EXPLICIT COMPUTATIONS ON THE DESINGULARIZED KUMMER SURFACE

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Abstract. We find formulas for the birational maps from a Kummer surface $\mathcal{K}$ and its dual $\mathcal{K}^*$ to their common minimal desingularization $S$. We show how the nodes of $\mathcal{K}$ and $\mathcal{K}^*$ blow up. Then we give a description of the group of linear automorphisms of $S$.

1. Introduction

In the 19th century a singular surface $\mathcal{K}$, called the Kummer surface, was attached to a quadratic line complex. A minimal desingularization $\Sigma$ of $\mathcal{K}$ and a birational map $\kappa : \mathcal{K} \to \Sigma$ were constructed by geometric methods. One may call this the classical construction of the Kummer surface, which we recall in Section 4.

Another construction is the following: Let $A$ be an abelian surface, and let $\sigma$ be the involution of $A$ given by $\sigma(x) = -x$. The quotient $\mathcal{K} = A/\sigma$ has 16 double points, and one defines a K3 surface $S$ to be $\mathcal{K}$ with these 16 nodes blown up ([1], Prop. 8.11). To be consistent with the historical point of view and with our main reference [3], we call $S$ the desingularized Kummer surface. If $A = \mathcal{J}(C)$, where $\mathcal{J}(C)$ is the Jacobian of a curve $C$ of genus 2, then $\mathcal{K}$ is called the Kummer surface belonging to $C$. The connection between the two constructions of the Kummer surface is explicitly established in [3], Chapter 17 (see Lemma 4.5).

A desingularization $S$ of $\mathcal{K}$ is constructed explicitly in [3], Chapter 16, by algebraic methods. Denote by $\mathcal{K}^*$ the projective dual of $\mathcal{K}$. There are birational maps $\kappa : \mathcal{K} \to S$ and $\kappa^* : \mathcal{K}^* \to S$ and morphisms extending $\kappa^{-1} : S \to \mathcal{K}$ and $\kappa^{*-1} : S \to \mathcal{K}^*$ to all of $S$. We denote these extensions also by $\kappa^{-1}$ and $\kappa^{*-1}$. They are minimal desingularizations of $\mathcal{K}$ and $\mathcal{K}^*$.

Origins. Cassels and Flynn explain that the surface $S$ comes from the behavior of six of the tropes (see Definition 2.3) under the duplication map. The existence of $S$ raises more far-reaching questions. Indeed, if the ground field $k$ is algebraically closed, one has a commutative diagram:

$\begin{align*}
\mathcal{J}(C) & \xrightarrow{d_0} \mathcal{J}(C)^0 \\
\pi & \downarrow \quad \downarrow \pi_0 \\
\mathcal{K} & \xrightarrow{d_0^*} \mathcal{K}^*
\end{align*}$

(1.1)
where $\mathcal{J}(C)^0$ is the dual of $\mathcal{J}(C)$ as an abelian variety. Here, the maps $d_0$ and $d_0'$ depend on the choice of a rational point on $C$. Thus the abelian varieties duality matches with the projective one. When $k$ is not algebraically closed, one has to enlarge the ground field to obtain such diagrams, yet $\mathcal{S}$ is a desingularization over $k$ of both $K$ and $K^*$. One may ask if there is a unifying object for $\mathcal{J}(C)$ and $\mathcal{J}(C)^0$, generalizing the abelian varieties duality.

Recent developments. The Jacobian $\mathcal{J}(C)$ can be embedded in $\mathbb{P}^{15}$ and is described by 72 quadratic equations \( (3) \). More computable objects, $\mathcal{S}$ and its twists, appeared in recent attempts by M. Stoll and N. Bruin, to compute the Mordell-Weil group of $\mathcal{J}(C)$. We give a brief account of it in Section 5 of \[8\].

Cassels and Flynn already suggested that the 2-Selmer group could be investigated by using twists of $\mathcal{S}$. In 2007 A. Logan and R. van Luijk \[7\] and P. Corn \[2\] made use of twists of $\mathcal{S}$ to find specific curves with nontrivial 2-torsion elements in the Tate-Shafarevich groups of their Jacobians.

**Our results and structure of this paper.** In Section 2 we give a background. The rare relevant facts, not included in this paper, are contained in \[3\] and \[8\].

This paper is structured along two computational ideas. First, to profit from the algebraic construction of $\mathcal{S}$ in \[8\] in order to describe its linear automorphism. Second, to link $\mathcal{S}$ and $\Sigma$ and thus bring line complexes into the picture. Our results achieve part of the program suggested in \[3\], at the end of Chapter 16.

In Section 3 we complete the construction in \[8\], enlarge the set of points where $\kappa : K \rightarrow \mathcal{S}$ is explicitly defined and find formulas for $\kappa$. These formulas allow one to describe how each singularity of $\mathcal{S}$ and $\Sigma$ blows up (see Lemma \[5.6\]).

In Section 4 we define an explicit isomorphism $\Theta$ between $\Sigma$ and $\mathcal{S}$. We observe that well-known projective isomorphisms $W_i : K \rightarrow K^*$ lift to $\Sigma$ to correspondences given by line complexes of degree 1. Using $\Theta$ we show that the $W_i$ lift to $\mathcal{S}$ to commuting involutions $\varepsilon^{(i)}$ (Corollary \[4.12\]). With the formulas for $\kappa$ at hand, one can now find formulas for $\kappa^*$. In Section 5 we give a description of the group of linear automorphisms of $\mathcal{S}$.

