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FOURIER DUALITY FOR FRACTAL MEASURES WITH AFFINE SCALES

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Abstract. For a family of fractal measures, we find an explicit Fourier duality. The measures in the pair have compact support in $\mathbb{R}^d$, and they both have the same matrix scaling; but the two use different translation vectors, one by a subset B in $\mathbb{R}^d$, and the other by a related subset L. Among other things, we show that there is then a pair of infinite discrete sets $\Gamma(L)$ and $\Gamma(B)$ in $\mathbb{R}^d$ such that the $\Gamma(L)$-Fourier exponentials are orthogonal in $L^2(\mu_B)$, and the $\Gamma(B)$-Fourier exponentials are orthogonal in $L^2(\mu_L)$. These sets of orthogonal "frequencies" are typically lacunary, and they will be obtained by scaling in the large. The nature of our duality is explored below both in higher dimensions and for examples on the real line.

Our duality pairs do not always yield orthonormal Fourier bases in the respective $L^2(\mu)$-Hilbert spaces, but depending on the geometry of certain finite orbits, we show that they do in some cases. We further show that there are new and surprising scaling symmetries of relevance for the ergodic theory of these affine fractal measures.

1. Introduction

Fractal scaling and self-similarity occurs in nature, and in applications, such as to large communication networks. To understand the nature of fractal scaling, it has proved useful to develop specific model cases. Here we focus on such a class, those given by a finite family of affine transformations. Iteration in the small yields fractal measures μ, and their support sets fall into one of the classes of compact fractals embedded in $\mathbb{R}^d$. Iteration of scale in the large, in turn, leads to a kind of fractal networks comprising of a lattice skeleton and lacunarity, degrees of sparsity. We study when these scales in the large lead to orthogonal families in $L^2(\mu)$.

There have been a number of recent papers dealing with and treating a variety of features of the harmonic analysis of fractal measures μ with affine scales; see for example [DJ06a, DJ07a, DJ07b, DJ07c, DJ07d, DJ08, DJ09, GIL09, HL08, IW08, Str99, SW99, Str00].

In this paper we study pairs of orthogonal complex exponentials in two Hilbert spaces $L^2(\mu)$ for a pair of fractal measures μ with affine scales. Each μ in the pair has the same scaling matrix, but the affine mappings making up the iterated function systems (IFSs) are different.

The particular pairs we study are selected from a certain axiom involving a fixed complex Hadamard matrix. It is indexed by a triple, a matrix and two sets of

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vectors; more specifically, by an expansive d by d matrix R with integer entries, and by two subsets B and L in $\mathbb{R}^d$ having the same cardinality. We show that when the data (R, B, L) are fixed, we then get a naturally defined pair of affine measures μ_B and μ_L, each with R-selfsimilarity. The two measures arise by taking scaling in the small with powers of the inverse R^{-1}, and initiating with the given sets B and L. The support of the measures is typically a Cantor fractal, e.g., a Cantor set on the line, or for example, a planar Sierpinski set in $\mathbb{R}^2$.

We offer a detailed Fourier duality for the pair, and we show that there is a pair of discrete sets $\Gamma(L)$ and $\Gamma(B)$ in $\mathbb{R}^d$ such that the $\Gamma(L)$-Fourier exponentials are orthogonal in $L^2(\mu_B)$, and the $\Gamma(B)$-Fourier exponentials are orthogonal in $L^2(\mu_L)$. These sets of orthogonal "frequencies" will be obtained by scaling in the large. The nature of this duality is explored below both in higher dimensions and for examples on the real line.

Our duality pairs do not always yield orthonormal Fourier bases in the respective $L^2(\mu)$-Hilbert spaces, but we show that they do in some cases. We further show that there are new and surprising scaling symmetries of relevance for the ergodic theory of these affine fractal measures.

The paper is organized as follows. The problem we consider here began with the question raised first by Fuglede [Fug74] for open subsets Ω in $\mathbb{R}^d$ concerning orthogonal Fourier bases for the corresponding Hilbert space $L^2(\Omega)$ with respect to Lebesgue measure, i.e., of orthogonal bases (ONBs) of complex exponentials. While the possibility of such Fourier bases is related to tiling properties for Ω, it is known not to be equivalent [Tao04]. For a brief summary of these questions, see for example [JP99, IKP99, IKT03, JP91, Jør82b, Jør82a].

However, the ideas from [JP99] and [DJ07a] still suggested intriguing connections between spectra and geometry, more specifically that Fourier bases should be tied to an intrinsic selfsimilarity, and further that this can be made precise with the use of Hadamard matrices (see Definition 2.5). But, in addition, this suggested algorithmic iterations of the similarity transformations, so the introduction of iterated function systems (IFSs) built from a finite family of affine transformations in $\mathbb{R}^d$ (see Definition 2.1). Thus, the measures μ arising in the limit can be expected to come from a dual affine iteration. Further [JP99] suggested a definite relationship between the two sides in such a duality.

In section 2 we introduce the Hadamard matrices, and we prove the stated relations for the two IFS measures in duality. In Proposition 2.6 we show that each of the Hilbert spaces $L^2(\mu)$ for the dual system of measures have a natural pair of infinite families of orthogonal Fourier frequencies. The fractal measures arise by iteration in the small, while the orthogonal families by iteration in the large.

The interplay between the two sides in the duality is made precise in Corollaries 2.8 and 2.9. In Theorem 2.10 we introduce a class of invertible matrices which define equivalence between the two measures. Furthermore, we offer examples in section 3 to the effect that the two Hilbert spaces $L^2(\mu)$ for dual measures may have different Fourier properties.

Furthermore, we show in Lemma 3.3 that the nature of the orthogonal families from Proposition 2.6 is determined by geometry. By this we mean that for infinite families of orthogonal Fourier frequencies, the obstruction to the ONB property is determined by whether a certain pair of dynamical systems have non-trivial orbits.

A further aspect of the duality is pointed out in Lemma 4.1 with the use of a pair of transfer operators.

We offer computable examples in sections 3 and 5 which we believe may be of independent interest. The example in section 5 has already been the subject of extensive work; see for example [JP98, DJ09, DHS09].

In section 5 we prove that $L^2(\mu)$ may have orthogonal Fourier bases ONBs (i.e., ONBs of complex exponential frequencies) having arbitrarily small upper Beurling density. Compared with what is usually expected in standard sample theory, this result reveals new features of the harmonic analysis of the Hilbert spaces $L^2(\mu)$.

2. Pairs of spectral fractal measures

There is a general construction of measures with intrinsic selfsimilarity due to Hutchinson [Hut81], but it is closely related to families of infinite product measures considered earlier. The starting point for this construction is a finite family F of contractive transformations in a complete metric space and an assignment of probabilities on F. Repeated iteration of the mappings in F, and taking averages, then leads to a unique Borel probability measure μ in the limit; see (2.2) below. A special case of this is the measure resulting from a Cantor iteration, and recursive rescaling. Motivated by this, we will be concerned here with the special case when the mappings in F are affine transformations in $\mathbb{R}^d$ for some d; see (2.4). In that case, there is a family of complex exponentials indexed by points in $\mathbb{R}^d$, and we will be interested in iterative algorithms for the construction of maximal orthogonal families in $L^2(\mu)$. Ideally we ask for these families to form orthonormal bases (ONBs) for $L^2(\mu)$, i.e., Fourier bases.

