SOME GENERALIZED EUCLIDEAN AND 2-STAGE EUCLIDEAN NUMBER FIELDS THAT ARE NOT NORM-EUCLIDEAN

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Abstract. We give examples of Generalized Euclidean but not norm-Euclidean number fields of degree greater than 2. In the same way we give examples of 2-stage Euclidean but not norm-Euclidean number fields of degree greater than 2. In both cases, no such examples were known.

1. Introduction

In 1985, Johnson, Queen and Sevilla [9] introduced a generalization of the classical notion of Euclidean number field.

Definition 1.1. A number field $K$ is said to be Generalized Euclidean or simply G.E. if for every $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that the ideal $(\alpha, \beta)$ is principal, there exists $\Upsilon \in \mathbb{Z}_K$ such that

$$|N_{K/Q}(\alpha - \Upsilon \beta)| < |N_{K/Q}(\beta)|.$$ 

If $(\alpha, \beta)$ is principal, we thus have at our disposal the Euclidian algorithm to compute a gcd of $\alpha$ and $\beta$ because it is easy to see that $(\beta, \alpha - \Upsilon \beta)$ is principal again, and so on. Note that if $K$ is norm-Euclidean, then $K$ is G.E. and, if $K$ is principal, i.e., has class number 1, then $K$ is G.E. if and only if $K$ is norm-Euclidean. If we want to illustrate the difference between “G.E.” and “norm-Euclidean”, the interesting case is when $K$ is G.E. but not principal (so not norm-Euclidean). The following result was established by Johnson, Queen and Sevilla in [9].

Theorem 1.1. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is G.E. but not norm-Euclidean for $d = 10$ and $d = 65$. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is not G.E. for $d = 15, 26, 30, 35, 39, 51, 78, 87, 102, 115, 195$ and $230$.

Furthermore, Johnson, Queen and Sevilla conjectured that $K = \mathbb{Q}(\sqrt{d})$ (with $d > 1$ squarefree) is G.E. if and only if $K$ is norm-Euclidean or $d = 10$ or 65.

Another variation on norm-Euclidean number fields has been introduced by Cooke [7].

Definition 1.2. Let $m$ be a rational integer $\geq 1$. The number field $K$ is $m$-stage Euclidean if and only if for every $\alpha \in \mathbb{Z}_K$ and every $\beta \in \mathbb{Z}_K \setminus \{0\}$ there exists a positive rational integer $k \leq m$ and $k$ pairs $(q_i, r_i)$ ($1 \leq i \leq k$) of elements of $\mathbb{Z}_K$
such that
\[
\alpha = \beta q_1 + r_1, \\
\beta = r_1 q_2 + r_2, \\
\vdots \\
r_{k-2} = r_{k-1} q_k + r_k,
\]
and \(|N_{K/Q}(r_k)| < |N_{K/Q}(\beta)|\).

When it is well defined, let us put
\[
[q_1, q_2, \ldots, q_k] = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots + \frac{1}{q_k}}},
\]
where \(a_k\) and \(b_k\) are given by
\[
a_1 = q_1, \quad b_1 = 1, \\
a_2 = a_1 q_2 + 1, \quad b_2 = q_2,
\]
and recursively for \(k \geq 3\) by
\[
a_k = a_{k-1} q_k + a_{k-2}, \quad b_k = q_k b_{k-1} + b_{k-2}.
\]

Since
\[
\frac{\alpha}{\beta} = \frac{a_k}{b_k} + (-1)^{k+1} \frac{r_k}{b_k \beta},
\]
this definition is equivalent to the following.

**Definition 1.3.** The number field \(K\) is \(m\)-stage Euclidean if and only if for every \(\xi \in K\), there exists a positive rational integer \(k \leq m\), and \(k\) elements \(q_1, q_2, \ldots, q_k \in \mathbb{Z}_K\) such that
\[
\left| N_{K/Q}(\xi - [q_1, q_2, \ldots, q_k]) \right| < \frac{1}{|N_{K/Q}(\beta)|}.
\]

As in the previous case, norm-Euclidean implies \(m\)-stage Euclidean, but contrary to what happens with the G.E. condition, we have the following result [7].

**Theorem 1.2.** A number field \(K\) with unit rank \(r \geq 1\) (i.e., \(r = \text{rank}(\mathbb{Z}_K^*) \geq 1\)) is principal if and only if \(K\) is \(m\)-stage Euclidean for some \(m\).

