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A POSTERIORI ERROR CONTROL OF DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC OBSTACLE PROBLEMS

THIRUPATHI GUDI AND KAMANA PORWAL

Abstract. In this article, we derive an *a posteriori* error estimator for various discontinuous Galerkin (DG) methods that are proposed in (Wang, Han and Cheng, SIAM J. Numer. Anal., 48:708–733, 2010) for an elliptic obstacle problem. Using a key property of DG methods, we perform the analysis in a general framework. The error estimator we have obtained for DG methods is comparable with the estimator for the conforming Galerkin (CG) finite element method. In the analysis, we construct a non-linear smoothing function mapping DG finite element space to CG finite element space and use it as a key tool. The error estimator consists of a discrete Lagrange multiplier associated with the obstacle constraint. It is shown for non-over-penalized DG methods that the discrete Lagrange multiplier is uniformly stable on non-uniform meshes. Finally, numerical results demonstrating the performance of the error estimator are presented.

1. Introduction

Variational inequalities form an important class of non-linear problems for which the obstacle problem may be considered as a prototype model. The convergence of finite element methods for the obstacle problem has been known for many years [27, 14, 37, 24]. *A priori* error estimates for conforming linear and quadratic finite element methods are derived in [27] and [14, 37], respectively. For general convergence analysis and error estimates, we refer to [24]. On the other hand, *a posteriori* error analysis of conforming finite element methods for obstacle problems has received attention during the past decade [19, 34, 6, 8, 31]. In [19], the first *a posteriori* error estimator is derived using a positivity preserving interpolation operator. The *a posteriori* error analysis in [34] is derived without using such a positivity preserving interpolation operator. The estimates in [19] and [34] are slightly different from each other, but it is shown therein that both the estimators are reliable and efficient. In [6], error estimates based on averaging techniques are derived for the conforming finite element method. Error estimates for conforming finite element methods in an abstract framework can be found in [8, 31] when the obstacle is a $P_1(\Omega)$ function (affine obstacle), where $P_1(\Omega)$ is the space of linear

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polynomials restricted to Ω. Recently, convergence of an adaptive conforming finite element method for the obstacle problem has been studied in [17].

During the past decade and a half, discontinuous Galerkin methods have become attractive for the numerical approximation of a variety of problems [3]. In this article, we are concerned with the application of DG methods for variational inequalities. Recently in [38], *a priori* error analysis of various discontinuous Galerkin (DG) methods has been derived for elliptic variational inequalities of the first and the second kinds. In this article, we derive *a posteriori* error estimates for various DG finite element methods for the second order elliptic obstacle problem.

The model problem we consider is the following: Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. Let the obstacle $\chi \in H^1(\Omega) \cap C^0(\overline{\Omega})$ be such that $\chi \leq 0$ on $\partial\Omega$. Define the closed and convex set $\mathcal{K}$ by

$$\mathcal{K} := \{v \in H_0^1(\Omega) : v \geq \chi \text{ a.e. in } \Omega\}.$$

The model problem is to find $u \in \mathcal{K}$ such that

$$a(u, v-u) \geq (f, v-u) \quad \forall v \in \mathcal{K}, \tag{1.1}$$

where

$$a(w, v) = \int_\Omega \nabla w \cdot \nabla v \, dx \tag{1.2}$$

and $(\cdot,\cdot)$ is the $L_2(\Omega)$ inner-product. We denote the $L_2(\Omega)$ norm by $\|\cdot\|$.

From the theory of elliptic variational inequalities [30, 24, 4], the model problem (1.1) has a unique solution. Let $\chi^+ = (\chi + |\chi|)/2$. Then $\chi^+ \in \mathcal{K}$ and the following estimate can easily be established by taking $v = \chi^+$ in (1.1) and using the Poincaré inequality:

$$\|u\|_{H^1(\Omega)} \leq C \left(\|f\| + \|\nabla \chi^+\|\right). \tag{1.3}$$

Define $\sigma(u) \in H^{-1}(\Omega)$ by

$$\langle \sigma(u), v\rangle = (f, v) - a(u, v), \quad \forall v \in H_0^1(\Omega), \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pair of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The norm $\|\cdot\|_{-1}$ on $H^{-1}(\Omega)$ is defined by

$$\|g\|_{-1} = \sup_{v \in H_0^1(\Omega), v \neq 0} \frac{\langle g, v\rangle}{\|\nabla v\|}.$$

From (1.4) and (1.1), it is easy to deduce that

$$\langle \sigma(u), v-u\rangle \leq 0, \quad \forall v \in \mathcal{K}. \tag{1.5}$$

Furthermore, the following stability bound is immediate from (1.3), (1.4) and the Poincaré inequality:

$$\|\sigma(u)\|_{H^{-1}(\Omega)} \leq C \left(\|f\| + \|\nabla \chi^+\|\right).$$

In [38], *a priori* error estimates for various DG methods have been derived for elliptic variational inequalities of the first and the second kinds. Therein, the authors have considered piecewise linear and quadratic discontinuous finite element spaces. In this article, we derive *a posteriori* error estimates for various DG methods and obtain the results in a unified framework. To this end, a key property of DG methods is identified (see (4.4) below). With the help of a non-linear smoothing function which we introduce later in the paper, we have derived the error estimator that is comparable with the estimator for conforming finite element methods in [34] and

[31]. As in the case of conforming finite element methods, the error estimator for DG methods involves a discrete Lagrange multiplier ($\sigma_h(u_h)$, an approximation to $\sigma(u)$). We have shown for non-over-penalized DG methods that the discrete Lagrange multiplier is stable uniformly on non-uniform meshes (Lemma 6.2 below). In the case of the conforming finite element method, this type of stability estimate is possible for the discrete Lagrange multiplier [6, Lemma 8].

The rest of the article is organized as follows. In Section 2, we present notation and preliminaries. In Section 3, we introduce a smoothing function which satisfies some constraints that are useful in our *a posteriori* error analysis. In Section 4, we formulate the DG methods in an abstract form with two key properties. Section 5 is devoted to the *a posteriori* error analysis. Applications of the underlying results to various DG methods are discussed in Section 6. Therein, stability estimates for the solution and the discrete Lagrange multiplier associated with the obstacle constraint are also derived. In Section 7, numerical results demonstrating the performance of the error estimator are presented. Finally, conclusions and future problems are discussed in Section 8.

2. Preliminaries

The following notation will be used throughout the article:

$$\begin{aligned}
\mathcal{T}_h &= \text{a regular simplicial triangulation of } \Omega \\
T &= \text{a triangle of } \mathcal{T}_h \\
h_T &= \text{diameter of } T \\
h &= \max\{h_T : T \in \mathcal{T}_h\} \\
\mathcal{V}_h^i &= \text{a set of all vertices in } \mathcal{T}_h \text{ that are in } \Omega \\
\mathcal{V}_h^b &= \text{a set of all vertices in } \mathcal{T}_h \text{ that are on } \partial\Omega \\
\mathcal{V}_h &= \mathcal{V}^i \cup \mathcal{V}^b \\
\mathcal{V}_T &= \text{a set of three vertices of } T \\
\mathcal{E}_h^i &= \text{a set of all interior edges of } \mathcal{T}_h \\
\mathcal{E}_h^b &= \text{a set of all boundary edges of } \mathcal{T}_h \\
\mathcal{E}_h &= \mathcal{E}_h^i \cup \mathcal{E}_h^b \\
h_e &= \text{length of an edge } e \in \mathcal{E}_h \\
\mathcal{T}_p &= \text{a set of all triangles sharing the vertex } p \\
\mathcal{T}_e &= \text{patch of two triangles sharing the edge } e \\
\mathcal{E}_p &= \text{a set of all edges connected to the vertex } p \\
\nabla_h &= \text{piecewise (element-wise) gradient.}
\end{aligned}$$

We assume that the triangulation $\mathcal{T}_h$ is regular by means that there are no hanging nodes in $\mathcal{T}_h$ and the triangles in $\mathcal{T}_h$ are shape-regular. As a consequence of this, the cardinality of $\mathcal{E}_p$ is uniformly bounded for all $p \in \mathcal{V}_h$ and for all $h > 0$. Furthermore, the triangles in $\mathcal{T}_h$ are assumed to be closed.

