Abstract. We determine the isogeny classes of abelian surfaces over $\mathbb{F}_q$ whose group of $\mathbb{F}_q$-rational points has order divisible by $q^2$. We also solve the same problem for Jacobians of genus-2 curves.

In a recent paper [4], Ravnshøj proved: if $C$ is a genus-2 curve over a prime field $\mathbb{F}_p$, and if one assumes that the endomorphism ring of the Jacobian $J$ of $C$ is the ring of integers in a primitive quartic CM-field, and that the Frobenius endomorphism of $J$ has a certain special form, then $p^2 \not| \#J(\mathbb{F}_p)$. Our purpose here is to deduce this conclusion under less restrictive hypotheses. We write $q = p^m$, where $p$ is prime, and for any abelian variety $J$ over $\mathbb{F}_q$ we let $P_J$ denote the Weil polynomial of $J$, namely the characteristic polynomial of the Frobenius endomorphism $\pi_J$ of $J$. As shown by Tate [6, Thm. 1], two abelian varieties over $\mathbb{F}_q$ are isogenous if and only if their Weil polynomials are identical. Thus, the following result describes the isogeny classes of abelian surfaces $J$ over $\mathbb{F}_q$ for which $q^2 \mid \#J(\mathbb{F}_q)$.

**Theorem 1.** The Weil polynomials of abelian surfaces $J$ over $\mathbb{F}_q$ satisfying $q^2 \mid \#J(\mathbb{F}_q)$ are as follows:

(1.1) $X^4 + X^3 - (q + 2)X^2 + qX + q^2$ (if $q$ is odd and $q > 8$);

(1.2) $X^4 - X^3 + q^2$;

(1.3) $X^4 - X^3 + qX^2 - qX + q^2$ (if $m$ is odd or $p \not\equiv 1 \mod 4$);

(1.4) $X^4 - 2X^3 + (2q + 1)X^2 - 2qX + q^2$;

(1.5) $X^4 + aX^3 + bX^2 + aqX + q^2$, where $(a, b)$ occurs in the same row as $q$ in the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>(9, 42)</td>
</tr>
<tr>
<td>9</td>
<td>(6, 20)</td>
</tr>
<tr>
<td>7</td>
<td>(4, 16)</td>
</tr>
<tr>
<td>5</td>
<td>(3, 6) or (8, 26)</td>
</tr>
<tr>
<td>4</td>
<td>(2, 5), (4, 11), or (6, 17)</td>
</tr>
<tr>
<td>3</td>
<td>(1, 4), (3, 5), or (4, 10)</td>
</tr>
<tr>
<td>2</td>
<td>(0, 3), (1, 0), (1, 4), (2, 5), or (3, 6)</td>
</tr>
</tbody>
</table>

The special form required of the Frobenius endomorphism in [4] has an immediate consequence for the shape of its characteristic polynomial, and by inspection the above polynomials do not have the required shape. Thus the main result of [4] follows from the above result.
Our proof of Theorem 1 relies on the classical results of Tate ([6] Thm. 1] and [8 Thm. 8]) and Honda [2] describing the Weil polynomials of abelian varieties over finite fields. An explicit version of their results in the case of simple abelian surfaces was given by Rück [5, Thm. 1.1]; together with the analogous results of Waterhouse [7, Thm. 4.1] for elliptic curves, this yields the following:

**Lemma 2.** The Weil polynomials of abelian surfaces over $\mathbb{F}_q$ are precisely the polynomials $X^4 + aX^3 + bX^2 + aqX + q^2$, where $a, b \in \mathbb{Z}$ satisfy $|a| \leq 4\sqrt{q}$ and $2|a|\sqrt{q} - 2q \leq b \leq \frac{a^2}{4} + 2q$, and where $a, b, \text{and the values } \Delta := a^2 - 4(b - 2q) \text{ and } \delta := (b + 2q)^2 - 4qa^2 \text{ satisfy one of the conditions (2.1)-(2.4) below:}