As a consequence, if we want to study the difference between \(m\)-stage Euclidean and norm-Euclidean, we have to look at number fields with class number 1 and find some example where \(K\) is principal, \(m\)-stage Euclidean, but not norm-Euclidean.

The following result was established by Cooke [7].

**Theorem 1.3.** For \(d = 14, 22, 23, 31, 38, 43, 46, 53, 61, 69, 89, 93, 97, \mathbb{Q}(\sqrt{d})\) is 2-stage Euclidean, but not norm-Euclidean.

Furthermore, Cooke and Weinberger [8] proved that, under GRH, every principal number field \(K\) with unit rank \(r \geq 1\) is 4-stage Euclidean, and even 2-stage Euclidean if \(K\) has at least one real place.
For both notions (G.E. and $m$-stage Euclidean), no examples of number fields of degree greater than 2 were known. Our main results are the following.

**Theorem 1.4.** None of the totally real number fields enumerated in Table 1 are principal. They all are G.E. except for the second cubic number field of discriminant 3969, defined by $x^3 - 21x - 35$, which is neither principal nor G.E.

**Table 1.** Here, $n$ is the degree of the field $K$, $D_K$ its discriminant, $P(x)$ its equation, $h$ its class number and $M(K)$ its Euclidean minimum.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_K$</th>
<th>$P(x)$</th>
<th>$h$</th>
<th>$M(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1957</td>
<td>$x^3 - x^2 - 9x + 10$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2597</td>
<td>$x^3 - x^2 - 9x + 8$</td>
<td>3</td>
<td>$5/2$</td>
</tr>
<tr>
<td>3</td>
<td>2777</td>
<td>$x^3 - x^2 - 14x + 23$</td>
<td>2</td>
<td>$5/3$</td>
</tr>
<tr>
<td>3</td>
<td>3969$^1$</td>
<td>$x^3 - 21x - 28$</td>
<td>3</td>
<td>$4/3$</td>
</tr>
<tr>
<td>3</td>
<td>3969</td>
<td>$x^3 - 21x - 35$</td>
<td>3</td>
<td>$7/3$</td>
</tr>
<tr>
<td>3</td>
<td>3981</td>
<td>$x^3 - x^2 - 11x + 12$</td>
<td>2</td>
<td>$3/2$</td>
</tr>
<tr>
<td>3</td>
<td>4212</td>
<td>$x^3 - 12x - 10$</td>
<td>3</td>
<td>$7/2$</td>
</tr>
<tr>
<td>3</td>
<td>4312</td>
<td>$x^3 - x^2 - 16x + 8$</td>
<td>3</td>
<td>$11/4$</td>
</tr>
<tr>
<td>3</td>
<td>5684</td>
<td>$x^3 - 14x - 14$</td>
<td>3</td>
<td>$9/2$</td>
</tr>
<tr>
<td>4</td>
<td>21025</td>
<td>$x^4 - 17x^2 + 36$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>32625</td>
<td>$x^4 - x^3 - 19x^2 + 4x + 76$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>46400</td>
<td>$x^4 - 22x^2 + 116$</td>
<td>2</td>
<td>$5/4$</td>
</tr>
<tr>
<td>4</td>
<td>51200</td>
<td>$x^4 - 20x^2 + 50$</td>
<td>2</td>
<td>$7/2$</td>
</tr>
</tbody>
</table>

**Theorem 1.5.** The totally real number fields of degree 3 and of discriminants < 15000 which are principal but not norm-Euclidean (82 cases) are 2-stage norm-Euclidean. The same is true for degree 4 and discriminants 18432, 34816, 35152 and for degree 5 and discriminant 390625. In all of these cases, the number field is principal, not norm-Euclidean, but 2-stage norm-Euclidean.

Details on the number fields appearing in Theorem 1.5 are available from [6] and are given in the online version of this paper. In Section 2, we recall other definitions and general results. In Sections 3 and 4, we study the case of Generalized Euclidean number fields and the case of 2-stage Euclidean number fields, respectively.

### 2. The Algorithm, Generalities

Let $K$ be a number field of degree $n$. We have designed an algorithm which allows us to compute the Euclidean minimum of $K$, in particular when $K$ is totally real [4], but also in the general case [3]. According to theoretical results [4], this algorithm can also give the upper part of the Euclidean spectrum of $K$ and this yields new examples of number fields with interesting properties.

