NEW RESULTS ON REVERSE ORDER LAW FOR \{1, 2, 3\}- AND \\
\{1, 2, 4\}-INVERSES OF BOUNDED OPERATORS

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Abstract. In this paper, using some block-operator matrix techniques, we give necessary and sufficient conditions for the reverse order law to hold for \{1, 2, 3\}- and \{1, 2, 4\}-inverses of bounded operators on Hilbert spaces. Furthermore, we present some new equivalents of the reverse order law for the Moore-Penrose inverse.

1. Introduction

Let \( \mathcal{H} \) and \( \mathcal{K} \) be complex Hilbert spaces and let \( B(\mathcal{H}, \mathcal{K}) \) denote the set of all bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \). For a given \( A \in B(\mathcal{H}, \mathcal{K}) \), the symbols \( N(A) \) and \( R(A) \) denote the null space and the range of \( A \), respectively. For given subsets \( M, N \) of \( B(\mathcal{H}, \mathcal{K}) \), by \( MN \) we denote the set consisting of all products \( XY \), where \( X \in M \) and \( Y \in N \).

Recall that \( A \in B(\mathcal{H}, \mathcal{K}) \) has the Moore-Penrose inverse if there exists an operator \( X \in B(\mathcal{K}, \mathcal{H}) \) such that

\[
\begin{align*}
(1) \quad AXA &= A, \\
(2) \quad XAX &= X, \\
(3) \quad (AX)^* &= AX, \\
(4) \quad (XA)^* &= XA.
\end{align*}
\]

The Moore-Penrose inverse of an operator \( A \in B(\mathcal{H}, \mathcal{K}) \) exists if and only if \( A \) has closed range and in this case it is unique. It is denoted by \( A^\dagger \).

For any \( A \in B(\mathcal{H}, \mathcal{K}) \), let \( A\{i, j, \ldots, k\} \) denote the set of all operators \( X \in B(\mathcal{K}, \mathcal{H}) \) which satisfy equations \((i), \ (j), \ldots, \ (k)\) of (1.1). In this case \( X \) is a \{i, j, \ldots, k\}-inverse of \( A \) and is denoted by \( A^{(i,j,\ldots,k)} \). Evidently, \( A\{1, 2, 3, 4\} = \{A^\dagger\} \), when \( A \) has closed range.

The reverse order law for generalized inverses plays an important role in theoretic research and numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see, for instance, [11, 9, 13, 14, 18]). These problems have attracted considerable attention since the mid-1960s, and many interesting results for generalized inverses of products of matrices or operators have been obtained (see [5, 6, 10, 11, 15–17]). T.E. Greville [8] first proved that \((AB)^\dagger = B^\dagger A^\dagger\) if and only if \( R(A^*AB) \subseteq R(B) \) and \( R(BB^*A^*) \subseteq R(A^\dagger) \), for matrices \( A \) and \( B \). This result was extended to linear bounded operators on Hilbert spaces in [11]. Later, the reverse order law for the Moore-Penrose inverse was considered in rings with involution (see [12]).
Xiong and Zheng [20] considered the reverse order law for \{1, 2, 3\}- and \{1, 2, 4\}-generalized inverses of products of two matrices. Their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. In [2] the authors considered the reverse order law for \(K\)-inverses in the cases \(K \in \{ \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\) for elements of a C*-algebra.

In this paper, using block-operator matrix techniques, we consider the reverse order law for \{1, 2, 3\}- and \{1, 2, 4\}-inverses of bounded linear operators on Hilbert spaces. We give necessary and sufficient conditions for

\[B\{1, 2, 3\} \cdot A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}\]

and

\[B\{1, 2, 4\} \cdot A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\} .\]

We generalized the results from [2] to the case of bounded linear operators on Hilbert spaces. This is of particular importance especially in statistics where bounded linear operators play a very important role. Furthermore, we present new equivalent conditions for the reverse order law for the Moore-Penrose inverse. It should be pointed out that when restricted to the set of matrices, our results for the reverse order law for the Moore-Penrose inverse yield facts not previously known.

