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EXPLICIT COMPUTATIONS ON THE DESINGULARIZED KUMMER SURFACE

V. G. LOPEZ NEUMANN AND CONSTANTIN MANOIL

ABSTRACT. We find formulas for the birational maps from a Kummer surface $\mathcal{K}$ and its dual $\mathcal{K}^*$ to their common minimal desingularization $\mathcal{S}$. We show how the nodes of $\mathcal{K}$ and $\mathcal{K}^*$ blow up. Then we give a description of the group of linear automorphisms of $\mathcal{S}$.

1. INTRODUCTION

In the 19th century a singular surface $\mathcal{K}$, called the Kummer surface, was attached to a quadratic line complex. A minimal desingularization Σ of $\mathcal{K}$ and a birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ were constructed by geometric methods. One may call this the *classical* construction of the Kummer surface, which we recall in Section 4.

Another construction is the following: Let A be an abelian surface, and let σ be the involution of A given by $\sigma(x) = -x$. The quotient $\mathcal{K} = A/\sigma$ has 16 double points, and one defines a K3 surface $\mathcal{S}$ to be $\mathcal{K}$ with these 16 nodes blown up ([1], Prop. 8.11). To be consistent with the historical point of view and with our main reference [3], we call $\mathcal{S}$ the *desingularized* Kummer surface. If $A = \mathcal{J}(\mathcal{C})$, where $\mathcal{J}(\mathcal{C})$ is the Jacobian of a curve $\mathcal{C}$ of genus 2, then $\mathcal{K}$ is called the Kummer surface *belonging to* $\mathcal{C}$. The connection between the two constructions of the Kummer surface is explicitly established in [3], Chapter 17 (see Lemma 4.5).

A desingularization $\mathcal{S}$ of $\mathcal{K}$ is constructed explicitly in [3], Chapter 16, by algebraic methods. Denote by $\mathcal{K}^*$ the projective dual of $\mathcal{K}$. There are birational maps $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ and $\kappa^* : \mathcal{K}^* \dashrightarrow \mathcal{S}$ and morphisms extending $\kappa^{-1} : \mathcal{S} \dashrightarrow \mathcal{K}$ and ${\kappa^*}^{-1} : \mathcal{S} \dashrightarrow \mathcal{K}^*$ to all of $\mathcal{S}$. We denote these extensions also by κ^{-1} and ${\kappa^*}^{-1}$. They are minimal desingularizations of $\mathcal{K}$ and $\mathcal{K}^*$.

Origins. Cassels and Flynn explain that the surface $\mathcal{S}$ comes from the behavior of six of the tropes (see Definition 2.3) under the duplication map. The existence of $\mathcal{S}$ raises more far-reaching questions. Indeed, if the ground field k is algebraically closed, one has a commutative diagram:

$$\begin{array}{ccc} \mathcal{J}(\mathcal{C}) & \xrightarrow{d_0} & \mathcal{J}(\mathcal{C})^0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ \mathcal{K} & \xrightarrow{d_0^*} & \mathcal{K}^* \end{array} \tag{1.1}$$

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where $\mathcal{J}(\mathcal{C})^0$ is the dual of $\mathcal{J}(\mathcal{C})$ as an abelian variety. Here, the maps d_0 and d_0^* depend on the choice of a rational point on $\mathcal{C}$. Thus the abelian varieties duality matches with the projective one. When k is not algebraically closed, one has to enlarge the ground field to obtain such diagrams, yet $\mathcal{S}$ is a desingularization *over* k of both $\mathcal{K}$ and $\mathcal{K}^*$. One may ask if there is a unifying object for $\mathcal{J}(\mathcal{C})$ and $\mathcal{J}(\mathcal{C})^0$, generalizing the abelian varieties duality.

Recent developments. The Jacobian $\mathcal{J}(\mathcal{C})$ can be embedded in $\mathbb{P}^{15}$ and is described by 72 quadratic equations ([4]). More computable objects, $\mathcal{S}$ and its twists, appeared in recent attempts by M. Stoll and N. Bruin, to compute the Mordell-Weil group of $\mathcal{J}(\mathcal{C})$. We give a brief account of it in Section 5 of [8].

Cassels and Flynn already suggested that the 2-Selmer group could be investigated by using twists of $\mathcal{S}$. In 2007 A. Logan and R. van Luijk ([7]) and P. Corn ([2]) made use of twists of $\mathcal{S}$ to find specific curves with nontrivial 2-torsion elements in the Tate-Shafarevich groups of their Jacobians.

Our results and structure of this paper. In Section 2 we give a background. The rare relevant facts, not included in this paper, are contained in [3] and [8].

This paper is structured along two computational ideas. First, to profit from the algebraic construction of $\mathcal{S}$ in [3] in order to describe its linear automorphism. Second, to link $\mathcal{S}$ and Σ and thus bring line complexes into the picture. Our results achieve part of the program suggested in [3], at the end of Chapter 16.

In Section 3 we complete the construction in [3], enlarge the set of points where $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ is explicitly defined and find formulas for κ. These formulas allow one to describe how each singularity of $\mathcal{K}$ and $\mathcal{K}^*$ blows up (see Lemma 3.6).

In Section 4 we define an explicit isomorphism Θ between Σ and $\mathcal{S}$. We observe that well-known projective isomorphisms $W_i : \mathcal{K} \longrightarrow \mathcal{K}^*$ lift to Σ to correspondences given by line complexes of degree 1. Using Θ we show that the W_i lift to $\mathcal{S}$ to commuting involutions $\varepsilon^{(i)}$ (Corollary 4.12). With the formulas for κ at hand, one can now find formulas for κ^*.

In Section 5 we give a description of the group of linear automorphisms of $\mathcal{S}$.

2. Preliminaries

We work over a field k with $\mathrm{char}(k) \neq 2$ and with more than 5 elements. By a curve, we mean a smooth projective irreducible variety of dimension 1. *Throughout this paper, $\mathcal{C}$ will be a curve of genus* 2. Such a curve admits an affine model:

$$(2.1) \qquad \mathcal{C}' : \quad Y^2 = F(X) = f_0 + f_1 X + \cdots + f_6 X^6 \in k[X], \qquad f_6 \neq 0$$

and F has distinct roots $\theta_1, \ldots, \theta_6$. The points $\mathfrak{a}_i = (\theta_i, 0)$ are the Weierstrass points. We denote by $\infty^{\pm}$ the points at infinity on the completion $\mathcal{C}$ of $\mathcal{C}'$. For a given point $\mathfrak{r} = (x, y)$ on $\mathcal{C}$, the *conjugate of* $\mathfrak{r}$ *under the* $\pm Y$ *involution* is the point $\bar{\mathfrak{r}} = (x, -y)$. For a divisor $\mathfrak{X} = \sum n_i \mathfrak{r}_i$ we denote by $\overline{\mathfrak{X}} = \sum n_i \bar{\mathfrak{r}}_i$. The class of a divisor $\mathfrak{X}$ is denoted by $[\mathfrak{X}]$ and a divisor in the canonical class by $K_{\mathcal{C}}$.

