MATHEMATICS OF COMPUTATION
Volume 81, Number 277, January 2012, Pages 467–479
S 0025-5718(2011)02512-3
Article electronically published on June 30, 2011

ANALYTICAL FORMULAE FOR EXTENDED ${}_3F_2$-SERIES OF WATSON–WHIPPLE–DIXON WITH TWO EXTRA INTEGER PARAMETERS

WENCHANG CHU

Abstract. By combining the linearization method with Dougall's sum for well–poised ${}_5F_4$-series, we investigate the generalized Watson series with two extra integer parameters. Four analytical formulae are established, which can also be used to evaluate the extended Whipple and Dixon series via the Thomae transformation. Twelve concrete formulae are presented as exemplification.

1. Introduction and motivation

For an indeterminate x and a natural number n, denote the shifted–factorials by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N},$$
$$\langle x\rangle_0 = 1 \quad \text{and} \quad \langle x\rangle_n = x(x-1)\cdots(x-n+1) \quad \text{for} \quad n \in \mathbb{N}.$$

Following Bailey [1], the hypergeometric series, for an indeterminate z and two nonnegative integers p and q, is defined by

$${}_{1+p}F_q\left[\begin{matrix} a_0,\, a_1,\, \cdots,\, a_p \\ b_1,\, \cdots,\, b_q \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_p)_k}{k!(b_1)_k \cdots (b_q)_k} z^k$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the right-hand side. For the sake of brevity, the product and fraction of shifted factorials will be abbreviated respectively as

$$\begin{aligned} [A,\, B,\, \cdots,\, C]_n &= (A)_n (B)_n \cdots (C)_n, \\ \begin{bmatrix} \alpha,\, \beta,\, \cdots,\, \gamma \\ A,\, B,\, \cdots,\, C \end{bmatrix}_n &= \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}. \end{aligned}$$

The Γ-function quotient will analogously be shortened to

$$\Gamma\begin{bmatrix} \alpha,\, \beta,\, \cdots,\, \gamma \\ A,\, B,\, \cdots,\, C \end{bmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

There are many hypergeometric series formulae in the literature of mathematics and physics. Among them, the following three formulae for nonterminating series have been fundamental in the theory of classical hypergeoemtric series (cf. Bailey [1, Chapter III] and Chu [5]).

Received by the editor October 8, 2010 and, in revised form, November 30, 2010 and December 24, 2010.

2010 *Mathematics Subject Classification.* Primary 33C20; Secondary 05A10.

Key words and phrases. Classical hypergeometric series, Dixon's theorem, Watson's formula, Whipple's formula, The Thomae transformation, Dougall's theorem for well–poised series.

- Dixon [6]: $\Re(1+\frac{a}{2}-b-c)>0$:

$$(1)\quad {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b, 1+a-c\end{matrix}\Big|1\right]=\Gamma\left[\begin{matrix}1+a-b, 1+a-c, 1+\frac{a}{2}, 1+\frac{a}{2}-b-c \\ 1+\frac{a}{2}-b, 1+\frac{a}{2}-c, 1+a, 1+a-b-c\end{matrix}\right].$$

- Watson [19]: $\Re(1-a-b+2c)>0$:

$$(2)\qquad {}_3F_2\left[\begin{matrix} a,\ b, & c \\ \frac{1+a+b}{2}, & 2c\end{matrix}\Big|1\right]=\Gamma\left[\begin{matrix}\frac{1}{2}, & \frac{1+a+b}{2}, & \frac{1}{2}+c, & \frac{1-a-b}{2}+c \\ \frac{1+a}{2}, & \frac{1+b}{2}, & \frac{1-a}{2}+c, & \frac{1-b}{2}+c\end{matrix}\right].$$

- Whipple [20]: $\Re(c)>0$:

$$(3)\qquad {}_3F_2\left[\begin{matrix} a, 1-a, c \\ e, 1+2c-e\end{matrix}\Big|1\right]=\Gamma\left[\begin{matrix}\frac{e}{2}, & \frac{1+e}{2}, & c+\frac{1-e}{2}, & c+\frac{2-e}{2} \\ \frac{e+a}{2}, & \frac{1+e-a}{2}, & c+\frac{1-e+a}{2}, & c+\frac{2-e-a}{2}\end{matrix}\right].$$

As shown by Bailey [1, Chapter III], these three ${}_3F_2(1)$-series are connected by the following useful transformation due to Kummer, Thomae, and Whipple, which can be reproduced as

$$(4)\qquad {}_3F_2\left[\begin{matrix} a,\ c,\ e \\ b,\ d\end{matrix}\Big|1\right]=\Gamma\left[\begin{matrix} b,\ d,\ \lambda \\ a,\ c+\lambda, e+\lambda\end{matrix}\right]{}_3F_2\left[\begin{matrix} b-a, d-a,\ \lambda \\ c+\lambda, e+\lambda\end{matrix}\Big|1\right]$$

where $\lambda=b+d-a-c-e$. This transformation has further been employed by Milgram [14], where a systematic examination has been carried out for numerous ${}_3F_2(1)$-series identities.

Wimp [21] has shown that Watson's formula cannot be generalized into the series ${}_3F_2\left[\begin{smallmatrix} a,b,c \\ d,2c\end{smallmatrix}\Big|1\right]$ with four free parameters $\{a,b,c,d\}$ (see Zeilberger [22] for a shorter proof). Since then, attention has been turned to extending the series displayed in (1)–(3) by adjusting the parameters with specific integers m and n:

$${}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b+m, 1+a-c+n\end{matrix}\Big|1\right], {}_3F_2\left[\begin{matrix} a,\ b, & c \\ \frac{1+a+b+m}{2}, & 2c+n\end{matrix}\Big|1\right], {}_3F_2\left[\begin{matrix} a, 1-a+m, b \\ c, 1+2b-c+n\end{matrix}\Big|1\right].$$

When $m=n=0$, they reduce to the original Dixon–Watson–Whipple series. By utilizing contiguous relations for the ${}_3F_2$-series (cf. Rainville [15, §48]), Lavoie *et al.* [10, 11, 12] made an extensive investigation into these three series when m and n are small integers. Their experimental computation leads to a collection of 25 formulae for generalized Watson series (the second one just displayed) in [10] and 38 formulae for generalized Dixon series (the first one just displayed) in [11], where the latter ones are further translated, via the Thomae transformation, into 38 formulae for generalized Whipple series (the third one just displayed) in [12]. For limiting relations, different proofs and other related works, the reader can see the following papers [2, 3, 4, 9, 16, 17, 18].

