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INEQUALITIES FOR ZEROS OF JACOBI POLYNOMIALS VIA OBRECHKOFF'S THEOREM

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Abstract. In this paper we obtain sharp limits for all the zeros of Jacobi polynomials. We employ Obrechkoff's theorem on generalized Descartes' rule of signs and certain elaborated connection formulae which involve Jacobi and Laguerre polynomials.

1. Introduction

The behavior of zeros of the classical continuous orthogonal polynomials has been studied extensively, mainly because of their beautiful electrostatic interpretation and their important role as nodes of Gaussian quadrature formulae. The Jacobi and Laguerre polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right)$$

and

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n; \alpha+1; x)$$

in terms of the hypergeometric functions

$${}_pF_q(a_1,\dots,a_p; b_1,\dots,b_q; z) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{z^j}{j!},$$

where the Pochhammer symbol $(a)_j$ takes the values $(a)_0 = 1$ and $(a)_j = a(a+1)\cdots(a+j-1)$ for $j \in \mathbb{N}$.

In what follows, we denote by $x_{n,k}(\alpha,\beta)$ and $x_{n,k}(\alpha)$ the zeros of $P_n^{(\alpha,\beta)}(x)$ and $L_n^{(\alpha)}(x)$, respectively, both sequences arranged in decreasing order. Therefore, when $\alpha, \beta > -1$, and $\alpha > -1$, we have

$$-1 < x_{n,n}(\alpha,\beta) < x_{n,n-1}(\alpha,\beta) < \cdots < x_{n,1}(\alpha,\beta) < 1$$

and

$$0 < x_{n,n}(\alpha) < x_{n,n-1}(\alpha) < \cdots < x_{n,1}(\alpha).$$

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We shall describe briefly a simple idea motivated by the following convolution formula obtained in [25] and [18, Corollary 3.6 (i)]:

$$(x_1+x_2)^j P_j^{(a,b)}\left(\frac{x_2-x_1}{x_1+x_2}\right)\frac{(-1)^j n!j!\, L_n^{(a+b+2j+1)}(x_1+x_2)}{(a+1)_j(n+j)!}$$
$$= \sum_{\ell=0}^{n+j} Q_j(\ell;a,b,n+j)L_\ell^{(a)}(x_1)L_{n+j-\ell}^{(b)}(x_2). \tag{1.1}$$

Set $n=0$ in it and recall that Hahn polynomials $Q_j(x;\alpha,\beta,N)$, $j=0,1\ldots,N$, are defined by (see [17]):

$$Q_j(x;\alpha,\beta,N) = {}_3F_2(-j,j+\alpha+\beta+1,-x;\alpha+1,-N;1), \quad \alpha,\beta>-1.$$

Then, by the Gauss identity,

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we obtain

$$\begin{aligned} Q_j(\ell;a,b,j) &= {}_3F_2(-j,j+a+b+1,-\ell;a+1,-j;1)\\ &= {}_2F_1(j+a+b+1,-\ell;a+1;1)\\ &= \frac{\Gamma(a+1)\Gamma(\ell-j-b)}{\Gamma(a+1+\ell)\Gamma(-j-b)}\\ &= (-1)^\ell\frac{(b+1)_j}{(a+1)_j(b+1)_{j-\ell}}. \end{aligned}$$

Therefore, if we apply (1.1) with $n=0$, $j=n$, $\ell=k$, $x_1=x$ and $x_2=y$, we obtain

$$(x+y)^n P_n^{(a,b)}\left(\frac{y-x}{x+y}\right) = \sum_{k=0}^{n}\frac{(a+1)_n(b+1)_n}{(a+1)_k(b+1)_{n-k}}L_k^{(a)}(x)\,(-1)^{n-k}L_{n-k}^{(b)}(y). \tag{1.2}$$

If we look at this formula, a rather straightforward observation can be made. If y is greater than all zeros of the polynomials $(-1)^{n-k}L_{n-k}^{(b)}(y)$, then these factors are positive. On the other hand, because of the normalization of the Laguerre polynomials, all $L_k^{(a)}(x)$ are also positive if x is smaller than all zeros of $L_k^{(a)}(x)$, $k=0,\ldots,n$. Since the zeros of the orthogonal polynomials interlace, we can immediately conclude that the sum on the right-hand side is positive for all values of x and y, for which $y\ge x_{n,1}(b)$ and $x\le x_{n,n}(a)$. Since $(y-x)/(x+y)$ decreases with x and increases with y when both are positive, the largest zero $x_{n,1}(a,b)$ of $P_n^{(a,b)}(x)$ does not exceed the quantity of the argument on the left-hand side for these extremal values of the variables. In other words, we have:

Proposition 1. *The inequalities*

$$x_{n,1}(a,b) \le \frac{x_{n,1}(b)-x_{n,n}(a)}{x_{n,1}(b)+x_{n,n}(a)}$$

hold for every $n\in\mathbb{N}$ and $a,b>-1$.