2. Preliminaries

Let \(\mathcal{H}, \mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\) have closed range. The operator \(A\) has the following matrix decomposition (see [4], [7])

\[(2.1)\quad A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \left[ \begin{array}{c} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{array} \right],\]

where \(A_1\) is invertible. Also, \(A^\dagger\) has the form

\[(2.2)\quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \left[ \begin{array}{c} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{array} \right].\]

If \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\) has closed range, then we can explicitly describe the sets \(A\{1, 2, 3\}\) and \(A\{1, 2, 4\}\) using the representation of \(A\) given by (2.1).

**Lemma 2.1.** Let \(\mathcal{H}, \mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\) have closed range. Then

\[A\{1, 2, 3\} = \left\{ \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix} : \left[ \begin{array}{c} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{array} \right] : X_3 \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(A)) \right\}\]

and

\[A\{1, 2, 4\} = \left\{ \begin{bmatrix} A_1^{-1} & X_2 \\ 0 & 0 \end{bmatrix} : \left[ \begin{array}{c} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{array} \right] : X_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(A)) \right\}.\]

**Proof.** Suppose that \(A\) and \(A^\dagger\) are given by (2.1) and (2.2), respectively. Since

\[A\{1, 2, 3\} = \{ A^\dagger + (I - A^\dagger A)XAA^\dagger : X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \},\]
we have that \( A^{(1,2,3)} \in A\{1,2,3\} \) if and only if
\[
A^{(1,2,3)} = A^\dagger + (I - A^\dagger A)XXA^\dagger
\]
\[
= \begin{bmatrix}
A_1^{-1} & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A_1^{-1} & 0 \\
X_3 & 0
\end{bmatrix},
\]
for some \( X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \). The proof for the case of \( \{1,2,4\}\)-inverses follows similarly. \( \square \)

**Lemma 2.2.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces. Let \( A \in \mathcal{B}(\mathcal{H},\mathcal{K}) \) and \( B \in \mathcal{B}(\mathcal{L},\mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed. Then \( \mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\} \) if and only if \( \mathcal{R}(B) \cap \mathcal{N}(A) = \{0\} \).

**Proof.** First, let us note that
\[
(2.3) \quad B|_{\mathcal{R}(B^*)} : \mathcal{R}(B^*) \rightarrow \mathcal{R}(B)
\]
is an invertible operator.

(\( \Rightarrow \)): Suppose that \( \mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\} \) and let \( x \in \mathcal{R}(B) \cap \mathcal{N}(A) \). By (2.3), there exists \( y \in \mathcal{R}(B^*) \) such that \( By = x \). Now, \( y \in \mathcal{R}(B^*) \cap \mathcal{N}(AB) \), i.e., \( y = 0 \), so \( x = By = 0 \).

(\( \Leftarrow \)): If we suppose that \( \mathcal{R}(B) \cap \mathcal{N}(A) = \{0\} \) and take \( u \in \mathcal{R}(B^*) \cap \mathcal{N}(AB) \), we get that \( Bu \in \mathcal{R}(B) \cap \mathcal{N}(A) \), i.e., \( Bu = 0 \). Using (2.3) it follows that \( u = 0 \). \( \square \)

Let us introduce the following notation: if a Hilbert space \( \mathcal{H} \) is decomposed as \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k \) where \( \mathcal{H}_i \perp \mathcal{H}_j \) for \( i \neq j \), then we shall write \( \mathcal{H} = \mathcal{H}_1 \uplus \cdots \uplus \mathcal{H}_k \). If \( \mathcal{U} \) is a complement space of a Hilbert space \( \mathcal{H} \) we shall denote by \( \mathcal{H} \uplus \mathcal{U} \) the unique subspace \( \mathcal{V} \) of \( \mathcal{H} \) such that \( \mathcal{H} = \mathcal{U} \uplus \mathcal{V} \).

