(x-x_j).$$

Hence the positivity of the λ_j's is equivalent to the interlacing property of the zeros of t_n and $t_n^{[1]}$. Taking into consideration the well-known fact (see e.g. [1]) that a polynomial t_n with n simple zeros generates a $(2n-1-m, n, d\sigma)$ qf if and only if it is orthogonal to $\mathbb{P}_{n-1-m}$ with respect to $d\sigma$, i.e.,

(1.5) $$\int_{-1}^{+1} x^j t_n(x)d\sigma(x) = 0 \quad \text{for } j = 0,\ldots,n-1-m,$$

we get the following first characterization of positive qf.

Lemma 1.1. *Let $n, m \in \mathbb{N}_0$. Then t_n generates a positive $(2n-1-m, n, d\sigma)$ qf if and only if t_n is orthogonal to $\mathbb{P}_{n-1-m}$ with respect to $d\sigma$, t_n and $t_n^{[1]}$ have all zeros in $(-1,1)$ and they strictly interlace.*

Recall the simple fact that (1.5) implies that t_n has a representation of the form

(1.6) $$t_n(x) = \sum_{j=0}^{m} \mu_j p_{n-j},$$

where $\mu_j \in \mathbb{R}$ and p_n denotes the monic polynomial of degree n which is orthogonal on $[-1,1]$ to $\mathbb{P}_{n-1}$ with respect to $d\sigma$, i.e.,

(1.7) $$\int_{-1}^{+1} x^j p_n(x)d\sigma(x) = 0 \quad \text{for } j = 0,\ldots,n-1.$$

It is well known that (p_n) satisfies a recurrence relation of the form

(1.8) $p_n(x) = (x-\alpha_n)p_{n-1}(x) - \beta_n p_{n-2}(x) \quad n = 1,2,\ldots,$

where $p_{-1} := 0$, $p_0 := 1$ and the β_n's are positive. Polynomials with property (1.5), respectively of the form (1.6), are called quasi-orthogonal polynomials also.

In the following we will also need the fact that the associated polynomials $(p_n^{(k)})$ of order k, $k \in \mathbb{N}_0$, defined by

(1.9) $p_n^{(k)}(x) = (x-\alpha_{n+k})p_{n-1}^{(k)}(x) - \beta_{n+k}p_{n-2}^{(k)}(x),$

$p_{-1}^{(k)} := 0, p_0^{(k)} := 1$ are, by Favard's Theorem (see [12]), orthogonal with respect to a positive measure, denoted by $\sigma^{(k)}$, the support of which is, since all zeros of $p_n^{(k)}$ lie in $(-1,+1)$, contained in $[-1,+1]$, i.e.,

(1.10) $$\int_{-1}^{+1} x^j p_n^{(k)}(x)d\sigma^{(k)}(x) = 0 \quad \text{for } j = 0,\ldots,n-1,$$

The measure $\sigma^{(k)}$ can be given explicitly; see [13] and [18, Thm. 3.9].

2. Review of the Characterization Theorem and applications

Theorem 2.1 (Characterization Theorem). *Let $n, m \in \mathbb{N}_0$, $n \geq m$, put $l := [(m+1)/2]$ and let t_n be a monic polynomial of degree n. The following statements are equivalent:*

(a) *t_n generates a positive $(2n - 1 - m, n, d\sigma)$ qf.*

(b) *t_n is orthogonal with respect to a sequence $\{c_j\}_0^{2n-1}$ which is positive definite on $[-1, 1]$ and satisfies the condition*

$$
(2.1) \qquad c_j = \int_{-1}^1 x^j d\sigma(x) \quad \text{for } j = 0, \ldots, 2n - 1 - m.
$$

(c) *t_n can be generated by a recurrence relation of the form*

$$
(2.2) \qquad t_j(x) = (x - \tilde{\alpha}_j)t_{j-1}(x) - \tilde{\beta}_j t_{j-2}(x) \quad j = 1, \ldots, n,
$$

$t_{-1}(x) := 0$, $t_0(x) := 1$, with $\tilde{\alpha}_j \in \mathbb{R}$, $\tilde{\beta}_j > 0$ and $\operatorname{sgn} t_j(\pm 1) = (\pm 1)^j$ for $j = 1, \ldots, n$, and

$$
(2.3) \qquad \tilde{\alpha}_j = \alpha_j \text{ for } j = 1, \ldots, n - [\tfrac{m+1}{2}] \text{ and } \tilde{\beta}_j = \beta_j \text{ for } j = 2, \ldots, n - [\tfrac{m}{2}].
$$

(d) *t_n has a representation of the form*

$$
(2.4) \qquad t_n(x) = g_l(x)p_{n-l}(x) - \tilde{\beta}_{n+1-l}g_{l-1}(x)p_{n-l-1}(x),
$$

where $\tilde{\beta}_{n+1-l} > 0$ and $\tilde{\beta}_{n+1-l} = \beta_{n+1-l}$ if $m = 2l - 1$ and the monic polynomials g_l and g_{l-1} are generated by a recurrence relation of the form

$$
(2.5) \qquad g_j(x) = (x - \tilde{\alpha}_{n+1-j})g_{j-1}(x) - \tilde{\beta}_{n+2-j}g_{j-2}(x) \qquad j = 1, \ldots, l,
$$

$g_{-1}(x) := 0$, $g_0(x) := 1$, with $\tilde{\alpha}_{n+1-j} \in \mathbb{R}$, $\tilde{\beta}_{n+2-j} > 0$ and $\operatorname{sgn} g_j(\pm 1) = (\pm 1)^j$ for $j = 1, \ldots, l$.

(e) *There are polynomials $g_l(x) = x^l + \ldots$, $g_{l-1}(x) = x^{l-1} + \ldots$, whose zeros are simple, strictly interlacing, and located in $(-1, 1)$ such that*

$$
(2.6) \qquad t_n(x) = g_l(x)p_{n-l}(x) - \tilde{\beta}_{n+1-l}g_{l-1}(x)p_{n-l-1}(x)
$$

with $\tilde{\beta}_{n+1-l} > 0$ and $\tilde{\beta}_{n+1-l} = \beta_{n+1-l}$ if $m = 2l - 1$ and $\operatorname{sgn} t_n(\pm 1) = (\pm 1)^n$.

(f) *There are polynomials r_l, s_{l-1} of degree l and $l - 1$ whose zeros are simple, strictly interlacing, and located in $(-1, 1)$ such that*

$$
(2.7) \qquad t_n(x) = r_l(x)p_{n-l}(x) - (1 - x^2)s_{l-1}(x)p_{n-l-1}^{(1-x^2)}(x)
$$

with $\operatorname{sgn} t_n(\pm 1) = (\pm 1)^n$, where $p_{n-l-1}^{(1-x^2)}$ denotes the monic polynomial of degree $n - l - 1$ orthogonal with respect to $(1 - x^2)d\sigma$. If $m = 2l - 1$, then r_l, respectively, s_{l-1} has leading coefficient $(1 - a_{2(n-l)-1})/2$ and $(1 + a_{2(n-l)-1})/2$, where $a_{2(n-l)-1}$ is defined in (3.6) below.

First, characterization (f) has been given by the author [15] (in terms of orthogonal polynomials on the unit circle) and about the same time, independently, a weaker version of (e) by Sottas and Wanner [28]. They describe positive qf having all their nodes real and simple but not necessarily located in a given interval $[a, b]$. A little bit later characterizations (c) and (d) were discovered by the author in [16, Thm. 2] (concerning (d) see the proof of Thm. 2); (b) has been added in [19]. H. J. Schmid [23] came up with an alternative approach to Theorem 2.1(b) and (c).

Y. Xu [33, 34] studied the representation of the nodes polynomial t_n as a characteristic polynomial of a symmetric tridiagonal matrix with positive subdiagonal entries. It might be worth mentioning that Theorem 2.1 can now be proved by elementary methods; see [19, Proof of Thm. 3.2].

By Theorem 2.1(c) and Favard's Theorem or directly by (b) it follows immediately that t_n is orthogonal to $\mathbb{P}_{n-1}$ with respect to a measure depending on n. This fact has been observed by several authors [16, 27, 34]. The explicit determination of such a measure was expected to be a difficult task; see e.g. [34]. Using the version of (f) in terms of orthogonal polynomials on the unit circle (abbreviated OPUC) and some facts on OPUC's, such a measure will be derived in Theorem 3.1 below in a relatively simple way. It is the basis for the derivation of asymptotics for the weights.

