THE EULER IMPLICIT/EXPLICIT SCHEME FOR THE
2D TIME-DEPENDENT NAVIER-STOKES EQUATIONS
WITH SMOOTH OR NON-SMOOTH INITIAL DATA

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Abstract. This paper considers the stability and convergence results for
the Euler implicit/explicit scheme applied to the spatially discretized two-
dimensional (2D) time-dependent Navier-Stokes equations. A Galerkin finite
element spatial discretization is assumed, and the temporal treatment is im-

plicit/explicit scheme, which is implicit for the linear terms and explicit for the
nonlinear term. Here the stability condition depends on the smoothness of the
initial data \( u_0 \in H^\alpha \), i.e., the time step condition is \( \tau \leq C_0 \) in the case of
\( \alpha = 2 \), \( \tau |\log h| \leq C_0 \) in the case of \( \alpha = 1 \) and \( \tau h^{-2} \leq C_0 \) in the case of \( \alpha = 0 \)
for mesh size \( h \) and some positive constant \( C_0 \). We provide the
\( H^2 \)-stability of the scheme under the stability condition with
\( \alpha = 0, 1, 2 \) and obtain the optimal \( H^1 - L^2 \) error estimate of the numerical velocity and the optimal \( L^2 \) error
estimate of the numerical pressure under the stability condition with \( \alpha = 1, 2 \).

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) assumed to have a Lipschitz continuous
boundary \( \partial \Omega \) and to satisfy a further condition stated in (A1) below. We consider
the time-dependent Navier–Stokes problem

\[
\begin{cases}
  u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \text{div} \ u = 0, \quad (x, t) \in \Omega \times (0, T); \\
  u(x, 0) = u_0(x), \quad x \in \Omega; \quad u(x, t)|_{\partial \Omega} = 0, \quad t \in [0, T],
\end{cases}
\]

where \( u = u(x, t) = (u_1(x, t), u_2(x, t)) \) represents the velocity vector, \( p = p(x, t) \)
the pressure, \( f = f(x, t) \) the prescribed body force, \( u_0(x) \) the initial velocity, \( \nu > 0 \)
the viscosity, and \( T > 0 \) a finite time.

There are numerous works devoted to the development of efficient schemes for
the Navier-Stokes equations, fully implicit, semi-implicit and implicit/explicit scheme. A key
issue is the stability conditions of schemes. Usually the fully implicit schemes are
unconditionally stable. However, at each time step, one has to solve a system of
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scheme or an explicit scheme for the nonlinear term. A semi-implicit scheme for the nonlinear term results in a linear system with a variable coefficient matrix of time, and an explicit treatment for the nonlinear term gives a constant matrix. Stability and convergence conditions of schemes have been studied by many authors. The main results are summarized below, where we set \( \Omega \subset \mathbb{R}^d \) with \( d = 2, 3 \), and \( 0 < h < 1 \) denotes the mesh size in the spatial direction and \( 0 < \tau = \frac{T}{N} < 1 \) denotes the step size in the time direction, which may change. However, \( T > 0 \) is fixed throughout this paper.

- For the Crank-Nicolson scheme (fully implicit), Heywood and Rannacher \[23\] proved that it is almost unconditionally stable and convergent, i.e. stable and convergent when

\[
\tau \leq C_0, \\
\tag{1.2}
\]

for some positive constant \( C_0 \) depending on the data \( (\nu, \Omega, T, u_0, f) \) in the case of \( d = 2, 3 \).

- For a two-step scheme (semi-implicit), He and Li \[14\] gave the following convergence condition:

\[
\tau h^{-1/2} \leq C_0. \\
\tag{1.3}
\]

- For the Crank-Nicolson extrapolation scheme (semi-implicit), He \[15\] has proved that (1.2) is the stability and convergence condition of the scheme in the case of \( d = 2 \).

- For the Crank-Nicolson/Adams-Bashforth scheme (implicit/explicit), Marion and Temam provided in \[32\] the following stability condition:

\[
\tau h^{-d} \leq C_0, \ d = 2, 3, \\
\tag{1.4}
\]

and recently, Tone \[37\] proved the convergence under the condition

\[
\tau h^{2-d/2} \leq C_0, \ d = 2, 3. \\
\tag{1.5}
\]

- A modified Crank-Nicolson/Adams-Bashforth scheme (implicit/explicit) was proposed by Johnston and Liu \[26\], in which the nonlinear term and pressure term are discretized explicitly. They claimed in their numerical simulations that the scheme is stable under the standard stability condition

\[
\|u\|_{L^\infty} \tau h^{-1} \leq 1, \ d = 2, 3. \\
\tag{1.6}
\]

No theoretical analysis has been given.

- For a three-step backward extrapolating scheme (implicit/explicit), Baker et al. \[4\] gave the convergence condition

\[
\tau h^{-4/7} \leq C_0, \\
\tag{1.7}
\]

in the case of \( d = 2, 3 \).

- Clearly, the time-step condition

\[
\tau h^{-r} \leq C_0, \\
\tag{1.8}
\]

for some \( r > 0 \) was imposed in these previous works when an implicit/explicit scheme is applied.

Recently, He and Sun \[19\] have improved the result of (1.8) and proved that the stability and convergence condition of the Crank-Nicolson/Adams-Bashforth scheme is (1.2).
This paper focuses on the Euler implicit/explicit scheme with a finite element approximation in spatial direction for solving the time-dependent Navier-Stokes equations in the case of $d = 2$, which were studied by Marion and Temam [32], Tone [37], Kim and Moin [27] and Issacson and Keller [25]. The scheme consists of using equations in the case of approximation in spatial direction for solving the time-dependent Navier-Stokes problem. Under the assumptions (A1), (A2) in §2 with $u^0 \in D(A^{\alpha/2})$, $\alpha = 0, 1, 2$ and (A3) in §3, we prove that the scheme is stable, i.e.,

$$\sigma^{2-\alpha}(t_m)(\|u^m - u^{m-1}\|_2^2 + \nu^2\|A_hu^m\|_0^2 + \|p^m\|_0^2) \leq \kappa, \quad 1 \leq m \leq N,$$

when the stability condition

$$\tau \leq C_0, \quad \alpha = 2,$$

$$\tau \log h \leq C_0, \quad \alpha = 1,$$

$$\tau h^{-2} \leq C_0, \quad \alpha = 0.$$  

is satisfied. Under the stability condition (1.10) with $\alpha = 1, 2$, we also provide the $H^1 - L^2$ optimal error estimate for the numerical velocity and the $L^2$-optimal error estimate for the numerical pressure:

$$\|u(t_m) - u^m\|_{H^1} \leq \kappa(\sigma^{-(2-\alpha)}(t_m)\tau^2 + \sigma^{-(2-\alpha)}(t_m)h^4),$$

$$\|u(t_m) - u^m\|_{L^2} \leq \kappa(\sigma^{-(3-\alpha)}(t_m)\tau^2 + \sigma^{-(3-\alpha)}(t_m)h^2),$$

$$\|p(t_m) - p^m\|_{L^2} \leq \kappa(\sigma^{-(4-\alpha)}(t_m)\tau^2 + \sigma^{-(4-\alpha)}(t_m)h^2),$$

for all $1 \leq m \leq N$. Here $\sigma(t) = \min\{1, t\}$, $\kappa$ is some positive constant depending on the data ($\nu, \Omega, T, u_0, f$), and $A_h$ is a discrete Stokes operator.

Moreover, similar results were proved for the Euler implicit/explicit scheme which is applied to the spatial discretization based on the spectral Galerkin method by He [11] [12].

**Remark 1.1.** In the case of $\alpha = 2$, for the first order scheme (the Euler implicit/explicit scheme) we obtain the same $H^1$-error bound of the numerical velocity and a better $L^2$-error bound of the numerical pressure than the second order scheme (Crank-Nicolson scheme), excepting the $L^2$-error estimate for the numerical velocity. Previously, Heywood and Rannacher in [23] provided the following error estimates for the numerical velocity and pressure:

$$\|u(t_m) - u^m\|_{H^1} \leq \kappa(\sigma^{-1}(t_m)\tau^2 + h^2), \quad 1 \leq m \leq N,$$

$$\|p(t_m) - p^m\|_{L^2} \leq \kappa(\sigma^{-3}(t_m)\tau^2 + \sigma^{-1}(t_m)h^2), \quad 1 \leq m \leq N,$$

and the $L^2$-error estimate for the numerical velocity:

$$\|u(t_m) - u^m\|_{L^2} \leq \kappa(\sigma^{-2}(t_m)\tau^4 + h^4), \quad t_m \in (0, T), \quad 1 \leq m \leq N.$$

This paper is organized as follows. In §2 an abstract functional setting of the Navier-Stokes problem is given together with some basic assumptions (A1) and (A2) with $\alpha = 0, 1, 2$. In §3 we set out our assumption (A3) concerning the finite element spaces $X_h$ and $M_h$, finite element Galerkin approximation in space and some properties on the trilinear form $b(\cdot, \cdot, \cdot)$. Section 3 contains the optimal error estimate and a priori estimate results of the finite element solution $(u_h(t), p_h(t))$. In §4 we describe the Euler implicit/explicit scheme and prove the stability result of the scheme. In §5 we describe the dual scheme and prove its stability result. In §6 we obtain the optimal $H^1 - L^2$-error estimate of the numerical velocity and the
optimal $L^2$-error estimate of the numerical pressure under the stability condition (1.10) with $\alpha = 1, 2$.

2. Functional setting of the Navier–Stokes equations

For the mathematical setting of problem (1.1), we introduce the following Hilbert spaces:

\[ X = H^1_0(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L^2_0(\Omega) = \left\{ q \in L^2(\Omega); \int_{\Omega} q dx = 0 \right\}. \]

The space $L^2(\Omega)^d$, $d = 1, 2, 4$, is associated with the usual $L^2$-scalar product $(\cdot, \cdot)$ and $L^2$-norm $\| \cdot \|$. The space $X$ is associated with its usual scalar product and equivalent norm

\[ ((u, v)) = (\nabla u, \nabla v), \quad \|u\|_X = \|\nabla u\|_0. \]

Next, let the closed subset $V$ of $X$ be given by

\[ V = \{ v \in X; \text{div } v = 0 \} \]

and denote by $H$ the closed subset of $Y$, i.e.,

\[ H = \{ v \in Y; \text{div } v = 0, v \cdot n|_{\partial \Omega} = 0 \}. \]

We refer readers to [1, 10, 22, 36] for details on these spaces. We denote the Stokes operator by $A = -P\Delta$, where $P$ is the $L^2$-orthogonal projection of $Y$ onto $H$ and the domain of $A$ by $D(A) = H^2(\Omega)^2 \cap V$. As mentioned above, we need a further assumption on $\Omega$ provided in [23].

(A1) Assume that $\Omega$ is smooth so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

\[ -\nu \Delta v + \nabla q = g, \quad \text{div} v = 0 \quad \text{in } \Omega, \quad v|_{\partial \Omega} = 0, \]

for any prescribed $g \in Y$, exists and satisfies

\[ \|v\|_{H^2} + \|q\|_{H^1} \leq c\|g\|_0, \]

where $c > 0$ is a generic constant depending on $\Omega$ and $\nu$, and may take different values at its different occurrences.

We remark that the validity of assumption (A1) is known (see [10, 22, 28, 36]) if $\partial \Omega$ is of $C^2$ or if $\Omega$ is a two-dimensional convex polygon. From the assumption (A1), it is well known [1, 10, 22, 29] that

\[ \|v\|_{H^2} \leq c\|Av\|_0, \quad v \in D(A), \]

(2.1)

\[ \|v\|_0 \leq \gamma_0\|\nabla v\|_0, \quad v \in X, \quad \|\nabla v\|_0 \leq \gamma_0\|Av\|_0, \quad v \in D(A), \]

(2.2)

where $\gamma_0$ is a positive constant depending only on $\Omega$. We usually make the following assumption about the prescribed data for problem (1.1).

(A2) The initial velocity $u_0(x)$ and the force $f(x, t)$ are such that $u_0 \in D(A^{\alpha/2})$, $f$, $f_t$, $f_{tt} \in L^\infty(0, T; Y)$ with

\[ \|A^{\alpha/2}u_0\|_0 + \sup_{0 \leq t \leq T} \{ \|f(t)\|_0 + \|f_t(t)\|_0 + \|f_{tt}(t)\|_0 \} \leq C \]

for some positive constant $C$, and $\alpha = 0, 1, 2$, where $D(A^{1/2}) = V$ and $D(A^0) = H$.

