MATHEMATICS OF COMPUTATION
Volume 82, Number 281, January 2013, Pages 1–24
S 0025-5718(2012)02601-9
Article electronically published on April 13, 2012

L^2-ESTIMATES FOR THE EVOLVING SURFACE FINITE ELEMENT METHOD

GERHARD DZIUK AND CHARLES M. ELLIOTT

ABSTRACT. In this paper we consider the evolving surface finite element method for the advection and diffusion of a conserved scalar quantity on a moving surface. In an earlier paper using a suitable variational formulation in time dependent Sobolev space we proposed and analysed a finite element method using surface finite elements on evolving triangulated surfaces. An optimal order H^1-error bound was proved for linear finite elements. In this work we prove the optimal error bound in $L^2(\Gamma(t))$ uniformly in time.

1. INTRODUCTION

Conservation of a scalar u subject to advection and diffusion on an evolving hypersurface $\Gamma(t) \subset \mathbb{R}^{n+1}, n = 1, 2, 3$ for time $t \in [0, T]$, $T > 0$, leads to the diffusion equation

$$\partial^\bullet u + u\,\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathcal{A}\nabla_\Gamma u) = 0 \tag{1.1}$$

on $\Gamma(t)$. Here ν is a normal vector field to the surface, v_ν is the normal velocity of the surface, v_τ is an advective tangential velocity field, $v = v_\nu + v_\tau$, ∇_Γ is the tangential surface gradient, and $\mathcal{A}$ is a given diffusion tensor which is positive definite on the tangent space of Γ. We set

$$\partial^\bullet u = \partial^\circ u + v_\tau \cdot \nabla_\Gamma u$$

to be the the material derivative and

$$\partial^\circ u = u_t + v_\nu \cdot \nabla u$$

to be the normal time derivative.

In [9] we proposed a finite element approximation using piecewise linear finite elements on a triangulated surface interpolating $\Gamma(t)$ whose vertices move with the velocity v. The finite element method is based on the variational form

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{A}\nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi \tag{1.2}$$

where φ is an arbitrary test function defined on the surface $\Gamma(t)$ for all t. For the error between the continuous solution u and spatially discrete solution u_h we proved

Received by the editor July 15, 2010 and, in revised form, July 5, 2011 and July 29, 2011.
2010 *Mathematics Subject Classification.* Primary 65M60, 65M15; Secondary 35K99, 35R01, 35R37, 76R99.
Key words and phrases. Surface finite elements, advection diffusion, moving surface, error analysis.
The work was supported by Deutsche Forschungsgemeinschaft via SFB/TR 71.
This research was also supported by the UK Engineering and Physical Sciences Research Council (EPSRC), Grant EP/G010404.

in [9] the bound

$$\sup_{t\in(0,T)} \|u(\cdot,t)-u_h(\cdot,t)\|^2_{L^2(\Gamma(t))} + \int_0^T \|\nabla_\Gamma(u(\cdot,t)-u_h(\cdot,t))\|^2_{L^2(\Gamma(t))}dt \le ch^2,$$

where the constant c depends on norms of the continuous solution and the data of the problem. This estimate gives optimal linear convergence for the error in the gradients. But for the L^2-error one expects (just as in the Cartesian case) convergence of order two. This is consistent with numerical experiments performed in [9] in Table 1. The purpose of this paper is to prove an optimal L^2-estimate for the error between continuous and discrete problem:

$$\sup_{t\in(0,T)} \|u(\cdot,t)-u_h(\cdot,t)\|_{L^2(\Gamma(t))} \le ch^2.$$

1.1. **Numerical methods for surface PDEs.** Let us review briefly the computation of surface partial differential equations and, in particular, parabolic equations on evolving surfaces.

Surface finite elements on triangulated hypersurfaces were proposed and analysed for the Laplace-Beltrami equation by [8] and extended to parabolic (including nonlinear and higher order) equations on stationary surfaces in [10]. Higher order finite element spaces for elliptic equations were analysed in [6] and an adaptive finite element method for stationary surfaces was considered in [7]. An extension of the idea of the surface finite element method (SFEM) is to use surface finite volumes. An analysis of elliptic equations using general meshes is given in [22, 23]. A method for parabolic equations on stationary surfaces using logically Cartesian grids is presented in [3].

The evolving surface finite element method (ESFEM) was proposed in [9] in order to treat diffusion and transport on moving surfaces. The key idea is to use the Leibniz formula for differentiating with respect to time integrals over moving surfaces in order to derive weak and variational formulations. The upshot is that the velocity and mean curvature of the surface do not appear explicitly in the formulations. Applications to complex physical models may be found in [13, 14]. The idea may also be used in a finite volume context, [24].

Another approach is to numerically solve bulk equations in one space dimension higher. This may be a natural approach when the surface is computed implicitly using phase field or level set methods [4, 32, 29]. The idea is then to exploit the implicit formulation and use a bulk triangulation rather than a surface triangulation. One idea is to solve the surface partial differential equations on all level sets of a prescribed function. This is inherently an Eulerian method and yields degenerate equations; see [2, 19, 18, 11] for stationary surfaces. For surface elliptic equations, [5] gave a discretisation error analysis for a narrow band level set method using the unfitted finite element method. Another method using a bulk unfitted mesh and finite element space independent of the surface which is given by a level set function has been proposed in [27, 28]. Eulerian approaches to transport and diffusion on evolving surfaces were given in [1, 33] where level set approximations to surface quantities were required. On the other hand, an elegant formulation avoiding the need to do this was provided in [12] using an implicit surface version of the Leibniz formula.

The closest point method for partial differential equations on stationary surfaces, [31, 25, 26], is based on considering $u(p(x))$ where $p(x) \in \Gamma$ is the point closest to

x. The surface partial differential equation is then embedded and discretised in a neighbourhood of Γ using $u(p(x))$. Implementation requires the knowledge or calculation of the closest point $p(x)$. In the cited references this approach has been used to solve a wide variety of equations on stationary surfaces.

The diffuse interface approach [15, 16, 30] is based on approximating the surface using a phase field type variable and solving a bulk advection diffusion equation with coefficients which are zero or small outside of a transition layer.

1.2. **Layout of the paper.** This paper is organised as follows. In the following section we fix some notation and develop the notions of material derivative and transport formulae for moving surfaces appropriate for our needs. Then in the third section we formulate the advection diffusion equation on an evolving surface. The evolving surface finite element method is presented in section 4. In the next section we derive useful estimates concerning interpolation bounds, the approximation of geometry and the perturbation of appropriate bilinear forms. In order to prove the error bound it is convenient to use the Ritz projection and, in particular, we need error bounds for the material derivative which are proved in section 6. Finally, in section 7 we prove the L^2-error bound.

2. Setting and notation

2.1. **Description of the surface.** For each $t \in [0,T]$, $T > 0$, let $\Gamma(t)$ be a compact smooth connected hypersurface in $\mathbb{R}^{n+1}$ $(n = 1,2,3)$ and $\Gamma_0 = \Gamma(0)$. We assume that $\Gamma(t)$ is the boundary of an open bounded set $\Omega(t)$. It follows that $\Gamma(t)$ has a representation defined by a smooth level set function. Let $d = d(x,t)$, $x \in \mathbb{R}^{n+1}, t \in [0,T]$ be the signed distance function so that $d(\cdot,t) < 0$ in $\Omega(t)$ and $d(\cdot,t) > 0$ in $\mathbb{R}^{n+1} \setminus \overline{\Omega}(t)$. Assume that

$$\Gamma(t) = \{x \in \mathcal{N}(t) |\, d(x,t) = 0\},$$

where $\mathcal{N}(t)$ is an open subset of $\mathbb{R}^{n+1}$ in which $\nabla d \neq 0$ and chosen so that

$$d, d_t, d_{x_i}, d_{x_i x_j} \in C^2(\mathcal{N}_T)\,(i,j = 1,\dots,n+1)$$

for $\mathcal{N}_T = \bigcup_{t\in[0,T]} \mathcal{N}(t) \times \{t\}$. The orientation of $\Gamma(t)$ is set by taking the normal ν to $\Gamma(t)$ to be in the direction of increasing d. Hence we define a normal vector field by

$$\nu(x,t) = \nabla d(x,t),\ x \in \mathcal{N}(t),$$

so that the normal velocity v_ν of $\Gamma(t)$ is given by

$$v_\nu(x,t) = -d_t(x,t)\nu(x,t),\ x \in \Gamma(t). \tag{2.1}$$

We denote by $P = P(x,t)$ the projection onto the tangent space at $x \in \Gamma(t)$, $P_{ij} = \delta_{ij} - \nu_i\nu_j$ and by $\mathcal{H} = \mathcal{H}(x,t)$ the (extended) Weingarten map

$$\mathcal{H}_{ij} = (\nu_i)_{x_j} = d_{x_i x_j}.$$

We will use the fact that

$$P\mathcal{H} = \mathcal{H}P = \mathcal{H}.$$

We choose $\mathcal{N}(t)$ so that for every $x \in \mathcal{N}(t)$ and $t \in [0,T]$ there exists a unique $p(x,t) \in \Gamma(t)$ such that

$$x = p(x,t) + d(x,t)\nu(p(x,t),t). \tag{2.2}$$

This is possible by the assumptions above; see for example [21]. Observe that

$$\nu(x,t) = \nu(p(x,t),t), \quad P(x,t) = P(p(x,t),t). \tag{2.3}$$

From this one easily gets the relation

$$\frac{\partial d}{\partial t}(x,t) = \frac{\partial d}{\partial t}(p(x,t),t). \tag{2.4}$$

2.2. **Surface gradients.** For any function $\eta \in C^1(\mathcal{N}(t))$ we define its (spatial) tangential gradient on $\Gamma(t)$ by

$$\nabla_\Gamma \eta = P\nabla\eta = \nabla\eta - \nabla\eta \cdot \nu\,\nu$$

where, for x and y in $\mathbb{R}^{n+1}$, $x \cdot y$ denotes the usual scalar product and $\nabla\eta$ denotes the usual gradient on $\mathbb{R}^{n+1}$. The tangential gradient $\nabla_\Gamma\eta$ only depends on the values of η restricted to $\Gamma(t)$ and $\nabla_\Gamma\eta \cdot \nu = 0$. The components of the tangential gradient will be denoted by

$$\nabla_\Gamma \eta = \left((\nabla_\Gamma)_1\,\eta, \ldots, (\nabla_\Gamma)_{n+1}\,\eta\right).$$

The Laplace-Beltrami operator on $\Gamma(t)$ is the tangential divergence of the tangential gradient:

$$\Delta_\Gamma \eta = \nabla_\Gamma \cdot \nabla_\Gamma \eta.$$

Let $\mathcal{M}(t) \subset \Gamma(t)$ have a boundary $\partial\mathcal{M}(t)$ whose intrinsic unit outer normal (conormal), tangential to $\Gamma(t)$, is denoted by μ. Then the formulae for integration by parts on $\mathcal{M}(t)$ and $\Gamma(t)$ are

$$\int_{\mathcal{M}(t)} (\nabla_\Gamma)_i \eta = \int_{\mathcal{M}(t)} \eta H \nu_i + \int_{\partial\mathcal{M}(t)} \eta \mu_i \tag{2.5}$$

and

$$\int_{\Gamma(t)} (\nabla_\Gamma)_i \eta = \int_{\Gamma(t)} \eta H \nu_i, \tag{2.6}$$

where H denotes the mean curvature of Γ with respect to ν, which is given by

$$H = \nabla_\Gamma \cdot \nu. \tag{2.7}$$

The orientation is such that for a sphere $\Gamma = \{x \in \mathbb{R}^{n+1} |\, |x - x_0| = R\}$ and the choice $d(x) = |x - x_0| - R$ the normal is pointing out of the ball $B_R(x_0) = \{x \in \mathbb{R}^{n+1} |\, |x - x_0| < R\}$ and the mean curvature of Γ is given by $H = \frac{n}{R}$. Note that H is the sum of the principle curvatures rather than the arithmetic mean and hence differs from the common definition by a factor n. The mean curvature vector $H\nu$ is invariant with respect to the choice of the sign of d.