**2. Preliminaries**

We work over a field $k$ with $\text{char}(k) \neq 2$ and with more than 5 elements. By a curve, we mean a smooth projective irreducible variety of dimension 1. Throughout this paper, $\mathcal{C}$ will be a curve of genus 2. Such a curve admits an affine model:

\[(2.1) \quad \mathcal{C}' : \quad Y^2 = F(X) = f_0 + f_1X + \cdots + f_6X^6 \in k[X], \quad f_6 \neq 0 \]

and $F$ has distinct roots $\theta_1, \ldots, \theta_6$. The points $a_i = (\theta_i, 0)$ are the Weierstrass points. We denote by $\infty^\perp$ the points at infinity on the completion $\mathcal{C}$ of $\mathcal{C}'$. For a given point $r = (x, y)$ on $\mathcal{C}$, the conjugate of $r$ under the $\pm Y$ involution is the point $\bar{r} = (x, -y)$. For a divisor $\mathcal{X} = \sum n_i r_i$ we denote by $\bar{\mathcal{X}} = \sum n_i \bar{r}_i$. The class of a divisor $\mathcal{X}$ is denoted by $[\mathcal{X}]$ and a divisor in the canonical class by $K_{\mathcal{C}}$.

One can regard a point of the Jacobian $\mathcal{J}(\mathcal{C})$ as the class of a divisor $\mathcal{X} = r + u$, where $r = (x, y), u = (u, v)$ is a pair of points on $\mathcal{C}$. Starting from this, Flynn constructs a projective embedding of the Jacobian as follows \( (4) \). For a point $\mathcal{X} = \{r, u\}$ on $\mathcal{C}'^2$ (symmetric product) with $r = (x, y), u = (u, v)$, define:

\[
\sigma_0 = 1, \quad \sigma_1 = x + u, \quad \sigma_2 = xu, \quad \beta_0 = \frac{F_0(x, u) - 2yv}{(x - u)^2},
\]
where
\[
F_0(x, u) = 2f_0 + f_1(x + u) + 2f_2xu + f_3xu(x + u) \\
+ 2f_4(xu)^2 + f_5(xu)^2(x + u) + 2f_6(xu)^3.
\]

The Jacobian is then the projective locus of \( z = (z_0 : \ldots : z_{15}) \) in \( \mathbb{P}^{15} \), where \( z_0 = \delta; z_1 = \gamma_1; z_2 = \gamma_0; z_i = \beta_{5-i}, i = 3, 4, 5; z_i = \alpha_{9-i}, i = 6, \ldots, 9; z_i = \sigma_{14-i}, i = 10, \ldots, 14; \) and \( z_{15} = \rho \). For the definition of the functions \( \alpha, \beta, \) etc. and details, see [3], Chapter 2.

**Definition 2.1.** The Kummer surface \( \mathcal{K} \) belonging to a curve of genus 2, is the projective locus in \( \mathbb{P}^3 \) of the elements \( \xi = (\xi_1 : \xi_2 : \xi_3) = (\sigma_0 : \sigma_1 : \sigma_2 : \beta_0) \).

The equation of the Kummer surface is
\[
(2.2) \quad \mathcal{K} : \quad K = K_2\xi_1^2 + K_1\xi_4 + K_0 = 0,
\]
where the \( K_i \) are forms of degree 4 – \( i \) in \( \xi_1, \xi_2, \xi_3 \) ([3], formula (3.1.9)). The natural map from \( J(C) \) to \( \mathcal{K} \) given by
\[
(z_0 : \ldots : z_{15}) \mapsto (z_{14} : z_{13} : z_{12} : z_5) = (\xi_1 : \ldots : \xi_4)
\]
is 2 to 1; the ramification points correspond to divisor classes \([\mathcal{X}]\) with \([\mathcal{X}] = [\mathcal{X}]\).

The images of these classes are the 16 nodes on \( \mathcal{K} \). These are the tropes; they cut conics on \( \mathcal{K} \). They correspond to the 16 singular points of \( \mathcal{K}^* \). The equations of the \( T_i \) are:
\[
(2.3) \quad T_i : \quad \theta_i^2\xi_1 - \theta_i\xi_2 + \xi_3 = 0,
\]

**Lemma 2.4.** For each Weierstrass point \( a_i = (\theta_i, 0) \) there is a projective map \( W_i : \mathcal{K} \to \mathcal{K}^* \) induced by the addition of \( a_i \) to a divisor of degree 2. One has \( W_i(N_0) = T_i \) (singular point of \( \mathcal{K}^* \)). Furthermore, \( W_i^{-1} \circ W_j(N_0) = N_{ij} \).

**Proof.** See [3], Lemma 4.5.1. \( \square \)

## 3. The desingularized Kummer surface

We recall the facts from [3], Chapter 16 we need, keeping the notation there.

Let \( \mathbf{p} = (p_0 : \ldots : p_5) \), where the \( p_j \) are indeterminates, and put \( P(X) = \sum_0^5 p_jX^j \). Let \( \mathcal{S} \) be the projective locus of the \( \mathbf{p} \) for which \( P(X)^2 \) is congruent to a quadratic in \( X \) modulo \( F(X) \). Put
\[
(3.1) \quad P_j(X) = \prod_{i \neq j}(X - \theta_i) = \sum_{k=0}^5 h_{jk}X^k
\]
and \( \omega_j = P_j(\theta_j) \neq 0 \). Since \( \theta_i \neq \theta_j \) for \( i \neq j \), we have \( \omega_j \neq 0 \) and the \( P_j \) span the vector space of polynomials of degree at most 5. We have
\[
(3.2) \quad P(X) = \sum_j \pi_j P_j(X), \quad \text{where} \quad \pi_j = \frac{P(\theta_j)}{\omega_j}.
\]
The $K3$ surface $S$ is the complete intersection in $\mathbb{P}^5$ of the quadrics $S_0$, $S_1$, $S_2$ where

$$(3.3) \quad S_i : S_i = 0 \quad \text{and} \quad S_i = \sum_j \theta_j^i \omega_j \pi_j^2 \quad \text{for} \quad i = 0, 1, 2.$$ 

It is a minimal desingularization of $K$ and of $K^*$. Here the $S_i$ are quadratic forms in $p$ with coefficients in $\mathbb{Z}[f_1, \ldots, f_6]$.

The following theorems hold ([3], Theorems 16.5.1 and 16.5.3):

**Theorem 3.1.** There is a birational map $\kappa : K \dasharrow S$ defined for general $\xi \in K$ as follows: Let $X = \{(x, y), (u, v)\}$ correspond to $\xi$. Put $G(X) = (X - x)(X - u)$ and let $M(X)$ be the cubic determined by the property that $Y - M(X)$ vanishes twice on $X$. Let $P(X) = \sum_3 p_j X^j$ be determined by $GP \equiv M \mod F$. Then $\kappa(\xi)$ is the point with projective coordinates $(p_0 : \ldots : p_5)$.