**Remark 2.1.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H},\mathcal{K}), B \in \mathcal{B}(\mathcal{L},\mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed. Denote by
\[
\begin{align*}
\mathcal{H}_1 &= \mathcal{R}(B) \cap \mathcal{N}(A), \\
\mathcal{H}_2 &= \mathcal{R}(B) \uplus \mathcal{H}_1, \\
\mathcal{H}_3 &= \mathcal{N}(B^*) \cap \mathcal{N}(A), \\
\mathcal{H}_4 &= \mathcal{N}(B^*) \uplus \mathcal{H}_3,
\end{align*}
\]
Hilbert spaces \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) can be decomposed as
\[
\mathcal{H} = \mathcal{R}(B) \uplus \mathcal{N}(B^*), \quad \mathcal{K} = \mathcal{R}(A) \uplus \mathcal{N}(A^*), \quad \mathcal{L} = \mathcal{R}(B^*) \uplus \mathcal{N}(B),
\]
where
\[
\mathcal{R}(B) = \mathcal{H}_1 \uplus \mathcal{H}_2, \quad \mathcal{N}(B^*) = \mathcal{H}_3 \uplus \mathcal{H}_4, \quad \mathcal{R}(A) = \mathcal{K}_1 \uplus \mathcal{K}_2, \quad \mathcal{R}(B^*) = \mathcal{L}_1 \uplus \mathcal{L}_2.
\]

We can prove that \( B^\dagger (\mathcal{R}(B) \cap \mathcal{N}(A)) = \mathcal{R}(B^*) \cap \mathcal{N}(AB) \). Indeed let \( x \in \mathcal{R}(B^*) \cap \mathcal{N}(AB) \). Then \( x \in \mathcal{R}(B^\dagger) = \mathcal{R}(B^\dagger B) = B^\dagger \mathcal{R}(B) \) and \( A B x = 0 \), i.e., \( B x \in \mathcal{N}(A) \). So we have \( x = B^\dagger B x \in B^\dagger \mathcal{N}(A) \). Finally, \( x \in B^\dagger (\mathcal{R}(B) \cap \mathcal{N}(A)) \). On the other hand, let \( y \in B^\dagger (\mathcal{R}(B) \cap \mathcal{N}(A)) \), i.e., \( y \in \mathcal{R}(B^\dagger B) = \mathcal{R}(B^*) \), then for some \( z \in \mathcal{R}(B) \cap \mathcal{N}(A) \) we have \( y = B^\dagger z \). So \( A B y = A B B^\dagger z = A z = 0 \), i.e., \( y \in \mathcal{N}(AB) \). Thus, by Lemma 2.2, we get
\[
\mathcal{H}_2 = \mathcal{R}(B) \iff \mathcal{H}_1 = \{0\} \iff \mathcal{N}(AB) = \mathcal{N}(B) \iff \mathcal{L}_1 = \{0\} \iff \mathcal{L}_2 = \mathcal{R}(B^*).
Furthermore,

\[ H_2 = \{0\} \iff K_1 = R(B) \iff R(B) \subset N(A) \iff L_1 = R(B^*) \iff L_2 \]
\[ = \{0\} \iff K_1 = \{0\} \iff K_2 = R(A). \]

Throughout the paper we will use the notation from the above remark.

A similar result to the following one but for the case of two Hilbert spaces has been presented in [17]. Now we give a different proof in the case of three Hilbert spaces.

**Lemma 2.3.** Let \( H, K \) and \( L \) be Hilbert spaces and let \( A \in \mathcal{B}(H, K) \), \( B \in \mathcal{B}(L, H) \) be such that \( R(A), R(B) \) and \( R(AB) \) are closed.

1. If \( AB \neq 0 \) and \( N(AB) \neq N(B) \), then \( A \) and \( B \) have the following operator matrix forms:

   \[
   A = \begin{bmatrix}
   0 & A_{12} & 0 & A_{14} \\
   0 & 0 & 0 & A_{24}
   \end{bmatrix}:
   \begin{bmatrix}
   H_1 \\
   H_2 \\
   H_3 \\
   H_4
   \end{bmatrix} \to \begin{bmatrix}
   K_1 \\
   K_2 \\
   N(A^*)
   \end{bmatrix}
   \] (2.4)

   and

   \[
   B = \begin{bmatrix}
   B_{11} & B_{12} & 0 \\
   0 & B_{22} & 0 \\
   0 & 0 & 0
   \end{bmatrix}:
   \begin{bmatrix}
   L_1 \\
   L_2 \\
   N(B)
   \end{bmatrix} \to \begin{bmatrix}
   H_1 \\
   H_2 \\
   H_3 \\
   H_4
   \end{bmatrix},
   \] (2.5)

   where \( A_{12}, B_{11}, B_{22} \) are invertible operators and \( A_{24} \) is a surjection.