One can regard a point of the Jacobian $J(\mathcal{C})$ as the class of a divisor $\mathfrak{X} = \mathfrak{r} + \mathfrak{u}$, where $\mathfrak{r} = (x, y)$, $\mathfrak{u} = (u, v)$ is a pair of points on $\mathcal{C}$. Starting from this, Flynn constructs a projective embedding of the Jacobian as follows ([4]). For a point $\mathfrak{X} = \{\mathfrak{r}, \mathfrak{u}\}$ on $\mathcal{C}^{(2)}$ (symmetric product) with $\mathfrak{r} = (x, y)$, $\mathfrak{u} = (u, v)$, define:

$$\sigma_0 = 1, \quad \sigma_1 = x + u, \quad \sigma_2 = xu, \quad \beta_0 = \frac{F_0(x, u) - 2yv}{(x - u)^2},$$

where

$$\begin{aligned} F_0(x,u) =& 2f_0 + f_1(x+u) + 2f_2xu + f_3xu(x+u) \\ &+ 2f_4(xu)^2 + f_5(xu)^2(x+u) + 2f_6(xu)^3. \end{aligned}$$

The Jacobian is then the projective locus of $\mathbf{z} = (z_0 : \ldots : z_{15})$ in $\mathbb{P}^{15}$, where $z_0 = \delta$; $z_1 = \gamma_1$; $z_2 = \gamma_0$; $z_i = \beta_{5-i}$, $i = 3,4,5$; $z_i = \alpha_{9-i}$, $i = 6,\ldots,9$; $z_i = \sigma_{14-i}$, $i = 10,\ldots,14$; and $z_{15} = \rho$. For the definition of the functions α, β, etc. and details, see [3], Chapter 2.

Definition 2.1. The Kummer surface $\mathcal{K}$ belonging to a curve of genus 2, is the projective locus in $\mathbb{P}^3$ of the elements $\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4) = (\sigma_0 : \sigma_1 : \sigma_2 : \beta_0)$.

The equation of the Kummer surface is

$$\mathcal{K}: \quad K = K_2\xi_4^2 + K_1\xi_4 + K_0 = 0, \tag{2.2}$$

where the K_i are forms of degree $4-i$ in ξ_1, ξ_2, ξ_3 ([3], formula (3.1.9)). The natural map from $J(\mathcal{C})$ to $\mathcal{K}$ given by

$$(z_0 : \ldots : z_{15}) \longmapsto (z_{14} : z_{13} : z_{12} : z_5) = (\xi_1 : \ldots : \xi_4)$$

is 2 to 1; the ramification points correspond to divisor classes $[\mathfrak{X}]$ with $[\mathfrak{X}] = [\overline{\mathfrak{X}}]$. The images of these classes are the 16 nodes on $\mathcal{K}$: $N_0 = (0:0:0:1)$ corresponds to $[K_{\mathcal{C}}]$ and 15 nodes N_{ij} correspond to classes $[\mathfrak{X}_{ij}] = [\mathfrak{a}_i + \mathfrak{a}_j]$ with $i \neq j$.

Definition 2.2. The dual Kummer surface $\mathcal{K}^* \subset (\mathbb{P}^3)^{\checkmark} = \mathbb{P}^3$ is the projective dual of $\mathcal{K}$, i.e., to the point $\xi \in \mathcal{K}$ corresponds the point $\eta \in \mathcal{K}^*$ such that $\eta = (\eta_1 : \eta_2 : \eta_3 : \eta_4) \in (\mathbb{P}^3)^{\checkmark}$ gives the tangent plane to $\mathcal{K}$ at ξ.

Definition 2.3. There are 6 planes T_i containing the 6 nodes N_0 and N_{ij}, $j \neq i$ and 10 planes T_{ijk} containing the 6 nodes N_{mn} for $\{m,n\} \subset \{i,j,k\}$ or $\{m,n\} \cap \{i,j,k\} = \emptyset$. These are the *tropes*; they cut conics on $\mathcal{K}$. They correspond to the 16 singular points of $\mathcal{K}^*$. The equations of the T_i are:

$$T_i: \quad \theta_i^2\xi_1 - \theta_i\xi_2 + \xi_3 = 0, \tag{2.3}$$

Lemma 2.4. *For each Weierstrass point $\mathfrak{a}_i = (\theta_i, 0)$ there is a projective map $W_i : \mathcal{K} \longrightarrow \mathcal{K}^*$ induced by the addition of $\mathfrak{a}_i$ to a divisor of degree 2. One has $W_i(N_0) = T_i$ (singular point of $\mathcal{K}^*$). Furthermore, $W_i^{-1} \circ W_j(N_0) = N_{ij}$.*

Proof. See [3], Lemma 4.5.1. □

3. The desingularized Kummer surface

We recall the facts from [3], Chapter 16 we need, keeping the notation there.

Let $\mathbf{p} = (p_0 : \ldots : p_5)$, where the p_j are indeterminates, and put $P(X) = \sum_0^5 p_jX^j$. Let $\mathcal{S}$ be the projective locus of the $\mathbf{p}$ for which $P(X)^2$ is congruent to a quadratic in X modulo $F(X)$. Put

$$P_j(X) = \prod_{i \neq j}(X - \theta_i) = \sum_{k=0}^{5} h_{jk}X^k \tag{3.1}$$

and $\omega_j = P_j(\theta_j) \neq 0$. Since $\theta_i \neq \theta_j$ for $i \neq j$, we have $\omega_j \neq 0$ and the P_j span the vector space of polynomials of degree at most 5. We have

$$P(X) = \sum_j \pi_j P_j(X), \qquad \text{where} \qquad \pi_j = \frac{P(\theta_j)}{\omega_j}. \tag{3.2}$$

The $K3$ surface $\mathcal{S}$ is the complete intersection in $\mathbb{P}^5$ of the quadrics $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$ where

$$\mathcal{S}_i : S_i = 0 \qquad \text{and} \qquad S_i = \sum_j \theta_j^i \omega_j \pi_j^2 \quad \text{for} \quad i = 0, 1, 2. \tag{3.3}$$

It is a minimal desingularization of $\mathcal{K}$ and of $\mathcal{K}^*$. Here the S_i are quadratic forms in $\mathbf{p}$ with coefficients in $\mathbb{Z}[f_1, \ldots, f_6]$.

The following theorems hold ([3], Theorems 16.5.1 and 16.5.3):

Theorem 3.1. *There is a birational map $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ defined for general $\xi \in \mathcal{K}$ as follows: Let $\mathfrak{X} = \{(x, y), (u, v)\}$ correspond to ξ. Put $G(X) = (X - x)(X - u)$ and let $M(X)$ be the cubic determined by the property that $Y - M(X)$ vanishes twice on $\mathfrak{X}$. Let $P(X) = \sum_0^5 p_j X^j$ be determined by $GP \equiv M \bmod F$. Then $\kappa(\xi)$ is the point with projective coordinates $(p_0 : \ldots : p_5)$.*

Let $\kappa^* : \mathcal{K}^* \dashrightarrow \mathcal{S}$ be the birational map defined in [3], Theorem 16.5.2.