Lewanowicz [13] made the first attempt to find an analytical formula for the last three series with m and n being general integer parameters, who derived general expressions in terms of three known formulae (1)–(3). However, for most of his formulae, the connection coefficients have to be determined implicitly by recurrence relations involving two variables, which are not convenient to practical evaluation.

The present paper aims at establishing analytical formulae explicitly for the generalized Watson–Whipple–Dixon series with two integer parameters m and n. In the next section, we shall prove four crucial preliminary lemmas that transform the generalized Watson series into Watson's original one (2). This will be accomplished

by combining the linearization method with the following summation theorem for terminating well–poised ${}_5F_4$-series due to Dougall [7] (cf. Bailey [1, §4.3]):

$$\left[\begin{matrix}1+a, 1+a-b-d\\ 1+a-b, 1+a-d\end{matrix}\right]_m = {}_5F_4\left[\begin{matrix}a,\ 1+\frac{a}{2}, & b, & d, & -m\\ & \frac{a}{2}, 1+a-b, & 1+a-d, & 1+a+m\end{matrix}\middle|1\right]. \tag{5}$$

According to the signs of m and n, four analytical formulae for the generalized Watson series with two extra integer parameters will be derived in Section 3. By means of the Thomae transformation (4), the generalized Whipple–Dixon series with two integer parameters m and n are converted to those for generalized Watson series in sections 4 and 5.

Based on our analytical formulae, the corresponding commands in *Mathematica* are written to evaluate the series of three classes (Watson, Whipple and Dixon) for any specific integers m and n. We are limited to the present twelve examples with four representatives for each class, even though it would be routine to produce a longer list such as those given by Lavoie *et al.* [10, 11, 12].

2. Four transformation lemmas

Define the two nonterminating ${}_3F_2$-series with four free parameters $\{a, b, c, d\}$ by

$$U_m(a,b,c,d) = {}_3F_2\left[\begin{matrix}a,\ b, & c\\ \frac{1+a+b+m}{2}, & d\end{matrix}\middle|1\right], \tag{6}$$

$$V_n(a,b,c,d) = {}_3F_2\left[\begin{matrix}a,\ b, & c\\ d, & 2c+n\end{matrix}\middle|1\right]. \tag{7}$$

Then Watson's series displayed in (2) results in the particular cases

$$U_0(a,b,c,2c) = V_0(a,b,c,\tfrac{1+a+b}{2}).$$

By means of the linearization method, we shall show four transformations for the series U_m and V_n with respect to the signs of m and n. They will be utilized, in the next section, to derive explicit formulae for the generalized Watson series with two integer parameter m and n.

2.1. **Transformation for U_m.** Observe that any $m+1$ polynomials of degree m in k are linearly dependent. There exist $m+1$ constants $\{X_m^i\}$ such that

$$1 = \sum_{i=0}^{m} X_m^i (a+k)_i (b+k)_{m-i}. \tag{8}$$

Equating the coefficients of k^j with $0 \le j \le m$, we get from (8) a system of equations in $\{X_m^i\}$, which can be resolved by the following *Mathematica* commands for any (not too large) specific natural number m:

```
pp[x_, n_] := Pochhammer[x, n]
uu[k_, m_] := Sum[xx[i, m]*pp[a+k, i]*pp[b + k, m - i], {i, 0, m}]
ee[m_]:=Table[Coefficient[uu[k,m],k,i] == If[i==0,1,0], {i, 0, m}]
ww[m_]:=Flatten[MapAll[Factor, Solve[ee[m], Table[xx[i,m], {i,0,m}]]]]
```

The first few outputs for "ww[m]" with $1 \le m \le 5$ have a nice pattern and suggest the following solution:

$$X_m^i = (-1)^i \binom{m}{i} \frac{b-a+m-2i}{(b-a-i)_{m+1}} \quad \text{for} \quad i = 0, 1, \cdots, m. \tag{9}$$

This is equivalent to the equality

$$\frac{(b-a)_m}{(b+k)_m} = {}_4F_3\left[\begin{matrix} a-b-m, 1+\frac{a-b-m}{2}, a+k, -m \\ \frac{a-b-m}{2}, 1-b-k-m, 1+a-b \end{matrix}\Big| -1\right], \tag{10}$$

which can be justified by the limiting case of Dougall's theorem for the ${}_5F_4$–series displayed in (5), specified by $a \to a-b-m$, $b \to a+k$ and $d \to \infty$.

Now considering the composite series and then interchanging the summation order, we have the double series expression

$$\begin{aligned} U_m(a,b,c,d) &= \sum_{k\geq 0}\left[\begin{matrix} a,\ b,\ c \\ 1, \frac{1+a+b+m}{2}, d \end{matrix}\right]_k \sum_{i=0}^{m} X_m^i (a+k)_i (b+k)_{m-i} \\ &= \sum_{i=0}^{m} X_m^i \sum_{k\geq 0}\left[\begin{matrix} a,\ b,\ c \\ 1, \frac{1+a+b+m}{2}, d \end{matrix}\right]_k (a+k)_i (b+k)_{m-i} \\ &= \sum_{i=0}^{m} X_m^i (a)_i (b)_{m-i} \sum_{k\geq 0}\left[\begin{matrix} a+i, b+m-i, c \\ 1, \frac{1+a+b+m}{2}, d \end{matrix}\right]_k \end{aligned}$$

which gives rise to the following equation:

$$U_m(a,b,c,d) = \sum_{i=0}^{m} X_m^i (a)_i (b)_{m-i} U_0(a+i, b+m-i, c, d). \tag{11}$$

Substituting the explicit expression of X_m^i into the last equation and then simplifying the result, we get the first transformation formula.

