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ON THE LOG-CONCAVITY OF A JACOBI THETA FUNCTION

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ABSTRACT. A new proof of the log-concavity of the Jacobi theta function, appearing in the Fourier representation of the Riemann Ξ function, is presented. An open problem, involving the normalized moments of log-concave kernels, is investigated. In particular, several Turán-type inequalities are established.

1. INTRODUCTION

A C^2- function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be *logarithmically concave* (or *log-concave* for short), if $\log(f(t))$ is concave or, equivalently, $(f'(t))^2 - f(t)f''(t) \geq 0$ for all $t \in \mathbb{R}$. The notion of concavity, in its various forms, plays an important role in probability theory (see, for example, [10] or [2] and the references therein), the theory of majorization and its applications ([12]) as well as in the study of the distribution of zeros of entire functions represented by the Fourier transforms of certain "admissible" kernels (cf. Definition 3.2). In this note, we are especially interested in investigating the kernel whose Fourier transform is the Riemann Ξ-function (cf. [13] or [14, p. 285]):

$$\Xi(x) := \int_0^\infty \Phi(t)\cos(xt)\,dt = \frac{1}{2}\int_{-\infty}^\infty \Phi(t)e^{ixt}\,dt, \qquad (x \in \mathbb{R}), \tag{1.1}$$

where the *Jacobi theta function*, $\Phi(t)$, is defined by

$$\Phi(t) := \sum_{n=1}^\infty a_n(t), \qquad \text{and}$$

$$a_n(t) = \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right)\exp\left(-\pi n^2 e^{4t}\right), \quad (n = 1, 2, \dots). \tag{1.2}$$

(In the above nomenclature, we have deleted the usual, inconsequential, factors of 2 in the definitions of Ξ and Φ.) Now, it is known that Ξ is a real entire function of order 1 ([16, p. 29]), of maximal type ([9]) and that the Riemann Hypothesis is the statement that all the zeros of $\Xi(x)$ are real (cf. [13] or [14, p. 278], [16, p. 254]).

In the sequel, we will often refer to some of the basic properties of the kernel, Φ, summarized below.

Theorem A ([6, Theorem A] and [7, Theorem 2.1])**.** *Consider the Jacobi theta function, Φ, defined by* (1.2)*. Then, the following are valid:*

(i) $\Phi(t) > 0$ *for all* $t \geq 0$;
(ii) $\Phi(z)$ *is analytic in the strip* $-\pi/8 < \Im z < \pi/8$;
(iii) $\Phi(t)$ *is an even function, so that* $\Phi^{(2m+1)}(0) = 0 \quad (m = 0, 1, \dots)$;

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(iv) *for any* $\varepsilon > 0$ *and for each* $n = 0, 1, \ldots,$

$$\lim_{t\to\infty} \Phi^{(n)}(t) \exp\left((\pi - \varepsilon)e^{4t}\right) = 0;$$

(v) $\Phi'(t) < 0$ *for all* $t > 0$*;*

(vi) $a_n'(t) < 0$ *for all* $t \geq 0$*, for each* $n = 2, 3 \ldots;$

(vii) $a_1'(t) \geq 0$ *for* $0 \leq t \leq t_0$*, and* $a_1'(t) < 0$ *for all* $t > t_0$*, where*

$$t_0 := \frac{1}{4} \log \left[\frac{15 + \sqrt{105}}{8\pi} \right] = 0.001 \cdots;$$

(viii) ([7, Theorem 2.1])

$$\left(\log \Phi(\sqrt{t})\right)'' < 0 \qquad \textit{for} \;\; t > 0. \tag{1.3}$$

Part (viii) of Theorem A states that $\Phi(\sqrt{t})$ is log-concave for $t > 0$. A calculation shows that this is equivalent to the inequality

$$g(t) := t\left[(\Phi'(t))^2 - \Phi(t)\Phi''(t)\right] + \Phi(t)\Phi'(t) > 0, \qquad t > 0. \tag{1.4}$$