Moreover, we define the continuous bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ on $X \times X$ and $X \times M$, respectively, by

\[ a(u, v) = \nu((u, v)), \quad u, v \in X, \quad d(v, q) = (q, \text{div } v), \quad v \in X, \quad q \in M, \]
and a trilinear form on $X \times X \times X$ by
\[
    b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\text{div}u)v, w)
    = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v), \quad u, v, w \in X.
\]

With the above notation, the variational formulation of problem (1.1) reads as follows: Find $(u, p) \in (X, M)$ for all $t \in [0, T]$ such that for all $(v, q) \in (X, M)$,
\[
    (u_t, v) + a(u, v) - d(v, p) + d(u, q) + b(u, u, v) = (f, v),
\]
(2.3)
\[
    u(0) = u_0.
\]
(2.4)

In order to proceed the theoretical and numerical analysis for the variational formulation (2.3)-(2.4), we need to introduce the following existence, uniqueness and modified regularity results.

**Theorem 2.1.** Under the assumptions (A1) and (A2), the problem (2.3)-(2.4) admits a unique solution $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ satisfying the following regularities:
\[
    \|u(t)\|_0^2 + \sigma \|u(t)\|_0^2 + \sigma^2(\|u_t(t)\|_0^2 + \|\nabla u(t)\|_0^2 + \|\nabla p(t)\|_0^2 + \|\nabla p_t(t)\|_0^2)^2
    \leq \kappa,
\]
(2.5)
\[
    \int_0^t (\|\nabla u\|_0^2 + \sigma \|u(t)\|_0^2 + \|Au(t)\|_0^2 + \|\nabla p(t)\|_0^2) ds
\]
(2.6)
\[
    + \int_0^t (\|\nabla u\|_0^2 + \sigma \|u(t)\|_0^2 + \|Au(t)\|_0^2 + \|\nabla p(t)\|_0^2 + \|\nabla p_t(t)\|_0^2) ds \leq k,
\]
for all $0 \leq t \leq T$.

**Proof.** For the existence and uniqueness of the solution in the case of $\alpha = 0$, the reader may refer to Temam [36]. For the regularity results related to $\alpha = 2$, the reader may refer to Heywood and Rannacher [22], and for the regularity results related to $\alpha = 1$, the reader may refer to Hill and Sül [24] and He [11] and He et al. [17].

The case $\alpha = 0$ has been proved in [12], except for the estimates of $\|\nabla p(t)\|_0^2$ and $\|\nabla p_t\|_0^2$. However, these can be done by using (1.1) and some nonlinear term estimates. \qed

3. Finite element Galerkin approximation

Let $h > 0$ be a real positive parameter. The finite element subspace $(X_h, M_h)$ of $(X, M)$ is characterized by $J_h = J_h(\Omega)$, a partitioning of $\Omega$ into triangles $K$ or quadrilaterals $K$, assumed to be uniformly regular as $h \to 0$. For further details, the reader may refer to Ciarlet [7] and Girault and Raviart [10].

We define the subspace $V_h$ of $X_h$ given by
\[
    V_h = \left\{ v_h \in X_h; d(v_h, q_h) = 0, \forall q_h \in M_h \right\}.
\]
(3.1)

Let $P_h : Y \rightarrow V_h$ denote the $L^2$-orthogonal projection defined by
\[
    (P_h v, v_h) = (v, v_h), \quad v \in Y, \quad v_h \in V_h.
\]
We assume that the couple \((X_h, M_h)\) satisfies the following approximation properties:

**A3** For each \(v \in H^2(\Omega)^2 \cap X\) and \(q \in H^1(\Omega) \cap M\), there exist approximations \(\pi_h v \in X_h\) and \(\rho_h q \in M_h\) such that

\[
\|\nabla (v - \pi_h v)\|_0 \leq c h \|Av\|_0, \quad \|q - \rho_h q\|_0 \leq c h \|\nabla q\|_0.
\]

For each \(v_h \in X_h\), one has the inverse inequality

\[
\|\nabla v_h\|_0 \leq c_1 h^{-1} \|v_h\|_0, \quad v_h \in X_h;
\]

and the so-called inf-sup inequality: For each \(q_h \in M_h\), there exists \(v_h \in X_h, v_h \neq 0\), such that

\[
d(v_h, q_h) \geq c_2 \|q_h\|_0 \|\nabla v_h\|_0,
\]

where \(c_1\) and \(c_2\) are positive constants depending on \(\Omega\).

We give an example of the spaces \(X_h\) and \(M_h\) such that the assumption (A3) is satisfied. Let \(\Omega\) be a convex, polygonal domain in plane and \(\Omega = J_h(\Omega)\), a partitioning of \(\Omega\) into triangles \(K\), assumed to be uniformly regular as \(h \to 0\). For any nonnegative integer \(l\), we denote by \(\mathcal{P}_l(K)\) the space of polynomials of degrees less than or equal to \(l\) on \(K\).

**Example 1** (Girault-Raviart \[10\]).

\[
X_h = \{v_h \in C^0(\Omega)^2 \cap X; v_h|_K \in P_2(K)^2, \quad \forall K \in J_h\},
\]

\[
M_h = \{q_h \in M; q_h|_K \in P_0(K), \quad \forall K \in J_h\}.
\]

**Example 2** (Bercovier-Pironneau \[5\]). We consider the triangulation \(J_{h/2}\) obtained by dividing each triangle of \(J_h\) into four triangles (by joining the mid-sides). We set

\[
X_h = \{v_h \in C^0(\Omega)^2 \cap X; v_h|_K \in P_1(K)^2, \quad \forall K \in J_{h/2}\},
\]

\[
M_h = \{q_h \in C^0(\Omega) \cap M; q_h|_K \in P_1(K), \quad \forall K \in J_h\}.
\]

The following properties are classical (see \[2, 10, 22, 24\]):

\[
\|\nabla P_h v\|_0 \leq c \|\nabla v\|_0, \quad v \in X,
\]

\[
\|v - P_h v\|_0 + h \|\nabla (v - P_h v)\|_0 \leq c h^2 \|Av\|_0, \quad v \in D(A),
\]

\[
\|v - P_h v\|_0 \leq c h \|\nabla (v - P_h v)\|_0, \quad v \in X.
\]

The standard finite element Galerkin approximation of (2.3)–(2.4) based on \((X_h, M_h)\) reads as follows: Find \((u_h, p_h) \in (X_h, M_h)\) such that for all \(0 < t \leq T\) and \((v_h, q_h) \in (X_h, M_h)\),

\[
(u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + b(u_h, u_h, v_h) = (f, v_h),
\]

\[
u_h(0) = u_{0h} = P_h u_0.
\]

With the above statements, a discrete analogue \(A_h = -P_h \Delta_h\) of the Stokes operator \(A\) is defined through the condition that \((-\Delta_h u_h, v_h) = ((u_{ht}, v_h))\) for all \(u_h, v_h \in X_h\). The restriction of \(A_h\) to \(V_h\) is invertible, with the inverse \(A_h^{-1}\). Since \(A_h^{-1}\) is self-adjoint and positive definite, we may define “discrete” Sobolev norms on \(V_h\), of any order \(r \in R\), by setting

\[
\|v_h\|_r = \|A_h^{r/2} v_h\|_0, \quad v_h \in V_h.
\]
These norms will be assumed to have various properties similar to their continuous counterparts, an assumption that implicitly imposes conditions on the structure of the spaces $X_h$ and $M_h$. In particular, it holds that

$$
\|v_h\|_1 = \|\nabla v_h\|_0, \quad \|v_h\|_2 = \|A_h v_h\|_0, \quad v_h \in V_h.
$$

By the way, we derive from (2.2) that

$$(3.10) \quad \|v_h\|_0 \leq \gamma_0 \|\nabla v_h\|_0, \quad \|\nabla v_h\|_0 \leq \gamma_0 \|A_h v_h\|_0, \quad v_h \in V_h,$$

where $\gamma_0 > 0$ is a constant depending only on $\Omega$.

This section considers preliminary estimates which are useful in the error estimates of finite element solution. Some estimates of the trilinear form $b$ are given in the following lemma and the proof can be found in [15,16,24].

**Lemma 3.1.** The trilinear form $b$ satisfies the following estimates:

$$(3.11) \quad b(u, v_h, w_h) = ((u \cdot \nabla)v_h, w_h) = -((u \cdot \nabla)w_h, v_h),$$

$$(3.12) \quad b(u_h, v_h, w_h) = -b(u_h, w_h, v_h),$$

$$|b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)| \leq c_0 \log h^{1/2} \|u_h\|_1 \|v_h\|_1 \|w_h\|_0,$$

$$(3.13) \quad |b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)| \leq \frac{c_0}{2} \|u_h\|_0^{1/2} \|u_h\|_1^{1/2} \|v_h\|_1 \|w_h\|_0^{1/2} \|w_h\|_1^{1/2},$$

$$|b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)| \leq \frac{c_0}{2} \|u_h\|_1 \|v_h\|_0^{1/2} \|v_h\|_1^{1/2} \|w_h\|_0^{1/2} \|w_h\|_1^{1/2},$$

for all $u \in V, v_h, w_h \in X_h$ and

$$(3.14) \quad \|v_h\|_0 \leq \gamma_0 \|\nabla v_h\|_0, \quad \|\nabla v_h\|_0 \leq \gamma_0 \|A_h v_h\|_0, \quad v_h \in V_h,$$

where $c_0 > 0$ is a constant depending only on $\Omega$.

Before we proceed further, we need some continuous and discrete Gagliardo-Nirenberg estimates (see Temam [36] and Hill and Suli [24]).

**Lemma 3.2.** It holds that

$$(3.15) \quad \frac{c_0}{2} \|A_h v_h\|_0^{1/2} \|v_h\|_1 \|w_h\|_0 \leq \frac{c_0}{2} \|A_h v_h\|_0^{1/2} \|v_h\|_1 \|w_h\|_0,$$

for all $u_h, v_h \in V_h, w_h \in X_h$, where $c_0 > 0$ is a constant depending only on $\Omega$.

In order to perform our error analysis for time discretization, we recall the following smooth properties of $(u_h, p_h)$. 
Theorem 3.3. Assume that assumptions (A1)-(A3) are valid. Then the finite element solution \((u_h, p_h)\) satisfies the following estimates:

\[
\|u_h(t)\|_0^2 + \sigma^{\frac{1}{2}}(1-\alpha)(\alpha-2)\|\nabla u_h(t)\|_0^2 + \sigma^{\frac{1}{2}}(\alpha-1)\|A_h u_h(t)\|_0^2
\]
\[
+ \int_0^t \left(\|\nabla u_h\|_0^2 + \sigma^{\frac{1}{2}}(1-\alpha)(\alpha-2)\|A_h u_h\|_0^2\right)ds \leq \kappa,
\]
\[
\sigma^{2+\alpha}(t)\|u_{ht}(t)\|_0^2 \leq \kappa, \quad \forall t, \quad r = 0, 1, 2,
\]
\[
\int_0^t \left(\sigma^{\frac{1}{2}}(\alpha-1)\|\nabla u_h\|_0^2 + \sigma^{1+\alpha}(\alpha-2)\|u_{ht}\|_0^2\right)ds \leq \kappa, \quad \forall t, \quad r = 1, 2,
\]

for all \(0 \leq t \leq T\).

For the proof of Theorem 3.3 in the case of \(\alpha = 2\), the reader is referred to Heywood and Rannacher [22] and He and Sun [19]. Theorem 3.3 with \(\alpha = 1, 0\) can be proved in a manner similar to the one used in [23] [19].

Next, we can provide some bounds of the error \((u - u_h, p - p_h)\).

Theorem 3.4. Under the assumptions (A1), (A2) with \(\alpha = 1, 2\) and (A3), it holds that

\[
\sigma^{\alpha-2}(t)\|u(t) - u_h(t)\|_0^2 + h^2\sigma^{\alpha-2}(t)\|\nabla(u(t) - u_h(t))\|_0^2
\]
\[
+ \sigma^{\alpha-2}(t)h^2\|p(t) - p_h(t)\|_0^2 \leq \kappa h^4,
\]

for all \(t \in (0, T]\).

Proof. For the case \(\alpha = 2\), Heywood and Rannacher [22] have proved [3.19]. For the case \(\alpha = 1\), Hill and Silly [24] have proved

\[
(\sigma(t) + h^2)\|u(t) - u_h(t)\|_0^2 + h^2\sigma(t)\|\nabla(u(t) - u_h(t))\|_0^2
\]
\[
+ h^2 \int_0^t \|\nabla(u - u_h)\|_0^2 ds \leq \kappa h^4,
\]

for all \(t \in (0, T]\).

Hence, it is sufficient to prove

\[
\sigma(t)\|p(t) - p_h(t)\|_0 \leq \kappa h, \quad \forall t \in (0, T],
\]

for \(\alpha = 1\).