Now we may use this rather simple but nice observation, have in mind the monotonicity of $(y-x)/(x+y)$ with respect to x and y, and employ limits for the extreme zeros of Laguerre polynomials to obtain sharp bounds for the zeros of Jacobi polynomials. This will be done in Section 5.

However, it is possible to employ formula (1.2) and similar ones, as well as a beautiful result, due to Obrechkoff [20], to derive inequalities for all zeros of $P_n^{(a,b)}(x)$. This is what we do in the present paper. We point out that Obrechkoff's theorem was used in [2, 3, 5] to obtain sharp inequalities for zeros of other sequences of orthogonal polynomials.

One of our main results, which is based on this idea, is:

Theorem 1. *Let $n \in \mathbb{N}$ and $a, b > -1$. Then the inequalities*

$$\frac{x_{k,k}(b) - x_{n,1}(a)}{x_{k,k}(b) + x_{n,1}(a)} \leq x_{n,k}(a,b) \leq \frac{x_{n,1}(b) - x_{n-k+1,n-k+1}(a)}{x_{n,1}(b) + x_{n-k+1,n-k+1}(a)}$$

hold for every $k \in \mathbb{N}$ with $1 \leq k \leq n$.

Interesting inequalities between zeros of Jacobi polynomials follow from formula (4.1) below which was obtained in [25] and [18]. Our result reads as follows:

Theorem 2. *Let $n \in \mathbb{N}$ and $a, b, c > -1$. Then the inequalities*

$$x_{n,k+1}(a+b+1,c) \leq \frac{1 + x_{n,1}(b,c)}{2} x_{n-k,1}(a, b+c+2k+1) + \frac{1 - x_{n,1}(b,c)}{2}$$

hold for every k with $0 \leq k \leq n-1$.

Another application of Obrechkoff's theorem and (4.1) yields:

Theorem 3. *Let $n \in \mathbb{N}$ and $a, b, c > -1$. Then the inequalities*

$$\frac{1 + x_{n,k+1}(a,b)}{1 - x_{n,k+1}(a,b)} \leq \frac{1 + x_{n,1}(c,b)}{2} \; \frac{1 + x_{n-k,1}(a, b+c+2k+1)}{1 - x_{n-k,1}(a, b+c+2k+1)}$$

hold for every k with $0 \leq k \leq n-1$.

Except for being surprisingly beautiful, the above inequalities yield precise limits for the zeros of Jacobi polynomials. Recall that the classical approach, using Sturm's comparison theorem, furnishes precise inequalities for all the zeros $x_{n,k}(a,b)$ only when $-1/2 \leq a, b \leq 1/2$; see Szegő's book [24, Theorem 6.3.2]. There is only one result, due to Elbert, Laforgia and Rodonó [11] where one finds limits for all the zeros of the Jacobi polynomials for large values of the parameters. The result in [11] is based on a very nice refinement of Szegő argument but the limits are given in an implicit form. Observe that our results provide estimates for all zeros of the Jacobi polynomials only in terms of the extreme zeros of either Laguerre or other families of Jacobi polynomials. Since precise limits for the extreme zeros are known, Theorems 1, 2 and 3 immediately imply explicit bounds for *all zeros* of $P_n^{(a,b)}(x)$, for the entire range of the parameters, $a, b > -1$. As it will be seen in the last section, the results are very sharp, especially when one of the parameters is much larger than the other one. When $\alpha = \beta$, that is, in the the ultraspherical case, the best known limits for the zeros of Gegenbauer polynomials were obtained in [3] where we employed Obrechkoff's theorem and another elaborate connection formula. Let us mention that our results provide bounds for the zeros of Laguerre polynomials via the asymptotic relation (see [24, formula (6.71.11)])

$$\lim_{\beta \to \infty} \frac{\beta}{2}(1 - x_{n,k}(\alpha, \beta)) = x_{n,n-k+1}(\alpha).$$

In Section 2 we recall the statement of Obrechkoff's theorem. Theorems 1, 2 and 3 are established in Sections 3 and 4. There we emphasize various interesting

inequalities which follow immediately from the main results. Finally, in Section 5 we employ those corollaries, together with limits for the extreme zeros of classical orthogonal polynomials, known in the literature, to obtain very sharp bounds for all zeros of Jacobi and Laguerre polynomials.

2. The Theorem of Obrechkoff

The statement of the theorem concerns generalized Descartes' rule of signs for a sequence of orthogonal polynomials. Denote by $Z(f;(a,b))$ the number of zeros, counting their multiplicities, of the function $f(x)$ in (a,b). Let $S(a_0,\dots,a_n)$ be the number of sign changes in the sequence $a_0,\dots,a_n$. The number $S(a_0,\dots,a_n)$ is obtained as follows: first we delete the zero entries in the sequence and then count the number of pairs of consecutive terms for which $a_{j-1}a_j<0$. For example, $S(-1,5,2,0,-3,4)=3$.