Let $\kappa^* : K^* \dasharrow S$ be the birational map defined in [3], Theorem 16.5.2.

**Theorem 3.2.** Let $\xi \in K$ and $\eta \in K^*$ be dual, that is, $\eta$ gives the tangent to $K$ at $\xi$. Then $\kappa(\xi) = \kappa^*(\eta)$.

Our first result is the following.

**Lemma 3.3.** The map $\kappa : K \dasharrow S$ from Theorem 3.1 is given by the formulas listed below.

**Proof.** The problem is to make effective the method given in [3], Chapter 16. For completeness and due to typing errors there, we recall it in [8], in the Appendix. As presented in [3], the method works for a general element $[X] = [(x, y) + (u, v)]$, where $yv \neq 0$ and $x \neq u$.

First, we put $y^2 = F(x)$, $v^2 = F(u)$, $yv = [F_0(x, u) - \beta_0(x - u)^2] / 2$ in the coefficients of $P(X)$, then as the resulting coefficients are symmetric functions of $x$ and $u$, we express them in terms of $\xi_2 = x + u$ and $\xi_3 = xu$. Finally, we homogenize the formulas with respect to $\xi_1 = 1$, $\xi_2$, $\xi_3$, $\xi_4$. One first obtains

$$\kappa(\xi) = (\tilde{p}_0(\xi) : \ldots : \tilde{p}_5(\xi)), \quad \text{where}$$

$$(3.4) \quad \tilde{p}_j(\xi) = \alpha_j K_2 + \beta_j (K_1 \xi_4 + K_0), \quad \text{for} \quad 0 \leq j \leq 5.$$ 

Here $\alpha_j$ and $\beta_j$ are homogeneous forms in $\xi$ of degree 4 and 2, respectively, and the $K_j$ are those in (2.2). Taking $p_j(\xi) = (\tilde{p}_j(\xi) - \beta_j K) / K_2 = \alpha_j - \beta_j \xi_j^2$, we obtain formulas of degree 4 for $\kappa$, which will be defined also for $K_2 = 0$, extending $\kappa$ to images of divisor classes $[X] = [(x, y) + (u, v)]$ with $x = u$ and $y = v \neq 0$. However, the formulas do not work for points with $F'(x) = 0$ and for the image of $[2 \infty^+]$. We will treat the case $y = 0$ or $v = 0$ in connection with nodes and tropes.
Polynomial definition of $\kappa$. 

$$p_0 = -f_3f_6\xi_1^3 + 1/2f_5^2\xi_2\xi_3^3 - 2\xi_3^2f_4f_6\xi_1 - 2\xi_3^2f_1f_6f_2 - 2\xi_3f_1\xi_2f_2 - 1/2\xi_3^2f_1\xi_4 - 1/2\xi_3^2f_1\xi_1f_4 - 3\xi_3^2f_6f_3 - 2\xi_3^2f_6f_1 - 1/2\xi_3^2f_6f_3 - 3\xi_3^2f_6f_1 - 1/2\xi_3f_6f_3 - 3\xi_3f_6f_1 - 3\xi_3f_6f_1$$

$$p_1 = 2\xi_4^2f_6 - 2\xi_3f_6\xi_4^2f_5f_6 = 1/2\xi_4^2f_6f_3 - 1/2f_4f_2f_6 - 3\xi_4f_2f_3 + 3/2\xi_3f_3f_4 - 3\xi_3f_3f_4 + 2\xi_2f_6\xi_4^2 + 3/2\xi_3\xi_4f_3 + 3/2f_6\xi_4^2$$

$$p_2 = 2\xi_4^2f_4f_3 - f_4f_6\xi_3 + 3\xi_4^2f_4f_6 - 3\xi_4f_4f_6 - 3\xi_4f_4f_6 - 3\xi_4f_4f_6$$

$$p_3 = -2\xi_6^2f_6\xi_1^3 - 2\xi_6^2f_6\xi_1^3 + 2\xi_6^2f_6\xi_1^3 - 2\xi_6^2f_6\xi_1^3$$

$$p_4 = \xi_3^2f_4f_3 + 2\xi_3\xi_1f_2f_1 - 2\xi_3^2f_2f_1 - 2\xi_3f_1f_2 - 2\xi_3f_1f_2$$

$$p_5 = 2f_6\xi_3^2f_3 - 2\xi_3f_6\xi_3^2f_1 - f_2f_1 + \xi_4f_4 + f_4f_6$$

Let the point $(p_0: \ldots: p_5)$ be represented by $P(X) = \sum_{i=0}^{5} p_iX^i$ and let

$$g_i(X) = 1 - 2\frac{P_i(X)}{P_i(\theta_i)}, \quad i = 1, \ldots, 6$$
where $P_i(X)$ is defined by (3.1). We see that $g_i(\theta_j) = (-1)^{\delta_{ij}}$, so $g_i(X)^2 \equiv 1 \mod F(X)$. There are 6 commuting involutions $\varepsilon(i)$ of $S$, defined as follows:

$$
(3.6) \quad \varepsilon(i)(P(X)) = g_i(X)P(X) \mod F(X), \quad i = 1, \ldots, 6.
$$

In terms of coordinates $\pi_j$, one has

$$
(3.7) \quad \varepsilon(i)(\pi_j) = (-1)^{\delta_{ij}}\pi_j.
$$

**Definition 3.4.** We define $\text{Inv}(S)$ to be the group of 32 commuting involutions of $S$ generated by the $\varepsilon(i)$.

The $p = (p_0 : p_1 : 0 : 0 : 0 : 0)$ form a rational line $\Delta_0 \subset S$. We shall often write $p_0 + p_1 \Xi \in \Delta_0$. Acting on $\Delta_0$ by the involutions gives 31 further lines.

