FOURIER EXPANSIONS FOR APOSTOL-BERNOULLI, APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

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ABSTRACT. We find Fourier expansions of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. We give a very simple proof of them.

1. Introduction and statement of main results

Let $w \in \mathbb{C}$ and $x$ a variable. The Apostol-Bernoulli polynomials $B_n(x;w)$, Apostol-Euler polynomials $E_n(x;w)$ and Apostol-Genocchi polynomials $G_n(x;w)$ are given by the generating functions

$$
\sum_{n \geq 0} B_n(x;w) \frac{t^n}{n!} = \frac{te^{xt}}{we^t - 1}, \quad |t + \log(w)| < 2\pi,
$$

(1.1)

$$
\sum_{n \geq 0} E_n(x;w) \frac{t^n}{n!} = \frac{2e^{xt}}{we^t + 1}, \quad |t + \log(w)| < \pi,
$$

(1.2)

$$
\sum_{n \geq 0} G_n(x;w) \frac{t^n}{n!} = \frac{2te^{xt}}{we^t + 1}, \quad |t + \log(w)| < \pi,
$$

(1.3)

where 

$$
w = |w|e^{i\theta}, -\pi \leq \theta < \pi \quad \text{and} \quad \log(w) = \log(|w|) + i\theta.
$$

These polynomials are a natural extension of the classical Bernoulli, Euler and Genocchi polynomials: $B_n(x) = B_n(x;1)$, $E_n(x) = E_n(x;1)$, $G_n(x) = G_n(x;1)$, see [3]. These polynomials have many applications in mathematics. Our main results are

Theorem 1.1. Let $w \in \mathbb{C}\{0\}$. For $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$. We have

$$
B_n(x;w) = \frac{-n!}{w^x(2\pi i)^n} \sum_{k \in \mathbb{Z}}^{*} e^{2\pi i k x} \left( k - \frac{\log(w)}{2\pi i} \right)^n,
$$

(1.4)

where $\sum_{k \in \mathbb{Z}}^{*} = \sum_{k \in \mathbb{Z}\{0\}}$ if $w = 1$ and $\sum_{k \in \mathbb{Z}}^{*} = \sum_{k \in \mathbb{Z}}$ if $w \neq 1$.  

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Theorem 1.2. Let $w \in \mathbb{C}\setminus\{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. We have

$$E_n(x; w) = \frac{2(n!)^2}{w^x(2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi i(k - \frac{1}{2})x}}{(k - \frac{1}{2} - \frac{\log(w)}{2\pi i})^{n+1}},$$

where $\sum_{k \in \mathbb{Z}}^{**} = \sum_{k \in \mathbb{Z} \setminus \{0\}}$ if $w = -1$ and $\sum_{k \in \mathbb{Z}}^{**} = \sum_{k \in \mathbb{Z}}$ if $w \neq -1$.

Theorem 1.3. Let $w \in \mathbb{C}\setminus\{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. We have

$$G_n(x; w) = \frac{2(n!)^2}{w^x(2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi i(k - \frac{1}{2})x}}{(k - \frac{1}{2} - \frac{\log(w)}{2\pi i})^{n+1}},$$

where $\sum_{k \in \mathbb{Z}}^{**} = \sum_{k \in \mathbb{Z} \setminus \{0\}}$ if $w = -1$ and $\sum_{k \in \mathbb{Z}}^{**} = \sum_{k \in \mathbb{Z}}$ if $w \neq -1$.

Remark 1.4. Luo’s proof [4], for Theorems [1.1 and 1.2] uses the Lipschitz summation formula [2] which is not easy to understand. In this paper we propose a very simple proof. On the other hand, Theorem 1.3 is new.

2. Proofs of main results

Proof of Theorem 1.1. We consider $\int_C f_n(t) \, dt$ with $f_n(t) = \frac{e^{-nt}}{w e^t - 1}$, the contour $C$ being a circle with radius $(2N + \epsilon)\pi$ ($\epsilon$ fixed real number such that $\epsilon \pi i \pm \log(w) \neq 0$ (mod $2\pi i$), centered at the origin. If $w \neq 1$, the poles of the integrand are $t_k = 2\pi ik - \log(w), k \in \mathbb{Z}$ and $t_\infty = 0$. The residues of the functions $f_n(t)$ for $k \in \mathbb{Z}$ are easily found to be $e^{-x(2\pi ik - \log(w))^{-1} e^{2\pi ikx}}$, and from Theorem 1.1 the residue at $z_\infty = 0$ is seen to be $\frac{\Re_n(x; w)}{n!}$. The integral around the circle $C$ tends to zero as $N \to \infty$ provided $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$, and by the theorem of residues we obtain

$$B_n(x; w) = \frac{-n!}{w^x(2\pi i)^n} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi ikx}}{(k - \frac{\log(w)}{2\pi i})^{n+1}}.$$

If $w = 1$, the poles of the integrand are $t_k = 2\pi ik, k \in \mathbb{Z}$. The residues of the functions $f_n(t)$ for $k \in \mathbb{Z} \setminus \{0\}$ are easily found to be $(2\pi ik)^{-1} e^{2\pi ikx}$, and from Theorem 1.1 the residue at $z_0 = 0$ is seen to be $\frac{\Re_n(x; w)}{n!}$. The integral around the circle $C$ tends to zero as $N \to \infty$ provided $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$, and by the theorem of residues we obtain

$$B_n(x; w) = \frac{-n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi ikx}}{k^n}.$$  

This yields the theorem. □

Proof of Theorem 1.2. We apply the same method to the function $g_n(t) = \frac{e^{-(n+1)t}}{w e^t + 1}$ the contour $C'$ being a circle with radius $(2N + 1 + \epsilon)\pi$ ($\epsilon$ fixed real number such that $\epsilon \pi i \pm \log(w) \neq 0$ (mod $\pi i$), centered at the origin. We omit the details. □

Proof of Theorem 1.3. We have $G_{n+1}(x; w) = (n+1)E_n(x; w)$. Thus we get Theorem 1.3 from Theorem 1.2. □
FOURIER EXPANSIONS FOR A-B, A-E AND A-G POLYNOMIALS

References


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