Theorem 3.2. *Let $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^*$ be dual, that is, η gives the tangent to $\mathcal{K}$ at ξ. Then $\kappa(\xi) = \kappa^*(\eta)$.*

Our first result is the following.

Lemma 3.3. *The map $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ from Theorem* 3.1 *is given by the formulas listed below.*

Proof. The problem is to make effective the method given in [3], Chapter 16. For completeness and due to typing errors there, we recall it in [8], in the Appendix. As presented in [3], the method works for a general element $[\mathfrak{X}] = [(x, y) + (u, v)]$, where $yv \neq 0$ and $x \neq u$.

First, we put $y^2 = F(x)$, $v^2 = F(u)$, $yv = [F_0(x, u) - \beta_0(x - u)^2]/2$ in the coefficients of $P(X)$, then as the resulting coefficients are symmetric functions of x and u, we express them in terms of $\xi_2 = x + u$ and $\xi_3 = xu$. Finally, we homogenize the formulas with respect to $\xi_1 = 1$, ξ_2, ξ_3, ξ_4. One first obtains

$$\kappa(\xi) = (\tilde{p}_0(\xi) : \ldots : \tilde{p}_5(\xi)),$$

where

$$\tilde{p}_j(\xi) = \alpha_j K_2 + \beta_j (K_1 \xi_4 + K_0), \quad \text{for } 0 \leq j \leq 5. \tag{3.4}$$

Here α_j and β_j are homogeneous forms in ξ of degree 4 and 2, respectively, and the K_j are those in (2.2). Taking $p_j(\xi) = (\tilde{p}_j(\xi) - \beta_j K)/K_2 = \alpha_j - \beta_j \xi_4^2$, we obtain formulas of degree 4 for κ, which will be defined also for $K_2 = 0$, extending κ to images of divisor classes $[\mathfrak{X}] = [(x, y) + (u, v)]$ with $x = u$ and $y = v \neq 0$. However, the formulas do not work for points with $F'(x) = 0$ and for the image of $[2\infty^+]$. We will treat the case $y = 0$ or $v = 0$ in connection with nodes and tropes.

Polynomial definition of κ.

$$\begin{aligned}
p_0 = &-f_3f_6\xi_1\xi_3^3+1/2f_5^2\xi_2\xi_3^3-2\xi_3^3f_4f_6\xi_2+2\xi_3^2\xi_1^2f_1f_6-\xi_3^2\xi_1^2f_5f_2\\
&-2\xi_3^2\xi_1\xi_2f_6f_2-1/2\xi_3^2\xi_1f_5\xi_4-1/2\xi_3^2\xi_1\xi_2f_5f_3-\xi_3^2\xi_2^2f_6f_3-2\xi_3^2\xi_2f_6\xi_4\\
&-1/2\xi_3f_3\xi_4\xi_1^2-3/2\xi_3\xi_1^2\xi_2f_5f_1-\xi_3\xi_2f_4\xi_4\xi_1-3\xi_3\xi_1\xi_2^2f_6f_1\\
&-1/2\xi_3\xi_2^2f_5\xi_4+f_1f_2\xi_1^4+\xi_1^3\xi_2f_1f_3+3/2\xi_1^3f_1\xi_4+\xi_2^2f_4f_1\xi_1^2+\xi_2^3f_5f_1\xi_1\\
&+\xi_2^4f_1f_6-1/2\xi_1\xi_2\xi_4^2
\end{aligned}$$

$$\begin{aligned}
p_1 = &\,2\xi_1^4f_2^2-2\xi_3\xi_1\xi_2^2f_6f_2+1/2\xi_1^2\xi_2^2f_5f_1-1/2\xi_1^4f_3f_1+2\xi_2^4f_2f_6+3\xi_1^3\xi_4f_2\\
&+1/2\xi_3f_3^2\xi_1^3+1/2\xi_2^3f_5\xi_4+\xi_3\xi_1^2\xi_2f_4f_3-1/2\xi_3^2\xi_2^2f_5^2+3/2\xi_3\xi_1\xi_2^2f_5f_3\\
&+2\xi_3^2f_4f_6\xi_2^2-\xi_3\xi_1^2\xi_2f_5f_2+\xi_3\xi_2^2f_6\xi_4+2\xi_3\xi_2^3f_6f_3+\xi_1^2\xi_4^2+2\xi_1^3\xi_2f_2f_3\\
&-\xi_3\xi_1^2\xi_2f_6f_1+2\xi_1^2f_2f_4\xi_2^2+3/2\xi_1^2\xi_2f_3\xi_4+\xi_1f_4\xi_4\xi_2^2+\xi_1\xi_2^3f_6f_1+2\xi_1\xi_2^3f_5f_2\\
&+2\xi_3^2\xi_1^2f_6f_2-1/2\xi_3^2\xi_1^2f_5f_3+\xi_3^2\xi_4f_6\xi_1
\end{aligned}$$

$$\begin{aligned}
p_2 = &\,2\xi_1^2\xi_2^2f_4f_3-f_6f_5\xi_1\xi_3^3+\xi_3^2\xi_1^2f_3f_6-\xi_3^2\xi_1^2f_4f_5+\xi_3^2\xi_1f_5^2\xi_2\\
&+\xi_3f_1f_6\xi_1^3+\xi_3\xi_1^3f_3f_4-2\xi_3f_5f_2\xi_1^3+2\xi_3\xi_1^2\xi_2f_4^2-2\xi_3f_5\xi_4\xi_1^2\\
&-\xi_1^4f_1f_4+2\xi_1^4f_3f_2-2\xi_3\xi_2f_6f_2\xi_1^2-5\xi_3\xi_2^2f_6f_3\xi_1+2\xi_3\xi_1\xi_2^2f_5f_4\\
&-3\xi_3f_6\xi_4\xi_2\xi_1-3\xi_3\xi_2f_5f_3\xi_1^2+2\xi_2^4f_3f_6+\xi_2^3f_6\xi_4+2f_3\xi_4\xi_1^3+2\xi_2f_3^2\xi_1^3\\
&+2\xi_2^3f_5f_3\xi_1-\xi_2f_5f_1\xi_1^3-\xi_2^2f_6f_1\xi_1^2+\xi_2^2f_5\xi_4\xi_1+\xi_2f_4\xi_4\xi_1^2+2\xi_3\xi_2^3f_6f_4\\
&+\xi_3^2\xi_2^2f_6f_5-4\xi_3^2\xi_1f_4f_6\xi_2
\end{aligned}$$