2. If \( AB \neq 0 \) and \( N(AB) = N(B) \), then \( A \) and \( B \) have the following operator matrix forms:

   \[
   A = \begin{bmatrix}
   A_{12} & 0 & A_{14} \\
   0 & 0 & A_{24}
   \end{bmatrix}:
   \begin{bmatrix}
   R(B) \\
   H_3 \\
   H_4
   \end{bmatrix} \to \begin{bmatrix}
   K_1 \\
   K_2 \\
   N(A^*)
   \end{bmatrix}
   \] (2.6)

   and

   \[
   B = \begin{bmatrix}
   B_{22} & 0 \\
   0 & 0 \\
   0 & 0
   \end{bmatrix}:
   \begin{bmatrix}
   R(B^*) \\
   N(B)
   \end{bmatrix} \to \begin{bmatrix}
   R(B) \\
   H_3 \\
   H_4
   \end{bmatrix},
   \] (2.7)

   where \( A_{12}, B_{22} \) are invertible operators and \( A_{24} \) is a surjection.

3. If \( AB = 0 \) and \( N(AB) \neq N(B) \), then \( A \) and \( B \) have the following operator matrix forms:

   \[
   A = \begin{bmatrix}
   0 & 0 & A_{24} \\
   0 & 0 & 0
   \end{bmatrix}:
   \begin{bmatrix}
   R(B) \\
   H_3 \\
   H_4
   \end{bmatrix} \to \begin{bmatrix}
   R(A) \\
   N(A^*)
   \end{bmatrix}
   \] (2.8)

   and

   \[
   B = \begin{bmatrix}
   B_{11} & 0 \\
   0 & 0 \\
   0 & 0
   \end{bmatrix}:
   \begin{bmatrix}
   R(B^*) \\
   N(B)
   \end{bmatrix} \to \begin{bmatrix}
   R(B) \\
   H_3 \\
   H_4
   \end{bmatrix},
   \] (2.9)

   where \( B_{11} \) and \( A_{24} \) are invertible.
Proof. We will assume that the spaces \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) are decomposed as in Remark 2.1 so that the conclusions of the remark also hold.

(1) Suppose that \( AB \neq 0 \) and \( N(AB) \neq N(B) \). We have that \( B \) can be represented by

\[
B = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{21} & B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
N(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\]

where \( \widehat{B} = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix} : \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2
\end{bmatrix} \) is invertible.

Since \( B\mathcal{L}_1 \subset \mathcal{H}_1 \), we get that \( B_{21} = 0 \). Now from the invertibility of \( \widehat{B} \), we get that \( B_{11} : \mathcal{L}_1 \rightarrow \mathcal{H}_1 \) and \( B_{22} : \mathcal{L}_2 \rightarrow \mathcal{K}_2 \) are invertible.

Now, we will prove that \( A \) has a matrix form given by (2.4). Suppose that

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
N(A^*)
\end{bmatrix}.
\]

The reductions \( A_{i1} \) and \( A_{i3} \) for \( i = 1, 2, 3 \) are null operators because \( \mathcal{K}_1, \mathcal{K}_3 \subset N(A) \). The range of the reductions \( A_{3j}, \{ j = 1, 2, 3, 4 \} \) is \( N(A^*) \), so \( A_{3j} = 0 \).

Now we will prove that \( A_{22} = 0 \): for any \( x \in \mathcal{H}_2 \subset \mathcal{R}(B) \), there exists \( y \in \mathcal{K} \) such that \( By = x \). Now, \( Ax = ABy \in \mathcal{K}_1 \) and \( Ax = A_{12}x + A_{22}x \). Since \( A_{12}x \in \mathcal{K}_1 \), we get that \( A_{22}x = 0 \).

In order to prove that \( A_{12} \) is bijective, first we will prove that \( N(A_{12}) = \{0\} \): let \( u \in \mathcal{K}_2 \) be such that \( A_{12}u = 0 \). Then \( u \in N(A) \) which implies that \( u \in \mathcal{K}_1 \cap \mathcal{H}_2 = \{0\} \).