Definition 2.1. Let (Y, d) be a complete metric space, and let $(\tau_i)_{i=1}^N$ be a finite system of contractive mappings, i.e., there is a constant c, $0 < c < 1$ such that

$$d(\tau_i(x), \tau_i(y)) \leq c d(x, y), \quad (x, y \in Y, i = 1, \dots, N). \tag{2.1}$$

Let $(p_i)_{i=1}^N$, $p_i \geq 0$ satisfy $\sum_{i=1}^N p_i = 1$. Then by [Hut81] there is a unique probability Borel measure μ such that

$$\mu = \sum_{i=1}^N p_i \mu \circ \tau_i^{-1} \tag{2.2}$$

where $\mu \circ \tau^{-1}$ is the measure given by $(\mu \circ \tau^{-1})(E) = \mu(\tau^{-1}(E))$, $E \in \mathfrak{B}(Y)$=Borel sets, and $\tau^{-1}(E) = \{x \in Y : \tau(x) \in E\}$.

Here we will study the case $Y = \mathbb{R}^d$ with its usual metric, and we will be interested in families (τ_i) consisting of transformations of the form

$$x \mapsto R^{-1}(x + b) \tag{2.3}$$

where R is a fixed expansive matrix (i.e., all eigenvalues λ satisfy $|\lambda| > 1$) and where b is in a finite subset B of $\mathbb{R}^d$.

We will further restrict to the case of equal probabilities $p_i = 1/N$ in (2.2).

The mappings in (2.3) will be denoted

$$\tau_b(x) = R^{-1}(x + b). \tag{2.4}$$

It is known that the corresponding measure $\mu = \mu_{R,B}$ solving (2.2) has as support a Cantor set $\mathbb{R}^d$. However, depending on B, the measure μ will vary. It is convenient to choose B such that $0 \in B$. In that case,

$$\operatorname{supp}(\mu) = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k : b_k \in B \right\} =: X(B). \tag{2.5}$$

Setting

$$e_t(x) := e^{2\pi i t \cdot x}, \quad (t, x \in \mathbb{R}^d)$$

we get

Definition 2.2. Let $\Gamma \subset \mathbb{R}^d$ be some discrete subset, and let

$$E(\Gamma) := \{ e_\gamma : \gamma \in \Gamma \}.$$

We say that Γ is orthogonal in $L^2(\mu)$ iff the functions in $E(\Gamma)$ are orthogonal, i.e.,

$$\langle e_{\gamma_1}, e_{\gamma_2} \rangle = \widehat{\mu}(\gamma_2 - \gamma_1) = 0, \text{ for all } \gamma_1 \neq \gamma_2 \in \Gamma,$$

where $\widehat{\mu}$ is defined as the Fourier transform

$$\widehat{\mu}(t) = \int_{\mathbb{R}^d} e_t(x)\, d\mu(x). \tag{2.6}$$

Definition 2.3. Set $N := \#B$ and

$$\chi_B(t) = \frac{1}{N} \sum_{b \in B} e_b(t), \tag{2.7}$$

$$\delta_B := \frac{1}{N} \sum_{b \in B} \delta_b \text{ (Dirac notation);} \tag{2.8}$$

then

$$\chi_B(t) = \widehat{\delta}_B(t). \tag{2.9}$$

Lemma 2.4. *Let R and B be as above, and let $(\tau_b)_{b \in B}$ be the IFS in* (2.4). *Let $\mu = \mu_{R,B}$ be the corresponding Hutchinson measure, and let R^T be the transposed matrix. Then*

$$\widehat{\mu}(t) = \prod_{k=1}^{\infty} \chi_B\left((R^T)^{-k}(t) \right) \tag{2.10}$$

where the infinite product is absolutely convergent.

Proof. Well known! □

Definition 2.5. Let a $d \times d$ matrix R be given. Assume $R \in M_d(\mathbb{Z})$ and that R is expansive. Let $B, L \subset \mathbb{R}^d$ be such that $0 \in B$, $0 \in L$, $N = \#B = \#L$ and assume

$$R^k b \cdot l \in \mathbb{Z}, \text{ for all } b \in B, l \in L, k \in \mathbb{Z}, k \geq 0 \tag{2.11}$$

and further that the matrix

$$\frac{1}{\sqrt{N}} \left(e^{2\pi i R^{-1} b \cdot l} \right)_{b \in B, l \in L} \tag{2.12}$$

is unitary.

Set

$$\Gamma(B) := \left\{ \sum_{k=0}^{n} R^k b_k : b_k \in B, n \in \mathbb{Z}_+ \right\}, \tag{2.13}$$

$$\Gamma(L) := \left\{ \sum_{k=0}^{n} (R^T)^k l_k : l_k \in L, n \in \mathbb{Z}_+ \right\}. \tag{2.14}$$

Proposition 2.6. *Let μ_B be the Hutchinson measure for $(\tau_b)_{b\in B}$ and let μ_L be the Hutchinson measure for the dual system*

$$\sigma_l(x) := (R^T)^{-1}(x + l), \quad (l \in L). \tag{2.15}$$

Then $\Gamma(B)$ is orthogonal in $L^2(\mu_L)$, and $\Gamma(L)$ is orthogonal in $L^2(\mu_B)$.

Proof. The conclusions of the proposition amount to the following:

$$\langle e_{\gamma_1}\, ,\, e_{\gamma_2} \rangle_{\mu_B} = \widehat{\mu}_B(\gamma_1 - \gamma_2) = \delta_{\gamma_1,\gamma_2} \text{ for all } \gamma_1, \gamma_2 \in \Gamma(L) \tag{2.16}$$

and

$$\langle e_{\xi_1}\, ,\, e_{\xi_2} \rangle_{\mu_L} = \widehat{\mu}_L(\xi_1 - \xi_2) = \delta_{\xi_1,\xi_2}, \text{ for all } \xi_1, \xi_2 \in \Gamma(B). \tag{2.17}$$

In view of (2.11) and (2.12) in the definition, it is easy to prove one of the stated properties and we offer the details in the verification of (2.17). Using Lemma 2.4, we get

$$\widehat{\mu}_L(\xi) = \prod_{k=1}^{\infty} \chi_L(R^{-k}\xi) \tag{2.18}$$

and we will be using this for $\xi = \xi_1 - \xi_2 \neq 0$, with pairs of points from the set $\Gamma(B)$. So we are concerned with

$$\xi = \sum_{k=0}^{n} R^k (b_k - \beta_k)$$

where $b_0, \dots, b_n, \beta_0, \dots, \beta_n \in B$.

Substitution into (2.18) shows that the product representation for $\widehat{\mu}_L(\xi_1 - \xi_2)$ will contain a factor of the form

$$\chi_L(R^{-1}(b - \beta)) \tag{2.19}$$

where $b, \beta \in B$ distinct. Then

$$\chi_L(R^{-1}(b - \beta)) = \frac{1}{N} \sum_{l \in L} \overline{e^{2\pi i R^{-1}\beta \cdot l}} e^{2\pi i R^{-1} b \cdot l} = \langle \mathrm{row}_\beta\, ,\, \mathrm{row}_b \rangle \tag{2.20}$$

where row_β is a notation for the row with index β in the Hadamard matrix (2.12), and where $\langle \cdot\, ,\, \cdot \rangle$ in (2.17) refers to the standard complex inner product in $\mathbb{C}^N$.

Since the matrix in (2.12) is assumed unitary, it follows that distinct rows are orthogonal. The desired conclusion follows. ☐

Corollary 2.7. *Let the data (R, B, L) be as stated in Proposition* 2.6*; then both of the orthogonal systems $\Gamma(B)$ and $\Gamma(L)$ are infinite.*

Proof. By symmetry, it is enough to prove this for $\Gamma(B)$. We will show that if $b_0, b_1, \dots$ and $\beta_0, \beta_1, \dots$ in B yield the same vector,

$$\gamma = \sum_{k=0}^{n} R^k b_k = \sum_{k=0}^{n} R^k \beta_k, \tag{2.21}$$

then $b_k = \beta_k$ for $0 \le k \le n$.