**Notation 3.5.** We denote:

$$
\Delta_i = \varepsilon(i)(\Delta_0), \quad \Delta_{ij} = \varepsilon(i) \circ \varepsilon(j)(\Delta_0) \quad \text{and} \quad \Delta_{ijk} = \varepsilon(i) \circ \varepsilon(j) \circ \varepsilon(k)(\Delta_0).
$$

The main result in this section describes how the nodes of $K$ and $K^*$ blow up.

**Lemma 3.6.** The map $\kappa$ blows up the node $N_0 = (0 : 0 : 0 : 0 : 1)$ of $K$ into the line $\Delta_0$ and the 15 nodes $N_{ij}$ into the lines $\Delta_{ij}$. The tropes $T_i$ and $T_{ijk}$ blow up by $\kappa^*$ into the lines $\Delta_i$ and $\Delta_{ijk}$.

**Proof.** The node $N_0$ corresponds to the canonical class, so we consider divisors of the type $\Xi = (x, y) + (u, v)$ with $u = x + h$, $h$ small and $v \approx -y \neq 0$. Then the local behavior of the Kummer coordinates is $\xi_1 = 1, \xi_2 = 2x + h \approx 2x, \xi_3 = x(x + h) \approx x^2$ and

$$
\xi_4 = \frac{F_0(x, x + h) - 2yv}{h^2} \approx \frac{4y^2}{h^2}.
$$

Replacing this in the formulas for $\kappa$ and clearing denominators, then taking the limit as $h \to 0$ and dividing by $y^4 \neq 0$, we obtain

$$
\kappa(\xi) \approx (-16xy^4 : 16y^4 : 0 : 0 : 0 : 0) \approx (-x : 1 : 0 : 0 : 0 : 0).
$$

Note that $\Delta_0 \cap \Delta_i = (-\theta_i : 1 : 0 : 0 : 0 : 0)$, since for $(X - \theta_i) \in \Delta_0$ we have

$$
\varepsilon(i)(X - \theta_i) \equiv g_i(X)(X - \theta_i) \equiv (X - \theta_i) \mod F(X).
$$

We now show that $\Delta_0 \cap \Delta_{ij} = \emptyset$ for $i \neq j$. Indeed, the intersection point $p$ should be invariant by $\varepsilon(i) \circ \varepsilon(j)$. A polynomial $P(X)$ represents such a point iff

$$
\alpha P(X) \equiv g_i(X)g_j(X)P(X) \mod F(X) \quad \text{for some } \alpha \in \bar{k}^* \text{ iff } \frac{F(X)}{P(X)(\alpha - g_i(X)g_j(X))}.
$$

Replacing $X$ by the roots of $F(X)$ one sees that $P(X)$ must have at least two roots among $\theta_i$, so it must be of degree at least 2 and therefore cannot represent a point on $\Delta_0$. Similarly, $\Delta_0 \cap \Delta_{ijk} = \emptyset$ for $i \neq j \neq k$.

The six $\Delta_i$ are strict transforms of the conics $\Gamma_i$ cut on $K$ by the tropes $T_i$. To see this and to define $\kappa$ for points corresponding to divisors $\Xi = \{x, y\} + \{\theta_i, 0\}$ with $y \neq 0$, write $F(X) = f_k(X - \theta_1)P_1(X)$. From this we get formulas for $f_k, k = 0, \ldots, 6$ depending on $\theta_i$ and $h_{ij}, j = 0, \ldots, 5$, the coefficients of $P_1(X)$, which we plug into $\xi_4 = (F_0(x, \theta_i)/(x - \theta_i)^2)$. We substitute then $\xi_1 = 1, \xi_2 = x + \theta_i,
that is, \( \xi_3 = x \theta_i \) and \( \xi_4 \) in the formulas for \( \kappa \). On multiplying by \((x - \theta_i)^2/(f_2^2 P_1(x))\) (note that \( P_1(x) \neq 0 \)), we obtain

\[
P(X) = 2(x - \theta_i)P_1(X) + P_1(\theta_i)(X - x),
\]

that is,

\[
\begin{align*}
p_0 & = 2h_{i0}(x - \theta_i) - P_1(\theta_i)x, \\
p_1 & = 2h_{i1}(x - \theta_i) + P_1(\theta_i), \\
p_j & = 2h_{ij}(x - \theta_i) \quad \text{for } 2 \leq j \leq 5.
\end{align*}
\]

Equation (3.8) shows that the points \((1 : x + \theta_i : x \theta_i : \xi_4)\) belong to the conic \( \Gamma_i \).

Formulas (3.9) give parametric equations (in \( x \)) of the strict transform of \( \Gamma_i \) by \( \kappa \).

To confirm that this is \( \Delta_i \), one verifies that

\[
P(X) \equiv P_1(\theta_i)g_i(X)(X - x) \mod F(X).
\]

Applying the results in Section 4 and especially Corollary 4.1.1 one concludes that:

1) the tropes \( T_i \) considered as singular points of \( \mathcal{K}^* \), blow up by \( \kappa^* \) into \( \Delta_i \);
2) each of the fifteen \( N_{ij} \) blows up into \( \Delta_{ij} \);
3) the tropes \( T_{ijk}, i \neq j \neq k \) blow up into \( \Delta_{ijk} \);
4) the ten \( \Delta_{ijk} (i \neq j \neq k) \) are strict transforms of the ten conics cut on \( \mathcal{K} \) by the tropes (planes) not containing \( N_0 \). Each of them intersects six \( \Delta_{ij} \) since each node is on six tropes.

4. Line complexes

Let \( u = (u_1 : u_2 : u_3 : u_4) \) and \( v = (v_1 : v_2 : v_3 : v_4) \) be distinct points in \( \mathbb{P}^3 \).