Define a broken Sobolev space

$$H^1(\Omega, \mathcal{T}_h) = \{v \in L_2(\Omega) : \, v_T = v|_T \in H^1(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

For the DG methods, we are required to define the jump and mean of discontinuous functions. For any $e \in \mathcal{E}_h^i$, there are two triangles T_+ and T_- such that $e = \partial T_+ \cap \partial T_-$. Let n_- be the unit normal of e pointing from T_- to T_+, and $n_+ = -n_-$. (cf. Figure 2.1). For any $v \in H^1(\Omega, \mathcal{T}_h)$, we define the jump and mean of v on e by

$$[\![v]\!] = v_- n_- + v_+ n_+, \text{ and } \{\!\{v\}\!\} = \frac{1}{2}(v_- + v_+), \text{ respectively,}$$

where $v_\pm = v\big|_{T_\pm}$. Similarly define for $w \in H^1(\Omega, \mathcal{T}_h)^2$ the jump and mean of w on $e \in \mathcal{E}_h^i$ by

$$[\![w]\!] = w_- \cdot n_- + w_+ \cdot n_+, \text{ and } \{\!\{w\}\!\} = \frac{1}{2}(w_- + w_+), \text{ respectively,}$$

where $w_\pm = w|_{T_\pm}$.

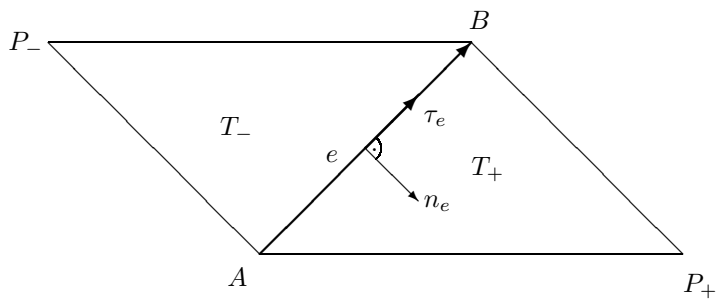

FIGURE 2.1. Two neighboring triangles T_- and T_+ that share the edge $e = \partial T_- \cap \partial T_+$ with initial node A and end node B and unit normal n_e. The orientation of $n_e = n_- = -n_+$ equals the outer normal of T_- and, hence, points into T_+.

For any edge $e \in \mathcal{E}_h^b$, there is a triangle $T \in \mathcal{T}_h$ such that $e = \partial T \cap \partial\Omega$. Let n_e be the unit normal of e that points outside T. For any $v \in H^1(T)$, we set on $e \in \mathcal{E}_h^b$

$$[\![v]\!] = v n_e \text{ and } \{\!\{v\}\!\} = v,$$

and for $w \in H^1(T)^2$,

$$[\![w]\!] = w \cdot n_e, \text{ and } \{\!\{w\}\!\} = w.$$

The discontinuous finite element space is defined by

$$V_h = \{v \in L_2(\Omega) : v|_T \in P_1(T),\, T \in \mathcal{T}_h\},$$

where $v|_T$ denotes the restriction of v to T and $P_1(T)$ is the space of polynomials of degree less than or equal to one defined on T. Subsequently, we also require the conforming finite element subspace $V_c = V_h \cap H_0^1(\Omega)$. Define the discrete analogue of $\mathcal{K}$ by

$$\mathcal{K}_h := \{v_h \in V_h : v_h|_T(p) \geq \chi_h(p),\ \ \forall p \in \mathcal{V}_T,\ \ \forall T \in \mathcal{T}_h\},$$

where $\chi_h \in V_c$ is the nodal interpolation of χ.

Throughout the article, we denote by C a generic positive constant that is independent of the mesh parameter h.

The following approximation result will be useful in our subsequent analysis [12]:

Lemma 2.1. *Let $\psi \in H_0^1(\Omega)$. Then, there is $\Pi_h\psi \in V_h \cap H_0^1(\Omega)$ such that on any $T \in \mathcal{T}_h$,*

$$\|\psi - \Pi_h\psi\|_{H^l(T)} \le Ch_T^{1-l}\|\psi\|_{H^1(\mathcal{T}_T)}, \quad l = 0, 1, \tag{2.1}$$

where $\mathcal{T}_T = \{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$.

We also require the following trace inequality [12]:

Lemma 2.2. *Let $\psi \in H^1(T)$ for $T \in \mathcal{T}_h$ and let $e \in \mathcal{E}_h$ be an edge of T. Then, it holds that*

$$\|\psi\|_{L_2(e)} \le C\big(h_e^{-1/2}\|\psi\|_{L_2(T)} + h_e^{l/2}\|\nabla\psi\|_{L_2(T)}\big). \tag{2.2}$$

Finally, we recall the following inverse inequality on V_h [12, 20]:

Lemma 2.3. *For $v_h \in V_h$,*

$$\|v_h\|_{L_2(e)} \le Ch_e^{-1/2}\|v_h\|_{L_2(T)} \ \forall T \in \mathcal{T}_h, \tag{2.3}$$

$$\|\nabla v_h\|_{L_2(T)} \le Ch_T^{-1}\|v_h\|_{L_2(T)} \ \forall T \in \mathcal{T}_h, \tag{2.4}$$

where e is an edge of T.

3. A smoothing function

In this section, we construct a smoothing map $E_h : V_h \to V_c$ with some useful properties. Our construction is motivated by the construction of an enriching operator introduced by Brenner [9, 10]. Let $v_h \in V_h$ and define $E_hv_h \in V_c$ by the following:

For a boundary vertex $p \in \mathcal{V}_h^b$, we take $E_hv_h(p) = 0$. For an interior vertex $p \in \mathcal{V}_h^i$, we define

$$E_hv_h(p) = \min\{v_h|_T(p) : T \in \mathcal{T}_p\}. \tag{3.1}$$

It is clear that $E_hv_h \in V_c$ for all $v_h \in V_h$.

Lemma 3.1. *It holds that*

$$v_h|_T(p) \ge E_hv_h(p) \quad \forall T \in \mathcal{T}_p, \quad \forall p \in \mathcal{V}_h^i, \quad \forall v_h \in V_h \tag{3.2}$$

and

$$E_hv_h(p) \ge \chi_h(p) \quad \forall v_h \in \mathcal{K}_h, \quad \forall p \in \mathcal{V}_h. \tag{3.3}$$

In particular, $E_hv_h \in \mathcal{K}_h \cap V_c$ for all $v_h \in \mathcal{K}_h$.

Proof. Let $v_h \in V_h$ and $T \in \mathcal{T}_h$. Then for any vertex $p \in \mathcal{V}_T \cap \mathcal{V}_h^i$,

$$v_h|_T(p) \ge \min\{v|_{T'}(p) : T' \in \mathcal{T}_p\} = E_hv_h(p),$$

which proves (3.2). But for $v_h \in \mathcal{K}_h$, there is some $T_* \in \mathcal{T}_p$ (see Figure 3.1) such that

$$E_hv_h(p) = \min\{v|_{T'}(p) : T' \in \mathcal{T}_p\} = v_h|_{T_*}(p) \ge \chi_h(p).$$

Since $\chi_h(p) \le 0$ for any boundary vertex $p \in \mathcal{V}_h^b$, we have $E_hv_h(p) = 0 \ge \chi_h(p)$. This proves (3.3). □

Remark 3.2. For any $T \in \mathcal{T}_h$ with $T \cap \partial\Omega = \emptyset$, we find from (3.2) that

$$(E_hv_h - v_h)(x) \le 0 \quad \forall x \in T.$$

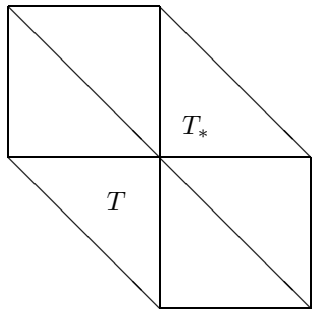

FIGURE 3.1. Set of triangles sharing a vertex

Lemma 3.3. *Let $v \in V_h$. It holds that*

$$\sum_{T\in\mathcal{T}_h} h_T^{-2}\|E_h v - v\|^2_{L_2(T)} \le C\left(\sum_{e\in\mathcal{E}_h}\int_e \frac{1}{h_e}[\![v]\!]^2 ds\right) \quad \forall\, v \in V_h \tag{3.4}$$

and

$$\sum_{T\in\mathcal{T}_h} \|\nabla(E_h v - v)\|^2_{L_2(T)} \le C\left(\sum_{e\in\mathcal{E}_h}\int_e \frac{1}{h_e}[\![v]\!]^2 ds\right) \quad \forall\, v \in V_h. \tag{3.5}$$