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1. In [2] and [10], the Euclidean minimum of the number field with discriminant 3969, defined by $x^3 - 21x - 28$ was erroneously announced to be 1.
2. See the “Table Supplement” link on line.
From now on, we suppose that $K$ is totally real and that $n > 2$. We denote by $\mathbb{Z}_K$ the ring of its integers and by $N_{K/Q}$ its absolute norm. The Euclidean minimum of an element $\xi \in K$ is

$$m_K(\xi) = \inf_{\gamma \in \mathbb{Z}_K} |N_{K/Q}(\xi - \gamma)|$$

and the Euclidean minimum of $K$ is

$$M(K) = \sup_{\xi \in K} m_K(\xi).$$

The set of values taken by $m_K$ is called the Euclidean spectrum of $K$. We know the following important result [4].

**Theorem 2.1.** The Euclidean spectrum of $K$ is the union of $\{0\}$ and of a strictly decreasing sequence of rationals $(r_i)_{i \geq 0}$ with limit 0. For each $i$, the set of $\xi \in K$ such that $m_K(\xi) = r_i$ is finite modulo $\mathbb{Z}_K$.

In fact, we have a stronger result, which can be formulated in terms of the inhomogeneous spectrum. However, we shall not need this in what follows.

**Corollary 2.2.** The set of $\xi \in K$ such that $m_K(\xi) \geq 1$ is finite modulo $\mathbb{Z}_K$.

Recall now that we have at our disposal an algorithm which can give us all the $\xi \in K$ such that $m_K(\xi) \geq 1$. Without going into detail—these can be found in [5]—let us give nevertheless the theorem which justifies the algorithm and the main ideas that are behind it. Let us choose a constant $k > 0$ and a let us embed $K$ into $K \otimes_{\mathbb{Q}} \mathbb{R} =: \mathbb{K}$, which we can identify with $\mathbb{R}^n$, in which $\mathbb{Z}_K$ is a lattice. Under this identification an element $\xi$ of $K$ is viewed as $(\sigma_i(\xi))_{1 \leq i \leq n}$, where the $\sigma_i$ are the embeddings of $K$ into $\mathbb{R}$. The map $m_K$ extends to a map $m_K^\mathbb{R}$ from $\mathbb{R}^n$ to $\mathbb{R}^+$ in a natural way:

$$m_K^\mathbb{R}(x) = \inf_{\gamma \in \mathbb{Z}_K} \left| \prod_{i=1}^n (x_i - \sigma_i(\gamma)) \right|.$$ 

Moreover, the product of two elements of $K$ is extended to the product coordinate by coordinate in $\mathbb{R}^n$. This new product of two elements $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$. Finally, let $\varepsilon$ be a nontorsion unit of $\mathbb{Z}_K$.

The main idea is to find in a fundamental domain $\mathcal{F}$ associated to $\mathbb{Z}_K$ in $\mathbb{R}^n$, $s$ distinct bounded sets $\mathcal{T}_i$ ($1 \leq i \leq s$) with the property that for each such $\mathcal{T}_i$ there exists an $X_i \in \mathbb{Z}_K$ and $s_i$ integers $n_{i,1}, \ldots, n_{i,s_i}$ ($s_i > 0$) such that

$$(\varepsilon \cdot \mathcal{T}_i - X_i) \cap \mathcal{H} \subseteq \bigcup_{1 \leq \ell \leq s_i} \mathcal{T}_{n_{i,\ell}} \quad (i = 1, \ldots, s),$$

where

$$\mathcal{H} = \{x \in \mathbb{R}^n \text{ such that } m_K^\mathbb{R}(x) \leq k\}.$$ 