We hasten to remark that inequality (1.4) implies that $\Phi(t)$ is log-concave for $t > 0$. However, the *tour de force* proof of (1.4) is not transparent. Indeed, in [7] the proof of (1.4) requires ten lemmas (cf. [7, pp. 184–197]) and involves a painstaking analysis of the first 6 (six) derivatives of Φ. One of our goals in this paper is to provide a transparent proof of the log-concavity of the simpler function; that is, the function $\Phi(t)$. We commence the next section with an outline of the proof of the main result of Section 2 (Theorem 2.4). In Section 3, we investigate a discrete analog of log-concavity (cf. Definition 3.1, the Turán-type inequality) and its relation to the moments of certain log-concave kernels (see Open Problem 3.3 and Theorem 3.6). In particular, we infer from the log-concavity of $\Phi(t)$, that its "normalized" moments satisfy a Turán-type inequality (Corollary 3.7).

2. Proof of the log-concavity of Φ

In order to facilitate the proofs of the log-concavity of Φ, we will adopt, throughout this section, the following nomenclature (which is consistent with that used in [7]). Set $\Phi(t) = a_1(t) + \Phi_1(t)$, where

$$\Phi_1(t) := \sum_{n=2}^{\infty} a_n(t) = \sum_{n=2}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right) \exp\left(-\pi n^2 e^{4t}\right). \tag{2.1}$$

Then an outline of the proof of our main theorem (Theorem 2.4) is as follows. Since $\Phi(t)$ is even (cf. Theorem A (iii)), we consider, for $t \geq 0$,

$$\begin{aligned} S(t) &:= (\Phi'(t))^2 - \Phi(t)\Phi''(t) \\ &= (a_1'(t) + \Phi_1'(t))^2 - (a_1(t) + \Phi_1(t))\,(a_1''(t) + \Phi_1''(t)) \\ &=: L(t) + U(t) + (\Phi_1'(t))^2, \end{aligned} \tag{2.2}$$

where

$$L(t) := (a_1'(t))^2 - a_1(t)a_1''(t) \qquad \text{and} \tag{2.3}$$

$$U(t) := 2a_1'(t)\Phi_1'(t) - a_1''(t)\Phi_1(t) - \Phi(t)\Phi_1''(t), \qquad \text{where} \tag{2.4}$$
$$\Phi(t) = a_1(t) + \Phi_1(t).$$

Thus, in order to prove that $S(t) = (\Phi'(t))^2 - \Phi(t)\Phi''(t) > 0$ $(t \in \mathbb{R})$, we will show that a positive lower bound of $L(t)$ (Lemma 2.2) is greater than an upper bound of $|U(t)|$ (Lemma 2.3). Observe that $L(t) > 0$ $(t \in \mathbb{R})$ means that $a_1(t)$ is log-concave on $\mathbb{R}$. Given the rapid decay of Φ_1, one expects that for "large" values of t, $L(t) > |U(t)|$. This inequality, however, is not obvious for "small" values of t. Striving for simplicity, our estimates are not necessarily sharp; but they will suffice to prove that $\log \Phi$ is concave.

We commence by establishing some upper estimates for $\Phi_1, |\Phi_1'|$, $|\Phi_1''|$ and Φ. For the sake of simplicity of presentation, we will set $y = \pi e^{4t}$, $t \geq 0$, so that $y \geq \pi$.

Proposition 2.1 (Upper bounds for $\Phi_1, |\Phi_1'|$, $|\Phi_1''|$ and Φ)**.** *Let* $\Phi_1(t) = \sum_{n=2}^{\infty} a_n(t)$ (cf. (2.1)) *and set* $y = \pi e^{4t}$, $t \geq 0$. *Then the following estimates hold:*

(i) $0 < \Phi_1(t) = |\Phi_1(t)| \leq 64 e^t y^2 e^{-4y}, \quad (t \geq 0)$.
(ii) ([6, Lemma 3.3]) $|\Phi_1'| \leq 565 e^t y^3 e^{-4y}, \quad (t \geq 0)$.
(iii) ([7, Lemma 3.1]) $|\Phi_1''| \leq (1.031) 2^{13} e^t y^4 e^{-4y}, \quad (t \geq 0)$.
(iv) ([6, p. 534, (3.41)]) $0 < \Phi(t) < \frac{203}{202} a_1(t), \quad (t \geq 0)$.