For computational purposes it might be more convenient to describe positive qf by the associated Jacobi matrices; compare [5, 7]. $J_n^G(d\sigma)$ denotes the Jacobi matrix associated with the Gauss qf with respect to $d\sigma$. We may reformulate the equivalence (a) $\Leftrightarrow$ (c) of the Characterization Theorem 2.1 with the help of Jacobi matrices as follows:

Corollary 2.2. $x_1, x_2, \ldots, x_n, -1 < x_1 < x_2 < \cdots < x_n < 1$, are the nodes of a positive $(2n + 1 - m, n, d\sigma)$ qf if and only if $x_1, \ldots, x_n \in \mathbb{R}$ are the eigenvalues of a Jacobi matrix J_n of the form, $l = [\frac{m+1}{2}]$,

$$(2.8) \qquad J_n = \begin{pmatrix} J_{n-l}^G(d\sigma) & \tilde{\beta}_{n-l+1}\vec{e}_{n-l+1} \\ \tilde{\beta}_{n-l+1}\vec{e}_{n-l} & \tilde{J}_l \end{pmatrix}$$

where $\tilde{J}_l$ is a Jacobi matrix with spectrum contained in $(-1, 1)$, $\tilde{\beta}_{n-l+1} > 0$ and such that the spectrum of J_n is contained in $(-1, 1)$ and $\tilde{\beta}_{n-l+1} = \beta_{n-l+1}$ if $m = 2l - 1$; $\vec{e}_j$ denotes the j-th coordinate vector.

Proof. Necessity follows by (2.2) and (2.3) and the well-known connection with Jacobi matrices. Note that $\tilde{J}_l$ is the Jacobi matrix associated with the orthogonal polynomial g_l from (2.5).

Sufficiency. Since J_n is a Jacobi matrix, it is associated with an orthogonal polynomial t_n with zeros $x_1, \ldots, x_n$, $x_1 < x_2 < \cdots < x_n$, which satisfies a recurrence relation of the form (2.2). By the form of the Jacobi matrix J_n it follows that (2.3) is satisfied. Hence, t_n is of the form (2.6), where g_l is associated with the Jacobi matrix $\tilde{J}_l$. Thus by assumption on the spectrum of $\tilde{J}_l$, g_l has all zeros in $(-1, 1)$ which strictly interlace with the zeros of g_{l-1} which is associated with $\tilde{J}_{l-1}$. $\square$

If in Corollary 2.2 only $\tilde{\beta}_{n-l+1} > 0$ is supposed, then the smallest or largest node may be outside of $(-1, 1)$. Naturally the polynomial $t_j, j = 0, \ldots, n$ from Theorem 2.1(c) can be written as a characteristic polynomial of the corresponding cutted Jacobi matrix J_n also.

In the following we denote by J_n^* the Jacobi matrix which is the reverse of the Jacobi matrix J_n. Theorem 2.1, respectively, Corollary 2.2 show how to generate simultaneously positive qf with respect to two given measures σ and $\tilde{\sigma}$. Indeed, let, $N \in \mathbb{N}$,

$$(2.9) \qquad t_{2N} = \tilde{p}_N p_N - \mu_{N+1}\tilde{p}_{N-1}p_{N-1},$$

where $\mu_{N+1} > 0$ and such that $t_{2N}(\pm 1) > 0$; or equivalently put in (2.8) $\tilde{J}_l = (J_N^G(d\tilde{\sigma}))^*$ and $\tilde{\beta}_{n-l+1} = \mu_{N+1}$. Then t_{2N} generates a positive $(2N - 1, 2N, d\sigma)$ as

well as a positive $(2N - 1, 2N, d\tilde{\sigma})$ qf. This follows immediately by (2.6) putting $g_N = \tilde{p}_N(p_N)$ and $g_{N-1} = \tilde{p}_{N-1}(p_{N-1})$, respectively.

For $n = 2N + 1, l = N$, even the simplest case $\tilde{J}_l = (J_N^G(d\sigma))^*$, that is, when in (2.2) $\tilde{\alpha}_{2N+2-j} = \alpha_j, j = 1, \ldots, N$, and $\tilde{\beta}_{2N+2-j} = \beta_{j+1}, j = 1, \ldots, l - 1$, are of special interest. In fact, as observed by M. Spalevic [29], this leads to generalized averaged Gaussian qf, which cover, in particular, the so-called nested and stratified qf, found in a different way by Laurie [10]. For a more detailed discussion see Corollary 2.5 and the following remarks.

Next let us show that the problem of positive quadrature can be reduced dramatically with respect to the degree with the help of associated measures.

First let us recall the known fact that

$$(2.10) \qquad p_m = p_j^{(m-j)} p_{m-j} - \beta_{m-j+1} p_{j-1}^{(m-j+1)} p_{m-j-1}$$

for $m \in \mathbb{N}, 0 \le j \le m$, where the representation is unique for $j \le [m/2]$. Thus, if t_n is orthogonal to $\mathbb{P}_{n-m-1}$ with respect to $d\sigma$, we get

$$(2.11) \quad t_n = \sum_{j=0}^{m} \mu_j p_{n-j} = (\sum_{j=0}^{m} \mu_j p_{m-j}^{(n-m)}) p_{n-m} - \beta_{n-m+1} (\sum_{j=0}^{m} \mu_j p_{m-j-1}^{(n-m+1)}) p_{n-m-1}$$

which yields by Theorem 2.1 the following characterization.

Corollary 2.3. *Let $n, m \in \mathbb{N}, n \ge 2m, \mu_0, \ldots, \mu_m \in \mathbb{R}, \mu_0 \ne 0$. Then the following statements are equivalent:*

(a) $t_n = \sum_{j=0}^{m} \mu_j p_{n-j}$ *generates a positive $(2n - 1 - m, n, d\sigma)$ qf.*

(b) $\sum_{j=0}^{m} \mu_j p_{m-j}^{(n-m)}$ *and $\sum_{j=0}^{m} \mu_j p_{m-j-1}^{(n-m+1)}$ have all zeros in $(-1, 1)$, they strictly interlace, and*

$$(2.12) \quad (\pm 1)^n \left(p_{n-m} \sum_{j=0}^{m} \mu_j p_{m-j}^{(n-m)} - \beta_{n+1-m} p_{n-m-1} \sum_{j=0}^{m} \mu_j p_{m-j-1}^{(n-m+1)} \right) (\pm 1) > 0.$$

(c) $\sum_{j=0}^{m} \mu_j p_{m-j}^{(n-m)}$ *generates a positive $(m-1, m, d\sigma^{(n-m)})$ qf and (2.12) holds.*

Condition (2.12) guarantees that the smallest and largest zero of t_n is in $(-1, 1)$.

Let us discuss briefly how to obtain on the basis of Corollary 2.3 (c) a simple, rather complete description of quasi-orthogonal polynomials of finite length which generate positive qf for $n \ge n_0$. This kind of problem has been pointed out in several papers; see [3, 8, 14, 17, 24, 33, 34]. Suppose that the recurrence coefficients of (p_k) satisfy

$$(2.13) \qquad \lim_n \alpha_n = 0 \text{ and } \lim_n \lambda_n = \frac{1}{4}$$

(for instance, by Rakhmanov's Theorem (2.13) holds if $\sigma' > 0$ a.e. on $[-1, +1]$). Then by (1.8), for fixed $k, m \in \mathbb{N}$, uniformly on compact subsets of $\mathbb{C}$,

$$(2.14) \qquad \lim_n p_k^{(n)}(x) = \hat{U}_k(x) \text{ and } \sigma^{(n-m)} \xrightarrow[n \to \infty]{*} \sqrt{1 - x^2}$$

where $\hat{U}_k(x) = 2^{-k} \sin(k + 1) \arccos x / \sin \arccos x$ is the monic Chebyshev polynomial of second kind of degree k and, as usual, $\xrightarrow{*}$ denotes weak convergence. Thus by Corollary 2.3(c) and (2.14), it is reasonable to expect (under additional conditions possibly) that $\sum_{j=0}^{m} \mu_j p_{n-j}$ generates for every $n \ge n_0$ a positive $(2n - 1 - m, n, d\sigma)$ qf if $\sum_{j=0}^{m} \mu_j \hat{U}_{m-j}$ generates a positive $(m - 1, m, \sqrt{1 - x^2})$ qf and

conversely. Now we know by [16, Corollary 2] or Theorem 4.1(a) below that $\sum_{j=0}^{m} \mu_j \hat{U}_{m-j}$ generates a positive $(m-1, m, \sqrt{1-x^2})$ qf if $\sum_{j=0}^{m} \mu_j z^j$ has all zeros in $|z| < 1/2$ and, indeed, as we have proved in [20, Corollary 3.4], for measures σ in the Szegő class the last condition is sufficient also that $\sum_{j=0}^{m} \mu_j p_{n-j}$ generates a positive $(2n-1-m, n, d\sigma)$ qf for every $n \geq n_0$. On the other hand, it can be shown [20] that $\sum_{j=0}^{m} \mu_j z^j$ has all zeros in $|z| \leq \frac{1}{2}$ if $\sum_{j=0}^{m} \mu_j p_{n-j}$ has all zeros in $(-1, 1)$ for every $n \geq n_0$, in particular, if it generates a positive $(2n-1-m, n, d\sigma)$ qf for every $n \geq n_0$.