Green's formula on the surface Γ is

$$\int_\Gamma \nabla_\Gamma \xi \cdot \nabla_\Gamma \eta = \int_{\partial\Gamma} \xi \nabla_\Gamma \eta \cdot \mu - \int_\Gamma \xi \Delta_\Gamma \eta. \tag{2.8}$$

If Γ is closed, then $\partial\Gamma$ is empty and the boundary terms do not appear. For these facts about tangential derivatives we refer to [17], pp. 389–391. Note that, in general, higher order tangential derivatives do not commute.

We shall use Sobolev spaces on surfaces Γ. For a given C^2 surface Γ we define

$$H^1(\Gamma) = \{\eta \in L^2(\Gamma) \mid \nabla_\Gamma \eta \in L^2(\Gamma)^{n+1}\}.$$

In this definition $\nabla_\Gamma \eta$ denotes the weak derivative defined via integration by parts with the help of (2.6). For smooth enough Γ we analoguously define the Sobolev spaces $H^k(\Gamma)$ for $k \in \mathbb{N}$. We set

$$\mathcal{G}_T = \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}.$$

2.3. **The material derivative.**

Definition 2.1 (Material derivatives)**.** The *normal time derivative* ∂° is defined by

$$\partial^\circ f := \frac{\partial f}{\partial t} + v_\nu \cdot \nabla f. \tag{2.9}$$

We say that v_τ is a tangential velocity field provided $v_\tau \cdot \nu = 0$ in $\mathcal{N}(t)$. Given a tangential velocity field v_τ we say that

$$v := v_\nu + v_\tau \tag{2.10}$$

is a *material velocity* field. Given a tangential vector field v_τ we denote the *material derivative* of a scalar function $f = f(x,t)$ defined on $\mathcal{N}_T$ by

$$\partial^\bullet f := \partial^\circ f + v_\tau \cdot \nabla_\Gamma f = \frac{\partial f}{\partial t} + v \cdot \nabla f. \tag{2.11}$$

Remark 2.2. The normal time derivative is an intrinsic surface derivative on $\mathcal{G}_T$ depending only on values of f on this surface and independent of any parameterisation of the surface; *cf.* [20]. For any x_0 on $\Gamma(t_0)$ we may define a curve $\{\gamma(t) \in \Gamma(t), t_0 - \epsilon < t < t_0 + \epsilon\}$ by setting $\gamma(t_0) = x_0$ and $\gamma'(t) = v_\nu(\gamma(t), t)$. Clearly,

$$\frac{d}{dt} d(\gamma, \cdot) = \frac{\partial d}{\partial t} + \nabla d \cdot \gamma' = \frac{\partial d}{\partial t} + \nu \cdot v_\nu = d_t + \nu \cdot v_\nu = 0$$

so that, indeed, the curve does lie on the surface. It follows that

$$\partial^\circ f = \frac{df(\gamma, \cdot)}{dt}. \tag{2.12}$$

The material derivative is also an intrinsic surface derivative on $\mathcal{G}_T$. Now consider a curve for any x_0 on $\Gamma(t_0)$ such that $\{\gamma(t) \in \Gamma(t), t_0 - \epsilon < t < t_0 + \epsilon\}, \gamma(t_0) = x_0$ with velocity

$$\gamma'(t) = (v_\nu + v_\tau)(\gamma(t), t).$$

Now we see that

$$\frac{d}{dt} d(\gamma, \cdot) = \frac{\partial d}{\partial t} + \nabla d \cdot \gamma' = \frac{\partial d}{\partial t} + \nu \cdot (v_\nu + v_\tau) = d_t + \nu \cdot v_\nu = 0$$

so that, indeed, the curve lies on the surface. Furthermore, we have that

$$\partial^\bullet f = \frac{df(\gamma, \cdot)}{dt}. \tag{2.13}$$

2.4. **Transport theorem.** The following formula for the differentiation of time dependent surface integrals will play a decisive role.

Lemma 2.1. *Let $\mathcal{M}(t)$ be an evolving surface with normal velocity v_ν. Let v_τ be a tangential velocity field on $\mathcal{M}(t)$. Let the boundary $\partial\mathcal{M}(t)$ evolve with the velocity $v := v_\nu + v_\tau$. Assume that f is a function such that all the following quantities exist. Then*

$$\frac{d}{dt}\int_{\mathcal{M}(t)} f = \int_{\mathcal{M}(t)} \partial^\bullet f + f\,\nabla_\Gamma \cdot v. \tag{2.14}$$

With the deformation tensor $D(v)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1}\left(\mathcal{A}_{ik}\left(\nabla_\Gamma\right)_k v_j + \mathcal{A}_{jk}\left(\nabla_\Gamma\right)_k v_i\right)(i,j = 1,\dots,n+1)$ and the tensor

$$\mathcal{B}(v) = \partial^\bullet \mathcal{A} + \nabla_\Gamma \cdot v\,\mathcal{A} - 2D(v), \tag{2.15}$$

we have the formula

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}(t)} \mathcal{A}\nabla_\Gamma f \cdot \nabla_\Gamma g = &\int_{\mathcal{M}(t)} \Big(\mathcal{A}\nabla_\Gamma \partial^\bullet f \cdot \nabla_\Gamma g + \mathcal{A}\nabla_\Gamma f \cdot \nabla_\Gamma \partial^\bullet g\Big)\\ &+ \int_{\mathcal{M}(t)} \mathcal{B}(v)\nabla_\Gamma f \cdot \nabla_\Gamma g.\end{aligned} \tag{2.16}$$

Proof. A proof of this Lemma for $f = g$ and $\mathcal{A} = I$ was given in the appendix of [9]. The more general form $f \neq g$ can be obtained easily by polarization. □

Remark 2.3. Observe that in the above Leibniz formula, (2.14), it is sufficient that f and $\partial^\bullet f \in L^1(\mathcal{G}_T)$, $v \in C^1(\mathcal{G}_T)$, and the equation is understood in the distributional sense that

$$-\int_0^T \psi' \int_{\mathcal{M}(t)} f\,dA\,dt = \int_0^T \psi \int_{\mathcal{M}(t)} \partial^\bullet f + f\,\nabla_\Gamma \cdot v\,dA\,dt \quad \forall \psi \in C_0^1\left((0,T)\right).$$

Remark 2.4. The formulae (2.14) and (2.16) are true when $\mathcal{M}(t)$ is without boundary and with any tangential velocity field v_τ including $v_\tau = 0$. In particular, the formulae are true with $v = v_\nu$ and the material derivative $\partial^\bullet$ replaced by the normal time derivative ∂°.

Remark 2.5. In the following we use these formulae to derive the advection diffusion equation. In this derivation we will take v_τ to be zero and consider mass conservation for a region $\mathcal{M}(t)$ evolving on the surface with normal velocity v_ν. On the other hand, our numerical discretization will be based on a triangulated surface whose vertices move with a velocity $v_\nu + v_\tau$ in which case we apply the above formulae element by element in order to derive transport formulae for a triangulated surface.

3. Formulation of the advection diffusion equation

3.1. **Conservation law.** Let u be the density of a scalar quantity on $\Gamma(t)$ (for example mass per unit area $n = 2$ or mass per unit length $n = 1$). Let q denote a surface flux. The basic conservation law we wish to consider can be formulated for an arbitrary portion $\mathcal{M}(t)$ of $\Gamma(t)$, which is the image of a portion $\mathcal{M}(0)$ of $\Gamma(0)$ evolving with the prescribed velocity $v := v_\nu$. The law is that, for every $\mathcal{M}(t)$,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = -\int_{\partial\mathcal{M}(t)} q \cdot \mu, \tag{3.1}$$

where, $\partial\mathcal{M}(t)$ is the boundary of $\mathcal{M}(t)$ (a curve if $n = 2$ and the end points of a curve if $n = 1$) and μ is the conormal on $\partial\mathcal{M}(t)$. Thus μ is the unit normal to $\partial\mathcal{M}(t)$ pointing out of $\mathcal{M}(t)$ and tangential to $\Gamma(t)$. Observing that components of q normal to $\mathcal{M}$ do not contribute to the flux, we assume that q is a tangent vector.

With the use of integration by parts, (2.5), we obtain

$$\int_{\partial\mathcal{M}(t)} q \cdot \mu = \int_{\mathcal{M}(t)} \nabla_\Gamma \cdot q - \int_{\mathcal{M}(t)} q \cdot \nu H = \int_{\mathcal{M}(t)} \nabla_\Gamma \cdot q.$$

On the other hand, by the transport formula (2.14) we have

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u = \int_{\mathcal{M}(t)} \left(\partial^\circ u + u\nabla_\Gamma \cdot v_\nu\right),$$

so that

$$\int_{\mathcal{M}(t)} (\partial^\circ u + u\,\nabla_\Gamma \cdot v_\nu + \nabla_\Gamma \cdot q) = 0,$$

which implies the pointwise conservation law

$$\partial^\circ u + u\,\nabla_\Gamma \cdot v_\nu + \nabla_\Gamma \cdot q = 0. \tag{3.2}$$

In this paper we wish to consider a diffusive flux $q_d := -\mathcal{A}\nabla_\Gamma u$ and an advective flux $q_a := uv_\tau$ where v_τ is an advective tangential velocity field, i.e., $v_\tau \cdot \nu = 0$, so that

$$q = q_d + q_a = -\mathcal{A}\nabla_\Gamma u + uv_\tau,$$

then we arrive at the PDE (1.1)

$$\partial^\bullet u + u\,\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathcal{A}\nabla_\Gamma u) = 0.$$

3.2. **Weak and variational formulations.** In the following we assume that $\mathcal{A}$ is a sufficiently smooth $(n+1)\times(n+1)$ symmetric matrix which maps the tangent space of Γ at a point into itself and is positive definite on the tangent space, i.e.,

$$\mathcal{A}\xi \cdot \xi \geq c_0|\xi|^2 \quad \forall \xi \in \mathbb{R}^{n+1}, \xi \cdot \nu = 0 \tag{3.3}$$

with some constant $c_0 > 0$. Consequently, $\mathcal{A}P = P\mathcal{A} = \mathcal{A}$. For the definition of a solution we assume that the elements of $\mathcal{A}$ belong to $L^\infty(\mathcal{G}_T)$. We set

$$H^1(\mathcal{G}_T) := \{\eta \in L^2(\mathcal{G}_T) \mid \nabla_\Gamma \eta \in L^2(\mathcal{G}_T), \partial^\circ \eta \in L^2(\mathcal{G}_T)\}.$$

Definition 3.1 (Weak solution)**.** A function $u \in H^1(\mathcal{G}_T)$ is a weak solution of (1.1), if for almost every $t \in (0,T)$,

$$\int_{\Gamma(t)} \partial^\bullet u\varphi + \int_{\Gamma(t)} u\varphi\,\nabla_\Gamma \cdot v + \int_{\Gamma(t)} \mathcal{A}\nabla_\Gamma u \cdot \nabla_\Gamma \varphi = 0$$

for every $\varphi(\cdot,t) \in H^1(\Gamma(t))$.