Proof. (i) For $t \geq 0$ and $y = \pi e^{4t}$, consider

$$\begin{aligned} 0 < \Phi_1(t) = \sum_{n=2}^{\infty} a_n(t) &= \sum_{n=2}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right) \exp\left(-\pi n^2 e^{4t}\right) \\ &\leq 2e^t \sum_{n=2}^{\infty} n^4 \pi^2 e^{8t} \exp\left(-\pi n^2 e^{4t}\right) \\ &= 2e^t \left(16 y^2 e^{-4y} + \sum_{n=3}^{\infty} y^2 n^4 e^{-n^2 y}\right). \end{aligned} \tag{2.5}$$

If $p(s) := y^2 s^4 e^{-ys^2}$, then $p'(s) = 2e^{-ys^2} s^3 y^2 (2 - s^2 y) < 0$ $(y \geq \pi, s \geq 2)$ and whence by the integral test,

$$\begin{aligned} \sum_{n=3}^{\infty} y^2 n^4 e^{-n^2 y} &\leq \int_2^{\infty} p(s)\, ds < \int_2^{\infty} y^2 s^5 e^{-ys^2}\, ds \qquad (s^4 < s^5, s \geq 2) \\ &= \frac{1}{y} e^{-4y} (1 + 4y + 8y^2) < 16 y^2 e^{-4y}. \end{aligned} \tag{2.6}$$

If we combine this (not necessarily optimal) upper bound (2.6) with (2.5), we obtain the desired estimate (i). Similar arguments, *mutatis mutandis*, establish the upper estimates (ii) and (iii). (For a slightly different proof of (ii) and (iii), we refer to the references cited above.)

(iv) The upper bound for $\Phi(t)$ is based on the readily established observation that $0 < \Phi_1(t) < \frac{1}{202} a_1(t)$ (cf. [6, (3.41)]). □

Lemma 2.2 (A positive lower bound for $L(t) = (a_1'(t))^2 - a_1(t) a_1''(t)$)**.** *Let* $y = \pi e^{4t}$ *for* $t \geq 0$. *Set* $a_1(t) = \left(2\pi^2 e^{9t} - 3\pi e^{5t}\right) \exp\left(-\pi e^{4t}\right)$ (cf. (1.2) with $n = 1$). *Then*

$$L(t) = (a_1'(t))^2 - a_1(t) a_1''(t) \geq 256 e^{2t} e^{-2y} y^3 \qquad (t \geq 0,\ y \geq \pi). \tag{2.7}$$

Proof. A computation shows that

$$\begin{aligned} L(t) &= 16 \exp(-2\pi e^{4t} + 14t) \pi^3 (15 - 12\pi e^{4t} + 4\pi^2 e^{8t}) \\ &= 16 e^{14t} e^{-2y} \pi^3 (15 - 12y + 4y^2) \\ &= 16 e^{2t} e^{-2y} y^3 (15 - 12y + 4y^2). \end{aligned} \tag{2.8}$$

Since $\theta(y) := (15 - 12y + 4y^2) = 4(y - \frac{3}{2})^2 + 6$ is an increasing function for $y \geq \pi$, it follows that $4(y - \frac{3}{2})^2 + 6 \geq 4(\pi - \frac{3}{2})^2 + 6 \geq 16$. Using this lower bound for the quadratic $\theta(y)$ in (2.8), we obtain the desired estimate, (2.7), for $L(t)$. □