2.1. Application to Gauss-Kronrod quadratures.
Let us demonstrate some consequences of the Characterization Theorem to Gauss-Kronrod qf, abbreviated by G-K qf which are $(4N+1-m, 2N+1, d\sigma)$ qf, $0 \leq m \leq N$, with nodes at the zeros of p_N. By Theorem 2.1 we obtain the following complete characterization of positive G-K qf given by the author in [16, Corollary 4]. Note that the added equivalence (c) and the last statement on the positivity, which we expect to be of importance in future studies of G-K qf, follow immediately by (2.15) and Theorem 2.1(e) and Lemma 1.1, respectively. Because the proof gives other important information needed in the following and is short and simple we reproduce it.

Corollary 2.4. *Let $N, k \in \mathbb{N}_0$, $N \geq 2k$. The following three statements are equivalent:*

(a) *There exist a positive $(4N + 1 - 2k, 2N + 1, d\sigma)$ qf which has N nodes at the zeros of p_N.*

(b) *There exist polynomials g_k, g_{k-1} which satisfy the conditions of the Characterization Theorem 2.1(d) such that*

$$(2.15) \qquad p_N = g_k p_{N-k}^{(N+1)} - \tilde{\beta}_{2N+2-k} g_{k-1} p_{N-k-1}^{(N+1)},$$

where $\tilde{\beta}_{2N+2-k} > 0$.

(c) *p_N generates a positive $(2N - 1 - 2k, N, d\sigma^{(N+1)})$ qf.*

Moreover, the G-K qf is positive if and only if p_N and $\int \frac{p_N(x) - p_N(t)}{x - t} d\sigma^{(N+1)}$ have strictly interlacing zeros on $(-1, 1)$.

Proof. Suppose that $t_{2N+1}(x) = p_N(x) E_{N+1}(x)$ generates a positive $(4N + 1 - 2k, 2N + 1, d\sigma)$ qf. Then it follows by Theorem 2.1 that

$$(2.16) \qquad p_N E_{N+1} = t_{2N+1} = g_k p_{2N+1-k} - \tilde{\beta}_{2N+2-k} g_{k-1} p_{2N-k}.$$

Plugging in (2.16) and the representations (2.10) for p_{2N+1-k} and p_{2N-k} we obtain

$$(2.17) \qquad \begin{aligned} p_N E_{N+1} = t_{2N+1} =& (g_k p_{N-k}^{(N+1)} - \tilde{\beta}_{2N+2-k} g_{k-1} p_{N-k-1}^{(N+1)}) p_{N+1} \\ &- \beta_{N+2} (g_k p_{N-k-1}^{(N+2)} - \tilde{\beta}_{2N+2-k} g_{k-1} p_{N-k-2}^{(N+2)}) p_N \end{aligned}$$

and thus, since p_N and p_{N+1} have no common zero, (2.15) follows.

The sufficiency part follows by putting t_{2N+1} as in (2.17) and using (2.10) which gives (2.16) and by Theorem 2.1 the assertion. $\square$

Obviously, recalling the equivalence of (e) and (d) in Theorem 2.1, representation (2.15) is equivalent to the fact that the Jacobi matrix

$$
(2.18) \quad
\begin{pmatrix}
\alpha_{N+2} & \sqrt{\beta_{N+3}} & & & & & \\
\sqrt{\beta_{N+3}} & \alpha_{N+3} & \sqrt{\beta_{N+4}} & & & & \\
\ddots & \ddots & \ddots & & & & \\
& \sqrt{\beta_{2N+1-k}} & \alpha_{2N+1-k} & \sqrt{\tilde{\beta}_{2N+2-k}} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \sqrt{\tilde{\beta}_{2N}} & \tilde{\alpha}_{2N} & \sqrt{\tilde{\beta}_{2N+1}} \\
& & & & \sqrt{\tilde{\beta}_{2N+1}} & \tilde{\alpha}_{2N+1}
\end{pmatrix}
$$

has the same eigenvalues as the Jacobi matrix $J_N^G(d\sigma)$. In this form our above characterization (2.15) of positive G-K qf was given by Laurie [10] about ten years later. In the sequel characterization (2.15), i.e., Corollary 4 from [16], was missed also in [2, p. 1038] and the survey papers [5] and [11, Theorem 2], where it is called Laurie's fundamental result on Gauss-Kronrod quadrature. It turned out that (2.15), respectively, (2.18) is important from the computational point of view also [2, 5].

We mention that by (2.17) and (2.15) the so-called Stieltjes polynomial E_{N+1} has a representation of the form

$$
(2.19) \qquad E_{N+1}(x) = p_{N+1}(x) - \beta_{N+2} \int \frac{p_N(x) - p_N(t)}{x - t} d\sigma^{(N+1)}(t)
$$

which can be derived also with the help of the representation

$$
E_{N+1} p_N = \sum_{j=0}^{N} \mu_j p_{2N+1-j}
$$

and (2.11); see [22]. Having in mind Corollary 2.4(c) and the discussion following Corollary 2.3 (for more information on associated polynomials and measures see [21, Section 3]) we conjecture that for every $N \geq N_0$ positive G-K quadrature is not possible, if $p_N(., w)$ does not admit positive $(N-1, N, \sqrt{1-x^2})$ quadrature for $N \geq N_0$, where w is supposed to be continuously differentiable.

2.2. Extensions of the Gauss qf with arbitrary degree of exactness.

Corollary 2.5. a) $p_N E_{N+1}$ *generates a positive* $(2N, 2N+1, d\sigma)$ *qf if and only if* $E_{N+1} = p_{N+1} + (\alpha_{N+1} - \tilde{\alpha}_{N+1}) p_N - \tilde{\beta}_{N+2} g_{N-1}$, *where* $g_{N-1} \in \mathbb{P}_{N-1}$ *is such that* g_{N-1} *and* p_N *have strictly interlacing zeros in* $(-1, 1)$ *and* $\tilde{\alpha}_{N+1} \in \mathbb{R}$ *and* $\tilde{\beta}_{N+2} > 0$ *are such that* $E_{N+1}(1) > 0$ *and* $(-1)^{N+1} E_{N+1}(-1) > 0$.

If $\tilde{\alpha}_{N+1} = \alpha_{N+1}$, *then the* $(2N, 2N+1, d\sigma)$ *qf becomes a* $(2N+1, 2N+1, d\sigma)$ *qf, and if in addition* $\tilde{\beta}_{N+2} = \beta_{N+2}$, *then a* $(2N+2, 2N+1, d\sigma)$ *qf.*

b) $t_{2N+1} = p_N E_{N+1}$ *generates a positive* $(4N+1-m, 2N+1, d\sigma)$, $0 \leq m \leq 2N+1$, *qf if and only if* $g_{N+1} := E_{N+1} + \beta_{N+1} p_{N-1}$ *and* $g_N := p_N$ *can be generated*

by a recurrence relation (2.5), $n = 2N + 1$, *with the constraints*

(2.20) $\tilde{\alpha}_{2N+1-[\frac{m+1}{2}]-\nu} = \alpha_{2N+1-[\frac{m+1}{2}]-\nu}$ $\nu = 0, 1, \ldots, N - [\frac{m+1}{2}]$,

(2.21) $\tilde{\beta}_{2N+1-[\frac{m}{2}]-\nu} = \beta_{2N+1-[\frac{m}{2}]-\nu}$ $\nu = 0, 1, \ldots, N - [\frac{m}{2}]$.