In [9] we proved the existence of a weak solution. Furthermore, for the initial data $u_0 \in H^1(\Gamma(0))$ and the elements of $\mathcal{A}$ and $v \in C^1(\mathcal{G}_T)$ the solution satisfies

$$\sup_{(0,T)} \|u\|^2_{L^2(\Gamma)} + \int_0^T \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} \leq c\|u_0\|^2_{L^2(\Gamma(0))},$$

$$\int_0^T \|\partial^\bullet u\|^2_{L^2(\Gamma)} + \sup_{(0,T)} \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} \leq c\|u_0\|^2_{H^1(\Gamma(0))}$$

where $c = c(\mathcal{A}, v, \mathcal{G}_T, T)$.

Lemma 3.1 (Variational form)**.** *A weak solution u satisfies the* variational form *of* (1.1)*: for almost every $t \in (0,T)$,*

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{A}\nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi \tag{3.4}$$

for every $\varphi \in H^1(\mathcal{G}_T)$.

Proof. This is a consequence of the Leibniz formula (2.14) and we understand the equation (3.4) in the sense described in Remark 2.3. □

Remark 3.2. Note that , since $\mathcal{A}\nabla_\Gamma u \cdot \nabla_\Gamma \varphi = P\mathcal{A}P\nabla_\Gamma u \cdot \nabla_\Gamma \varphi$, in the weak form (3.4) we do not have to assume that $\mathcal{A}$ maps tangent space to tangent space.

3.3. **Abstract framework.** For $\varphi, \psi \in H^1(\Gamma)$ we define the bilinear forms

$$a(\varphi,\psi) = \int_\Gamma \mathcal{A}\nabla_\Gamma \varphi \cdot \nabla_\Gamma \psi, \tag{3.5}$$

$$m(\varphi,\psi) = \int_\Gamma \varphi\psi, \tag{3.6}$$

$$g(v;\varphi,\psi) = \int_\Gamma \varphi\psi\nabla_\Gamma \cdot v. \tag{3.7}$$

The variational form (3.4) becomes

$$\frac{d}{dt}m(u,\varphi) + a(u,\varphi) = m(u,\partial^\bullet\varphi). \tag{3.8}$$

Remark 3.3 (Abstract transport formulae). Note that from the transport formulae, Lemma 2.1, we have for $\varphi, \psi \in H^1(\mathcal{G}_T)$,

$$\frac{d}{dt}m(\varphi,\psi) = m(\partial^\bullet\varphi,\psi) + m(\varphi,\partial^\bullet\psi) + g(v;\varphi,\psi) \tag{3.9}$$

and

$$\frac{d}{dt}a(\varphi,\psi) = a(\partial^\bullet\varphi,\psi) + a(\varphi,\partial^\bullet\psi) + b(v;\varphi,\psi) \tag{3.10}$$

with the bilinear form

$$b(v;\varphi,\psi) = \int_\Gamma \mathcal{B}(v)\nabla_\Gamma\varphi \cdot \nabla_\Gamma\psi. \tag{3.11}$$

4. Surface finite element method

4.1. **Triangulated surface.** The smooth evolving surface $\Gamma(t)$ ($\partial\Gamma(t) = \emptyset$) is approximated by an evolving surface

$$\Gamma_h(t) \subset \mathcal{N}(t)\ ,\ (\partial\Gamma_h(t) = \emptyset),$$

which for each t is of class $C^{0,1}$ and in time is smooth. In particular, for $n = 2$, $\Gamma_h(t)$ is a triangulated (and hence polyhedral) surface consisting of simplices $E(t) \in \mathcal{T}_h(t)$, which form an admissible triangulation. We suppose that the maximum diameter of the simplices in $\mathcal{T}_h(t)$ are bounded uniformly in time by h. Note that by (2.2) for every triangle $E(t) \subset \Gamma_h(t)$ there is a unique curved triangle $e(t) = p(E(t),t) = \{p(x,t) |\, x \in E(t)\} \subset \Gamma(t)$. We assume that $\Gamma_h(t)$ is homeomorphic to $\Gamma(t)$. This implies that there is a bijective correspondence between the triangles on Γ_h and the induced curvilinear triangles on Γ.

We begin by setting up extensions, η_h^l and η^{-l}, of functions η_h and η defined, respectively, on $\Gamma_h(t)$ and $\Gamma(t)$. For any continuous function η_h on $\Gamma_h(t)$ we define the lift onto $\Gamma(t)$ by

$$\eta_h^l(p,t) = \eta_h(x(p,t),t) \text{ for } p \in \Gamma(t) \tag{4.1}$$

where by (2.2) and our assumptions, $x(p,t)$ is uniquely defined by

$$x(p,t) = p + d(x(p,t),t)\nu(p,t). \tag{4.2}$$

For any continuous function η defined on the surface $\Gamma(t)$ we define a constant extension to $\mathcal{N}(t)$ in the normal direction by

$$\eta^{-l}(x,t) := \eta(p(x,t),t). \tag{4.3}$$

By definition (see (2.2),(2.3))

$$\eta^{-l}(x,t) = \eta(x - d(x,t)\nu(x,t),t), \quad x \in \Gamma_h(t).$$

For any function η we define its (spatial) tangential gradient on $E(t) \subset \Gamma_h(t)$ by

$$\nabla_{\Gamma_h}\eta = P_h\nabla\eta = \nabla\eta - \nabla\eta\cdot\nu_h\,\nu_h$$

where ν_h is the normal to $E(t)$ oriented in the direction of increasing d and $(P_h)_{ij} := \delta_{ij} - (\nu_h)_i(\nu_h)_j$.

Remark 4.1. The chain rule together with the definition of the tangential gradients on smooth and discrete surface and (2.3) gives

$$\nabla_{\Gamma_h}\eta(x,t) = P_h(x,t)\,(I - d(x,t)\mathcal{H}(x,t))\,P(x,t)\nabla_\Gamma\eta^l(p(x,t),t), \quad x \in \Gamma_h(t).$$

4.2. **Finite element space.** We use piecewise linear finite elements on the evolving surface $\Gamma_h(t)$ and set

$$\mathcal{G}_T^h = \bigcup_{t\in[0,T]} \Gamma_h(t)\times\{t\}.$$

Definition 4.2 (Finite element spaces)**.** For each t we have the finite element spaces

$$S_h(t) = \left\{\phi_h \in C^0(\Gamma_h(t)) |\, \phi_h|_E \text{ is linear affine for each } E \in \mathcal{T}_h(t)\right\},$$
$$S_h^l(t) = \left\{\varphi_h = \phi_h^l |\, \phi_h \in S_h(t)\right\}.$$

Note that $S_h^l(t) \subset H^1(\Gamma(t))$ and that for each $\varphi_h \in S_h^l$ there is a unique $\phi_h \in S_h$ such that $\varphi_h = \phi_h^l$.

4.3. **Evolving triangulations and the discrete material derivative.** In this paper we consider triangulated surfaces for which the vertices $\{X_j(t)\}_{j=1}^N$ of the triangles sit on $\Gamma(t)$ so that $\Gamma_h(t)$ is an interpolation. Furthermore, we advect the nodes in the tangential direction with the advective velocity v_τ as well as keeping them on the surface using the normal velocity v_ν so that

$$\dot{X}_j(t) := \frac{dX_j}{dt}(t) = v(X_j(t),t) \quad (j = 1,\dots,N). \tag{4.4}$$

Note that it follows from the above definition and Remark 2.2 that $X_j(t) \in \Gamma(t)$ provided $X_j(0) \in \Gamma(0)$. We denote by $\{\chi_j, j = 1,2,..N\}$ the piecewise linear basis functions from $S_h(t)$ such that $\chi_j(X_i(t),t) = \delta_{ij}$.

Definition 4.3. We define a material velocity for $x = X(t)$ on the surface $\Gamma_h(t)$ by

$$\dot{X}(t) = V_h(X(t),t)\ ,\ V_h(x,t) := \sum_{j=1}^{N} \dot{X}_j(t)\chi_j(x,t) \text{ for } x \in S_h(t), \tag{4.5}$$

and the associated material velocity on $\Gamma(t)$ by

$$\dot{Y}(t) = v_h(Y(t),t) = \frac{\partial p}{\partial t}(X(t),t) + V_h(X(t),t)\cdot\nabla p(X(t),t) \tag{4.6}$$

so that for $x \in \Gamma_h(t)$,

$$v_h(p(x,t),t) = (P - d\mathcal{H})(x,t)V_h(x,t) - d_t(x,t)\nu(x,t) - d(x,t)\nu_t(x,t), \tag{4.7}$$

where $Y(t) = p(X(t),t)$ is defined by (2.2).

Remark 4.4. Observe that:

(i) The edges of $e(t)$ are the projections onto $\Gamma(t)$ of the edges of $E(t) \subset \Gamma_h(t)$ and thus evolve with the material velocity $v_h(\cdot,t)$.

(ii) The discrete material velocity v_h is *not* the interpolation of v in $S_h^l(t)$ which is given by

$$I_h v(p(x,t),t) = \sum_{j=1}^{N} \dot{X}_j(t)\chi_j^l(p(x,t),t) = \sum_{j=1}^{N} \dot{X}_j(t)\chi_j(x,t) = V_h(x,t)$$

for $x \in \Gamma_h(t)$.

Definition 4.5. Given the discrete velocity field $V_h \in (S_h)^{n+1}$ and the associated velocity field v_h on $\Gamma(t)$, (4.7), we define the *discrete material derivatives* on $\Gamma_h(t)$ and $\Gamma(t)$ element by element through the equations

$$\partial_h^\bullet \phi_h|_{E(t)} := (\phi_{ht} + V_h\cdot\nabla\phi_h)|_{E(t)}, \tag{4.8}$$

$$\partial_h^\bullet \varphi_h|_{e(t)} := (\varphi_{ht} + v_h\cdot\nabla\varphi_h)|_{e(t)}. \tag{4.9}$$

The material derivative and lifting process do commute in a suitable sense. This means that the material derivatives $\partial_h^\bullet$ for functions from S_h and from S_h^l survive the lifting process.