We consider the $\mathcal{T}_i$ as the vertices of a directed graph $\mathcal{G}$ and represent (1) by $s_i$ directed edges whose tail is $\mathcal{T}_i$ and whose respective heads are the $\mathcal{T}_{n_{i,\ell}}$ ($1 \leq \ell \leq s_i$). To describe such an edge of $\mathcal{G}$ we shall use the notation $\mathcal{T}_i \rightarrow \mathcal{T}_{n_{i,\ell}}(X_i)$. The set $\mathcal{C}$ of simple cycles of $\mathcal{G}$ is nonempty and finite. Each element $c$ of $\mathcal{C}$ of length $j$ is in the form of the circular path, $\mathcal{T}_0' \rightarrow \mathcal{T}_1'(X_0') \cdots \rightarrow \mathcal{T}_{j-1}'(X_{j-2}') \rightarrow \mathcal{T}_j'(X_{j-1}')$, for some subset $\{\mathcal{T}_1', \ldots, \mathcal{T}_{j-1}'\} \subseteq \{\mathcal{T}_1, \ldots, \mathcal{T}_s\}$, where $X_i'$ denotes the element $X \in \mathbb{Z}_K$
associated to $T'_i$. This defines, in a unique way, $j$ elements of $K$, $\xi_0, \ldots, \xi_{j-1}$ by the formula
\[
\xi_r = \frac{\varepsilon^j - 1}{\varepsilon^j - 1} X'_r + \varepsilon^{j-2} X'_{r+1} + \cdots + X'_{j-1+r} \quad (r = 0, \ldots, j - 1);
\]
the indices being read modulo $j$. In this context, we say that $\xi_0, \ldots, \xi_{j-1}$ are associated to the cycle $c$.

We denote by $E$ the finite set of all elements of $K$ associated to the elements of $\mathcal{C}$. The $\xi_1$ associated to a cycle $c$ are in the same orbit modulo $\mathbb{Z}_K$ under the action of $\mathbb{Z}_K^*$ (in fact $\xi_{r+1} = \varepsilon \cdot \xi_r - X'_r$) and satisfy
\[
m_{\mathcal{M}}(\xi_0) = \cdots = m_{\mathcal{M}}(\xi_{j-1}) = m(c),
\]
which is a rational number. Finally, define
\[
m(G) = \max_{c \in \mathcal{C}} m(c) = \max_{\xi \in E} m_{\mathcal{M}}(\xi).
\]

Let us say that $G$ is convenient if every infinite path of $G$ is ultimately periodic. The essential result is the following.

**Theorem 2.3.** Assume that $G$ is convenient and that there exists $T \in \{T_1, \ldots, T_s\}$ and $x \in \mathbb{R}^n$ such that $m_{\mathcal{M}}(x) > k$. Then:

i) $m_{\mathcal{M}}(x) \leq m(G)$.

ii) If $x \in K$, there exists $\xi \in E$ such that $x \equiv \xi \mod \mathbb{Z}_K$.

In this situation we know all the potential $\xi \in K$ such that $m_K(\xi) > k$, and since computing $m_K(\xi)$ is possible (again see [5] for more details), we know in fact that all of the $\xi \in K$ such that $m_K(\xi) > k$. To identify the elements $\xi \in K$ such that $m_K(\xi) \geq 1$, it is sufficient to run the algorithm with $k = 0.999$, for instance.

### 3. Generalized Euclidean number fields

#### 3.1. Generalities

From the definition of a G.E. number field and the definition of the map $m_K$, we have the following result.

**Proposition 3.1.** The field $K$ is G.E. if and only if for every $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$, the ideal $(\alpha, \beta)$ is not principal.

**Remark 1.** Suppose that we have at our disposal the finite set $S$ of all $\xi \in K$ (modulo $\mathbb{Z}_K$) such that $m_K(\xi) \geq 1$, and that for each such $\xi$ we have a representative $u/v$ where $(u, v) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$. Let $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$. Then there exists $\xi \equiv u/v$ in $S$ such that $\alpha/\beta = u/v + \gamma$ with $\gamma \in \mathbb{Z}_K$. Since
\[
(\alpha, \beta) = (\beta u/v + \gamma \beta, \beta) = (\beta u/v, \beta) = \beta / v(u, v),
\]
it is sufficient, for proving that $K$ is G.E., to check that for every $\xi \equiv u/v \in S$, $(u, v)$ is not principal.

#### 3.2. A first example

The purpose of this subsection is to study in detail a particular case. Other results, obtained in another way, will be given in the next subsection. Let $K$ be the normal quartic field generated by any one of the roots of
\[
P(X) = X^4 - 20X^2 + 50.
\]
The field $K$ is totally real, its discriminant is 51200, its class number is 2, and a $\mathbb{Z}$-basis of $\mathbb{Z}_K$ is $(e_1, e_2, e_3, e_4)$ with
\[
e_1 = 1, \quad e_2 = \sqrt{2}, \quad e_3 = \sqrt{10 + 5\sqrt{2}}, \quad e_4 = \sqrt{10 - 5\sqrt{2}}.
\]
Our algorithm shows that
\[ M(K) = \frac{7}{2}, \]
and that there is a unique \( \xi \in K \) (modulo \( \mathbb{Z}_K \)) such that \( m_K(\xi) \geq 1 \). More precisely,
\[ \xi = \frac{1}{2}(e_3 + e_4). \]
According to Remark 1 if we want to establish that \( K \) is G.E., we just have to prove that the ideal \((2, e_3 + e_4)\) is not principal.