Lemma 1 ($m \in \mathbb{N}_0$)**.**

$$\begin{aligned} U_m(a,b,c,d) = \frac{(b)_m}{(b-a)_m} \sum_{i=0}^{m} \binom{m}{i} U_0(a+i, b+m-i, c, d) \\ \times \frac{a-b-m+2i}{a-b-m}\left[\begin{matrix} a, a-b-m \\ 1+a-b, 1-b-m \end{matrix}\right]_i. \end{aligned}$$

The special case $d = 2c$ of this lemma has been discovered by Lewanowicz [13, Equation 2.2]. To our knowledge, this is the only case that the explicit coefficients are previously determined for expressing the generalized Watson series with two integer parameters m and n in terms of Watson's original one (2).

2.2. **Transformation for U_{-m}.** In this case, there exist $m+1$ constants $\{\mathcal{X}_m^i\}$ independent of k such that

$$\Big(\frac{1+a+b-m}{2}+k\Big)_m = \sum_{i=0}^{m} \mathcal{X}_m^i (a+k)_i (b+k)_{m-i}. \tag{12}$$

Analogously, with the help of the *Mathematica* commands below,

```
uu[k_,m_]:=pp[k+(1+a+b-m)/2,m]-Sum[xx[i,m]*pp[a+k,i]*pp[b+k,m-i],{i,0,m}]
ee[m_] := Table[Coefficient[uu[k, m], k, i] == 0, {i, 0, m}]
ww[m_] := Flatten[MapAll[Factor, Solve[ee[m], Table[xx[i, m], {i,0,m}]]]]
```

the coefficients $\mathcal{X}_m^i$ are detected (but to be verified) as follows:

$$\mathcal{X}_m^i = \Big(\frac{1-a+b-m}{2}\Big)_m \binom{m}{i} \frac{b-a+m-2i}{(b-a-i)_{m+1}} \quad \text{for} \quad i = 0, 1, \cdots, m. \tag{13}$$

This is equivalent to the equality

$$\left[\begin{matrix} b-a, \frac{1+a+b-m}{2}+k \\ b+k, \frac{1-a+b-m}{2} \end{matrix}\right]_m = {}_4F_3\left[\begin{matrix} a-b-m, 1+\frac{a-b-m}{2}, a+k, -m \\ \frac{a-b-m}{2}, 1-b-k-m, 1+a-b \end{matrix}\Big|1\right], \tag{14}$$

which turns out to be a special case of Dougall's formula (5) under the parameter specifications $a \to a-b-m$, $b \to a+k$ and $d \to (1+a-b-m)/2$.

Then we can manipulate the composite series

$$\begin{aligned} U_{-m}(a,b,c,d) &= \sum_{k\geq 0}\left[\begin{matrix} a,\ b,\ c \\ 1, \frac{1+a+b-m}{2}, d \end{matrix}\right]_k \sum_{i=0}^{m} \mathcal{X}_m^i \frac{(a+k)_i(b+k)_{m-i}}{\left(\frac{1+a+b-m}{2}+k\right)_m} \\ &= \sum_{i=0}^{m} \mathcal{X}_m^i \sum_{k\geq 0}\left[\begin{matrix} a,\ b,\ c \\ 1, \frac{1+a+b-m}{2}, d \end{matrix}\right]_k \frac{(a+k)_i(b+k)_{m-i}}{\left(\frac{1+a+b-m}{2}+k\right)_m} \\ &= \sum_{i=0}^{m} \mathcal{X}_m^i \frac{(a)_i(b)_{m-i}}{\left(\frac{1+a+b-m}{2}\right)_m} \sum_{k\geq 0}\left[\begin{matrix} a+i, b+m-i, c \\ 1, \frac{1+a+b+m}{2}, d \end{matrix}\right]_k, \end{aligned}$$

which results in the equation

$$U_{-m}(a,b,c,d) = \sum_{i=0}^{m} \mathcal{X}_m^i \frac{(a)_i(b)_{m-i}}{\left(\frac{1+a+b-m}{2}\right)_m} U_0(a+i, b+m-i, c, d). \tag{15}$$

Substituting the explicit expression of $\mathcal{X}_m^i$ into the last equation and then simplifying the result, we find the second transformation formula.

Lemma 2 ($m \in \mathbb{N}_0$)**.**

$$\begin{aligned} U_{-m}(a,b,c,d) = \left[\begin{matrix} b, \frac{1-a+b-m}{2} \\ b-a, \frac{1+a+b-m}{2} \end{matrix}\right]_m \sum_{i=0}^{m}(-1)^i\binom{m}{i} U_0(a+i, b+m-i, c, d) \\ \times \frac{a-b-m+2i}{a-b-m}\left[\begin{matrix} a, a-b-m \\ 1+a-b, 1-b-m \end{matrix}\right]_i. \end{aligned}$$

2.3. **Transformation for V_n.** Now that any $n+1$ polynomials of degree n in k are linearly dependent, there exist $n+1$ constants $\{Y_n^j\}$ such that

$$1 = \sum_{j=0}^{n} Y_n^j \langle k\rangle_j \langle 2c+n+k-1\rangle_{n-j}. \tag{16}$$

Equating the coefficients of k^i with $0 \leq i \leq n$, the *Mathematica* commands

```
qq[x_, n_] := (-1)^n*Pochhammer[-x, n]
vv[k_, n_] := Sum[yy[j, n]*qq[k, j]*pp[2c + k + j, n - j], {j, 0, n}]
ee[n_] :=Table[Coefficient[vv[k, n], k, j] == If[j == 0,1,0], {j, 0, n}]
ww[n_] :=Flatten[MapAll[Factor, Solve[ee[n], Table[yy[j, n], {j,0,n}]]]]
```

reveal the following experimental solution:

$$Y_n^j = (-1)^j\binom{n}{j}\frac{2j-1+2c}{(j-1+2c)_{n+1}} \quad \text{for} \quad j = 0, 1, \cdots, n. \tag{17}$$

This is equivalent to the equality

$$\frac{(2c)_n}{(2c+k)_n} = {}_4F_3\left[\begin{matrix} 2c-1, c+\frac{1}{2}, -k, -n \\ c-\frac{1}{2}, 2c+k, 2c+n \end{matrix}\Big| -1\right], \tag{18}$$

which can be justified again by the limiting case of Dougall's sum (5), corresponding to the parameter settings $a \to 2c-1$, $b \to -k$ and $d \to \infty$.