Lemma 4.1 (Transport property of the basis functions)**.** *The basis functions satisfy the transport property that*

$$\partial_h^\bullet \chi_j = 0\ ,\ \partial_h^\bullet \chi_j^l = 0 \tag{4.10}$$

from which it follows that for $\phi_h \in S_h(t), \Phi_j(t) := \phi_h(X_j(t),t)$,

$$\partial_h^\bullet \phi_h = \sum_{j=1}^{N} \dot{\Phi}_j(t)\chi_j \in S_h(t) \tag{4.11}$$

and for $\varphi_h = \phi_h^l \in S_h^l$,

$$\partial_h^\bullet \varphi_h = (\partial_h^\bullet \phi_h)^l = \sum_{j=1}^{N} \dot{\Phi}_j(t)\chi_j^l \in S_h^l(t). \tag{4.12}$$

Proof. It was shown in [9] that $\partial_h^\bullet \chi_j = 0$. Using $\chi_j(x,t) = \chi_j^l(p(x,t),t)$ and (2.2) we find that

$$0 = \partial_h^\bullet \chi_j = (\chi_{j\,t} + V_h \cdot \nabla \chi_j) = (\chi_{j\,t}^l + (p_t + (V_h \cdot \nabla)p) \cdot \nabla \chi_j^l)(p,\cdot) = \partial_h^\bullet \chi_j^l(p,\cdot).$$

The remaining equations follow immediately from

$$\partial_h^\bullet \phi_h = \partial_h^\bullet \sum_{j=1}^N \Phi_j(t)\chi_j(x,t)\ ,\ \partial_h^\bullet \varphi_h = \partial_h^\bullet \sum_{j=1}^N \Phi_j(t)\chi_j^l(p(x,t),t)$$

and for $x \in \Gamma_h(t)$,

$$\left(\partial_h^\bullet \phi_h\right)^l (p(x,t),t) = \sum_{j=1}^N \dot{\Phi}_j(t)\chi_j^l(p(x,t),t) = \partial_h^\bullet \varphi_h(p(x,t),t),$$

because by (4.10), $\partial_h^\bullet \chi_j^l = 0$. □

4.4. **Discrete bilinear forms.** As discrete analogues of the bilinear forms (3.5), (3.6) and (3.7) we define

$$a_h(\phi_h, W_h) = \int_{\Gamma_h} \mathcal{A}^{-l} \nabla_{\Gamma_h} \phi_h \cdot \nabla_{\Gamma_h} W_h, \tag{4.13}$$

$$m_h(\phi_h, W_h) = \int_{\Gamma_h} \phi_h W_h, \tag{4.14}$$

$$g_h(V_h; \phi_h, W_h) = \int_{\Gamma_h} \phi_h W_h \nabla_{\Gamma_h} \cdot V_h \tag{4.15}$$

for $\phi_h, W_h \in S_h$. Here, the discrete tangential gradients have to be understood in a piecewise sense. Note also that on the discrete surface it is necessary to evaluate an approximation of the diffusion tensor. We choose the natural extension defined by the lift from the surface $\Gamma(t)$ to a neighbourhood as defined in (4.3).

4.5. **Transport theorems on triangulated surfaces.**

Lemma 4.2 (Transport theorems on triangulated surfaces)**.** *Let $\Gamma_h(t)$ be an evolving admissable triangulation with material velocity V_h. Then*

$$\frac{d}{dt}\int_{\Gamma_h(t)} f = \int_{\Gamma_h(t)} \partial_h^\bullet f + f\, \nabla_{\Gamma_h} \cdot V_h. \tag{4.16}$$

For $\phi \in S_h(t), W_h \in S_h(t)$,

$$\frac{d}{dt} m_h(\phi, W_h) = m_h(\partial_h^\bullet \phi, W_h) + m_h(\phi, \partial_h^\bullet W_h) + g_h(V_h; \phi, W_h), \tag{4.17}$$

$$\frac{d}{dt} a_h(\phi, W_h) = a_h(\partial_h^\bullet \phi, W_h) + a_h(\phi, \partial_h^\bullet W_h) + b_h(V_h; \phi, W_h), \tag{4.18}$$

with the bilinear form

$$b_h(V_h; \phi, W_h) = \sum_{E(t) \in \mathcal{T}_h(t)} \int_{E(t)} \mathcal{B}_h(V_h) \nabla_{\Gamma_h} \phi \cdot \nabla_{\Gamma_h} W_h \tag{4.19}$$

where

$$\mathcal{B}_h(V_h) = \partial_h^\bullet \mathcal{A}^{-l} + \nabla_{\Gamma_h} \cdot V_h\, \mathcal{A}^{-l} - 2D_h(V_h),$$

$$D_h(V_h)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1} \left(\mathcal{A}_{ik}^{-l} \left(\nabla_\Gamma\right)_k V_{hj} + \mathcal{A}_{jk}^{-l} \left(\nabla_\Gamma\right)_k V_{hi}\right), \quad i,j = 1,\ldots,n+1.$$

Let $\Gamma(t)$ be an evolving surface decomposed into curved elements $e(t)$ whose edges move with velocity v_h. Then

$$\frac{d}{dt}\int_{\Gamma(t)} f = \int_{\Gamma(t)} \partial_h^\bullet f + f\,\nabla_{\Gamma_h}\cdot v_h. \tag{4.20}$$

For $\varphi, w, \partial_h^\bullet\varphi, \partial_h^\bullet w \in H^1(\Gamma(t))$,

$$\frac{d}{dt}m(\varphi,w) = m(\partial_h^\bullet\varphi,w) + m(\varphi,\partial_h^\bullet w) + g(v_h;\varphi,w), \tag{4.21}$$

$$\frac{d}{dt}a(\varphi,w) = a(\partial_h^\bullet\varphi,w) + a(\varphi,\partial_h^\bullet w) + b(v_h;\varphi,w). \tag{4.22}$$

Proof. In order to prove (4.16) we write

$$\frac{d}{dt}\int_{\Gamma_h(t)} f = \sum_{E(t)\subset\Gamma_h(t)} \frac{d}{dt}\int_{E(t)} f$$

and apply the Leibniz formula (2.14) to each element $E(t)$. Using (2.16) the proof of (4.18) follows in the same way. Decomposing $\Gamma(t)$ into the union of the curved elements $e(t)$ and applying (2.14), (2.16) to each element $e(t)$ we obtain (4.20) and (4.21), (4.22). □

Remark 4.6. Note that since V_h is obtained by interpolation that, in general, on each element it will have non-zero normal and tangential components even if v_τ is zero. Thus we need the particular forms of the Leibniz formulae (2.14) and (2.16) involving the discrete material derivative.

4.6. **Finite element method and convergence theorem.**

Definition 4.7 (Evolving Surface Finite Element Method)**.** Given $U_{h0} \in S_h(0)$ find $U_h \in S_h^T = \{\phi_h \text{ and } \partial_h^\bullet\phi_h \in C^0(\mathcal{G}_T^h) | \phi_h(\cdot,t) \in S_h(t)\ t \in [0,T]\}$ such that for all $\phi_h \in S_h^T$ and $t \in (0,T]$,

$$\frac{d}{dt}m_h(U_h,\phi_h) + a_h(U_h,\phi_h) = m_h(U_h,\partial_h^\bullet\phi_h), \quad U_h(\cdot,0) = U_{h0}. \tag{4.23}$$

Using the transport property Lemma 4.1 it follows that this definition is equivalent to:

$$\frac{d}{dt}m_h(U_h,\chi_j) + a_h(U_h,\chi_j) = 0 \quad U_h(\cdot,0) = U_{h0}, \tag{4.24}$$

for all $j = 1,\dots,N$.

Remark 4.8 (Matrix version of the finite element method). Setting $M(t)$ to be the evolving mass matrix

$$M(t)_{jk} = \int_{\Gamma_h(t)} \chi_j\chi_k,$$

$\mathcal{S}(t)$ to be the evolving stiffness matrix

$$\mathcal{S}(t)_{jk} = \int_{\Gamma_h(t)} \mathcal{A}^{-l}\nabla_{\Gamma_h}\chi_j\nabla_{\Gamma_h}\chi_k,$$

and $U_h = \sum_{j=1}^N \alpha_j\chi_j$, $\alpha = (\alpha_1,\dots,\alpha_N)$, we arrive at the following simple version of the finite element approximation

$$\frac{d}{dt}\left(M(t)\alpha\right) + \mathcal{S}(t)\alpha = 0, \tag{4.25}$$

which does not explicitly involve the velocity of the surface.

Since the mass matrix $M(t)$ is uniformly positive definite for $t \in [0,T]$ and the stiffness matrix $\mathcal{S}(t)$ is positive semi-definite, we get existence and uniqueness of the semi-discrete finite element solution.

Remark 4.9. Observe that our method and analysis includes the case of advection diffusion on a stationary surface in which $v_\nu = 0$ and the vertices are moved with the tangential velocity v_τ.

The estimates of the following lemma are proved in [9].

Lemma 4.3. *There exists a unique finite element solution of* (4.23). *The solution satisfies the a priori bounds*

$$\sup_{(0,T)} \|U_h\|^2_{L^2(\Gamma_h)} + \int_0^T \|\nabla_{\Gamma_h} U_h\|^2_{L^2(\Gamma_h)} \le c\|U_{h0}\|^2_{L^2(\Gamma_h(0))},$$

$$\int_0^T \|\partial_h^\bullet U_h\|^2_{L^2(\Gamma_h)} + \sup_{(0,T)} \|\nabla_{\Gamma_h} U_h\|^2_{L^2(\Gamma_h)} \le c\|U_{h0}\|^2_{L^2(\Gamma_h(0))} + c\|\nabla_{\Gamma_h} U_{h0}\|^2_{L^2(\Gamma_h(0))}.$$

Theorem 4.4 (Convergence)**.** *Let u be a sufficiently smooth solution of* (3.4) *satisfying*

$$\int_0^T \|u\|^2_{H^2(\Gamma)} + \|\partial^\bullet u\|^2_{H^2(\Gamma)} dt < \infty \tag{4.26}$$

and let $u_h(,t) = U_h^l(\cdot,t), t \in [0,T]$ be the spatially discrete solution from Lemma 4.3 *with initial data $u_{h0} = U_{h0}^l$ satisfying*

$$\|u(\cdot,0) - u_{h0}\|_{L^2(\Gamma(0))} \le ch^2.$$

Then the error estimate

$$\sup_{t\in(0,T)} \|u(\cdot,t) - u_h(\cdot,t)\|_{L^2(\Gamma(t))} \le ch^2 \tag{4.27}$$

holds for a constant c independent of h but depending on the norms (4.26).

Remark 4.10. Under suitable assumptions on $\mathcal{G}_T$, $\mathcal{A}$, v and u_0 it can be shown that (4.26) holds; see [9].

5. Approximation estimates

5.1. **Approximation of the surface.** Let $P_h = P_h(x,t)$ be the projection $P_{h,ij} = \delta_{ij} - \nu_{h,i}\nu_{h,j}$. In order to relate integrals on $\Gamma_h(t)$ to integrals on $\Gamma(t)$, we use the quotient, δ_h, between the smooth and discrete surface measures dA and dA_h, defined by $\delta_h dA_h = dA$. The proof of the following lemma can be found from Lemma 5.1 in [9].

Lemma 5.1. *Assume $\Gamma(t)$ and $\Gamma_h(t)$ are as above. Then*

$$\sup_{(0,T)} \|d\|_{L^\infty(\Gamma_h)} \le ch^2, \tag{5.1}$$

$$\sup_{(0,T)} \|1 - \delta_h\|_{L^\infty(\Gamma_h)} \le ch^2, \tag{5.2}$$

$$\sup_{(0,T)} \|P - PP_hP\|_{L^\infty(\Gamma_h)} \le ch^2. \tag{5.3}$$

Some calculations then lead to the following lemma about the comparison of Sobolev norms on discrete and continuous surfaces, [8].

Lemma 5.2. *For $t \in [0,T]$ let $\eta_h : \Gamma_h(t) \to \mathbb{R}$ with lift $\eta_h^l : \Gamma(t) \to \mathbb{R}$. Then for the corresponding plane simplex, $E \subset \Gamma_h(t)$, and curvilinear simplex, $e = p(E,t) \subset \Gamma(t)$, the following estimates hold if the norms exist. There is a constant $c > 0$ independent of h and t such that*

$$\frac{1}{c}\|\eta_h\|_{L^2(E)} \le \|\eta_h^l\|_{L^2(e)} \le c\|\eta_h\|_{L^2(E)}, \tag{5.4}$$

$$\frac{1}{c}\|\nabla_{\Gamma_h}\eta_h\|_{L^2(E)} \le \|\nabla_\Gamma \eta_h^l\|_{L^2(e)} \le c\|\nabla_{\Gamma_h}\eta_h\|_{L^2(E)}, \tag{5.5}$$

$$\|\nabla^2_{\Gamma_h}\eta_h\|_{L^2(E)} \le c\|\nabla^2_\Gamma \eta_h^l\|_{L^2(e)} + ch\|\nabla_\Gamma \eta_h^l\|_{L^2(e)}. \tag{5.6}$$

5.2. **Finite element interpolation estimates.** An interpolant is constructed in the obvious way. For $\eta \in C^0(\Gamma(t))$ the pointwise linear interpolation $\tilde{I}_h\eta \in S_h$ is well defined. The vertices of $\Gamma_h(t)$ lie on the smooth surface $\Gamma(t)$ and so the nodal values of η are well defined for this interpolation, since $n \le 3$. We then lift $\tilde{I}_h\eta$ onto $\Gamma(t)$ by the process (4.1) to obtain $I_h\eta = (\tilde{I}_h\eta)^l$ for which the following approximation lemma holds, [8].

