COVERS OF THE INTEGERS WITH ODD MODULI AND THEIR APPLICATIONS TO THE FORMS $x^m - 2^n$ AND $x^2 - F_{3n}/2$

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Abstract. In this paper we construct a cover \(\{a_s(\text{mod } n_s)\}_{s=1}^k\) of \(\mathbb{Z}\) with odd moduli such that there are distinct primes \(p_1, \ldots, p_k\) dividing \(2^{n_1} - 1, \ldots, 2^{n_k} - 1\) respectively. Using this cover we show that for any positive integer \(m\) divisible by none of 3, 5, 7, 11, 13 there exists an infinite arithmetic progression of positive odd integers the \(m\)th powers of whose terms are never of the form \(2^n \pm p^a\) with \(a, n \in \{0, 1, 2, \ldots\}\) and \(p\) a prime. We also construct another cover of \(\mathbb{Z}\) with odd moduli and use it to prove that \(x^2 - F_{3n}/2\) has at least two distinct prime factors whenever \(n \in \{0, 1, 2, \ldots\}\) and \(x \equiv a \pmod{M}\), where \(\{F_i\}_{i \geq 0}\) is the Fibonacci sequence, and \(a\) and \(M\) are suitable positive integers having 80 decimal digits.

1. Introduction

For \(a \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}\) we let
\[
a(n) = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\}
\]
which is a residue class modulo \(n\). A finite system
\[
A = \{a_s(n_s)\}_{s=1}^k
\]
of residue classes is said to be a cover of \(\mathbb{Z}\) if every integer belongs to some members of \(A\). Obviously (1.1) covers all the integers if it covers 0, 1, \ldots, \(N_A - 1\) where \(N_A = [n_1, \ldots, n_k]\) is the least common multiple of the moduli \(n_1, \ldots, n_k\). The reader is referred to [Gu] for problems and results on covers of \(\mathbb{Z}\) and to [FFKPY] for a recent breakthrough in the field. In this paper we are only interested in applications of covers.

By a known result of Bang [B] (see also Zsigmondy [Z] and Birkhoff and Vandiver [BV]), for each integer \(n > 1\) with \(n \neq 6\), there exists a prime factor of \(2^n - 1\) not dividing \(2^m - 1\) for any \(0 < m < n\); such a prime is called a primitive prime divisor of \(2^n - 1\). P. Erdős, who introduced covers of \(\mathbb{Z}\) in the early 1930s, constructed the following cover (cf. [E])
\[
A_0 = \{0(2), 0(3), 1(4), 3(8), 7(12), 23(24)\}
\]

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whose moduli are distinct, greater than one and different from 6. It is easy to check that $2^2 - 1, 2^3 - 1, 2^4 - 1, 2^8 - 1, 2^{12} - 1, 2^{24} - 1$ have primitive prime divisors $3, 7, 5, 17, 13, 241$ respectively. Using the cover $A_0$ and the Chinese Remainder Theorem, Erdős showed that any integer $x$ satisfying the congruences

\[
\begin{align*}
&x \equiv 2^0 \pmod{3}, \\
&x \equiv 2^0 \pmod{7}, \\
&x \equiv 2^1 \pmod{5}, \\
&x \equiv 2^3 \pmod{17}, \\
&x \equiv 2^7 \pmod{13}, \\
&x \equiv 2^{23} \pmod{241},
\end{align*}
\]

and the additional congruences $x \equiv 1 \pmod{2}$ and $x \equiv 3 \pmod{31}$ cannot be written in the form $2^n + p$ with $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $p$ a prime. The reader may consult [SY] for a refinement of this result. By improving the work of Cohen and Selfridge [CS], Sun [S00] showed that for any integer $n$ having at least two distinct prime factors, and he was able to prove this when $n \equiv \pm 1 \pmod{12}$. The conjecture is particularly difficult when $n$ is a high power of 2. In a recent paper [FFK], Filaseta, Finch and Kozek confirmed the conjecture.

A famous conjecture of Erdős and J. L. Selfridge states that there does not exist a cover of $\mathbb{Z}$ to show that if $n$ is sufficiently large and $n \equiv 1807873 \pmod{3543120}$, then $F_n \neq p^a + q^b$ with $p, q$ prime numbers and $a, b \in \mathbb{N}$, where the Fibonacci sequence $\{F_n\}_{n \geq 0}$ is given by

\[
F_0 = 0, \ F_1 = 1, \text{ and } F_{n+1} = F_n + F_{n-1} \text{ for } n = 1, 2, 3, \ldots.
\]

A recent paper [FFK], Filaseta, Finch and Kozek confirmed the conjecture.

Theorem 1.1. There exists a cover $A_1 = \{a_s(n_s)\}_{s=1}^{173}$ of $\mathbb{Z}$ with all the moduli greater than one and dividing the odd number

\[
3^3 \times 5^2 \times 7 \times 11 \times 13 = 675675,
\]

for which there are distinct primes $p_1, \ldots, p_{173}$ greater than 5 such that each $p_s (1 \leq s \leq 173$) is a primitive prime divisor of $2^{n_s} - 1$.

Theorem 1.1 has the following application.

Theorem 1.2. Let $N$ be any positive integer. Then there is a residue class consisting of odd numbers such that for each nonnegative $x$ in the residue class and each $m \in \{1, \ldots, N\}$ divisible by none of $3, 5, 7, 11, 13$, the number $x^m - 2^m$ with $n \in \mathbb{N}$ always has at least two distinct prime factors.

Remark 1.1. Let $m \in \mathbb{Z}^+$. Chen [C] conjectured that there are infinitely many positive odd numbers $x$ such that $x^m - 2^m$ with $n \in \mathbb{Z}^+$ always has at least two distinct prime factors, and he was able to prove this when $m \equiv 1 \pmod{2}$ or $m \equiv \pm 2 \pmod{12}$. The conjecture is particularly difficult when $m$ is a high power of 2. In a recent paper [FFK], Filaseta, Finch and Kozek confirmed the conjecture.
for $m = 4, 6$ with the help of a deep result of Darmon and Granville [DG] on
generalized Fermat equations; they also showed that there exist infinitely many
integers $x \in \{1, 3^8, 5^8, \ldots\}$ such that $x^m 2^n + 1$ with $n \in \mathbb{Z}^+$ always has at least two
distinct prime divisors.