$$\begin{aligned}
p_3 = &-2f_6^2\xi_1\xi_3^3-\xi_3^2\xi_1^2f_5^2+2\xi_3^2\xi_1^2f_4f_6-\xi_3^2\xi_2f_6f_5\xi_1+2\xi_3^2f_6^2\xi_2^2\\
&+\xi_3\xi_1^3f_5f_3-2\xi_3\xi_1^3f_2f_6-\xi_3\xi_1^2\xi_2f_6f_3-2\xi_3\xi_1^2f_6\xi_4+2\xi_3\xi_1\xi_2^2f_5^2\\
&-4\xi_3\xi_1f_4f_6\xi_2^2+2\xi_3f_6f_5\xi_2^3-\xi_1^4f_1f_5+2\xi_1^4f_4f_2+2\xi_1^3f_4\xi_4+2\xi_1^3\xi_2f_4f_3\\
&-\xi_1^3\xi_2f_6f_1+\xi_1^2\xi_2f_5\xi_4+2\xi_1^2f_4^2\xi_2^2+\xi_1\xi_2^2f_6\xi_4+2\xi_1\xi_2^3f_5f_4+2\xi_2^4f_6f_4
\end{aligned}$$

$$\begin{aligned}
p_4 = &\,\xi_3^2\xi_1^2f_6f_5-2\xi_3^2\xi_2f_6^2\xi_1+\xi_3f_3f_6\xi_1^3-2\xi_3\xi_2f_5^2\xi_1^2+2\xi_3f_4f_6\xi_2\xi_1^2\\
&-2\xi_3\xi_2^2f_6f_5\xi_1+2\xi_3f_6^2\xi_2^3-f_6f_1\xi_1^4+2f_5f_2\xi_1^4+2f_5\xi_4\xi_1^3+2\xi_2f_5f_3\xi_1^3\\
&+2\xi_2^2f_5f_4\xi_1^2+\xi_2f_6\xi_4\xi_1^2+2\xi_2^3f_5^2\xi_1+2\xi_2^4f_6f_5
\end{aligned}$$

$$\begin{aligned}
p_5 = &\,2(f_6\xi_1^2\xi_3^2-\xi_3f_5\xi_2\xi_1^2-2\xi_3f_6\xi_2^2\xi_1+f_2\xi_1^4+\xi_1^3\xi_4+\xi_1^3f_3\xi_2+f_4\xi_2^2\xi_1^2\\
&+\xi_2^3f_5\xi_1+f_6\xi_2^4)f_6. \qquad \square
\end{aligned}$$

Let the point $(p_0 : \dots : p_5)$ be represented by $P(X) = \sum_{i=0}^{5} p_i X^i$ and let

$$g_i(X) = 1 - 2\frac{P_i(X)}{P_i(\theta_i)}, \qquad i = 1, \dots, 6 \tag{3.5}$$

where $P_i(X)$ is defined by (3.1). We see that $g_i(\theta_j) = (-1)^{\delta_{ij}}$, so $g_i(X)^2 \equiv 1 \mod F(X)$. There are 6 commuting involutions $\varepsilon^{(i)}$ of $\mathcal{S}$, defined as follows:

$$\varepsilon^{(i)}(P(X)) = g_i(X)P(X) \mod F(X), \qquad i = 1, \ldots, 6. \tag{3.6}$$

In terms of coordinates π_j, one has

$$\varepsilon^{(i)}(\pi_j) = (-1)^{\delta_{ij}}\pi_j. \tag{3.7}$$

Definition 3.4. We define $\mathrm{Inv}(\mathcal{S})$ to be the group of 32 commuting involutions of S generated by the $\varepsilon^{(i)}$.

The $\mathbf{p} = (p_0 : p_1 : 0 : 0 : 0 : 0)$ form a rational line $\Delta_0 \subset \mathcal{S}$. We shall often write $p_0 + p_1 X \in \Delta_0$. Acting on Δ_0 by the involutions gives 31 further lines.

Notation 3.5. We denote:

$$\Delta_i = \varepsilon^{(i)}(\Delta_0)\,, \quad \Delta_{ij} = \varepsilon^{(i)} \circ \varepsilon^{(j)}(\Delta_0) \quad \text{and} \quad \Delta_{ijk} = \varepsilon^{(i)} \circ \varepsilon^{(j)} \circ \varepsilon^{(k)}(\Delta_0)\,.$$

The main result in this section describes how the nodes of $\mathcal{K}$ and $\mathcal{K}^*$ blow up.

Lemma 3.6. *The map κ blows up the node $N_0 = (0 : 0 : 0 : 1)$ of $\mathcal{K}$ into the line Δ_0 and the 15 nodes N_{ij} into the lines Δ_{ij}. The tropes T_i and T_{ijk} blow up by κ^* into the lines Δ_i and Δ_{ijk}.*

Proof. The node N_0 corresponds to the canonical class, so we consider divisors of the type $\mathfrak{X} = (x, y) + (u, v)$ with $u = x + h$, h small and $v \approx -y \neq 0$. Then the local behavior of the Kummer coordinates is $\xi_1 = 1$, $\xi_2 = 2x + h \approx 2x$, $\xi_3 = x(x+h) \approx x^2$ and

$$\xi_4 = \frac{F_0(x, x+h) - 2yv}{h^2} \approx \frac{4y^2}{h^2}.$$

Replacing this in the formulas for κ and clearing denominators, then taking the limit as $h \to 0$ and dividing by $y^4 \neq 0$, we obtain

$$\begin{array}{rcl} \kappa(\xi) & \approx & (-16xy^4 : 16y^4 : 0 : 0 : 0 : 0) \\ & \approx & (-x : 1 : 0 : 0 : 0 : 0). \end{array}$$

Note that $\Delta_0 \cap \Delta_i = (-\theta_i : 1 : 0 : 0 : 0 : 0)$, since for $(X - \theta_i) \in \Delta_0$ we have

$$\varepsilon^{(i)}(X - \theta_i) \equiv g_i(X)(X - \theta_i) \equiv (X - \theta_i) \mod F(X).$$

We now show that $\Delta_0 \cap \Delta_{ij} = \emptyset$ for $i \neq j$. Indeed, the intersection point $\mathfrak{p}$ should be invariant by $\varepsilon^{(i)} \circ \varepsilon^{(j)}$. A polynomial $P(X)$ represents such a point iff

$$\begin{array}{c} \alpha P(X) \equiv g_i(X)g_j(X)P(X) \mod F(X) \quad \text{for some } \alpha \in \bar{k}^* \\ \text{iff} \\ F(X) \mid P(X)(\alpha - g_i(X)g_j(X)). \end{array}$$

Replacing X by the roots of $F(X)$ one sees that $P(X)$ must have at least two roots among θ_k, so it must be of degree at least 2 and therefore cannot represent a point on Δ_0. Similarly, $\Delta_0 \cap \Delta_{ijk} = \emptyset$ for $i \neq j \neq k$.