The finite sequence of functions $f_0(x),\dots,f_n(x)$ obeys Descartes' rule of signs in the interval (a,b) if the number of zeros in (a,b) of any real nonzero linear combination

$$a_0f_0(x)+\cdots+a_nf_n(x)$$

does not exceed the number of sign changes in the sequence $a_0,\dots,a_n$, that is,

$$Z(a_0f_0(x)+\cdots+a_nf_n(x);(a,b))\le S(a_0,\dots,a_n). \tag{2.1}$$

Then Obrechkoff's theorem reads as follows:

Theorem A (Obrechkoff [20, 1918])**.** *Let the sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$p_{-1}(x)=0,\ p_0(x)=1,\ xp_n(x)=\alpha_np_{n+1}(x)+\beta_np_n(x)+\gamma_np_{n-1}(x), n\ge 0, \tag{2.2}$$

where $\alpha_n,\beta_n,\gamma_n\in\mathbb{R}$, $\alpha_n,\gamma_n>0$. If ζ_n is the largest zero of $p_n(x)$, then the sequence of polynomials $p_0(x),\dots,p_n(x)$ obeys Descartes' rule of signs in (ζ_n,∞).

Observe that Favard's theorem [12] (see also [4, Theorem 4.4]) implies that the requirements of Theorem A are equivalent to the fact that $\{p_n(x)\}$ is a sequence of orthogonal polynomials. Then Obrechkoff's theorem can be reformulated as:

Corollary 1. *Let the orthogonal polynomials $p_k(x)$, $k=1,\dots,n$, be normalized in such a way that their leading coefficients are all of the same sign and let ζ_n be the largest zero of $p_n(x)$. Then, for any sequence $a_0,\dots,a_n$, which is not identically zero,*

$$Z(a_0p_0(x)+\cdots+a_np_n(x);(\zeta_n,\infty))\le S(a_0,\dots,a_n). \tag{2.3}$$

It is essential that the polynomials are normalized in such a way that their leading coefficients are all of the same sign.

We refer the reader to [5] and to [8], where historical facts, the original proof of Obrechkoff and another proof of Theorem A, due to Schoenberg [23], can be found.

3. Inequalities between zeros of Jacobi and Laguerre polynomials

In this section we establish Theorem 1 and some of its corollaries. We apply Obrechkoff's theorem to the expansion (1.2), considering both sides as functions of y. In other words, we look at (1.2) as an expression of the form

$$\widehat{P}_n(y)=\sum_{k=0}^{n}\alpha_kp_k(y), \tag{3.1}$$

with

$$\widehat{P}_n(y) = \frac{(x+y)^n}{(a+1)_n(b+1)_n} P_n^{(a,b)}\left(\frac{y-x}{y+x}\right),$$

$$p_k(y) = \frac{(-1)^k}{(a+1)_{n-k}(b+1)_k} L_k^{(b)}(y),$$

whose leading coefficients are positive and

$$\alpha_k = L_{n-k}^{(a)}(x).$$

Therefore, Obrechkoff's theorem implies that the number of zeros of $\widehat{P}_n(y)$ that are greater than $x_{n,1}(b)$, does not exceed the number of sign changes in the sequence

$$(L_0^{(a)}(x), L_1^{(a)}(x), \ldots, L_n^{(a)}(x)).$$

Set $x = x_{k,k}(a)$, that is, the smallest zero of $L_k^{(a)}(x)$, in (1.2), or equivalently, in (3.1). We shall count the number os sign changes in the sequence

$$\{L_j^{(a)}(x_{k,k}(a))\} = \left(L_0^{(a)}(x_{k,k}(a)), L_1^{(a)}(x_{k,k}(a)), \ldots, L_n^{(a)}(x_{k,k}(a))\right).$$

Since $L_j^{(a)}(x)$, $j = 0, \ldots, n$, are positive at the origin and their zeros interlace, then

$$L_0^{(a)}(x_{k,k}(a)) > 0, L_1^{(a)}(x_{k,k}(a)) > 0, \ldots, L_{k-1}^{(a)}(x_{k,k}(a)) > 0, L_k^{(a)}(x_{k,k}(a)) = 0.$$

This yields

$$S(\{L_j^{(a)}(x_{k,k}(a))\}) = S(L_{k-1}^{(a)}(x_{k,k}(a)), L_{k+1}^{(a)}(x_{k,k}(a)), \ldots, L_n^{(a)}(x_{k,k}(a)))$$

and then the sequence $\{L_j^{(a)}(x_{k,k}(a))\}$ may have at most $n-k$ sign changes. Observe now that, if η is a zero of

$$\widehat{P}_n(y; x_{k,k}(a)) = \frac{(y + x_{k,k}(a))^n}{(a+1)_n(b+1)_n} P_n^{(a,b)}\left(\frac{y - x_{k,k}(a)}{y + x_{k,k}(a)}\right),$$

then

$$\frac{\eta - x_{k,k}(a)}{\eta + x_{k,k}(a)}$$

is a zero of $P_n^{(a,b)}(x)$. Recall that its zeros $x_{n,k}(a,b)$ are arranged in decreasing order. Since $(y-x)/(y+x)$ increases with y, then the zeros $\eta_{n,j}(a,b)$ of $\widehat{P}_n(y; x_{k,k}(a))$ are also arranged in decreasing order.