Now considering the composite series and then interchanging the summation order, we have the double series expression

$$\begin{aligned} V_n(a,b,c,d) &= \sum_{k\geq 0}\begin{bmatrix} a,\ b, & c \\ 1,\ d, & 2c+n \end{bmatrix}_k \sum_{j=0}^{n} Y_n^j \langle k\rangle_j \langle 2c+n+k-1\rangle_{n-j} \\ &= \sum_{j=0}^{n} Y_n^j \sum_{k\geq 0}\begin{bmatrix} a,\ b, & c \\ 1,\ d, & 2c+n \end{bmatrix}_k \langle k\rangle_j \langle 2c+n+k-1\rangle_{n-j} \\ &= \sum_{j=0}^{n} Y_n^j \frac{(a)_j(b)_j(c)_j}{(d)_j(2c+n)_{2j-n}} \sum_{k\geq 0}\begin{bmatrix} a+j,\ b+j, & c+j \\ 1,\ d+j, & 2c+2j \end{bmatrix}_k, \end{aligned}$$

which results in the equation

$$V_n(a,b,c,d) = \sum_{j=0}^{n} Y_n^j \frac{(a)_j(b)_j(c)_j}{(d)_j(2c+n)_{2j-n}} V_0(a+j,b+j,c+j,d+j). \tag{19}$$

Substituting the explicit expression of Y_n^j into the last equation and then simplifying the result, we have the third transformation formula.

Lemma 3 ($n \in \mathbb{N}_0$)**.**

$$V_n(a,b,c,d) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(2c-1)_j}{(2c-1)_{2j}} \begin{bmatrix} a,b,c \\ d,2c+n \end{bmatrix}_j V_0(a+j,b+j,c+j,d+j).$$

2.4. **Transformation for V_{-n}.** Analogously, there exist $n+1$ constants $\{\mathcal{Y}_n^j\}$ independent of k such that

$$(c-n+k)_n = \sum_{j=0}^{n} \mathcal{Y}_n^j \langle k\rangle_j \langle 2c-n+k-1\rangle_{n-j}. \tag{20}$$

By utilizing the *Mathematica* commands

```
vv[k_, n_] := pp[c-n+k,n]-Sum[yy[j,n]*qq[k,j]*pp[2c-2n+k+j,n-j], {j,0,n}]
ee[n_] := Table[Coefficient[vv[k, n], k, j] == 0, {j, 0, n}]
ww[n_] := Flatten[MapAll[Factor, Solve[ee[n], Table[yy[j, n], {j,0,n}]]]]
```

we can similarly figure out the coefficients $\mathcal{Y}_n^j$ as follows:

$$\mathcal{Y}_n^j = \binom{n}{j} \frac{2c+2j-2n-1}{2c+j-n-1} \frac{(1-c)_n}{(2+n-j-2c)_n} \quad \text{for} \quad j = 0,1,\cdots,n. \tag{21}$$

This is equivalent to the equality

$$\begin{bmatrix} c-n+k, 2c-2n \\ c-n, 2c-2n+k \end{bmatrix}_n = {}_4F_3\left[\begin{matrix} 2c-2n-1, c-n+\frac{1}{2}, -k, -n \\ c-n-\frac{1}{2}, 2c-2n+k, 2c-n \end{matrix}\middle| 1\right], \tag{22}$$

which is, in fact, a particular case of Dougall's theorem (5) with the parameters being specified by $a \to 2c-2n-1$, $b \to -k$ and $d \to c-n$.

For the following composite series, it can be reformulated as

$$\begin{aligned} V_{-n}(a,b,c,d) &= \sum_{k\geq 0}\begin{bmatrix} a,\ b, & c \\ 1,\ d, & 2c-n \end{bmatrix}_k \sum_{j=0}^{n} \mathcal{Y}_n^j \frac{\langle k\rangle_j \langle 2c-n+k-1\rangle_{n-j}}{(c-n+k)_n} \\ &= \sum_{j=0}^{n} \mathcal{Y}_n^j \sum_{k\geq 0}\begin{bmatrix} a,\ b, & c \\ 1,\ d, & 2c-n \end{bmatrix}_k \frac{\langle k\rangle_j \langle 2c-n+k-1\rangle_{n-j}}{(c-n+k)_n} \\ &= \sum_{j=0}^{n} \mathcal{Y}_n^j \frac{(a)_j(b)_j(c)_{j-n}}{(d)_j(2c-n)_{2j-n}} \sum_{k\geq 0}\begin{bmatrix} a+j,\ b+j, & c+j-n \\ 1,\ d+j, & 2c+2j-2n \end{bmatrix}_k, \end{aligned}$$

which leads to the equation

$$V_{-n}(a,b,c,d) = \sum_{j=0}^{n} \mathcal{Y}_n^j \frac{(a)_j(b)_j(c)_{j-n}}{(d)_j(2c-n)_{2j-n}} V_0(a+j,b+j,c+j-n,d+j). \tag{23}$$

Substituting the explicit expression of $\mathcal{Y}_n^j$ into the last equation and then simplifying the result, we obtain the fourth transformation formula.