Proof. Let $T \in \mathcal{T}_h$. Using scaling,

$$h_T^{-2}\|E_h v - v\|^2_{L_2(T)} \le C\sum_{p\in\mathcal{V}_T}(E_h v - v)|^2_T(p).$$

Let $p \in \mathcal{V}_T \cap \mathcal{V}_h^i$. By the definition of E_h, there is some $T_* \in \mathcal{T}_p$ (see Figure 3.1) such that $E_h v(p) = v|_{T_*}(p)$. Then

$$(E_h v - v)|_T(p) = v|_{T_*}(p) - v|_T(p).$$

Further, there are $T_i \in \mathcal{T}_p$ for $i = 1, \cdots, n$ (for some $n \ge 1$) such that $T_1 = T$ and $T_n = T_*$. Therefore,

$$\begin{aligned} v|_{T_*}(p) - v|_T(p) = {} & v|_{T_*}(p) - v|_{T_{n-1}}(p) + v|_{T_{n-1}}(p) - v|_{T_{n-2}}(p) + v|_{T_{n-2}}(p) \\ & + \cdots + v|_{T_3}(p) - v|_{T_2}(p) + v|_{T_2}(p) - v|_T(p). \end{aligned}$$

Using the following inverse inequality [12, 20]:

$$\|v\|_{L_\infty(e)} \le C h_e^{-1/2}\|v\|_{L_2(e)}, \tag{3.6}$$

we find

$$|v|_{T_*}(p) - v|_T(p)|^2 \le C\sum_{e\in\mathcal{E}_p}\|[\![v]\!]\|^2_{L_\infty(e)} \le C\sum_{e\in\mathcal{E}_p}\frac{1}{h_e}\|[\![v]\!]\|^2_{L_2(e)}. \tag{3.7}$$

For $p \in \mathcal{V}_T \cap \mathcal{V}_h^b$, we have $E_h v(p) = 0$ and

$$(E_h v - v)|_T(p) = -v|_T(p).$$

Similar to the previous case, there are $T_i \in \mathcal{T}_p$ for $i = 1, \cdots, m$ (for some $m \ge 1$) such that $T_1 = T$ and $T_m = T_*$, where T_* is such that $\partial T_* \cap \partial\Omega$ is an edge in $\mathcal{E}_h^b$.

Therefore,

$$\begin{aligned}-v|_T(p) = -v|_{T_*} + v|_{T_*}(p) - v|_{T_{m-1}}(p) + v|_{T_{m-1}}(p) - v|_{T_{m-2}}(p) + v|_{T_{m-2}}(p)\\ + \cdots + v|_{T_3}(p) - v|_{T_2}(p) + v|_{T_2}(p) - v|_T(p).\end{aligned}$$

Using the inverse inequality (3.6), we find

$$|(E_h v - v)|_T(p)|^2 \le \|v\|^2_{L_\infty(e)} \le C \sum_{e\in\mathcal{E}_p} \frac{1}{h_e} \|[\![v]\!]\|^2_{L_2(e)}. \tag{3.8}$$

From (3.7) and (3.8),

$$h_T^{-2}\|E_h v - v\|^2_{L_2(T)} \le C \sum_{p\in\mathcal{V}_T} \sum_{e\in\mathcal{E}_p} \frac{1}{h_e} \|[\![v]\!]\|^2_{L_2(e)}.$$

Sum over all $T \in \mathcal{T}_h$ and obtain

$$\sum_{T\in\mathcal{T}_h} h_T^{-2}\|E_h v - v\|^2_{L_2(T)} \le C \sum_{e\in\mathcal{E}_h} \frac{1}{h_e} \|[\![v]\!]\|^2_{L_2(e)},$$

where we have used the assumption that the cardinality of $\mathcal{E}_p$ is uniformly bounded. This proves (3.4). Using the inverse inequality (2.4), we obtain (3.5). □

4. Discontinuous Galerkin methods

In this section, we introduce the DG methods. Define

$$\mathcal{A}_h(w,v) = a_h(w,v) + b_h(w,v), \tag{4.1}$$

where

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \; dx \tag{4.2}$$

and b_h is a bilinear form that consists of all consistency and stability terms.

The discontinuous Galerkin approximation $u_h \in \mathcal{K}_h$ is the solution of

$$\mathcal{A}_h(u_h, v_h - u_h) \ge (f, v_h - u_h) \quad \forall v_h \in \mathcal{K}_h. \tag{4.3}$$

We assume the following:

(A1) The discrete problem (4.3) has a unique solution.

(A2) For $w_h \in V_h$ and $v_h \in V_c$, it holds that

$$|\, b_h(w_h, v_h)| \le C \left(\sum_{e\in\mathcal{E}_h} \int_e \frac{1}{h_e} [\![w_h]\!]^2 \, ds \right)^{1/2} \|\nabla v_h\|. \tag{4.4}$$

Some choices of $b_h(\cdot,\cdot)$ for various DG methods satisfying (A2) are listed in Section 6. Moreover, if the bilinear form $\mathcal{A}_h$ is coercive and bounded with respect to some norm on V_h, then the discrete problem (4.3) has a unique solution [38].

For the rest of the analysis, we define $\sigma_h(u_h) \in V_h$ by

$$\langle \sigma_h(u_h), v_h\rangle_h = (f, v_h) - \mathcal{A}_h(u_h, v_h) \quad \forall v_h \in V_h, \tag{4.5}$$

where $\langle\cdot,\cdot\rangle_h$ is defined by

$$\langle w_h, v_h\rangle_h = \sum_{T\in\mathcal{T}_h} \frac{|T|}{3} \sum_{p\in\mathcal{V}_T} w_h|_T(p) v_h|_T(p),$$

and $|T|$ is the area of T.

It is easy to verify that

$$C_1\|v_h\|^2_{L_2(\Omega)} \le \langle v_h, v_h\rangle_h \le C_2\|v_h\|^2_{L_2(\Omega)}, \tag{4.6}$$

for two positive constants C_1 and C_2 independent of h.

Below we denote the restriction of $\sigma_h(u_h)$ on $T \in \mathcal{T}_h$ by $\sigma_h|_T(u_h)$. Also we denote by $\sigma_h|_T(u_h;p)$ the nodal value of $\sigma_h|_T(u_h)$ at node p of T. The following error estimate of the lumped mass quadrature is well known [33, Lemma 15.1], [34].

Lemma 4.1. *It holds that*

$$\left|\int_T \sigma_h(u_h)v_h\,dx - \frac{|T|}{3}\sum_{p\in\mathcal{V}_T}\sigma_h|_T(u_h;p)v_h|_T(p)\right| \le Ch_T^2\|\nabla\sigma_h(u_h)\|_{L_2(T)}\,\|\nabla v_h\|_{L_2(T)}.$$

We now derive some properties of $\sigma_h(u_h)$. Let $T \in \mathcal{T}_h$ and λ_T^p be the barycentric coordinate of T associated with the node $p \in \mathcal{V}_T$. Define $\phi_h \in V_h$ by

$$\phi_h := \begin{cases} \lambda_T^p & \text{on } T, \\ 0 & \text{otherwise.} \end{cases}$$

We find by taking $v_h = u_h + \phi_h$ in (4.3) that

$$\sigma_h|_T(u_h;p) \le 0 \quad \text{for all } p \in \mathcal{V}_T. \tag{4.7}$$

From (4.7), we note that $\sigma_h|_T(u_h) \le 0$ for every $T \in \mathcal{T}_h$. Suppose for a node $p \in \mathcal{V}_T$ that $u_h|_T(p) > \chi_h(p)$. Then for sufficiently small $\delta > 0$, we substitute $v_h = u_h - \delta\phi_h$ in (4.3) and obtain using (4.7) that $\sigma_h|_T(u_h;p) = 0$. This implies

$$\sigma_h|_T(u_h;p) = 0 \quad \text{for all } p \in \mathcal{V}_h \text{ such that } u_h|_T(p) > \chi_h(p). \tag{4.8}$$

Since $\sigma_h(u_h) \in V_h \subset L_2(\Omega)$, it defines a continuous linear functional $F_{\sigma_h(u_h)} \in H^{-1}(\Omega)$ by

$$\langle F_{\sigma_h(u_h)}, v\rangle = \int_\Omega \sigma_h(u_h)v\,dx, \quad \forall v \in H_0^1(\Omega). \tag{4.9}$$

5. Reliable and efficient a posteriori error estimate

In this section, we derive *a posteriori* error estimates for the DG methods. We define a Galerkin functional $G_h \in H^{-1}(\Omega)$ by

$$\langle G_h, v\rangle = a_h(u_h - u, v) + \langle F_{\sigma_h(u_h)} - \sigma(u), v\rangle \quad \forall v \in H_0^1(\Omega), \tag{5.1}$$

where a_h is defined by (4.2). The functional G_h may be treated as residual.