Lemma 5.3 (Interpolation)**.** *For given $\eta \in H^2(\Gamma(t))$ there exists a unique $I_h\eta \in S_h^l(t)$ such that*

$$\|\eta - I_h\eta\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma(\eta - I_h\eta)\|_{L^2(\Gamma(t))} \le ch^2\left(\|\nabla^2_\Gamma\eta\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma\eta\|_{L^2(\Gamma(t))}\right).$$

5.3. **Material derivative and geometric perturbation errors.**

Lemma 5.4. *Assume $\Gamma(t)$ and $\Gamma_h(t)$ are as above. Then*

$$\sup_{(0,T)} \|\partial_h^\bullet d\|_{L^\infty(\Gamma_h)} \le ch^2, \tag{5.7}$$

$$\sup_{(0,T)} \|\partial_h^\bullet (P_h\nu)\|_{L^\infty(\Gamma_h)} \le ch, \tag{5.8}$$

$$\sup_{(0,T)} \|\partial_h^\bullet \delta_h\|_{L^\infty(\Gamma_h)} \le ch^2. \tag{5.9}$$

Proof. It is sufficient to consider a single element $E(t) \subset \Gamma_h(t)$. In order to facilitate calculations we consider an orthogonal transformation $C(t) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $E(t) = C(t)\tilde{E}(t)$ where $\tilde{E} \subset \mathbb{R}^n \times \{y_0(t)\}$. Using the equation

$$x = p(x,t) + d(x,t)\nu(x,t) \tag{5.10}$$

for $x \in \mathcal{N}(t)$, we set $\tilde{p}(y,t) := C^*(t)p(C(t)y,t)$ and observe that for $\tilde{e}(t) := \tilde{p}(\tilde{E},t)$ we have $C(t)\tilde{e}(t) = e(t)$. We set $\tilde{d}(y,t) := d(C(t)y,t)$ and observe that $\tilde{\nu}(y,t) := \nabla\tilde{d}(y,t) = C^*(t)\nu(C(t)y,t)$. Observing that for $z \in e(t)$, $\tilde{z} \in \tilde{e}(t)$ and $x = C(t)y \in \mathcal{N}(t)$ we have $|y - \tilde{z}| = |x - z|$ so that in our new coordinate system we have the relation

$$y = \tilde{p}(y,t) + \tilde{d}(y,t)\tilde{\nu}(y,t) \tag{5.11}$$

where $\tilde{p}(y,t) \in \tilde{e}$ and $\tilde{d}(y,t)$ is the distance function to $\tilde{e}(t)$ and $\tilde{\nu}(y,t)$ is normal to $\tilde{e}(t)$. Denoting by $\tilde{v}$ the velocity field in the transformed coordinates we observe that

$$\tilde{v}_3(Y_k,t) = \frac{dy_0}{dt}(t)$$

where $\{Y_k(t)\}_{k=1}^{n+1}$ are the vertices of the simplex $\tilde{E}(t)$.

Let us treat the case $n = 2$ in the following. We use the original variables x etc. in order to make the presentation simpler. First we observe that (5.7) holds. This comes from the fact that $\partial_h^\bullet d = 0$ in the vertices of the triangle E. Then the interpolant $I\partial_h^\bullet d$ vanishes identically and the common interpolation on E gives the estimate

$$\|\partial_h^\bullet d\|_{L^\infty(E)} = \|\partial_h^\bullet d - I\partial_h^\bullet d\|_{L^\infty(E)} \le ch^2\|\partial_h^\bullet d\|_{H^{2,\infty}(E)} \le ch^2.$$

With a similar argument we get the estimates

$$\|\partial_h^\bullet \nu_j\|_{L^\infty(E)} \le ch \quad (j = 1, 2). \tag{5.12}$$

Thus we have proved the estimates (5.7) and (5.8).

We prove (5.9) for $n = 2$ by estimating

$$\partial_h^\bullet(|p_{x_1} \wedge p_{x_2}|) = \nu \cdot (\partial_h^\bullet p_{x_1} \wedge p_{x_2} + p_{x_1} \wedge \partial_h^\bullet p_{x_2}).$$

From (5.11) respectively (5.10) we deduce

$$p_{x_j} = e_j - \nu_j\nu - d\nu_{x_j} \quad (j = 1, 2)$$

with the jth standard basis vector $e_j \in \mathbb{R}^3$. Then

$$\partial_h^\bullet p_{x_j} = -\nu\partial_h^\bullet\nu_j - \nu_j\partial_h^\bullet\nu - \nu_{x_j}\partial_h^\bullet d - d\partial_h^\bullet\nu_{x_j} = -\nu\partial_h^\bullet\nu_j - \nu_j\partial_h^\bullet\nu + O(h^2),$$

because $|\partial_h^\bullet d| + |d| = O(h^2)$ by (5.7). With this we calculate the material derivative of the surface element:

$$\begin{aligned}
\partial_h^\bullet |p_{x_1} \wedge p_{x_2}| &= \nu \cdot (\partial_h^\bullet p_{x_1} \wedge p_{x_2} + p_{x_1} \wedge \partial_h^\bullet p_{x_2}) \\
&= \nu \cdot \big(\big(-\nu\partial_h^\bullet\nu_1 - \nu_1\partial_h^\bullet\nu + O(h^2)\big) \wedge \big(e_2 - \nu_2\nu + O(h^2)\big) \\
&\qquad + \big(e_1 - \nu_1\nu + O(h^2)\big) \wedge \big(-\nu\partial_h^\bullet\nu_2 - \nu_2\partial_h^\bullet\nu + O(h^2)\big)\big) \\
&= \nu \cdot \big(-\nu_1\partial_h^\bullet\nu \wedge e_2 - e_1 \wedge \nu_2\partial_h^\bullet\nu + O(h^2)\big) \\
&= -\nu_1\nu \cdot (\partial_h^\bullet\nu \wedge e_2) - \nu_2\nu \cdot (e_1 \wedge \partial_h^\bullet\nu) + O(h^2) \\
&= \nu \cdot \partial_h^\bullet\nu \wedge (\nu_2 e_1 - \nu_1 e_2) + O(h^2) \\
&= -(\nu_1\partial_h^\bullet\nu_1 + \nu_2\partial_h^\bullet\nu_2)\,\nu_3 + O(h^2).
\end{aligned}$$

Since from (5.12) we know that $|\nu_j| + |\partial_h^\bullet\nu_j| = O(h)$ for $j = 1, 2$, the estimate (5.9) follows. Note that from the last step we also get that $\partial_h^\bullet\nu_3 = O(h^2)$. □

We begin with the bounding of the geometric perturbation errors in the bilinear forms.

Lemma 5.5. *For any $(W_h, \phi_h) \in S_h \times S_h$ with corresponding lifts $(w_h, \varphi_h) \in S_h^l \times S_h^l$ the following bounds hold:*

$$|m(w_h, \varphi_h) - m_h(W_h, \phi_h)| \le ch^2\|w_h\|_{L^2(\Gamma(t))}\|\varphi_h\|_{L^2(\Gamma(t))}, \tag{5.13}$$

$$|a(w_h, \varphi_h) - a_h(W_h, \phi_h)| \le ch^2\|\nabla_\Gamma w_h\|_{L^2(\Gamma(t))}\|\nabla_\Gamma\varphi_h\|_{L^2(\Gamma(t))}, \tag{5.14}$$

$$|g(v_h; w_h, \varphi_h) - g_h(V_h; W_h, \phi_h)| \le ch^2\|w_h\|_{H^1(\Gamma(t))}\|\varphi_h\|_{H^1(\Gamma(t))}. \tag{5.15}$$

Proof. The bound (5.13) follows by noting

$$\begin{aligned}
&|m(w_h, \varphi_h) - m_h(W_h, \phi_h)| \\
&\quad = |\int_{\Gamma(t)} (1 - \frac{1}{\delta_h^l}) w_h\varphi_h| \le \|1 - \frac{1}{\delta_h}\|_{L^\infty(\Gamma_h)}\|w_h\|_{L^2(\Gamma)}\|\varphi_h\|_{L^2(\Gamma)}
\end{aligned}$$

and using Lemma 5.1.

In order to prove (5.14) it is convenient to introduce the notation

$$\mathcal{Q}_h = \frac{1}{\delta_h}(\mathcal{A}^{-l})^{-1}(I - d\mathcal{H})PP_h\mathcal{A}^{-l}P_hP(I - d\mathcal{H})$$

on $\Gamma_h(t)$ and its lifted version, $\mathcal{Q}_h^l$ on $\Gamma(t)$. Using Remark 4.1 we may write on $\Gamma_h(t)$,

$$\begin{aligned}
\mathcal{A}^{-l}\nabla_{\Gamma_h}W_h \cdot \nabla_{\Gamma_h}\phi_h &= \mathcal{A}^{-l}P_h(P - d\mathcal{H})P\nabla_\Gamma w_h(p,\cdot) \cdot P_h\,(P - d\mathcal{H})\,P\nabla_\Gamma\varphi_h(p,\cdot) \\
&= \mathcal{A}^{-l}P_hP(I - d\mathcal{H})\nabla_\Gamma w_h(p,\cdot) \cdot P_hP(I - d\mathcal{H})\nabla_\Gamma\varphi_h(p,\cdot) \\
&= \delta_h\mathcal{A}^{-l}\mathcal{Q}_h\nabla_\Gamma w_h(p,\cdot) \cdot \nabla_\Gamma\varphi_h(p,\cdot).
\end{aligned}$$

We use the geometry estimates from Lemma 5.1 and the fact that $\mathcal{A}P = P\mathcal{A} = \mathcal{A}$ and get the estimate

$$\left|\mathcal{A}^{-l} - \delta_h\mathcal{A}^{-l}\mathcal{Q}_h\right| \le \left|P\mathcal{A}^{-l}P - PP_hP\mathcal{A}^{-l}PP_hP\right| + ch^2 \le ch^2.$$

Hence the bound (5.14) follows from

$$a(w_h,\varphi_h) - a_h(W_h,\phi_h) = \int_{\Gamma(t)} \mathcal{A}\nabla_\Gamma w_h\nabla_\Gamma\varphi_h - \int_{\Gamma_h(t)} \delta_h\mathcal{A}^{-l}\mathcal{Q}_h\nabla_\Gamma w_h(p,\cdot)\nabla_\Gamma\varphi_h(p,\cdot).$$

For the bound (5.15) we use Lemma 5.4. The transport formula gives for related simplices $e \subset \Gamma$ and $E \subset \Gamma_h$,

$$\int_e \nabla_\Gamma \cdot v_h = \frac{d}{dt}|e|, \quad \int_E \nabla_{\Gamma_h} \cdot V_h = \frac{d}{dt}|E|. \tag{5.16}$$