Proof. Concerning a). Necessity. By Theorem 2.1(e),

(2.22) $t_{2N+1} = g_{N+1}p_N - \beta_{N+1}g_N p_{N-1}.$

By (2.22) and $t_{2N+1} = p_N E_{N+1}$ we get

$$g_N = p_N \qquad \text{and} \qquad E_{N+1} = g_{N+1} - \beta_{N+1}p_{N-1}.$$

Sufficiency. Put

(2.23) $g_{N+1} = (x - \tilde{\alpha}_{N+1})p_N - \tilde{\beta}_{N+2}g_{N-1}.$

Then, setting $g_N = p_N$,

$$t_{2N+1} = p_N E_{N+1} = g_{N+1}p_N - \beta_{N+1}p_{N-1}g_N.$$

Since g_{N-1} and p_N have interlacing zeros by (2.23), g_{N+1} and $g_N = p_N$ have interlacing zeros which implies by Theorem 2.1(e) the assertion.

Concerning b). Necessity follows as in the proof of part a) and by (2.3).

Sufficiency. For simplicity of writing, let $m = 2k$. By (2.20) and (2.21) and induction arguments, g_{N+1} and g_N can be written in the form

$$g_N = g_k p_{N-k}^{(N+1)} - \tilde{\beta}_{2N+2-k}g_{k-1}p_{N-k-1}^{(N+1)},$$

and

$$g_{N+1} = g_k p_{N+1-k}^{(N)} - \tilde{\beta}_{2N+2-k}g_{k-1}p_{N-k}^{(N)},$$

where $\tilde{\beta}_{2N+2-k} > 0$, which implies (compare (2.16) and (2.17)) that

$$g_{N+1}p_N - \beta_{N+1}g_N p_{N-1} = g_k p_{2N+1-k} - \tilde{\beta}_{2N+2-k}g_{k-1}p_{2N-k},$$

which is the assertion. $\square$

Note the difference of G-K quadrature, where the extension polynomial E_{N+1} is uniquely determined by (2.19).

If in the first statement of Corollary 2.5 we put $g_{N-1} = p_{N-1}$ and let $\tilde{\beta}_{N+2} \in \mathbb{R}^+$ and $\tilde{\alpha}_{N+1} \in (-1, 1)$ be such that

(2.24) $\tilde{\beta}_{N+2} + (\tilde{\alpha}_{N+1} - \alpha_{N+1})\dfrac{p_N}{p_{N-1}}(\pm 1) < \dfrac{p_{N+1}}{p_{N-1}}(\pm 1),$

then it follows that the polynomial

(2.25) $t_{2N+1} = p_N(p_{N+1} + (\alpha_{N+1} - \tilde{\alpha}_{N+1})p_N - \tilde{\beta}_{N+2}p_{N-1})$

generates a positive $(2N, 2N + 1, d\sigma)$ qf. We note that there always exist $\tilde{\alpha}_{N+1} \in (-1, 1)$ and $\tilde{\beta}_{N+2} > 0$ such that (2.24) is satisfied. The weights $\lambda_{\nu, 2N+1}(d\sigma)$ associated with the ν-th zero of p_N, $\nu = 1, \ldots, N$, are given by

(2.26) $\lambda_{\nu, 2N+1}(d\sigma) = \left(\dfrac{\tilde{\beta}_{N+2}}{\tilde{\beta}_{N+2} + \beta_{N+1}} \right) \lambda_{\nu, N}^G(d\sigma).$

(2.26) follows by (1.3) and the fact that by a straightforward calculation

(2.27) $(p_N E_{N+1})^{[1]}(y) = E_{N+1}(y)p_N^{[1]}(y) + \displaystyle\int p_N^2 d\sigma$

and that at the zeros x_ν of p_N by (2.25)

$$E_{N+1}(x_\nu) = -(\beta_{N+1} + \tilde{\beta}_{N+2})p_{N-1}(x_\nu)$$

(2.28)

$$= -(\beta_{N+1} + \tilde{\beta}_{N+2}) \int p_{N-1}^2 d\sigma / p_N^{[1]}(x_\nu).$$

If we put in addition to (2.25) $\tilde{\beta}_{N+2} = \beta_{N+1}$ and $\tilde{\alpha}_{N+1} = \alpha_{N+1}$, then the degree of exactness is raised by one and the $(2N+1, 2N+1, d\sigma)$ qf becomes the so-called averaged Gaussian qf introduced by Laurie in [9]. By (2.26) the weights associated with the zeros of p_N become half of the Gaussian weights.

The case when in (2.25) $\tilde{\alpha}_{N+1} = \alpha_{N+1}$ and $\tilde{\beta}_{N+2} = \beta_{N+2}$, which yields degree of exactness $2N + 2$, was studied recently by Spalevic in [29]. His starting point was Theorem 2.1(c) by putting there, $\tilde{\alpha}_{2N+2-j} = \alpha_j, j = 1, \ldots, N$, and $\tilde{\beta}_{2N+2-j} = \beta_{j+1}, j = 1, \ldots, l - 1, n = 2N + 1$.

Let us mention that in the last two cases condition (2.24) need not be satisfied anymore, that is, that two nodes may be outside.

3. Explicit weight function for positive qf

Next let us demonstrate how to obtain a representation of positive qf with the help of orthogonal polynomials on the unit circle, given in [15]; see also [16, 19]. Denote by $\Phi_n(z) = z^n + \ldots, n \in \mathbb{R}_0$, the polynomial orthogonal on $[0, 2\pi]$ with respect to the positive measure

(3.1)
$$\psi(\varphi) = \begin{cases} -\sigma(\cos\varphi) & \text{for } \varphi \in [0, \pi], \\ \sigma(\cos\varphi) & \text{for } \varphi \in (\pi, 2\pi], \end{cases}$$

i.e.,

(3.2)
$$\int_0^{2\pi} e^{-ik\varphi} \Phi_n(e^{i\varphi}) d\psi(\varphi) = 0 \text{ for } k = 0, \ldots, n - 1.$$

Note that if σ is absolutely continuous on $[-1, +1]$ and $\sigma'(x) = w(x)$, then ψ is absolutely continuous with $\psi'(\varphi) = w(\cos\varphi)|\sin\varphi|$ for $\varphi \in [0, 2\pi]$. It is well known (see e.g. [25, 31]) that the Φ_n's satisfy a recurrence relation of the type

(3.3)
$$\Phi_n(z) = z\Phi_{n-1}(z) - a_{n-1}\Phi_{n-1}^*(z) \quad \text{for } n \in \mathbb{N},$$

where $a_n \in (-1, 1)$ for $n \in \mathbb{N}_0$ and where $\Phi_n^*(z) = z^n \Phi_n(z^{-1})$ denotes the reciprocal polynomial of Φ_n. The polynomials p_n and $p_{n-1}^{(1-x^2)}$ orthogonal on $[-1, +1]$ with respect to $d\sigma$ and $(1-x^2)d\sigma$, respectively, can be given by OPUC's as follows (see [31, Section 11.5] or [26])

(3.4)
$$2^{n-1}p_n(x) = \text{Re}\{z^{-n+1}\Phi_{2n-1}(z)\} = \frac{\text{Re}\{z^{-n}\Phi_{2n}(z)\}}{1 - a_{2n-1}},$$

(3.5)
$$2^{n-1}p_{n-1}^{(1-x^2)}(x) = \frac{\text{Im}\{z^{-n+1}\Phi_{2n-1}(z)\}}{\sin\varphi} = \frac{\text{Im}\{z^{-n}\Phi_{2n}(z)\}}{(1 + a_{2n-1})\sin\varphi},$$

where $x = \frac{1}{2}(z + z^{-1}), z = e^{i\varphi}, \varphi \in [0, \pi]$. The parameters are given by (see [26])

(3.6)
$$a_{2n-1} = 1 - (u_n + v_n) \quad \text{and} \quad a_{2n} = (v_n - u_n)/(v_n + u_n),$$

where

(3.7)
$$u_n = p_{n+1}(1)/p_n(1) \quad \text{and} \quad v_n = -p_{n+1}(-1)/p_n(-1).$$