Now Obrechkoff's theorem implies that at most $n-k$ of the numbers $\eta_{n,j}(a,b)$ may exceed $x_{n,1}(b)$. Thus,

$$\eta_{n,n-k+1}(a,b) < x_{n,1}(b). \tag{3.2}$$

On the other hand, $\eta_{n,n-k+1}(a,b)$ is uniquely defined by

$$\frac{\eta_{n,n-k+1}(a,b) - x_{k,k}(a)}{\eta_{n,n-k+1}(a,b) + x_{k,k}(a)} = x_{n,n-k+1}(a,b),$$

which is equivalent to

$$\eta_{n,n-k+1}(a,b) = x_{k,k}(a)\frac{1 + x_{n,n-k+1}(a,b)}{1 - x_{n,n-k+1}(a,b)}.$$

Setting this expression in (3.2), we obtain

$$x_{n,n-k+1}(a,b) \le \frac{x_{n,1}(b) - x_{k,k}(a)}{x_{n,1}(b) + x_{k,k}(a)}, \tag{3.3}$$

which is equivalent to

$$x_{n,k}(a,b) \leq \frac{x_{n,1}(b) - x_{n-k+1,n-k+1}(a)}{x_{n,1}(b) + x_{n-k+1,n-k+1}(a)}. \tag{3.4}$$

Changing the roles of a and b in (3.3) and using the fact that $x_{n,k}(a,b) = -x_{n,n-k+1}(b,a)$, we obtain the lower limit

$$x_{n,k}(a,b) \geq \frac{x_{k,k}(b) - x_{n,1}(a)}{x_{k,k}(b) + x_{n,1}(a)}. \tag{3.5}$$

It is worth pointing out that the latter inequalities can be obtained by Obrechkoff's theorem by similar arguments that we have performed if one considers both sides of (1.2) as polynomials in x. Finally, we write

$$\frac{x_{k,k}(b) - x_{n,1}(a)}{x_{k,k}(b) + x_{n,1}(a)} \leq x_{n,k}(a,b) \leq \frac{x_{n,1}(b) - x_{n-k+1,n-k+1}(a)}{x_{n,1}(b) + x_{n-k+1,n-k+1}(a)}, \tag{3.6}$$

which is exactly the statement of Theorem 1.

Now we comment on some consequences of these inequalities. For $k = 1$ we obtain

$$\frac{x_{1,1}(b) - x_{n,1}(a)}{x_{1,1}(b) + x_{n,1}(a)} \leq x_{n,1}(a,b) \leq \frac{x_{n,1}(b) - x_{n,n}(a)}{x_{n,1}(b) + x_{n,n}(a)}, \tag{3.7}$$

and, for $k = n$,

$$\frac{x_{n,n}(b) - x_{n,1}(a)}{x_{n,n}(b) + x_{n,1}(a)} \leq x_{n,n}(a,b) \leq \frac{x_{n,1}(b) - x_{1,1}(a)}{x_{n,1}(b) + x_{1,1}(a)}. \tag{3.8}$$

Therefore, for every $n \in \mathbb{N}$ and each k, $1 \leq k \leq n$, we have

$$\frac{x_{n,n}(b) - x_{n,1}(a)}{x_{n,n}(b) + x_{n,1}(a)} \leq x_{n,k}(a,b) \leq \frac{x_{n,1}(b) - x_{n,n}(a)}{x_{n,1}(b) + x_{n,n}(a)}.$$

Other very interesting consequences of (3.7) and (3.8) are obtained when we consider the left-hand side of inequality (3.7) and the right-hand side of inequality (3.8), having in mind that $x_{1,1}(\alpha) = \alpha + 1$:

$$x_{n,1}(a,b) \geq 1 - \frac{2\, x_{n,1}(a)}{b + 1 + x_{n,1}(a)}$$

and

$$x_{n,n}(a,b) \leq -1 + \frac{2\, x_{n,1}(b)}{a + 1 + x_{n,1}(b)}.$$

The latter inequalities show that the largest zero of $P_n^{(a,b)}(x)$ goes to 1 very fast when b converges to infinity while its smallest zero goes to -1 as a converges to infinity. These facts are intuitively clear from the electrostatic interpretation of the zeros of Jacobi polynomials (see [24, Section 6.7]) but here we have quantitative results which show the minimal speed of convergence.

Let us mention that another family of inequalities relating the zeros of Jacobi and Laguerre polynomials follows from Theorem 1 and Corollary 1 in [7]. If we set

$$f_n(\alpha,\beta) = 2n(n + \alpha + \beta + 1) + (\alpha + 1)(\beta + 1),$$

then the results, obtained in [7] can be written in the from

$$1 - \frac{2(2n + \alpha + 1)}{f_n(\alpha,\beta)}\, x_{n,n-k+1}(\alpha) \leq x_{n,k}(\alpha,\beta) \leq \frac{2(2n + \beta + 1)}{f_n(\alpha,\beta)}\, x_{n,k}(\beta) - 1.$$

If we apply formula (1.1) with $j = 0$, we derive the relation

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(y),$$

which, by a similar argument, yields the inequalities

$$x_{n,n-k+1}(\alpha+\beta+1) \le x_{n1}(\alpha) + x_{k1}(\beta), \quad 1 \le k \le n,$$

and this result was obtained in [2].