We set \(e_h = P_h u - u_h\) and subtract (3.18) from (2.3) with \(v = v_h\) to obtain

\[
(u_t - u_{ht}, v_h) + a(u - u_h, v_h) - h(d(v_h, p - p_h) + b(u, u - u_h, v_h))
\]
\[
+ b(u - u_h, u, v_h) - b(u - u_h, u - u_h, v_h) = 0, \quad \forall v_h \in X_h.
\]

Taking \(v_h = 2e_{ht} \in V_h\) in (3.22) yields

\[
2\|e_{ht}\|_0^2 + \frac{d}{dt}\|\nabla(u - u_h)\|_0^2 + 2b(u, u - u_h, e_{ht})
\]
\[
+ 2b(u - u_h, u, e_{ht}) - 2b(u - u_h, u - u_h, e_{ht})
\]
\[
= 2a(u - u_h, u_t - P_h u_t) + 2\frac{d}{dt}(e_h, p - \rho_h p) - 2d(e_h, p_t - \rho_h p_t).
\]
Due to (3.22), (3.23), (3.24), (3.25) and Lemmas 3.1 and 3.2, we have
\[
2a(u - u_h, u_t - P_h u_t) \leq 2\nu \|\nabla (u - u_h)\|_0 \|\nabla (u_t - P_h u_t)\|_0
\leq ch \|\nabla (u - u_h)\|_0 \|Au\|_0,
\]
\[
2|d(e_h, p_t - \rho_h p_t)| \leq 2\sqrt{2}\|\nabla e_h\|_0 \|p_t - \rho_h p_t\|_0
\leq ch(\|\nabla (u - u_h)\|_0 + h\|Au\|_0)\|\nabla p_t\|_0,
\]
\[
2|b(u - u_h, e_h)| + 2|b(u - u_h, u, e_h)|
\leq 4(\|u\|_L^2 \|\nabla (u - u_h)\|_0 + \|\nabla u\|_{L^2} \|u - u_h\|_{L^2}) |e_h|_0
\leq \frac{1}{2} \|e_h\|_0^2 + c\|Au\|_0^2 \|\nabla (u - u_h)\|_0^2,
\]
\[
2|b(u - u_h, u - u_h, e_h)| \leq c\|\nabla (u - u_h)\|_0^2 \|\nabla e_h\|_0
\leq \frac{1}{2} \|e_h\|_0^2 + ch^{-2} \|\nabla (u - u_h)\|_0^4.
\]
Combining this inequality with (3.23) gives
\[
\|e_h\|_0^2 + 2\nu \frac{d}{dt} \|\nabla (u - u_h)\|_0^2 \leq 2\nu \frac{d}{dt} (e_h, p - \rho_h p)
+ ch \|\nabla (u - u_h)\|_0 \|Au\|_0 + ch(\|\nabla (u - u_h)\|_0 + h\|Au\|_0)\|\nabla p_t\|_0
\leq \frac{1}{2} \|e_h\|_0^2 + c\|Au\|_0^2 \|\nabla (u - u_h)\|_0^2.
\]
(3.24)
Multiplying (3.24) by \(\sigma(t)\), and integrating with respect to time and then using Theorem 2.1 and (3.20), we obtain
\[
\int_0^t \sigma(s) \|e_h\|_0^2 ds \leq 2\sigma(t)d(e_h(t), p(t) - \rho_h p(t))
+ 2 \int_0^t |d(e_h, p - \rho_h p)| ds + 2\nu \int_0^t \|\nabla (u - u_h)\|_0^2 ds + \kappa h^2
\leq \sigma(t)h(\|\nabla (u - u_h)\|_0 + h\|Au\|_0)\|\nabla p\|_0
+ h \int_0^t (\|\nabla (u - u_h)\|_0 + h\|Au\|_0)\|\nabla p\|_0 ds + \kappa h^2 \leq \kappa h^2,
\]
(3.25)
for all \(t \in (0, T]\).

Differentiating (3.22) with respect to time gives
\[
(u_t - u_{htt}, v_h) + a(u_t - u_{ht}, v_h) - d(v_h, p_t - \rho_h p_t) + b(u_t, u - u_h, v_h)
+ b(u - u_h, u_t - u_{ht}, v_h) + b(u_t - u_{ht}, u, v_h)
- b(u - u_h, u - u_h, v_h) - b(u - u_h, u_t - u_{ht}, v_h) = 0, \forall v_h \in V_h.
\]
(3.26)
Taking \(v_h = 2e_{ht} \in V_h\) in (3.26) and using Lemma 3.1, one finds
\[
\frac{d}{dt} \|e_{ht}\|_0^2 + 2\nu \|\nabla (u_t - u_{ht})\|_0^2 + 2\nu \|\nabla e_{ht}\|_0^2 + 2b(u_t, u - u_h, e_{ht}) + 2b(u - u_h, u_t, e_{ht})
+ 2b(u, u_t - P_h u_t, e_{ht}) + 2b(u_t - P_h u_t, u, e_{ht}) + 2b(e_{ht}, u, e_{ht})
- 2b(u - P_h u_t, u - u_h, e_{ht}) - 2b(u - u_h, u_t - P_h u_t, e_{ht})
= \nu \|\nabla (u_t - P_h u_t)\|_0^2 + 2d(e_{ht}, p_t - \rho_h p_t).
\]
(3.27)
Due to (3.2), (3.22), (3.26) and Lemma 3.1, we have
\[ 2|b(u_t, u - u_h, e_{ht})| + 2|b(u - u_h, u_t, e_{ht})| \]
\[ \leq 8\gamma_0 \|
abla u_t\|_0 \|
abla (u - u_h)\|_0 \|
abla e_{ht}\|_0 \]
\[ \leq \frac{\nu}{8} \|
abla e_{ht}\|_0^2 + c \|
abla u_t\|_0^2 \|
abla (u - u_h)\|_0^2 , \]
\[ 2|b(e_{ht}, u_h, e_{ht})| \leq 4 \|e_{ht}\|_0^{1/2} \|\nabla e_{ht}\|_0^{1/2} \|u_h\|_0^{1/2} \|\nabla u_h\|_0^{1/2} \]
\[ \leq \frac{\nu}{8} \|
abla e_{ht}\|_0^2 + c \|u_h\|_0^2 \|
abla u_h\|_0^2 \|
abla e_{ht}\|_0^2 , \]
\[ 2|b(u, u_t - P_h u_t, e_{ht})| + 2|b(u_t - P_h u_t, u, e_{ht})| \]
\[ \leq 8\gamma_0 \|
abla (u_t - P_h u_t)\|_0 \|
abla u\|_0 \|
abla e_{ht}\|_0 \]
\[ \leq \frac{\nu}{8} \|
abla e_{ht}\|_0^2 + c^2 \|\nabla u\|_0^2 \|Au_t\|_0^2 , \]
\[ 2\|u_t - P_h u_t\|_0 + 2\|u - u_h\|_0 - \rho_h p_t \leq c^2 \|Au_t\|_0^2 + c^2 \|\nabla e_{ht}\|_0 \|\nabla p_t\|_0 \]
\[ \leq \frac{\nu}{8} \|
abla e_{ht}\|_0^2 + c^2 \|\nabla u\|_0^2 + \|\nabla p_t\|_0^2 . \]

Combining (3.27) with the above estimates yields
\[ \frac{d}{dt} \|e_{ht}\|_0^2 \leq c \|u_h\|_0^2 \|
abla u_h\|_0^2 \|
abla e_{ht}\|_0^2 + c \|
abla u_t\|_0^2 \|
abla (u - u_h)\|_0^2 \]
\[ + \nu \|
abla (u_t - P_h u_t)\|_0^2 + 2d(e_{ht}, p_t - \rho_h p_t) \leq c^2 \|Au_t\|_0^2 + c^2 \|\nabla e_{ht}\|_0 \|\nabla p_t\|_0 \].

Multiplying (3.28) by \(\sigma^2(t)\), and integrating with respect to time, we obtain
\[ \sigma^2(t) \|e_{ht}(t)\|_0^2 \leq c \int_0^t \sigma(s)(1 + \|u_h\|_0^2 + \|\nabla u_h\|_0^2)(\|Au_t\|_0^2 + \|\nabla p_t\|_0^2)ds \]
\[ + \int_0^t \sigma^2(s) \|
abla u_t\|_0^2 \|
abla (u - u_h)\|_0^2 ds \]
\[ + \int_0^t \int_0^t \sigma^2(s)(1 + \|u_h\|_0^2 + \|\nabla u_h\|_0^2)(\|Au_t\|_0^2 + \|\nabla p_t\|_0^2)ds . \]

Using (3.20), (3.25), Theorem 2.1 and Theorem 3.3 in (3.29), we obtain
\[ \sigma^2(t) \|e_{ht}(t)\|_0^2 \leq \kappa h^2 , \]
which yields
\[ \sigma^2(t) \|u_t - u_h\|_0^2 \leq 2\sigma^2(t) \|e_{ht}(t)\|_0^2 + 2\sigma^2(t) \|u_t(t) - P_h u_t(t)\|_0^2 \]
\[ \leq 2\sigma^2(t) \|e_{ht}(t)\|_0^2 + c^2 \sigma^2(t) \|
abla u_t\|_0^2 \leq \kappa h^2 . \]

Finally, by using (3.22), (3.22), (3.24), (3.25) and Lemma 3.2, one finds
\[ \sigma(t) \|p(t) - \rho_h p(t)\|_0 \leq \sigma(t)(||\rho_h p(t) - p(t)||_0 + \|\rho_h p(t) - \rho_h p(t)\|_0) \]
\[ \leq c\sigma(t) \|u_t(t) - u_h(t)\|_0 + c\sigma(t) \|\nabla p(t)\|_0 \]
\[ + c\sigma(t) \|1 + \|\nabla u\|_0 + \|\nabla (u - u_h)\|_0\|\|\nabla (u(t) - u_h(t))\|_0 . \]

Using (3.20), (3.20) and Theorem 2.1 in (3.31), we obtain (3.21). \(\square\)
We will frequently use a discrete version of the Gronwall lemmas used in [13] and [34].

**Lemma 3.5.** Let $C$, $\tau$, and $a_n$, $b_n$, $d_n$, for integers $n \geq 0$, be nonnegative numbers such that

\begin{equation}
(3.32) \quad a_m + \tau \sum_{n=1}^{m} b_n \leq \tau \sum_{n=0}^{m-1} a_n d_n + C, \ m \geq 1.
\end{equation}

Then

\begin{equation}
(3.33) \quad a_m + \tau \sum_{n=1}^{m} b_n \leq C \exp \left( \tau \sum_{n=0}^{m-1} d_n \right), \ m \geq 1.
\end{equation}

**Theorem 3.6.** Under the assumptions (A1), (A2) with $\alpha = 1, 2$ and (A3), $u_{ht}$ and $u_{hht}$ satisfy the following bounds:

\begin{equation}
(3.34) \quad \int_{0}^{t} \sigma^{3-r-\alpha}(s)\|A^{1-r/2}_h u_{htt}\|_0^2 ds \leq \kappa, \ r = 0, 1, 2, \ \alpha = 1 \text{ or } r = 0, 1, \ \alpha = 2,
\end{equation}

\begin{equation}
(3.35) \quad \sigma^{4-\alpha}(t)\|u_{htt}(t)\|_0^2 + \int_{0}^{t} \sigma^{4-\alpha}(s)(\|u_{htt}\|_1^2 + \|A^{1/2}_h u_{htt}\|_0^2) ds \leq \kappa,
\end{equation}

for all $0 \leq t \leq T$.

**Proof.** Differentiating (3.3) with respect to $t$ gives

\begin{equation}
(3.36) \quad (u_{htt}, v_h) + a(u_{ht}, v_h) + b(u_{ht}, u_h, v_h) + b(u_h, u_{ht}, v_h) = (f_t, v_h),
\end{equation}

for all $v_h \in V_h$.

In view of (3.10) and Lemma 3.1 we deduce from (3.36) that

\[
\|A^{1-r/2}_h u_{htt}\|_0 \leq (\nu + c_0 \gamma_0 \|
abla u_h\|_0)\|A^{1-r/2}_h u_{ht}\|_0 + \gamma_0^2 \|f_t\|_0,
\]

which yields

\begin{equation}
(3.37) \quad \int_{0}^{t} \sigma^{3-r-\alpha}(s)\|A^{1-r/2}_h u_{htt}\|_0^2 ds \leq c \int_{0}^{t} (1 + \|
abla u_h\|_0^2)\sigma^{3-r-\alpha}(s)\|A^{1-r/2}_h u_{htt}\|_0^2 ds
\end{equation}

\begin{equation}
\quad + c \int_{0}^{t} \|f_t\|_0^2 ds,
\end{equation}

for $r = 0, 1, 2, \ \alpha = 1$ or $r = 0, 1, \ \alpha = 2$. Using Theorem 3.3 in (3.27) gives (3.31).

Furthermore, by differentiating (3.3) with respect to $t$ gives

\begin{equation}
(3.38) \quad (u_{htt}, v_h) + a(u_{ht}, v_h) + 2b(u_{ht}, u_h, v_h) + b(u_h, u_{ht}, v_h) = (f_t, v_h),
\end{equation}

for all $v_h \in V_h$.