To prove that \( A_{12} : \mathcal{H}_2 \rightarrow \mathcal{K}_1 \) is surjective take any \( k \in \mathcal{K}_1 = \mathcal{R}(AB) \). There exists \( k' \in \mathcal{K} \) such that \( ABk' = k \). Since \( BK' \in \mathcal{R}(B) = \mathcal{K}_1 \oplus \mathcal{K}_2 \), there exist \( h_1 \in \mathcal{H}_1 \) and \( h_2 \in \mathcal{H}_2 \) such that \( Bk' = h_1 + h_2 \). Now, \( Ah_2 = A(Bk' - h_1) = k \), i.e., \( A_{12}h_2 = k \).

The surjective property of \( A_{24} : \mathcal{H}_4 \rightarrow \mathcal{K}_2 \) follows from the fact that for any \( u \in \mathcal{K}_2 \), there exists \( v \in \mathcal{K} \) such that \( Av = u \). Let us decompose \( v = \sum_{i=1}^{4} v_i \), where \( v_i \in \mathcal{H}_i \). It is evident that \( A_{24}v_4 = u \).

The proof of (2) and (3) is analogous. \( \square \)

3. Main Results

Z. Xiong and B. Zheng [20] presented necessary and sufficient conditions for (3.1)

\[
B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\},
\]

in the case when \( A \) and \( B \) are matrices. Here, we give another characterization of (3.1) for linear bounded operators on Hilbert spaces using techniques which are completely different from those used in [20]. First, we will give the following remark:

Remark 3.1. Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed, \( AB \neq 0 \) and \( N(AB) \neq N(B) \). Then we can suppose that the operators \( A \) and \( B \) are represented by (2.4) and (2.5),
respectively. By Lemma 2.1, \( X \in B\{1, 2, 3\} \) if and only if there exist operators \( F_{11} \) and \( F_{12} \) such that

\[
(3.2) \quad X = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ F_{11} & F_{12} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}.
\]

To describe the set \( A\{1, 2, 3\} \), suppose that an arbitrary \( Y \in A\{1, 2, 3\} \) is given by

\[
(3.3) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \\ Y_{41} & Y_{42} & Y_{43} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}.
\]

Since \( AY \) is hermitian, we get that

\[
AY = \begin{bmatrix} A_{12}Y_{21} + A_{14}Y_{41} & A_{12}Y_{22} + A_{14}Y_{42} & 0 \\ A_{24}Y_{41} & A_{24}Y_{42} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix},
\]

where \( A_{12}Y_{22} + A_{14}A_{42} = (A_{42}Y_{41})^* \) and \( A_{12}Y_{21} + A_{14}Y_{41}, A_{24}Y_{42} \) are hermitian. Since \( AY \) is an orthogonal projection on \( \Re(A) \), from the definition of the subspaces \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) we can conclude that \( A_{12}Y_{21} + A_{14}Y_{41} = I, A_{24}Y_{42} = I, A_{12}Y_{22} + A_{14}A_{42} = 0 \) and \( A_{24}Y_{41} = 0 \). Now, from \( YAY = 0 \), we get that \( Y_{i3} = 0 \), for \( i = 1, 2, 3 \). Hence, \( Y \in A\{1, 2, 3\} \) if and only if

\[
(3.4) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ Y_{31} & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
\]

where the \( Y_{ij} \) satisfy the following equalities:

\[
(3.5) \quad \begin{cases} Y_{2i}A_{24}Y_{42} = Y_{i2}, & i = 1, 2, 3, \\ A_{12}Y_{21} + A_{14}Y_{41} = I_{\mathcal{K}_1}, \\ A_{12}Y_{22} + A_{14}Y_{42} = 0, \\ A_{24}Y_{42} = I_{\mathcal{K}_2}, A_{24}Y_{41} = 0. \end{cases}
\]

Since

\[
(3.6) \quad AB = \begin{bmatrix} 0 & A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix},
\]

we get that \( Z \in (AB)\{1, 2, 3\} \) if and only if there exist operators \( N_1 \) and \( N_2 \) such that

\[
(3.7) \quad Z = \begin{bmatrix} N_1 & 0 & 0 \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}.
\]
Theorem 3.1. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$.