Put \( p_{ij} = u_iv_j - u_jv_i \). The Grassmann coordinates of the line \( \langle u, v \rangle \subset \mathbb{P}^3 \) are

\[
p = (p_{43} : p_{24} : p_{41} : p_{21} : p_{31} : p_{32}) = (X_1 : \ldots : X_6).
\]

The Grassmannian quadric \( \mathcal{G} \subset \mathbb{P}^5 \), representing the lines in \( \mathbb{P}^3 \) has the equation:

\[
G(X_1, \ldots, X_6) = 2X_1X_4 + 2X_2X_5 + 2X_3X_6 = 0.
\]

**Definition 4.1.** A line complex of degree \( d \) is a set of lines in \( \mathbb{P}^3 \) whose Grassmann coordinates satisfy a homogeneous equation \( Q(X_1, \ldots, X_6) = 0 \) of degree \( d \).

If \( d = 1 \), this is called a linear complex, and if \( d = 2 \), it is a quadratic complex.

A line \( L \in \mathcal{G} \) parametrizes a pencil of lines in \( \mathbb{P}^3 \). The lines of a pencil \( L \) all pass through a point \( f(L) = u \), called the *focus* of the pencil, and lie in one plane \( \mathfrak{h}(L) = \pi_u \), the *plane* of the pencil.

All lines in a linear complex \( \mathcal{L} \) passing through a given point \( u \) (respectively, lying in a plane \( \pi \)), form a pencil \( L_u \) (respectively, \( L_\pi \)). Each linear complex \( \mathcal{L} \) establishes a *correspondence* between points and planes in \( \mathbb{P}^3 \),

\[
I(u) = \mathfrak{h}(L_u), \quad I(\pi) = f(L_\pi), \quad I^2 = 1,
\]

which is also defined for lines; if \( l \subset \mathbb{P}^3 \) is the line \( \langle u, u' \rangle \), then \( I(l) = I(u) \cap I(u') \).

The line \( I(l) \) is the *polar* line of \( l \) with respect to the given linear complex.

**Definition 4.2.** Two linear complexes are called apolar if the correspondences they define commute.
Let $H$ be any quadratic form in six variables such that the quadrics $G = 0$ and $H = 0$ intersect transversely, and denote by $\mathcal{H} = \{ x \in \mathbb{P}^3 \mid H(x) = 0 \}$. Let $\mathcal{W} = \mathcal{G} \cap \mathcal{H}$ and $\mathcal{A} = \text{the set of lines on } \mathcal{W}$. The points in $\mathcal{W}$ represent the lines in $\mathbb{P}^3$ whose Grassmann coordinates $p$ satisfy $H(p) = 0$. A line $L \in \mathcal{A}$ represents a pencil of lines in $\mathbb{P}^3$ of the quadratic complex defined by $H$.

**Definition 4.3.** The Kummer surface $\mathcal{K} \subset \mathbb{P}^3$ associated to the quadratic complex $\mathcal{H}$ is the locus of focuses of such pencils: $\mathcal{K} = \{ f(L) \mid L \in \mathcal{A} \}$.

**Definition 4.4.** The dual Kummer surface $\mathcal{K}^* \subset \mathbb{P}^3$ associated to the quadratic complex $\mathcal{H}$ is the locus of planes of such pencils.

From now on we suppose $f_6 = 1$.

**Lemma 4.5.** For any curve $\mathcal{C}$ of genus 2, the Kummer surface belonging to the curve $\mathcal{C}$ given by (2.1) coincides with the Kummer surface just defined, if one takes the quadratic complex $\mathcal{H}$ to be given by

$$H = -4X_1X_5 - 4X_2X_6 - X_3^2 + 2f_5X_3X_6 + 4f_0X_4^2 + 4f_1X_4X_5 + 4f_2X_2^2 + 4f_3X_5X_6 + (4f_4 - f_5^2)X_6^2.$$

**Proof.** See [3], Lemma 17.3.1 and pages 182–183. \qed

If a point $\xi \in \mathbb{P}^3$ is the focus of the pencil corresponding to the line $L_\xi \in \mathcal{A}$, then $L_\xi$ lies in the plane $\Pi_\xi \subset \mathcal{G}$ corresponding to lines in $\mathbb{P}^3$ passing through $\xi$. But then the conic $\Pi_\xi \cap \mathcal{H}$ contains $L_\xi$, so it is degenerate; $\Pi_\xi$ is tangent to $\mathcal{H}$ and $\Pi_\xi \cap \mathcal{H} = L_\xi \cup L'_\xi$. The lines of the quadratic complex passing through $\xi$ are in the two pencils $L_\xi$ and $L'_\xi$, each with focus $\xi$, lying in the planes $\pi_\xi$ and $\pi'_\xi$ in $\mathbb{P}^3$. The line $l_\xi = \pi_\xi \cap \pi'_\xi$ is represented on $\mathcal{G}$ by the point $p_\xi = L_\xi \cap L'_\xi$ and is called a singular line of the quadratic complex. If $L_\xi \neq L'_\xi$ the pencils are distinct and $\xi$ is a simple point of the Kummer; there is a one-to-one correspondence $\xi \leftrightarrow p_\xi$. However, if $L_\xi = L'_\xi$, then $\pi_\xi = \pi'_\xi$ and all the lines in $L_\xi$ are singular lines. The point $\xi$ is a singular point of the Kummer, because the map $f : \mathcal{A} \longrightarrow \mathcal{K}$ is algebraic. Therefore, the variety $\Sigma$ parametrizing singular lines is a desingularization of the Kummer.

**Definition 4.6.** The birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ is defined by $\kappa_1(\xi) = p_\xi$.

**Definition 4.7.** The birational map $\kappa_1^* : \mathcal{K}^* \dashrightarrow \Sigma$ associates to a plane $\pi$ tangent to $\mathcal{K}$ the intersection point of the lines in $\mathcal{A}$ parametrizing the two pencils in $\mathcal{H}$ contained in $\pi$.

The maps $\kappa_1^{-1}$ and $\kappa_1^*^{-1}$ extend to minimal desingularizations $\kappa_1^{-1} : \Sigma \longrightarrow \mathcal{K}$ and $\kappa_1^*^{-1} : \Sigma \longrightarrow \mathcal{K}^*$ (see [3] and [6]). The following is proved in [3], page 181:

**Lemma 4.8.** The surface $\mathcal{K}^*$ is the projective dual of $\mathcal{K}$; that is, if $\xi = f(L) \in \mathcal{K}$, then $\eta = f(L) \in \mathcal{K}^*$ is the tangent plane of $\mathcal{K}$ at $\xi$. Therefore $\kappa_1(\xi) = \kappa_1^*(\eta)$.