**Theorem 3.2.** The field \( K \) is not norm-Euclidean, but it is G.E.

**Proof.** First, we note that \( e_3 + e_4 = e_2 \cdot e_3 \), so that we are reduced to proving that the ideal \((e_2, e_3)\) is not principal. Suppose on the contrary that it is principal so that we have
\[ e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \nu \mathbb{Z}_K, \]
with \( \nu \in \mathbb{Z}_K \). Since \( N_{K/\mathbb{Q}}(e_2) = 4 \) and \( N_{K/\mathbb{Q}}(e_3) = 50 \), we have
\[ N_{K/\mathbb{Q}}(\nu) \mid 2 = \gcd(4, 50), \]
so that we have two possibilities: either \( \nu \in \mathbb{Z}_K^\times \) or \( N_{K/\mathbb{Q}}(\nu) = \pm 2 \).

**First case:** \( \nu \) is a unit and we have, in fact, \( e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \mathbb{Z}_K \).
In this case, there exist \( u, v \in \mathbb{Z}_K \) such that
\[ 1 = e_2 \cdot u + e_3 \cdot v. \]
Let us write
\[
\begin{align*}
  u &= a + be_2 + ce_3 + de_4, \\
  v &= a' + b'e_2 + c'e_3 + d'e_4,
\end{align*}
\]
where \( a, b, c, d, a', b', c', d' \in \mathbb{Z} \).
Since \( e_2 \cdot e_3 = e_3 + e_4 \), \( e_2 \cdot e_4 = e_3 - e_4 \) and \( e_3 \cdot e_4 = 5e_2 \), if we substitute \( 3 \) into \( 2 \) we obtain, by identification of the coefficients in our \( \mathbb{Z} \)-basis, that \( 2b + 10c' = 1 \), which is clearly impossible.

**Second case:** \( \nu \) has norm \( \pm 2 \).
If \( \nu = a + be_2 + ce_3 + de_4 \) where \( a, b, c, d \in \mathbb{Z} \), an easy computation leads to
\[
\pm 2 = N_{K/\mathbb{Q}}(\nu) = a^4 + 4b^4 + 50c^4 + 50d^4 - 4a^2b^2 - 20a^2c^2 - 20a^2d^2 - 40b^2c^2 - 40b^2d^2 + 100ac^2d^2 + 40abc^2d^2 + 200dcd^3 - 200dcd^3 + 80ab. \]
This implies that
\[ \pm 2 \equiv (a^2 - 2b^2)^2 \pmod{5}, \]
which is impossible as neither of \( \pm 2 \) are quadratic residues \( \pmod{5} \). \( \Box \)
3.3. The Dedekind-Hasse criterion. In this subsection, we study the link between G.E. and a Euclidean-type map that we shall deduce from the Dedekind-Hasse criterion. This will lead us to define an easy test which allows us to find new examples, without requiring detailed calculations as above. First, recall the Dedekind-Hasse criterion (see for instance [11]).

**Theorem 3.3.** A number field $K$ has class number 1 if and only if for every $\alpha, \beta \in \mathbb{Z}_K \setminus \{0\}$ such that $\beta \nmid \alpha$, there exist $\gamma, \delta \in \mathbb{Z}_K$ such that
\begin{equation}
0 < |N_{K/\mathbb{Q}}(\alpha \gamma - \beta \delta)| < |N_{K/\mathbb{Q}}(\beta)|.
\end{equation}

This leads to the following natural definition.

**Definition 3.1.** For every $\xi \in K \setminus \mathbb{Z}_K$ we shall denote by $h_K(\xi)$ the real number defined by
\[ h_K(\xi) = \inf \{ m_K(\Upsilon \xi); \ \Upsilon \in \mathbb{Z}_K \text{ and } \Upsilon \xi \notin \mathbb{Z}_K \}. \]

This map has the following elementary properties, which we give here without proof.

**Proposition 3.4.** For every $\xi \in K \setminus \mathbb{Z}_K$ we have:
\begin{enumerate}
  
  \item $0 < h_K(\xi) \leq m_K(\xi)$.
  