(2.1) $v_p(b) = 0$;

(2.2) $v_p(b) \geq m/2$ and $v_p(a) = 0$, and either $\delta = 0$ or $\delta$ is a non-square in the ring $\mathbb{Z}_p$ of $p$-adic integers;

(2.3) $v_p(b) \geq m$ and $v_p(a) \geq m/2$ and $\Delta$ is a square in $\mathbb{Z}$, and if $q$ is a square and we write $a = \sqrt{qa}$ and $b = qb'$ then

\[
p \not\equiv 1 \mod 4 \quad \text{if } b' = 2,
\]
\[
p \not\equiv 1 \mod 3 \quad \text{if } a' \not\equiv b' \mod 2;
\]

(2.4) the conditions in one of the rows of the following table are satisfied:

<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>Conditions on $p$ and $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>$q$ is a square and $p \not\equiv 1 \mod 8$, or $q$ is a non-square and $p \not\equiv 2$</td>
</tr>
<tr>
<td>$(0, -q)$</td>
<td>$q$ is a square and $p \not\equiv 1 \mod 12$, or $q$ is a non-square and $p \not\equiv 3$</td>
</tr>
<tr>
<td>$(0, q)$</td>
<td>$q$ is a non-square</td>
</tr>
<tr>
<td>$(0, -2q)$</td>
<td>$q$ is a non-square</td>
</tr>
<tr>
<td>$(0, 2q)$</td>
<td>$q$ is a square and $p \equiv 1 \mod 4$</td>
</tr>
<tr>
<td>$(\pm \sqrt{q}, q)$</td>
<td>$q$ is a square and $p \not\equiv 1 \mod 5$</td>
</tr>
<tr>
<td>$(\pm 2\sqrt{q}, q)$</td>
<td>$q$ is a non-square and $p = 2$</td>
</tr>
<tr>
<td>$(\pm 2\sqrt{q}, 3q)$</td>
<td>$q$ is a square and $p \equiv 1 \mod 3$</td>
</tr>
<tr>
<td>$(\pm 5\sqrt{q}, 3q)$</td>
<td>$q$ is a non-square and $p = 5$</td>
</tr>
</tbody>
</table>

Moreover, the surface $J$ is simple if and only if either

- $\Delta$ is a non-square in $\mathbb{Z}$; or
- $(a, b) = (0, 2q)$ and $q$ is a square and $p \equiv 1 \mod 4$; or
- $(a, b) = (\pm 2\sqrt{q}, 3q)$ and $q$ is a square and $p \equiv 1 \mod 3$.

The $p$-rank of $J$ (namely, the rank of the $p$-torsion subgroup of $J(\mathbb{F}_q)$) is 2 in (2.1), 1 in (2.2), and 0 in (2.3) and (2.4).

**Proof of Theorem 1.** As shown by Weil [9], for any abelian surface $J$ over $\mathbb{F}_q$, the Weil polynomial $P_J$ is a monic quartic in $\mathbb{Z}[X]$ whose complex roots have absolute value $\sqrt{q}$. In particular, $\#J(\mathbb{F}_q) = \deg(\pi_J - 1) = P_J(1) \leq (\sqrt{q} + 1)^4$, so if $\#J(\mathbb{F}_q) = cq^2$ with $c \in \mathbb{Z}$ then $c \leq (1 + q^{-1/2})^4$. It follows that $c = 1$ unless $q \leq 27$. In light of the above lemma, there are just finitely many cases to consider with $c > 1$; we treated these cases using the computer program presented at the end of this paper, which gave rise to precisely the solutions in (1.5). Henceforth assume $c = 1$. 