Recall that $\{F_n\}_{n \geq 0}$ is the Fibonacci sequence. Set $u_n = F_{3n}/2$ for $n \in \mathbb{N}$. Clearly, $u_0 = 0$, $u_1 = 1$, and
\[
\begin{align*}
    u_{n+1} &= \frac{F_{3n+3}}{2} = \frac{F_{3n+1} + (F_{3n+1} + F_{3n})}{2} \\
         &= F_{3n+1} + u_n = F_{3n-1} + 3u_n \\
         &= 4u_n + \frac{2F_{3n-1} - F_{3n}}{2} \\
         &= 4u_n + \frac{F_{3n-1} - F_{3n-2}}{2} \\
         &= 4u_n + u_{n-1}
\end{align*}
\]
for every $n = 1, 2, 3, \ldots$.

Now we give the third theorem which is of a new type and will be proved on the
basis of a certain cover of $\mathbb{Z}$ with odd moduli.

**Theorem 1.3.** Let
\[
a = 31207386885274502188173522132023665167365193670823768234185354856354918873864275
\]
and
\[
M = 36812852443922071184402498913076070503146229820861211558347078871354783744850778.
\]
Then, for any $x \equiv a \pmod{M}$ and $n \in \mathbb{N}$, the number $x^2 - F_{3n}/2$ has at least two
distinct prime divisors.

**Remark 1.2.** (a) Actually our proof of Theorem 1.3 yields the following stronger result: Whenever $y \in a^2(M)$ and $n \in \mathbb{N}$, the number $y - F_{3n}/2$ has at least two
distinct prime divisors.

(b) In view of Theorem 1.3, it is interesting to study the diophantine equation
\[x^2 - F_{3n}/2 = \pm p^a\] with $a, n, x \in \mathbb{N}$ and $p$ a prime, or the equation $F_{3n} = 2x^2 \pm dy^2$ with $d$ equal to 1 or 2 or twice an odd prime. The related equation $F_n = x^2 \pm dy^2$
has been investigated by Ballot and Luca [BL].

The second author has the following conjecture.

**Conjecture 1.1.** Let $m$ be any positive integer. Then there exist $b, d \in \mathbb{Z}^+$ such
that whenever $x \in b^m(d)$ and $n \in \mathbb{N}$ the number $x - F_n$ has at least two distinct prime divisors. Also, there are odd integer $b$ and even number $d \in \mathbb{Z}^+$ such that
whenever $x \in b^m(d)$ and $n \in \mathbb{N}$ the number $x - 2^n$ has at least two distinct prime
divisors.

**Remark 1.3.** (a) We are unable to prove Conjecture 1.1 since it is difficult for us to
construct a suitable cover of $\mathbb{Z}$ for the purpose.

(b) In 2006, Bugeaud, Mignotte and Siksek [BMS] showed that the only powers
in the Fibonacci sequence are
\[
F_0 = 0, \quad F_1 = F_2 = 1, \quad F_6 = 2^3 \quad \text{and} \quad F_{12} = 12^2.
\]
It seems challenging to solve the diophantine equation \( x^m - F_n = \pm p^a \) with \( a, n, x \in \mathbb{N}, m > 1, \) and \( p \) a prime.

We are going to show Theorems 1.1–1.3 in Sections 2–4 respectively.

2. Proving Theorem 1.1 via constructions

Proof of Theorem 1.1. Let \( a_1(n_1), \ldots, a_{173}(n_{173}) \) be the following 173 residue classes respectively.

\[
\begin{align*}
0(3), & \quad 1(5), \quad 0(7), \quad 1(9), \quad 7(11), \quad 8(11), \quad 7(13), \quad 8(15), \quad 19(21), \quad 17(25), \quad 22(25), \\
25(27), & \quad 23(33), \quad 29(35), \quad 30(35), \quad 14(39), \quad 17(39), \quad 4(45), \quad 13(45), \quad 0(55), \\
25(55), & \quad 50(55), \quad 25(63), \quad 52(63), \quad 9(65), \quad 2(75), \quad 32(75), \quad 13(77), \quad 41(91), \\
62(91), & \quad 76(91), \quad 5(99), \quad 65(99), \quad 86(99), \quad 44(105), \quad 59(105), \quad 89(105), \quad 31(117), \\
43(117), & \quad 83(117), \quad 103(117), \quad 35(135), \quad 43(135), \quad 88(135), \quad 26(143), \quad 86(143), \\
125(143), & \quad 35(165), \quad 37(175), \quad 87(175), \quad 162(175), \quad 34(189), \quad 53(189), \quad 155(195), \\
85(225), & \quad 130(225), \quad 157(225), \quad 202(225), \quad 137(231), \quad 158(231), \quad 104(273), \\
146(273), & \quad 188(273), \quad 65(275), \quad 175(275), \quad 152(297), \quad 218(297), \quad 79(315), \\
284(315), & \quad 295(315), \quad 87(325), \quad 112(325), \quad 162(325), \quad 16(351), \quad 44(351), \\
97(351), & \quad 286(351), \quad 313(351), \quad 15(385), \quad 225(385), \quad 290(385), \quad 191(429), \\
203(429), & \quad 284(429), \quad 34(455), \quad 454(455), \quad 130(495), \quad 230(495), \quad 395(495), \\
179(525), & \quad 362(525), \quad 445(525), \quad 494(525), \quad 335(585), \quad 355(585), \quad 412(585), \\
490(585), & \quad 7(675), \quad 232(675), \quad 277(675), \quad 502(675), \quad 200(693), \quad 257(693), \\
515(693), & \quad 445(715), \quad 500(715), \quad 555(715), \quad 356(819), \quad 538(819), \quad 629(819), \\
100(825), & \quad 145(825), \quad 265(825), \quad 475(825), \quad 179(945), \quad 494(945), \quad 562(975), \\
637(975), & \quad 662(975), \quad 862(975), \quad 937(975), \quad 115(1001), \quad 808(1001), \quad 5(1155), \\
809(1155), & \quad 845(1155), \quad 950(1155), \quad 614(1287), \quad 742(1287), \quad 1010(1287), \\
767(1365), & \quad 977(1365), \quad 1235(1365), \quad 350(1485), \quad 220(1575), \quad 662(1575), \\
1012(1575), & \quad 1390(1575), \quad 470(1755), \quad 580(1755), \quad 610(1755), \quad 880(1755), \\
564(1925), & \quad 949(1925), \quad 1089(1925), \quad 1334(1925), \quad 1474(1925), \quad 1859(1925), \\
202(2079), & \quad 895(2079), \quad 911(2079), \quad 1105(2145), \quad 1670(2145), \quad 1012(2275), \\
1362(2275), & \quad 1537(2275), \quad 647(2457), \quad 853(2457), \quad 1210(2457), \quad 1214(2457), \\
2365(2457), & \quad 2384(2457), \quad 670(2457), \quad 2245(2475), \quad 2290(2475), \\
2264(3003), & \quad 1390(3465), \quad 416(3861), \quad 3195(5005), \quad 1600(5775), \\
2920(6435), & \quad 7825(10395), \quad 583939(675675).
\end{align*}
\]

It is easy to check that the least common multiple of \( n_1, \ldots, n_{173} \) is the odd number

\[
3^3 \times 5^2 \times 7 \times 11 \times 13 = 675675.
\]

Since \( A_1 = \{ a_*(n_*) \}_{i=1}^{173} \) covers \( 0, \ldots, 675674, \) it covers all the integers.