Let us assume that t_n has a representation of the form (2.6). Following [16, 19] we put

$$(3.8) \qquad s_{l-1}(x) = \frac{e_{n-l}^+ - e_{n-l}^-}{2} \tilde{\beta}_{n+1-l} g_{l-1}(x)$$

and

$$(3.9) \qquad r_l(x) = 2(g_l(x) - (x + \frac{e_{n-l}^+ + e_{n-l}^-}{e_{n-l}^+ - e_{n-l}^-}) s_{l-1}(x)),$$

where

$$(3.10) \qquad e_{n-l}^{\pm} = \frac{p_{n-l-1}(\pm 1)}{p_{n-l}(\pm 1)}.$$

Representing $(x^2 - 1) p_{n-l-1}^{(1-x^2)}(x)$ in the form (see [31, Thm.2.5])

$$(3.11) \qquad (x^2 - 1) p_{n-l-1}^{(1-x^2)}(x) = p_{n+1-l}(x) - \mu_{1,n-l} p_{n-l}(x) - \mu_{2,n-l} p_{n-l-1}(x)$$
$$= (x - \alpha_{n+1-l} - \mu_{1,n-l}) p_{n-l}(x) - (\mu_{2,n-l} + \beta_{n+1-l}) p_{n-l-1}(x)$$

we get by considering (3.11) at the points ± 1,

$$(3.12) \quad \mu_{2,n-l} + \beta_{n+1-l} = \frac{2}{e_{n-l}^+ - e_{n-l}^-} \quad \text{and} \quad \alpha_{n+1-l} + \mu_{1,n-l} = -\frac{e_{n-l}^+ + e_{n-l}^-}{e_{n-l}^+ - e_{n-l}^-},$$

which gives again by straightforward calculation that

$$(3.13) \quad r_l(x) p_{n-l}(x) - (1 - x^2) s_{l-1}(x) p_{n-l-1}^{(1-x^2)}(x) = g_l p_{n-l} - \tilde{\beta}_{n+1-l} g_{l-1} p_{n-l-1}.$$

Furthermore, we put

$$(3.14) \quad q_m(z) = (2z)^l \{ r_l(\frac{1}{2}(z + z^{-1})) + \frac{(z - z^{-1})}{2} s_{l-1}(\frac{1}{2}(z + z^{-1})) \} \text{ if } m = 2l$$

and
(3.15)
$$q_m(z) = 2(2z)^{l-1} \{ \frac{r_l(\frac{1}{2}(z + z^{-1}))}{1 - a_{2(n-l)-1}} + \frac{(z - z^{-1})}{2} \frac{s_{l-1}(\frac{1}{2}(z + z^{-1}))}{1 + a_{2(n-l)-1}} \} \text{ if } m = 2l - 1,$$

i.e., for $m = 2l$,

$$(3.16) \qquad r_l(x) = 2^{-l} \operatorname{Re}\{ e^{-il\varphi} q_m(e^{i\varphi}) \} \text{ and } s_{l-1}(x) = 2^{-l} \frac{\operatorname{Im}\{ e^{-il\varphi} q_m(e^{i\varphi}) \}}{\sin \varphi},$$

and for $m = 2l - 1$ (see [16, Lemma 2]),

$$(3.17) \qquad r_l(x) = (1 - a_{2(n-l)-1}) 2^{-l} \operatorname{Re}\{ e^{-i(l-1)\varphi} q_m(e^{i\varphi}) \}$$

and

$$(3.18) \qquad s_{l-1}(x) = (1 + a_{2(n-l)-1}) 2^{-l} \frac{\operatorname{Im}\{ e^{-i(l-1)\varphi} q_m(e^{i\varphi}) \}}{\sin \varphi}.$$

Then the representation (2.6) becomes

$$(3.19) \qquad \begin{aligned} t_n(x) &= r_l(x) p_{n-l}(x) - (1 - x^2) s_{l-1}(x) p_{n-l-1}^{(1-x^2)}(x) \\ &= \operatorname{Re}\{ z^{-n+1} q_m(z) \Phi_{2n-1-m}(z) \} \end{aligned}$$

where $x = \cos\varphi$, $z = e^{i\varphi}$, $\varphi \in [0, \pi]$.

The point is now, if $g_l(x)$ and $g_{l-1}(x)$ have all zeros in $(-1, 1)$ and if they are all simple and interlaced, then it can be shown by Cauchy's Index Theorem (see [16])

that $q_m(z)$ has all zeros in the open unit disk $|z| < 1$; the converse statement holds true also. Hence $q_m\Phi_{2n-1-m}$, $n \in \mathbb{N}$, is a polynomial which has all zeros in $|z| < 1$ which has many important consequences as we shall see.

Recall that by Characterization Theorem 2.1 t_n is orthogonal with respect to some positive measure depending on n; see also [27, Thm. 3.8] concerning the existence of such measures having a special form. In the following theorem we give such a measure explicitly. We mention that there may be several positive measures with respect to which t_n is orthogonal; uniqueness is guaranteed only if the second kind of polynomials also coincide.

Theorem 3.1. *Let $n, m \in \mathbb{N}_0$, $n \geq m$, and let t_n be a monic polynomial of degree n. The following statements are equivalent:*

(a) *t_n generates a positive $(2n - 1 - m, n, d\sigma)$ qf.*

(b) *$t_n(x)$ has a representation of the form*

(3.20) $$2^{n-1}t_n(x) = \text{Re}\{z^{-n+1}q_m(z)\Phi_{2n-1-m}(z)\};$$

$x = \frac{1}{2}(z + z^{-1})$, $z = e^{i\varphi}$, $\varphi \in [0, \pi]$, where $q_m(z) = z^m + \ldots$ is a real polynomial with all zeros in $|z| < 1$.

(c) *$t_n(x)$ is orthogonal on $[-1, 1]$ to $\mathbb{P}_{n-1}$ with respect to a weight function of the form, $x = \cos\varphi$, $\varphi \in [0, \pi]$,*

(3.21) $$v_{m,n}(x) = \frac{1}{|q_m(e^{i\varphi})|^2|\Phi_{2n-1-m}(e^{i\varphi})|^2\sqrt{1 - x^2}}$$

where $q_m(z) = z^m + \ldots$ is a real polynomial with all zeros in $|z| < 1$.

Proof. (a)$\Leftrightarrow$(b) has been shown in [15, Thm. 2]; see also [16].

Concerning (b)$\Rightarrow$(c). Since $(q_m\Phi_{2n-1-m})(z)$ has all zeros in $|z| < 1$ it follows [4] that $(q_m\Phi_{2n-1-m})(z)$ is orthogonal on the unit circle to $\{e^{-ik\varphi}\}_{k=0}^{2n-2}$ with respect to the weight function $f(\varphi) = 1/|q_m\Phi_{2n-1-m}(e^{i\varphi})|^2$. Thus (see [31, Section 11.5]), $\text{Re}\{z^{-n+1}(q_m\Phi_{2n-1-m})(z)\}$ is orthogonal on $[-1, 1]$ to $\mathbb{P}_{n-1}$ with respect to $v_{m,n}(x)dx$.

(c)$\Rightarrow$(a). Since $|q_m(e^{i\varphi})|^2$ is a cosine polynomial of degree m, it follows that t_n is on $[-1, 1]$ orthogonal to $\mathbb{P}_{n-m-1}$ with respect to $dx/|\Phi_{2n-m-1}(e^{i\varphi})|^2\sqrt{1 - x^2}$. Now $\Phi_{2n-m-1}(z)$ is the polynomial orthogonal on the unit circumference to $\{e^{-ik\varphi}\}_{k=0}^{2n-m-2}$ with respect to $d\psi(\varphi)$, where $\psi(\varphi)$ is the measure from (3.1). It is known (see [4, p. 200]) that

(3.22) $$\int_{-\pi}^{+\pi} \frac{e^{ik\varphi}}{|\Phi_{2n-1-m}(e^{i\varphi})|^2}d\varphi = \int_{-\pi}^{+\pi} e^{ik\varphi}d\psi(\varphi) \qquad k = 0, \ldots, 2n - 1 - m,$$

hence

(3.23) $$\int_{-1}^{+1} \frac{x^k}{|\Phi_{2n-1-m}(e^{i\varphi})|^2\sqrt{1 - x^2}}dx = \int_{-1}^{+1} x^k d\sigma(x) \qquad k = 0, \ldots, 2n - 1 - m.$$

Thus it follows that t_n is on $[-1, 1]$ orthogonal to $\mathbb{P}_{n-m-1}$ with respect to $d\sigma(x)$. Since t_n is orthogonal with respect to $v_{m,n}$, it has n simple zeros $x_j = \cos\varphi_j, j = 1, \ldots, n$. Now, by the Gaussian formula, recall that $|q_m(e^{i\varphi})|^2$ is a cosine polynomial of degree $\leq m$, we have that

(3.24) $$\int_{-1}^{+1} p(x)|q_m(e^{i\varphi})|^2 v_{m,n}(x)dx = \sum_{j=0}^{n} \lambda_j^G(v_{m,n})|q_m(e^{i\varphi_j})|^2 p(x_j)$$

for any $p \in \mathbb{P}_{2n-1-m}$ and thus

$$(3.25) \qquad \lambda_j(w) = \lambda_j^G(v_{m,n})|q_m(e^{i\varphi_j})|^2 > 0 \qquad j = 1, \ldots, n,$$

which proves the implication. $\qquad\qquad\square$

Note that $|(q_m\Phi_{2n-1-m})(e^{i\varphi})|^2$ in the denominator of (3.21) is a positive cosine polynomial of degree $\leq 2n - 1$.