For ease of reference and in order to expedite the calculations below, we proceed and express $a_1(t) = \left(2\pi^2 e^{9t} - 3\pi e^{5t}\right) \exp\left(-\pi e^{4t}\right)$, $a_1'(t)$ and $a_1''(t)$ in terms of $y = \pi e^{4t}$, $(t \geq 0)$. Thus,

$$a_1(t) = e^t e^{-y} q(y), \qquad \text{where} \quad q(y) = y(2y - 3), \tag{2.9}$$

$$a_1'(t) = -e^t e^{-y} q_1(y), \qquad \text{where} \quad q_1(y) = y(15 - 30y + 8y^2), \quad \text{and} \tag{2.10}$$

$$a_1''(t) = e^t e^{-y} q_2(y), \qquad \text{where} \quad q_2(y) = y(-75 + 330y - 224y^2 + 32y^3). \tag{2.11}$$

Preliminaries aside, we next establish an upper bound for $|U(t)|$ (see (2.4)).

Lemma 2.3 (An upper bound for $|U(t)|$)**.** *For $y = \pi e^{4t}$, $t \geq 0$, let*

$$E(y) := e^{2t} e^{-2y} y^3, \qquad y = \pi e^{4t}, t \geq 0. \tag{2.12}$$

Then an upper bound of $|U(t)|$ (cf. (2.4)) *is*

$$\begin{aligned} |U(t)| \leq & |2a_1'(t)\Phi_1'(t)| + |a_1''(t)\Phi_1(t)| + |\Phi(t)\Phi_1''(t)| \\ \leq & 56,424 e^{-3y} E(y) y^3. \end{aligned} \tag{2.13}$$

Proof. We proceed and separately estimate the summands in (2.13). By Proposition 2.1(ii), (2.10) and (2.12), we have

$$\begin{aligned} |2a_1'(t)\Phi_1'(t)| \leq & |2e^t e^{-y} q_1(y)||565 e^t e^{-4y} y^3| \\ \leq & E(y) e^{-3y} 1,130(15y + 30y^2 + 8y^3) =: E(y)A(y). \end{aligned} \tag{2.14}$$

Using Proposition 2.1(i), (2.11) and (2.12), we obtain the estimate

$$\begin{aligned} |a_1''(t)\Phi_1(t)| \leq & |e^t e^{-y} q_2(y)||64 e^t e^{-4y} y^2| \\ \leq & E(y) e^{-3y} 64(75 + 330y + 224y^2 + 32y^3) =: E(y)B(y). \end{aligned} \tag{2.15}$$

Finally, we invoke Proposition 2.1(iii), (iv), (2.9) and (2.12), to deduce that

$$\begin{aligned} |\Phi(t)\Phi_1''(t)| \leq & |\frac{203}{202} e^t e^{-y} q(y)||(1.031) 2^{13} e^t y^4 e^{-4y}| \\ \leq & E(y) e^{-3y} 8,562(2y^3 + 3y^2) =: E(y)C(y). \end{aligned} \tag{2.16}$$

Therefore, by (2.14), (2.15) and (2.16), an upper bound for $|U(t)|$ is

$$\begin{aligned} |U(t)| &\leq E(y)\left[A(y) + B(y) + C(y)\right] \\ &\leq E(y) 2\, e^{-3y}[(2,400 + 19,035y + 36,961\, y^2 + 14,206y^3)] \\ &= E(y) 2\, e^{-3y}(14,206y^3) \frac{(2,400 + 19,035y + 36,961\, y^2 + 14,206y^3)}{14,206y^3} \\ &\leq E(y) 2\, e^{-3y}(14,206y^3) \left[1 + \frac{400}{2,351y^3} + \frac{6,345}{4,702y^2} + \frac{36,961}{14,106y}\right] \\ &\leq E(y) 2\, e^{-3y}(14,206y^3)(1.97) < 56,424 E(y)\, e^{-3y} y^3. \end{aligned}$$