If not, then let k be the first term in (2.21) with $b_k \ne \beta_k$. Then, from (2.21), we get

$$R^{-1}(b_k - \beta_k) = \sum_{i=k+1}^{n} R^{i-k-1}(\beta_i - b_i)$$

and, using (2.11) this implies that

$$e^{2\pi i R^{-1} b_k \cdot l} = e^{2\pi i R^{-1} \beta_k \cdot l}, \text{ for all } l \in L.$$

But then the rows b_k and β_k in the Hadamard matrix (2.12) coincide, so they cannot be orthogonal. The contradiction implies the corollary. $\square$

Corollary 2.8. *Setting*

$$\sigma^{(B)}_{\Gamma(L)}(t) = \sum_{\gamma \in \Gamma(L)} |\widehat{\mu}_B(t+\gamma)|^2 \tag{2.22}$$

and

$$\sigma^{(L)}_{\Gamma(B)}(t) = \sum_{\xi \in \Gamma(B)} |\widehat{\mu}_L(t+\gamma)|^2, \tag{2.23}$$

we note that the two functions are entire analytic of $t = (t_1, t_2, \dots, t_d)$, i.e., have entire analytic extensions to $\mathbb{C}^d$.

Further, we have

$$\sigma^{(B)}_{\Gamma(L)}(t) \le 1, \text{ and } \sigma^{(L)}_{\Gamma(B)}(t) \le 1 \text{ for all } t \in \mathbb{R}^d.$$

Proof. While most of the conclusions follow from earlier papers, we include here the proof for $\sigma^{(B)}_{\Gamma(L)}$ in (2.22). The analogous formula holds for $\sigma^{(L)}_{\Gamma(B)}$ in (2.23).

Let P_L denote the orthogonal projection in $L^2(\mu_B)$ onto the closed subspace $\mathcal{H}(L)$ spanned by $E(\Gamma(L))$, i.e., by the exponentials

$$E(\Gamma(L)) := \{e_\gamma : \gamma \in \Gamma(L)\} \subset L^2(\mu_B).$$

Then

$$\begin{aligned}\sigma^{(B)}_{\Gamma(L)}(t) &= \sum_{\gamma \in \Gamma(L)} |\widehat{\mu}_B(t+\gamma)|^2 = \sum_{\gamma \in \Gamma(L)} |\langle e_{-t}, e_\gamma \rangle_{\mu_B}|^2 = \|P_L e_{-t}\|^2_{L^2(\mu_B)} \\ &= \langle e_{-t}, P_L e_{-t} \rangle_{L^2(\mu_B)} = \langle e_0, U(t) P_L U(-t) e_0 \rangle_{L^2(\mu_B)},\end{aligned}$$

where, for $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, $U(t)$ denotes multiplication by $e^{2\pi i(t_1 x_1 + \cdots + t_d x_d)}$. Since $\operatorname{supp}(\mu_B) = X(B)$ is compact, $U(t)P_L U(-t)$ is entire analytic.

The computation further shows that

$$\sigma^{(B)}_{\Gamma(L)}(t) \le \|U(t) P_L U(-t)\| \|e_0\|^2_{L^2(\mu_B)} = \|P_L\| = 1$$

where we used the fact that P_L is a projection, and so has norm 1. $\square$

Corollary 2.9. *The following conclusions hold:*

(i) $\Gamma(L)$ *is total in* $L^2(\mu_B)$ *(i.e., it is an ONB) iff* $\sigma^{(B)}_{\Gamma(L)} \equiv 1$ *in* $\mathbb{R}^d$.

(ii) $\Gamma(B)$ *is total in* $L^2(\mu_L)$ *iff* $\sigma^{(L)}_{\Gamma(B)} \equiv 1$ *in* $\mathbb{R}^d$.

Proof. See [JP99], [DJ08]. □

Theorem 2.10. *Let* R, B, L *be as stated in Definition* 2.5. *Assume further that* $R^T = R$. *If there is a* $G \in GL_d(\mathbb{R})$ *such that* $G = G^T$,

$$GR = RG, \tag{2.24}$$

$$G(B) = L, \tag{2.25}$$

then the two spectral functions (2.22), (2.23) *in Corollary* 2.8 *satisfy*

$$G\Gamma(B) = \Gamma(L) \tag{2.26}$$

and

$$\sigma^{(L)}_{\Gamma(B)}(t) = \sigma^{(B)}_{\Gamma(L)}(Gt) \text{ for all } t \in \mathbb{R}^d. \tag{2.27}$$

Proof. It follows from (2.24) and (2.25) that (2.26) is satisfied.

Since $G(B) = L$, we have

$$\chi_L(t) = \frac{1}{N}\sum_{b\in B} e_{G(b)}(t) = \frac{1}{N}\sum_{b\in B} e_b(Gt) = \chi_B(Gt)$$

and

$$\widehat{\mu}_L(t) = \prod_{k=1}^{\infty} \chi_L(R^{-k}t) = \prod_{k=1}^{\infty} \chi_B(GR^{-k}t) = \prod_{k=1}^{\infty} \chi_B(R^{-k}Gt) = \widehat{\mu}_B(Gt).$$

Furthermore, for $t \in \mathbb{R}^d$,

$$\begin{aligned}\sigma^{(L)}_{\Gamma(B)}(t) &= \sum_{\gamma\in\Gamma(B)} |\widehat{\mu}_L(t+\gamma)|^2 = \sum_{\gamma\in\Gamma(B)} |\widehat{\mu}_B(G(t+\gamma))|^2 = \sum_{\gamma\in\Gamma(B)} |\widehat{\mu}_B(Gt+G\gamma)|^2 \\ &= \sum_{\xi\in\Gamma(L)} |\widehat{\mu}_B(Gt+\xi)|^2 = \sigma^{(B)}_{\Gamma(L)}(Gt)\end{aligned}$$

which is the desired formula (2.27). □

Corollary 2.11. *If* $d = 1$, *and* $\#(B) = \#(L) = 2$, *then there exists* $g \in \mathbb{R} \setminus \{0\}$ *such that*

$$\sigma^{(L)}_{\Gamma(B)}(t) = \sigma^{(B)}_{\Gamma(L)}(gt) \text{ for all } t \in \mathbb{R}.$$

3. Hadamard matrices and extreme cycles

In this section we prove that iteration by some fixed scaling matrix and its transposed leads to a new Fourier duality for a pair of IFS measures. Each of the two measures arises from an affine system, but the systems are linked by a complex Hadamard matrix; see Definition 2.5 and Example 3.1 below. Lemma 3.3 offers a geometric tool which allows us to test when our iterative algorithms from section 2 for the maximal orthogonal Fourier families in $L^2(\mu)$ in fact lead to orthonormal Fourier bases (ONBs) in the respective $L^2(\mu)$-spaces.

The basic building block for our examples is the matrix (2.12). An $N \times N$ matrix $\mathcal{H}$ over $\mathbb{C}$ is called *Hadamard* if $|\mathcal{H}_{j,k}| = 1/\sqrt{N}$ for all $j, k = 1, \dots, N$.

Complex Hadamard matrices serve as tools in combinatorics, in algebra, and in applications, see e.g. [CHK97, Den09, Dit03, Dit04, DFdGtHR04, DvA08, Ped04, XCQ05]. The simplest are the unitary $N \times N$ matrices which define the Fourier transform on the finite cyclic groups $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$. If ζ is the N-th root of 1, $\xi = e^{2\pi i/N}$, then

$$\frac{1}{\sqrt{N}}(\zeta^{j\cdot k})_{j,k\in\mathbb{Z}_N} \tag{3.1}$$

is a complex Hadamard matrix. Moreover, the complex Hadamard matrices are closed under the following operations:

(i) permutation of rows;
(ii) permutation of columns;
(iii) multiplication of a fixed row by a fixed phase;
(iv) tensor product.