Define the following estimators:

$$\eta_1 = \left(\sum_{T\in\mathcal{T}_h} h_T^2\|f - \sigma_h(u_h)\|^2_{L_2(T)}\right)^{1/2}, \qquad \eta_2 = \left(\sum_{e\in\mathcal{E}_h^i} h_e\|[\![\nabla u_h]\!]\|^2_{L_2(e)}\right)^{1/2},$$

$$\eta_3 = \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![u_h]\!]\|^2_{L_2(e)}\right)^{1/2} \quad \text{and} \quad \eta_4 = \left(\sum_{T\in\mathcal{T}_h} h_T^4\|\nabla\sigma_h(u_h)\|^2_{L_2(T)}\right)^{1/2}.$$

Using the residual functional G_h, we derive below an estimate for the error.

Lemma 5.1. *It holds that*

$$\|\nabla_h(u-u_h)\|^2 + \|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1}^2 \\ \leq C\left(\|G_h\|_{-1}^2 + \eta_3^2\right) - 2\langle F_{\sigma_h(u_h)} - \sigma(u), E_h u_h - u\rangle.$$

Proof. Let $\tilde{u}_h = E_h u_h$. Then using (5.1),

$$a_h(u_h - u, \tilde{u}_h - u) = \langle G_h, \tilde{u}_h - u\rangle - \langle F_{\sigma_h(u_h)} - \sigma(u), \tilde{u}_h - u\rangle.$$

Therefore,

$$a_h(\tilde{u}_h - u, \tilde{u}_h - u) = \langle G_h, \tilde{u}_h - u\rangle - \langle F_{\sigma_h(u_h)} - \sigma(u), \tilde{u}_h - u\rangle \\ + a_h(\tilde{u}_h - u_h, \tilde{u}_h - u)$$

and

$$\|\nabla(\tilde{u}_h - u)\|^2 \leq \left(\|G_h\|_{-1} + \|\nabla_h(\tilde{u}_h - u_h)\|\right)\|\nabla(\tilde{u}_h - u)\| \\ - \langle F_{\sigma_h(u_h)} - \sigma(u), \tilde{u}_h - u\rangle.$$

Young's inequality $ab \leq a^2/2 + b^2/2$, triangle inequality and (3.5) complete the estimate for $\|\nabla_h(u-u_h)\|$. To find the estimate for $\|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1}$, we note that

$$\langle F_{\sigma_h(u_h)} - \sigma(u), v\rangle = \langle G_h, v\rangle - a_h(u_h - u, v). \tag{5.2}$$

Using triangle inequality, (3.5) and the estimate for $\|\nabla_h(u-u_h)\|$, we complete the proof. □

In the following lemma, we derive an estimate for the residual G_h.

Lemma 5.2. *Assume that* (A1)-(A2) *hold true. Then, it holds that*

$$\|G_h\|_{-1} \leq C\left(\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2\right)^{1/2}.$$

Proof. Let $v \in H_0^1(\Omega)$ and $v_h \in V_c$ be the approximation of v as in Lemma 2.1. Then

$$\langle G_h, v\rangle = \langle G_h, v - v_h\rangle + \langle G_h, v_h\rangle. \tag{5.3}$$

First, using (5.1), (4.9), (1.4), (4.1) and (4.5), we find

$$\begin{aligned}\langle G_h, v_h\rangle &= a_h(u_h - u, v_h) + \langle F_{\sigma_h(u_h)} - \sigma(u), v_h\rangle \\ &= a_h(u_h, v_h) + (\sigma_h(u_h), v_h) - a(u, v_h) - \langle \sigma(u), v_h\rangle \\ &= (\sigma_h(u_h), v_h) - \langle \sigma_h(u_h), v_h\rangle_h - b_h(u_h, v_h).\end{aligned}$$

Using Lemma 4.1, (4.4) and Lemma 2.1, we find

$$\langle G_h, v_h\rangle \leq C\left(\eta_4^2 + \eta_3^2\right)^{1/2}\|\nabla v_h\| \leq C\left(\eta_4^2 + \eta_3^2\right)^{1/2}\|\nabla v\|.$$

Second, using (5.1), (4.9), (4.5), (4.2), integration by parts, Lemma 2.2 and Lemma 2.1, we find

$$\begin{aligned}\langle G_h, v - v_h\rangle &= a_h(u_h - u, v - v_h) + \langle F_{\sigma_h(u_h)} - \sigma(u), v - v_h\rangle \\ &= a_h(u_h, v - v_h) + (\sigma_h(u_h), v - v_h) - (f, v - v_h) \\ &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial u_h|_T}{\partial n_T}(v - v_h)\,ds + (\sigma_h(u_h), v - v_h) - (f, v - v_h) \\ &= \sum_{e\in\mathcal{E}_h^i}\int_e [\![\nabla u_h]\!]\{\!\{v - v_h\}\!\}\,ds + \sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla u_h\}\!\}\cdot[\![v - v_h]\!]\,ds + (\sigma_h(u_h) - f, v - v_h) \\ &\leq C\left(\eta_2^2 + \eta_1^2\right)^{1/2}\|\nabla v\|.\end{aligned}$$

Here we have used the fact that $[\![v - v_h]\!] = 0$ on all $e \in \mathcal{E}_h$. This completes the proof. $\square$

Note that if we establish a computable lower bound for $\langle F_{\sigma_h(u_h)} - \sigma(u), E_h u_h - u\rangle$, then it is clear from Lemma 5.1 that an error estimate follows.

For this, we define the following sets:

$$\mathbb{F}_h = \{T \in \mathcal{T}_h : \exists\, p_1,\ p_2 \in \mathcal{V}_T \text{ such that } u_h|_T(p_1) = \chi_h(p_1) \text{ and } u_h|_T(p_2) > \chi_h(p_2)\},$$

$$\mathbb{C}_h = \{T \in \mathcal{T}_h : \text{For all } p \in \mathcal{V}_T,\ u_h|_T(p) = \chi_h(p)\},$$

and

$$\mathbb{N}_h = \{T \in \mathcal{T}_h : \text{For all } p \in \mathcal{V}_T,\ u_h|_T(p) > \chi_h(p)\}.$$

We call $\mathbb{F}_h$, $\mathbb{C}_h$ and $\mathbb{N}_h$ a free boundary set, a contact set and a non-contact set, respectively. We also define

$$\partial\mathbb{F}_h^i = \{e \in \mathcal{E}_h^i \cap \mathcal{E}_p : p \in \mathcal{V}_T \text{ with } u_h|_T(p) = \chi_h(p),\ T \in \mathbb{F}_h\}$$

and

$$\mathbb{C}_{\partial\Omega} = \{T \in \mathbb{C}_h : T \cap \partial\Omega \neq \emptyset,\ \chi_h(p) < 0 \text{ for at least one } p \in \mathcal{V}_T \cap \mathcal{V}_h^b\}.$$

Now we are ready to establish a lower bound for $\langle F_{\sigma_h(u_h)} - \sigma(u), E_h u_h - u\rangle$. In the proof of the following lemma, we will see that the smoothing function E_h in Section 3 plays an important role.