□

Theorem 2.4. *The Jacobi theta function, $\Phi(t)$, is (strictly) log-concave on $\mathbb{R}$. That is,*

$$S(t) = (\Phi'(t))^2 - \Phi(t)\Phi''(t) > 0 \quad \textit{for} \quad t \in \mathbb{R}. \tag{2.17}$$

Proof. Since $\Phi(t)$ is an even function (cf. Theorem A (iii)), $S(t)$ is even and thus it suffices to show that $S(t) > 0$ for $t \geq 0$. By (2.2), $S(t) \geq L(t)+U(t)$ for $t \geq 0$. Now, for $t \geq 0$, the estimates $L(t) \geq 256e^{2t}e^{-2y}y^3 = 256E(y)$, where $E(y) = e^{2t}e^{-2y}y^3$, (Lemma 2.2) and $|U(t)| \leq 56,424E(y)e^{-3y}y^3$ (Lemma 2.3), yield

$$\begin{aligned} S(t) \geq L(t) - |U(t)| >& E(y)\left[256 - 56,424e^{-3y}y^3\right] \\ \geq& E(y)\left[256 - 56,424e^{-3\pi}\pi^3\right] > 114E(y) > 0, \end{aligned}$$

where we used the fact that $56,424e^{-3y}y^3$ is a decreasing function of y, $y \geq \pi$. □

We conclude this section by noting that an argument, analogous to the one used to prove Lemma 2.2, shows that each $a_n(t)$, $(n \geq 1)$ (defined in (1.2)) is log-concave for $t \geq 0$; that is,

$$(a_n'(t))^2 - a_n(t)\, a_n''(t) \geq 0, \qquad (t \geq 0, n \in \mathbb{N}). \tag{2.18}$$

The techniques in this section, in conjunction with (2.18), suggest the following conjecture.

Conjecture 2.5. *The derivatives of the Jacobi theta function, $\Phi(t)$, are (strictly) log-concave on $\mathbb{R}$. That is, for each $n \in \mathbb{N}$,*

$$S_n(t) = (\Phi^{(n)}(t))^2 - \Phi^{(n-1)}(t)\Phi^{(n+1)}(t) > 0 \quad \textit{for} \quad t \in \mathbb{R}. \tag{2.19}$$

Since $\Phi(t)$ is an even function (cf. Theorem A (iii)), $S_n(t)$ is even and whence it suffices to establish (2.19) for $t \geq 0$.

3. Log-concavity and Turán-type inequalities

In order to expedite our presentation (and for the sake of simplicity), it will be convenient to introduce here the following definitions.

Definition 3.1 (A discrete analog of log-concavity)**.** A sequence of real numbers, $\{\gamma_k\}_{k=0}^{\infty}$, is said to be *log-concave* or is said to satisfy the *Turán inequalities*, if

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \qquad k = 1, 2, 3, \ldots \quad \text{(Turán inequalities)}. \tag{3.1}$$

Definition 3.2. A function $K : \mathbb{R} \longrightarrow \mathbb{R}^+$ is termed an *admissible kernel*, if $K(t)$ satisfies (i) $K(t) > 0$ for all $t \geq 0$, (ii) $K(t)$ is even, (iii) $K(t) \in C^{\infty}(\mathbb{R})$, and (iv) $K^{(n)}(t) = O\left(\exp\left(-|t|^{2+\alpha}\right)\right)$ as $t \longrightarrow \infty$ for some $\alpha > 0$ and $n = 0, 1, 2, \ldots$.

