

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme accessible from www.ams.org/msc/.

1[11N35, 11N36, 11N32]—*A higher-dimensional sieve method with procedures for computing sieve functions*, by Harold G. Diamond, H. Halberstam, and William F. Galway, Cambridge University Press, Cambridge, 2008, xxi+266 pp., hardcover, US\$82.00, ISBN 978-0-521-89487-6

The *Twin Prime Conjecture* is a difficult unsolved problem in number theory; it states that there are infinitely many primes p such that $p + 2$ is also prime. This is a special case of a more general statement known as *Dickson's Conjecture*, which is that if h_1, h_2, \dots, h_k are distinct integers satisfying some obviously necessary conditions, then there are infinitely many natural numbers n such that

$$(1) \quad n + h_1, n + h_2, \dots, n + h_k$$

are all prime. The “obviously necessary condition” is that there are no local obstructions; i.e., that for every fixed prime p there is some integer m such that p does not divide $(m + h_1)(m + h_2) \dots (m + h_k)$.

No special cases of Dickson's Conjecture are known to be true for any $k \geq 2$. Nonetheless, number theorists have made partial progress by means of sieve methods. The first sieve was of course the Sieve of Eratosthenes, but modern sieve methods began with V. Brun starting in 1915. Brun used an upper bound sieve with the number of twin primes strong enough to show that the sum of the reciprocals of the twin primes converges. He later employed a lower bound sieve to show that there are infinitely many n such that both n and $n + 2$ have at most 9 prime factors. Many efforts have been made to improve the latter result; the current best theorem on this topic is due to Chen Jing-Run, who proved in 1973 that there are infinitely many primes p such that $p + 2$ has at most two prime factors.

A sieve problem starts with a finite set of integers \mathcal{A} and a set of primes \mathcal{P} . The object is to estimate $S(\mathcal{A}, \mathcal{P}, z)$, which is the number of elements in \mathcal{A} that are not divisible by any prime $p \in \mathcal{P}$ with $p < z$. The term “dimension” refers to the average number of residue classes sifted out for each prime. For example, in the twin prime problem, one starts with the set $\mathcal{A} = \{n(n + 2) : n \leq X\}$. For each prime $p < z$, one removes all elements from \mathcal{A} such that $p|n(n + 2)$. The latter condition is equivalent to $n \equiv 0$ or $2 \pmod{p}$. We are therefore removing 2 residue classes mod p for each odd prime p , so the dimension is $\kappa = 2$. In the general case of the Dickson Conjecture enunciated in (1), the dimension is k .

This book is devoted to the study of the Jurkat-Richert sieve. The original version of this, developed by W.B. Jurkat and H.-E. Richert in 1965, was limited to dimension 1. It starts with two basic ideas. The first is the Buchstab identity, which states that

$$S(\mathcal{A}, \mathcal{P}, z) = |\mathcal{A}| - \sum_{p < z, p \in \mathcal{P}} S(\mathcal{A}_p, \mathcal{P}, p).$$

This follows easily by noting that each element of \mathcal{A} is counted exactly once in the sum on the right; i.e., is such an element n counted when p is the largest prime dividing n . The second idea is Selberg's Λ^2 upper bound sieve, which relies on the fact that the square of a real number is non-negative.

Of course, the explication of these two basic ideas entails some intricate details, and the details get considerably more complicated for higher dimensions. The results for $\kappa > 1$ were developed by Diamond and Halberstam in a series of papers published between 1988 and 2002. (Richert, who died in 1993, was a co-author on some of the early papers.) The Buchstab identity leads to difference-differential equations, and considerable work is required to analyze the solutions of these equations. At the outset, the authors expected that the most complicated part of the analysis would be in large dimensions. In fact, it turns out that the dimensions between 1 and 2 are the most difficult. There is a fundamental bifurcation in the behavior of the underlying sieve functions that happens in the neighborhood of $\kappa = 1.8344\dots$, and some delicate analysis is required to understand this transition.

Here, the authors have collected the results into one self-contained volume. They made an editorial decision to confine the exposition to half-integer values of κ . This allows them to address all of the most interesting applications, and at the same time, they avoid the analytic complications coming from those κ a little bit below 2.

The first six chapters of this book are devoted to developing these two basic ideas in suitable form. Chapter 7 presents the linear sieve, that is, the sieve in dimension one. Chapter 8 applies the linear sieve to prove that there are infinitely many primes p such that $p+2$ has at most three prime factors. This is weaker than Chen's theorem, but the proof given here requires considerably less machinery. The clever proof given here is originally due to R.C. Vaughan.

Chapters 9 and 10 are devoted to the Jurkat-Richert sieve in half-integer dimensions greater than one. The proofs are given modulo "The Main Analytic Theorem." The proof of the latter occupies Chapters 12 through 18.

Chapter 11 is devoted to the "weighted sieve," which is a variant of the sieve that counts elements with a limited number of prime factors in the sifting set \mathcal{P} . This variant is important in applications.

The accurate computation of sieve functions is an important but difficult subtopic within sieve methods. William Galway has prepared a *Mathematica* package that can be downloaded from www.math.uiuc.edu/SieveTheoryBook and used for these computations. This package is described in an appendix of the book.

The material is technical, but the authors have organized it very clearly. They have worked with this material for over twenty years, and their presentation has clearly benefited from successive refinements. The writing is clear and lively. I recommend this book for any graduate student or researcher who wants to see the development and application of sieve methods. The book will also be of interest to those who work with differential-difference equations. Finally, I also recommend this volume as an excellent example of how to make an elegant presentation of complicated technical material.

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