Taking $v_h = 2u_{ht}$ in (3.38) and using (3.10) and Lemma 3.1, we deduce

\begin{equation}
(3.39) \quad \frac{d}{dt} \|u_{htt}\|_0^2 \leq 2\nu \|u_{htt}\|_1^2 + 4\nu \|u_{htt}\|_0^2 + 4\nu^{-1} \gamma_0^2 \|f_t\|_0^2.
\end{equation}
In view of (3.10) and Lemma 3.1 we deduce from (3.36) that
\[
4|b(u_{ht}, u_{ht}, u_{ht})| \leq 2c_0\gamma_0\|u_{ht}\|^2\|u_{ht}\|_1 + \nu^2\|u_{ht}\|^2_1 + 4\nu^{-1}c_0^2\gamma_0^2\|u_{ht}\|^4_1,
\]
\[
2|b(u_{ht}, u_h, u_{ht})| \leq c_0\gamma_0\|u_{ht}\|_0\|u_{ht}\|^2_1\|A_hu_h\|_0
\begin{align*}
& \leq \nu^2\|u_{ht}\|^2_1 + \nu^{-1}c_0^2\gamma_0^2\|A_hu_h\|^2\|u_{ht}\|^2_0.
\end{align*}
\]
Combining these inequalities with (3.39) gives
\[
\frac{d}{dt}\|u_{ht}\|^2_0 + \nu\|u_{ht}\|^2_1 \leq 4\nu^{-1}\gamma_0^2\|f_{ht}\|^2_0 + 4\nu^{-1}c_0^2\gamma_0^2\|u_{ht}\|^4_1 + \nu^{-1}c_0^2\gamma_0^2\|A_hu_h\|^2\|u_{ht}\|^2_0.
\]
(4.20)
Multiply (4.20) by \(\sigma^{4-\alpha}(t)\) yields
\[
\frac{d}{dt}(\sigma^{4-\alpha}(t)\|u_{ht}\|^2_0) + \nu\sigma^{4-\alpha}(t)\|u_{ht}\|^2_1 \leq 4\nu^{-1}\gamma_0^2\|f_{ht}\|^2_0 + \sigma^{4-\alpha}(t)\|u_{ht}\|^2_1 + (4 - \alpha + \sigma(t)\|A_hu_h\|^2)\sigma^{4-\alpha}(t)\|u_{ht}\|^2_0.
\]
Integrating (4.21) from 0 to \(t\) and using (3.34) and Theorem 3.3, we deduce
\[
\sigma^{4-\alpha}(t)\|u_{ht}\|^2_0 + \nu\int_0^t \sigma^{4-\alpha}(s)\|u_{ht}\|^2_1 ds \leq \kappa, \forall t \in (0, T].
\]
Finally, it follows from (3.38) and (3.10) and Lemma 3.1 that
\[
\int_0^t \sigma^{4-\alpha}(s)\|A_h^{-1/2}u_{ht}\|^2_0 ds \leq c\int_0^t (1 + \|u_{ht}\|_1^2)\sigma^{4-\alpha}(s)\|u_{ht}\|^2_1 ds
\]
\begin{align*}
& + c\int_0^t \{\sigma^{4-\alpha}(s)\|u_{ht}\|^2_1 + \|f_{ht}\|^2_0\} ds.
\end{align*}
(4.33)
Using Theorem 3.3 in (3.33), together with (3.42), gives (3.35) for \(\alpha = 1, 2\). \(\square\)

4. The Euler implicit/explicit scheme

In this section we consider the time discretization of the finite element Galerkin approximation (3.3)-(3.4). Usually for the fully implicit scheme, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easier in computation. But it suffers the severely restricted time step size from stability requirement. A popular approach is based on an implicit scheme for the linear terms and an explicit scheme for the nonlinear term. An explicit scheme for the nonlinear term results in a linear system with a constant coefficient matrix such that the computation is easy and the time step restriction is \(\tau \leq C_0\) which will be proved in this section and Section 6.

Let \(t_n = n\tau(n = 0, 1, \ldots, N)\), \(\tau = \frac{T}{N}\) the time step size, and \(N\) an integer. We define \(u_0^n = u_{0h} = P_h u_0\) and \((u_1^n, p_1^n) \in (X_h, M_h)\) by the Euler implicit/explicit scheme:
\[
\begin{align*}
& (d_n u_n^n, v_h) + a(u_n^n, v_h) - d(v_h, p_n^n) + d(u_n^n, q_h) + b(u_n^{n-1}, v_h) = (f(t_n), v_h),
\end{align*}
\]
here \(d_n u_n^n = \frac{1}{2}(u_n^n - v_n^{n-1})\).

We see from (3.33) and (3.34)-(3.36) that
\[
\|u_0^n\|_0 = \|u_{0h}\|_0 = \|P_h u_0\|_0 \leq c_\alpha\|A^{\alpha/2}u_0\|_0,
\]
if \(u_0 \in D(A^{\alpha/2})\) for some constants \(c_\alpha\) with \(\alpha = 0, 1, 2\).
The following theorem provides the stability of the scheme (4.1).

**Theorem 4.1.** Suppose that the assumptions (A1)-(A3) are valid and $0 < \tau < 1$ satisfies the following stability condition:

$$G_h \tau \leq \nu, \quad G_h = \begin{cases} 4^2 \nu^{-3/2} e_0^2 \gamma_1 \gamma_2 / \kappa_1 / \kappa_2, & \alpha = 2, \\ 4^2 \sigma^2 \nu^{-1} \kappa_1 | \log h|, & \alpha = 1, \\ 4^2 \sigma^2 \kappa_0 h^{-2}, & \alpha = 0. \end{cases}$$

Then the following hold:

$$\|u_h^m\|^2_0 + \nu \tau \sum_{n=1}^{m} \|u_h^n\|^2_1 \leq \kappa_0,$$

$$\tau \sum_{n=1}^{m} \sigma^{2(1-\alpha)(2-\alpha)} (t_n) (\nu^2 \|A_h u_h^n\|^2_0 + \nu \|d_t u_h^n\|^2_1 \tau + \|d_t u_h^n\|^2_0) + \sigma^{2-\alpha} (t_m) (\nu^2 \|A_h u_h^m\|^2_0 \leq \kappa_1,$$

$$\|u_h^m\|^2_1 \leq \kappa_2,$$

for all $0 \leq m \leq N$, where $\kappa_\alpha \geq 2^2 \nu^\alpha \|A^\alpha/2 u_0\|^2_0$ are some positive constants depending on the data $(\nu, \Omega, T, u_0, f)$.

**Proof.** First, taking $v_h = 2w_h^n \tau \in V_h$ and $v_h = A_h u_h^n \tau + \nu^{-1} d_t u_h^n \tau \in V_h$, respectively, and $q_h = 0$ in (4.1) and using (3.12) and the relation

$$2(x - y)x = |x|^2 - |y|^2 + |x - y|^2, \quad \forall x, y \in R^2,$$

we obtain

$$\|u_h^m\|^2_0 = \|u_h^{n-1}\|^2_0 + \|d_t u_h^n\|^2_1 \nu^{-2} + 2 \nu \|u_h^n\|^2_1 \tau + 2b(u_h^{n-1}, u_h^n, d_t u_h^n) \tau^2$$

$$\|u_h^m\|^2_1 = \|u_h^{n-1}\|^2_1 + \|d_t u_h^n\|^2_1 \tau^{-2} + \nu^{-1} \|d_t u_h^n\|^2_0 \tau + \nu \|A_h u_h^n\|^2_0 \tau + b(u_h^{n-1}, u_h^{-1}, A_h u_h^n + \nu^{-1} d_t u_h^n) \tau$$

In view of Lemma 3.1 and (3.10), it holds that

$$2|b(u_h^{n-1}, u_h^{n-1}, d_t u_h^n)\tau = 2|b(u_h^{n-1}, u_h^n, d_t u_h^n)\tau^2 \leq \frac{1}{2} G^{1/2}(u_h^{n-1}) \|u_h^{n-1}\|_1 \|d_t u_h^n\|_0 \tau^2$$

$$\leq \frac{1}{2} \|d_t u_h^n\|_0 \tau^2 + \frac{\nu}{4} G(u_h^{n-1}) \|u_h^n\|_1 \tau, \quad 2|(f(t_n), u_h^n)\tau \leq \frac{\nu}{4} \|u_h^n\|_1 \tau + \nu^{-1} \gamma_0 \|f(t_n)\|_0 \tau,$$
and
\[
|b(u_h^{n-1}, u^n - u_h^{n-1}, A_h u_h^n + \nu^{-1} d_t u_h^n)| \tau
\]
\[
\leq \frac{1}{2} G^{1/2}(u_h^{n-1}) \| d_t u_h^n \|_1 \| A_h u_h^n + \nu^{-1} d_t u_h^n \|_0 \tau^2
\]
\[
\leq \frac{1}{2} \| d_t u_h^n \|_1^2 \tau^2 + \frac{1}{2} G(u_h^{n-1}) (\| A_h u_h^n \|_0^2 + \nu^{-2} \| d_t u_h^n \|_0^2) \tau^2,
\]
\[
|b(u_h^{n-1}, u_h^n, A_h u_h^n + \nu^{-1} d_t u_h^n)| \tau
\]
\[
\leq c_0(\| u_h^{n-1} \|_0^{1/2} \| u_h^{n-1} \|_1^{1/2} \| u_h^n \|_1^{1/2} + \| u_h^{n-1} \|_1 \| u_h^n \|_0^{1/2})
\times \| A_h u_h^n \|_0^{1/2} (\| A_h u_h^n \|_0 + \nu^{-1} \| d_t u_h^n \|_0) \tau
\]
\[
\leq \frac{\nu}{8} \| A_h u_h^n \|_0^2 \tau + \frac{1}{8 \nu} \| d_t u_h^n \|_0^2 \tau
\]
\[
+ 2(\frac{4}{\nu})^3 c_0(\| u_h^{n-1} \|_0^2 \| u_h^n \|_0^2 + \| u_h^{n-1} \|_0^2 \| u_h^n \|_0^2) \| u_h^{n-1} \|_0^2 \| u_h^n \|_0^2 \tau,
\]
\[
|(f(t_n), A_h u_h^n + \nu^{-1} d_t u_h^n)| \tau \leq \frac{\nu}{8} \| A_h u_h^n \|_0^2 \tau + \frac{1}{8 \nu} \| d_t u_h^n \|_0^2 \tau + 4 \nu^{-1} \| f(t_n) \|_0^2 \tau,
\]
where
\[
G(u_h^n) = \begin{cases} 4^2 c_0^2 \| u_h^n \|_1 \| A_h u_h^n \|_0, & \alpha = 2, \\ 4^2 c_0^2 \log h \| u_h^n \|_0^2, & \alpha = 1, \\ 4^2 c_0^2 h^{-2} \| u_h^n \|_0^2, & \alpha = 0. \end{cases}
\]
Combining these inequalities with (3.8) and (4.9) yields
\[
\| u_h^n \|_0^2 - \| u_h^{n-1} \|_0^2 + \frac{1}{2} \| d_t u_h^n \|_0^2 \tau^2 + \nu \| u_h^n \|_1^2 \tau + \frac{1}{2} (\nu - G(u_h^{n-1}) \tau) \| u_h^{n} \|_1^2 \tau
\]
\[
\leq 4 \nu^{-1} \| f(t_n) \|_0^2 \tau,
\]
\[
2 \nu \| u_h^n \|_1^2 - 2 \nu \| u_h^{n-1} \|_1^2 + \| d_t u_h^n \|_0^2 \tau^2 + \| d_t u_h^n \|_0^2 \tau + \nu^2 \| A_h u_h^n \|_0^2 \tau
\]
\[
+ \frac{\nu}{2} (\nu - G(u_h^{n-1}) \tau) (\| A_h u_h^n \|_0^2 + \nu^{-2} \| d_t u_h^n \|_0^2) \tau
\]
\[
\leq d_{n-1} \nu \| u_h^{n-1} \|_1^2 \tau + 8 \nu^{-1} \| f(t_n) \|_0^2 \tau,
\]
where
\[
d_{n-1} = 4(\frac{4}{\nu})^3 c_0(\| u_h^{n-1} \|_0^2 \| u_h^n \|_0^2 + \| u_h^{n-1} \|_0^2 \| u_h^n \|_0^2).
\]
Now, we define \( d_t u_h^0 = \lim_{t \to 0} u_h(t) \) through (3.8), i.e.,
\[
(d_t u_h^0, v_h) + a(u_h^0, v_h) + b(u_h^0, u_h^0, v_h) = (f(t_0), v_h),
\]
for all \( v_h \in V_h \). Then, we deduce from (4.11) and (4.13) that
\[
(d_t u_h^1, v_h) + a(d_t u_h^1, v_h) = \frac{1}{\tau} \int_{t_0}^{t_1} (f_t(t), v_h) dt,
\]
and
\[
(d_t u_h^n, v_h) + a(d_t u_h^n, v_h) + b(d_t u_h^{n-1}, u_h^{n-1}, v_h) + b(u_h^{n-2}, d_t u_h^{n-1}, v_h)
\]
\[
= \frac{1}{\tau} \int_{t_{n-1}}^{t_n} (f_t(t), v_h) dt,
\]
for all $2 \leq n \leq N$. Hence, it follows from (4.14) that

$$
(4.16) \quad \|d_t u_h^n\|^2_0 + \|d_t u_h^{n-1}\|^2_0 + \nu \|d_t u_h^n\|^2_0 \leq \|d_t u_h^0\|^2_0 + \frac{\gamma^2}{\nu} \int_{t_0}^{t_1} \|f_t(t)\|^2_0 dt.
$$