Consulting Theorem A, we readily see that $\Phi(t)$ is an admissible kernel. We remark that, today, there are no known necessary *and* sufficient conditions that a "nice" kernel (such as an admissible kernel) must satisfy in order that its Fourier transform have only real zeros. It is this central issue that motivates us to consider admissible kernels and questions related to the distribution of zeros of real entire functions represented by Fourier transforms. Specifically, for suitably "nice" kernels $K(t)$ (cf. Definition 3.2), we consider the following real entire functions

$$F(x) := \int_0^{\infty} K(t)\cos(xt)\, dt \quad \text{and} \quad \text{(with the change of variables } y = -x^2) \tag{3.2}$$

$$F_c(y) := \int_0^{\infty} K(t)\cosh(t\sqrt{y})\, dt = \sum_{k=0}^{\infty} \frac{\mu_k}{(2k)!} y^k, \tag{3.3}$$

where the μ_k's are the moments (associated with the kernel K) defined by

$$\mu_k := \int_0^\infty t^{2k}\, K(t)\, dt, \quad k = 0, 1, 2, \ldots, \qquad \gamma_k := \frac{k!\mu_k}{(2k)!}. \tag{3.4}$$

We remark that the growth condition imposed on the admissible kernel, K, implies that the entire function F has order at most $(2+\alpha)/(1+\alpha) < 2$ and whose genus is at most 1 [14, p. 269]. Moreover, the entire function F has only real zeros if and only if the entire function F_c has only real negative zeros. Now, it is known that a *necessary condition* for the entire function F_c to have only real zeros is that the γ_k's, defined in (3.3), satisfy the Turán inequalities (3.1) (see, for example, [1, p. 24], [8], [15], or [11, p. 337]). Thus, if $\gamma_k := \frac{k!\mu_k}{(2k)!}$, then in terms of the moments μ_k, we have the equivalent inequalities

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \quad \text{if and only if} \quad \mu_k^2 - \tau_k \mu_{k-1}\mu_{k+1} \geq 0, \tag{3.5}$$

where $\tau_k := (2k-1)/(2k+1)$ will be referred to as a *Turán constant* for the sequence of moments $\{\mu_k\}_{k=0}^\infty$. The fundamental open problem we are concerned with may be stated as follows.

Open Problem 3.3. Let $K(t)$ be an admissible kernel with moments μ_k (see (3.4)). If $K(t)$ is log-concave on $[0,\infty)$, then, is the sequence $\{\gamma_k\}_{k=0}^\infty$ log-concave? In other words, does the sequence of moments, $\{\mu_k\}_{k=0}^\infty$, satisfy the Turán inequalities (3.5) with Turán constant τ_k?

The following example shows that the moments of a well-behaved, log-concave kernel, which however is not an admissible kernel, need *not* satisfy the Turán inequalities (3.5) with Turán constant τ_k.

Example 3.4. Let $K(t) := t\exp(-t^{3/2})$, $t \geq 0$. Then $(K'(t))^2 - K(t)K''(t) \geq (1/4)(4 + 3t^{3/2})\exp(-2t^{3/2})$ for $t \geq 0$ and hence $K(t)$ is log-concave on $[0,\infty)$. Now, calculations of the moments, in terms of the gamma function, Γ, show that

$$\mu_1 = \frac{2}{3}\Gamma\left(\frac{8}{3}\right), \quad \mu_2 = 4, \quad \text{and} \quad \mu_3 = \frac{2}{3}\Gamma\left(\frac{16}{3}\right).$$

Hence, $\mu_2^2 - \dfrac{((2)(2)-1)}{(2)(2)+1}\mu_1\mu_3 = -0.1005\ldots$; that is, the Turán inequality (with Turán constant τ_2 in case $k=2$) fails to hold.

Remark 3.5. The following elementary observation will be useful in the sequel. If a sequence $\{c_k\}_{k=0}^\infty$, of positive numbers is log-concave, then the sequence $\{c_{2k}\}_{k=0}^\infty$ is also log-concave. Indeed, by assumption (cf. Definition 3.1), $c_k^2 - c_{k-1}c_{k+1} \geq 0$ for all $k \geq 1$. Hence, in particular, $c_{2k-1} \geq \sqrt{c_{2k-2}c_{2k}}$ and $c_{2k+1} \geq \sqrt{c_{2k+2}c_{2k}}$. Thus, $c_{2k}^2 \geq c_{2k-1}c_{2k+1} \geq \sqrt{c_{2k-2}}\sqrt{c_{2k+2}}c_{2k}$ and whence , $c_{2k} \geq \sqrt{c_{2k-2}}\sqrt{c_{2k+2}}$. That is, $\{c_{2k}\}_{k=0}^\infty$ is also log-concave.