4. Positive quadrature formulas of Radau and Lobatto type

The statements given in Theorem 3.1 can be extended easily to measures of the form $(1-x)^\alpha(1+x)^\beta d\sigma(x)$, $\alpha, \beta \in \{0, 1\}$, which are of importance in connection with positive qf of Radau and Lobatto type. We call an interpolation qf with n nodes and degree of exactness $2n - 1 - m$ a $(2n - 1 - m, n, d\sigma)$ qf of Radau, resp. Lobatto type, if one, respectively, two nodes coincide with the boundary points ± 1 and all other nodes are simple and in $(-1, 1)$.

Theorem 4.1. *Suppose that $q_m(z)$ has all zeros in $|z| < 1$.*

a)

$$(4.1) \qquad 2^{n-1}t_{n-1}^{(1-x^2)}(\cos\varphi) := \frac{\mathrm{Im}\{z^{-n+1}q_m(z)\Phi_{2n-1-m}(z)\}}{\sin\varphi},$$

   where $x = \frac{1}{2}(z + z^{-1})$, $z = e^{i\varphi}$, $\varphi \in [0, \pi]$, is orthogonal on $[-1, 1]$ to $\mathbb{P}_{n-2}$ with respect to the weight function $(1 - x^2)v_{m,n}(x)$ and generates a positive $(2n - 3 - m, n - 1, (1 - x^2)d\sigma)$ qf. $v_{m,n}$ is defined in (3.21).
   Moreover, $(x^2 - 1)t_{n-1}^{(1-x^2)}$ generates the Lobatto qf with respect to $v_{m,n}$ and a positive $(2n - 1 - m, n + 1, d\sigma)$ qf of Lobatto type.

b)

$$(4.2) \qquad 2^n t_n^{(1+x)}(x) := \frac{\mathrm{Re}\{z^{-n+1/2}q_m(z)\Phi_{2n-1-m}(z)\}}{\cos\varphi/2},$$

   respectively,

$$(4.3) \qquad 2^n t_n^{(1-x)}(x) := \frac{\mathrm{Im}\{z^{-n+1/2}q_m(z)\Phi_{2n-1-m}(z)\}}{\sin\varphi/2}$$

   is orthogonal to $\mathbb{P}_{n-1}$ with respect to $(1 \pm x)v_{m,n}(x)$ and generates a positive $(2n - 1 - m, n, (1 \pm x)d\sigma)$ qf.
   Moreover, $(1 \pm x)t_n^{(1\pm x)}$ generates the Radau qf with respect to $v_{m,n}$ and a positive $(2n - m, n + 1, d\sigma)$ qf of Radau type.

Proof. The orthogonality property follows by [31, Section 11.5] again. As in the proof of the implication (c) $\Rightarrow$ (a) from Theorem 3.1 it follows that it generates a corresponding positive qf with respect to $(1-x^2)d\sigma$. Since $(x^2 - 1)t_{n-1}^{(1-x^2)}$ generates the Lobatto qf if and only if $t_{n-1}^{(1-x^2)}$ is orthogonal with respect to $(1 - x^2)d\sigma$, part a) is proved.

Analogously part b) follows. $\qquad\qquad\square$

5. ASYMPTOTICS OF WEIGHTS

In the following, $p_n^{(1\pm x)}$ denotes the monic polynomial of degree n orthogonal with respect to $(1 \pm x)d\sigma$ as $p_n^{(1-x^2)}$ denotes that one is orthogonal with respect to $(1 - x^2)d\sigma$.

Lemma 5.1. *Denote by* $R_n^{(x^2-1)}$, $R_n^{(x\pm 1)}$, *and* R_{n-1} *the polynomials of the second kind of* $(y^2 - 1)p_{n-1}^{(1-x^2)}(y), (y \pm 1)p_n^{(1\pm x)}(y)$ *and of* $p_n(y)$ *with respect to* $d\sigma$. *Then the following relations hold:*

$$(5.1) \qquad p_n(y)R_n^{(x^2-1)}(y) - (y^2 - 1)p_{n-1}^{(1-x^2)}(y)R_{n-1}(y) = c_n$$

where

$$(5.2) \qquad c_n = \int_{-1}^{+1} p_n^2(t)d\sigma(t) + \int_{-1}^{+1} [p_{n-1}^{(1-x^2)}(t)]^2(1 - t^2)d\sigma(t)$$

and

$$(5.3) \qquad (y + 1)p_n^{(1+x)}(y)R_n^{(x-1)}(y) - (y - 1)p_n^{(1-x)}(y)R_n^{(x+1)}(y) = d_n$$

where

$$(5.4) \qquad d_n = \int_{-1}^{+1} [p_n^{(1-x)}(t)]^2(1 - t)d\sigma(t) + \int_{-1}^{+1} [p_n^{(1+x)}(t)]^2(1 + t)d\sigma(t).$$

Proof. To show that (5.1) holds it suffices obviously to demonstrate that the polynomial of the second kind of $(y^2 - 1)p_{n-1}^{(1-x^2)}(y)p_n(y)$ with respect to $d\sigma$ has the following two representations:

$$(5.5) \qquad (y^2 - 1)p_{n-1}^{(1-x^2)}(y)R_{n-1}(y) + \int_{-1}^{+1} p_n^2(t)d\sigma(t)$$

and

$$(5.6) \qquad p_n(y)R_n^{(x^2-1)}(y) - \int_{-1}^{+1} [p_{n-1}^{(1-x^2)}(t)]^2(1 - t^2)d\sigma.$$

Indeed, on the one hand, the polynomial of the second kind can be written in the form

$$(5.7) \qquad \begin{aligned} &(y^2 - 1)p_{n-1}^{(1-x^2)}(y) \int_{-1}^{+1} \frac{p_n(y) - p_n(t)}{y - t}d\sigma(t) \\ &+ \int_{-1}^{+1} \frac{(y^2 - 1)p_{n-1}^{(1-x^2)}(y) - (t^2 - 1)p_{n-1}^{(1-x^2)}(t)}{y - t}p_n(t)d\sigma \end{aligned}$$

and, on the other hand, in the form

$$(5.8) \qquad \begin{aligned} &p_n(y) \int \frac{(y^2 - 1)p_{n-1}^{(1-x^2)}(y) - (t^2 - 1)p_{n-1}^{(1-x^2)}(t)}{y - t}d\sigma \\ &+ \int \frac{p_n(y) - p_n(t)}{y - t}p_{n-1}^{(1-x^2)}(t)(t^2 - 1)d\sigma(t). \end{aligned}$$

Using the orthogonality property of p_n respectively, $p_{n-1}^{(1-x^2)}$, (5.5) and (5.6) follow. Relation (5.3) is proved similarly by showing that the polynomials of the

second kind of $(y^2 - 1)p_n^{(1+x)}(y)p_n^{(1-x)}(y)$ with respect to $d\sigma$ has the following two representations:

$$(5.9) \qquad (y+1)p_n^{(1+x)}(y)R_n^{(x-1)}(y) - \int [p_n^{(1-x)}(t)]^2(1-x)d\sigma$$

and

$$(5.10) \qquad (y-1)p_n^{(1-x)}(y)R_n^{(x+1)}(y) + \int [p_n^{(1+x)}(t)]^2(1+x)d\sigma. \qquad \square$$

Lemma 5.2. *Let c_n, d_n be given by (5.1) and (5.3) and let $x_{j,n}$ and $y_{j,n-1}$ be the j-th zero of p_n and $p_{n-1}^{(1-x^2)}$, respectively. Then*

$$(5.11) \qquad \lambda_{j,n}^G(d\sigma) = \frac{c_n}{(1-x_{j,n}^2)p_{n-1}^{(1-x^2)}(x_{j,n})p_n'(x_{j,n})}$$

$$(5.12) \qquad \lambda_{j,n-1}^G((1-x^2)d\sigma) = \frac{-c_n}{p_n(y_{j,n-1})(p_{n-1}^{(1-x^2)})'(y_{j,n-1})}.$$

Furthermore,

$$(5.13) \qquad \lambda_{j,n}^G((1 \pm x)d\sigma) = \frac{-d_n}{(u_{j,n} \mp 1)p_n^{(1\mp x)}(u_{j,n})(p_n^{(1\pm x)})'(u_{j,n})}$$

where $u_{j,n}$ denotes the j-th zero of $p_n^{(1\pm x)}$, respectively.