Next, by taking $v_h = 2d_t u_h^n$ in (4.13) and using (4.14), we deduce

$$
||d_t u_h^n|_0^2 - \|d_t u_h^{n-1}\|^2_0 + \frac{2}{\nu} \|d_t u_h^n\|^2_0 \leq 2b(d_t u_h^{-1}, u_h^{-1}, d_t u_h^n) + 2b(u_h^{-2}, d_t u_h^n, d_t u_h^n)\tau^2,
$$

$$
(4.17) = 2(\int_{t_{n-1}}^{t_n} f_t(t), d_t u_h^n) dt.
$$

In view of Lemma 3.1 and (4.10), it holds that

$$
2|b(d_t u_h^n, u_h^{n-1}, d_t u_h^n)| \leq c_0 \|u_h^{n-1}\|_0^2 \|u_h^{-1}\|_0^2 \|d_t u_h^n\|_0^2 \|d_t u_h^n\|_0^2 \|d_t u_h^n\|_0^2
$$

$$
\leq \frac{\nu}{4} \|d_t u_h^n\|^2_0 + \left(\frac{2}{\nu}\right)^3 c_0 \|u_h^{n-1}\|^2_0 \|u_h^{-1}\|^2_0 \|d_t u_h^n\|^2_0
$$

$$
2|b(d_t u^n - d_t u_h^{n-1}, u_h^{n-1}, d_t u_h^n)| \leq \frac{1}{2} \|G^{1/2}(u_h^{n-1})\|(d_t u_h^n)_1 \|d_t u_h^n\|_0^2
$$

$$
\leq \frac{1}{4} \|d_t u_h^n\|^2_0 + \frac{1}{4} \|G(u_h^{n-1})\| \|d_t u_h^n\|_0^2
$$

$$
2|b(u_h^{n-2}, d_t u^n, d_t u_h^n)| \leq \frac{1}{2} \|G^{1/2}(u_h^{n-2})\| \|d_t u_h^n\|_0^2
$$

$$
\leq \frac{1}{4} \|d_t u_h^n\|^2_0 + \frac{1}{4} \|G(u_h^{n-2})\| \|d_t u_h^n\|_0^2
$$

$$
2|\int_{t_{n-1}}^{t_n} (f_t(t), d_t u_h^n) dt| \leq \frac{\nu}{4} \|d_t u_h^n\|^2_0 + \frac{4\gamma^2}{\nu} \int_{t_{n-1}}^{t_n} \|f_t(t)\|^2_0 dt.
$$

Combining these inequalities with (4.17) yields

$$
||d_t u_h^n|_0^2 - \|d_t u_h^{n-1}\|^2_0 + \frac{1}{2} \|d_t u_h^n\|^2_0 \leq \frac{1}{2} \|d_t u_h^{n-1}\|_0^2 \|d_t u_h^n\|_0^2
$$

$$
+ \frac{1}{4} \{(2\nu - G(u_h^{n-1}))\|d_t u_h^n\|_0^2\}
$$

$$
\leq \frac{1}{2} \|d_t u_h^{n-1}\|_0^2 \|d_t u_h^n\|_0^2 + \frac{4\gamma^2}{\nu} \int_{t_{n-1}}^{t_n} \|f_t(t)\|^2_0 dt.
$$

(4.18)

for all $2 \leq n \leq N$.

Next, we deduce from (4.11) and Lemma 3.1 that

$$
2\nu \|A_h u_h^n\|_0 \leq 2\|d_t u_h^n\|_0 + 2\|f(t_n)\|_0 + \nu \|d_t u_h^{n-1}\|_0 \|u_h^{n-1}\|_0 \|d_t u_h^{n-1}\|_0
$$

$$
(4.19) \leq 2\|d_t u^n\|_0 + 2\|f(t_n)\|_0 + \nu \|A_h u_h^{n-1}\|_0 \|d_t u_h^{n-1}\|_0 \|u_h^{n-1}\|_0 \|u_h^{n-1}\|_0.
$$

Moreover, we deduce from (2.2), (3.4), (4.1) and Lemma 3.1 that

$$
(4.20) \|p_h^n\|_0 \leq c\|u_h^n\|_0 + c\|d_t u^n\|_0 + c\|f(t_n)\|_0 + c\|u_h^{n-1}\|_0.
$$

Now, we will prove (4.11)-(4.16) by induction. For $\alpha = 0, 1, 2$, we deduce from (4.1) that

$$
(4.21) \quad G(u_h^0) \tau \leq G_h \tau \leq \nu.
$$

Due to (4.12), (4.4)-(4.6) hold for $m = 0$. For $\alpha = 0, 1$, we can obtain (4.4)-(4.6) with $m = 1$ by using (4.11)-(4.12), (4.20)-(4.21). For $\alpha = 2$, (4.13) and Lemma 3.1.
can yield

\[ \| d_t u_h^n \|_0 \leq 2\nu \| A_h u_h^n \|_0 + \| f(t_0) \|_0 + G^{1/2}(u_h^n) \| u_h^n \|_1. \]

Hence, we imply (4.4)-(4.6) with (4.22), (4.11), (4.12), (4.10) and (4.19)-(4.22). Assuming that (4.4)-(4.6) hold for \( m = 0, 1, \ldots, J \), we want to prove that they hold for \( m = J + 1 \).

**Proof of (4.4).** In view of the induction assumption and (4.3), it holds that

\[ G(u_h^{n-1}) \tau \leq G_h \tau \leq \nu, \quad \text{for } 1 \leq n \leq J + 1, \quad G(u_h^{n-2}) \tau \leq G_h \tau \leq \nu, \quad 2 \leq n \leq J, \]

for \( \alpha = 0, 1, 2 \). Summing (4.11) from \( n = 1 \) to \( J + 1 \) and using (4.23), we obtain (4.4) for \( m = J + 1 \) in the case of \( \alpha = 0, 1, 2 \).

**Proof of (4.5).** For \( \alpha = 1, 2 \), by summing (4.12) from \( n = 1 \) to \( n = J + 1 \) and using (4.23), we obtain

\[ 2\nu \| u_h^{n-1} \|^2 - \nu \| u_h^{n-1} \|_1^2 + \sigma(t_n)(\| d_t u_h^n \|^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0) \]

\[ \leq 2\nu \| u_h^{n-1} \|^2 \tau + d_n \| u_h^n \|^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0. \]

We set

\[ a_n = \nu \| u_h^n \|_1^2, \quad C = 8\nu^{-1}T \sup_{0 \leq t \leq T} \| f(t) \|^2_0 + 2\nu \| u_h^n \|^2_1, \]

\[ b_n = \| d_t u_h^n \|^2_0 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0. \]

Applying Lemma 3.5 to (4.24) and using (4.3), we obtain (4.4) with \( m = J + 1 \).

For \( \alpha = 0 \), multiplying (4.12) by \( \sigma(t_n) \), using (4.23) and noting \( \sigma(t_n) \leq \sigma(t_{n-1}) + \tau \), which will often be used later, we obtain

\[ 2\nu \| u_h^{n-1} \|^2 - \nu \| u_h^{n-1} \|_1^2 + \sigma(t_n)(\| d_t u_h^n \|^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0) \]

\[ \leq 2\nu \| u_h^{n-1} \|^2 \tau + d_n \| u_h^n \|^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0. \]

for all \( 1 \leq n \leq J + 1 \). Summing (4.25) from \( n = 1 \) to \( n = J + 1 \), we deduce

\[ \sigma(t_{J+1}) \| u_h^{J+1} \|^2_1 + \sum_{n=1}^{J+1} \sigma(t_n)(\| d_t u_h^n \|^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0) \]

\[ \leq 4\tau \sum_{n=0}^{J} d_n \sigma(t_n) \| u_h^n \|^2_1 + 2\nu \sum_{n=0}^{J} \| u_h^n \|^2_1 \]

\[ + 8\nu^{-1}T \sup_{0 \leq t \leq T} \| f(t) \|^2_0 + 2\nu \tau^2 d_0 \| u_h^1 \|^2_1. \]

Setting

\[ a_n = \sigma(t_n) \| u_h^n \|^2_1, \quad C = 8\nu^{-1}T \sup_{0 \leq t \leq T} \| f(t) \|^2_0 \tau, \]

\[ b_n = \sigma(t_n)(\| d_t u_h^n \|^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|^2_0). \]

Applying Lemma 3.5 to (4.26) and using (4.3)-(4.4), we arrive at (4.5) for \( m = J + 1 \).
Proof of (1.6). If \( \|A_h u_{h}^{J+1}\|_0 \leq \|A_h u_{h}^{J}\|_0 \), then the induction assumption yields
\[
\sigma^{2-\alpha} (t) \nu ^2 \| A_h u_{h}^{J+1} (t) \|_0^2 \leq \kappa_2, \quad 1 \leq J \leq N - 1,
\]
for \( \alpha = 0, 1, 2 \). Hence, we always assume that
\[
(4.27) \quad \| A_h u_{h}^{J+1}\|_0 \geq \| A_h u_{h}^{J}\|_0, \quad 1 \leq J \leq N - 1.
\]
For \( \alpha = 2 \), summing (4.13) from \( n = 2 \) to \( n = J + 1 \), adding (4.16) and using (4.14)-(4.15) and (4.23), we deduce
\[
(4.28) \quad \| d_t u_{h}^{J+1} \|_0^2 + \tau \sum_{n=1}^{J+1} (\nu \| d_t u_{h}^0 \|_1^2 + \| d_t u_{h}^n \|_0^2)
\]
Thus, by combining (4.27)-(4.28) with (4.19)-(4.20) with \( \alpha = 2 \), summing from \( n = 2 \) to \( n = J + 1 \), we find
\[
\sigma(t_{J+1}) \| d_t u_{h}^{J+1} \|_0^2 + \nu \tau \sum_{n=1}^{J+1} \sigma(t_n) \| d_t u_{h}^n \|_1^2 \leq \tau \sum_{n=1}^{J+1} (1 + 2(\frac{4}{\nu})^4 c_0^4 \nu \sigma(t_{J+1}) \| d_t u_{h}^0 \|_1^2 + \| d_t u_{h}^n \|_0^2)
\]
\[
(4.29) \quad + \frac{4 \nu^3}{c_0^4 \nu} \sum_{n=1}^{J+1} \int_0^T \| f_i(t) \|_0^2 dt + 2(1 + d_0 \tau) \nu \| d_t u_{h}^0 \|_1^2 + 16 \nu^{-1} \| f(t) \|_0^2.
\]
Now, by using (4.27), (4.28) and (4.19)-(4.20) in (4.29), we obtain (4.6) for \( m = J + 1 \).

Finally, for \( \alpha = 0 \), by multiplying (4.18) by \( \sigma(t_{n}) \), noting \( \sigma^2(t_{n}) \leq \sigma^2(t_{n-1}) + 3 \sigma(t_{n-1}) \tau \), which will often be used later, summing from \( n = 2 \) to \( n = J + 1 \) and using (4.12) with \( n = 1 \), we find
\[
\sigma^2(t_{J+1}) \| d_t u_{h}^{J+1} \|_0^2 + \nu \tau \sum_{n=1}^{J+1} \sigma^2(t_n) \| d_t u_{h}^n \|_1^2 \leq \tau \sum_{n=1}^{J+1} \sigma(t_n) (1 + 2(\frac{4}{\nu})^4 c_0^4 \nu \| u_{h}^{n-1} \|_0^2 \| u_{h}^{n-1} \|_1^2) \| d_t u_{h}^n \|_0^2
\]
\[
(4.30) \quad + \frac{4 \nu^3}{c_0^4 \nu} \sum_{n=1}^{J+1} \int_0^T \| f_i(t) \|_0^2 dt + 2(1 + d_0 \tau) \nu \| d_t u_{h}^0 \|_1^2 + 16 \nu^{-1} \| f(t) \|_0^2.
\]
Hence, by using (4.30), (4.26)-(4.27) and (4.19)-(4.20), we obtain (4.6) for \( m = J + 1 \).