Using the software Mathematica and the main tables of [BLSTW] pp. 1–59, below we associate each \( n \in \{ n_1, \ldots, n_{173} \} \) with \( m_n \) distinct primitive prime divisors \( p_{n,1}, \ldots, p_{n,m_n} \) of \( 2^n - 1 \) and write \( n : p_{n,1}, \ldots, p_{n,m_n} \) for this, where \( m_n \) is the
number of occurrences of \(n\) among the moduli \(n_1, \ldots, n_{173}\). For those

\[ n \in \{1485, 3003, 3465, 3861, 5005, 5775, 6435, 10395, 675675\}, \]

as \(m_n = 1\) we just need one primitive prime divisor of \(2^n - 1\) whose existence is guaranteed by Bang’s theorem; but they are too large to be included in the following list.

819: 2681001528674743, 2195163179301930391319-0133458460354695385029575839412973526577742148160962406275456512257;
825: 702948566745151, 9115784422509601, 4108316654247271397904298251217-568560929751, 10124924126024061560521761223037698180014266941;
945: 12433952107854694991430452149922241, 89371283318924988713544642-723090246780404381895167300604125955649427240114665899126768182601;
975: 1951, 8837728285481551, 26159666847897288501, 16637633811923086317158263-5718252801, 4294500770439625509689707482827620567912171434677816776993-9979855730352201;
1001: 6007, 695274469463696085141217909349909207;
1155: 2311, 6250631311, 494224324441, 2600788923312052743240883667728867-90199621606534384599607578416912079166019131929370820827703893645393-545946152505751;
1287: 216217, 71477407, 141968533929529744009;
1365: 469561, 5239016292934591, 22249810017228257182560929751;
1575: 82013401, 32758188751, 7664145826926901, 764384916291005220555242-939647951;
1755: 3511, 196911, 424273477248635891, 854886551940900951;
1925: 11551, 13167001, 1891705201, 5591298184498951, 292615400703113951,
5627063970437389360344951;
2079: 4159, 16633, 80932047967;
2145: 96001053721, 3478786868881;
2275: 218401, 2831920001, 1970116360308855665077130351;
2457: 56511, 1410319, 21287449, 41194063, 1675116877566248927, 178631307-4995391292297656133027144291751;
2475: 4951, 143551, 1086033846151.

Observe that \( p_{n,j} > 5 \) for all \( n \in \{n_1, \ldots, n_{173}\} \) and \( 1 \leq j \leq m_n \). In view of the above, Theorem 1.1 has been proved. \( \square \)

3. Proof of Theorem 1.2

Recall that an odd prime \( p \) is called a Wieferich prime if \( 2^{p-1} \equiv 1 \pmod{p^2} \).
The only known Wieferich primes are 1093 and 3511, and there are no others below \( 1.25 \times 10^{15} \) (cf. \( [R] \) p. 230).

Suppose that \( n \neq 6 \) is an integer greater than one, and \( p \) is a primitive prime divisor of \( 2^n - 1 \). Then \( n \) is the order of 2 mod \( p \) and hence \( p - 1 \) is a multiple of \( n \) by Fermat’s little theorem. Thus \( 2^n - 1 \mid 2^{p-1} - 1 \), and hence \( p^2 \nmid 2^n - 1 \) if \( p \) is not a Wieferich prime.

Let \( A_1 = \{a_s(n_s)\}_{s=1}^{173} \) and \( p_1, \ldots, p_{173} \) be as described in Theorem 1.1. For each \( s = 1, \ldots, 173 \) let \( q_s \) be a primitive prime divisor of \( 2^{p_s^2} - 1 \). Then \( p_1, \ldots, p_{173}, q_1, \ldots, q_{173} \) are distinct odd primes since \( \{p_1^2, \ldots, p_{173}^2\} \cap \{n_1, \ldots, n_{173}\} = \emptyset \).

For each \( s = 1, \ldots, 173 \) let \( \alpha_s \) be the largest positive integer with \( p_s^{\alpha_s} \mid 2^{n_s} - 1 \). Since 3511 is the only Wieferich prime in the set \( \{p_1, \ldots, p_{173}\} \), we have \( \alpha_s = 1 \) if \( p_s \neq 3511 \). In the case \( p_s = 3511 \), we have \( \alpha_s = 2 \) since \( 3511^2 \mid 2^{3510} - 1 \), but \( 3511^3 \mid 2^{3510} - 1 \).

Let \( M = 2^{2L} \prod_{s=1}^{173} p_s^{\alpha_s+2} q_s \), where \( L \) is the smallest positive integer satisfying
\[
2^L - 1 > \max\{16N, p_1^{\alpha_1+1}, \ldots, p_{173}^{\alpha_{173}+1}\}.
\]
By the Chinese Remainder Theorem, there exists a unique \( a \in \{1, \ldots, M\} \) such that
\[
1 + 3 \cdot 2^L \cdot (2^{2L}) \cap \bigcap_{s=1}^{173} (x_s^{b_s} (p_s^{a_s+2}) \cap y_s^{b_s} (q_s)) = a(M).
\]

Let \( m \leq N \) be a positive integer relatively prime to \( 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 15015 \), and write \( m = 2^a m_0 \) with \( a \in \mathbb{N} \), \( m_0 \in \mathbb{Z}^+ \) and \( 2 \not| \ m_0 \). Let \( s \in \{1, \ldots, 173\} \). Since \( n_s \) is a divisor of \( 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = 675675 \), we have \( \gcd(m, n_s) = 1 \) and hence \( m_0 b_s \equiv a_s \pmod{n_s} \) for some \( b_s \in \mathbb{N} \).