Although at present, we are unable to provide an affirmative answer to Open Problem 3.3, nonetheless, we are able to deduce a somewhat weaker result, as the following theorem shows.

Theorem 3.6. *Let $K(t)$ be an admissible, log-concave kernel $(t \geq 0)$ with moments*

$$\mu_k = \int_0^\infty t^{2k} K(t)\, dt, \qquad k = 0, 1, 2, \ldots. \tag{3.6}$$

Then the moments satisfy the following Turán-type inequalities

$$\mu_n^2 \geq \frac{n}{(n+1)}\left(\frac{2n-1}{2n+1}\right)\mu_{n-1}\mu_{n+1}, \qquad n = 1, 2, 3, \ldots. \tag{3.7}$$

Proof. In case of the "normalized" moments

$$\lambda_x := \frac{1}{\Gamma(x+1)}\int_0^\infty t^x\, K(t)\, dt, \qquad (x > -1), \tag{3.8}$$

it is known that if $K(t)$ is log-concave on $[0, \infty)$, then $\log \lambda_x$ is concave for $x \geq 0$ [12, p. 77 and p. 494]. By Remark 3.5, the sequence $\{\lambda_{2k}\}_{k=0}^\infty$ is also log-concave; that is, $\lambda_{2n}^2 \geq \lambda_{2n-2}\lambda_{2n+2}$. Then, rewriting this inequality in terms of (3.8) and (3.6), we obtain the required Turán-type inequalities (3.7). □

An alternate proof of Theorem 3.6 (at least for $n \geq 2$ in (3.7)) can be based on the *reverse Lyaponov inequality* (see, for example, [2, p. 7] or [3, p. 384]):

$$\lambda_b^{a-c} \geq \lambda_c^{a-b}\lambda_a^{b-c}, \quad a \geq b \geq c \geq 1, c \in \mathbb{N} \text{ (Reverse Lyaponov Inequality).} \tag{3.9}$$

Indeed, if we set $a = 2n+2, b = 2n$ and $c = 2n-2, n \geq 2$, then we infer that the Turán-type inequalities (3.6) hold for $n \geq 2$ (on account of the restriction that $c \geq 1$).

As a consequence of Theorem 2.4 and Theorem 3.6, we obtain the following corollary.

Corollary 3.7. *Let*

$$b_k := \int_0^\infty t^{2k}\, \Phi(t)\, dt, \quad k = 0, 1, 2, \ldots,$$

denote the moments of the Jacobi theta function Φ. *Then the following Turán-type inequalities hold:*

$$b_n^2 \geq \frac{n}{(n+1)}\left(\frac{2n-1}{2n+1}\right) b_{n-1}b_{n+1}, \qquad n = 1, 2, 3, \ldots. \tag{3.10}$$

In conclusion, we remark that in [6, pp. 539-540] it is reported that the moments b_k, $(k = 0, 1, \ldots, 20)$ have been calculated to 110 digits of precision. In fact, the calculated values of the moments are available for $k = 0, 1, \ldots, 500$. Using these numerical values, we find that

$$\Delta_k := b_k^2 - \frac{k}{k+1} b_{k-1}b_{k+1} > 0 \qquad \text{for } k = 3, 4, \ldots, 499; \tag{3.11}$$

but that, curiously, $\Delta_k < 0$ for $k = 1, 2$. The empirical evidence (3.11) suggests that for $k \geq 3$ stronger Turán inequalities hold. The asymptotic estimates and techniques in [5], Corollary 3 (see also [4]) could shed light on this interesting phenomenon, at least for large k.

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