4. Inequalities for zeros of Jacobi polynomials with different parameters

In this section we prove Theorems 2 and 3. In order to do this we obtain inequalities for the zeros of Jacobi polynomials via the following formula which was established in [25] and [18, Corollary 3.15 (i)]:

$$\begin{aligned}(4.1)\qquad & (x_1+x_2)^n P_n^{(a,b)}\left(\frac{x_2-x_1}{x_1+x_2}\right) P_j^{(a+b+2n+1,c)}(1-2(x_1+x_2)) \\ &= \sum_{\ell=0}^{n+j} \binom{j+n}{n} \frac{(b+1)_n (c+1)_j (a+b+c+j+n+2)_\ell}{(c+1)_\ell (b+c+\ell+1)_\ell (b+c+2\ell+2)_{j+n-\ell}} \\ &\times R_\ell(\lambda(n); b, c, -j-n-1, a+b+j+n+1) \\ &\times P_{n+j-\ell}^{(a,b+c+2\ell+1)}(1-2x_1)(1-x_1)^\ell P_\ell^{(b,c)}\left(\frac{1-x_1-2x_2}{1-x_1}\right),\end{aligned}$$

where the Racah polynomials $R_\ell(\lambda(n); \alpha, \beta, \gamma, \delta)$, with $\lambda(n) = n(n+\gamma+\delta+1)$, are defined by [17, formula (1.2.1)].

We apply (4.1) in the particular cases when either $n = 0$ or $j = 0$ and these yield the inequalities stated in Theorems 2 and 3, respectively.

4.1. The case $n = 0$.

It is easy to see that

$$R_\ell(\lambda(0); b, c, -j-1, a+b+j+1) = 1.$$

Then, setting $j = n$, $\ell = k$, $x_1 = x$, $x_2 = -y$ in (4.1), we obtain

$$P_n^{(a+b+1,c)}(1+2(y-x)) = \sum_{k=0}^{n} A_k P_{n-k}^{(a,b+c+2k+1)}(1-2x)\,(1-x)^k P_k^{(b,c)}\left(1+\frac{2y}{1-x}\right),$$

where

$$A_k = \frac{(c+1)_n (a+b+c+n+2)_k}{(c+1)_k (b+c+k+1)_k (b+c+2k+2)_{n-k}} > 0.$$

Suppose that x is fixed. Then the number of zeros of

$$\widehat{P}_n(y) = P_n^{(a+b+1,c)}(1+2(y-x)),$$

which are greater than the largest zero of

$$P_k^{(b,c)}\left(1+\frac{2y}{1-x}\right),$$

does not exceed the number of sign changes in the sequence

$$\left(P_n^{(a,b+c+1)}(1-2x), P_{n-1}^{(a,b+c+3)}(1-2x), \ldots, P_0^{(a,b+c+2n+1)}(1-2x)\right).$$

Driver, Jordaan and Mbuyi [10, Theorem 2.1] proved that, for any fixed $k \in \mathbb{N}$ and $\alpha, \beta > -1$ the zeros of $P_k^{(\alpha,\beta)}(x)$ and $P_{k-1}^{(\alpha,\beta+2)}(x)$ interlace. Therefore the zeros of the polynomials in the above sequence also interlace. Their leading coefficients alternate in sign so that they are positive when $x = -1$.

Consider the case when ξ is the smallest value for which $P_{n-k}^{(a,b+c+2k+1)}(1-2\xi)$ vanishes. This means that

$$\xi = \frac{1 - x_{n-k,1}(a, b+c+2k+1)}{2}.$$

Then the number of sign changes is at most k. Therefore, if the zeros of this polynomial are arranged in decreasing order, the zero $\widehat{y}_{n,k+1}$ of $\widehat{P}_n(y)$ does not exceed the largest zero of $P_n^{(b,c)}(1 + (2/(1-\xi))y)$.

The largest zero $\widehat{y}$ of $\widehat{P}_n(y)$ is obtained by $1 + 2(y - \xi) = x_{n,1}(a+b+1, c)$ which yields

$$\widehat{y}_{n,k+1} = \frac{x_{n,k+1}(a+b+1,c) - x_{n-k,1}(a, b+c+2k+1)}{2}.$$

Similarly, the largest zero of $P_n^{(b,c)}(1+(2/(1-\xi))y)$ is derived from $1+(2/(1-\xi))y = x_{n,1}(b,c)$ and this is equivalent to

$$y = \frac{(x_{n,1}(b,c) - 1)\ (1 + x_{n-k,1}(a, b+c+2k+1))}{4}.$$

Therefore, for $k = 0, \ldots, n-1$, we have

$$x_{n,k+1}(a+b+1,c) \le \frac{1 + x_{n,1}(b,c)}{2} x_{n-k,1}(a, b+c+2k+1) + \frac{1 - x_{n,1}(b,c)}{2}. \tag{4.2}$$

Thus, we have proved Theorem 2.