Lemma 5.3. *Let $\epsilon > 0$ be an arbitrary number. Then, it holds that*

$$\begin{aligned}
&\langle F_{\sigma_h(u_h)} - \sigma(u), E_h u_h - u\rangle \\
&\geq -\epsilon\|F_{\sigma_h(u_h)} - \sigma(u)\|_{-1}^2 - \epsilon\|\nabla_h(u - u_h)\|^2 - \frac{C}{\epsilon}\|\nabla(\chi - E_h u_h)^+\|^2 - \frac{C}{\epsilon}\big(\eta_3^2 + \eta_4^2\big) \\
&\quad - C\left(\sum_{e\in\partial\mathbb{F}_h^i}\int_e h_e[\![\nabla(u_h - \chi_h)]\!]^2\, ds\right) + \left(\sum_{T\in\mathbb{F}_h, T\cap\partial\Omega=\emptyset}\int_T \sigma_h(u_h)(\chi - \chi_h)^-\, dx\right) \\
&\quad + \left(\sum_{T\in\mathbb{C}_h}\int_T \sigma_h(u_h)(\chi - \chi_h)^-\, dx\right) + \left(\sum_{T\in\mathbb{C}_{\partial\Omega}}\int_T \sigma_h(u_h)(\tilde{u}_h - u_h)\, dx\right),
\end{aligned}$$

where $v^+ = (v + |v|)/2$ and $v^- = (|v| - v)/2$ for a given function v.

Proof. Let $\tilde{u}_h = E_h u_h$ and let $u_h^* = \max\{\tilde{u}_h, \chi\}$. Then $u_h^* \in \mathcal{K}$ and using (1.5),

$$\langle \sigma(u), u_h^* - u\rangle \leq 0.$$

First note that

$$\begin{aligned}
-\langle \sigma(u), \tilde{u}_h - u\rangle &= -\langle \sigma(u), \tilde{u}_h - u_h^*\rangle - \langle \sigma(u), u_h^* - u\rangle \\
&\geq \langle \sigma(u), u_h^* - \tilde{u}_h\rangle = \langle \sigma(u), (\chi - \tilde{u}_h)^+\rangle \\
&= \langle \sigma(u) - F_{\sigma_h(u_h)}, (\chi - \tilde{u}_h)^+\rangle + \langle F_{\sigma_h(u_h)}, (\chi - \tilde{u}_h)^+\rangle \\
&= \langle \sigma(u) - F_{\sigma_h(u_h)}, (\chi - \tilde{u}_h)^+\rangle + (\sigma_h(u_h), (\chi - \tilde{u}_h)^+) \\
&\geq (\sigma_h(u_h), (\chi - \tilde{u}_h)^+) - \epsilon\|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1} - \frac{1}{\epsilon}\|\nabla(\chi - \tilde{u}_h)^+\|,
\end{aligned}$$

where $\epsilon > 0$ is an arbitrary number. Therefore,

$$\langle F_{\sigma_h(u_h)} - \sigma(u), \tilde{u}_h - u\rangle \geq \big(\sigma_h(u_h), \big[\tilde{u}_h - u + (\chi - \tilde{u}_h)^+\big]\big) - \epsilon\|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1} - \frac{1}{\epsilon}\|\nabla(\chi - \tilde{u}_h)^+\|. \tag{5.4}$$

It is enough now to find a lower bound for

$$\big(\sigma_h(u_h), \big[\tilde{u}_h - u + (\chi - \tilde{u}_h)^+\big]\big) = \sum_{T\in\mathcal{T}_h}\int_T \sigma_h(u_h)\big[\tilde{u}_h - u + (\chi - \tilde{u}_h)^+\big]\,dx. \tag{5.5}$$

We split the sum on the right-hand side of (5.5) into four parts and consider the cases one by one.

Case 1 (*Non-contact set*, $\mathbb{N}_h$): Let $T \in \mathbb{N}_h$. Then $u_h > \chi_h$ on T. Using (4.8), we obtain $\sigma_h(u_h) \equiv 0$ on T and

$$\int_T \sigma_h(u_h)\big[\tilde{u}_h - u + (\chi - \tilde{u}_h)^+\big]\,dx = 0.$$

Case 2 (*Free boundary set near* $\partial\Omega$, $\mathbb{F}_h$): Let $T \in \mathbb{F}_h$ and $T \cap \partial\Omega \neq \emptyset$. Then since there is a node $p \in T$ such that $\sigma_h|_T(u_h; p) = 0$, we find by using the equivalence of norms on finite dimensional spaces and scaling that

$$\|\sigma_h(u_h)\|_{L_2(T)} \leq Ch_T\|\nabla\sigma_h(u_h)\|_{L_2(T)}. \tag{5.6}$$

If $T \cap \partial\Omega$ is an edge in $\mathcal{E}^\partial$, then since $\tilde{u}_h - u \in H_0^1(\Omega)$ and $(\chi - \tilde{u}_h)^+ \in H_0^1(\Omega)$, we find using Poincaré inequality that

$$\|\tilde{u}_h - u\|_{L_2(T)} \leq Ch_T\|\nabla(\tilde{u}_h - u)\|_{L_2(T)} \quad \text{and} \tag{5.7}$$

$$\|(\chi - \tilde{u}_h)^+\|_{L_2(T)} \leq Ch_T\|\nabla(\chi - \tilde{u}_h)^+\|_{L_2(T)}. \tag{5.8}$$

In this case if $T \cap \partial\Omega$ is a node, say p, in $\mathcal{V}_h^b$, then since $\Big(\Pi_h(\tilde{u}_h - u)\Big)(p) = 0$, we find as in (5.6) and using Lemma 2.1 that

$$\|\Pi_h(\tilde{u}_h - u)\|_{L_2(T)} \leq Ch_T\|\nabla\Pi_h(\tilde{u}_h - u)\|_{L_2(T)} \leq Ch_T\|\nabla(\tilde{u}_h - u)\|_{L_2(\mathcal{T}_T)},$$

where Π_h and $\mathcal{T}_T$ are defined as in Lemma 2.1. Again using Lemma 2.1,

$$\begin{aligned}\|\tilde{u}_h - u\|_{L_2(T)} &\leq \|(\tilde{u}_h - u) - \Pi_h(\tilde{u}_h - u)\|_{L_2(T)} + \|\Pi_h(\tilde{u}_h - u)\|_{L_2(T)}\\ &\leq Ch_T\|\nabla(\tilde{u}_h - u)\|_{L_2(\mathcal{T}_T)}.\end{aligned} \tag{5.9}$$

Similarly, we find

$$\|(\chi - \tilde{u}_h)^+\|_{L_2(T)} \leq Ch_T\|\nabla(\chi - \tilde{u}_h)^+\|_{L_2(\mathcal{T}_T)}. \tag{5.10}$$

Therefore using (5.6), (5.7)-(5.10), we find that

$$\int_T \sigma_h(u_h)\big[\tilde{u}_h - u + (\chi - \tilde{u}_h)^+\big]\,dx \geq -\epsilon\|\nabla(\tilde{u}_h - u)\|^2_{L_2(\mathcal{T}_T)} - \|\nabla(\chi - \tilde{u}_h)^+\|^2_{L_2(\mathcal{T}_T)} - \frac{C}{\epsilon}h_T^4\|\nabla\sigma_h(u_h)\|^2_{L_2(T)}. \tag{5.11}$$

For the other cases using $u \geq \chi$ a.e. in Ω and $\sigma_h(u_h) \leq 0$, we note that

$$\int_T \sigma_h(u_h)(\tilde{u}_h - u) = \int_T \sigma_h(u_h)(\tilde{u}_h - \chi) + \int_T \sigma_h(u_h)(\chi - u) \geq \int_T \sigma_h(u_h)(\tilde{u}_h - \chi).$$