As the order of \( 2 \pmod{p_s} \) is the odd number \( n_s \), \( n_s \) divides \( (p_s-1)/\gcd(2^a, p_s-1) \) and hence
\[
2^{(p_s-1)/\gcd(2^a, p_s-1)} \equiv 1 \pmod{p_s}, \quad 2^{p_s (p_s-1)/\gcd(2^a, p_s-1)} \equiv 1 \pmod{p_s}, \quad \ldots
\]

Since there is a primitive root modulo \( p_s^{a_s+2} \) and
\[
2^{\varphi(p_s^{a_s+2})/\gcd(2^a, \varphi(p_s^{a_s+2}))} = 2^{p_s^{a_s+1} (p_s-1)/\gcd(2^a, p_s-1)} \equiv 1 \pmod{p_s^{a_s+2}}
\]
(where \( \varphi \) is Euler’s totient function), by [IR Proposition 4.2.1] there exists \( x_s \in \mathbb{Z} \) with \( x_s^{2^{n_s}} \equiv 2 \pmod{p_s^{a_s+2}} \). Similarly, the order \( p_s^2 \) of \( 2 \pmod{q_s} \) divides \( (q_s-1)/\gcd(2^a, q_s-1) \), therefore \( 2^{(q_s-1)/\gcd(2^a, q_s-1)} \equiv 1 \pmod{q_s} \) and hence \( y_s^{2^{n_s}} \equiv 2 \pmod{q_s} \) for some \( y_s \in \mathbb{Z} \).

Let \( x \geq 0 \) be an element of \( a(M) \). As \( A_1 \) is a cover of \( \mathbb{Z} \), for any \( n \in \mathbb{N} \) there is an \( s \in \{1, \ldots, 173\} \) such that \( n \equiv a_s \pmod{n_s} \). Clearly,
\[
x^m \equiv (x_s^{b_s})^m = (x_s^{2^{n_s}})^{m_0 b_s} \equiv 2^{m_0 b_s} (\pmod{p_s^{a_s+2}}),
\]
thus
\[
x^m \equiv x_s^{2^{n_s} m_0 b_s} \equiv 2^{m_0 b_s} (\pmod{p_s^{a_s+2}})
\]
since \( 2^{m_0} \equiv 1 \pmod{p_s^{a_s+2}} \) and \( m_0 b_s \equiv a_s \pmod{n_s} \).

As \( 16m \leq 16N < 2^L - 1 \) and \( x \equiv 1 + 3 \cdot 2^L \pmod{2^{2L}} \), we have \( |x^m - 2^n| \geq 2^L - 1 > p_s^{a_s+1} \) by [C, Lemma 1]. So \( |x^m - 2^n| \neq 0, p_s^{a_s}, p_s^{a_s+1}, 1 \). If \( x^m - 2^n \) is not divisible by \( p_s^{a_s+2} \), then it must have at least two distinct prime divisors.

Now we assume that \( x^m - 2^n \equiv 0 \pmod{p_s^{a_s+2}} \). Note that \( 2^n \equiv x^m \equiv 2^{m_0 b_s} (\pmod{p_s^{a_s+2}}) \). Since \( n_s \) is the order of \( 2 \pmod{p_s^{a_s}} \) and not the order of \( 2 \pmod{p_s^{a_s+1}} \), by [C, Corollary 3] we have \( 2^n \equiv 2^{m_0 b_s} (\pmod{q_s}) \). Thus
\[
x^m - 2^n \equiv (y_s b_s)^{2^n m_0} - 2^{m_0 b_s} \equiv 0 \pmod{q_s}
\]
and so the nonzero integer \( x^m - 2^n \) has at least two distinct prime divisors (including \( p_s \) and \( q_s \)).

By the above, we have proved the desired result.

Remark 3.1. Given \( m, n \in \mathbb{Z}^+ \) and an odd prime \( p \), the equation \( x^m = 2^n \) has finitely many solutions. As observed by the referee, this is a consequence of the Darmon-Granville theorem in [DG]. In the case \( m = 2 \), all the finitely many solutions are effectively computable by the algorithms given by Weger [W].
4. Proof of Theorem 1.3

Lemma 4.1. Let $c \in \mathbb{Z}^+$, and define $\{U_n\}_{n \geq 0}$ by

$$U_0 = 0, \ U_1 = 1, \text{ and } U_{n+1} = cU_n + U_{n-1} \text{ for } n = 1, 2, 3, \ldots.$$

Suppose that $n > 0$ is an integer with $n \equiv 2 \pmod{4}$ and $p$ is a prime divisor of $U_n$ which divides none of $U_1, \ldots, U_{n-1}$. Then $U_{kn+r} \equiv U_r \pmod{p}$ for all $k \in \mathbb{N}$ and $r \in \{0, \ldots, n-1\}$.

Proof. By [HS, Lemma 2], $U_{n+1} \equiv -(-1)^{n/2} = 1 \pmod{p}$. If $k \in \mathbb{N}$ and $r \in \{0, \ldots, n-1\}$, then $U_{kn+r} \equiv U_{k_r} = U_r \pmod{p}$, where $k_r$ is the residue of $k$ modulo $p$.

Proof of Theorem 1.3. Let $b_1(m_1), \ldots, b_{24}(m_{24})$ be the following 24 residue classes:

$$1(3), \ 2(5), \ 3(5), \ 4(7), \ 5(11), \ 6(7), \ 7(9), \ 8(9), \ 9(15), \ 10(15), \ 11(15), \ 12(21),$$

$$13(21), \ 14(31), \ 15(31), \ 16(35), \ 17(35), \ 18(35), \ 19(45), \ 20(45), \ 21(45), \ 22(45),$$

It is easy to check that $\{b_t(m_t)\}_{t=1}^{24}$ forms a cover of $\mathbb{Z}$ with odd moduli. Set $m_0 = 1$. Then

$$B = \{1(b_1(m_0)), 2b_1(2m_1), \ldots, 2b_{24}(2m_{24})\}$$

is a cover of $\mathbb{Z}$ with all the moduli congruent to 2 mod 4.

Let $u_n = F_{3n}/2$ for $n \in \mathbb{N}$. As we mentioned in Section 1, $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = 4u_n + u_{n-1}$ for $n = 1, 2, 3, \ldots$. For a prime $p$ and an integer $n > 0$, we call $p$ a primitive prime divisor of $u_n$ if $p \mid u_n$, but $p \nmid u_k$ for those $0 < k < n$.