Observe that the right-hand side of (4.2) can be seen as a convex combination of -1 and $x_{n,1}(a, b+c+1)$. This immediately yields:

Corollary 2. *Let $n \in \mathbb{N}$ and $a, b, c > -1$. Then the inequalities*

$$x_{n,k+1}(a+b+1,c) \le x_{n-k,1}(a, b+c+2k+1) \tag{4.3}$$

hold for every k with $0 \le k \le n-1$.

Let us point out that (4.3) reduces to an equality when $k = 0$ and $b = -1$. The same holds true for (4.2).

4.2. **The case $j = 0$.** We shall show for $\ell \le n$ that

$$R_\ell(\lambda(n); b, c, -n-1, a+b+n+1) = (-1)^\ell \frac{(c+1)_\ell (a+1+n-\ell)_\ell}{(b+1)_\ell (a+b+c+n+2)_\ell}. \tag{4.4}$$

Indeed, it follows for the definition of Racah polynomials, given in [17, (1.2.1)], that

$$\begin{aligned} &R_\ell(\lambda(n); b, c, -n-1, a+b+n+1) \\ &\quad = {}_4F_3\left(\begin{matrix} -\ell, \ell+b+c+1, -n, a+b+n+1 \\ b+1, a+b+c+n+2, -n \end{matrix} \middle| 1 \right) \\ &\qquad = {}_3F_2\left(\begin{matrix} -\ell, \ell+b+c+1, a+b+n+1 \\ b+1, a+b+c+n+2 \end{matrix} \middle| 1 \right). \end{aligned}$$

An application of the theorem of Pfaff and Saalschütz, which was proved by Pfaff [21] in 1797 rediscovered by Saalschütz [22] in 1890 (see [1, Theorem 2.2.6] and [14, (1.7.1)]), yields

$$R_\ell(\lambda(n); b, c, -n-1, a+b+n+1) = \frac{(-c-\ell)_\ell\,(-a-n)_\ell}{(b+1)_\ell\,(-a-b-c-n-\ell-1)_\ell}.$$

By using the definition of Pochhammer's symbol we obtain (4.4).

Thus, we obtain

$$(x+y)^n P_n^{(a,b)}\left(\frac{y-x}{y+x}\right) = \sum_{k=0}^{n} C_k P_{n-k}^{(a,b+c+2k+1)}(1-2x)\,(1-x)^k P_k^{(c,b)}\left(\frac{2y}{1-x}-1\right),$$

where $C_k > 0$ and we used $P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x)$.

Suppose that x is fixed. Then the number of zeros of

$$\widehat{P}_n(y) = (x+y)^n P_n^{(a,b)}\left(\frac{y-x}{y+x}\right),$$

which are greater than the largest zero of

$$P_n^{(c,b)}\left(\frac{2y}{1-x}-1\right),$$

does not exceed the number of sign changes in the sequence

$$\left(P_n^{(a,b+c+1)}(1-2x), P_{n-1}^{(a,b+c+3)}(1-2x), \ldots, P_0^{(a,b+c+2n+1)}(1-2x)\right),$$

the same as before. Again by [10, Theorem 2.1] the zeros of these polynomials interlace, their leading coefficients alternate in sign so that they are positive when $x=-1$, and the smallest value of ξ for which $P_{n-k}^{(a,b+c+2k+1)}(1-2\xi)$ vanishes is

$$\xi = \frac{1 - x_{n-k,1}(a, b+c+2k+1)}{2}.$$

Then the number of sign changes in the sequence as at most k. Therefore, the zero $\widehat{y}_{n,k+1}$ of $\widehat{P}_n(y)$ does not exceed the largest zero y_n of $P_n^{(c,b)}((2/(1-\xi))y-1)$.

The zero $\widehat{y}_{n,k+1}$ of $\widehat{P}_n(y)$ is

$$\widehat{y}_{n,k+1} = \xi\frac{1+x_{n,k+1}(a,b)}{1-x_{n,k+1}(a,b)},$$

and the largest zero of $P_n^{(c,b)}((2/(1-\xi))y-1)$ is

$$y_n = \frac{(1-\xi)(1+x_{n,1}(c,b))}{2}.$$

Therefore, for $k = 0, \ldots, n-1$, we have

$$\frac{1+x_{n,k+1}(a,b)}{1-x_{n,k+1}(a,b)} \le \frac{1+x_{n,1}(c,b)}{2}\;\frac{1+x_{n-k,1}(a,b+c+2k+1)}{1-x_{n-k,1}(a,b+c+2k+1)}. \tag{4.5}$$

This completes the proof of Theorem 3.