Denote by $G(\vec{X}, \vec{Y})$ the bilinear form associated to the Grassmannian $G$. Make the change of coordinates

$$\zeta_i = \frac{G(\vec{X}, \vec{Y}(\theta_i))}{\sqrt{\omega_i}}, \quad (4.1)$$
with vectors $\hat{v}(\theta_i)$ as in \cite{8} formula (17.4.3). Let $K$ be the Kummer surface associated to the quadratic complex $H$ of Lemma 4.3. By \cite{8}, Section 31 or \cite{8} formula (17.4.2), a minimal desingularization of $K$ is the $K3$ surface in $\mathbb{P}^5$:

$$\Sigma = \Sigma_0 \cap \Sigma_1 \cap \Sigma_2,$$

where $\Sigma_i : \sum_j \theta_i^j \zeta_j^2 = 0$ for $i = 0, 1, 2$.

**Proposition 4.9.** There is an explicit isomorphism $\Theta : \Sigma \rightarrow S$.

*Proof.* Let $\Theta : \Sigma \rightarrow S$ be defined by

$$\Theta(\zeta_1 : \ldots : \zeta_6) = \left(\frac{\zeta_1}{\sqrt{\omega_1}} : \ldots : \frac{\zeta_6}{\sqrt{\omega_6}}\right) = (\pi_1 : \ldots : \pi_6).$$

To pass to variables $X_j$ and $p_j$, recall that $P(X) = \sum_{i=1}^5 p_i X^i$. By (4.2) and (4.1):

$$\frac{P(\theta_1)}{\omega_1} = \pi_1 = \frac{\zeta_1}{\sqrt{\omega_1}} = G(\hat{X}, \hat{v}(\theta_1)).$$

Now, as polynomials in $X$, we have $G(\hat{X}, \hat{v}(X)) = P(X)$, because both have degree 5 and agree on the six $\theta_i$. Explicit formulas for $\Theta$ are:

$$p_0 = X_1 + f_1 X_4 \quad \quad p_2 = X_3 + 2f_3 X_5 + 2f_3 X_4 + f_5 X_6 \quad \quad p_4 = 2f_5 X_4 + 2X_5$$

$$p_1 = X_2 + 2f_2 X_4 + f_3 X_5 \quad \quad p_3 = 2f_3 X_4 + 2f_5 X_5 + 2X_6 \quad \quad p_5 = 2X_4.$$

**Proposition 4.10.** Denoting by $\kappa^{-1}$ and $\kappa_1^{-1}$ the blow-downs from $S$, respectively, $\Sigma$ to $K$ one has $\kappa_1^{-1} = \kappa^{-1} \circ \Theta$.

*Proof.* Pick a point $\xi \in \mathbb{P}^3$ and write the equations of the plane $\Pi_\xi \subset G$ of lines through $\xi$ (see equations (4.7) of \cite{8}). Take $\mathcal{H}$ to be defined as in Lemma 4.3. Now, $\Pi_\xi$ is tangent to $\mathcal{H}$ iff the intersection consists of two lines: $\Pi_\xi \cap \mathcal{H} = L_\xi \cup L'_\xi$.

Computing in terms of $\xi$ the coordinates of $p_\xi = L_\xi \cap L'_\xi$, we find homogeneous formulas for $X_i$ in $\xi$ of degree 4: $p_\xi = (X_1(\xi) : \ldots : X_6(\xi)) = \kappa_\xi(\xi)$. Comparing $\kappa(\xi) = (p_0(\xi) : \ldots : p_5(\xi))$ from Lemma 4.3 with

$$\Theta \circ \kappa_\xi(\xi) = (\tilde{p}_0(\xi) : \ldots : \tilde{p}_5(\xi)) : K \rightarrow S$$

yields $\tilde{p}_i p_5 - \tilde{p}_5 p_i = \delta_i K$, with $K$ given by (2.2), for $\delta_i$ a homogeneous polynomial in $\xi$.

Associated with a quadratic complex $\mathcal{H} : H = 0$ there is a set of 6 mutually apolar linear complexes $L_k$, such that the polar of any line in $H$ with respect to $L_k$ is in $\mathcal{H}$. If $G$ and $H$ are written in diagonal form, these complexes are $L_k : \zeta_k = 0$ for $k = 1, \ldots, 6$. The action of the correspondences $I_k$ on lines in $\mathbb{P}^3$ translates in coordinates $\xi = (\zeta_1 : \ldots : \zeta_6)$ by

$$I_k(\zeta) = (-1)^{\delta_k} \zeta,$$

which restricts to $\Sigma$. The Kummer surface is determined by $\mathcal{H}$, so it is invariant under the transformation $I_k$. The set of nodes and tropes is invariant (see \cite{8}, Section 30).

**Proposition 4.11.** Let $W_k$ be as in Lemma 2.4. For any $k$, the map $I_k$ is the unique automorphism of $\Sigma$ such that $\kappa_1^{-1} \circ I_k = W_k \circ \kappa_1^{-1}$.
Proof. Let \( \xi \in \mathcal{K} \) be a simple point and denote \( p_\xi = \kappa_1(\xi) \). For a subset \( V \subset \mathcal{G} \), put \( I_k(V) = \{I_k(l) \in \mathcal{G} \mid l \in V \} \). The pencils \( I_k(L_\xi) \) and \( I_k(L'_\xi) \) are both contained in the polar plane of \( \xi \) with respect to \( L_k \), which by Lemma 4.13 is \( W_k(\xi) \). The plane in \( \mathbb{P}^3 \) parametrizing lines in \( W_k(\xi) \) is therefore tangent to \( \mathcal{H} \) at \( I_k(L_\xi) \cap I_k(L'_\xi) = I_k(L_\xi \cap L'_\xi) = I_k(p_\xi) = I_k(\kappa_1(\xi)) \). By definition of \( \kappa^*_1 \) we have \( \kappa^*_1(W_k(\xi)) = I_k \circ \kappa_1(\xi) \).