**Theorem 4.2.** Under the assumptions of Theorem 4.1, it holds that
\[
(4.31) \quad \sigma^{3-\alpha} (t_{m}) \| d_t u_{h}^m \|_1^2 + \nu \tau \sum_{n=2}^{m} \sigma^{3-\alpha} (t_{n}) \| A_h d_t u_{h}^n \|_0^2 \leq \kappa_3,
\]
\[
(4.32) \quad \tau \sum_{n=2}^{m} \sigma^{3-\alpha} (t_{n}) \| d_t u_{h}^n \|_0^2 \leq \kappa_4,
\]
for all \(2 \leq m \leq N\) and \(\alpha = 1, 2\), where \(\kappa_3\) and \(\kappa_4\) are some positive constants depending on the data \((\nu, \Omega, T, u_0, f)\).

**Proof.** First, taking \(u_h = 2A_h d_t u_h^n \tau \in V_h\) in (4.15), we deduce

\[
\begin{align*}
\|d_t u_h^n\|^2_0 &- \|d_t u_h^{n-1}\|^2_0 + \|d_t u_h^n\|^2_0 \tau + 2 \nu \|A_h d_t u_h^n\|^2_0 \tau \\
&+ 2b(d_t u_h^{n-1}, u_h^{n-1}, A_h d_t u_h^n) \tau + 2b(u_h^{n-2}, d_t u_h^{n-1}, A_h d_t u_h^n) \tau \\
&\leq 2 \int_{t_{n-1}}^{t_n} (f(t), A_h d_t u_h^n) dt.
\end{align*}
\]

(4.33)

In view of Lemma \([4.1\) and \((3.10)\), it holds that

\[
\begin{align*}
2b(d_t u_h^{n-1}, u_h^{n-1}, A_h d_t u_h^n) \tau &\leq 2c_0 \gamma_0 \|A_h u_h^{n-1}\|_0 (d_t u_h^{n-1})_1 \|A_h d_t u_h^n\|_0 \tau \\
&\leq \nu \|A_h d_t u_h^n\|^2_0 \tau + 8 \nu^{-1} c_0^2 \gamma_0^2 \|A_h u_h^{n-1}\|^2_1 \|d_t u_h^{n-1}\|^2_0 \tau \\
\end{align*}
\]

(4.34)

\[
\begin{align*}
2b(u_h^{n-2}, d_t u_h^n, A_h d_t u_h^n) \tau &\leq 2c_0 \gamma_0^{1/2} \|u_h^{n-2}\|_1 \|d_t u_h^n\|_1^{1/2} \|A_h d_t u_h^n\|^{3/2}_0 \tau \\
&\leq \nu \|A_h d_t u_h^n\|^2_0 \tau + 2 \left( \frac{2}{\nu} \right)^3 c_0^3 \gamma_0^2 \|u_h^{n-2}\|^2_1 \|d_t u_h^n\|_1^2 \tau \\
\end{align*}
\]

\[
\begin{align*}
2b(u_h^{n-2}, d_t u_h^n - d_t u_h^{n-1}, A_h d_t u_h^n) \tau &\leq 2 G^{1/2}(u_h^{n-2}) \|d_t u_h^n\|_1 \|A_h d_t u_h^n\|_0 \tau \\
&\leq 1 \frac{1}{4} \|G(u_h^{n-2})\|_0 \|A_h d_t u_h^n\|_0 \tau \\
\end{align*}
\]

(4.35)

\[
\begin{align*}
2 \int_{t_{n-1}}^{t_n} (f(t), A_h d_t u_h^n) dt &\leq \nu \|A_h d_t u_h^n\|^2_0 \tau + \frac{8}{\nu} \int_{t_{n-1}}^{t_n} \|f(t)\|^2_0 dt.
\end{align*}
\]

Combining these inequalities with (4.33) yields

\[
\begin{align*}
\|d_t u_h^n\|^2_0 &- \|d_t u_h^{n-1}\|^2_0 + \nu \|A_h d_t u_h^n\|^2_0 \tau + \frac{1}{4} (\nu - G(u_h^{n-2}) \|A_h d_t u_h^n\|^2_0 \tau \\
&\leq 8 \nu^{-1} \gamma_0^2 \|A_h u_h^{n-1}\|^2_1 \|d_t u_h^{n-1}\|^2_0 \tau + 2 \left( \frac{2}{\nu} \right)^3 c_0^3 \gamma_0^2 \|u_h^{n-2}\|^2_1 \|d_t u_h^n\|_1^2 \tau \\
\end{align*}
\]

(4.36)

\[
\begin{align*}
&+ \frac{8}{\nu} \int_{t_{n-1}}^{t_n} \|f(t)\|^2_0 dt,
\end{align*}
\]

for all \(2 \leq n \leq N\). Multiplying (4.14) by \(\sigma^{3-\alpha}(t_n)\) and using (4.23), we deduce

\[
\begin{align*}
\sigma^{3-\alpha}(t_n) \|d_t u_h^n\|_1^2 - \sigma^{3-\alpha}(t_{n-1}) \|d_t u_h^{n-1}\|_1^2 + \nu \sigma^{3-\alpha}(t_n) \|A_h d_t u_h^n\|^2_0 \tau \\
&\leq c \sigma^{3-\alpha}(t_{n-1}) \|A_h u_h^{n-1}\|_0^2 \|d_t u_h^{n-1}\|^2_1 \tau \\
&+ c \sigma^{3-\alpha}(t_n) \|u_h^{n-2}\|^2_1 \|d_t u_h^n\|_1^2 \tau \\
&+ c \sigma^{3-\alpha}(t_{n-1}) \|d_t u_h^{n-1}\|^2_1 \tau + c \int_{t_{n-1}}^{t_n} \|f(t)\|^2_0 dt,
\end{align*}
\]

(4.35)

for all \(2 \leq n \leq N\). Summing (4.35) from \(n = 2\) to \(n = m\) and using (4.6), we obtain (4.33).

Then, we deduce from (4.15), (3.10) and Lemma 3.1 that

\[
\begin{align*}
\|d_t u_h^n\|_0 &\leq \nu \|A_h d_t u_h^n\|_0 + c \|d_t u_h^{n-1}\|_1 \|A_h u_h^{n-1}\|_0 + H(n-3) \|A_h u_h^{n-2}\|_0 \\
&+ \frac{1}{2} H(2-n) G^{1/2}(u_0) \|d_t u_h^1\|_1 + \tau^{-1/2} \left( \int_{t_{n-1}}^{t_n} \|f(t)\|^2_0 dt \right)^{1/2}
\end{align*}
\]

(4.36)
for all $2 \leq n \leq N$, where $H(t) = 1$, as $t \geq 0$ and $H(t) = 0$, as $t < 0$. Thus, we deduce from (4.30) that
\[
\sigma^{3-\alpha}(t_n)\|dt u_n^h\|_0^2 \leq c\sigma^{3-\alpha}(t_n)\|A_h dt u_n^h\|_0^2 + H(2-n)G(u_h^0)(2\tau)^{3-\alpha}\|dt u_n^h\|_1^2 + c\int_{t_n}^{t_{n+1}} \|f_t\|_0^2 dt + c\sigma^{2-\alpha}(t_{n+1})\|dt u_{n+1}^h\|_0^2 + H(n-3)\sigma(t_{n+1})\|A_h u_{n+1}^h\|_0^2\tau.
\]
Summing the above inequality from $n = 2$ to $n = m$ and using Theorem 4.1, (4.21) and (4.31), we get (4.32). \[\Box\]

5. Dual Euler scheme: Stability analysis

In order to derive the $L^2$-bound on the error $u_h(t_n) - u^n_h$ in the case of $\alpha = 1$, we employ a parabolic argument that has already been used in [23] for the Crank-Nicolson scheme of the time-dependent Navier-Stokes equation. Let $1 \leq m \leq N$ be given. We consider the linearized “backward” counterpart of the discrete Navier-Stokes (4.1): For $\xi^n \in V_h$, $1 \leq n \leq m$, find $\Phi^{n-1}_h \in V_h$ such that
\[
(\Phi^{n-1}_h, v_h, d_t \Phi^{n-1}_h - a(v_h, \Phi^{n-1}_h) - b(u^n_h, v_h, \Phi^{n-1}_h) - b(v_h, u^n_h, \Phi^{n-1}_h) = (v_h, \xi^n),
\]
for $v_h \in V_h$ with an initial value $\Phi^m_h = 0$.

Here, we need to introduce the following discrete dual Gronwall lemma provided in [11].

**Lemma 5.1.** Let $C > 0$ and let $a_n, b_n, d_n$, for integers $0 \leq n \leq m$, be nonnegative numbers such that
\[
a_k + \tau \sum_{n=k}^{m} b_n \leq \tau \sum_{n=k+1}^{m} d_n a_n + C, \quad 0 \leq k \leq m.
\]
Then
\[
a_k + \tau \sum_{n=k}^{m} b_n \leq C\exp(\tau \sum_{n=k+1}^{m} d_n), \quad 0 \leq k \leq m,
\]
where we assume that $\tau \sum_{n=m+1}^{m} d_n = 0$.

The following lemma provides the stability of the scheme (5.1).

**Lemma 5.2.** Under the assumptions of Theorem 4.1, the following a priori estimate holds:
\[
\|\Phi^k_h\|_1^2 + \nu \tau \sum_{n=k}^{m} \|A_h \Phi^n_h\|_0^2 \leq \kappa \tau \sum_{n=1}^{m} \|\xi^n\|_0^2,
\]
for all $0 \leq k \leq m$.

**Proof.** The proof follows the line of argument used in the proofs of Theorem 4.1. In view of Lemma 3.1 and (4.3), we can prove that (5.1) admits a unique solution sequence $\{\Phi^k_h\}_0^m$. 

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Moreover, by taking $v_h = -2A_h\Phi_h^{n-1}\tau$ in (5.1), we obtain
\[
\|\Phi_h^{n-1}\|^2_1 - \|\Phi_h^n\|^2_1 + \|d_t\Phi_h^n\|^2_2 + 2\nu\|A_h\Phi_h^{n-1}\|^2_0\tau \\
+ 2\nu^{-1}\|A_h\Phi_h^{n-1}, u_h^n, \Phi_h^{n-1}\tau + 2b(u_h^n, A_h\Phi_h^{n-1}, \Phi_h^{n-1})\tau
\]
(5.5)

From Lemma 3.1 and (3.10), we have
\[
2b(A_h\Phi_h^{n-1}, u_h^n, \Phi_h^n)\tau + 2b(u_h^n, A_h\Phi_h^{n-1}, \Phi_h^{n-1})\tau \\
\leq \frac{\nu}{4}\|A_h\Phi_h^{n-1}\|^2_0\tau + 4\nu^{-1}\|\Phi_h^n\|^2_1 + 4\nu^{-1}\|\xi^n\|^2_0\tau,
\]
(5.6)

for all $1 \leq n \leq m$. Using (4.3) and Theorem 4.1 with $\alpha = 1, 2$, we have
\[
\nu - G(u_h^n)\tau \geq \nu - G_h\tau \geq 0, \forall 0 \leq n \leq N.
\]
(5.7)

Summing (5.6) from $k + 1$ to $m$ and using (5.7) and Theorem 4.1 we obtain
\[
\|\Phi_h^n\|^2_1 + \nu\tau\sum_{n=k}^{m-1} \|A_h\Phi_h^n\|^2_0 \\
\leq 4\tau\sum_{n=k+1}^{m} \nu^{-1}\|\Phi_h^n\|^2_1 + 2\|\Phi_h^n\|^2_1 + 4\nu^{-1}\tau\sum_{n=1}^{m} \|\xi^n\|^2_0,
\]
(5.8)

for all $0 \leq k \leq m - 1$. Applying Lemma 5.1 to (5.8) and using Theorem 4.1 yields (5.4).