Thus,

$$\int_T \sigma_h(u_h)\left[\tilde{u}_h - u + (\chi - \tilde{u}_h)^+\right] dx \geq \int_T \sigma_h(u_h)\left[\tilde{u}_h - \chi + (\chi - \tilde{u}_h)^+\right] dx$$
$$= \int_T \sigma_h(u_h)(\chi - \tilde{u}_h)^- \, dx,$$

and for the remaining cases we find a lower bound for this term.
Case 3 (*Free boundary set away from* $\partial\Omega$, $\mathbb{F}_h$): Let $T \in \mathbb{F}_h$ and $T \cap \partial\Omega = \emptyset$. Note that for two functions f and g with $g \geq 0$, we have the inequality

$$(f - g)^- \leq f^- + g.$$

Using this together with $\sigma_h(u_h) \leq 0$ and $\tilde{u}_h \geq \chi_h$,

$$\int_T \sigma_h(u_h)(\chi - \tilde{u}_h)^- \, dx = \int_T \sigma_h(u_h)\big((\chi - \chi_h) - (\tilde{u}_h - \chi_h)\big)^- \, dx$$
$$\geq \int_T \sigma_h(u_h)(\chi - \chi_h)^- \, dx + \int_T \sigma_h(u_h)(\tilde{u}_h - \chi_h) \, dx. \tag{5.12}$$

Since $T \in \mathbb{F}_h$, there is some $p \in \mathcal{V}_T$ such that $u_h|_T(p) = \chi_h(p)$. By the definition of $\tilde{u}_h$,

$$\tilde{u}_h(p) = \min\{u_h|_{T'}(p) : T' \in \mathcal{T}_p\} \geq \chi_h(p).$$

But since $u_h|_T(p) = \chi_h(p)$, we have $\tilde{u}_h|_T(p) = \chi_h(p)$ and also $\tilde{u}_h \geq \chi_h$. By [34, Lemma 3.6],

$$\|\tilde{u}_h - \chi_h\|_{L_2(T)} \leq Ch_T \left(\sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h^i} \int_e h_e [\![\nabla(\tilde{u}_h - \chi_h)]\!]^2 \, ds \right)^{1/2}.$$

Using inverse inequality (2.3) and (3.5), we find

$$\|\tilde{u}_h - \chi_h\|_{L_2(T)} \leq Ch_T \left(\sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h^i} \int_e h_e [\![\nabla(u_h - \chi_h)]\!]^2 \, ds \right)^{1/2}$$
$$+ \quad Ch_T \left(\sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h^i} \int_e \frac{1}{h_e} [\![u_h]\!]^2 \, ds \right)^{1/2}. \tag{5.13}$$

Using (5.6) and (5.13), we estimate the second term on the right-hand side of (5.12) as follows:

$$\Big| \int_T \sigma_h(u_h)(\tilde{u}_h - \chi_h) \, dx \Big| \leq \|\sigma_h(u_h)\|_{L_2(T)} \|\tilde{u}_h - \chi_h\|_{L_2(T)}$$
$$\leq Ch_T^2 \|\nabla \sigma_h(u_h)\|_{L_2(T)} \left(\sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h^i} \int_e h_e [\![\nabla(u_h - \chi_h)]\!]^2 \, ds \right)^{1/2}$$
$$+ Ch_T^2 \|\nabla \sigma_h(u_h)\|_{L_2(T)} \left(\sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h^i} \int_e \frac{1}{h_e} [\![u_h]\!]^2 \, ds \right)^{1/2},$$

which implies that

$$\int_T \sigma_h(u_h)(\tilde{u}_h - \chi_h)\,dx \geq -C\left(h_T^4\|\nabla\sigma_h(u_h)\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_p\cap\mathcal{E}_h^i}\int_e \frac{1}{h_e}[\![u_h]\!]^2\,ds\right)$$
$$- C\sum_{e\in\mathcal{E}_p\cap\mathcal{E}_h^i}\int_e h_e[\![\nabla(u_h-\chi_h)]\!]^2\,ds. \tag{5.14}$$

Case 4 (*Contact set*, $\mathbb{C}_h$): Let $T\in\mathbb{C}_h$. Then, we have $T\subset\{x\in\bar{\Omega} : u_h(x)=\chi_h(x)\}$, and using (5.12),

$$\int_T \sigma_h(u_h)(\chi-\tilde{u}_h)^-\,dx \geq \int_T \sigma_h(u_h)(\chi-\chi_h)^-\,dx + \int_T \sigma_h(u_h)(\tilde{u}_h-\chi_h)\,dx$$
$$= \int_T \sigma_h(u_h)(\chi-\chi_h)^-\,dx + \int_T \sigma_h(u_h)(\tilde{u}_h-u_h)\,dx. \tag{5.15}$$

Here we consider two cases. In the first case, let $T\subset\Omega$. Then using (3.2), we have $(\tilde{u}_h-u_h|_T)(p)\leq 0$ for all $p\in\mathcal{V}_T$ which implies $(\tilde{u}_h-u_h|_T)(x)\leq 0$ for all $x\in T$. Also, since $\sigma_h(u_h)\leq 0$, we have

$$\int_T \sigma_h(u_h)(\tilde{u}_h-u_h)\,dx \geq 0$$

and

$$\int_T \sigma_h(u_h)(\chi-\tilde{u}_h)^-\,dx \geq \int_T \sigma_h(u_h)(\chi-\chi_h)^-\,dx, \tag{5.16}$$

which is a part of the error estimator. In the second case, let $T\cap\partial\Omega\neq\emptyset$. Then there is at least one $p\in\mathcal{V}_T\cap\mathcal{V}_h^i$ for which using (3.2), we have $(\tilde{u}_h-u_h|_T)(p)\leq 0$. Also there is at least one $p\in\mathcal{V}_T\cap\mathcal{V}_h^b$. For all $p\in\mathcal{V}_T\cap\mathcal{V}_h^b$, if we have $\chi_h(p)=0$, then since T is in contact set, we have $u_h|_T(p)=0$ and $(\tilde{u}_h-u_h|_T)(p)=0$. This implies that we obtain again $\tilde{u}_h-u_h\leq 0$ on T and

$$\int_T \sigma_h(u_h)(\chi-\tilde{u}_h)^-\,dx \geq \int_T \sigma_h(u_h)(\chi-\chi_h)^-\,dx. \tag{5.17}$$

Suppose for some $p\in\mathcal{V}_T\cap\mathcal{V}_h^b$ that we have $\chi_h(p)<0$. Then $u_h|_T(p)=\chi_h(p)<0$. This will not happen if $u_h=0$ on the boundary, i.e., when u_h satisfies the homogeneous boundary condition exactly, the contact set is interior to Ω. But in the DG methods $u_h\neq 0$ on the boundary, we will retain the term on the right-hand side of (5.15) as part of the estimator on the set $\mathbb{C}_{\partial\Omega}$. Now combining (5.4), (5.11), (5.14), (5.15), (5.16) and (5.17), we complete the proof. □

Using Lemma 5.1, Lemma 5.2 and Lemma 5.3, we state and deduce the main result of the paper.

Theorem 5.4. *Assume that* (A1)-(A2) *hold true. Then, it holds that*

$$\|\nabla_h(u-u_h)\| + \|F_{\sigma_h(u_h)}-\sigma(u)\|_{-1} \leq C\left(\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2\right)^{1/2} + C\|\nabla(\chi-\tilde{u}_h)^+\|$$
$$+ C\left(\sum_{e\in\partial\mathbb{F}_h^i}\int_e h_e[\![\nabla(u_h-\chi_h)]\!]^2\,ds + \sum_{T\in\mathbb{F}_h, T\cap\partial\Omega=\emptyset}\int_T [-\sigma_h(u_h)]\,(\chi-\chi_h)^-\,dx\right)^{1/2}$$
$$+ C\left(\sum_{T\in\mathbb{C}_h}\int_T [-\sigma_h(u_h)]\,(\chi-\chi_h)^-\,dx + \sum_{T\in\mathbb{C}_{\partial\Omega}}\int_T [-\sigma_h(u_h)]\,(\tilde{u}_h-u_h)\,dx\right)^{1/2}.$$

Remark 5.5. For sufficiently small h (or on a sufficiently refined mesh), the set $\mathbb{C}_{\partial\Omega}$ may be empty since we can expect that the solution u_h becomes sufficiently small on the boundary, i.e. $0 \approx u_h > \chi_h$ on the boundary.

Remark 5.6. From Theorem 5.4 we note that when the solution $u_h \in V_c$ (in which case $\tilde{u}_h = u_h$), the error estimator coincides with the error estimator for the conforming finite element method [34].