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The Weil polynomials of abelian surfaces over \( \mathbb{F}_q \) are the polynomials \( P(X) := X^4 + aX^3 + bX^2 + aqX + q^2 \) occurring in the above lemma. We must determine which of these polynomials satisfy \( P(1) = q^2 \), or equivalently, \( b = -1 - a(q + 1) \). The inequality \( -1 - a(q + 1) = b \leq a^2/4 + 2q \) says that \( q^2 \leq (a/2 + q + 1)^2 \), and since \( a/2 + q + 1 \geq -2\sqrt{q} + q + 1 > 0 \), this is equivalent to \( q \leq a/2 + q + 1 \), or in other words \( -2 \leq a \). The inequality \( 2|a|\sqrt{q} - 2q \leq b = -1 - a(q + 1) \) always holds if \( a \in \{0, -1, -2\} \), and if \( a \geq 1 \) it is equivalent to \( a(\sqrt{q} + 1)^2 \leq 2q - 1 \); since \( 2q - 1 < 2q < 2(\sqrt{q} + 1)^2 \), this implies \( a = 1 \), in which case \( (\sqrt{q} + 1)^2 \leq 2q - 1 \) is equivalent to \( q \geq 8 \).

Condition (2.1) holds if and only if \( a \neq -1 \mod p \), or equivalently either \( a \in \{0, -2\} \) or both \( a = 1 \) and \( p \neq 2 \). This accounts for (1.1), (1.2), and (1.4).

Condition (2.3) cannot hold, since \( p \mid a \) implies \( b \equiv -1 \mod p \).

The condition \( v_p(b) \geq m/2 \) says that \( a \equiv -1 \mod p^{[m/2]} \), or equivalently \( a = -1 \). In this case, \( b = q \) and \( \delta = 9q^2 - 4q \), so \( \delta \neq 0 \). If \( q \) is odd, then \( \delta \) is a square in \( \mathbb{Z}_p \), and if only if \( \delta \) is a square modulo \( pq \), or equivalently, \( m \) is even and \(-4\) is a square modulo \( p \), which means that \( p \equiv 1 \mod 4 \). If \( q \) is even, then \( \delta \) is not a square in \( \mathbb{Z}_2 \), since for \( q \leq 8 \) we have \( \delta \in \{28, 128, 544\} \), and for \( q > 8 \) we have \( \delta \equiv -4q \mod 16q \). Thus (2.2) gives rise to (1.3).

Finally, if \( a = -2 \) then \( b = 2q + 1 \), and if \( a = 0 \) then \( b = -1 \), so in either case \( q \nmid b \). Thus (2.4) cannot hold, and the proof is complete.

Next we determine which of the Weil polynomials in (1.1)–(1.5) occur for Jacobians. We use the classification of Weil polynomials of Jacobians of genus-2 curves. This classification was achieved by the combined efforts of many mathematicians, culminating in the following result [3, Thm. 1.2]:

**Lemma 3.** Let \( P_J = X^4 + aX^3 + bX^2 + aqX + q^2 \) be the Weil polynomial of an abelian surface \( J \) over \( \mathbb{F}_q \).

(1) **If** \( J \) **is simple**, **then** \( J \) **is not isogenous to a Jacobian if and only if the conditions in one of the rows of the following table are met:**

<table>
<thead>
<tr>
<th>Condition on ( p ) and ( q )</th>
<th>Conditions on ( a ) and ( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^2 - b = q ) and ( b &lt; 0 ) and all prime divisors of ( b ) are ( 1 \mod 3 )</td>
<td>( a = 0 ) and ( b = 1 - 2q )</td>
</tr>
<tr>
<td>( p &gt; 2 )</td>
<td>( a = 0 ) and ( b = 2 - 2q )</td>
</tr>
<tr>
<td>( p \equiv 11 \mod 12 ) and ( q ) square</td>
<td>( a = 0 ) and ( b = -q )</td>
</tr>
<tr>
<td>( p = 3 ) and ( q ) square</td>
<td>( a = 0 ) and ( b = -q )</td>
</tr>
<tr>
<td>( p = 2 ) and ( q ) non-square</td>
<td>( a = 0 ) and ( b = -q )</td>
</tr>
<tr>
<td>( q = 2 ) or ( q = 3 )</td>
<td>( a = 0 ) and ( b = -2q )</td>
</tr>
</tbody>
</table>