By (iv) we mean this: if U is $N \times N$ and V is $M \times M$ complex Hadamard matrices, then $W := U \otimes V$ is also a $NM \times NM$ complex Hadamard matrix. To see this, note that $U \otimes V$ has entries of modulus $1/\sqrt{NM}$. With the definition of $U \otimes V$ on $\mathbb{C}^N \otimes \mathbb{C}^M \cong \mathbb{C}^{NM}$,

$$(U \otimes V)(z \otimes w) := Uz \otimes Vw$$

we see that $U \otimes V$ is again unitary.

Example 3.1. If

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ and } V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

then

$$U \otimes V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ 1 & 1 & -1 & -1 \\ i & -i & -i & i \end{pmatrix}.$$

By contrast, the Fourier transform of the group $\mathbb{Z}_4$ is

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

The main distinction between duality pairs μ_B and μ_L in the case $d = 1$ and $d > 1$ has to do with the following:

Definition 3.2. Let $\tau_b(x) = R^{-1}(x + b)$ be an IFS as specified in (2.4). If $X(B)$ is defined as in (2.5), a *finite B-cycle point* is a point in $X(B)$ of the form $(w, w, \ldots)$ obtained by the repetition of a fixed finite word $w = (b_1 \ldots b_n)$. When w is fixed, we denote by $x(w)$ the corresponding finite cycle point, i.e., setting

$$x_n := R^{-1}b_1 + \cdots + R^{-n}b_n$$

we get

$$x(w) = x_n + R^{-n}x_n + R^{-2n}x_n + \cdots = (I - R^{-n})^{-1}x_n = R^n(R^n - I)^{-1}x_n.$$

Note that $x(w)$ can also be defined by the property

$$\tau_{b_1} \ldots \tau_{b_n} x(w) = x(w).$$

We call the set $\{x(w), \tau_{b_n}x(w), \tau_{b_{n-1}}\tau_{b_n}x(w), \ldots, \tau_{b_2}\ldots\tau_{b_n}x(w)\}$ the *B-cycle generated by $x(w)$*.

Similarly, one can define L-cycle points and L-cycles, associated to the IFS $(\sigma_l)_{l\in L}$ (see (2.15)). Consider a system R, B, L in $\mathbb{R}^d$ satisfying the conditions in Definition 2.5. A finite B-cycle C in $X(B)$ is said to be *L-extreme* if $|\chi_L(x)| = 1$ for all $x \in C$. A finite L-cycle C is said to be *B-extreme* if $|\chi_B(x)| = 1$ for all $x \in C$.

Lemma 3.3. *Let R, B, L in $\mathbb{R}$ (so the dimension $d = 1$) be as specified in Definition 2.5. In particular, we assume $0 \in B$ and $0 \in L$.*

(i) *Then $\Gamma(L)$ is an ONB in $L^2(\mu_B)$ iff the only B-extreme cycles in $X(L)$ are the singleton $\{0\}$.*
(ii) *Moreover, the set $\Gamma(B)$ is an ONB in $L^2(\mu_L)$ iff the only L-extreme cycles in $X(B)$ are the singleton $\{0\}$*

For dimensions $d > 1$, the condition on the extreme cycles is only necessary, but not always sufficient for the corresponding Γ set to be an ONB.

Proof. [ŁW02, DJ06b] □

In the analysis of extreme B-cycles, the following lemma will be useful. See [DJ07a, Theorem 4.1] for more details.

Lemma 3.4. *If C is a B-extreme cycle, then for all $x \in C$, and all $b \in B$, $b \cdot x \in \mathbb{Z}$.*

Proof. For all $x \in C$, we must have $|\chi_B(x)| = 1$. Therefore

$$\left| \sum_{b\in B} e^{2\pi i b\cdot x} \right| = N.$$

All the terms in the sum have absolute value 1. There are N of them, and one of them is 1, since $0 \in B$. Therefore we have equality in the triangle inequality and this implies that $e^{2\pi i b\cdot x} = 1$ for all $b \in B$. Therefore $b \cdot x \in \mathbb{Z}$ for all $b \in B$. □

In the following, we point out the simplifications resulting from specialization to $d = 1$. In particular (Proposition 3.5) we point out that the dual systems from section 2 when specialized to the particular Hadamard matrices defining finite Fourier transforms yield intriguing pairs of fractal measures in duality.

In this section, we consider a family of examples which are related to the finite cyclic group $\mathbb{Z}_N$, but they display fractal features and fractal duality which has not been studied earlier in duality theorems involving the finite cyclic groups.

Setting 4.1. Fix integers M and N in $\mathbb{Z}_+$, and assume $N|M$, i.e., $M = qN$ for some $q \in \mathbb{Z}_+$, $q > 1$. We will consider the following instance of a system (R, B, L) in $\mathbb{R}^d$ subject to the conditions in Definition 2.5. Set $R = M$ and $L = \{0, 1, \dots, N-1\}$, and $B = qL = \{0, q, 2q, \dots, (N-1)q\}$.

As a result, we see that the Hadamard matrix for R, B, L is

$$\frac{1}{\sqrt{N}} \left(\zeta^{k\cdot l}\right)_{k,l\in\mathbb{Z}_N}, \quad \zeta = e^{2\pi i/N}. \tag{3.2}$$

Proposition 3.5. *Let (R, B, L) be a system constructed from the numbers N and M as in Setting 4.1. Then*

(i) *$\Gamma(L)$ is an ONB in $L^2(\mu_B)$;*
(ii) *$\Gamma(B)$ is an ONB in $L^2(\mu_L)$.*

Proof. We will establish (i) and (ii) as an application of Lemma 3.3. Note that the IFS which generates μ_B is

$$\tau_k^{(B)}(x) = \frac{x + kq}{M}, \quad k \in \mathbb{Z}_N, \tag{3.3}$$

and for μ_L it is

$$\tau_l^{(L)}(x) = \frac{x + l}{M}, \quad l \in \mathbb{Z}_N. \tag{3.4}$$

For the two functions χ_B and χ_L we have

$$\chi_L(t) = \frac{1}{N} \sum_{k=0}^{N-1} e_k(t) \tag{3.5}$$

and

$$\chi_B(t) = \chi_L(qt). \tag{3.6}$$

In exploring the extreme cycles we note that

$$|\chi_L(t)| = 1 \text{ iff } t \in \mathbb{Z} \tag{3.7}$$

and

$$|\chi_B(t)| = 1 \text{ iff } t \in \frac{1}{q}\mathbb{Z}. \tag{3.8}$$

Since $X(B) \subset [0, \frac{qN-1}{M-1}]$, $X(L) \subset [0, \frac{N-1}{M-1}]$ and $M = qN$, $q > 1$, it follows that the only L-cycle in $X(B)$ is $\{0\}$; similarly, the only B-extreme cycle in $X(L)$ is $\{0\}$. The result now follows from Lemma 3.3. ☐

The next example relates to the pair of fractals in Proposition 3.5 naturally associated with the finite Fourier transform of $\mathbb{Z}_n$. Since $\mathbb{Z}_n$ is its own Fourier dual, the corresponding B–L duality may be phrased in the language of the $\mathbb{Z}_n$-Fourier transform, with the two sets B and L essentially being a copy of $\mathbb{Z}_n$. Example 3.6 below shows that if one of the sets is changed by one point, then the ONB conclusions from Proposition 3.5 no longer holds.

Example 3.6. The following example shows that if the numbers in the sets B or L in Proposition 3.5 are modified, then the ONB conclusion may fail.

Let $R = 8$ and $B = 2 \cdot \{0, 1, 2, 3\} = \{0, 2, 4, 6\}$ as in Proposition 3.5, but for L now choose $L = \{0, 1, 2, 7\}$, i.e., change 3 to 7 as compared to Proposition 3.5 with $N = 4$ and $q = 2$. Now $x = 1$ satisfies $\tau_7^{(L)} x = x$ and $|\chi_B(x)| = 1$. Therefore $\{1\}$ is a cycle in $X(L)$ which is B-extreme. Combining this with the earlier observations we see that the following hold: $\Gamma(L)$ is not an ONB in $L^2(\mu_B)$, but $\Gamma(B)$ is an ONB in $L^2(\mu_L)$.