(ii) $\mathcal{R}(B) = \mathcal{R}(A^*AB) \oplus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$, $\mathcal{R}(AB) = \mathcal{R}(A)$.

Proof. We use the decompositions of the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ and the matrix decompositions of operators $A$, $B$ given in Lemma [23]. We distinguish two cases:

(1) First, suppose that $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that the operators $A$ and $B$ are represented by (3.2) and (3.3), respectively. Also, in Remark 3.1 we have given characterizations of the sets $A\{1, 2, 3\}$, $B\{1, 2, 3\}$ and $(AB)\{1, 2, 3\}$ which we will use in this proof.

(i) $\Rightarrow$ (ii): Let arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ be given by (3.2) and (3.3), respectively. Then

$$XY = \begin{bmatrix}
M_1 & M_2 & 0 \\
B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\
F_{11}Y_{11} + F_{12}Y_{21} & F_{11}Y_{12} + F_{12}Y_{22} & 0
\end{bmatrix},$$

(3.8)

where $M_1 = B_{11}^{-1}Y_{11} - B_{12}^{-1}B_{12}B_{22}^{-1}Y_{21}$, $M_2 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}$.

Since $XY \in (AB)\{1, 2, 3\}$, we conclude that $XY$ must be of the form $Z$ given by (3.7) for some operators $N_1$ and $N_2$. If we compare (3.8) and (3.7), we get that $Y_{12} = 0$, $Y_{22} = 0$ and $Y_{21} = A_{12}^{-1}$. Hence, it follows that the system of the operator equations (3.5) is such that $Y_{12}$, $Y_{22}$ and $Y_{21}$ are uniquely determined. Since $A_{24}$ is surjective and $A_{24}Y_{22} = I_{\mathcal{K}_2}$ we get that $Y_{12} = 0$ if and only if $\mathcal{K}_2 = \{0\}$, i.e., $A_{24} = 0$. If this were not true, then $Y_{12}$ could be taken to be an arbitrary operator on an appropriate subspace, which is not the case. Now, from the first equation of (3.5), we get that $Y_{21} = A_{12}^{-1}$ it must be that $A_{14} = 0$. It is evident that $A_{24} = 0$ is equivalent to $\mathcal{R}(AB) = \mathcal{R}(A)$.

Now, simple computation shows that

$$A^*AB = \begin{bmatrix}
0 & 0 & 0 \\
0 & A_{12}^*A_{12}B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},$$

and finally we get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

(ii) $\Rightarrow$ (i): Suppose $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(AB) = \mathcal{R}(A)$. We must show that for arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ there exists $Z \in (AB)\{1, 2, 3\}$ such that $XY = Z$.

From $\mathcal{R}(AB) = \mathcal{R}(A)$, we get that $\mathcal{K}_2 = \{0\}$, i.e., $A_{24} = 0$. Also by $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and the fact that

$$A^*AB = \begin{bmatrix}
0 & 0 & 0 \\
0 & A_{12}^*A_{12}B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},$$
where $A_{12}$ and $B_{22}$ are invertible, we have $A_{14} = 0$.

Now, we get that $Y \in A \{1, 2, 3\}$ if and only if

$$
(3.9) \quad Y = \begin{bmatrix}
Y_{11} & 0 & 0 \\
A_{12}^1 & 0 & 0 \\
Y_{31} & 0 & 0 \\
Y_{41} & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3
\end{bmatrix},
$$

where $Y_{11}, Y_{31}, Y_{14}$ are arbitrary. It is evident that for arbitrary $X \in B \{1, 2, 3\}$ and $Y \in A \{1, 2, 3\}$ there exists $Z \in (AB) \{1, 2, 3\}$ such that $XY = Z$, i.e.,

$$
B \{1, 2, 3\} A \{1, 2, 3\} \subseteq (AB) \{1, 2, 3\}.
$$

(2) When $N(AB) = N(B)$, the operators $A$ and $B$ are represented by (2.6) and (2.7), respectively, and the proof is analogous to case (1). □