(2) **If** \( J \) **is not simple**, **then** there are integers \( s, t \) **such that** \( P_J = (X^2 - sX + q)(X^2 - tX + q) \), **and** \( s \) **and** \( t \) **are unique if we require that** \( |s| \geq |t| \) **and that if** \( s = -t \) **then** \( s \geq 0 \). **For such** \( s \) **and** \( t \), \( J \) **is not isogenous to a Jacobian if and only if the conditions in one of the rows of the following table are met:**


<table>
<thead>
<tr>
<th>p-rank of J</th>
<th>Condition on p and q</th>
<th>Conditions on s and t</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>s = 1 and t = −1</td>
</tr>
<tr>
<td>2</td>
<td>s = t and t^2 − 4q ∈ {−3, −4, −7}</td>
<td>q = 2</td>
</tr>
</tbody>
</table>

Theorem 4. The polynomials in (1.1)–(1.5) which are not Weil polynomials of Jacobians are precisely the polynomials \( X^4 + aX^3 + bX^2 + aqX + q^2 \), where q and \( (a, b) \) satisfy the conditions in one of the rows of the following table:

<table>
<thead>
<tr>
<th>q</th>
<th>(a, b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(8, 26)</td>
</tr>
<tr>
<td>4</td>
<td>(6, 17)</td>
</tr>
<tr>
<td>2</td>
<td>(−2, 5), (0, 3), (1, 4), (2, 5), or (3, 6)</td>
</tr>
</tbody>
</table>

Proof. Let \( J \) be an abelian surface over \( \mathbb{F}_q \) whose Weil polynomial \( P_J = X^4 + aX^3 + bX^2 + aqX + q^2 \) satisfies one of (1.1)–(1.5). In each case, \( a^2 − b \neq q \), and if \( a = 0 \) then \( b \in \{-1, 3\} \), so if \( J \) is simple then Lemma 3 implies \( J \) is isogenous to a Jacobian.

Henceforth assume \( J \) is not simple, so \( P_J = (X^2 − sX + q)(X^2 − tX + q) \) where \( s, t \in \mathbb{Z}_q \); we may assume that \( |s| ≥ |t| \), and that \( s ≥ 0 \) if \( s = −t \). Note that \( a = −s − t \) and \( b = 2q + st \), so \( (X − s)(X − t) = X^2 + aX + b − 2q \). In particular, \( Δ := a^2 − 4(b − 2q) \) is a square, say \( Δ = z^2 \) with \( z ≥ 0 \).

Suppose \( P_J \) satisfies (1.1), so \( Δ = 12q + 9 \). Then \( (z − 3)(z + 3) = 12q \) is even, so \( z − 3 \) and \( z + 3 \) are even and incongruent mod 4, whence their product is divisible by 8, so \( q \) is even, a contradiction.

Now suppose \( P_J \) satisfies (1.2), so \( Δ = 8q + 4 \). Then \( (z − 2)(z + 2) = 8q \), so at least one of \( z − 2 \) and \( z + 2 \) is divisible by 4; but these numbers differ by 4, so they are both divisible by 4, whence their product is divisible by 16, so \( q \) is even. Thus \( 8q \) is a power of 2 which is the product of two positive integers that differ by 4, so \( q = 4 \). In this case, \( (q, a, b, s, t) = (4, 0, −1, 3, −3) \), which indeed satisfies (1.2).

Moreover, (2.1) holds, so Lemma 3 implies \( J \) has p-rank 2. Since \( |s − t| = 6 \notin \{0, 1\} \) and \( q \neq 2 \), Lemma 3 implies \( J \) is isogenous to a Jacobian.