The next example shows how the two spectral functions in Corollary 2.8 will typically be quite different when the conditions in Theorem 2.10 are not satisfied. In the example the non-zero vectors in B are linearly independent while they are proportional in L. As a result there cannot be an invertible matrix G satisfying the conditions in Theorem 2.10.

Example 3.7. Let

$$R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad L = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}.$$

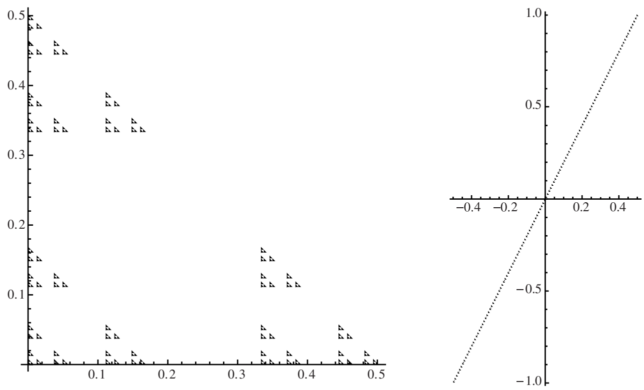

FIGURE 1. The attractors $X(B)$ and $X(L)$

We prove that $\Gamma(L)$ is a spectrum for μ_B but $\Gamma(B)$ is not a spectrum for μ_L.

Set $\zeta := \zeta_3 = e^{2\pi i/3}$. Then the matrix in (2.12) is

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \zeta & \zeta^2\\ 1 & \zeta^2 & \zeta\end{pmatrix},$$

which is the matrix of the Fourier transform on $\mathbb{Z}_3$.

It is easy to see that $X(B) \subset [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and $X(L) \subset \left\{t\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) : -\frac{1}{2} \le t \le \frac{1}{2}\right\}$ are contained in a line.

Even though the dimension here is 2, and Lemma 3.3 applies only to dimension 1, we will still be able to use it for the pair $(\mu_B, \Gamma(L))$, since the attractor $X(L)$ is contained in a line and therefore χ_B has only finitely many zeros in $X(L)$ (see [DJ07b]).

Note that

$$\chi_B\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{3}(1 + e^{2\pi i x} + e^{2\pi i y}), \quad \chi_L\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{3}(1 + e^{2\pi i(x+2y)} + e^{-2\pi i(x+2y)}).$$

Then $\left|\chi_B\begin{pmatrix}x\\y\end{pmatrix}\right| = 1$ iff $x, y \in \mathbb{Z}$, and $\left|\chi_L\begin{pmatrix}x\\y\end{pmatrix}\right| = 1$ iff $x + 2y \in \mathbb{Z}$.

This implies that there are no extreme B-cycles in $X(L)$, and so $\Gamma(L)$ is an ONB for μ_B. Also, we have that $\left(\begin{smallmatrix}0\\1/2\end{smallmatrix}\right)$ is a fixed point for the map $\tau_{(0,1)^T}$ and $\left|\chi_L\left(\begin{smallmatrix}0\\1/2\end{smallmatrix}\right)\right| = 1$, and therefore $\left\{\left(\begin{smallmatrix}0\\1/2\end{smallmatrix}\right)\right\}$ is a non-trivial L-extreme cycle. Hence $\Gamma(B)$ is not an ONB for μ_L.

The next proposition will show that if the set L' is an integer multiple of the set L, then it can only have more B-extreme cycles. This implies that $\Gamma(L')$ will have fewer chances of being an ONB.

Proposition 3.8. *Suppose (R, B, L) and (R, B, L') are as specified in Definition 2.5 and $L' = qL$ for some non-zero integer q. If there are some non-trivial B-extreme cycles in $X(L)$, then there are non-trivial B-extreme cycles in $X(L')$. Thus, if the dimension is 1, and if $\Gamma(L)$ is not an ONB, then $\Gamma(L')$ is not an ONB either.*

Proof. Let C be a B-extreme cycle in $X(L)$. Then, by Lemma 3.4, $b \cdot x \in \mathbb{Z}$ for all $b \in B$ and $x \in C$. Then $b \cdot qx \in \mathbb{Z}$ for $b \in B, x \in C$. Therefore $|\chi_B(qx)| = 1$

for all $x \in C$. Also, if $x_0, x_1 \in C$ and $(R^T)^{-1}(x_0 + l_0) = x_1$, for some $l_0 \in L$, then $(R^T)^{-1}(qx_0 + ql_0) = x_1$. This shows that qC is a B-extreme cycle in $X(L')$. $\square$

4. Transfer operators

Recall (Corollary 2.8) that each of the affine fractal measures under consideration has a well-defined spectral function. Here we show (Lemma 4.1) that the pair of spectral functions corresponding to our paired fractal measures are fixed under an associated pair of transfer operators. Since transfer operators have a rich spectral theory, this throws light on the harmonic analysis of affine fractal measures.

Let R, B, L in $\mathbb{R}^d$ be as specified in Definition 2.5, and assume $0 \in B$, $0 \in L$. Let $(\tau_b)_{b\in B}$ and $(\tau_l)_{l\in L}$ be the dual IFSs from (2.4) and (2.15), respectively.

We will consider the following transfer operators:

$$(T_B f)(t) = \sum_{l\in L} |\chi_B(\tau_l(t))|^2 f(\tau_l(t)) \tag{4.1}$$

and

$$(T_L f)(t) = \sum_{b\in B} |\chi_L(\tau_b(t))|^2 f(\tau_b(t)). \tag{4.2}$$

Set

$$E_+(T_B) := \left\{ f : \mathbb{R} \to [0,1] : T_B f = f, f \in C^1 \text{ and } f(0) = 1 \right\}, \tag{4.3}$$

and similarly for $E_+(T_L)$.

Lemma 4.1. *Let T_B and T_L be the transfer operators in (4.1) and (4.2). Let $\mathbf{1}$ be the constant function one. Then*

$$\mathbf{1} \in E_+(T_B) \cap E_+(T_L), \tag{4.4}$$

$$\sigma^{(B)}_{\Gamma(L)} \in E_+(T_B), \tag{4.5}$$

and

$$\sigma^{(L)}_{\Gamma(B)} \in E_+(T_L). \tag{4.6}$$

Moreover, each space $E_+(T_B)$ and $E_+(T_L)$ is a convex order interval; specifically, the following implication holds:

$$\text{If } f \in E_+(T_B), \text{ then } \sigma^{(B)}_{\Gamma(L)}(t) \le f(t) \le 1, \tag{4.7}$$

and similarly for $E_+(T_L)$.

Proof. Details are included in [DJ06b, DJ07a, DJ07b, DJ07c]. We sketch the proof of (4.5). Using the definition of $\Gamma(L)$ we get

$$\Gamma(L) = L + R^T \Gamma(L). \tag{4.8}$$

Substitution into (2.22) then yields

$$\begin{aligned}
\sigma^{(B)}_{\Gamma(L)}(t) &= \sum_{l\in L} \sum_{\gamma\in\Gamma(L)} |\widehat{\mu}_B(t + l + R^T\gamma)|^2 \\
&= \sum_{l\in L} |\chi_B(\sigma_l(t))|^2 \sum_{\gamma\in\Gamma(L)} |\widehat{\mu}_B((R^T)^{-1}(t+l) + \gamma)|^2 \\
&= \sum_{l\in L} |\chi_B(\tau_l(t))|^2 \sigma^{(B)}_{\Gamma(L)}(\tau_l(t)) = T_B(\sigma^{(B)}_{\Gamma(L)})(t),
\end{aligned}$$

as claimed.