Corollary 5.7. *Let $\chi \in P_1(\Omega)$. Assume that* (A1)-(A2) *hold true. Then, it holds that*

$$\|\nabla_h(u-u_h)\| + \|F_{\sigma_h(u_h)} - \sigma(u)\|_{-1} \le C\left(\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2\right)^{1/2} + C\left(\sum_{T\in\mathbb{C}_{\partial\Omega}} \int_T \left[-\sigma_h(u_h)\right](\tilde{u}_h - u_h)\,dx\right)^{1/2}.$$

For the rest of the section, we introduce two notation below. The oscillations of f over $D \subset \Omega$, $Osc(f,D)$, is defined by

$$Osc(f,D) = \left(\sum_{T\subset D} h_T^2 \|f - f_T\|^2_{L_2(T)}\right)^{1/2},$$

where f_T is the integral mean of f over T. We denote the $H^{-1}(D)$ norm by $\|\cdot\|_{-1,D}$.

The proofs of the following local efficiency estimates are similar to the ones in [34]; for brevity we omit the proof.

Theorem 5.8. *The following estimates hold:*
For all $T \in \mathcal{T}_h$,

$$h_T\|f-\sigma_h(u_h)\|_{L_2(T)} + h_T^2\|\nabla\sigma_h(u_h)\|_{L_2(T)} \le C\left(\|\nabla(u-u_h)\|_{L_2(T)} + \|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1,T} + Osc(f,T)\right), \quad \textit{for all } e \in \mathcal{E}_h^i,$$

$$h_e^{1/2}\|[\![\nabla u_h]\!]\|_{L_2(e)} \le C\big(\|\nabla_h(u-u_h)\|_{L_2(\mathcal{T}_e)} + \|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1,\mathcal{T}_e} + Osc(f,\mathcal{T}_e)\big), \quad \textit{for all } e \in \partial\mathbb{F}_h^i,$$

$$h_e^{1/2}\|[\![\nabla(u_h-\chi_h)]\!]\|_{L_2(e)} \le C\big(\|\nabla_h(u-u_h)\|_{L_2(\mathcal{T}_e)} + \|\sigma(u) - F_{\sigma_h(u_h)}\|_{-1,\mathcal{T}_e} + h_e^{1/2}\|[\![\nabla\chi_h]\!]\|_{L_2(e)} + Osc(f,\mathcal{T}_e)\big), \quad \textit{for all } T \in \mathcal{T}_h,$$

$$\|\nabla(\chi-\tilde{u}_h)^+\|_{L_2(T)} \le C\big(\|\nabla(\chi-\chi_h)\|_{L_2(T\cap\{\chi>u_h\})} + \sum_{p\in\mathcal{V}_T}\sum_{e\in\mathcal{E}_p\cap\mathcal{E}_h^i} h_e^{1/2}\|[\![\nabla(\chi_h-\tilde{u}_h)]\!]\|_{L_2(e)}\big), \quad \textit{for all } T \in \mathbb{C}_h \cup \mathbb{F}_h \textit{ with } T\cap\partial\Omega = \emptyset,$$

$$\Big(\int_T \big(-\sigma_h(u_h)\big)(\chi-\chi_h)^- dx\Big)^{1/2}$$
$$\le C\big(\|\nabla_h(u-u_h)\|_{L_2(T)} + \|\sigma(u)-F_{\sigma_h(u_h)}\|_{-1,T} + Osc(f,T)$$
$$+ h_T^{-1}\|(\chi-\chi_h)^-\|_{L_2(T)}\big) + C\left(\int_T f^-(\chi-\chi_h)^- dx\right)^{1/2},$$
$$\textit{and for all } T\in\mathbb{C}_{\partial\Omega},\ \Big(\int_T \big(-\sigma_h(u_h)\big)(\tilde{u}_h-u_h)dx\Big)^{1/2}$$
$$\le C\big(\|\nabla_h(u-u_h)\|_{L_2(T)} + \|\sigma(u)-F_{\sigma_h(u_h)}\|_{-1,T} + Osc(f,T)$$
$$+ h_T^{-1}\|(\tilde{u}_h-u_h)\|_{L_2(T)}\big) + C\left(\int_T f^-(\chi-\chi_h)^- dx\right)^{1/2}.$$

6. Examples of DG methods

In this section, we list some choices of $b_h(\cdot,\cdot)$ for various DG methods that have appeared in the literature and verify the assumptions (A1)-(A2) in Section 4:

First, we consider the non-over-penalized interior penalty methods:

SIPG Method [23, 39, 2]**:** For $\gamma \ge \gamma_0 > 0$,

$$b_h(w,v) = -\sum_{e\in\mathcal{E}_h}\int_e \big(\{\!\{\nabla w\}\!\}[\![v]\!] + \{\!\{\nabla v\}\!\}[\![w]\!]\big)ds + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[\![w]\!][\![v]\!]\,ds. \tag{6.1}$$

NIPG Method [32]**:** For $\gamma > 0$,

$$b_h(w,v) = -\sum_{e\in\mathcal{E}_h}\int_e \big(\{\!\{\nabla w\}\!\}[\![v]\!] - \{\!\{\nabla v\}\!\}[\![w]\!]\big)ds + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[\![w]\!][\![v]\!]\,ds. \tag{6.2}$$

IIPG Method [26]**:** $\gamma \ge \gamma_0 > 0$,

$$b_h(w,v) = -\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\nabla w\}\!\}[\![v]\!]ds + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[\![w]\!][\![v]\!]\,ds. \tag{6.3}$$

LDG Method [21, 18]**:** For $\gamma > 0$ and $\beta \in \mathbb{R}^2$,

$$\begin{aligned} b_h(w,v) = &-\sum_{e\in\mathcal{E}_h}\int_e \big(\{\!\{\nabla w\}\!\}[\![v]\!] + \{\!\{\nabla v\}\!\}[\![w]\!]\big)ds \\ &+ \sum_{e\in\mathcal{E}_h^i}\int_e \big(\beta\cdot[\![w]\!][\![\nabla v]\!] + [\![\nabla w]\!]\beta\cdot[\![v]\!]\big)ds + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[\![w]\!][\![v]\!]\,ds \\ &+ \int_\Omega \big[r([\![w]\!]) + \ell(\beta\cdot[\![w]\!])\big]\cdot\big[r([\![v]\!]) + \ell(\beta\cdot[\![v]\!])\big]\,dx. \end{aligned} \tag{6.4}$$

Bassi *et al.* [7]**:** For $\gamma > 3$,

$$b_h(w,v) = -\sum_{e\in\mathcal{E}_h}\int_e \big(\{\!\{\nabla w\}\!\}[\![v]\!] + \{\!\{\nabla v\}\!\}[\![w]\!]\big)ds + \sum_{e\in\mathcal{E}_h}\int_\Omega \gamma r_e([\![w]\!])r_e([\![v]\!])\,ds. \tag{6.5}$$

Brezzi *et al.* [15]**:** For $\gamma > 0$,

$$b_h(w,v) = -\sum_{e\in\mathcal{E}_h}\int_e \big(\{\!\{\nabla w\}\!\}[\![v]\!] + \{\!\{\nabla v\}\!\}[\![w]\!]\big)\,ds + \int_\Omega r\big([\![w]\!]\big)\cdot r\big([\![v]\!]\big)\,dx$$
$$+\sum_{e\in\mathcal{E}_h}\int_\Omega \gamma r_e\big([\![w]\!]\big) r_e\big([\![v]\!]\big)\,ds. \tag{6.6}$$

In the definitions of b_h above, r and l denote the global lifting operators while r_e denotes the local lifting operator which is defined in [3].

For all above six DG methods (6.1)-(6.6), define the norm $\|\cdot\|_h$ on V_h by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h}\|\nabla v\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_h}\frac{1}{h_e}\|[\![v]\!]\|^2_{L_2(e)}. \tag{6.7}$$

Then from [3, 25], there are positive constants C_1 and C_2 independent of h such that

$$\mathcal{A}_h(v_h,v_h) \geq C\|v_h\|_h^2 \quad \forall v_h\in V_h, \tag{6.8}$$
$$\mathcal{A}_h(w_h,v_h) \leq C\|w_h\|_h\,\|v_h\|_h \quad \forall v_h,\, w_h\in V_h. \tag{6.9}$$

While assumption (A2) follows immediately from the forms of b_h and inverse inequality (2.3), assumption (A1) follows from inequalities (6.8)-(6.9) (see [38]).