Let $p_0, \ldots, p_{24}$ be the following 25 distinct primes respectively:

$$2, \ 19, \ 31, \ 11, \ 211, \ 29, \ 5779, \ 541, \ 181, \ 31249, \ 1009, \ 767131, \ 21211, \ 911,$$

One can easily verify that each $p_t$ $(0 \leq t \leq 24)$ is a primitive prime divisor of $u_{2m_t}$.

The residue class $a(M)$ in Theorem 1.3 is actually the intersection of the following 25 residue classes with the moduli $p_0, \ldots, p_{24}$ respectively:

$$1(2), \ 2(19), \ 14(31), \ 4(11), \ 94(211), \ 5(29), \ 0(5779), \ 156(541), \ 76(181), \ 10727(31249), \ 501(1009), \ 2(767131), \ 7199(21211), \ 257(911), \ 30(71),$$

It is known that the only solutions of the diophantine equation $F_n = 2x^2$ with $n, x \in \mathbb{N}$ are $(n, x) = (0, 0), (3, 1), (6, 2)$. (Cf. [CG, Theorem 4].) Let $x$ be any integer in the residue class $a(M)$. Then $|x| > 2$ and hence $x^2 \neq u_n = F_{3n}/2$ for all $n \in \mathbb{N}$. With the help of Lemma 4.1 in the case $c = 4$, one can check that $x^2 \equiv u_1 = 1 \pmod{p_0}$ and $x^2 \equiv u_{2t} \pmod{p_t}$ for all $t = 1, \ldots, 24$.

Let $n$ be any nonnegative integer. As $B$ forms a cover of $\mathbb{Z}$, it holds that $n \equiv 1 \pmod{2m_0}$ or $n \equiv 2b_t \pmod{2m_t}$ for some $1 \leq t \leq 24$. By Lemma 4.1 with $c = 4$, if $n \equiv 1 \pmod{2m_0}$, then $u_n \equiv u_1 = 1 \pmod{p_0}$ and hence $x^2 - u_n \equiv x^2 - 1 \equiv 0 \pmod{p_0}$; if $n \equiv 2b_t \pmod{2m_t}$, then $u_n \equiv u_{2b_t} \pmod{p_t}$ and hence $x^2 - u_n \equiv x^2 - u_{2b_t} \equiv 0 \pmod{p_t}$. Thus, it remains to show that for any given $a, b \in \mathbb{N}$ we can deduce a contradiction if $x^2 - u_1 = \pm 2^b$ or $x^2 - u_{2b_t} = \pm p_t^b$ for some $1 \leq t \leq 24$. 

Case 4.0. \( x^2 - u_{1+2a} = \pm 2^b \).
As \( p_2 = 31 \) and \( p_3 = 11 \) are primitive prime divisors of \( u_{2a_2} = u_{2a_3} = u_{10} \), and
\[
u_1 = 1, \quad u_3 = 17, \quad u_5 = 305, \quad u_7 = 5473, \quad u_9 = 98209
\]
have residues \(-14, -5, -14, 1 \) modulo 31 and residues \(-5, -3, -5, 1 \) modulo 11 respectively. If \( 2a + 1 \not\equiv 5 \) (mod 10), then by Lemma 4.1 we have
\[
x^2 - u_{1+2a} \equiv 10 - 1, 10 - (-14) \not\equiv \pm 1, \pm 2, \pm 4, \pm 8, \pm 16 \pmod{31}
\]
which contradicts \( x^2 - u_{1+2a} = \pm 2^b \). (Note that \( 2^5 \equiv 1 \pmod{31} \).) So \( 2a + 1 \equiv 5 \) (mod 10). It follows that
\[
x^2 - u_{1+2a} \equiv 10 - (5) \equiv -2^4 \pmod{31}
\]
and \( x^2 - u_{1+2a} \equiv 5 - (-3) = 2^3 \pmod{11} \).
Thus \( x^2 - u_{1+2a} \) can only be \(-2^b \) with \( b \equiv 4 \) (mod 5), which cannot be congruent to \( 2^3 \) mod 11. (Note that \( 2^5 \equiv -1 \pmod{11} \).) So we have a contradiction.

Case 4.1. \( x^2 - u_{2+6a} = \pm 19^b \).
Observe that
\[
u_0 = 0, \quad u_2 = 4, \quad u_4 = 72, \quad u_6 = 1292, \quad u_8 = 23184
\]
have residues \(-5, 5, -4 \) modulo 11 and \( 0, 4, 10, -10, -4 \) modulo 31 respectively. Also, \( 19^b \equiv 2^{3b} \equiv \pm 1, \pm 2, \pm 4, \pm 3, \pm 5 \pmod{11} \) and \( 19^b \equiv (-2^2 \cdot 3)^b \equiv -3^b \equiv 5 \pmod{31} \).
If \( 2 + 6a \equiv 0 \pmod{10} \), then
\[
x^2 - u_{2+6a} \equiv 5 - 0 \equiv 19^b, -19^b \pmod{11}
\]
and hence \( x^2 - u_{2+6a} = (1)^{d+1}19^b \pm 5d \) for some \( d \in \mathbb{N} \), this leads to a contradiction since \( x^2 - u_{2+6a} \equiv 10 - 0 \pmod{31} \), but
\[
19^b \pm 5d \equiv 8 \times 5^d \equiv 8, 9, 14 \not\equiv -10 \pmod{31}.
\]
Now we handle the case \( 2 + 6a \equiv 2 \pmod{10} \). Since \( 181 \) is a primitive prime divisor of \( u_{30} \), and \( 6a \equiv 0 \pmod{30} \) and \( 19^6 \equiv -1 \pmod{181} \), we have
\[
x^2 - u_{2+6a} = 76^2 - u_2 \equiv -20 \not\equiv \pm 19^b \pmod{181}
\]
which leads a contradiction.
If \( 2 + 6a \equiv 4 \pmod{10} \), then \( x^2 - u_{2+6a} \equiv 10 - 10 = 0 \pmod{31} \). If \( 2 + 6a \equiv 6 \pmod{10} \), then \( x^2 - u_{2+6a} \equiv 5 - 5 = 0 \pmod{11} \). So, when \( 2 + 6a \equiv 4, 6 \pmod{10} \), we get a contradiction since \( x^2 - u_{2+6a} = \pm 19^b \).
If \( 2 + 6a \equiv 8 \pmod{10} \), then \( x^2 - u_{2+6a} = 5 - (-4) \equiv 19^2, -19^7 \pmod{11} \) and hence \( x^2 - u_{2+6a} = (-1)^d19^2 \pm 5d \) for some \( d \in \mathbb{N} \), this leads to a contradiction since \( x^2 - u_{2+6a} \equiv 10 - 4 \equiv -11 \times 10 \pmod{31} \), but
\[
19^2 \pm 5d \equiv -11 \times 5^d \equiv -11, -11 \times 5, -11 \times (-6) \not\equiv \pm 11 \times 10 \pmod{31}.
\]
Case 4.2. \( x^2 - u_{4+10a} = \pm 31^b \).
As \( x^2 - u_{4+10a} \equiv 5 - (-5) = -1 \pmod{11} \) and \( 31^b \equiv (-2)^b \equiv 1, -2, 4, -8, 16 \pmod{11} \), we must have \( x^2 - u_{4+10a} = -31^b \) with \( b \equiv 0 \pmod{5} \).
As \( 31^5 \equiv 2^5 \equiv 8 \pmod{19} \), \( 31^b \equiv 8^{b/5} \equiv \pm 1, \pm 8, \pm 7 \pmod{19} \). If \( 3 \not| a \), then \( 4 + 10a \equiv 0, 2 \pmod{6} \) and hence
\[
x^2 - u_{4+10a} \equiv 4 - u_0, 4 - u_2 \equiv 4, 0 \not\equiv -31^b \pmod{19}.
\]
Thus \( a = 3c \) for some \( c \in \mathbb{N} \). As
\[
-8^{b/5} \equiv -31^b = x^2 - u_{4+10a} \equiv 4 - u_4 = 4 - 72 \equiv 8 \pmod{19},
\]
we have $b/5 - 1 \equiv 3 \pmod{6}$ and hence $b = 20 + 30d$ for some $d \in \mathbb{N}$. As $31^{10} \equiv -1 \pmod{181}$, we have $31^b = 31^{20+30d} \equiv (-1)^{2+3d} = (-1)^d \pmod{181}$. On the other hand, 