Proof. Relation (5.11) follows immediately by (1.3) and (5.1); note that by notation $p_n^{[1]} \equiv R_{n-1}$.

Concerning (5.12). Using the fact that by Christoffel's formula [31, Section 2.5] $(y^2 - 1)p_{n-1}^{(1-x^2)}(y) = p_{n+1}(y) - \mu_{1,n}p_n(y) - \mu_{2,n}p_{n-1}(y), \mu_{1,n}, \mu_{2,n} \in \mathbb{R}$, one obtains by straightforward calculation that at the zeros y of $p_{n-1}^{(1-x^2)}$ the relation $(p_{n-1}^{(1-x^2)})^{[1]}(y) = -R_n^{(x^2-1)}(y)$ holds, which gives by (1.3) and (5.1) the assertion.

Relation (5.13) follows similarly by (5.3) and (1.3) taking into consideration the fact that with the help of Christoffel's formula at the zero u of $p_n^{(1\pm x)}$ the relation $(p_n^{(1\pm x)})^{[1]}(u) = \pm R_n^{(y\pm 1)}(u)$ holds. $\qquad \square$

Theorem 5.3. *Let $d\sigma(x) = w(x)dx$ be such that $f(\varphi) = w(\cos\varphi)|\sin\varphi|$ is positive and from Lip γ, $0 < \gamma \le 1$, on $[\alpha, \beta] \subseteq [0, \pi]$ and that $\log f(\varphi)$ is integrable on $[0, \pi]$. Suppose that (t_n) generates a positive $(2n-m(n)-1, n, w)$ qf with quadrature weights $\lambda_{j,n}(w)$ and nodes $x_{j,n} = \cos\varphi_{j,n}$ and that the associated $q_{m(n)}$ satisfy uniformly on $[\alpha, \beta]$ the limit relation*

$$(5.14) \qquad \mathrm{Re}\{\frac{e^{i\varphi}}{n}\frac{q_{m(n)}^{*'}(e^{i\varphi})}{q_{m(n)}^*(e^{i\varphi})}\} \xrightarrow[n\to\infty]{} h(\varphi).$$

Then

$$(5.15) \qquad \frac{\pi \sin\varphi_{j,n}w(\cos\varphi_{j,n})}{n\lambda_{j,n}(w)} = 1 - h(\varphi_{j,n}) + o(1)$$

uniformly for all $\varphi_{j,n} \in [\alpha + \varepsilon, \beta - \varepsilon], \varepsilon > 0$.

Furthermore, the weights $\lambda_{j,n-1}((1-x^2)w)$ *of the positive* $(2n-3-m, n-1,$
$(1-x^2)w)$ *qf, whose nodes* $x_{j,n-1} = \cos \psi_{j,n-1}$ *are the zeros of* $t_{n-1}^{(1-x^2)}$ *from (4.1),*
are given asymptotically by

$$(5.16) \qquad \frac{\pi \sin^3 \psi_{j,n-1} w(\cos \psi_{j,n-1})}{n\lambda_{j,n-1}((1-x^2)w)} = 1 - h(\psi_{j,n-1}) + o(1)$$

uniformly for all $\psi_{j,n-1} \in [\alpha + \varepsilon, \beta - \varepsilon], \varepsilon > 0.$
   Finally, the weights $\lambda_{j,n}((1 \pm x)w)$ *of the positive* $(2n - 1 - m, n, (1 \pm x)w)$
qf, whose nodes $x_{j,n} = \cos \eta_{j,n}$ *are the zeros of* $t_n^{(1\pm x)}$ *from (4.2) and (4.3), are*
asymptotically given by

$$(5.17) \qquad \frac{\pi(1 \pm y_{j,n}) \sin \eta_{j,n} w(\cos \eta_{j,n})}{n\lambda_{j,n}((1 \pm x)w)} = 1 - h(\eta_{j,n}) + o(1)$$

uniformly for all $\eta_{j,n} \in [\alpha + \varepsilon, \beta - \varepsilon], \varepsilon > 0.$

Proof. For brevity let us put $g_{2n-1}(z) = zq_m(z)\Phi_{2n-1-m}(z)$, hence $g_{2n-1}^*(z) = q_m^*(z)\Phi_{2n-1-m}^*(z)$, and let

$$(5.18) \qquad t_n(\cos \varphi) = \mathrm{Re}\{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})\}$$

and

$$(5.19) \qquad -\sin \varphi t_{n-1}^{(1-x^2)}(\cos \varphi) = \mathrm{Im}\{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})\}.$$

Since

$$\mathrm{Im}\{\overline{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})}\}\frac{d}{d\varphi}\mathrm{Re}\{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})\}$$

$$- \mathrm{Re}\{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})\}\frac{d}{d\varphi}\mathrm{Im}\{\overline{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})}\}$$

$$(5.20) \qquad = \mathrm{Im}\{(\overline{e^{-in\varphi}g_{2n-1}^*(e^{i\varphi})})\frac{d}{d\varphi}(e^{-in\varphi}g_{2n-1}^*(e^{i\varphi}))\}$$

$$= \mathrm{Im}\{(-in)|g_{2n-1}^*(e^{i\varphi})|^2 + ie^{i\varphi}g_{2n-1}^{*'}(e^{i\varphi})\overline{g_{2n-1}^*(e^{i\varphi})}\},$$

it follows that at the zeros of $t_n(\cos \varphi)$,

$$(5.21) \qquad \frac{\sin^2 \varphi \; t_{n-1}^{(1-x^2)}(\cos \varphi)t_n'(\cos \varphi)}{n|g_{2n-1}^*(e^{i\varphi})|^2} = 1 - \mathrm{Re}\{\frac{e^{i\varphi}}{n}\frac{g_{2n-1}^{*'}(e^{i\varphi})}{g_{2n-1}^*(e^{i\varphi})}\}.$$

Recalling the orthogonal properties of t_n and $t_{n-1}^{(1-x^2)}$ with respect to $v_{n,m}$ and
$(1-x^2)v_{n,m}$ (see Theorem 3.1 and 4.1) we may apply (5.11) to $d\sigma = v_{n,m}(x)dx$.
Using the fact that $c_n = \pi$, which follows by

$$(5.22) \qquad t_n^2(x) + (1-x^2)t_{n-1}^{(1-x^2)}(x) = |g_{2n-1}^*(e^{i\varphi})|^2 = 1/v_{n,m}(x)\sqrt{1-x^2}$$

and (5.2), we obtain by (5.21) that

$$(5.23) \qquad \frac{\pi}{n\lambda_{j,n}^G(v_{n,m})|q_m^*(e^{i\varphi})|^2|\Phi_{2n-1-m}^*|^2}$$

$$= 1 - (\mathrm{Re}\{\frac{e^{i\varphi}}{n}\frac{q_m^{*'}(e^{i\varphi})}{q_m^*(e^{i\varphi})}\} + \mathrm{Re}\{\frac{e^{i\varphi}}{n}\frac{\Phi_{2n-1-m}^{*'}(e^{i\varphi})}{\Phi_{2n-1-m}^*(e^{i\varphi})}\}).$$

To simplify asymptotically (5.23) we need to recall the following fact. Since $f \in$ Lip γ, we know that

$$(5.24) \qquad \Phi^*_{2n-m-1}(e^{i\varphi}) = 1/D(e^{i\varphi}) + O(\frac{1}{n^\gamma})$$

uniformly on $[\alpha, \beta]$ where D is the so-called Szegő function, i.e., $D(z)$ is analytic on $|z| < 1$ and on $[\alpha, \beta]$ satisfies

$$(5.25) \qquad w(\cos\varphi)|\sin\varphi| = |D(e^{i\varphi})|^2,$$

since $w(\cos\varphi)$ is continuous there. Furthermore, let us prove that (5.24) implies that uniformly on $[\alpha + \varepsilon, \beta - \varepsilon], \varepsilon > 0$,

$$(5.26) \qquad \frac{1}{n}|\frac{\Phi^{*'}_{2n-1-m}(e^{i\varphi})}{\Phi^*_{2n-1-m}(e^{i\varphi})}| \underset{n\to\infty}{\longrightarrow} 0.$$