6. Error analysis

In this section, we establish the $H^1$- and $L^2$-bounds of the error $e^n = u_h(t_n) - u_h^n$ and the $L^2$-bound of the error $\eta^n = p_h(t_n) - p_h^n$ for all $1 \leq n \leq N$. To do this, we take $t = t_n$ in (5.8) and note
\[
u_h(t_n) - d_tu_h(t_n) = \tau^{-1}\int_{t_{n-1}}^{t_n} (u_{ht}(t_n) - u_{ht}(t)) dt = \tau^{-1}\int_{t_{n-1}}^{t_n} (t - t_{n-1})u_{ht}dt,
\]
to obtain
\[
(d_tu_h(t_n), v_h) + a(u_h(t_n), v_h) - d(v_h, p_h(t_n)) + d(u_h(t_n), q_h) + b(u_h(t_n), u_h(t_n), v_h) \\
= (f(t_n), v_h) - \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(u_{ht}(t), v_h) dt.
\]
(6.1)
Subtracting (4.1) from (6.1), we obtain
\[
(d_t e^n, v_h) + a(e^n, v_h) - d(v_h, e^n) + d(e^n, q_h) + b(e^n, u_h(t_n), v_h)
\]
\[+ b(u_h^n, e^n, v_h) = (E_n, v_h),
\]
for all \((v_h, q_h) \in (X_h, M_h)\) with
\[
(E_n, v_h) = -\frac{1}{\tau} \int_{t_{n-1}}^{t_n} (u_{h\text{htt}}(t), v_h)dt
\]
\[+ b(u_h^{n-1} - u_h^n, u_h^{n-1}, v_h) + b(u_h^n, u_h^{n-1} - u_h^n, v_h).
\]

**Lemma 6.1.** Under the assumptions of Theorem 4.1 with \(\alpha = 1, 2\), the error \(E_n\) satisfies the following bounds:

\[
\tau \sum_{n=1}^{m} \|A_h^{-1} P_h E_n\|_0^2 \leq \kappa \tau^2,
\]
\[
\tau \sum_{n=1}^{m} \|A_h^{-1/2} P_h E_n\|_0^2 \leq \kappa \tau^\alpha,
\]
\[
\tau \sum_{n=1}^{m} \|A_h^{-1/2} P_h d_t E_n\|_0^2 \leq \kappa \tau^2,
\]
for all \(1 \leq m \leq N\), and

\[
\tau \sum_{n=2}^{m} \sigma^{2-\alpha}(t_n) \|A_h^{-1/2} P_h E_n\|_0^2 \leq \kappa \tau^2, \hspace{1cm} 2 \leq m \leq N,
\]
\[
\tau \sum_{n=3}^{m} \sigma^{4-\alpha}(t_n) \|A_h^{-1/2} P_h d_t E_n\|_0^2 \leq \kappa \tau^2, \hspace{1cm} 3 \leq m \leq N.
\]

**Proof.** First, it follows from (6.3), (3.10) and Lemma 3.1 that

\[
\|A_h^{-1} P_h E_n\|_0^2 \tau \leq c \tau^\alpha \int_{t_{n-1}}^{t_n} \|A_h^{-1} u_{h\text{htt}}\|_0^2 dt
\]
\[+ c\|d_t u_h^n\|_0^2 (\|u_h^n\|_1^2 + \|u_h^{n-1}\|_1^2) \tau^3,
\]
\[
\|A_h^{-1/2} P_h E_n\|_0^2 \tau \leq c \tau^\alpha \int_{t_{n-1}}^{t_n} \sigma^{2-\alpha}(t) \|A_h^{-1/2} u_{h\text{htt}}\|_0^2 dt
\]
\[+ c\|d_t u_h^n\|_0^2 (\|u_h^{n-1}\|_1^2 + \|u_h^n\|_1^2) \tau^3,
\]
\[
\sigma^{2-\alpha}(t_n) \|A_h^{-1/2} P_h E_n\|_0^2 \tau \leq c \tau^\alpha \int_{t_{n-1}}^{t_n} \sigma^{2-\alpha}(t) \|A_h^{-1/2} u_{h\text{htt}}\|_0^2 dt
\]
\[+ c\sigma^{2-\alpha}(t_n) \|d_t u_h^n\|_0^2 (\|u_h^{n-1}\|_1^2 + \|u_h^n\|_1^2) \tau^3.
\]

Summing (6.9), (6.10) and (6.11) from 1 to \(m\), respectively, noting \(\tau^2 \leq \sigma^{2-\alpha}(t_n) \tau^\alpha\) and using (2.2), Theorem 3.6 and Theorem 4.1, we deduce (6.4)–(6.6) for \(\alpha = 1, 2\).

Next, by using (3.10) and Lemma 3.1 we deduce from (6.3) that

\[
\|E_n\|_0 \leq c \tau^{-1/2} \int_{t_{n-1}}^{t_n} (t - t_{n-1})^2 \|u_{h\text{htt}}\|_0^2 dt \right)^{1/2}
\]
\[+ c(\|A_h u_h^n\|_0 + \|A_h u_h^{n-1}\|_0) \|d_t u_h^n\|_0 \tau,
\]

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for all $2 \leq n \leq N$. Hence, we deduce from (6.12) that
\begin{equation}
\sigma^{3-\alpha}(t_n)\|E_n\|^2_0\tau \leq c\tau^2 \int_{t_{n-1}}^{t_n} \sigma^{3-\alpha}(t)\|u_{htt}\|^2_0 dt
\end{equation}
(6.13)
\begin{equation}
\alpha
\sigma^4\alpha(1-t_n)\|E_n\|^2_0 \leq c\tau^2 \int_{t_{n-1}}^{t_n} \sigma^{3-\alpha}(t)\|u_{htt}\|^2_0 dt
\end{equation}
(6.14)
for all $2 \leq n \leq N$. Summing (6.13) from $n = 2$ to $n = m$ and using (6.14) and Theorems 3.6, 4.1 and 4.2, we deduce (6.7).

Moreover, we deduce from (6.3) that
\begin{equation}
(d_t E_n, v_h) = -\tau^2 \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \int_{t-\tau}^{t} (u_{httt}(s), v_h) ds dt - b(d_t u_h^n, u_h^{n-1}, v_h)\tau + c(d_t u_h^n, v_h)\tau
\end{equation}
(6.15)
for all $3 \leq n \leq N$. Using (3.10) and Lemma 3.1, we deduce from (6.15) that
\begin{equation}
\|A_h^{1/2} P_h d_t E_n\|_0 \leq c\tau^{-3/2} \left(\int_{t_{n-1}}^{t_n} (t - t_{n-1})^2 \int_{t-\tau}^{t} u_{httt}(s) ds dt\right)^{1/2}
\end{equation}
\begin{equation*}
+ c\|d_t u_h^n\|_0\|A_h u_h^{n-1}\|_0 \tau + c\|d_t u_h^n\|_0\|A_h d_t u_h^n\|_0 \tau
\end{equation*}
\begin{equation*}
+ c\|d_t u_h^{n-1}\|_0\|A_h d_t u_h^{n-1}\|_0 \tau,
\end{equation*}
which yields
\begin{equation}
\sigma^{4-\alpha}(t_n)\|A_h^{1/2} P_h d_t E_n\|^2_0 \tau \leq c\tau^2 \int_{t_{n-2}}^{t_n} \sigma^{4-\alpha}(t)\|A_h^{1/2} u_{httt}\|^2_0 dt
\end{equation}
(6.16)
\begin{equation}
+ c\tau^2 \sigma^{3-\alpha}(t_n)\|A_h^{1/2} P_h E_n\|^2_0 \tau
\end{equation}
for all $3 \leq n \leq N$. Summing (6.16) from $3$ to $m$ and using Theorems 3.6, 4.1 and 4.2, we deduce (6.8). \hfill \Box

Lemma 6.2. Under the assumptions of Theorem 4.1 with $\alpha = 1, 2$, we have
\begin{equation}
\|e^m\|^2_0 + \tau \sum_{n=1}^{m} (\|d_t e^n\|^2_0 + \nu\|e^n\|^2_1) \leq \kappa \tau^\alpha,
\end{equation}
(6.17)
for all $1 \leq m \leq N$.

Proof. Taking $v_h = 2e^n\tau \in V_h$ and $q_h = 0$ in (6.2), we obtain
\begin{equation}
\|e^n\|^2_0 - \|e^{n-1}\|^2_0 + \|d_t e^n\|^2_0 2\nu\|e^n\|^2_1 \tau + 2b(e^n, u_h^n, e^n)\tau
\end{equation}
(6.18)
\begin{equation}
\leq \frac{\nu}{4}\|e^n\|^2_0 + 4\nu \|A_h^{-1/2} P_h E_n\|^2_0 \tau.
\end{equation}
Using Lemma 3.1 and (3.10), one finds
\begin{align*}
2|b(e^{n-1}, u^n_h, e^n)|\tau & \leq 2c_0\gamma^2 \|e^{n-1}\|_0^2 \|e^{n-1}\|_1^\frac{1}{2} \|u^n_h\|_1 \|e^n\|_1 \tau \\
& \leq \frac{\nu}{4} (\|e^n\|_1^2 + \|e^{n-1}\|_1^2) \tau + 2\left(\frac{2}{\nu}\right)^3 c_0^3 \gamma^3 \|u^n_h\|_1 \|e^{n-1}\|_1 \|e^n\|_1 \tau,
\end{align*}

(6.23)
\begin{align*}
2|b(e^n - e^{n-1}, u^n_h, e^n)|\tau & \leq \frac{1}{2} G^{1/2}(u_h^n) \|e^n\|_1 \|d_t e^n\|_0 \tau^2 \\
& \leq \frac{1}{2} \|d_t e^n\|_0^2 \tau^2 + \frac{1}{4} G(u_h^n) \|e^n\|_1^2 \tau^2.
\end{align*}

Hence, by combining the above inequalities with (6.18), we obtain
\begin{align*}
\|e^n\|_0^2 - \|e^{n-1}\|_0^2 & + \frac{1}{2} \|d_t e^n\|_0^2 \tau^2 + \nu \|e^n\|_1^2 \tau \\
& + \frac{\nu}{4} (\|e^n\|_1^2 - \|e^{n-1}\|_1^2) \tau + \frac{1}{4} (\nu - G(u_h^n) \tau) \|e^n\|_1^2 \tau
\end{align*}
(6.19)
\begin{align*}
& \leq 2\left(\frac{2}{\nu}\right)^3 c_0^3 \gamma^3 \|u^n_h\|_1 \|e^{n-1}\|_1 \|e^n\|_1 \|e^n\|_1 \tau + 4\nu^{-1} \|A_h^{-1/2} P_h E_n\|_0^2 \tau,
\end{align*}
for all 1 \leq n \leq N. Moreover, summing (6.19) from 1 to m and using (5.7), we have
\begin{align*}
\|e^m\|_0^2 & + \tau \sum_{n=1}^m \frac{1}{2} \|d_t e^n\|_0^2 \tau + \nu \|e^n\|_1^2 \tau \\
\end{align*}
(6.20)
\begin{align*}
& \leq \tau \sum_{n=0}^{m-1} d_n \|e^n\|_0^2 + 4\nu^{-1} \tau \sum_{n=1}^N \|A_h^{-1/2} P_h E_n\|_0^2,
\end{align*}
where \(d_n = 2\left(\frac{2}{\nu}\right)^3 c_0^3 \gamma^3 \|u^{n+1}_h\|_1^4\). We set
\[a_n = \|e^n\|_0^2, \quad b_n = \frac{1}{2} \|d_t e^n\|_0^2 \tau + \nu \|e^n\|_1^2, \quad C = 4\nu^{-1} \tau \sum_{n=1}^N \|A_h^{-1/2} P_h E_n\|_0^2,\]
and apply Lemma 3.5 to (6.20) and use Theorem 4.1 and Lemma 6.1 to deduce (6.17).

With the aid of Lemma 6.2, we obtain the following error estimate.

**Lemma 6.3.** Under the assumptions of Theorem 3.1 with \(\alpha = 1, 2\), we have
\begin{align*}
\sigma^{2-\alpha} (t_m) \|e^m\|_0^2 + \tau \sum_{n=1}^m \sigma^{2-\alpha} (t_n) \|e^n\|_1^2 \leq \kappa \tau^2,
\end{align*}
(6.21)
for all 1 \leq m \leq N.

**Proof.** For \(\alpha = 2\), Lemma 6.2 yields (6.21). For \(\alpha = 1\), we let \(\{\Phi_h^n\}_0^m\) be the solution of (6.1), corresponding to the initial value \(\Phi_h^n = 0\) and the right-hand side of \(\xi^n\)_0^m\)\(_1^m\). Then, by construction, it holds that
\begin{align*}
\|e^n\|_0^2 & = (e^n, d_t \Phi_h^n) \tau - a(e^n, \Phi_h^{n-1}) \tau \\
& - b(u^n_h, e^n, \Phi_h^{n-1}) \tau - b(e^n, u^n_h, \Phi_h^{n-1}) \tau.
\end{align*}
(6.22)
Taking \(v_h = \Phi_h^{n-1} \tau\) in (6.2) and adding (6.22), we obtain
\begin{align*}
\|e^n\|_0^2 & = (e^n, \Phi_h^n) - (e^{n-1}, \Phi_h^{n-1}) - (E_n, \Phi_h^{n-1}) \tau + b(e^n, e^n, \Phi_h^{n-1}) \tau.
\end{align*}
(6.23)
Summing (6.23) for 1 \leq n \leq m and using Lemmas 5.2, 6.1 and 6.2, we have
\[
\tau \sum_{n=1}^{m} \|e^n\|_0^2 \leq (\tau \sum_{n=1}^{m} \|A_{h}^{-1}P_h E_n\|_0^2)^{1/2} \left(\tau \sum_{n=1}^{m} \|A_h \Phi_{h}^{-1}\|_0^2\right)^{1/2} + c(\tau \sum_{n=1}^{m} \|e^n\|_0^2)^{1/2} \left(\tau \sum_{n=1}^{m} \|A_h \Phi_{h}^{-1}\|_0^2\right)^{1/2} \leq \kappa \tau + \frac{1}{2} \tau \sum_{n=1}^{m} \|e^n\|_0^2 + \kappa \tau^2. \tag{6.24}
\]

Next, multiplying (6.19) by \(\sigma(t_n)\), we deduce
\[
\sigma(t_n)||e^n||_0^2 - \sigma(t_{n-1})||e^{n-1}||_0^2 + \sigma(t_n)\nu||e^n||_1^2 \tau + \frac{\nu}{4}\sigma(t_n)||e^n||_1^2 - \sigma(t_{n-1})||e^{n-1}||_0^2 \tau 
\leq ||e^{n-1}||_0^2 \tau + \frac{\nu}{4}||e^{n-1}||_1^2 \tau^2 + 2\left(\frac{2}{\nu}\right)^3 c_0^3 \gamma_0 \|u_h\|_1^2 \|e^{n-1}\|_0^2 \tau + 4\nu^{-1}\sigma(t_n)||A_h^{-1/2}P_h E_n\|_0^2, \tag{6.25}
\]
for all 1 \leq n \leq N. Summing (6.25) from n = 1 to n = m, we have
\[
\sigma(t_m)||e^m||_0^2 + \nu \tau \sum_{n=1}^{m} \sigma(t_n)||e^n||_1^2 + \frac{\nu}{4}\sigma(t_m)||e^m||_1^2 \tau 
\leq \tau \sum_{n=1}^{m} \left(||e^n||_0^2 + \tau \nu||e^n||_1^2\right) + 2\tau \sum_{n=1}^{m} \left(\frac{2}{\nu}\right)^3 c_0^3 \gamma_0 \|u_h\|_1^2 ||e^{n-1}||_0^2 \tau + 4\tau \sum_{n=1}^{m} \nu^{-1}\sigma(t_n)||A_h^{-1/2}P_h E_n\|_0^2.
\]
Using (6.24), Theorem 4.1 and Lemmas 6.1 and 6.2 in the above inequality gives (6.21) for \(\alpha = 1\).