$$-31^b = x^2 - u_{4+10a} = x^2 - u_{4+30c} \equiv 76^2 - u_4 \equiv -16 - 72 = -88 \pmod{181}.$$ 

So we get a contradiction.

**Case 4.3.** $x^2 - u_{6+10a} = \pm 11^b$.

As $x^2 - u_{6+10a} \equiv 10 - (-10) \equiv -11 \pmod{31}$, and the order of 11 mod 31 is 30, we have $x^2 - u_{6+10a} = (-1)^{d-1}11^{1+15d}$ for some $d \in \mathbb{N}$. Since $11^{15} \equiv (-8)^{15} = (-2^9)^5 \equiv 1 \pmod{19}$, we have $x^2 - u_{6+10a} \equiv 11 \pmod{19}$.

If $6 + 10a \equiv 0, 2 \pmod{6}$, then 

$$x^2 - u_{6+10a} \equiv 4 - u_0, 4 - u_2 \not\equiv \pm 11 \pmod{19}.$$ 

So $6 + 10a \equiv 4 \pmod{6}$, i.e., $a = 1 + 3c$ for some $c \in \mathbb{N}$. Therefore, 

$$x^2 - u_{6+10a} = x^2 - u_{16+30c} \equiv -16 - u_{16} \equiv -16 - 47 \equiv -11 \times 88 \pmod{181}.$$ 

Note that 

$$(-11)^{15d} \equiv (-49)^d \equiv 1, -49, 48 \not\equiv 88 \pmod{181}.$$ 

As $x^2 - u_{6+10a} = (-11)^{1+15d}$, we get a contradiction.

**Case 4.4.** $x^2 - u_{8+14a} = \pm 211^b$.

As $p_5 = 29$ is a primitive divisor of $u_{2m_5} = u_{14}$, we have $x^2 - u_{8+14a} \equiv 25 - u_8 \equiv 25 - 13 = 12 \pmod{29}$.

Since 2 is a primitive root mod 29, also $211 \equiv 2^3 \pmod{29}$, $2^{3 \times 21} \equiv 2^{7} \equiv 12 \pmod{29}$, and $2^{3x7} \equiv 2^{12} \equiv -12 \pmod{29}$, we have $x^2 - u_{8+14a} = (-1)^{d-1}1211^7+14d$ for some $d \in \mathbb{N}$.

Observe that 

$$x^2 - u_{8+14a} \equiv 10 - u_0, 10 - u_2, 10 - u_4, 10 - u_6, 10 - u_8,$$ 

$$\equiv 10 - 0, 10 - 4, 10 - 10, 10 - (-10), 10 - (-4) \pmod{31}.$$ 

Clearly, $211 \equiv 5^2 \pmod{31}$ and $5^3 \equiv 1 \pmod{31}$, thus 

$$211^7+14d \equiv 5^2 + 28d \equiv 5^{2+d} \equiv -6, 1, 5 \pmod{31}.$$ 

Therefore, $2 \mid d, 3 \mid d$ and $8 + 14a \equiv 2 \pmod{10}$. It follows that $a = 1 + 5c$ for some $c \in \mathbb{N}$ and $d = 6e$ for some $e \in \mathbb{N}$.

Note that 

$$x^2 - u_{8+14(1+5c)} \equiv x^2 - u_2 \equiv 5 - 4 = 1 \pmod{11}$$ 

and 

$$(-1)^{d-1}211^7+14d \equiv -2^7(1+2e) \equiv -2^7(1+2e) \pmod{11}.$$ 

So $2^{7(1+2e)} \equiv -1 \equiv 2^7 \pmod{11}$, hence $7(1 + 2e) \equiv 5 \equiv 35 \pmod{10}$ and thus $e \equiv 2 \pmod{5}$. Therefore, $7 + 14d \equiv 7 + 84 \times 2 \equiv 35 \pmod{140}$ and hence 

$$211^7+14d \equiv (-2)^{35} \equiv \left(\frac{-2}{71}\right) = -\left(\frac{2}{71}\right) = -1 \pmod{71}$$ 

by the theory of quadratic residues, but 

$$x^2 - u_{8+14a} = x^2 - u_{22+70c} \equiv 30^2 - u_{22} = 900 - 13888945017644 \equiv 14 \pmod{71},$$ 

so we get a contradiction from the equality $x^2 - u_{8+14a} = -211^7+14d$. 