The following corollary illustrates how the projective duality (over \( k(\theta_k) \)) between \( \mathcal{K} \) and \( \mathcal{K}^* \) lifts to \( \mathcal{S} \).

**Corollary 4.12.** For any \( k \), the map \( \varepsilon^{(k)} \) is the unique automorphism of \( \mathcal{S} \) such that \( \kappa^{*-1} \circ \varepsilon^{(k)} = W_k \circ \kappa^{-1} \).

**Proof.** Let \( \xi, \eta \in \mathcal{K} \) be dual. We have:

\[
\begin{align*}
\Theta \circ \kappa_1^{-1}(\eta) & \overset{4.3}{=} \Theta \circ \kappa_1(\xi) \overset{4.10}{=} \kappa(\xi) \overset{3.5}{=} \kappa^*(\eta).
\end{align*}
\]

Note that \( \Theta \circ I_k \circ \Theta^{-1} = \varepsilon^{(k)} \), by \( 4.12 \), \( 4.13 \) and \( 3.7 \). Therefore,

\[
\kappa^*-1 \circ \varepsilon^{(k)} \overset{4.4}{=} \kappa^{-1} \circ \Theta \circ I_k \circ \Theta^{-1} \overset{4.11}{=} W_k \circ \kappa^{-1} \circ \Theta^{-1} \overset{4.10}{=} W_k \circ \kappa^{-1}.
\]

This is summarized in the following diagram, where the arrows to \( \mathcal{K} \) and \( \mathcal{K}^* \) are the minimal desingularizations:

\[
\begin{array}{cccc}
\mathcal{S} & \overset{\Theta}{\rightarrow} & \mathcal{K} & \overset{W_k}{\rightarrow} & \mathcal{K}^* \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma & \overset{I_k}{\rightarrow} & \Sigma & \overset{\Theta}{\rightarrow} & \mathcal{S}
\end{array}
\]

Now Corollary 4.12 is useful for finding explicit formulas for \( \kappa^* \), because

\[
\kappa^* = \kappa^* \circ W_i \circ \kappa^{-1} \circ \kappa \circ W_i^{-1} = \varepsilon^{(i)} \circ \kappa \circ W_i^{-1}
\]
on an open dense set in \( \mathcal{K}^* \). We find formulas for \( W_i \) in [5], Lemma 4.13. The resulting formulas for \( \kappa^* \) are huge and not listed in this paper, since on a given example it is easier to apply successively each map involved.

**Lemma 4.13.** For any point \( \xi \in \mathbb{P}^3 \) the plane with dual coordinates \( W_i(\xi) \) is the polar plane of \( \xi \) with respect to \( L_i \).

**Proof.** See [5], Lemma 4.14.

5. Linear automorphisms of \( \mathcal{S} \)

Keeping Notation 3.5 we let

\[
\begin{align*}
p_i &= \Delta_0 \cap \Delta_i, \\
p_{ij} &= \Delta_i \cap \Delta_{ij} = \varepsilon^{(i)}(p_j), \\
p_{ijk} &= \Delta_{ij} \cap \Delta_{ijk} = \varepsilon^{(i)}(p_{jk}).
\end{align*}
\]

There are no other lines on \( \mathcal{S} \) (5, page 775), so this is the whole structure of line intersections on \( \mathcal{S} \). Now let \( GL(S) \) be the group of linear automorphisms of \( \mathcal{S} \).

**Lemma 5.1.** Let \( A, B \in GL(S) \) such that \( A_{|\Delta_0} = B_{|\Delta_0} \). Then \( A = B \).
Proof. Let $I \in \text{GL}(S)$ be the identity. If $A \in \text{GL}(S)$ and $A|_{\Delta_0} = I|_{\Delta_0}$, then $A$ fixes the $p_i$, so invaries the $\Delta_i$. But then by $A$ invaries also $\Delta_{ij}$, the unique line other than $\Delta_0$ which meets $\Delta_i$ and $\Delta_j$, so $A$ fixes $p_{ij}$, $j = 1, \ldots, 6$. Hence $A|_{\Delta_1} = I|_{\Delta_1}$. Similarly, one sees that $A$ is the identity on any of the 32 lines on $S$, so $A = I$. \hfill \Box

Let $A \in \text{GL}(S)$. Since $A(\Delta_0)$ is a line, by Lemma 5.1 there exists a unique involution $\epsilon \in \text{Inv}(S)$ such that $\epsilon \circ A(\Delta_0) = \Delta_0$. We associate to $A$ the permutation $\sigma \in S_6$ such that

$$\tag{5.2} \epsilon \circ A(p_i) = p_{\sigma(i)} \quad \text{for } i = 1, \ldots, 6.$$  

Note that $\sigma = \text{id}$ iff $\epsilon \circ A_{|\Delta_0} = I_{|\Delta_0}$ iff $\epsilon \circ A = I$ (by Lemma 5.1) iff $A \in \text{Inv}(S)$.

Definition 5.2. $\text{GL}_0(S)$ is the subgroup of $\text{GL}(S)$ of linear automorphisms $A$ such that $A(\Delta_0) = \Delta_0$.