Lemma 4 ($n \in \mathbb{N}_0$)**.**

$$V_{-n}(a,b,c,d) = \sum_{j=0}^{n} \binom{n}{j} \frac{(2c-2n-1)_j}{(2c-2n-1)_{2j}} \begin{bmatrix} a, b, c-n \\ d, 2c-n \end{bmatrix}_j V_0(a+j,b+j,c+j-n,d+j).$$

3. Extension of Watson's ${}_3F_2$–series

For the two integer parameters m and n, the extended Watson series is defined by

$$W_{m,n}(a,b,c) := {}_3F_2\left[\begin{matrix} a,\ b, & c \\ \frac{1+a+b+m}{2}, & 2c+n \end{matrix}\middle| 1\right] \tag{24}$$

which reduces to Watson's original one displayed in (2) when $m = n = 0$. According to Lemmas 1 and 2, the series

$$W_{m,n}(a,b,c) = U_m(a,b,c,2c+n) \tag{25}$$

can be written as the linear sum of

$$U_0(a',b',c',2c'+n) = V_n(a',b',c',\tfrac{1+a'+b'}{2}) \tag{26}$$

which can further be expressed, in view of Lemmas 3 and 4, in terms of Watson's original series

$$W_{0,0}(a'',b'',c'') = V_0(a'',b'',c'',\tfrac{1+a''+b''}{2}). \tag{27}$$

This leads to the double sum expression of $W_{m,n}(a,b,c)$ in $(|m|+1)(|n|+1)$ terms of $W_{0,0}(a'',b'',c'')$, which permits us to evaluate $W_{m,n}(a,b,c)$ for any specific integers m and n. The results are displayed in the following four theorems with respect to the signs of m and n.

For $m \geq 0$ and $n \geq 0$, combining Lemma 1 with Lemma 3 yields the first summation theorem, which expresses $W_{m,n}$ explicitly in terms of $W_{0,0}$.

Theorem 5 ($m,\ n\in\mathbb{N}_0$)**.**

$$W_{m,n}(a,b,c)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\frac{a-b-m+2i}{a-b-m+i}\frac{(a)_{i+j}(b)_{m-i+j}}{(b-a-i)_m(2c-1)_{2j}}$$
$$\times\left[\begin{matrix}c,\ 2c-1\\ \frac{1+a+b+m}{2},2c+n\end{matrix}\right]_j W_{0,0}(a+i+j,b+m-i+j,c+j).$$

For $m\geq 0$ and $n\leq 0$, combining Lemma 1 with Lemma 4 gives the second summation theorem, which expresses $W_{m,n}$ explicitly in terms of $W_{0,0}$.

Theorem 6 ($m,\ n\in\mathbb{N}_0$)**.**

$$W_{m,-n}(a,b,c)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i}\binom{m}{i}\binom{n}{j}\frac{a-b-m+2i}{a-b-m+i}\frac{(a)_{i+j}(b)_{m-i+j}}{(b-a-i)_m(2c-2n-1)_{2j}}$$
$$\times\left[\begin{matrix}c-n,2c-2n-1\\ \frac{1+a+b+m}{2},2c-n\end{matrix}\right]_j W_{0,0}(a+i+j,b+m-i+j,c+j-n).$$

For $m\leq 0$ and $n\geq 0$, combining Lemma 2 with Lemma 3 results in the third summation theorem, which expresses $W_{m,n}$ explicitly in terms of $W_{0,0}$.

Theorem 7 ($m,\ n\in\mathbb{N}_0$)**.**

$$W_{-m,n}(a,b,c)=\frac{(\frac{1-a+b-m}{2})_m}{(\frac{1+a+b-m}{2})_m}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{j}\binom{m}{i}\binom{n}{j}\frac{a-b-m+2i}{a-b-m+i}\frac{(b)_{m-i+j}}{(b-a-i)_m}$$
$$\times\frac{(a)_{i+j}}{(2c-1)_{2j}}\left[\begin{matrix}c,2c-1\\ \frac{1+a+b+m}{2},2c+n\end{matrix}\right]_j W_{0,0}(a+i+j,b+m-i+j,c+j).$$

For $m\leq 0$ and $n\leq 0$, combining Lemma 2 with Lemma 4 leads to the fourth summation theorem, which expresses $W_{m,n}$ explicitly in terms of $W_{0,0}$.

Theorem 8 ($m,\ n\in\mathbb{N}_0$)**.**

$$W_{-m,-n}(a,b,c)=\frac{(\frac{1-a+b-m}{2})_m}{(\frac{1+a+b-m}{2})_m}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{a-b-m+2i}{a-b-m+i}\left[\begin{matrix}c-n,2c-2n-1\\ \frac{1+a+b+m}{2},2c-n\end{matrix}\right]_j$$
$$\times\frac{(a)_{i+j}(b)_{m-i+j}}{(b-a-i)_m(2c-2n-1)_{2j}}W_{0,0}(a+i+j,b+m-i+j,c+j-n).$$

These four analytical formulae constitute a complete scheme to evaluate the generalized Watson series $W_{m,n}$ for any two specific integers m and n. Four formulae are presented here as exemplification.

Example 9.

$$\begin{aligned}W_{1,1}(a,b,c)\ &=\ \frac{2^{4c-2}\Gamma(1+\frac{a+b}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{a+b}{2})/(a-b)}{\pi^{3/2}\Gamma(a)\Gamma(b)\Gamma(1+2c-a)\Gamma(1+2c-b)}\\ &\times\ \Big[(2c-a+b)\Gamma\Big(\frac{1+a}{2}\Big)\Gamma\Big(\frac{b}{2}\Big)\Gamma\Big(\frac{1}{2}+c-\frac{a}{2}\Big)\Gamma\Big(1+c-\frac{b}{2}\Big)\\ &\quad-(2c+a-b)\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{1+b}{2}\Big)\Gamma\Big(1+c-\frac{a}{2}\Big)\Gamma\Big(\frac{1}{2}+c-\frac{b}{2}\Big)\Big].\end{aligned}$$

Example 10.