Case 4.5. $x^2 - u_{12+14a} = \pm 29^b$.
As $29^b \equiv (-2)^b \equiv \pm 1, \pm 2, \pm 4, \pm 8, \pm 16 \pmod{31}$, $x^2 \equiv 14^2 \equiv 10 \pmod{31}$ and
$u_{12+14a} \equiv u_0, u_2, u_4, u_6, u_8 \equiv 0, 4, 10, -10, -4 \pmod{31}$,
we have $x^2 - u_{12+14a} \neq \pm 29^b \pmod{31}$. So, a contradiction occurs.

Case 4.6. $x^2 - u_{0+18a} = \pm 5779^b$.
As $x^2 - u_{18a} \equiv 2^2 - u_0 = 4 \pmod{19}$, $5779 \equiv 3 \pmod{19}$ and the order of 3
mod 19 equals 18, we have $x^2 - u_{18a} = (-1)^d - 5779^{3+9d} = (-5779)^{5+9d}$ for some
$d \in \mathbb{N}$.

Note that

\[( -5779)^{3d} \equiv (-13)^{3d} \equiv (2^3)^{3d} \equiv 2^{9d} \equiv 1, 2, 4, 8, 16 \pmod{31}\]

and $5779^5 \equiv 13^5 \equiv 6 \pmod{31}$. Thus

\[x^2 - (-5779)^{5+9d} \equiv 10 + 6 \times 2^d \equiv -15, -9, 3, -4, 13 \pmod{31}.
\]
while $u_{18a} \equiv u_0, u_2, u_4, u_6, u_8 \equiv 0, 4, 10, -10, -4 \pmod{31}$. As $u_{18a} = x^2 - (-5779)^{5+9d}$, we must have $18a \equiv 8 \pmod{10}$ and $d = 3 + 5e$ for some $e \in \mathbb{N}$.

Observe that $x^2 - u_{18a} = 5 - u_8 \equiv -2 \pmod{11}$, but

\[(-5779)^{5+9d} \equiv (-2^5)^{3+9(3+5e)} = (-1)^e 2^{64+90e} \equiv (-1)^e 2^{16} \equiv -2 \pmod{11}.
\]

So a contradiction occurs.

Case 4.7. $x^2 - u_{10+30a} = \pm 541^b$.
As $x^2 - u_{10+30a} = 5 - u_0 \equiv 5 \pmod{11}$ and

$541^b \equiv 2^b \equiv \pm 1, \pm 2, \pm 3, \pm 4, \pm 8, \pm 16 \pmod{11}$,
x^2 - u_{10+30a} = (-1)^d 541^{4+5d}$ for some $d \in \mathbb{N}$, and hence we have a contradiction
since $x^2 - u_{10+30a} \equiv 10 - u_0 = 10 \pmod{31}$, but

$541^{4+5d} \equiv (2 \times 7)^{4+5d} \equiv 7 \times 5^d \equiv 7, 7 \times 5, 7 \times (-6) \equiv \pm 10 \pmod{31}$.

Case 4.8. $x^2 - u_{22+30a} = \pm 181^b$.
As $x^2 - u_{22+30a} = 5 - u_2 = 5 - 4 \pmod{11}$ and $181^b \equiv 5^b \equiv 1, 5, 3, 4, -2 \pmod{11}$, we have $x^2 - u_{22+30a} = 181^b$ with $b \equiv 5d \pmod{11}$.

Since $x^2 - u_{22+30a} \equiv 5 - u_2 = \equiv 10 - 4 \equiv 6 \pmod{31}$ and $181^{5d} \equiv (-5)^{5d} \equiv 6^d \equiv 1, 6, 5, -1, -6, -5 \pmod{31}$, $d = 1 + 6e$ for some $e \in \mathbb{N}$. Note that $x^2 - u_{22+30a} = 4 - u_4 = 4 = 72 \equiv 8 \pmod{19}$, but

$181^{5d} \equiv (-9)^{5d} \equiv (-3^{10})^d \equiv 3^d \equiv 3 \times 7^e \equiv 3, 2, -5 \not\equiv 8 \pmod{19}$.

Case 4.9. $x^2 - u_{18+42a} = \pm 31249^b$.
Note that $31249^b \equiv 1^b \equiv 1 \pmod{31}$, $x^2 \equiv 10 \pmod{31}$ and also

$u_{18+42a} \equiv u_0, u_2, u_4, u_6, u_8 \equiv 0, 4, 10, -10, -4 \pmod{31}$.

Therefore, $x^2 - u_{18+42a} \neq \pm 31249^b \pmod{31}$.

Case 4.10. $x^2 - u_{24+42a} = \pm 1009^b$.
As $x^2 - u_{24+42a} \equiv 4 - u_0 = 4 \pmod{19}$, $1009 \equiv 2 \pmod{19}$ and 2 is a primitive
root modulo 19, we have $x^2 - u_{24+42a} = (-1)^d 1009^{2+9d} = (-1009)^{2+9d}$ for some
$d \in \mathbb{N}$. Observe that $u_{10} = 416020$ and $x^2 - u_{24+42a} = 25 - u_{10} = 10 \pmod{29}$, but
$6^7 \equiv -1 \pmod{29}$ and hence

$(-1009)^{2+9d} \equiv 6^2-9d \equiv \pm 1, \pm 6, \pm 7, \pm 13, \pm 9, \pm 4, \pm 5 \not\equiv 10 \pmod{29}$.

So we get a contradiction.
Case 4.11. $x^2 - u_{2+70a} = \pm 767131^b$.
Observe that $x^2 - u_{2+70a} = 5^2 - u_2 \equiv -8 \pmod{29}$ and $767131^b \equiv (-6)^b \equiv 1, -6, 7, -13, -9, -4, -5 \pmod{29}$.

So a contradiction occurs.

Case 4.12. $x^2 - u_{28+70a} = \pm 21211b$.
As $28 + 70a \equiv 0 \pmod{14}$, we have $x^2 - u_{28+70a} = 5^2 - u_0 \equiv -4 \pmod{29}$. On the other hand,

$$21211^b \equiv \pm 12b \equiv \pm 1, \pm 12 \pmod{29}.$$ 

Thus, we have a contradiction.

Case 4.13. $x^2 - u_{48+70a} = \pm 911^b$.
Note that $x^2 - u_{48+70a} = 5^2 - u_6 = 25 - 1292 \equiv 9 \pmod{29}$, but $911^b \equiv 12b \equiv \pm 1, \pm 12 \pmod{29}$.

Case 4.14. $x^2 - u_{58+70a} = \pm 71^b$.
Observe that $x^2 - u_{58+70a} = 5^2 - u_2 \equiv -8 \pmod{29}$, but $71^b \equiv 13b \equiv \pm 1, \pm 13, \pm 5, \pm 7, \pm 4, \pm 6, \pm 9 \pmod{29}$.