Since $|e| = \int_E \delta_h$ we get

$$\frac{d}{dt}|e| = \int_E \partial_h^\bullet\delta_h + \int_E \delta_h\nabla_{\Gamma_h} \cdot V_h = \int_E \partial_h^\bullet\delta_h + \int_E (\delta_h - 1)\nabla_{\Gamma_h} \cdot V_h + \frac{d}{dt}|E|,$$

so that by Lemma 5.4 and (5.16)

$$\Big|\int_e \nabla_\Gamma \cdot v_h - \int_E \nabla_{\Gamma_h} \cdot V_h\Big| \le \|\partial_h^\bullet\delta_h\|_{L^\infty(E)}|E| + \|\delta_h - 1\|_{L^\infty(E)}|E| \le ch^2|E|. \tag{5.17}$$

We use this estimate for the proof of (5.15) as follows:

$$\begin{aligned}
\int_e w_h\varphi_h\nabla_\Gamma \cdot v_h - \int_E W_h\phi_h\nabla_{\Gamma_h} \cdot V_h &= \int_e w_h\varphi_h\nabla_\Gamma \cdot v_h - \int_E W_h\phi_h\frac{1}{|E|}\int_E \nabla_{\Gamma_h} \cdot V_h \\
&\le \int_e w_h\varphi_h\nabla_\Gamma \cdot v_h - \int_e \Big(w_h\varphi_h\frac{1}{\delta_h^l|E|}\int_e \nabla_\Gamma \cdot v_h\Big) + ch^2\int_e |w_h||\varphi_h|.
\end{aligned}$$

Since we know from Lemma 5.1 that $||e| - |E|| \le ch^2$ we can continue the estimate by

$$\begin{aligned}
&\le \int_e w_h\varphi_h\Big(\nabla_\Gamma \cdot v_h - \frac{1}{|e|}\int_e \nabla_\Gamma \cdot v_h\Big) + ch^2\Big(\frac{1}{|e|}\int_e |\nabla_\Gamma \cdot v_h| + 1\Big)\int_e |w_h||\varphi_h| \\
&\le \int_e \Big(w_h\varphi_h - \frac{1}{|e|}\int_e w_h\varphi_h\Big)\Big(\nabla_\Gamma \cdot v_h - \frac{1}{|e|}\int_e \nabla_\Gamma \cdot v_h\Big) + ch^2\int_e |w_h||\varphi_h|.
\end{aligned}$$

We use Poincaré's inequality for functions with mean value zero on e, which is easily proved by transformation to E, and get

$$\begin{aligned}\int_e w_h\varphi_h\nabla_\Gamma\cdot v_h - \int_E W_h\phi_h\nabla_{\Gamma_h}\cdot V_h \\ \le ch^2\|\nabla_\Gamma(w_h\varphi_h)\|_{L^2(e)}\|\nabla_\Gamma\nabla_\Gamma\cdot v_h\|_{L^2(e)} + ch^2\|w_h\|_{L^2(e)}\|\varphi_h\|_{L^2(e)} \\ \le ch^2\|w_h\|_{H^1(e)}\|\varphi_h\|_{H^1(e)}.\end{aligned}$$

The second derivatives of v_h on e are easily estimated by a constant, when one uses the definition (4.7) together with the smoothness of the surface Γ, the smoothness of v and the fact that

$$|(\nabla_\Gamma)_i(\nabla_\Gamma)_j I_h v| \le c|\nabla_\Gamma I_h v|$$

since $I_h v$ is the lift of a linear affine function on E. Summing up over the elements e this finally proves the estimate (5.15) for the bilinear forms g and g_h. □

For later use we estimate the difference between continuous and discrete material derivative.

Lemma 5.6. *For the difference between the continuous velocity v (2.10), and the discrete velocity on the smooth surface, v_h, (4.7), we have the estimates*

$$|v - v_h| + h\,|\nabla_\Gamma(v - v_h)| \le ch^2 \quad \text{on } \Gamma. \tag{5.18}$$

Proof. For $x \in \Gamma_h$ we have from (4.7) that with $p = p(x,t) \in \Gamma(t)$,

$$\begin{aligned} v(p,\cdot) - v_h(p,\cdot) &= v(p,\cdot) - P(I - d\mathcal{H})V_h + d_t\nu + d\nu_t \\ &= P(v - I_h v)(p,\cdot) + d(\mathcal{H}I_h v(p,\cdot) + \nu_t). \end{aligned} \tag{5.19}$$

Here we have used Remark 4.4(ii), (2.3), (2.4) and $v_\nu = -d_t\nu$. This immediately gives

$$|v - v_h| \le ch^2$$

on $\Gamma(t)$ with a constant which depends on a bound for the second derivatives of v.

For the ith component of the tangential gradient we have

$$(\nabla_\Gamma)_i\,(v - v_h) = ((\nabla_\Gamma)_i\,P)\,(v - I_h v) + P\,(\nabla_\Gamma)_i\,(v - I_h v) + d\,(\nabla_\Gamma)_i\,(\mathcal{H}I_h v + \nu_t),$$

where we have used that $\nabla_\Gamma d = 0$. So,

$$|\nabla_\Gamma(v - v_h)| \le c|v - I_h v| + c|\nabla_\Gamma(v - I_h v)| + ch^2 \le ch$$

and the lemma is proved. □

The previous lemma allows us to estimate the difference between continuous and discrete material derivative.

Corollary 5.7.

$$\|\partial^\bullet z - \partial_h^\bullet z\|_{L^2(\Gamma)} \le ch^2\|z\|_{H^1(\Gamma)} \;\; z \in H^1(\Gamma), \tag{5.20}$$

$$\|\nabla_\Gamma(\partial^\bullet z - \partial_h^\bullet z)\|_{L^2(\Gamma)} \le ch\|z\|_{H^2(\Gamma)} \;\; z \in H^2(\Gamma). \tag{5.21}$$

Proof. The definitions of the material derivatives together with the fact that $v - v_h$ is a tangent vector (see (5.19)) give

$$\partial^\bullet z - \partial_h^\bullet z = (v - v_h)\cdot\nabla z = (v - v_h)\cdot\nabla_\Gamma z.$$

We use Lemma 5.6 for the estimate (5.20) of the $L^2(\Gamma)$-norm of this quantity. We then may apply the tangential gradient to the previous equation and again use Lemma 5.6 to obtain a bound for the gradient

$$\|\nabla(\partial^\bullet z - \partial_h^\bullet z)\|_{L^2(\Gamma)} \le ch\|z\|_{H^1(\Gamma)} + ch^2\|z\|_{H^2(\Gamma)}. \qquad \square$$

Lemma 5.8. *Assume that $W_h(\cdot,t), \phi_h(\cdot,t) \in S_h(t)$ and $w_h = W_h^l, \varphi_h = \phi_h^l \in S_h^l$. Then*

$$|m(\partial_h^\bullet w_h, \varphi_h) - m_h(\partial_h^\bullet W_h, \phi_h)| \le ch^2\|\partial_h^\bullet w_h\|_{L^2(\Gamma)}\|\varphi_h\|_{L^2(\Gamma)}. \tag{5.22}$$

Proof. From Lemma 4.1 we know that $\partial_h^\bullet w_h = (\partial_h^\bullet W_h)^l$ and consequently we have

$$\begin{aligned} m(\partial_h^\bullet w_h, \varphi_h) - m_h(\partial_h^\bullet W_h, \phi_h) &= m((\partial_h^\bullet W_h)^l, \varphi_h) - m_h(\partial_h^\bullet W_h, \phi_h) \\ &\le ch^2\|\partial_h^\bullet w_h\|_{L^2(\Gamma)}\|\varphi_h\|_{L^2(\Gamma)}. \end{aligned}$$

Here we use (5.13) from the Lemma 5.5. $\square$

6. Ritz projection and error analysis

It is convenient in the error analysis to use the Ritz projection $\mathcal{R}_h : H^1(\Gamma) \to S_h^l$ defined by

Definition 6.1. For given $z \in H^1(\Gamma)$, $\int_\Gamma z = 0$, there is a unique $\mathcal{R}_h z \in S_h^l$ such that

$$a(\mathcal{R}_h z, \varphi_h) = a(z, \varphi_h) \qquad \forall \varphi_h \in S_h^l \tag{6.1}$$

and $\int_\Gamma \mathcal{R}_h z = 0$.

6.1. Error in the Ritz projection.

Theorem 6.1. *The error in the Ritz projection satisfies the bounds*

$$\|z - \mathcal{R}_h z\|_{L^2(\Gamma)} + h\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)} \le ch^2\|z\|_{H^2(\Gamma)}. \tag{6.2}$$

Proof. For this we use the interpolation estimates (5.3).

$$\begin{aligned} c_0\|\nabla_\Gamma(z - \mathcal{R}_h z)\|^2_{L^2(\Gamma)} &\le a(z - \mathcal{R}_h z, z - \mathcal{R}_h z) = a(z - \mathcal{R}_h z, z - I_h z) \\ &\le c\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)}\|\nabla_\Gamma(z - I_h z)\|_{L^2(\Gamma)} \\ &\le ch\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)}\|z\|_{H^2(\Gamma)}. \end{aligned}$$

This implies

$$\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)} \le ch\|z\|_{H^2(\Gamma)}.$$

The $L^2(\Gamma)$ estimate comes with the Aubin-Nitsche duality argument. Solve the problem $w \in H^2(\Gamma)$ with $\int_\Gamma w = 0$,

$$-\nabla_\Gamma \cdot (\mathcal{A}\nabla_\Gamma w) = z - \mathcal{R}_h z.$$

Then $\|w\|_{H^2(\Gamma)} \le c\|z - \mathcal{R}_h z\|_{L^2(\Gamma)}$ and we get

$$\begin{aligned} \|z - \mathcal{R}_h z\|^2_{L^2(\Gamma)} &= a(w, z - \mathcal{R}_h z) = a(w - I_h w, z - \mathcal{R}_h z) \\ &\le c\|\nabla_\Gamma(w - I_h w)\|_{L^2(\Gamma)}\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)} \\ &\le ch^2\|w\|_{H^2(\Gamma)}\|z\|_{H^2(\Gamma)} \\ &\le ch^2\|z - \mathcal{R}_h z\|_{L^2(\Gamma)}\|z\|_{H^2(\Gamma)}. \end{aligned}$$

Thus

$$\|z - \mathcal{R}_h z\|_{L^2(\Gamma)} \le ch^2\|z\|_{H^2(\Gamma)},$$

and the theorem is proved. $\square$

6.2. **Error in the material derivative of the Ritz projection.** The estimate of the error for the material derivative of the Ritz projection requires more work, because the surface Γ depends on time. Taking the time derivative of (6.1) with (4.22) and using the definition of the Ritz projection (6.1) we have that

$$a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, \varphi_h) = -b(v_h; z - \mathcal{R}_h z, \varphi_h) \ \forall \varphi_h \in S_h^l. \tag{6.3}$$

From the above equation we see that, in general, $\partial_h^\bullet \mathcal{R}_h z \neq \mathcal{R}_h \partial_h^\bullet z$. It follows that we need to prove the following bounds for the time derivative.