  \item For every $\alpha \in \mathbb{Z}_K$, $h_K(\xi + \alpha) = h_K(\xi)$.
  
  \item For every $\epsilon \in \mathbb{Z}_K^*$, $h_K(\epsilon \xi) = h_K(\xi)$.
\end{enumerate}

We can now reformulate the Dedekind-Hasse criterion as follows.

**Theorem 3.5.** A number field $K$ has class number 1 if and only if for every $\xi \in K \setminus \mathbb{Z}_K$ we have $h_K(\xi) < 1$.

**Proof.** The norm being multiplicative, (1) can be reformulated as follows: For every $\xi \in K \setminus \mathbb{Z}_K$ there exist $\gamma, \delta \in \mathbb{Z}_K$ such that
\begin{equation}
0 < |N_{K/\mathbb{Q}}(\gamma \xi - \delta)| < 1,
\end{equation}

which leads to $m_K(\xi) < 1$. Since (5) cannot be true if $\gamma \xi \in \mathbb{Z}_K$, we have $h_K(\xi) < 1$. Conversely, since $|N_{K/\mathbb{Q}}(\gamma \xi - \delta)| = 0$ implies $\gamma \xi \in \mathbb{Z}_K$, which is excluded in the definition of $h_K$, we see that if $h_K(\xi) < 1$, then (5) is true. $\square$

Now consider a number field $K$ and put
\[ S = \{ \xi \in K; \ m_K(\xi) \geq 1 \}. \]

Suppose that $K$ is not norm-Euclidean so that $S \neq \emptyset$. We have the following result.

**Theorem 3.6.** One of the following three possibilities holds:
\begin{enumerate}
  
  \item For every $\xi \in S$, $h_K(\xi) < 1$. Then $K$ has class number 1 and is not G.E.
  
  \item For every $\xi \in S$, $h_K(\xi) \geq 1$. Then $K$ is G.E. (and not principal).
  
  \item There exist $\xi, \mu \in S$ such that $h_K(\xi) < 1$ and $h_K(\mu) \geq 1$. Then $K$ is not principal. If, in addition, there exists $\xi = \alpha/\beta \in S$ (with $\alpha, \beta \in \mathbb{Z}_K$) with $h_K(\xi) < 1$ and such that $(\alpha, \beta)$ is principal, then $K$ is not G.E. Otherwise it is G.E.
\end{enumerate}

**Proof.** Clearly, we have the three cases.

**Case 1.** The result is a consequence of Theorem 3.3 and of the fact that when the field is principal norm-Euclidean and G.E. are synonymous.
Theorem 3.5 indicates that \( K \) is not principal. By Proposition 3.1 it is sufficient to prove that for every \( \xi = \alpha/\beta \in S \) where \( \alpha, \beta \in \mathbb{Z}_K \), the ideal \((\alpha, \beta)\) is not principal. Otherwise, we have \((\alpha, \beta) = \nu \mathbb{Z}_K\) with \( \nu \in \mathbb{Z}_K \). By hypothesis \( h_K(\xi) \geq 1 \) so that for every \( X, Y \in \mathbb{Z}_K \) with \( X\xi \notin \mathbb{Z}_K \) we have

\[
|N_{K/\mathbb{Q}}(X\alpha - Y\beta)| \geq |N_{K/\mathbb{Q}}(\beta)|.
\]

Now \( \nu \) can be written \( \nu = X\alpha - Y\beta \) with \( X, Y \in \mathbb{Z}_K \) and \( X\xi \notin \mathbb{Z}_K \). Otherwise \( \nu \in \beta \mathbb{Z}_K \) so that \( \beta | \nu \). But this implies that \( \nu \) and \( \beta \) are associates and we have \( (\alpha, \beta) = \beta \mathbb{Z}_K \) which implies \( \beta | \alpha \) and \( \xi \in \mathbb{Z}_K \), which is impossible. We deduce from this that \( |N_{K/\mathbb{Q}}(\nu)| \geq |N_{K/\mathbb{Q}}(\beta)| \). Since \( N_{K/\mathbb{Q}}(\nu) | N_{K/\mathbb{Q}}(\beta) \) we have \( |N_{K/\mathbb{Q}}(\nu)| = |N_{K/\mathbb{Q}}(\beta)| \), and since \( \nu | \beta, \nu \) and \( \beta \) are associates, which is impossible by the previous argument.