**Lemma 6.4.** Under the assumptions of Theorem 4.1 with \(\alpha = 1, 2\), we have
\[
\sigma^{3-\alpha}(t_m)||e^m||_1^2 + \tau \sum_{n=2}^{m} \sigma^{3-\alpha}(t_n)(||d_t e^n||_0^2 + \nu^2||A_h e^n||_0^2) \leq \kappa \tau^2, \tag{6.26}
\]
for all 1 \leq m \leq N.

**Proof.** Taking \(v_h = 2A_h e^n \tau \in V_h\) and \(q_h = 0\) in (6.2), we obtain
\[
||e^n||_1^2 - ||e^{n-1}||_1^2 + ||d_t e^n||_1^2 \tau + 2\nu||A_h e^n||_0^2 \tau + 2b(e^n, u_h(t_n), A_h e^n) \tau 
\leq 2b(u_h^n, e^n, A_h e^n) \tau + 2b(e^n, u_h(t_n), A_h e^n) \tau \leq \frac{\nu}{4}||A_h e^n||_1^2 \tau + 4\nu^{-1}||E_n||_0^2 \tau. \tag{6.27}
\]
In view of Lemma 3.3 and (5.10), we have
\[
2|b(e^n, u_h(t_n), e^n)| \tau + 2|b(u_h^n, e^n, A_h e^n)| \tau \leq 2c_0 \gamma_0 \|e^n\|_0^2 \left(||A_h u_h^n||_0 + ||A_h u_h(t_n)||_0\right)||A_h e^n||_0^2 \tau \leq \frac{\nu}{4}||A_h e^n||_1^2 \tau + c(||A_h u_h(t_n)||_0^2 + ||A_h u_h^n||_0^2)\|e^n\|_1^2 \tau.
\]
Hence, by combining the above inequality with (6.27), we obtain
\[
||e^n||_1^2 - ||e^{n-1}||_1^2 + \nu||A_h e^n||_0^2 \tau 
\leq c(||A_h u_h^n||_0^2 + ||A_h u_h(t_n)||_0^2)\|e^n\|_1^2 \tau + 4\nu^{-1}||E_n||_0^2 \tau, \tag{6.28}
\]
for all $1 \leq n \leq N$. Multiplying (6.29) by $\sigma^{3-\alpha}(t_n)$, we find
\begin{align*}
\sigma^{3-\alpha}(t_n)\|e^n\|_1^2 - \sigma^{3-\alpha}(t_{n-1})\|e^{n-1}\|_1^2 + \nu\sigma^{3-\alpha}(t_n)\|A_h e^n\|_0^2 \tau & \\
& \leq c_0\sigma^{3-\alpha}(t_n)(\|A_h u_h^n\|_0^2 + \|A_h u_h(t_n)\|_0^2)\|e^n\|_0^2 \tau
& \quad + c_0\sigma^{3-\alpha}(t_{n-1})\|e^{n-1}\|_0^2 \tau + 4\nu^{-1}\sigma^{3-\alpha}(t_n)\|E_n\|_0^2 \tau,
\end{align*}
(6.29)
for all $1 \leq n \leq N$. Summing (6.29) from 2 to $m$, and using Theorems 3.3 and 4.1 and Lemmas 6.1, 6.2 and 6.3, we deduce
\begin{equation}
\sigma^{3-\alpha}(t_m)\|e^m\|_1^2 + \nu\tau \sum_{n=2}^{m}\sigma^{3-\alpha}(t_n)\|A_h e^n\|_0^2 \leq \kappa \tau^2,
\end{equation}
(6.30)
for all $1 \leq m \leq N$.

Finally, we deduce from (6.2), (3.10) and Lemma 3.1 that
\begin{align*}
\sigma^{3-\alpha}(t_n)\|d_t e^n\|_0^2 & \leq c_0\sigma^{3-\alpha}(t_n)(1 + \|u_h^n\|_1^2 + \|e^n\|_1^2)\|A_h e^n\|_0^2 \tau \\
& \quad + c_0\sigma^{3-\alpha}(t_n)\|E_n\|_0^2 \tau,
\end{align*}
(6.31)
for all $2 \leq n \leq N$. Summing (6.31) from $n = 2$ to $n = m$ and using (6.30), Theorem 4.1 and Lemmas 6.1 and 6.2, we deduce (6.26).

It remains to prove the error estimate for the discrete pressure $p_h^n$. To do this, we need to estimate $d_t e^n$. It follows from (6.2) that
\begin{align*}
(d_t e^n, v_h) & + a(d_t e^n, v_h) + b(d_t e^n, u_h(t_n), v_h) + b(e^{n-1}, d_t u_h(t_n), v_h) \\
& + b(d_t u_h^n, e^n, v_h) + b(u_h^{n-1}, d_t e^n, v_h) = (d_t E_n, v_h),
\end{align*}
(6.32)
for all $v_h \in V_h$ and $1 \leq n \leq N$. Taking $v_h = 2d_t e^n$ in (6.32) and using (3.12), we get
\begin{align*}
\|d_t e^n\|_0^2 - \|d_t e^{n-1}\|_0^2 + 2\nu\|d_t e^n\|_0^2 \tau + 2b(d_t e^n, u_h(t_n), d_t e^n) \tau
& \quad + 2b(d_t u_h^n, e^n, d_t e^n) \tau \\
& \leq \frac{\nu}{4}\|d_t e^n\|_0^2 \tau + 4\nu^{-1}\|A_h^{-1/2}P_h d_t E_n\|_0^2 \tau.
\end{align*}
(6.33)
In view of (3.10) and Lemma 3.1, we deduce
\begin{align*}
2|b(d_t e^n, u_h(t_n), d_t e^n)| \tau & \leq c_0 \gamma_0^{1/2}\|d_t e^n\|_0^{1/2}\|d_t e^n\|_1^{1/2}\|u_h(t_n)\|_{-1} \tau \\
& \leq \frac{\nu}{4}\|d_t e^n\|_0^2 \tau + (\frac{\nu}{2c_0})^{1/2}\|u_h(t_n)\|_1^{1/2}\|d_t e^n\|_0^2 \tau,
\end{align*}
(6.34)
\begin{align*}
2|b(e^{n-1}, d_t u_h^n, d_t e^n)| \tau & \leq 2c_0 \gamma_0 (\|A_h e^{n-1}\|_0 + \|A_h e^n\|_0)\|d_t u_h^n\|_0\|d_t e^n\|_1 \tau \\
& \leq \frac{\nu}{4}\|d_t e^n\|_0^2 \tau + 8\nu^{-1}c_0^2 \gamma_0^2 (\|A_h e^{n-1}\|_0^2 + \|A_h e^n\|_0^2)\|d_t u_h^n\|_0^2 \tau.
\end{align*}
Combining these inequalities with (6.33) gives
\begin{align*}
\sigma^{4-\alpha}(t_n)\|d_t e^n\|_0^2 - \sigma^{4-\alpha}(t_{n-1})\|d_t e^{n-1}\|_0^2 & \leq \sigma^{4-\alpha}(t_n)(\frac{\nu}{2})^{1/2}c_0 A_0^2 \|u_h(t_n)\|_1^2\|d_t e^n\|_0^2 \tau \\
& \quad + (\sigma^{3-\alpha}(t_{n-1})\|A_h e^{n-1}\|_0^2 + \sigma^{3-\alpha}(t_n)\|A_h e^n\|_0^2)\|d_t u_h^n\|_0^2 \tau \\
& \quad + c_0\sigma^{3-\alpha}(t_{n-1})\|d_t e^{n-1}\|_0^2 \tau + c_0\sigma^{4-\alpha}(t_n)\|A_h^{-1/2}P_h d_t E_n\|_0^2 \tau.
\end{align*}
(6.34)
Summing (6.34) from 3 to $m$ and using Theorem 3.3, Theorem 4.1, Lemma 6.1, Lemma 6.3 with $m = 1, 2$ and Lemma 6.4, we obtain
\begin{equation}
\sigma^{4-\alpha}(t_m)\|d_t e^m\|_0^2 \leq \kappa \tau^2, \quad 1 \leq m \leq N.
\end{equation}
(6.35)
Moreover, we deduce from (6.2), (3.10) and Lemma 3.1 that
\[
\|E_1\|_0 \leq \|u_{ht}(t_1)\|_0 + \tau^{-1/2} \left( \int_{t_0}^{t_1} \|u_{ht}\|_0^2 dt \right)^{1/2} + \frac{1}{2} \left( G^{1/2}(u_h^0) + G^{1/2}(u_h^1) \right) \|d_1u_1^h\|_1 \tau,
\]
which yields
\[
\sigma^{4-\alpha}(t_1)\|E_1\|^2_0 \leq c\sigma^{4-\alpha}(t_1)\|u_{ht}(t_1)\|_0^2 + c\sigma^{3-\alpha}(t_1) \int_{t_0}^{t_1} \|u_{ht}\|_0^2 dt + c\tau^2 \left( G(u_h^0)\tau + G(u_h^1)\tau \right) \|d_1u_1^h\|_1^2.
\]
(6.36)

Using (5.7), Theorem 3.3 and Theorem 4.2 in (6.36), we obtain
\[
\sigma^{4-\alpha}(t_1)\|E_1\|^2_0 \leq \kappa\tau^2.
\]
(6.37)

By (3.4), (3.10), (6.2) and Lemma 3.1 we deduce
\[
\|\eta^m\|_0 \leq c(|d_1e^m|_0 + \|e^m\|_1) + c\|e^m\|_1(\|u_h(t_m)\|_1 + \|u_m^h\|_1)
\]
\[
+ c\|E_m\|_0,
\]
which together with Theorems 3.3 and 4.1 yield
\[
\sigma^{4-\alpha}(t_m)\|\eta^m\|^2_0 \leq \kappa\sigma^{4-\alpha}(t_m)\|\eta^m\|^2_0 + \kappa\sigma^{3-\alpha}(t_m)\|e^m\|^2_1 + \sigma^{4-\alpha}(t_m)\|E_m\|^2_0.
\]
(6.38)

Using (6.35), (6.37), Lemma 6.1 and Lemma 6.4 in (6.38) yields
\[
\sigma^{4-\alpha}(t_m)\|\eta^m\|^2_0 \leq \kappa\tau^2, \quad 1 \leq m \leq N.
\]
(6.39)

Combining (6.39) with Lemma 6.3 and Lemma 6.4 yields the following error estimates results.

**Theorem 6.5.** Under the assumptions of Theorem 4.1, the following error estimates hold:

\[
\sigma^{2-\alpha}(t_m)\|u_h(t_m) - u_h^m\|_0^2 + \sigma^{3-\alpha}(t_m)\|u_h(t_m) - u_h^m\|_1^2 \leq \kappa\tau^2, \quad t_m \in (0, T],
\]
(6.40)

\[
\sigma^{4-\alpha}(t_m)\|p_h(t_m) - p_h^m\|_0^2 \leq \kappa\tau^2, \quad t_m \in (0, T].
\]
(6.41)

**Remark.** Combining Theorem 6.5 with (3.14) yields (1.11)–(1.13).

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**References**


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