Now suppose \( P_J \) satisfies (1.3), so \( Δ = 4q + 1 \). Then \( (z − 1)(z + 1) = 4q \), so \( z − 1 \) and \( z + 1 \) are even and incongruent mod 4, whence their product is divisible by 8, so \( q \) is even. Thus \( 4q \) is a power of 2 which is the product of two positive integers that differ by 2, so \( q = 2 \). In this case, \( (q, a, b, s, t) = (2, −1, 2, 2, −1) \), which indeed satisfies (1.3). Moreover, (2.2) holds, so Lemma 3 implies \( J \) has p-rank 1. Since \( |s − t| = 3 \notin \{0, 1\} \) and \( q \) is a non-square, Lemma 3 implies \( J \) is isogenous to a Jacobian.

Now suppose \( P_J \) satisfies (1.4), so \( Δ = 0 \) and \( q \notin \{0, ±2\sqrt{q}\} \), and thus Lemma 3 implies \( J \) is non-simple. Here \( (a, b, s, t) = (−2, 2q + 1, 1, 1) \), so Lemma 3 implies \( J \) has p-rank 2. Since \( s = t = 1 \), Lemma 3 implies \( J \) is isogenous to a Jacobian if
and only if \(1 - 4q \not\in \{-3, -4, -7\}\), or equivalently \(q \neq 2\). This gives rise to the first entry in the last line of the table.

Finally, if \(P_j\) satisfies (1.5), then the result follows from Lemma \(\text{Lemma 3}\) and Lemma \(\text{Lemma 2}\) via a straightforward computation. \(\square\)

**Remark 5.** The result announced in the abstract of [4] is false, since its hypotheses are satisfied by every two-dimensional Jacobian over \(F_p\). This is because the abstract of [4] does not mention the various hypotheses assumed in the theorems of that paper.

We used the following Magma [1] program in the proof of Theorem [1].

```
for q in [2..27] do if IsPrimePower(q) then
    Q:=Floor(4*Sqrt(q)); M:=Floor((Sqrt(q)+1)^4/q^2);
    for c in [2..M] do
        for a in [-Q..Q] do b:=-1-a*(q+1)+(c-1)*q^2;
        if b le (a^2/4)+2*q and 2*Abs(a)*Sqrt(q)-2*q le b then
            p:=Factorization(q)[1,1]; m:=Factorization(q)[1,2];
            Delta:=a^2-4*(b-2*q); delta:=(b+2*q)^2-4*q*a^2;
            if GCD(b,p) eq 1 then <q,a,b,c>;
            elif GCD(b,q) ge Sqrt(q) and GCD(a,p) eq 1 and
                (delta eq 0 or not IsSquare(pAdicRing(p)!delta)) then
                    <q,a,b,c>;
                elif IsDivisibleBy(b,q) and GCD(a,q) ge Sqrt(q) and
                    IsSquare(Delta) then
                        if not IsSquare(q) then <q,a,b,c>;
                        else sq:=p^((m div 2)); ap:=a div sq; bp:=b div q;
                            if not ((bp eq 2 and IsDivisibleBy(p-1,4)) or
                                (IsDivisibleBy(ap-bp,2) and IsDivisibleBy(p-1,3)))
                                then <q,a,b,c>;
                            end if;
                        end if;
                    end if;
                elif (a eq 0 and b eq 0) then
                    if ((IsSquare(q) and not IsDivisibleBy(p-1,8)) or
                        (not IsSquare(q) and p ne 2)) then <q,a,b,c>;
                    end if;
                elif (a eq 0 and b eq -q) then
                    if ((IsSquare(q) and not IsDivisibleBy(p-1,12)) or
                        (not IsSquare(q) and p ne 3)) then <q,a,b,c>;
                    end if;
                elif a eq 0 and b in \{-q,-2*q\} and not IsSquare(q) then
                    <q,a,b,c>;
                elseif a eq 0 and b eq 2*q and IsSquare(q) and
                    IsDivisibleBy(p-1,4) then <q,a,b,c>;
                end if;
            end if;
        end for
    end for
end if;
```

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elif Abs(a) eq p^((m+1) div 2) and b eq 3*q and not IsSquare(q) and p eq 5 then <q,a,b,c>;
end if;
end if;
end for;
end for;
end if;
end for;
end if;
end for;
end for;

REFERENCES


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