$$\begin{aligned} W_{1,-1}(a,b,c) &= \frac{2^{4c-5}\Gamma(1+\frac{a+b}{2})\Gamma(c-\frac{1}{2})\Gamma(c-\frac{a+b}{2})/(a-b)}{\pi^{3/2}\Gamma(a)\Gamma(b)\Gamma(2c-a-1)\Gamma(2c-b-1)} \\ &\times \Big[\Gamma\Big(\frac{1+a}{2}\Big)\Gamma\Big(\frac{b}{2}\Big)\Gamma\Big(c-\frac{1+a}{2}\Big)\Gamma\Big(c-\frac{b}{2}\Big) \\ &\quad -\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{1+b}{2}\Big)\Gamma\Big(c-\frac{a}{2}\Big)\Gamma\Big(c-\frac{1+b}{2}\Big)\Big]. \end{aligned}$$

Example 11.

$$\begin{aligned} W_{-1,1}(a,b,c) &= \frac{2^{4c-2}\Gamma(\frac{a+b}{2})\Gamma(c+\frac{1}{2})\Gamma(1+c-\frac{a+b}{2})}{\pi^{3/2}\Gamma(a)\Gamma(b)\Gamma(1+2c-a)\Gamma(1+2c-b)} \\ &\times \Big[\Gamma\Big(\frac{1+a}{2}\Big)\Gamma\Big(\frac{b}{2}\Big)\Gamma\Big(\frac{1}{2}+c-\frac{a}{2}\Big)\Gamma\Big(1+c-\frac{b}{2}\Big) \\ &\quad +\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{1+b}{2}\Big)\Gamma\Big(1+c-\frac{a}{2}\Big)\Gamma\Big(\frac{1}{2}+c-\frac{b}{2}\Big)\Big]. \end{aligned}$$

Example 12.

$$\begin{aligned} W_{-1,-1}(a,b,c) &= \frac{2^{4c-7}\Gamma(\frac{a+b}{2})\Gamma(c-\frac{1}{2})\Gamma(c-1-\frac{a+b}{2})}{\pi^{3/2}\Gamma(a)\Gamma(b)\Gamma(2c-a-1)\Gamma(2c-b-1)} \\ &\times \Big[(2c-a+b-2)\Gamma\Big(\frac{1+a}{2}\Big)\Gamma\Big(\frac{b}{2}\Big)\Gamma\Big(c-\frac{1+a}{2}\Big)\Gamma\Big(c-\frac{b}{2}\Big) \\ &\quad +(2c+a-b-2)\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{1+b}{2}\Big)\Gamma\Big(c-\frac{a}{2}\Big)\Gamma\Big(c-\frac{1+b}{2}\Big)\Big]. \end{aligned}$$

4. Extension of Whipple's ${}_3F_2$–series

For the two integer parameters m and n, define the extended Whipple series by

$$\Omega_{m,n}(a,b,c) = {}_3F_2\left[\begin{matrix} a, 1-a+m, b \\ c, 1+2b-c+n \end{matrix}\Big|1\right] \tag{28}$$

which reduces to Whipple's original one displayed in (3) when $m=n=0$. Applying the Thomae transformation (4), we have

$$\begin{aligned} \Omega_{m,n}(a,b,c) &= \Gamma\left[\begin{matrix} c, b-m+n, 1+2b-c+n \\ a, 1-a+b+n, 2b-m+n \end{matrix}\right] \\ &\quad \times {}_3F_2\left[\begin{matrix} c-a,\ 1+2b-a-c+n,\ b-m+n \\ 1-a+b+n,\quad 2b-m+n \end{matrix}\Big|1\right]. \end{aligned} \tag{29}$$

Identifying the last ${}_3F_2$-series with $W_{n,m-n}$, we get the following relation:

$$\begin{aligned} \Omega_{m,n}(a,b,c) &= \Gamma\left[\begin{matrix} c, b-m+n, 1+2b-c+n \\ a, 1-a+b+n, 2b-m+n \end{matrix}\right] \\ &\quad \times W_{n,m-n}(c-a, 1+2b-a-c+n, b-m+n). \end{aligned} \tag{30}$$

This transformation formula converts the generalized Whipple series into the generalized Watson series. Combining it with Theorems 5–8, we can evaluate the

generalized Whipple series $\Omega_{m,n}(a,b,c)$ for any specific integers m and n. Four examples are displayed below.

Example 13.

$$\begin{aligned}\Omega_{1,1}(a,b,c) &= \frac{2^{2b-3}\Gamma(c)\Gamma(2b-c+2)/\{(1-a)(1+b-c)\}}{\pi\Gamma(a+c-2)\Gamma(c-a)\Gamma(a+2b-c)\Gamma(2+2b-a-c)}\\ &\times \Big[\Gamma\Big(\frac{a+c-2}{2}\Big)\Gamma\Big(\frac{1+c-a}{2}\Big)\Gamma\Big(b+\frac{1+a-c}{2}\Big)\Gamma\Big(b+\frac{2-a-c}{2}\Big)\\ &\quad -\Gamma\Big(\frac{a+c-1}{2}\Big)\Gamma\Big(\frac{c-a}{2}\Big)\Gamma\Big(b+\frac{a-c}{2}\Big)\Gamma\Big(b+\frac{3-a-c}{2}\Big)\Big].\end{aligned}$$

Example 14.

$$\begin{aligned}\Omega_{-1,-1}(a,b,c) &= \frac{2^{2b-3}\Gamma(c)\Gamma(2b-c)}{\pi\Gamma(a+c)\Gamma(c-a)\Gamma(a+2b-c)\Gamma(2b-a-c)}\\ &\times \Big[\Gamma\Big(\frac{c-a}{2}\Big)\Gamma\Big(\frac{1+a+c}{2}\Big)\Gamma\Big(b+\frac{1-a-c}{2}\Big)\Gamma\Big(b+\frac{a-c}{2}\Big)\\ &\quad +\Gamma\Big(\frac{a+c}{2}\Big)\Gamma\Big(\frac{1-a+c}{2}\Big)\Gamma\Big(b+\frac{1+a-c}{2}\Big)\Gamma\Big(b-\frac{a+c}{2}\Big)\Big].\end{aligned}$$

Example 15.