The six Δ_i are strict transforms of the conics Γ_i cut on $\mathcal{K}$ by the tropes T_i. To see this and to define κ for points corresponding to divisors $\mathfrak{X} = \{x, y\} + \{\theta_i, 0\}$ with $y \neq 0$, write $F(X) = f_6(X - \theta_i)P_i(X)$. From this we get formulas for f_k, $k = 0, \ldots, 6$ depending on θ_i and h_{ij}, $j = 0, \ldots, 5$, the coefficients of $P_i(X)$, which we plug into $\xi_4 = (F_0(x, \theta_i)/(x - \theta_i)^2)$. We substitute then $\xi_1 = 1$, $\xi_2 = x + \theta_i$,

$\xi_3 = x\theta_i$ and ξ_4 in the formulas for κ. On multiplying by $(x-\theta_i)^2/(f_6^2 P_i(x))$ (note that $P_i(x) \neq 0$), we obtain

$$P(X) = 2(x-\theta_i)P_i(X) + P_i(\theta_i)(X-x), \tag{3.8}$$

that is,

$$\begin{aligned} p_0 &= 2h_{i0}(x-\theta_i) - P_i(\theta_i)x, \\ p_1 &= 2h_{i1}(x-\theta_i) + P_i(\theta_i), \\ p_j &= 2h_{ij}(x-\theta_i) \qquad \text{for } 2 \leq j \leq 5. \end{aligned} \tag{3.9}$$

Equation (2.3) shows that the points $(1 : x+\theta_i : x\theta_i : \xi_4)$ belong to the conic Γ_i. Formulas (3.9) give parametric equations (in x) of the strict transform of Γ_i by κ. To confirm that this is Δ_i, one verifies that

$$P(X) \equiv P_i(\theta_i)g_i(X)(X-x) \mod F(X).$$

Applying the results in Section 4 and especially Corollary 4.12, one concludes that:

1) the tropes T_i considered as singular points of $\mathcal{K}^*$, blow up by κ^* into Δ_i;
2) each of the fifteen N_{ij} blows up into Δ_{ij};
3) the tropes T_{ijk}, $i \neq j \neq k$ blow up into Δ_{ijk};
4) the ten Δ_{ijk} $(i \neq j \neq k)$ are strict transforms of the ten conics cut on $\mathcal{K}$ by the tropes (planes) not containing N_0. Each of them intersects six Δ_{ij} since each node is on six tropes. □

4. Line complexes

Let $u = (u_1 : u_2 : u_3 : u_4)$ and $v = (v_1 : v_2 : v_3 : v_4)$ be distinct points in $\mathbb{P}^3$. Put $p_{ij} = u_i v_j - u_j v_i$. The Grassmann coordinates of the line $\langle u, v\rangle \subset \mathbb{P}^3$ are

$$\mathfrak{p} = (p_{43} : p_{24} : p_{41} : p_{21} : p_{31} : p_{32}) = (X_1 : \ldots : X_6).$$

The Grassmannian quadric $\mathcal{G} \subset \mathbb{P}^5$, representing the lines in $\mathbb{P}^3$ has the equation:

$$G(X_1, \ldots, X_6) = 2X_1X_4 + 2X_2X_5 + 2X_3X_6 = 0.$$

Definition 4.1. A line complex of degree d is a set of lines in $\mathbb{P}^3$ whose Grassmann coordinates satisfy a homogeneous equation $Q(X_1, \ldots, X_6) = 0$ of degree d.

If $d = 1$, this is called a linear complex, and if $d = 2$, it is a quadratic complex.

A line $L \in \mathcal{G}$ parametrizes a pencil of lines in $\mathbb{P}^3$. The lines of a pencil L all pass through a point $\mathfrak{f}(L) = u$, called the *focus* of the pencil, and lie in one plane $\mathfrak{h}(L) = \pi_u$, the *plane* of the pencil.

All lines in a linear complex $\mathcal{L}$ passing through a given point u (respectively, lying in a plane π), form a pencil L_u (respectively, L_π). Each linear complex $\mathcal{L}$ establishes a *correspondence* between points and planes in $\mathbb{P}^3$,

$$I(u) = \mathfrak{h}(L_u), \quad I(\pi) = \mathfrak{f}(L_\pi), \quad I^2 = 1,$$

which is also defined for lines; if $l \subset \mathbb{P}^3$ is the line $\langle u, u'\rangle$, then $I(l) = I(u) \cap I(u')$. The line $I(l)$ is the *polar* line of l with respect to the given linear complex.

Definition 4.2. Two linear complexes are called apolar if the correspondences they define commute.

Let H be any quadratic form in six variables such that the quadrics $G = 0$ and $H = 0$ intersect transversely, and denote by $\mathcal{H} = \{x \in \mathbb{P}^5 \mid H(x) = 0\}$. Let $\mathcal{W} = \mathcal{G} \cap \mathcal{H}$ and $\mathcal{A} =$ the set of lines on $\mathcal{W}$. The points in $\mathcal{W}$ represent the lines in $\mathbb{P}^3$ whose Grassmann coordinates $\mathfrak{p}$ satisfy $H(\mathfrak{p}) = 0$. A line $L \in \mathcal{A}$ represents a pencil of lines in $\mathbb{P}^3$ of the quadratic complex defined by H.

Definition 4.3. The Kummer surface $\mathcal{K} \subset \mathbb{P}^3$ associated to the quadratic complex $\mathcal{H}$ is the locus of focuses of such pencils: $\mathcal{K} = \{\mathfrak{f}(L) \mid L \in \mathcal{A}\}$.

Definition 4.4. The dual Kummer surface $\mathcal{K}^* \subset \mathbb{P}^{3\checkmark}$ associated to the quadratic complex $\mathcal{H}$ is the locus of planes of such pencils.

From now on we suppose $f_6 = 1$.

Lemma 4.5. *For any curve $\mathcal{C}$ of genus 2, the Kummer surface belonging to the curve $\mathcal{C}$ given by* (2.1) *coincides with the Kummer surface just defined, if one takes the quadratic complex $\mathcal{H}$ to be given by*

$$\begin{aligned} H = &- 4X_1X_5 - 4X_2X_6 - X_3^2 + 2f_5X_3X_6 + 4f_0X_4^2 \\ &+ 4f_1X_4X_5 + 4f_2X_5^2 + 4f_3X_5X_6 + (4f_4 - f_5^2)X_6^2. \end{aligned}$$

Proof. See [3], Lemma 17.3.1 and pages 182–183. □

If a point $\xi \in \mathbb{P}^3$ is the focus of the pencil corresponding to the line $L_\xi \in \mathcal{A}$, then L_ξ lies in the plane $\Pi_\xi \subset \mathcal{G}$ corresponding to lines in $\mathbb{P}^3$ passing through ξ. But then the conic $\Pi_\xi \cap \mathcal{H}$ contains L_ξ, so it is degenerate; Π_ξ is tangent to $\mathcal{H}$ and $\Pi_\xi \cap \mathcal{H} = L_\xi \cup L'_\xi$. The lines of the quadratic complex passing through ξ are in the two pencils L_ξ and L'_ξ, each with focus ξ, lying in the planes π_ξ and π'_ξ in $\mathbb{P}^3$. The line $l_\xi = \pi_\xi \cap \pi'_\xi$ is represented on $\mathcal{G}$ by the point $\mathfrak{p}_\xi = L_\xi \cap L'_\xi$ and is called a *singular line* of the quadratic complex.