**Remark 3.2.** 1° If $AB = 0$, then $(AB) \{1, 2, 3\} = \{0\}$. In the case when $A = 0$ or $B = 0$, evidently $B \{1, 2, 3\} A \{1, 2, 3\} \subseteq (AB) \{1, 2, 3\}$. If it is not the case, we have that $AB = 0 \iff \mathcal{H}_2 = \{0\} \iff \mathcal{H}_1 = \mathcal{K}(B) \iff \mathcal{L}_2 = \{0\} \iff \mathcal{L}_1 = \mathcal{K}(B^*) \iff \mathcal{K}_1 = \{0\} \iff \mathcal{K}_2 = \mathcal{K}(A)$. Also, $A$ and $B$ are represented by (2.8) and (2.9), respectively, so arbitrary $X \in B \{1, 2, 3\}$ and $Y \in A \{1, 2, 3\}$ are represented by

$$
X = \begin{bmatrix}
B_{11}^{-1} & 0 & 0 \\
F_1 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{N}(B)
\end{bmatrix}
$$

and

$$
Y = \begin{bmatrix}
F_2 & 0 \\
F_3 & 0 \\
A_{24}^1 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_2 \\
\mathcal{N}(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
$$

for some operators $F_1, F_2$ and $F_3$.

By simple computation, we observe that

$$
XY = \begin{bmatrix}
B_{11}^{-1}F_2 & 0 \\
F_1F_2 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{N}(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_2 \\
\mathcal{N}(A^*)
\end{bmatrix} \neq 0,
$$

i.e., $B \{1, 2, 3\} A \{1, 2, 3\} \neq \{0\}$.

Hence,$AB = 0, A \neq 0, B \neq 0 \Rightarrow B \{1, 2, 3\} A \{1, 2, 3\} \notin (AB) \{1, 2, 3\}$.

2° From Theorem 3.1 we conclude that the condition

$$(ABB)^\dagger ABB^\dagger = BB^\dagger \text{ or } (AB)(AB)^\dagger = AA^\dagger$$

from [2] Theorem 3.3 can be replaced by the sole condition $(AB)(AB)^\dagger = AA^\dagger$, i.e., $\mathcal{R}(AB) = \mathcal{R}(A)$.

A similar result in the case $K = \{1, 2, 4\}$ follows from Theorem 3.1 by reversal of products:

**Theorem 3.2.** Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $B \{1, 2, 4\} A \{1, 2, 4\} \subseteq (AB) \{1, 2, 4\}$.

(ii) $\mathcal{R}(A^*) = \mathcal{R}(B B^* A^*) \oplus ^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)], \mathcal{N}(AB) = \mathcal{N}(B)$. 

Remark 3.3. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed, $AB \neq 0$ and $N(AB) \neq N(B)$. We have that operator $B$ is represented by (2.3), so

$$B^\dagger = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}.$$  

(3.10)

Also, if we suppose that the operator $A$ is represented by (2.3), by Remark 3.1 we get that $A^\dagger = Y$ is represented by (3.4), where $Y_{ij}$ satisfy (3.5). Now, since $YA$ is an orthogonal projection on $N(A)^\perp$, we get that

$$YA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

which implies that $Y_{21}A_{12} = I_{3\ell_2}$, $Y_{41}A_{12} = 0$, $Y_{21}A_{14} + Y_{22}A_{24} = 0$, $Y_{41}A_{14} + Y_{42}A_{24} = I_{3\ell_4}$. Now by $AYA = 0$ we get that $Y_{11} = 0$, $Y_{12} = 0$, $Y_{31} = 0$ and $Y_{32} = 0$. Hence,

$$A^\dagger = \begin{bmatrix} 0 & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N}(A^\ast) \end{bmatrix}$$

(3.11)

where

$$\begin{align*}
Y_{i2}A_{24}Y_{42} = Y_{i2}, & \quad i = 1, 4, \\
A_{12}Y_{21} + A_{14}Y_{41} = I_{3\ell_1}, \\
A_{12}Y_{22} + A_{14}Y_{42} = 0,
\end{align*}$$

(3.12)

Simple computation shows that

$$B^\dagger A^\dagger = \begin{bmatrix} -B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21} & M_3 & 0 \\ B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^\ast) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix},$$

(3.13)

where $M_3 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}.$

Using the previous remark, we obtain the following result:

**Theorem 3.3.** Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}.$

(ii) $B\{1, 2, 3\}A^\dagger \subseteq (AB)\{1, 2, 3\}.$

(iii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus \{\mathcal{R}(B) \cap \mathcal{N}(A)\}.$
3.1. We distinguish two cases: we obtain

\[ \text{Proof.} \]

We will use the same decompositions of spaces \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) as in Theorem 3.1. We distinguish two cases:

(1) Let \( N(AB) \neq N(B) \).