$$\begin{aligned}\Omega_{2,1}(a,b,c) &= \frac{2^{2b-4}\Gamma(c)\Gamma(2b-c+2)/\{(a-1)(a-2)(b-1)(b-c+1)\}}{\pi\Gamma(a+c-3)\Gamma(c-a)\Gamma(a+2b-c-1)\Gamma(2b-a-c+2)}\\ &\times \Big[(c-2b)\Gamma\Big(\frac{a+c-2}{2}\Big)\Gamma\Big(\frac{1-a+c}{2}\Big)\Gamma\Big(b+\frac{a-c-1}{2}\Big)\Gamma\Big(b+\frac{2-a-c}{2}\Big)\\ &\quad +(c-2)\,\Gamma\Big(\frac{a+c-3}{2}\Big)\Gamma\Big(\frac{c-a}{2}\Big)\Gamma\Big(b+\frac{a-c}{2}\Big)\Gamma\Big(b+\frac{3-a-c}{2}\Big)\Big].\end{aligned}$$

Example 16.

$$\begin{aligned}\Omega_{-2,-1}(a,b,c) &= \frac{2^{2b-2}\Gamma(c)\Gamma(2b-c)}{\pi\Gamma(a+c+1)\Gamma(c-a)\Gamma(a+2b-c+1)\Gamma(2b-a-c)}\\ &\times \Big[(2b-c)\Gamma\Big(\frac{2+a+c}{2}\Big)\Gamma\Big(\frac{1+c-a}{2}\Big)\Gamma\Big(b+\frac{1+a-c}{2}\Big)\Gamma\Big(b-\frac{a+c}{2}\Big)\\ &\quad +c\,\Gamma\Big(\frac{1+a+c}{2}\Big)\Gamma\Big(\frac{c-a}{2}\Big)\Gamma\Big(b+\frac{2+a-c}{2}\Big)\Gamma\Big(b+\frac{1-a-c}{2}\Big)\Big].\end{aligned}$$

5. Almost–poised ${}_3F_2$–series

For the two integer parameters m and n, define the almost–poised ${}_3F_2$-series by

$$\mathcal{D}_{m,n}(a,b,c) = {}_3F_2\left[\begin{matrix}a, & b, & c\\ & 1+a-b+m, & 1+a-c+n\end{matrix}\Big|1\right] \tag{31}$$

which reduces to Dixon's original one displayed in (1) when $m = n = 0$. Applying the Thomae transformation (4), we get

$$\begin{aligned}\mathcal{D}_{m,n}(a,b,c) &= \Gamma\left[\begin{matrix}1+a-b+m, 1+a-c+n, 2+a-2b-2c+m+n\\ b, 2+a-2b-c+m+n, 2+2a-2b-2c+m+n\end{matrix}\right]\\ &\times {}_3F_2\left[\begin{matrix}1+a-2b+m, 2+a-2b-2c+m+n, 1+a-b-c+n\\ 2+a-2b-c+m+n, 2+2a-2b-2c+m+n\end{matrix}\middle|1\right].\end{aligned} \tag{32}$$

Identifying the last ${}_3F_2$-series with $W_{n,m-n}$, we obtain the following relation:

$$\begin{aligned}\mathcal{D}_{m,n}(a,b,c) &= \Gamma\left[\begin{matrix}1+a-b+m, 1+a-c+n, 2+a-2b-2c+m+n\\ b, 2+a-2b-c+m+n, 2+2a-2b-2c+m+n\end{matrix}\right]\\ &\times W_{n,m-n}(1+a-2b+m, 2+a-2b-2c+m+n, 1+a-b-c+n).\end{aligned} \tag{33}$$

This transformation formula converts the almost–poised ${}_3F_2$-series into the generalized Watson series again. Combining it with Theorems 5–8, we can evaluate the almost–poised ${}_3F_2$-series $\mathcal{D}_{m,n}(a,b,c)$ for any specific integers m and n. Four formulae are exhibited as examples.

Example 17.

$$\begin{aligned}\mathcal{D}_{1,1}(a,b,c) &= \frac{2^{1+2a-2b-2c}\Gamma(a-b+2)\Gamma(a-c+2)/\{(b-1)(1-c)\}}{\pi\Gamma(a)\Gamma(a-2b+2)\Gamma(a-2c+2)\Gamma(a-b-c+2)}\\ &\times \Big[\Gamma\Big(\frac{1+a}{2}\Big)\Gamma\Big(\frac{2+a}{2}-b\Big)\Gamma\Big(\frac{2+a}{2}-c\Big)\Gamma\Big(\frac{5+a}{2}-b-c\Big)\\ &\quad -\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{3+a}{2}-b\Big)\Gamma\Big(\frac{3+a}{2}-c\Big)\Gamma\Big(\frac{4+a}{2}-b-c\Big)\Big].\end{aligned}$$

Example 18.

$$\begin{aligned}\mathcal{D}_{-1,-1}(a,b,c) &= \frac{2^{2a-2b-2c-3}\Gamma(a-b)\Gamma(a-c)}{\pi\Gamma(a)\Gamma(a-2b)\Gamma(a-2c)\Gamma(a-b-c)}\\ &\times \Big[\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{1+a}{2}-b\Big)\Gamma\Big(\frac{1+a}{2}-c\Big)\Gamma\Big(\frac{a}{2}-b-c\Big)\\ &\quad +\Gamma\Big(\frac{a+1}{2}\Big)\Gamma\Big(\frac{a}{2}-b\Big)\Gamma\Big(\frac{a}{2}-c\Big)\Gamma\Big(\frac{1+a}{2}-b-c\Big)\Big].\end{aligned}$$

Example 19.

$$\begin{aligned}\mathcal{D}_{2,2}(a,b,c) &= \frac{4^{b+c-a-3}\pi\Gamma(a-b+3)\Gamma(a-c+3)\Gamma(a-2b-2c+6)}{(b-1)(b-2)(c-2)(c-1)\Gamma(a-b-c+3)}\\ &\times \Bigg[\frac{a^2-2ab-2ac+2bc+5a-2b-2c+2}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{4+a}{2}-b\right)\Gamma\left(\frac{4+a}{2}-c\right)\Gamma\left(\frac{7+a}{2}-b-c\right)}\\ &\quad -\frac{4}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{3+a}{2}-b\right)\Gamma\left(\frac{3+a}{2}-c\right)\Gamma\left(\frac{6+a}{2}-b-c\right)}\Bigg].\end{aligned}$$