Lemma 5.3. Let $A \in \text{GL}(S)$ and $\sigma \in S_6$ be the permutation associated to $A$ by \ref{5.2}. Then, for any $1 \leq i \leq 6$ we have:

$$\tag{5.3} A \circ \epsilon (i) = \epsilon (\sigma(i)) \circ A.$$  

Proof. Let $B = \epsilon \circ A$. Then $B(\Delta_0) = \Delta_0$ and $B(p_i) = p_{\sigma(i)}$, so $B(\Delta_i) = \Delta_{\sigma(i)}$. The unique line cutting $\Delta_{\sigma(i)}$ and $\Delta_{\sigma(j)}$ is $\Delta_{\sigma(i)\sigma(j)}$, hence $B(\Delta_{ij}) = \Delta_{\sigma(i)\sigma(j)}$. Then

$$B(p_{ij}) = B(\Delta_i \cap \Delta_{ij}) = B(\Delta_i) \cap B(\Delta_{ij}) = \Delta_{\sigma(i)} \cap \Delta_{\sigma(i)\sigma(j)} = p_{\sigma(i)\sigma(j)}.$$  

Now one sees that $(\epsilon \circ A)^{-1} \circ \epsilon (\sigma(i)) \circ (\epsilon \circ A)$ acts like $\epsilon (i)$ on $p_j$. By Lemma 5.1 and knowing that $\text{Inv}(S)$ is commutative, we conclude $A \circ \epsilon (i) = \epsilon (\sigma(i)) \circ A$. \hfill \Box

Proposition 5.4. Let $\psi : \text{GL}(S) \rightarrow \text{GL}_0(S)$ be the map $A \mapsto \epsilon \circ A$ defined by formula \ref{5.2}. We have an exact sequence of groups

$$1 \rightarrow \text{Inv}(S) \rightarrow \text{GL}(S) \xrightarrow{\psi} \text{GL}_0(S) \rightarrow 1.$$  

By Proposition 5.4, $\text{Inv}(S) = \ker(\psi)$ is a normal subgroup of $\text{GL}(S)$.

Corollary 5.5. For any linear automorphism $A$ of $S$ not in $\text{Inv}(S)$, the centralizer of $A$ in $\text{Inv}(S)$ is not equal to $\text{Inv}(S)$.

We now show that $\text{GL}_0(S)$ is in bijection with the group of linear automorphisms of $\Delta_0$ which invaries the set $\{p_i, i = 1, \ldots, 6\}$. Lemma 5.1 and Proposition 5.6 below give necessary and sufficient conditions for the existence of nontrivial elements of $\text{GL}_0(S)$. For the existence of noncommuting involutions of $S$, see \cite{8}, Section 6.

Proposition 5.6. Let $\sigma \in S_6$ and let $B : \Delta_0 \rightarrow \Delta_0$ be a linear automorphism of $\Delta_0$ such that for $1 \leq i \leq 6$, we have $B(p_i) = p_{\sigma(i)}$. Then there exists a unique $A \in \text{GL}_0(S)$ such that $A|_{\Delta_0} = B$.

Proof. Suppose $\sigma$ and $B$ given. If $A$ exists, it is unique by Lemma 5.1 and $\sigma$ is the permutation associated to $A$ defined by \ref{5.2}. Let $A$ be the linear operator of $\mathcal{P}_5$ (polynomials of degree $\leq 5$) associated to $A$. Let $a, b, c, d \in \overline{k}$ such that

$$\tilde{A}(1) = aX + b \quad \text{and} \quad \tilde{A}(X) = cX + d.$$
After some linear algebra and using (5.3), we find that the image of a point \( p \in \mathcal{S} \) represented by \( P(X) = \sum \pi_j P_j(X) \) is

\[
A(P(X)) = \sum_j \left( \frac{\omega_j}{\omega_{\theta(j)}} (a \theta_{\sigma(j)} + b) \right) P_{\sigma(j)}(X).
\]

We have to prove that the point \( (\pi_1' : \ldots : \pi_n') \) satisfies equations (3.3).

Write \( \tilde{A}(X - \theta_j) \) in two ways, using the fact that \( A(p_i) = p_{\sigma(i)} \) or linearity of \( \tilde{A} \):

\[
\mu_j(X - \theta_{\sigma(j)}) = \tilde{A}(X - \theta_j) = cX + d - \theta_j(aX + b) \quad \text{where } \mu_j \in \mathcal{k}.
\]

Replacing \( X = \theta_{\sigma(j)} \), we obtain the formula

\[
\theta_j = \frac{c \theta_{\sigma(j)} + d}{a \theta_{\sigma(j)} + b},
\]

which gives the relations between the roots of \( F(X) \) necessary for the existence of the linear automorphism \( B \).

Now, we calculate \( \omega_j \) replacing each \( \theta_j \) by the formula (5.5):

\[
\omega_j = \prod_{i \neq j} (\theta_i - \theta_j) = \prod_{i \neq j} \left( \frac{c \theta_{\sigma(i)} + d}{a \theta_{\sigma(i)} + b} - \frac{c \theta_{\sigma(j)} + d}{a \theta_{\sigma(j)} + b} \right)
\]

\[
= \frac{1}{(a \theta_{\sigma(j)} + b)^2} \prod_{i \neq j} \left( \theta_{\sigma(i)} - \theta_{\sigma(j)} \right) \frac{bc - ad}{\text{constant}}.
\]

Call \( \gamma \) the constant part of the equation:

\[
\frac{\omega_j}{\omega_{\theta(j)}} = \frac{\gamma}{(a \theta_{\sigma(j)} + b)^4}.
\]

Replacing (5.6) in \( \alpha_j \), we have:

\[
\alpha_j = k_{\sigma(j)}(a \theta_{\sigma(j)} + b)^2 \frac{\gamma}{(a \theta_{\sigma(j)} + b)^4} = \gamma \frac{k_{\sigma(j)}}{(a \theta_{\sigma(j)} + b)^2}.
\]

To see that \( \alpha_j \) is quadratic in \( \theta_j \) (for each \( k_j \)), we use formula (5.5) to obtain:

\[
a \theta_j - c = \frac{ad - bc}{a \theta_{\sigma(j)} + b} \quad \text{which gives the result for } k_{\sigma(j)} = 1;
\]

\[
a^2 \theta_j - c^2 = \frac{2ac(ad - bc) \theta_{\sigma(j)} + a^2d^2 - b^2c^2}{(a \theta_{\sigma(j)} + b)^2} \quad \text{which gives the result for } k_{\sigma(j)} = \theta_{\sigma(j)};
\]

\[
b \theta_j - d = \frac{(bc - ad) \theta_{\sigma(j)}}{a \theta_{\sigma(j)} + b} \quad \text{which gives the result for } k_{\sigma(j)} = \theta_{\sigma(j)}^2. \quad \square
DUAL KUMMER SURFACE

References


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