Case 4.15. $x^2 - u_{12+90a} = \pm 119611^b$.
Since $x^2 - u_{12+90a} \equiv 4 - u_0 \equiv 4 \pmod{19}$ and

$$119611^b \equiv 6b \equiv 1, 6, -2, 7, 4, 5, -8, 9, -3 \pmod{29},$$

we must have $x^2 - u_{12+90a} = 119611^b$ with $b = 4 + 9d$ for some $d \in \mathbb{N}$. Note that $x^2 - u_{12+90a} \equiv 10 - u_2 = 6 \equiv 10 \times 13 \pmod{31}$, but

$$119611^{4+9d} \equiv 13^{4+9d} \equiv 10(-2)^d \pmod{31}$$

with $(-2)^d \equiv \pm 1, \pm 2, \pm 4, \pm 8, \pm 16 \not\equiv 13 \pmod{31}$. So we have a contradiction.

Case 4.16. $x^2 - u_{30+90a} = \pm 42391^b$.
As $x^2 - u_{30+90a} \equiv 4 - u_0 \equiv 4 \pmod{19}$, $42391 \equiv 2 \pmod{19}$ and $2$ is a primitive root mod 19, we have $x^2 - u_{30+90a} = (-1)^d 42391^{2+9d}$ for some $d \in \mathbb{N}$.

Note that $x^2 - u_{30+90a} \equiv 10 - 0 \pmod{31}$ and

$$(-42391)^{2+9d} \equiv (-14)^{2+9d} \equiv 10(-2)^{3d} \equiv 10(-1)^d 2^{2d} \pmod{31}.$$ 

Since the only residues of powers of 2 modulo 31 are 1, 2, 4, 8, 16, we must have $x^2 - u_{30+90a} = (-42391)^{2+9d}$ with $d$ divisible by both 5 and 2. Write $d = 10e$ with $e \in \mathbb{N}$. Then

$$x^2 - u_{30+90a} = 42391^{2+90e} \equiv (-3)^{2+90e} \equiv 9 \pmod{11},$$

which contradicts the fact $x^2 - u_{30+90a} \equiv 5 - u_0 = 5 \pmod{11}$.

Case 4.17. $x^2 - u_{58+90a} = \pm 271^b$.
Note that $x^2 - u_{58+90a} \equiv 10 - u_8 \equiv 14 \pmod{31}$ while

$$271^b \equiv (-2)^{3b} \equiv \pm 1, \pm 2, \pm 4, \pm 8, \pm 16 \pmod{31}.$$ 

Case 4.18. $x^2 - u_{60+90a} = \pm 811^b$.
As $x^2 - u_{60+90a} \equiv 10 - u_0 \equiv 10 \pmod{31}$ and $811^b \equiv 5^b \equiv 1, 5, 25 \pmod{31}$, we have a contradiction.

Case 4.19. $x^2 - u_{10+120a} = \pm 379^b$.
Note that $x^2 - u_{10+120a} \equiv 2^2 - u_4 = 4 - 72 \equiv 8 \pmod{19}$ but $379^b \equiv (-1)^b \equiv \pm 1 \pmod{19}$. 


Case 4.20. \( x^2 - u_{46+126a} = \mp 912871^b \).

Since \( x^2 - u_{46+126a} \equiv 2^3 - u_4 \equiv 3^3 \) (mod 19), 912871 \( \equiv 2^{4b} \) (mod 19) and the order of 2 mod 19 is 18, we must have \( x^2 - u_{46+126a} = -912871^b \) with \( b = 3 + 9d \) for some \( d \in \mathbb{N} \). Note that \( x^2 - u_{46+126a} \equiv 5^2 - u_4 = 25 - 72 \equiv 11 \) (mod 29), but
\[ 912871^{3+9d} \equiv 3^{2(3+9d)} \equiv 4^{1+3d} \equiv \pm 1, \pm 4, \pm 13, \pm 6, \pm 5, \pm 9, \pm 7 \) (mod 29).

So we have a contradiction.

Case 4.21. \( x^2 - u_{88+126a} = \mp 85429^b \).

Observe that \( x^2 - u_{88+126a} \equiv 5^2 - u_4 \equiv 11 \) (mod 29), but
\[ 85429^b \equiv (-5)^b \equiv 1, -5, -4, -9, -13, 7, -6 \) (mod 29).

So a contradiction occurs.

Case 4.22. \( x^2 - u_{132+210a} = \mp 631^b \).

Note that \( x^2 - u_{132+210a} \equiv 4^2 - u_2 \equiv 1 \) (mod 11) and 631 \( \equiv 2^2 \) (mod 11). Since \( 2^5 \equiv -1 \) (mod 11) and \( 2^{10} \equiv 1 \) (mod 11), we must have \( x^2 - u_{132+210a} = 631^b \) with \( b = 5d \) for some \( d \in \mathbb{N} \). As \( x^2 - u_{132+210a} \equiv 10 - u_2 = 6 \) (mod 31), 631 \( \equiv (-2^2 \times 5)^b \equiv -5^2 \equiv 6 \) (mod 31) and the order of 6 mod 31 is 6, we can write \( d = 1 + 6e \) with \( e \in \mathbb{N} \). Thus
\[ x^2 - u_{132+210a} = 631^{5+30e} \equiv (2^5)^{5+30e} \equiv (-2)^{1+6e} \equiv -2, 5, -3 \) (mod 19).

On the other hand, \( x^2 - u_{132+210a} \equiv 4 - u_0 = 4 \) (mod 19). This leads to a contradiction.

Case 4.23. \( x^2 - u_{42+630a} = \mp 69931^b \).

As \( 42 + 630a \equiv 0 \) (mod 6), we have \( x^2 - u_{42+630a} \equiv 2^2 - u_0 = 4 \) (mod 19). On the other hand, 69931 \( \equiv (-2)^{36} \equiv 1, -8, 7 \) (mod 19). So we get a contradiction.

Case 4.24. \( x^2 - u_{178+630a} = \pm 17011^b \).

Since \( 178 + 630a \equiv 10 \) (mod 14), we have
\[ x^2 - u_{178+630a} \equiv 5^2 - u_{10} = 25 - 416020 \equiv 10 \) (mod 29).

Note that \( 17011^b \equiv (-12)^b \equiv \pm 1, \pm 12 \) (mod 29). So a contradiction occurs.

In view of the above, we have completed the proof of Theorem 1.3.

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