(i)\( \Rightarrow \) (iii) If \( B^\dagger A^\dagger \in (AB)\{1,2,3\} \), then there exists an operator \( Z \in (AB)\{1,2,3\} \) such that \( B^\dagger A^\dagger = Z \), where \( Z \) is represented by (3.7). Comparing (3.7) with (3.13), we obtain \( Y_{21} = A_{12}^{-1}, Y_{22} = 0, Y_{12} = 0 \).

We have that (3.12) implies \( Y_{21} = A_{12}^{-1} \) only if \( A_{14} = 0 \) which implies the invertibility of \( A_{24} \). Hence

\[
A = \begin{bmatrix}
0 & A_{12} & 0 & 0 \\
0 & 0 & 0 & A_{24} \\
0 & 0 & 0 & 0
\end{bmatrix}; \quad \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
N(A^*)
\end{bmatrix}.
\]

It is easy to get \( \mathcal{R}(A^*AB) = \mathcal{R}(B) \odot ^\perp [\mathcal{R}(B) \cap N(A)] \).

(iii)\( \Rightarrow \) (i) Since \( \mathcal{R}(A^*AB) = \mathcal{R}(B) \odot ^\perp [\mathcal{R}(B) \cap N(A)] \) is equivalent to \( A_{14} = 0 \), we obtain from (3.12) that \( Y_{21} = A_{12}^{-1}, Y_{22} = 0, Y_{12} = 0 \). Hence, \( B^\dagger A^\dagger \in (AB)\{1,2,3\} \).

(ii)\( \Leftrightarrow \) (i) Using the representation of an arbitrary \( X \in B\{1,2,3\} \) given by (3.2), we get that \( XA^\dagger \in (AB)\{1,2,3\} \) if and only if \( B^\dagger A^\dagger \in (AB)\{1,2,3\} \).

(2) If \( N(AB) = N(B) \), the proof is analogous to case (1). \( \square \)

The case \( K = \{1,2,4\} \) is treated completely analogously, and the corresponding result follows by taking adjoints, or by reversal of products:

**Theorem 3.4.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB) \) are closed and \( AB \neq 0 \). Then the following statements are equivalent:

(i) \( B^\dagger A^\dagger \in (AB)\{1,2,4\} \).

(ii) \( B^\dagger A\{1,2,4\} \subseteq (AB)\{1,2,4\} \).

(iii) \( \mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \odot ^\perp [\mathcal{R}(A^*) \cap N(B^*)] \).

From the above two theorems, we get the following equivalent condition for the Moore-Penrose inverse.

**Theorem 3.5.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB) \) are closed and \( AB \neq 0 \). Then the following statements are equivalent:

(i) \( (AB)^\dagger = B^\dagger A^\dagger \).

(ii) \( \mathcal{R}(A^*AB) = \mathcal{R}(B) \odot ^\perp [\mathcal{R}(B) \cap N(A)] \) and \( \mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \odot ^\perp [\mathcal{R}(A^*) \cap N(B^*)] \).

**Remark 3.4.** The conditions (ii) from Theorem 3.5 are equivalent to the conditions \( R(A^*AB) \subseteq \mathcal{R}(B) \) and \( R(BB^*A^*) \subseteq \mathcal{R}(A^*) \) given in the paper by Greville [5] for matrices. Also, they are equivalent to those given in [5] Theorem 2.2 (c)] in the case of bounded linear operators on Hilbert space.

**ACKNOWLEDGEMENT**

The authors wish to thank the anonymous reviewers for very valuable comments and suggestions concerning an earlier version of this paper.
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