For $k=0$ and $c=-1$ this is an equality.

Observe that for $k=0$ the last inequality implies

$$\frac{1+x_{n,1}(a,b)}{1-x_{n,1}(a,b)} \le \frac{1+x_{n,1}(a,b+c+1)}{1-x_{n,1}(a,b+c+1)},$$

which is also a consequence of the facts that the function $f(x) = (1+x)/(1-x)$ increases in $(-1,1)$ and that $x_{n,1}(a,\beta)$ increases with β.

5. Sharp bounds for all zeros of Jacobi and Laguerre polynomials

In what follows we combine the inequalities obtained in the previous sections and some known estimates for the extreme zeros of Laguerre and Jacobi polynomials, to obtain new bounds for all the zeros of Jacobi polynomials via the formulas (3.6) and (4.3). Moreover, the limit relation (see [24, formula (6.71.11)])

$$\lim_{\beta\to\infty} \frac{\beta}{2}(1 - x_{n,k}(\alpha,\beta)) = x_{n,n-k+1}(\alpha)$$

yields estimates for all the zeros of Laguerre polynomials. Then we provide numerical evidence for the sharpness of the new bounds obtained in the present paper.

5.1. **Bounds for all zeros of $P_n^{(\alpha,\beta)}(x)$ via Theorem 1.** We recall various results about estimates for the extreme zeros of the Laguerre polynomials which we apply to obtain bounds for all zeros of Jacobi polynomials. The first limits are due to Krasikov [19, Theorem 6] and read as follows:

$$x_{n,n}(\alpha) > r^2 + 3r^{4/3}(s^2 - r^2)^{-1/3} =: K_{n,n}(\alpha),$$
$$x_{n,1}(\alpha) < s^2 - 3s^{4/3}(s^2 - r^2)^{-1/3} + 2 =: K_{n,1}(\alpha),$$

where $r = \sqrt{n+\alpha+1} - \sqrt{n}$ and $s = \sqrt{n+\alpha+1} + \sqrt{n}$. Another bounds established by Ismail and Li [16, Theorem 4], are

$$x_{n,n}(\alpha) > 2n + \alpha - 2 - \sqrt{1 + 4(n-1)(n+\alpha-1)} =: I_{n,n}(\alpha),$$

$$x_{n,1}(\alpha) < 2n + \alpha - 2 + \sqrt{1 + 4(n-1)(n+\alpha-1)} =: I_{n,1}(\alpha).$$

We refer the reader to the recent paper [6, Theorem 1] where one finds the estimates

$$x_{n,n}(\alpha) \geq \frac{2n^2 + n(\alpha-1) + 2(\alpha+1) - 2(n-1)\sqrt{n^2 + (n+2)(\alpha+1)}}{n+2} =: N_{n,n}(\alpha)$$

and

$$x_{n,1}(\alpha) \leq \frac{2n^2 + n(\alpha-1) + 2(\alpha+1) + 2(n-1)\sqrt{n^2 + (n+2)(\alpha+1)}}{n+2} =: N_{n,1}(\alpha).$$

Observe that [9, Corollary 1] contains the limits

$$x_{n,1}(\alpha) \leq 2n + \alpha - 1 + \sqrt{2(n+\alpha-1)}h_{n,1} =: D_{n,1}(\alpha)$$

for the largest zero of $L_n^{(\alpha)}(x)$ and using the same method one can obtain the lower bound

$$x_{n,n}(\alpha) \geq -2n + \alpha + 1 - \sqrt{2(n+\alpha-1)}h_{n,1} =: D_{n,n}(\alpha).$$

Now the result in Theorem 1, together with the fact that $(y-x)/(x+y)$ decreases with x and increases with y when both are positive, and the limits we have just recalled, we obtain:

Corollary 3. *The inequalities*

$$l_{n,k}^{(j)}(a,b) \leq x_{n,k}(a,b) \leq u_{n,k}^{(j)}(a,b), \quad 1 \leq j \leq 4,$$

where

$$l_{n,k}^{(1)}(a,b) = \frac{K_{k,k}(b) - K_{n,1}(a)}{K_{k,k}(b) + K_{n,1}(a)}, \qquad u_{n,k}^{(1)}(a,b) = \frac{K_{n,1}(b) - K_{n-k+1,n-k+1}(a)}{K_{n,1}(b) + K_{n-k+1,n-k+1}(a)},$$

$$l_{n,k}^{(2)}(a,b) = \frac{I_{k,k}(b) - I_{n,1}(a)}{I_{k,k}(b) + I_{n,1}(a)}, \qquad u_{n,k}^{(2)}(a,b) = \frac{I_{n,1}(b) - I_{n-k+1,n-k+1}(a)}{I_{n,1}(b) + I_{n-k+1,n-k+1}(a)},$$

$$l_{n,k}^{(3)}(a,b) = \frac{N_{k,k}(b) - N_{n,1}(a)}{N_{k,k}(b) + N_{n,1}(a)}, \qquad u_{n,k}^{(3)}(a,b) = \frac{N_{n,1}(b) - N_{n-k+1,n-k+1}(a)}{N_{n,1}(b) + N_{n-k+1,n-k+1}(a)},$$

$$l_{n,k}^{(4)}(a,b) = \frac{D_{k,k}(b) - D_{n,1}(a)}{D_{k,k}(b) + D_{n,1}(a)}, \qquad u_{n,k}^{(4)}(a,b) = \frac{D_{n,1}(b) - D_{n-k+1,n-k+1}(a)}{D_{n,1}(b) + D_{n-k+1,n-k+1}(a)},$$

hold for every $n \in \mathbb{N}$ *and each* $k = 1, \ldots, n$.