Example 20.

$$\begin{aligned}\mathcal{D}_{-2,-2}(a,b,c) &= \frac{4^{1+b+c-a}\pi\Gamma(a-b-1)\Gamma(a-c-1)\Gamma(a-2b-2c-2)}{\Gamma(a-b-c-1)}\\ &\times \left[\frac{a^2-2ab-2ac+2bc-3a+2b+2c+2}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b\right)\Gamma\left(\frac{a}{2}-c\right)\Gamma\left(\frac{a-1}{2}-b-c\right)}\right.\\ &\left.+\frac{4}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a-1}{2}-b\right)\Gamma\left(\frac{a-1}{2}-c\right)\Gamma\left(\frac{a-2}{2}-b-c\right)}\right].\end{aligned}$$

Before concluding the paper, we would like to point out that there exists a common generalization of bilateral series due to M. Jackson [8] for both Watson's theorem (2) and Whipple's theorem (3). The interested reader is encouraged to pursue the corresponding bilateral counterparts for the formulae displayed in this paper.

Acknowledgement

The author is sincerely grateful to the anonymous referee for the critical comments and valuable suggestions, that helped, in particular, to improve the presentation of the four transformation lemmas in Section 2. Thanks are also due to Dr. Xiaojing Chen for her careful proofreading.

References

[1] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935.

[2] S. M.P. Chaudhary, *Certain summation formulae associated to Gauss second summation theorem*, Global Journal of Science Frontier Research 10:3 (2010), 30–35.

[3] J. Choi, A. K. Rathie, S. Malani, *A summation formula of* ${}_6F_5(1)$, Commun. Korean Math. Soc. 19:4 (2004), 775–778. MR2096599 (2005f:33023)

[4] J. Choi, A. K. Rathie, B. Bhojak, *On Preece's identity and other contiguous results*, Commun. Korean Math. Soc. 20:1 (2005), 169–178. MR2167086 (2006j:33015)

[5] W. Chu, *Inversion techniques and combinatorial identities: A quick introduction to hypergeometric evaluations*, Math. Appl. 283 (1994), 31–57. MR1292876 (96a:33006)

[6] A. C. Dixon, *Summation of a certain series*, Proc. London Math. Soc. (1) 35 (1903), 285–289.

[7] J. Dougall, *On Vandermonde's theorem and some more general expansions*, Proc. Edinburgh Math. Soc. **25** (1907), 114–132.

[8] M. Jackson, *A generalization of the theorems of Watson and Whipple on the sum of the series* ${}_3F_2$, J. London Math. Soc. 24 (1949), 238–240. MR0032049 (11:246c)

[9] J. L. Lavoie, *Some summation formulas for the series* ${}_3F_2(1)$, Math. Comp. 49:179 (1987), 269–274. MR890268 (88f:33002)

[10] J. L. Lavoie, F. Grondin, A. K. Rathie, *Generalizations of Watson's theorem on the sum of a* ${}_3F_2$, Indian J. Math. 34:1 (1992), 23–32. MR1211517 (94c:33020)

[11] J. L. Lavoie, F. Grondin, A. K. Rathie, K. Arora, *Generalizations of Dixon's theorem on the sum of a* ${}_3F_2$, Math. Comp. 62:205 (1994), 267–276. MR1185246 (94e:33004)

[12] J. L. Lavoie, F. Grondin, A. K. Rathie, *Generalizations of Whipple's theorem on the sum of a* ${}_3F_2$, J. Comput. Appl. Math. 72:2 (1996), 293–300. MR1406215 (97j:33005)

[13] S. Lewanowicz, *Generalized Watson's summation formula for* ${}_3F_2(1)$, J. Comput. Appl. Math. 86 (1997), 375–386. MR1491313 (98m:33014)

[14] M. Milgram, *On hypergeometric* ${}_3F_2(1)$*, A review*, ArXiv:math.CA/0603096; Updated version, 2009.

[15] E. D. Rainville, *Special functions*, Chelsea Publishing Co., Bronx, N.Y., 1971. MR0393590 (52:14399)

[16] A. K. Rathie, M. A. Rakha, *A study of new hypergeometric transformations*, J. Phys. A: Math. Theor. 41 (2008), #445202 (15pp). MR2453177 (2010a:33008)

[17] T. K. Pogany, A. K. Rathie, *Another derivation method of summation formulae for the series* ${}_3F_2(-1)$ *and* ${}_4F_3(1)$, Math. Maced. 5 (2007), 63–67. MR2721608

[18] A. K. Rathie, R. B. Paris, *A New Proof of Watson's Theorem for the Series* ${}_3F_2(1)$, Appl. Math. Sci. 3:4 (2009), 161–164. MR2480594 (2010e:33009)

[19] G. N. Watson, *A note on generalized hypergeometric series*, Proc. London Math. Soc. (2) 23 (1925), xiii-xv.

[20] F. J. W. Whipple, *A group of generalized hypergeometric series: relations between 120 allied series of the type* $F[a, b, c; d, e]$, Proc. London Math. Soc. (2) 23 (1925), 104-114.

[21] J. Wimp, *Irreducible recurrences and representation theorems for* ${}_3F_2(1)$, Int. J. Comput. Math. Appl. 9 (1983), 663–678. MR726815 (85b:33005)

[22] D. Zeilberger, *Gauss's* ${}_2F_1(1)$ *cannot be generalized to* ${}_2F_1(x)$ J. Comput. Appl. Math. 39 (1992), 379–382. MR1164298 (93i:33002)

Hangzhou Normal University, Institute of Combinatorial Mathematics, Hangzhou 310036, People's Republic of China

Current address: Dipartimento di Matematica, Università del Salento, Lecce–Arnesano, P.O. Box 193, Lecce 73100, Italy

E-mail address: chu.wenchang@unisalento.it