Verification of $\sigma^{(B)}_{\Gamma(L)}(0) = 1$: this is a direct computation using (2.22), and the Hadamard axiom in (2.12). See also the formula

$$\sigma^{(B)}_{\Gamma(L)}(t) = \langle e_0\,,\, U(t)P_L U(-t)e_0\rangle_{L^2(\mu_B)} \tag{4.9}$$

from Corollary 2.8.

Indeed, setting $t = 0$ into (4.9) yields

$$\sigma^{(B)}_{\Gamma(L)}(0) = \langle e_0\,,\, U(0)P_L U(0)e_0\rangle_{L^2(\mu_B)} = \langle e_0\,,\, P_L e_0\rangle_{L^2(\mu_B)} = \langle e_0\,,\, e_0\rangle_{L^2(\mu_B)} = 1$$

since e_0 is in the range of the projection P_L. Recall the assumption $0 \in L$.

For the proof of (4.7), see [DJ06b, DJ08]. □

5. The Cantor measure with scale-similarity factor 4

In this section we revisit a particular fractal measure μ. It is a Cantor measure supported by a compact fractal contained in the real line. Its harmonic analysis was studied first in [JP98] where the authors proved that it has a Fourier basis. In addition to the realization of μ as a Hutchinson measure, it is also (see [JP98]) the result of a recursive Cantor construction with gap-spacing and subdivision 4.

Here is the specific algorithm: begin with the unit interval [0, 1], subdivide by 4, and leave two gaps. Then renormalize the restriction of Lebesgue measure to two of the four subintervals that are retained. Now continue recursively. The resulting sequence of measures has a unique limit. It is the Cantor measure with fractal dimension $\frac{1}{2}$.

Indeed, the limit measure μ has Hausdorff dimension $\frac{1}{2}$. And the Hausdorff dimension coincides with the scaling dimension. The authors of [JP98] showed that $L^2(\mu)$ has an explicit orthonormal basis (ONB) of complex exponentials. The ONB found in [JP98] is here called $\Gamma(\{0, 1\})$.

In the section below we explore the possibility of scaling the earlier known ONBs by integral multiples p. The fact that such a scaling may even lead to new ONBs is rather surprising as the scaled sets become more sparse with increasing values of the integer p. Surprisingly, the arithmetic properties of p explain and account for when the result is again an ONB. For example, we show that if p is divisible by 3, then $\Gamma(\{0, p\})$ is not an ONB in $L^2(\mu)$. When p does not contain the prime factor 3, we show that $\Gamma(\{0, p\})$ may or may not an ONB. For example we prove (Proposition 5.1) that the case $p = 5^k$ is affirmative, i.e., that $\Gamma(\{0, 5^k\})$ is an ONB in $L^2(\mu)$.

Proposition 5.1. *Let $R = 4$, $B = \{0, 2\}$. Then for every integer $k \geq 0$ and $L(5^k) := \{0, 5^k\}$, the set $\Gamma(L(5^k))$ is an ONB for $L^2(\mu_B)$.*

Proof. Using Lemma 3.3, we will show that for any integer $k \geq 0$, there are no non-trivial B-extreme cycles in $X(L(5^k))$. We will do this by induction on k. For $k = 0, 1$, this is easy to check; see the conditions below.

Fix $k \geq 2$. Let $x_0 \in X(L(5^k))$ be a point in a B-extreme cycle, $x_0 \neq 0$. We have

$$\chi_B(t) = \frac{1}{2}(1 + e^{2\pi i 2\cdot t}).$$

Then $|\chi_B(x_0)| = 1$ implies that $x_0 \in \frac{1}{2}\mathbb{Z}$.

We claim that any such B-extreme cycle point must be in $\mathbb{Z}$; but if x_0 is not in $\mathbb{Z}$, it must be of the form $x_0 = a/2$ with a odd. Let x_1 be the next point in the cycle, so $x_1 = (x_0 + l_0)/4$ for some $l_0 \in \{0, 5^k\}$. Since x_1 is a B-extreme cycle point we also have $x_1 \in \frac{1}{2}\mathbb{Z}$. If $l_0 = 5^k$, then $x_1 = (a + 2 \cdot 5^k)/8$. Since the numerator is odd, this cannot be in $\frac{1}{2}\mathbb{Z}$. If $l_0 = 0$ we must have $\frac{1}{2}\mathbb{Z} \ni x_1 = \frac{a}{8}$, which is again impossible. Thus $x_0 \in \mathbb{Z}$.

Let $x_0, x_1, \dots, x_{n-1}$ be the points in this B-extreme cycle, x_0 and let $l_0, \dots, l_{n-1} \in \{0, 5^k\}$ such that

$$\frac{x_0 + l_0}{4} = x_1, \frac{x_1 + l_1}{4} = x_2, \dots, \frac{x_{n-2} + l_{n-2}}{4} = x_{n-1}, \frac{x_{n-1} + l_{n-1}}{4} = x_0. \tag{5.1}$$

Then

$$x_0 \equiv 4x_1 \bmod 5^k, x_1 \equiv 4x_2 \bmod 5^k, \dots, x_{n-1} \equiv 4x_0 \bmod 5^k.$$

This implies that

$$(4^n - 1)x_i \equiv 0 \bmod 5^k, \quad (i \in \{0, \dots, n-1\}). \tag{5.2}$$

Let l be the largest power such that 5^l divides $4^n - 1$. In the case when $l < k$, it follows from (5.2) that all elements of the cycle x_i must be divisible by 5. Then $x_i = 5y_i$ for some $y_i \in \mathbb{Z}$. Dividing (5.1) by 5, it follows that y_i form a B-extreme cycle for $L = \{0, 5^{k-1}\}$, which contradicts the inductive hypothesis.

So we can assume $l = k$ so $4^n - 1$ is divisible by 5^k. We will prove that in this case n is divisible by 5^{k-1}. For this it is easy to prove by induction that for any $k \geq 0$:

$$2^{4 \cdot 5^k} \equiv 1 + 3 \cdot 5^{k+1} \bmod 5^{k+2}. \tag{5.3}$$

For $k = 0$ this is clear; assuming (5.3) for k, write $2^{4 \cdot 5^k} 1 + 3 \cdot 5^{k+1} + 5^{k+2}t$ then raise to the fifth power. Using the multinomial formula and keeping track of the terms not divisible by 5^{k+3}, it follows that (5.3) is true for $k+1$ as well.

Now consider the multiplicative group $(\mathbb{Z}_{5^k})^*$ of elements in $\mathbb{Z}_{5^k}$ that are relatively prime to 5^k. Equation (5.3) for $k-1$, shows that the order of 2 in this group divides $4 \cdot 5^{k-1}$, and (5.3) for $k-2$ shows that the order of 2 in this group cannot divide $4 \cdot 5^{k-2}$. Therefore the order of 2 is divisible by 5^{k-1}.

Then, if $2^{2n} \equiv 1 \bmod 5^k$ this implies that the order of 2 must divide $2n$ so 5^{k-1} divides n.

But if 5^{k-1} divides n, then $n \geq 5^{k-1}$. On the other hand, n is the length of the cycle. This means that we have at least 5^{k-1} points in $\mathbb{Z} \cap X(L)$.

However, $X(L)$ is contained in the interval $[0, \sum_{i=0}^\infty 5^k/4^i] = [0, 5^k/3]$. There are at most $5^k/3$ non-zero integers in this interval. But, as above, for all points in the cycle we must have $x_i + l_i$ divisible by 4, so $x_i \equiv 0 \bmod 4$ or $x_i \equiv -5^k \bmod 4$. These are only 2 equivalence classes mod 4, so the number of integers in $X(L)$ that can be on an extreme cycle is at most $(5^k/3)/2 < 5^{k-1}$. But since the length of the cycle, n, is at least 5^{k-1}, we reach a contradiction.