Theorem 6.2. *The time derivative of the Ritz projection satisfies the bounds*

$$\begin{aligned} &\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z\|_{L^2(\Gamma)} + h\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\|_{L^2(\Gamma)} \\ &\qquad \leq ch^2 \left(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{H^2(\Gamma)}\right). \end{aligned} \tag{6.4}$$

Proof. We estimate the term on the right-hand side of (6.3) using (6.1) to obtain

$$a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, \varphi_h) \leq ch\|z\|_{H^2(\Gamma)}\|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma)} \tag{6.5}$$

for all $\varphi_h \in S_h^l$. Thus we obtain

(6.6)

$$\begin{aligned} c_0\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\|^2_{L^2(\Gamma)} &\leq a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, \partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z) \\ &= a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, \partial_h^\bullet z - \partial^\bullet z) + a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, \partial^\bullet z - I_h \partial^\bullet z) \\ &\quad + a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, I_h \partial^\bullet z - \partial_h^\bullet \mathcal{R}_h z) \\ &\leq c\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\|_{L^2(\Gamma)} \left(\|\nabla_\Gamma(\partial^\bullet z - \partial_h^\bullet z)\|_{L^2(\Gamma)} + \|\nabla_\Gamma(\partial^\bullet z - I_h \partial^\bullet z)\|_{L^2(\Gamma)}\right) \\ &\qquad + ch\|z\|_{H^2(\Gamma)}\|\nabla_\Gamma(I_h \partial^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\|_{L^2(\Gamma)} \end{aligned}$$

using (6.5) on the last term. By Corollary 5.7 we have that

$$\|\nabla_\Gamma(\partial^\bullet z - \partial_h^\bullet z)\|_{L^2(\Gamma)} \leq ch\|z\|_{H^2(\Gamma)}, \tag{6.7}$$

from Lemma 5.3 we get

$$\|\nabla_\Gamma(\partial^\bullet z - I_h \partial^\bullet z)\|_{L^2(\Gamma)} \leq ch\|\partial^\bullet z\|_{H^2(\Gamma)}, \tag{6.8}$$

and with similar arguments,

$$\begin{aligned} &\|\nabla_\Gamma(I_h \partial^\bullet z - \partial_h^\bullet R_h z)\|_{L^2(\Gamma)} \\ &\qquad \leq ch\|\partial^\bullet z\|_{H^2(\Gamma)} + ch\|z\|_{H^2(\Gamma)} + \|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet R_h z)\|_{L^2(\Gamma)}. \end{aligned} \tag{6.9}$$

We use the estimates (6.7), (6.8) and (6.9) in (6.6) and get the estimate

$$\begin{aligned} &\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet R_h z)\|^2_{L^2(\Gamma)} \\ &\quad \leq \quad ch\left(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{H^2(\Gamma)}\right)\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet R_h z)\|_{L^2(\Gamma)} \\ &\qquad +ch\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet R_h z)\|^2_{L^2(\Gamma)}. \end{aligned}$$

Finally, for $h \leq h_0$,

$$\|\nabla(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\|^2_{L^2(\Gamma)} \leq ch^2(\|z\|^2_{H^2(\Gamma)} + \|\partial^\bullet z\|^2_{H^2(\Gamma)}),$$

and this implies the gradient estimate in (6.4).

We still have to prove that the $L^2(\Gamma)$ error for the time derivative of the Ritz projection decays quadratically in the grid size. For this we again use the Aubin-Nitsche trick and solve the problem

$$-\nabla_\Gamma \cdot (\mathcal{A}\nabla_\Gamma w) = \partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z - c_0 \text{ on } \Gamma, \int_\Gamma w = 0,$$

where $c_0 = \frac{1}{|\Gamma|}\int_\Gamma(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)$. We note that from $\int_\Gamma z = \int_\Gamma \mathcal{R}_h z = 0$ we have with Lemma 2.1 that

$$\begin{aligned} |\Gamma||c_0| &= \Big|\int_\Gamma(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\Big| = \Big|\int_\Gamma (z - \mathcal{R}_h z)\nabla_\Gamma \cdot v_h\Big| \le c\|z - \mathcal{R}_h z\|_{L^2(\Gamma)} \\ &\le ch^2\|z\|_{H^2(\Gamma)}, \end{aligned}$$

where we used (6.2). Then for arbitrary $w_h \in S_h^l$,

$$\begin{aligned} &\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z\|^2_{L^2(\Gamma)} - |\Gamma|c_0^2 = a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, w) \\ &\quad = a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, w - w_h) + a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, w_h) \\ &\quad \le c\|\nabla_\Gamma(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z)\|_{L^2(\Gamma)}\|\nabla_\Gamma(w - w_h)\|_{L^2(\Gamma)} + a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, w_h). \end{aligned}$$

For the last term on the right-hand side of this inequality we have with (6.3):

$$a(\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z, w_h) = -b(v_h; z - \mathcal{R}_h z, w_h).$$

For the bilinear form $b(v_h; \cdot, \cdot)$ we proceed as follows:

$$\begin{aligned} -b(v_h; z - \mathcal{R}_h z, w_h) &= b(v_h; z - \mathcal{R}_h z, w - w_h) - b(v_h; z - \mathcal{R}_h z, w) \\ &\le c\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)}\|\nabla_\Gamma(w - w_h)\|_{L^2(\Gamma)} - b(v_h; z - \mathcal{R}_h z, w) \\ &\le ch\|z\|_{H^2(\Gamma)}\|\nabla_\Gamma(w - w_h)\|_{L^2(\Gamma)} - b(v_h; z - \mathcal{R}_h z, w). \end{aligned}$$

Now with Lemma 5.6,

$$\begin{aligned} &b(v; z - \mathcal{R}_h z, w) - b(v_h; z - \mathcal{R}_h z, w) \\ &\quad \le \int_\Gamma |\mathcal{B}(v) - \mathcal{B}(v_h)|\,|\nabla_\Gamma(z - \mathcal{R}_h z)||\nabla_\Gamma w| \\ &\quad \le ch\|\nabla_\Gamma(z - \mathcal{R}_h z)\|_{L^2(\Gamma)}\|\nabla_\Gamma w\|_{L^2(\Gamma)} \\ &\quad \le ch^2\|z\|_{H^2(\Gamma)}\|\nabla_\Gamma w\|_{L^2(\Gamma)}. \end{aligned}$$

The remaining term is treated as follows:

$$\begin{aligned} b(v; z-\mathcal{R}_h z, w) &= \int_\Gamma \mathcal{B}(v)\nabla_\Gamma(z-\mathcal{R}_h z)\cdot\nabla_\Gamma w \\ &= \sum_{i,j=1}^{n+1}\int_\Gamma \mathcal{B}(v)_{ij}(\nabla_\Gamma)_j(z-\mathcal{R}_h z)(\nabla_\Gamma)_i w \\ &= \sum_{i,j=1}^{n+1}\int_\Gamma (\nabla_\Gamma)_j\left(\mathcal{B}(v)_{ij}(z-\mathcal{R}_h z)(\nabla_\Gamma)_i w\right) \\ &- \int_\Gamma (z-\mathcal{R}_h z)\sum_{i,j=1}^{n+1}(\nabla_\Gamma)_j\left(\mathcal{B}(v)_{ij}(\nabla_\Gamma)_i w\right) \\ &= \int_\Gamma \sum_{i,j=1}^{n+1} H\nu_j\mathcal{B}(v)_{ij}(z-\mathcal{R}_h z)(\nabla_\Gamma)_i w \\ &- \int_\Gamma (z-\mathcal{R}_h z)\sum_{i,j=1}^{n+1}(\nabla_\Gamma)_j\left(\mathcal{B}(v)_{ij}(\nabla_\Gamma)_i w\right). \end{aligned}$$

This leads to the estimate

$$b(v; z-\mathcal{R}_h z, w) \geq -c\|z-\mathcal{R}_h z\|_{L^2(\Gamma)}\|w\|_{H^2(\Gamma)} \geq -ch^2\|z\|_{H^2(\Gamma)}\|w\|_{H^2(\Gamma)}.$$

Collecting terms and choosing $w_h = I_h w$ we obtain

$$\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z\|^2_{L^2(\Gamma)} \leq ch^2\left(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{H^2(\Gamma)}\right)\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{R}_h z\|_{L^2(\Gamma)}.$$

This finally proves the estimate (6.4). □

7. Proof of the error bound

7.1. **Error decomposition and error equation.** It is convenient to introduce the error decomposition

$$u - u_h = \rho + \theta\ ,\ \rho := u - \mathcal{R}_h u\ ,\ \theta := \mathcal{R}_h u - u_h \in S_h^l.$$

We begin by rewriting the finite element equation (4.23) as

$$\begin{aligned} \frac{d}{dt}m(u_h,\varphi_h) + a(u_h,\varphi_h) - m(u_h,\partial_h^\bullet\varphi_h) &= \frac{d}{dt}\left(m(u_h,\varphi_h) - m_h(U_h,\phi_h)\right) \\ &+ \left(a(u_h,\varphi_h) - a_h(U_h,\phi_h)\right) + \left(m_h(U_h,\partial_h^\bullet\phi_h) - m(u_h,\partial_h^\bullet\varphi_h)\right), \end{aligned}$$

and using the transport formulae (4.17), (4.21) we have that u_h satisfies

$$\frac{d}{dt}m(u_h,\varphi_h) + a(u_h,\varphi_h) - m(u_h,\partial_h^\bullet\varphi_h) = F_1(\varphi_h)\ \forall\varphi_h \in S_h^l \tag{7.1}$$

where with $\phi_h^l = \varphi_h \in S_h^l$,

$$\begin{aligned} F_1(\varphi_h) &= \left(m(\partial_h^\bullet u_h,\varphi_h) - m_h(\partial_h^\bullet U_h,\phi_h)\right) \\ &+ \left(g(v_h;u_h,\varphi_h) - g_h(V_h;U_h,\phi_h)\right) + \left(a(u_h,\varphi_h) - a_h(U_h,\phi_h)\right). \end{aligned} \tag{7.2}$$

On the other hand, using the definition (6.1) of the Ritz projection and the equation (3.8) for the solution u we obtain

$$\frac{d}{dt}m(\mathcal{R}_h u,\varphi_h) + a(\mathcal{R}_h u,\varphi_h) - m(\mathcal{R}_h u,\partial_h^\bullet\varphi_h) = F_2(\varphi_h) \quad \forall\varphi_h \in S_h^l \tag{7.3}$$

where

$$F_2(\varphi_h) = m(\rho, \partial_h^\bullet \varphi_h) - \frac{d}{dt} m(\rho, \varphi_h) + m(u, \partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h). \tag{7.4}$$

It follows by subtraction of (7.1) from (7.3) that θ satisfies the finite element error equation

$$\frac{d}{dt} m(\theta, \varphi_h) + a(\theta, \varphi_h) - m(\theta, \partial_h^\bullet \varphi_h) = F_2(\varphi_h) - F_1(\varphi_h) \quad \forall \varphi_h \in S_h^l. \tag{7.5}$$

This error equation is the basis for the error estimate which we prove in the next section.