If $L_\xi \neq L'_\xi$ the pencils are distinct and ξ is a simple point of the Kummer; there is a one-to-one correspondence $\xi \leftrightarrow \mathfrak{p}_\xi$. However, if $L_\xi = L'_\xi$, then $\pi_\xi = \pi'_\xi$ and all the lines in L_ξ are singular lines. The point ξ is a singular point of the Kummer, because the map $\mathfrak{f} : \mathcal{A} \longrightarrow \mathcal{K}$ is algebraic. Therefore, the variety Σ parametrizing singular lines is a desingularization of the Kummer.

Definition 4.6. The birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ is defined by $\kappa_1(\xi) = \mathfrak{p}_\xi$.

Definition 4.7. The birational map $\kappa_1^* : \mathcal{K}^* \dashrightarrow \Sigma$ associates to a plane π tangent to $\mathcal{K}$ the intersection point of the lines in $\mathcal{A}$ parametrizing the two pencils in $\mathcal{H}$ contained in π.

The maps κ_1^{-1} and κ_1^{*-1} extend to minimal desingularizations $\kappa_1^{-1} : \Sigma \longrightarrow \mathcal{K}$ and $\kappa_1^{*-1} : \Sigma \longrightarrow \mathcal{K}^*$ (see [3] and [6]). The following is proved in [3], page 181:

Lemma 4.8. *The surface $\mathcal{K}^*$ is the projective dual of $\mathcal{K}$; that is, if $\xi = \mathfrak{f}(L) \in \mathcal{K}$, then $\eta = \mathfrak{h}(L) \in \mathcal{K}^*$ is the tangent plane of $\mathcal{K}$ at ξ. Therefore $\kappa_1(\xi) = \kappa_1^*(\eta)$.*

Denote by $G(\overrightarrow{X}, \overrightarrow{Y})$ the bilinear form associated to the Grassmannian G. Make the change of coordinates

$$\zeta_i = \frac{G(\overrightarrow{X}, \overrightarrow{v}(\theta_i))}{\sqrt{\omega_i}}, \tag{4.1}$$

with vectors $\overrightarrow{v}(\theta_i)$ as in [3] formula (17.4.3). Let $\mathcal{K}$ be the Kummer surface associated to the quadratic complex H of Lemma 4.5. By [6], Section 31 or [3] formula (17.4.2), a minimal desingularization of $\mathcal{K}$ is the $K3$ surface in $\mathbb{P}^5$:

$$\Sigma = \Sigma_0 \cap \Sigma_1 \cap \Sigma_2, \qquad \text{where} \qquad \Sigma_i : \quad \sum_j \theta_j^i \zeta_j^2 = 0 \quad \text{for} \quad i = 0, 1, 2.$$

Proposition 4.9. *There is an explicit isomorphism* $\Theta : \Sigma \longrightarrow \mathcal{S}$.

Proof. Let $\Theta : \Sigma \longrightarrow \mathcal{S}$ be defined by

$$\Theta(\zeta_1 : \ldots : \zeta_6) = \left(\frac{\zeta_1}{\sqrt{\omega_1}} : \ldots : \frac{\zeta_6}{\sqrt{\omega_6}} \right) = (\pi_1 : \cdots : \pi_6). \tag{4.2}$$

To pass to variables X_j and p_j, recall that $P(X) = \sum_0^5 p_j X^j$. By (3.2) and (4.1):

$$\frac{P(\theta_i)}{\omega_i} = \pi_i = \frac{\zeta_i}{\sqrt{\omega_i}} = \frac{G(\overrightarrow{X}, \overrightarrow{v}(\theta_i))}{\omega_i}.$$

Now, as polynomials in X, we have $G(\overrightarrow{X}, \overrightarrow{v}(X)) = P(X)$, because both have degree 5 and agree on the six θ_i. Explicit formulas for Θ are:

$$\begin{array}{lll} p_0 = X_1 + f_1 X_4 & p_2 = X_3 + 2f_4X_5 + 2f_3X_4 + f_5X_6 & p_4 = 2f_5X_4 + 2X_5 \\ p_1 = X_2 + 2f_2X_4 + f_3X_5 & p_3 = 2f_4X_4 + 2f_5X_5 + 2X_6 & p_5 = 2X_4. \end{array}$$ ☐

Proposition 4.10. *Denoting by* κ^{-1} *and* κ_1^{-1} *the blow-downs from* $\mathcal{S}$*, respectively,* Σ *to* $\mathcal{K}$ *one has* $\kappa_1^{-1} = \kappa^{-1} \circ \Theta$.

Proof. Pick a point $\xi \in \mathbb{P}^3$ and write the equations of the plane $\Pi_\xi \subset \mathcal{G}$ of lines through ξ (see equations (4.7) of [8]). Take $\mathcal{H}$ to be defined as in Lemma 4.5. Now, Π_ξ is tangent to $\mathcal{H}$ iff the intersection consists of two lines: $\Pi_\xi \cap \mathcal{H} = L_\xi \cup L'_\xi$. Computing in terms of ξ the coordinates of $\mathfrak{p}_\xi = L_\xi \cap L'_\xi$, we find homogeneous formulas for X_i in ξ_i of degree 4: $\mathfrak{p}_\xi = (X_1(\xi) : \ldots : X_6(\xi)) = \kappa_1(\xi)$. Comparing $\kappa(\xi) = (p_0(\xi) : \ldots : p_5(\xi))$ from Lemma 3.3 with

$$\Theta \circ \kappa_1(\xi) = (\hat{p}_0(\xi) : \cdots : \hat{p}_5(\xi)) : \mathcal{K} \longrightarrow \mathcal{S}$$

yields $\hat{p}_i p_5 - \hat{p}_5 p_i = \delta_i K$, with K given by (2.2), for δ_i a homogeneous polynomial in ξ. ☐

Associated with a quadratic complex $\mathcal{H} : H = 0$ there is a set of 6 mutually apolar linear complexes $\mathcal{L}_k$, such that the polar of any line in $\mathcal{H}$ with respect to $\mathcal{L}_k$ is in $\mathcal{H}$. If G and H are written in diagonal form, these complexes are $\mathcal{L}_k : \zeta_k = 0 \quad \text{for} \quad k = 1, \ldots, 6$. The action of the correspondences I_k on lines in $\mathbb{P}^3$ translates in coordinates $\zeta = (\zeta_1 : \ldots : \zeta_6)$ by

$$I_k(\zeta_i) = (-1)^{\delta_{ik}} \zeta_i, \tag{4.3}$$

which restricts to Σ. The Kummer surface is determined by $\mathcal{H}$, so it is invariant under the transformation I_k. The set of nodes and tropes is invariant (see [6], Section 30).

Proposition 4.11. *Let* W_k *be as in Lemma* 2.4. *For any* k*, the map* I_k *is the unique automophism of* Σ *such that* ${\kappa_1^*}^{-1} \circ I_k = W_k \circ {\kappa_1}^{-1}$.