Thus, there are no non-trivial B-extreme cycles, and therefore $\Gamma(\{0, 5^k\})$ is an ONB. $\square$

Remark 5.2. Proposition 5.1 shows that an ONB for a fractal measure can have the corresponding fractional upper Beurling density arbitrarily small.

The fractional upper Beurling density is defined as follows (see [CKS08]): for a discrete subset Λ of $\mathbb{R}^d$, and for $\alpha > 0$, the α-upper Beurling density of Λ is defined by

$$\mathcal{D}_\alpha^+(\Lambda) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap (x + h[-1,1]^d))}{h^\alpha}$$

and the upper Beurling dimension of Λ is defined by

$$\dim^+(\Lambda) = \sup\{\alpha > 0 : \mathcal{D}_\alpha(\Lambda) > 0\}.$$

It was proved in [DHSW09] that, under some mild assumptions, for any set Λ such that the exponentials $E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$ form a Bessel sequence in $L^2(\mu)$ (in particular ONBs), the upper Beurling dimension is equal to the Hausdorff dimension, which in this case is $\ln 2/\ln 4 = 1/2$, and the 1/2-upper Beurling density $\mathcal{D}^+_{1/2}(\Lambda)$ is finite.

Thus, we have $\mathcal{D}^+_{1/2}(\Gamma_1) < \infty$, where $\Gamma_1 := \Gamma(\{0,1\})$. We check also that $\mathcal{D}^+_{1/2}(\Gamma_1) > 0$.

Take $x = 0$ and $h = \sum_{k=0}^{n-1} 4^k = (4^n - 1)/3$ in the definition of the Beurling density. Then $\#(\Gamma_1 \cap [-h, h]) = 2^n$ (since we have two digits in $\{0,1\}$ and n positions to write the elements in Γ_1). Then

$$\mathcal{D}^+_{1/2}(\Gamma_1) \geq \limsup_{n\to\infty} \frac{2^n}{\left(\frac{4^n-1}{3}\right)^{1/2}} = \sqrt{3} > 0.$$

Thus,

$$0 < \mathcal{D}^+_{1/2}(\Gamma_1) < \infty.$$

Also, it is easy to see that

$$\mathcal{D}^+_{1/2}(5^k\Gamma_1) = \frac{1}{(5^k)^{1/2}}\mathcal{D}^+_{1/2}(\Gamma_1),$$

and therefore

$$\lim_{k\to\infty} \mathcal{D}^+_{1/2}(5^k\Gamma_1) = 0.$$

Let $d = 1$, $R = 4$ and $B = \{0,2\}$, so the measure μ_B satisfies

$$\widehat{\mu}_B(t) = e^{i\frac{2\pi t}{3}} \prod_{k=1}^{\infty} \cos\left(\frac{2\pi t}{4^k}\right). \tag{5.4}$$

Set

$$\Gamma := \Gamma_1 := \left\{\sum_{k=0}^{n} a_k 4^k : a_k \in \{0,1\}, n \in \mathbb{Z}_+\right\}, \tag{5.5}$$

i.e., $\Gamma = \Gamma(L_1)$ with $L_1 = \{0,1\}$.

Proposition 5.3. *Let $p \in \mathbb{Z}_+$, $p > 1$, and $L_p := \{0,p\}$. Then the following conditions are equivalent:*

(i) *$\Gamma(L_p) = p\Gamma_1$ is orthogonal (not necessarily ONB) in $L^2(\mu_B)$;*
(ii) *$\Gamma(B) = 2\Gamma_1$ is orthogonal in $L^2(\mu_{L_p})$;*
(iii) *p is odd.*

Proof. See Proposition 2.6. □

Corollary 5.4. *Let B and $L_p = \{0,p\}$ be as in Proposition 5.3, i.e., assume p is odd. Then the following conditions are equivalent:*

(i) *The only B-extreme cycles in $X(L_p)$ are the trivial singleton $\{0\}$.*
(ii) *The only L_p-extreme cycles in $X(B)$ are the trivial singleton $\{0\}$.*
(iii) *The orthogonal sets in Proposition 5.3 are ONBs.*

Proof. Combine Lemma 3.3 and Theorem 2.10. □

Here is a list of non-trivial B-extreme cycles for all the values of $p \leq 100$. This consists of all odd multiples of 3, and the only such p not divisible by 3 is 85. For the odd values of p that do not appear in the table, there are no such cycles so $\Gamma(L_p)$ is an ONB. Therefore, the list of positive integers p less than 100 such that $\Gamma(\{0, p\})$ is an ONB in $L^2(\mu_B)$ is: 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67, 71, 73, 77, 79, 83, 89, 91, 95, 97.

p	cycles
3	{1}
9	{3}
15	{4,1},{5}
21	{7}
27	{9}
33	{11}
39	{13}
45	{12,3},{15}
51	{13,16,4,1},{17}
57	{19},
63	{16,4,1},{17,20,5},{21}
69	{23}
75	{20,5},{25}
81	{27}
85	{23,27,28,7}
87	{29}
93	{31}
99	{33}

Remark 5.5. It follows from the analysis of the extreme cycles introduced in Proposition 5.3 and Corollary 5.4 that when p is divisible by 3, then there are non-trivial B-extreme cycles in $X(\{0, p\})$, and as a result, the orthogonal sets from Proposition 5.3 cannot be total in the respective $L^2(\mu)$-Hilbert spaces. However, this implication only goes one way as is illustrated in the table for the case of $p = 85$: Even when p does not have 3 as a prime factor there may indeed be non-trivial B-extreme cycles in $X(\{0, p\})$. For $p = 85$, there is a B-extreme cycle in $X(\{0, 85\})$ of length four.

6. Finite cycles

In this section we explore variations in the lists of B-extreme cycles for our family of 1D-examples, algebraic and geometric properties; and we identify two classes of such cycles. Our conclusions have direct relevance to orthogonal harmonic analysis of fractal measures as we developed it in the earlier sections of this paper. But these finite cycles are of independent interest as they have wider significance for factorizations of Mersenne numbers (among other topics); a subject with a multitude of applications: combinatorics, number theory, and encryption; see for example [FLS09, Mum94, MP04, Odl78, Vas06].

In addition, we mention that finite cycles that may be distributed on particular lattices have uses in other problems in analysis, for example in the study of representations of the Cuntz C^*-algebras. In [BJ99], the authors introduced a family of permutative representations of each of the Cuntz algebras $\mathcal{O}_n$. They showed that finite affine cycles distributed on certain associated lattices account for the orthogonal decompositions of these representations. A representation of $\mathcal{O}_n$ acting in a Hilbert space $\mathcal{H}$ is said to be permutative if it permutes the vectors in some orthonormal basis for $\mathcal{H}$. The papers [DJ07b, DJ07c] offer yet other applications of finite cycles with lattice coordinates. For related questions about representations of $\mathcal{O}_n$, see also [SZ08].

Note that in the study of fractals and dynamics, there are multitudes of families of finite cycles; but our present restriction that the particular cycles be B-extreme cuts down the number of cases, and for the discussion below, the B-extremality of a particular cycle $\mathcal{C}$ turns out to be equivalent to requiring $\mathcal{C}$ to be contained in a certain lattice.

We consider a Hadamard system in Definition 2.5 of the following form in one dimension, i.e., $d = 1$.

Setting. Let $R = 2n$, $B = \{0, 2\}$, $p \in \mathbb{Z}_+$ odd, and set $L = L_{n,p} = \{0, np/2\}$.

Then the conditions in Definition 2.5 are satisfied and the Hadamard matrix is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In this section we compute some special τ_L-cycles which are B-extreme. This is of interest since the ONB condition in Lemma 3.3 holds iff the only B-extreme τ_L-cycle is the singleton $\{0\}$.