Indeed,

$$(5.27) \qquad \begin{aligned} |\frac{d}{d\varphi}\Phi^*_{2n-m-1}(e^{i\varphi})| \leq & |\frac{d}{d\varphi}(\Phi^*_{2n-m-1}(e^{i\varphi}) - \Phi^*_{[\sqrt{2n-m-1}]}(e^{i\varphi}))| \\ & + |\frac{d}{d\varphi}\Phi^*_{[\sqrt{2n-m-1}]}(e^{i\varphi})|; \end{aligned}$$

hence, by the local version of Bernstein's inequality we get

$$(5.28) \qquad \begin{aligned} \max_{\varphi\in[\alpha+\varepsilon,\beta-\varepsilon]} & |\frac{d}{d\varphi}\Phi^*_{2n-m-1}(e^{i\varphi})| \\ & \leq \mathrm{const}((2n-m-1)\max_{\varphi\in[\alpha,\beta]}|\Phi^*_{2n-m-1}(e^{i\varphi}) - \Phi^*_{[\sqrt{2n-m-1}]}(e^{i\varphi})| \\ & + \sqrt{2n-m-1}\max_{\varphi\in[\alpha,\beta]}|\Phi^*_{[\sqrt{2n-m-1}]}(e^{i\varphi})|). \end{aligned}$$

Using the obvious fact that $|\frac{d}{d\varphi}\Phi^*_{2n-m-1}(e^{i\varphi})| = |\Phi^{*'}_{2n-1-m}(e^{i\varphi})|$ and (5.24) in conjunction with the facts that $D(e^{i\varphi}) \neq 0$ on $[\alpha, \beta]$ and $m(n) \leq n$ relation, (5.26) follows.

Thus the right hand side of (5.23) becomes by (5.14), the right hand side of (5.15). With the help of (5.24), (5.25) and (3.25) the left hand side of (5.23) takes the form of the left hand side of (5.15) which proves (5.15).

For (5.16) we first observe that by (5.20) at the zeros ψ of $\mathrm{Im}\{e^{in\varphi}g^*_{2n-1}(e^{-i\varphi})\}$ satisfies

$$(5.29) \qquad \begin{aligned} & \frac{-\sin^2\psi \; t_n(\cos\psi)(t_{n-1}^{(1-x^2)})'(\cos\psi)}{n|g^*_{2n-1}(e^{i\psi})|^2} \\ & = 1 - (\mathrm{Re}\{\frac{e^{i\psi}}{n}\frac{q^{*'}_m(e^{i\psi})}{q^*_m(e^{i\psi})}\} + \mathrm{Re}\{\frac{e^{i\psi}}{n}\frac{\Phi^{*'}_{2n-1-m}(e^{i\psi})}{\Phi^*_{2n-1-m}(e^{i\psi})}\}). \end{aligned}$$

(5.16) now follows as above by (5.12); recall that $c_n = \pi$, and by (5.24), (5.25) and the fact that

$$(5.30) \qquad \lambda^G_{j,n}((1-x^2)v_{m,n}) = |q^*_m(e^{i\psi})|^2\lambda_{j,n}((1-x^2)w).$$

To prove (5.17) we first observe that at the zeros η of $\cos\frac{\varphi}{2}t_n^{(1+x)}(\cos\varphi) = \mathrm{Re}\{e^{-i(n-\frac{1}{2})\varphi}g^*_{2n-1}(e^{i\varphi})\}$ or of $\sin\frac{\varphi}{2}t_n^{(1-x)}(\cos\varphi) = \mathrm{Im}\{e^{-i(n-\frac{1}{2})\varphi}g^*_{2n-1}(e^{i\varphi})\}$ the

following equality holds:

$$\operatorname{Im}\overline{\{e^{-i(n-\frac{1}{2})\eta}g^*_{2n-1}(e^{i\eta})\}}\frac{d}{d\eta}\operatorname{Re}\{e^{-i(n-\frac{1}{2})\eta}g^*_{2n-1}(e^{i\eta})\}$$

$$- \operatorname{Re}\{e^{-i(n-\frac{1}{2})\eta}g^*_{2n-1}(e^{i\eta})\}\frac{d}{d\eta}\operatorname{Im}\overline{\{e^{-i(n-\frac{1}{2})\eta}g^*_{2n-1}(e^{i\eta})\}}$$

(5.31)

$$= \operatorname{Im}\{\overline{(e^{-i(n-\frac{1}{2})\eta}g^*_{2n-1}(e^{i\eta}))}\frac{d}{d\eta}(e^{-i(n-\frac{1}{2})\eta}g^*_{2n-1}(e^{i\eta}))\}$$

$$= -(n-\frac{1}{2})|g^*_{2n-1}(e^{i\eta})|^2 + \operatorname{Re}\{e^{i\eta}g^{*'}_{2n-1}(e^{i\eta})\overline{g^*_{2n-1}(e^{i\eta})}\}.$$

Proceeding as above we obtain with the help of (5.13) that

(5.32)

$$\frac{(1\pm y)d_n}{(2n-1)\lambda_{j,n}((1\pm x)w)|\Phi^*_{2n-m-1}(e^{i\eta})|^2}$$

$$= 1 - 2\operatorname{Re}\{\frac{e^{i\eta}}{2n-1}\frac{g^{*'}_{2n-1}(e^{i\eta})}{g^*_{2n-1}(e^{i\eta})}\}.$$

Since

(5.33)
$$\frac{(1+x)}{2}(t_n^{(1+x)}(x))^2 + \frac{(1-x)}{2}(t_n^{(1-x)}(x))^2 = |g^*_{2n-1}(e^{i\varphi})|^2,$$

it follows that $d_n = 2\pi$. Using (5.24) and (5.25) and the boundedness of the $\lambda_{j,n}((1\pm x)w)$'s the assertion follows. $\square$

If the sequence $(m(n))$ is bounded, then $h(\varphi) \equiv 0$ on $[\alpha, \beta]$ if $0 < c_1 \leq |q^*_{m(n)}(e^{i\varphi})| \leq c_2$ on $[\alpha, \beta]$. For unbounded sequences $(m(n))$ the following corollary may be convenient.

Corollary 5.4. *If there holds uniformly for $\varphi \in [\alpha, \beta]$,*

(5.34)
$$\lim_m q^*_{m(n)}(e^{i\varphi}) = Q(e^{i\varphi}) \quad with \quad Q(e^{i\varphi}) \neq 0,$$

then in (5.14) $h(\varphi) \equiv 0$, and thus the asymptotic formulas (5.15)-(5.17) satisfy the circle law which reads in the standard case (5.15) as

(5.35)
$$n\lambda_{j,n}(w) = \pi\sqrt{1 - x^2_{j,n}}w(x_{j,n}) + o(1)$$

uniformly for all $x'_{j,n}$ with $\arccos x_{j,n} \in [\alpha + \varepsilon, \beta - \varepsilon]$, $\varepsilon > 0$, and correspondingly in the other cases.

In particular, (5.34) is satisfied if $(q_{m(n)}(z))$ is a sequence of OPUC's with respect to some weight functions $\tilde{f}(\varphi)$, where $\tilde{f}(\varphi)$ is positive and from $\operatorname{Lip}\gamma$ on $[\alpha, \beta]$, and $\log\tilde{f}(\varphi)$ is integrable on $[0, \pi]$.

Proof. By (5.34) it follows as in the proof of (5.26) that uniformly on $[\alpha + \varepsilon, \beta - \varepsilon]$,

(5.36)
$$\frac{1}{m}|\frac{q^{*'}_m(e^{i\varphi})}{q^*_m(e^{i\varphi})}|\underset{m\to\infty}{\longrightarrow} 0,$$

which gives by $m(n) \leq n$ and Theorem 5.3 the assertion. Concerning the last statement see (5.24). $\square$

For Gaussian qf with respect to weight functions which satisfy the assumption that $w(x)\sqrt{1-x^2}$ is positive on $[-1, 1]$ and is from $\operatorname{Lip}\gamma$, the circle law dates back to Szegő [30]. Recently V. Totik [32] has shown that it holds a.e. for general

measures, correspondingly modified with respect to the equilibrium measure. For the classical Radau and Lobatto qf the circle law was recently investigated in [6].

Naturally there are many classes of polynomials which satisfy (5.14) but not (5.34) in general; for instance, Fekete or Faber polynomials for simple connected domains contained in the open unit disk. Let us give a simple example.

Example 5.5. Let, $\alpha \in (-1, +1)$, and $\lim_n \frac{m(n)}{n} = 1$. Then an explicit asymptotic expression for the weights follows immediately by (5.15) and

$$1 - h(\varphi) = \text{Re}\{\frac{1}{1 - \alpha e^{i\varphi}}\} = \frac{1 - \alpha \cos \varphi}{1 + \alpha^2 - 2\alpha \cos \varphi}.$$

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