Proof. Let $\xi \in \mathcal{K}$ be a simple point and denote $\mathfrak{p}_\xi = \kappa_1(\xi)$. For a subset $V \subset \mathcal{G}$, put $I_k(V) = \{I_k(l) \in \mathcal{G} \mid l \in V\}$. The pencils $I_k(L_\xi)$ and $I_k(L'_\xi)$ are both contained in the polar plane of ξ with respect to $\mathcal{L}_k$, which by Lemma 4.13 is $W_k(\xi)$. The plane in $\mathbb{P}^5$ parametrizing lines in $W_k(\xi)$ is therefore tangent to $\mathcal{H}$ at $I_k(L_\xi) \cap I_k(L'_\xi) = I_k(L_\xi \cap L'_\xi) = I_k(\mathfrak{p}_\xi) = I_k \circ \kappa_1(\xi)$. By definition of κ_1^* we have $\kappa_1^*(W_k(\xi)) = I_k \circ \kappa_1(\xi)$. □

The following corollary illustrates how the projective duality (over $k(\theta_k)$) between $\mathcal{K}$ and $\mathcal{K}^*$ lifts to $\mathcal{S}$.

Corollary 4.12. *For any k, the map $\varepsilon^{(k)}$ is the unique automophism of $\mathcal{S}$ such that $\kappa^{*-1} \circ \varepsilon^{(k)} = W_k \circ \kappa^{-1}$.*

Proof. Let $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^*$ be dual. We have:

$$\Theta \circ \kappa_1^*(\eta) \overset{(4.8)}{=} \Theta \circ \kappa_1(\xi) \overset{(4.10)}{=} \kappa(\xi) \overset{(3.2)}{=} \kappa^*(\eta). \tag{4.4}$$

Note that $\Theta \circ I_k \circ \Theta^{-1} = \varepsilon^{(k)}$, by (4.2), (4.3) and (3.7). Therefore,

$$\kappa^{*-1} \circ \varepsilon^{(k)} \overset{(4.4)}{=} \kappa_1^{*-1} \circ \Theta^{-1} \circ \Theta \circ I_k \circ \Theta^{-1} \overset{(4.11)}{=} W_k \circ \kappa_1^{-1} \circ \Theta^{-1} \overset{(4.10)}{=} W_k \circ \kappa^{-1}.$$

This is summarized in the following diagram, where the arrows to $\mathcal{K}$ and $\mathcal{K}^*$ are the minimal desingularizations:

$$\begin{array}{ccccccc} \mathcal{S} & \overset{\Theta}{\longleftarrow} & \Sigma & \overset{I_k}{\longrightarrow} & \Sigma & \overset{\Theta}{\longrightarrow} & \mathcal{S} \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & \mathcal{K} & \underset{W_k}{\longrightarrow} & K^* & & \end{array}$$

□

Now Corollary 4.12 is useful for finding explicit formulas for κ^*, because

$$\kappa^* = \kappa^* \circ W_i \circ \kappa^{-1} \circ \kappa \circ W_i^{-1} = \varepsilon^{(i)} \circ \kappa \circ W_i^{-1}$$

on an open dense set in $\mathcal{K}^*$. We find formulas for W_i in [8], Lemma 4.13. The resulting formulas for κ^* are huge and not listed in this paper, since on a given example it is easier to apply successively each map involved.

Lemma 4.13. *For any point $\xi \in \mathbb{P}^3$ the plane with dual coordinates $W_i(\xi)$ is the polar plane of ξ with respect to $\mathcal{L}_i$.*

Proof. See [8], Lemma 4.14. □

5. Linear automorphisms of $\mathcal{S}$

Keeping Notation 3.5, we let

$$\begin{aligned} \mathfrak{p}_i &= \Delta_0 \cap \Delta_i, \\ \mathfrak{p}_{ij} &= \Delta_i \cap \Delta_{ij} = \varepsilon^{(i)}(\mathfrak{p}_j), \\ \mathfrak{p}_{ijk} &= \Delta_{ij} \cap \Delta_{ijk} = \varepsilon^{(i)}(\mathfrak{p}_{jk}). \end{aligned} \tag{5.1}$$

There are no other lines on $\mathcal{S}$ ([5], page 775), so this is the whole structure of line intersections on $\mathcal{S}$. Now let $\mathrm{GL}(\mathcal{S})$ be the group of linear automorphisms of $\mathcal{S}$.

Lemma 5.1. *Let $A, B \in GL(\mathcal{S})$ such that $A_{|\Delta_0} = B_{|\Delta_0}$. Then $A = B$.*

Proof. Let $I \in \mathrm{GL}(\mathcal{S})$ be the identity. If $A \in \mathrm{GL}(\mathcal{S})$ and $A_{|\Delta_0} = I_{|\Delta_0}$, then A fixes the $\mathfrak{p}_i$, so invaries the Δ_i. But then by A invaries also Δ_{ij}, the unique line other than Δ_0 which meets Δ_i and Δ_j, so A fixes $\mathfrak{p}_{ij}$, $j = 1, \ldots, 6$. Hence $A_{|\Delta_i} = I_{|\Delta_i}$. Similarly, one sees that A is the identity on any of the 32 lines on $\mathcal{S}$, so $A = I$. □

Let $A \in \mathrm{GL}(\mathcal{S})$. Since $A(\Delta_0)$ is a line, by Lemma 5.1 there exists a unique involution $\varepsilon \in \mathrm{Inv}(\mathcal{S})$ such that $\varepsilon \circ A(\Delta_0) = \Delta_0$. We associate to A the permutation $\sigma \in S_6$ such that

$$\varepsilon \circ A(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)} \quad \text{for } i = 1, \ldots, 6. \tag{5.2}$$

Note that $\sigma = \mathrm{id}$ iff $\varepsilon \circ A_{|\Delta_0} = I_{|\Delta_0}$ iff $\varepsilon \circ A = I$ (by Lemma 5.1) iff $A \in \mathrm{Inv}(\mathcal{S})$.

Definition 5.2. $\mathrm{GL}_0(\mathcal{S})$ is the subgroup of $\mathrm{GL}(\mathcal{S})$ of linear automorphisms A such that $A(\Delta_0) = \Delta_0$.

Lemma 5.3. *Let $A \in GL(\mathcal{S})$ and $\sigma \in S_6$ be the permutation associated to A by* (5.2). *Then, for any $1 \leq i \leq 6$ we have:*

$$A \circ \varepsilon^{(i)} = \varepsilon^{(\sigma(i))} \circ A. \tag{5.3}$$

Proof. Let $B = \varepsilon \circ A$. Then $B(\Delta_0) = \Delta_0$ and $B(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$, so $B(\Delta_i) = \Delta_{\sigma(i)}$. The unique line cutting $\Delta_{\sigma(i)}$ and $\Delta_{\sigma(j)}$ is $\Delta_{\sigma(i)\sigma(j)}$, hence $B(\Delta_{ij}) = \Delta_{\sigma(i)\sigma(j)}$. Then

$$B(\mathfrak{p}_{ij}) = B(\Delta_i \cap \Delta_{ij}) = B(\Delta_i) \cap B(\Delta_{ij}) = \Delta_{\sigma(i)} \cap \Delta_{\sigma(i)\sigma(j)} = \mathfrak{p}_{\sigma(i)\sigma(j)}.$$