Recall from Lemma 3.4 that a finite τ_L-cycle $\mathcal{C}$ is B-extreme iff $\mathcal{C} \subset \frac{1}{2}\mathbb{Z}$; see also Definition 3.2.

Set

$$\tau_0(t) = \frac{t}{2n} \text{ and } \tau_1(t) = \frac{t + \frac{np}{2}}{2n} = \frac{t}{2n} + \frac{p}{4} \tag{6.1}$$

and let $\omega = (\omega_1 \omega_2 \dots \omega_l)$, $\omega_i \in \{0, 1\}$ be a finite word. Set

$$\tau_\omega = \tau_{\omega_1} \circ \tau_{\omega_2} \circ \dots \circ \tau_{\omega_l}. \tag{6.2}$$

Lemma 6.1. *If p is divisible by $2n-1$, then there are non-trivial B-extreme $\tau_{L_{n,p}}$-cycles of length one.*

Proof. Let $p = m(2n-1)$, $m \in \mathbb{Z}$. Then the solution t to $\tau_1(t) = t$ is

$$t = \frac{np}{2(2n-1)} = \frac{mn}{2} \in \frac{1}{2}\mathbb{Z}. \tag{6.3}$$

Hence $\left\{\frac{mn}{2}\right\}$ is a B-extreme $\tau_{L_{n,p}}$-cycle. ∎

Corollary 6.2. (i) *For $R = 4$, $B = \{0, 2\}$, and $L = \{0, 3\}$, the singleton $\mathcal{C} = \{1\}$ is a B-extreme τ_L-cycle; and so, in particular, $\Gamma(\{0, 3\})$ is not an ONB in $L^2(\mu_{1/4})$.*

(ii) *For $R = 6$, $B = \{0, 2\}$ and $L = \left\{0, \frac{15}{2}\right\}$, the singleton $\mathcal{C} = \left\{\frac{3}{2}\right\}$ is a B-extreme τ_L-cycle.*

(iii) *For $R = 8$, $B = \{0, 2\}$, and $L = \{0, 14\}$, the singleton $\mathcal{C} = \{2\}$ is a B-extreme τ_L-cycle.*

Definition 6.3. Let $n \in \mathbb{Z}_+$ be given, and set $R = 2n$, $B = \{0, 2\}$ and let $\mu = \mu_{1/2n} = \mu_B$ be the fractal measure from section 2. We say that $p \in \mathbb{Z}_+$ odd is *admissible* iff there are non-trivial B-extreme τ_L-cycles. Here $L = L_{n,p} = \{0, np/2\}$.

Note that if $n = 2$, then $p = 3$ is admissible. A consequence of Lemma 6.1 is that for $n \in \mathbb{Z}_+$, $p = 2n - 1$ is admissible, and, with Proposition 3.8, we obtain that any p divisible by $2n - 1$ is admissible.

Question. Are there admissible values of p not divisible by $2n-1$?

The next theorem offers an affirmative answer.

Theorem 6.4. *Let $n \in \mathbb{Z}_+$ be given. Set*

$$p = \sum_{i=0}^{2n-1} (2n)^i. \tag{6.4}$$

Then p is admissible and not divisible by $2n-1$. There are associated B-extreme cycles of length $2n$.

Proof. Let p be given by (6.4). Then $p \equiv 1 \bmod 2n-1$, so it is not divisible by $2n-1$. We shall need

$$p^* := \sum_{i=0}^{2n-2} (2n)^i. \tag{6.5}$$

Note that $p^* \equiv 0 \bmod 2n-1$. So $p^* = m(2n-1)$ for some $m \in \mathbb{Z}$.

Consider the solution t of the fixed point equation

$$\tau_0\tau_1^{2n-1}(t) = t.$$

It is easy to see that

$$t = \frac{\frac{np}{2}\sum_{i=0}^{2n-2}(2n)^i}{(2n)^{2n}-1} = \frac{npp^*}{2(2n-1)p} = \frac{np^*}{2(2n-1)} = \frac{nm}{2} \in \frac{1}{2}\mathbb{Z}.$$

We check by induction that $\tau_1^k(t) \in \frac{1}{2}\mathbb{Z}$ for $k \in \{0, 1, \dots, 2n-1\}$. We use that $p = p^* + (2n)^{2n-1}$. We have

$$\begin{aligned}\tau_1(t) = \frac{t + \frac{np}{2}}{2n} = \frac{\frac{mn}{2} + \frac{np}{2}}{2n} &= \frac{n}{2}\frac{\frac{p^*}{2n-1} + p^* + (2n)^{2n-1}}{2n} \\ &= \frac{n}{2}\left(\frac{(2n)p^*}{2n(2n-1)} + (2n)^{2n-2}\right).\end{aligned}$$

So,

$$\tau_1(t) = \frac{mn}{2} + \frac{n}{2}(2n)^{2n-2} = t + \frac{n}{2}(2n)^{2n-2}.$$

Then

$$\tau_1^2(t) = \tau_1\left(t + \frac{n}{2}(2n)^{2n-2}\right) = \tau_1(t) + \frac{1}{2n}\frac{n}{2}(2n)^{2n-2} = t + \frac{n}{2}(2n)^{2n-2} + \frac{n}{2}(2n)^{2n-3},$$

where we used the previous step in the last equality.

By induction we get

$$\tau_1^k(t) = t + \frac{n}{2}(2n)^{2n-2} + \frac{n}{2}(2n)^{2n-3} + \cdots + \frac{n}{2}(2n)^{2n-k-1}$$

for all $k \leq 2n-1$, and this shows that $\tau_1^k(t)$ is in $\frac{1}{2}\mathbb{Z}$. Since $\tau_0\tau_1^{2n-1}(t) = t$, this implies that the entire cycle is in $\frac{1}{2}\mathbb{Z}$, so it is a B-extreme $\tau_{L_{n,p}}$-cycle. □

Remark 6.5. An application of Theorem 6.4 to $R = 4$ and $p = 1 + 4 + 4^2 + 4^3 = 85$ accounts for the cycle $\mathcal{C} = \{23, 27, 28, 7\}$ from the table before Remark 5.5. It is the smallest admissible p not divisible by 3.

For $R = 6$ and $p = 1 + 6 + 6^2 + 6^3 + 6^4 + 6^5 = 9331$ one has $L = \{0, 27993/2\}$ and the B-extreme cycle of length 6 is

$$\mathcal{C} = \left\{ \frac{4821}{2}, \frac{5469}{2}, \frac{5577}{2}, \frac{5595}{2}, 2799, \frac{933}{2} \right\}.$$

It is also the smallest admissible p not divisible by 5.

For $R = 8$ and $p = \sum_{i=0}^{7} 8^i = 2396745$ one has $L = \{0, 4793490\}$ and the B-extreme cycle of length 8 is

$$\mathcal{C} = \{609886, 675422, 683614, 684638, 684766, 684782, 684784, 85598\}.$$

We used the following Mathematica program to check for extreme cycles.

```
r = 8;
NextCycle[x_, p_] :=
  If[IntegerQ[2*x/r], x/r, If[IntegerQ[2*(x + p)/r], (x + p)/r, -1]];
pmax = r*(r^r - 1)/(4*(r - 1))
For[p = pmax - r/2, p < pmax, p = p + r/2;
 If[! IntegerQ[2*p/(r - 1)],
  For[ix = 1/2, ix <= IntegerPart[2*p/(r - 1)]/2, ix = ix + 1/2,
   x = NextCycle[ix, p]; flag = 0;
   While[x != -1 && x != ix, x = NextCycle[x, p]];
   If[x != -1, cycle = {}; If[flag == 0, Print["p= ", p]; flag = 1];
    Print["Cycle pt "]; i = 1;
    While[i == 1, x = NextCycle[x, p]; Print[" ", x];
     If[x == ix, i = 0]]; Print["End Cycle "]
    ]
   ]
  ]
 ]
```

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