**Case 3.** Theorem 3.5 indicates that \( K \) is not principal. The second assertion is a consequence of Proposition 3.1. Indeed, as previously, if \( h_K(\xi) \geq 1 \) and \( \xi = \alpha/\beta \), then \((\alpha, \beta)\) is not principal and this case is not an obstruction for \( K \) to be G.E. Finally, the only possibilities for contradicting G.E. come from the \( \xi = \alpha/\beta \in S \) such that \( h_K(\xi) < 1 \) and \((\alpha, \beta)\) is principal.

**Corollary 3.7.** Suppose that \( K \) is not norm-Euclidean and that, with the above notation, \( S \) modulo \( \mathbb{Z}_K \) is composed of a single orbit under the (multiplicative) action of \( \mathbb{Z}_K^\times \) modulo \( \mathbb{Z}_K^\times \), i.e., that if \( \xi, \mu \in S \) there exists an \( \varepsilon \in \mathbb{Z}_K^\times \) and an \( \alpha \in \mathbb{Z}_K \) such that \( \mu = \varepsilon \xi + \alpha \). Then either \( K \) is principal and not G.E. or \( K \) is not principal but is G.E.

**Proof.** If \( K \) is principal, we are in case 1. Otherwise, since all the elements of \( S \), which are in the same orbit, have the same image by \( h_K \) (Proposition 3.4), we cannot be in case 3 of Theorem 3.6. Finally, we are in case 2 and \( K \) is G.E.

**Remark 2.** To simplify notation and vocabulary, we shall often, by abuse of language, speak of \( \xi \in K \) to mean \( \xi \in K \mod \mathbb{Z}_K \). For instance, we shall speak of orbits in \( S \) under the action of \( \mathbb{Z}_K^\times \); in this context \( S \) and these orbits should be understood modulo \( \mathbb{Z}_K \).

**Corollary 3.8.** The totally real number fields of degree 3 and discriminants 1957, 2777, 3981 (see Table 1) are G.E. The totally real number fields of degree 4 and discriminants 46400 and 51200 (see Table 1) are G.E.

**Proof.** In fact, in all of these cases, our algorithm establishes that we are under the previous hypotheses. For discriminant 1957, we have \( M(K) = 2 \) and one orbit with one element in \( S \). For discriminant 2777, we have \( M(K) = 5/3 \) and one orbit with 2 elements in \( S \). For discriminant 3981, we have \( M(K) = 3/2 \) and one orbit with one element in \( S \). For discriminant 46400, we have \( M(K) = 5/4 \) and one orbit with 3 elements in \( S \). For discriminant 51200, we have \( M(K) = 7/2 \) and one orbit with one element in \( S \).

Now, if there are several orbits in \( S \), and we want to use Theorem 3.6, we have to see whether, for one element \( \xi \) by orbit, and for every orbit, we have \( h_K(\xi) \geq 1 \), in which case necessarily \( K \) is G.E. The problem is now: How can we compute \( h_K(\xi) \)? Our algorithm gives us every such \( \xi \) by its coordinates in a \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \). These coordinates are of the form \( (a_1/d, a_2/d, \ldots, a_n/d) \) where \( a_i \in \mathbb{Z} \) for every \( i \) and \( d \in \mathbb{Z}_{>0} \). Furthermore, we can compute \( m_K(\mu) \) for every \( \mu \in K \). Hence, it is
easy to see that, to compute \( h_K(\xi) \), it is sufficient to compute \( m_K(\Upsilon \xi) \) for every \( \Upsilon \) with coordinates in \( \{0, 1, \ldots, d - 1\} \) for our basis, such that \( \Upsilon \xi \not\in \mathbb{Z}_K \). This is easy to check. By definition, the value of \( h_K(\xi) \) will be the minimum of these \( m_K(\Upsilon \xi) \). Of course if for every \( \xi \) and every such \( \Upsilon \) we have \( \Upsilon \xi \in S \text{ mod } \mathbb{Z}_K \), then \( K \) is G.E.

Using this last approach we have established the following result.

**Theorem 3.9.** The following totally real number fields of degree \( n \) are G.E. but not norm-Euclidean:

- when \( n = 3 \), the fields in Table 1 with discriminants 2597, 4212, 4312, 5684;
- when \( n = 4 \), the fields in Table 1 with discriminants 21025, 32625.