The following tables show the results of numerical experiments which suggest that some of these bounds for the zeros of $P_n^{(a,b)}(x)$ are very sharp, especially when the parameter b is much larger than a.

TABLE 1. Bounds for $x_{4,k}(-0.5, 3000)$, $1 \le k \le 4$.

k	$l_{4,k}^{(1)}(-0.5, 3000)$	$x_{4,k}(-0.5, 3000)$	$u_{4,k}^{(1)}(-0.5, 3000)$
1	0.992500	0.999903	0.999946
2	0.992359	0.999109	0.999929
3	0.992256	0.997387	0.999898
4	0.992171	0.994291	0.999814

TABLE 2. Bounds for $x_{5,k}(-0.5, 3000)$, $1 \le k \le 5$.

k	$l_{5,k}^{(1)}(-0.5, 3000)$	$x_{5,k}(-0.5, 3000)$	$u_{5,k}^{(1)}(-0.5, 3000)$
1	0.990222	0.999732	0.999957
2	0.990039	0.999464	0.999947
3	0.989905	0.998095	0.99993
4	0.989793	0.995746	0.999899
5	0.989696	0.992157	0.999816

TABLE 3. Bounds for $x_{6,k}(-0.5, 3000)$, $1 \le k \le 6$.

k	$l_{6,k}^{(1)}(-0.5, 3000)$	$x_{6,k}(-0.5, 3000)$	$u_{6,k}^{(1)}(-0.5, 3000)$
1	0.987906	0.999934	0.999964
2	0.987679	0.999402	0.999957
3	0.987513	0.998303	0.999947
4	0.987376	0.996546	0.999931
5	0.987256	0.993939	0.999900
6	0.987147	0.989959	0.999818

5.2. **Bounds for the zeros of Jacobi polynomials via Corollary 2.** An upper bound for the zeros $x_{n,k}(a,b)$ can be obtained using (4.3) with $b=-1$ and the following estimate for the largest zero $x_{n,1}(\alpha,\beta)$ given in [6, Theorem 1]:

$$x_{n,1}(\alpha,\beta) < \frac{B+4(n-1)\sqrt{\Delta}}{A} =: \mu_n(\alpha,\beta), \tag{5.1}$$

where

$$A = (2n+\alpha+\beta)(n(2n+\alpha+\beta)+2(\alpha+\beta+2)),$$

$$B = (\beta-\alpha)((\alpha+\beta+6)n+2(\alpha+\beta)),$$

$$\Delta = n^2(n+\alpha+\beta+1)^2+(\alpha+1)(\beta+1)(n^2+(\alpha+\beta+4)n+2(\alpha+\beta)).$$

The new bounds are:

Corollary 4. *If $n\in\mathbb{N}$, $a,c>-1$, then the inequalities*

$$x_{n,k+1}(a,c) \le x_{n-k,1}(a,c+2k) < \mu_{n-k}(a,c+2k) =: \mu_{n,k+1}(a,c)$$

hold for every k with $0\le k\le n-1$.

Needless to say, for $k=0$, this result reduces to Theorem 1 in [6], so that Corollary 4 can be considered as a natural extension of that theorem.

The following three tables show how sharp these limits for the zeros of $P_n^{(a,c)}(x)$ are, especially in the case when the difference between the parameters a and c is large.

TABLE 4. Upper bound for $x_{4,k}(a,c)$, $1\le k\le 4$.

k	$x_{4,k}(-0.5,3000)$	$\mu_{4,k}(-0.5,3000)$
1	0.999903	0.999816
2	0.999109	0.999876
3	0.997387	0.999817
4	0.994291	0.999667

TABLE 5. Upper bound for $x_{5,k}(a,c)$, $1\le k\le 5$.

k	$x_{5,k}(-0.5,3000)$	$\mu_{5,k}(-0.5,3000)$
1	0.999732	0.999925
2	0.999464	0.999906
3	0.998095	0.999876
4	0.995746	0.999817
5	0.992157	0.999668

TABLE 6. Upper bound for $x_{6,k}(a,c)$, $1\le k\le 6$.

k	$x_{6,k}(-0.5,3000)$	$\mu_{6,k}(-0.5,3000)$
1	0.999934	0.999937
2	0.999402	0.999925
3	0.998303	0.999906
4	0.996546	0.999876
5	0.993939	0.999817
6	0.989959	0.999668

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