7.2. **Proof of Theorem 4.4.** We begin with estimating the terms F_1 and F_2 on the right-hand side of (7.5). Applying Lemma 5.5 and Lemma 5.8 yields the estimate

$$|F_1(\varphi_h)| \le ch^2 \|\partial_h^\bullet u_h\|_{L^2(\Gamma)} \|\varphi_h\|_{L^2(\Gamma)} + ch^2 \|\nabla_\Gamma u_h\|_{L^2(\Gamma)} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma)}. \tag{7.6}$$

For F_2 we first observe that from (3.9) we have

$$F_2(\varphi_h) = -m(\partial_h^\bullet \rho, \varphi_h) - g(v_h; \rho, \varphi_h) + m(u, \partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h),$$

and consequently from Theorem 6.1, Theorem 6.2 and Corollary 5.7,

(7.7)

$$\begin{aligned} &|F_2(\varphi_h)| \\ &\le \|\partial_h^\bullet \rho\|_{L^2(\Gamma)} \|\varphi_h\|_{L^2(\Gamma)} + c\|\rho\|_{L^2(\Gamma)} \|\varphi_h\|_{L^2(\Gamma)} + \|u\|_{L^2(\Gamma)} \|\partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h\|_{L^2(\Gamma)} \\ &\le ch^2 \left(\|u\|_{H^2(\Gamma)} + \|\partial^\bullet u\|_{H^2(\Gamma)} \right) \|\varphi_h\|_{L^2(\Gamma)} + ch^2 \|u\|_{L^2(\Gamma)} \|\varphi_h\|_{H^1(\Gamma)}. \end{aligned}$$

We insert $\varphi_h = \theta$ in (7.5), observe that by the transport formula we have

$$m(\theta, \partial_h^\bullet \theta) = \frac{1}{2} \frac{d}{dt} m(\theta, \theta) - \frac{1}{2} g(v_h; \theta, \theta),$$

and arrive at

$$\frac{1}{2} \frac{d}{dt} m(\theta, \theta) + a(\theta, \theta) + g(v_h; \theta, \theta) = F_1(\theta) - F_2(\theta).$$

We use the estimates (7.6), (7.7) and the ellipticity condition (3.3). Upon integrating in time we obtain

$$\begin{aligned} \frac{1}{2} \|\theta\|_{L^2(\Gamma)}^2 + c_0 \int_0^t \|\nabla_\Gamma \theta\|_{L^2(\Gamma)}^2 dt &\le \frac{1}{2} \|\theta_0\|_{L^2(\Gamma_0)}^2 + ch^2 \int_0^t \|u_h\|_{H^1(\Gamma)} \|\theta\|_{H^1(\Gamma)} dt \\ &+ ch^2 \int_0^t \left(\|\partial^\bullet u_h\|_{L^2(\Gamma)} + \|u\|_{H^2(\Gamma)} + \|\partial^\bullet u\|_{H^2(\Gamma)} \right) \|\theta\|_{L^2(\Gamma)} dt \\ &+ ch^2 \int_0^t \|u\|_{L^2(\Gamma)} \|\theta\|_{H^1(\Gamma)} dt + c \int_0^t \|\theta\|_{L^2(\Gamma)}^2 dt. \end{aligned}$$

We use Young's inequality for $\varepsilon > 0$ to arrive at

$$\begin{aligned} \frac{1}{2} \|\theta\|_{L^2(\Gamma)}^2 + (c_0 - \varepsilon) \int_0^t \|\nabla_\Gamma \theta\|_{L^2(\Gamma)}^2 dt &\le \frac{1}{2} \|\theta_0\|_{L^2(\Gamma_0)}^2 + c(\varepsilon) \int_0^t \|\theta\|_{L^2(\Gamma)}^2 dt \\ &+ c(\varepsilon) h^4 \int_0^t \|u_h\|_{H^1(\Gamma)}^2 + \|\partial^\bullet u_h\|_{L^2(\Gamma)}^2 + \|u\|_{H^2(\Gamma)}^2 + \|\partial^\bullet u\|_{H^2(\Gamma)}^2 dt, \end{aligned}$$

and a Gronwall argument then leads to

$$\sup_{(0,T)} \|\theta\|^2_{L^2(\Gamma)} + \int_0^T \|\nabla_\Gamma \theta\|^2_{L^2(\Gamma)} dt \le c\|\theta_0\|^2_{L^2(\Gamma_0)} + Ch^4, \tag{7.8}$$

where the quantity

$$C = \int_0^T \|u_h\|^2_{H^1(\Gamma)} + \|\partial^\bullet u_h\|^2_{L^2(\Gamma)} + \|u\|^2_{H^2(\Gamma)} + \|\partial^\bullet u\|^2_{H^2(\Gamma)} dt$$

still has to be estimated independently of the grid size h, but this follows from the a priori estimates for the discrete solution in Lemma 4.3. Thus, finally, we have the estimate (7.8) with a constant C which only depends on norms of the continuous solution u. Theorem 4.4 now follows from

$$\|u - u_h\|_{L^2(\Gamma)} \le \|\rho\|_{L^2(\Gamma)} + \|\theta\|_{L^2(\Gamma)}$$

together with the estimates for the Ritz projection in Theorem 6.1.

References

1. D. Adalsteinsson and J. A. Sethian, *Transport and diffusion of material quantities on propagating interfaces via level set methods*, Journal of Computational Physics **185** (2003), no. 1, 271–288. MR2010161 (2004i:65072)
2. M. Bertalmío, Li-Tien Cheng, S. J. Osher, and G. Sapiro, *Variational problems and partial differential equations on implicit surfaces*, J. Comput. Phys. **174** (2001), no. 2, 759–780. MR1868103 (2002j:65105)
3. D. A. Calhoun and C. Helzel, *A finite volume method for solving parabolic equations on logically cartesian curved surface meshes*, SIAM J. Sci. Comput. **31** (2009), no. 6, 4066–4099. MR2566584 (2011a:65276)
4. K. Deckelnick, G. Dziuk, and C. M. Elliott, *Computation of geometric partial differential equations and mean curvature flow*, Acta Numerica **14** (2005), 139–232. MR2168343 (2006h:65159)
5. K. Deckelnick, G. Dziuk, C. M. Elliott, and C.-J. Heine, *An h-narrow band finite element method for implicit surfaces*, IMA J. Numer. Anal. **30** (2010), 351–376. MR2608464 (2011c:65256)
6. A. Demlow, *Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces*, SIAM J. Numer. Anal. **47** (2009), 805–827. MR2485433 (2010a:65233)
7. A. Demlow and G. Dziuk, *An adaptive finite element method for the Laplace-Beltrami operator on surfaces*, SIAM Journal on Numerical Analysis **45** (2007), 421–442. MR2285862 (2008c:65320)
8. G. Dziuk, *Finite elements for the Beltrami operator on arbitrary surfaces*, Partial differential equations and calculus of variations (S. Hildebrandt and R. Leis, eds.), Lecture Notes in Mathematics, vol. 1357, Springer Berlin Heidelberg New York London Paris Tokyo, 1988, pp. 142–155. MR976234 (90i:65194)
9. G. Dziuk and C. M. Elliott, *Finite elements on evolving surfaces*, IMA Journal Numerical Analysis **27** (2007), 262–292. MR2317005 (2008c:65253)
10. ———, *Surface finite elements for parabolic equations*, J. Comp. Mathematics **25** (2007), 385–407. MR2337402 (2008j:74065)
11. ———, *Eulerian finite element method for parabolic pdes on implicit surfaces*, Interfaces and Free Boundaries **10** (2008), 119–138. MR2383539 (2009a:65250)
12. ———, *An Eulerian approach to transport and diffusion on evolving implicit surfaces*, Computing and Visualization in Science **13** (2010), 17–28. MR2565107 (2010k:76109)
13. C. Eilks and C. M. Elliott, *Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method*, Journal of Computational Physics **227** (2008), no. 23, 9727–9741. MR2469030 (2009i:80013)
14. C. M. Elliott and B. Stinner, *Modeling and computation of two phase geometric biomembranes using surface finite elements*, J. Comput. Phys. **229** (2010), 6585–6612. MR2660322 (2011j:74092)

15. ———, *Analysis of a diffuse interface approach to an advection diffusion equation on a moving surfacei*, Math. Mod. Meth. Appl. Sci. **5** (2009), 787–802. MR2531040 (2011a:35251)
16. Charles M. Elliott, Björn Stinner, Vanessa Styles, and Richard Welford, *Numerical computation of advection and diffusion on evolving diffuse interfaces*, IMA J. Numer. Anal. **31** (2011), 786–812. MR2832780
17. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1998.
18. John B. Greer, *An improvement of a recent Eulerian method for solving PDEs on general geometries*, J. Sci. Comput. **29** (2006), no. 3, 321–352. MR2272322
19. John B. Greer, Andrea L. Bertozzi, and Guillermo Sapiro, *Fourth order partial differential equations on general geometries*, J. Comput. Phys. **216** (2006), no. 1, 216–246. MR2223442 (2007e:65090)
20. M. E. Gurtin, *Configurational forces as basic concepts in continuum physics*, Applied Mathematical Sciences, vol. 137, Springer, Berlin-Heidelberg, 2000. MR1735282 (2002f:74001)
21. S. Hildebrandt, *Analysis 2*, Springer-Verlag, Berlin, Heidelburg, New York, 1982.
22. Lili Ju and Qiang Du, *A finite volume method on general surfaces and its error estimates*, J. Math. Anal. Appl. **352** (2009), no. 2, 645–668. MR2501909 (2010a:65212)
23. Lili Ju, Li Tian, and Deshang Wang, *A posteriori error estimates for finite volume approximations of elliptic equations on general surfaces*, Comput. Methods Appl. Mech. Engrg. **198** (2009), 716–726. MR2498524 (2010b:65234)
24. M. Lenz, S. F. Nemadjieu, and M. Rumpf, *Finite volume method on moving surfaces v. Proceedings of the 5th International Symposium held in Aussois, June 2008.*, Finite volumes for complex applications (Hoboken, NJ,) (R.Eymard and J.-M. Hérard, eds.), John Wiley & Sons, Inc., 2008, pp. Hoboken, NJ,. MR2451453
25. C. B. Macdonald and S. J. Ruuth, *Level set equations on surfaces via the Closest Point Method*, Journal of Scientific Computing **35** (2008), 219–240. MR2429939 (2009i:65141)
26. ———, *The implicit closest point method for the numerical solution of partial differential equations on surfaces.*, SIAM J. Sci. Comput. **31** (2009), 4330–4350. MR2594984 (2011c:65166)
27. M. A. Olshanskii, A. Reusken, and J. Grande, *A finite element method for elliptic equations on surfaces*, SIAM J. Numer. Anal. **47** (2009), 3339–3358. MR2551197 (2010k:65265)
28. Maxim A. Olshanskii and Arnold Reusken, *A finite element method for surface pdes: matrix properties*, Numer. Math. **114** (2010), no. 3, 491–520 (English). MR2570076 (2010m:65289)
29. S. Osher and R. Fedkiw, *Level set methods and dynamic implicit surfaces*, Appl. Math. Sci, vol. 153, Springer-Verlag, 2003. MR1939127 (2003j:65002)
30. A. Ratz and A. Voigt, *Pdes on surfaces - a diffuse interface approach*, Communications in Mathematical Sciences **4** (2006), no. 3, 575–590. MR2247931 (2007g:35120)
31. Steven J. Ruuth and Barry Merriman, *A simple embedding method for solving partial differential equations on surfaces*, Journal of Computational Physics **227** (2008), no. 3, 1943–1961. MR2450979 (2009h:35239)
32. J. A. Sethian, *Level set methods and fast marching methods,*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 1999. MR1700751 (2000c:65015)
33. Jian-Ju Xu and H.-K. Zhao, *An Eulerian formulation for solving partial differential equations along a moving interface*, Journal of Scientific Computing **19** (2003), 573–594. MR2028859 (2004j:65129)

Abteilung für Angewandte Mathematik, University of Freiburg, Hermann-Herder-Strasse 10, D–79104 Freiburg i. Br., Germany
E-mail address: gerd@mathematik.uni-freiburg.de

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom
E-mail address: c.m.elliott@warwick.ac.uk