**Proof.** We just give a typical example. For \( n = 3 \) and discriminant 2597, we have two orbits in \( S \), the first one \( O_1 \) with two elements \( (\pm(e_1 + 2e_2 + 2e_3))/3 \) modulo \( \mathbb{Z}_K \) where \( (e_i) \) is the \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \) returned by PARI [1] and the second one \( O_2 \) with one element \( (e_1 + e_2 + e_3)/2 \) modulo \( \mathbb{Z}_K \). Then we can easily check that \( \mathbb{Z}_K \cdot O_1 = O_1 \cup \{0\} \) and that \( \mathbb{Z}_K \cdot O_2 = O_2 \cup \{0\} \). The same thing happens in other cases with sometimes more complicated equalities, but always with \( \mathbb{Z}_K \cdot O \subseteq S \cup \{0\} \).

\( \square \)

**Remark 3.** If we want to treat all the nonprincipal number fields of degree 3 and discriminant < 6000, it remains to study the two number fields with discriminant 3969. In these cases, our previous method does not work, because we have some \( \xi = \alpha/\beta \in S \) such that \( h_K(\xi) < 1 \). The first one, \( K_1 \), is defined by \( x^3 - 21x - 28 \). For this field, \( S \) is composed of five orbits \( O_i \), \( 1 \leq i \leq 5 \). For four of them, say for \( 1 \leq i \leq 4 \), we have \( \mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\} \); but for the last one, \( O_5 \), this is not true. Take an element \( \alpha/\beta \) of \( O_5 \); here we can take \( \alpha = 3e_1 + 2e_2 + 2e_3 \) and \( \beta = 6 \) where \( (e_1, e_2, e_3) \) is the \( \mathbb{Z} \)-basis returned by PARI [1]. We can then prove directly as in Section 3.2 that the ideal \( (\alpha, \beta) \) is not principal. We conclude that \( K_1 \) is G.E.

For the second field, \( K_2 \), defined by \( x^3 - 21x - 35 \) the situation is different. Here \( S \) is composed of seven orbits \( O_i \), \( 1 \leq i \leq 7 \) and four of them, say \( O_i \) with \( 1 \leq i \leq 4 \), are such that \( \mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\} \). Now if we look at the three others, we find that two of them contain an \( \alpha/\beta \) for which \( (\alpha, \beta) \) is principal. For completeness these \( (\alpha, \beta) \) are \( (7e_1 + 12e_2 + 4e_3, 21) \) and \( (7e_1 + 5e_2 + 11e_3, 21) \) with the usual notation. Consequently, \( K_2 \) is not G.E. All the computations, which are long and complicated—in particular for \( K_2 \)—have been done by hand and checked using PARI [1]. We do not give them here for lack of space; anyway they are not especially enlightening.

Finally, we put all these results together to give us Theorem 3.2.

4. The 2-stage Euclidean number fields

Let us begin with an example. Let \( K \) be the totally real cubic number field with discriminant 3988, defined by \( x^3 - 16x - 4 \). Using our algorithm we see that the upper part of the Euclidean spectrum of \( K \) has five elements:

\[
\text{sp}(K) \cap [1, \infty) = \{19/8, 11/8, 5/4, 19/16, 133/128\}.
\]

The set \( S \) is composed of five orbits, respectively, the orbits of \( ae_1 + be_2 + ce_3 \) with \( (a, b, c) = (0, 1/2, 1/2), (1/2, 1/2, 0), (1/2, 1/2, 1/2), (0, 3/4, 1/2) \) and \( (0, 3/8, 1/2) \), where \( (e_1, e_2, e_3) \) is the \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \) returned by PARI [1]. These orbits have,
respectively, 1, 1, 1, 2 and 4 elements. For one element $\xi$ by orbit, we try to find $q_1, q_2 \in \mathbb{Z}_K$ such that

$$\left|N_{K/Q}(\xi - q_1 - \frac{1}{q_2})\right| < \frac{1}{|N_{K/Q}(q_2)|},$$

by testing “small” $q_1 \in \mathbb{Z}_K$ and “small” $q_2 \in \mathbb{Z}_K \setminus \{0\}$. In each case this is possible, so that for every $\xi \in S$, (6) is true. Finally, this implies that $K$ is 2-stage norm-Euclidean. Using exactly the same approach we have established the results of Theorem 1.5.

Remark 4. Obviously these fields, which are principal and not norm-Euclidean, are not G.E.

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References