Now one sees that $(\varepsilon \circ A)^{-1} \circ \varepsilon^{(\sigma(i))} \circ (\varepsilon \circ A)$ acts like $\varepsilon^{(i)}$ on $\mathfrak{p}_j$. By Lemma 5.1 and knowing that $\mathrm{Inv}(\mathcal{S})$ is commutative, we conclude $A \circ \varepsilon^{(i)} = \varepsilon^{(\sigma(i))} \circ A$. □

Proposition 5.4. *Let $\psi : GL(\mathcal{S}) \longrightarrow GL_0(\mathcal{S})$ be the map $A \mapsto \varepsilon \circ A$ defined by formula* (5.2). *We have an exact sequence of groups*

$$1 \longrightarrow \mathrm{Inv}(\mathcal{S}) \longrightarrow GL(\mathcal{S}) \xrightarrow{\psi} GL_0(\mathcal{S}) \longrightarrow 1.$$

By Proposition 5.4, $\mathrm{Inv}(\mathcal{S}) = \ker(\psi)$ is a normal subgroup of $\mathrm{GL}(\mathcal{S})$.

Corollary 5.5. *For any linear automorphism A of $\mathcal{S}$ not in* $\mathrm{Inv}(\mathcal{S})$, *the centralizer of A in* $\mathrm{Inv}(\mathcal{S})$ *is not equal to* $\mathrm{Inv}(\mathcal{S})$.

We now show that $\mathrm{GL}_0(\mathcal{S})$ is in bijection with the group of linear automorphisms of Δ_0 which invariate the set $\{p_i,\ i = 1, \ldots, 6\}$. Lemma 5.1 and Proposition 5.6 below give necessary and sufficient conditions for the existence of nontrivial elements of $\mathrm{GL}_0(\mathcal{S})$. For the existence of noncommuting involutions of $\mathcal{S}$, see [8], Section 6.

Proposition 5.6. *Let $\sigma \in S_6$ and let $B : \Delta_0 \longrightarrow \Delta_0$ be a linear automorphism of Δ_0 such that for $1 \leq i \leq 6$, we have $B(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$. Then there exists a unique $A \in GL_0(\mathcal{S})$ such that $A_{|\Delta_0} = B$.*

Proof. Suppose σ and B given. If A exists, it is unique by Lemma 5.1 and σ is the permutation associated to A defined by (5.2). Let $\widetilde{A}$ be the linear operator of $\mathcal{P}_5$ (polynomials of degree ≤ 5) associated to A. Let $a, b, c, d \in \bar{k}$ such that

$$\widetilde{A}(1) = aX + b \quad \text{and} \quad \widetilde{A}(X) = cX + d\,.$$

After some linear algebra and using (5.3), we find that the image of a point $\mathfrak{p} \in \mathcal{S}$ represented by $P(X) = \sum_j \pi_j P_j(X)$ is

$$\widetilde{A}(P(X)) = \sum_j \underbrace{\left(\pi_j \frac{\omega_j}{\omega_{\sigma(j)}}(a\theta_{\sigma(j)} + b)\right)}_{\pi'_{\sigma(j)}} P_{\sigma(j)}(X)\,. \tag{5.4}$$

We have to prove that the point $(\pi'_1 : \ldots : \pi'_6)$ satisfies equations (3.3).

We show that $k_{\sigma(j)}\omega_{\sigma(j)}{\pi'}^2_{\sigma(j)} = \alpha_j\omega_j\pi_j^2$ for a quadratic polynomial α_j in θ_j. We have:

$$k_{\sigma(j)}\omega_{\sigma(j)}{\pi'}^2_{\sigma(j)} = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\omega_j}{\omega_{\sigma(j)}}\omega_j\pi_j^2\,,$$

and then

$$\alpha_j = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\omega_j}{\omega_{\sigma(j)}}\,.$$

Write $\widetilde{A}(X - \theta_i)$ in two ways, using the fact that $A(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$ or linearity of $\widetilde{A}$:

$$\mu_j(X - \theta_{\sigma(j)}) = \widetilde{A}(X - \theta_j) = cX + d - \theta_j(aX + b) \quad \text{where } \mu_j \in \bar{k}\,.$$

Replacing $X = \theta_{\sigma(j)}$, we obtain the formula

$$\theta_j = \frac{c\theta_{\sigma(j)} + d}{a\theta_{\sigma(j)} + b}\,, \tag{5.5}$$

which gives the relations between the roots of $F(X)$ necessary for the existence of the linear automorphism B.

Now, we calculate ω_j replacing each θ_j by the formula (5.5):

$$\omega_j = \prod_{i\neq j}(\theta_i - \theta_j) = \prod_{i\neq j}\left(\frac{c\theta_{\sigma(i)} + d}{a\theta_{\sigma(i)} + b} - \frac{c\theta_{\sigma(j)} + d}{a\theta_{\sigma(j)} + b}\right)$$

$$= \frac{1}{(a\theta_{\sigma(j)} + b)^4}\underbrace{\frac{1}{\prod_i(a\theta_{\sigma(i)} + b)}}_{\text{constant}}\prod_{i\neq j}\left((\theta_{\sigma(i)} - \theta_{\sigma(j)})\underbrace{(bc - ad)}_{\text{constant}}\right)\,.$$

Call γ the constant part of the equation:

$$\frac{\omega_j}{\omega_{\sigma(j)}} = \frac{\gamma}{\left(a\theta_{\sigma(j)} + b\right)^4}\,. \tag{5.6}$$

Replacing (5.6) in α_j, we have:

$$\alpha_j = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\gamma}{\left(a\theta_{\sigma(j)} + b\right)^4} = \gamma\frac{k_{\sigma(j)}}{\left(a\theta_{\sigma(j)} + b\right)^2}\,.$$

To see that α_j is quadratic in θ_j (for each k_j), we use formula (5.5) to obtain:

$$a\theta_j - c = \frac{ad - bc}{a\theta_{\sigma(j)} + b} \quad \text{which gives the result for } k_{\sigma(j)} = 1;$$

$$a^2\theta_j^2 - c^2 = \frac{2ac(ad - bc)\theta_{\sigma(j)} + a^2d^2 - b^2c^2}{(a\theta_{\sigma(j)} + b)^2}$$

$$\text{which gives the result for } k_{\sigma(j)} = \theta_{\sigma(j)};$$

$$b\theta_j - d = \frac{(bc - ad)\theta_{\sigma(j)}}{a\theta_{\sigma(j)} + b} \qquad \text{which gives the result for } k_{\sigma(j)} = \theta^2_{\sigma(j)}\,. \quad \square$$

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Faculdade de Matemática, Universidade Federal de Uberlândia, MG, Brazil
E-mail address: gonzalo@famat.ufu.br

Section de Mathématiques, Université de Genève, CP 64, 1211 Geneva 4, Switzerland
E-mail address: constantin.manoil@edu.ge.ch
Current address: Collège Sismondi, 3 Chemin Rigot, 1202 Genève (Geneva), Switzerland