We next consider the over-penalized interior penalty methods:

Babuska-Zlamaal [5]: For $\gamma > 0$,

$$b_h(w,v) = \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e^3}[\![w]\!][\![v]\!]\,ds. \tag{6.10}$$

Brezzi *et al.* [16]**:** For $\gamma > 0$,

$$b_h(w,v) = \sum_{e\in\mathcal{E}_h}\int_\Omega \frac{\gamma}{h_e^2} r_e\big([\![w]\!]\big) r_e\big([\![v]\!]\big)\,ds. \tag{6.11}$$

WOPSIP [13]**:** For $\gamma > 0$,

$$b_h(w,v) = \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e^3}\Pi_e\big([\![w]\!]\big)\Pi_e\big([\![v]\!]\big)\,ds, \tag{6.12}$$

where $\Pi_e : L_2(e)\to P_0(e)$ is the L_2-projection.

For the above three over-penalized methods (6.10)-(6.12), the norm we consider is induced by the symmetric positive definite form $\mathcal{A}_h$. Define the norm $\|\cdot\|_h$ on V_h by

$$\|v\|_h^2 = \mathcal{A}_h(v,v). \tag{6.13}$$

Then clearly assumptions (A1) and (A2) hold for these three methods.

Therefore the error estimate in Theorem 5.4 holds for all the DG methods that are listed above.

6.1. **Stability bounds.** In this section, we derive uniform stability estimates for the solution u_h and for the discrete Lagrange multiplier $\sigma_h(u_h)$. In the analysis below, we require the following Poincaré type inequality for piece-wise H^1 functions [11]:

$$\|v\| \leq C\|v\|_h \quad \forall v \in H^1(\Omega, \mathcal{T}_h). \tag{6.14}$$

First, we derive the following stability result for the solution u_h:

Lemma 6.1. *Let the obstacle function $\chi \leq 0$ in a neighborhood of $\partial\Omega$. Then, there exists some positive constant C which is independent of h such that*

$$\|u_h\|_h \leq C, \tag{6.15}$$

where u_h is the solution of any of the DG methods in (6.1)-(6.6) *and* (6.10)-(6.12). *The norm $\|\cdot\|_h$ is defined by* (6.7) *for the DG methods in* (6.1)-(6.6), *and by* (6.13) *for the DG methods in* (6.10)-(6.12).

Proof. From the assumption $\chi \leq 0$ in a neighborhood of $\partial\Omega$, it is known from [24, Lemma 2.4] that $\mathcal{D}(\Omega) \cap \mathcal{K}$ is dense in $\mathcal{K}$. It implies that there exists a function $\phi \in H^2(\Omega) \cap \mathcal{K}$. Fixing such a ϕ, the nodal interpolation $I_h\phi \in V_c$ of ϕ is well-defined and satisfies $\|I_h\phi\|_h \leq C$, where the positive constant C depends on ϕ but is independent of h. Also since $\phi \in \mathcal{K}$, we have that $I_h\phi \in V_c \cap \mathcal{K}_h$. Set $v_h = I_h\phi$ in (4.3). Then using (6.8), (6.9), (6.13) and (6.14), we find

$$\begin{aligned} c\|u_h\|_h^2 &\leq \mathcal{A}_h(u_h, u_h) \leq \mathcal{A}_h(u_h, I_h\phi) + (f, u_h - I_h\phi) \\ &\leq C\left(\|u_h\|_h \|I_h\phi\|_h + \|f\| \|I_h\phi\|_h + \|f\| \|u_h\|_h\right). \end{aligned} \tag{6.16}$$

The proof follows from $\|I_h\phi\|_h \leq C$. □

For the DG methods (6.1)-(6.6), we find in the following a uniform upper bound for $\sigma_h(u_h)$ on unstructured meshes: For the rest of the section, let u_h be the solution of any of the DG methods (6.1)-(6.6) and norm $\|\cdot\|_h$ is defined by (6.7).

Lemma 6.2. *Let the obstacle function $\chi \leq 0$ in a neighborhood of $\partial\Omega$. Let u_h be the solution of any of the DG methods* (6.1)-(6.6) *and the norm $\|\cdot\|_h$ is defined by* (6.7). *Then it holds that*

$$\sum_{T\in\mathcal{T}_h} h_T^2 \|\sigma_h(u_h)\|_{L_2(T)}^2 \leq C, \tag{6.17}$$

where $\sigma_h(u_h)$ is defined by (4.5) *and C is a positive constant independent of h. Further, the distribution $F_{\sigma_h(u_h)}$ defined by* (4.9) *satisfies*

$$\|F_{\sigma_h(u_h)}\|_{-1} \leq C. \tag{6.18}$$

Proof. Define $w_h \in V_h$ element-wise by $w_h|_T = |T|\sigma_h|_T(u_h)$. Using (4.6),

$$\langle \sigma_h(u_h), w_h\rangle_h = \frac{1}{3}\sum_{T\in\mathcal{T}_h}\sum_{p\in\mathcal{V}_T} |T|^2\, \sigma_h|_T(u_h;p)^2 \geq C\sum_{T\in\mathcal{T}_h} |T|\|\sigma_h(u_h)\|_{L_2(T)}^2.$$

We substitute $v_h = w_h$ in (4.5) and then use (6.9) and (6.14) to find,

$$\langle \sigma_h(u_h), w_h\rangle_h = (f, w_h) - \mathcal{A}_h(u_h, w_h) \leq C\left(\|f\| + \|u_h\|_h\right)\|w_h\|_h.$$

Using the inverse inequalities (2.3)-(2.4), we find that

$$\begin{aligned}\|w_h\|_h^2 &= \sum_{T\in\mathcal{T}_h} |T|^2 \int_T |\nabla\sigma_h(u_h)|^2\,dx + \sum_{e\in\mathcal{E}_h^i} \int_e \frac{1}{h_e}\left(|T_1|\,\sigma_h|_{T_1}(u_h) - |T_2|\,\sigma_h|_{T_2}(u_h)\right)^2 \\ &\quad + \sum_{e\in\mathcal{E}_h^b} \int_e \frac{1}{h_e}\left(|T_e|\,\sigma_h|_{T_e}(u_h)\right)^2 \\ &\le C \sum_{T\in\mathcal{T}_h} |T| \|\sigma_h(u_h)\|_{L_2(T)}^2.\end{aligned}$$

Finally using Lemma 6.1, we obtain the estimate (6.17). Using (6.17), we derive the bound (6.18) for $F_{\sigma_h(u_h)}$. Let $v \in H_0^1(\Omega)$ and $v_h \in V_h$ be an approximation of v as in Lemma 2.1. Then

$$\int_\Omega \sigma_h(u_h) v\,dx = \int_\Omega \sigma_h(u_h)(v - v_h)\,dx + \int_\Omega \sigma_h(u_h) v_h\,dx. \tag{6.19}$$

The first term on the right-hand side of (6.19) is estimated as

$$|\big(\sigma_h(u_h), (v - v_h)\big)| \le \left(\sum_{T\in\mathcal{T}_h} h_T^2 \|\sigma_h(u_h)\|_{L_2(T)}^2\right)^{1/2} \|\nabla v\|. \tag{6.20}$$

The second term is split as

$$\int_\Omega \sigma_h(u_h) v_h\,dx = \int_\Omega \sigma_h(u_h) v_h\,dx - \langle \sigma_h(u_h), v_h\rangle_h + \langle \sigma_h(u_h), v_h\rangle_h.$$

Using (6.14) and Lemma 6.1, we find

$$\begin{aligned}|\langle \sigma_h(u_h), v_h\rangle_h| &= |(f, v_h) - \mathcal{A}_h(u_h, v_h)| \\ &\le C\left(\|f\| + \|u_h\|_h\right)\|v_h\|_h \le C\left(\|f\| + C\right)\|\nabla v\|.\end{aligned}$$

Using Lemma 4.1 and inverse inequality (2.4), we derive

$$\|F_{\sigma_h(u_h)}\|_{-1} \le C,$$

where the constant C is independent of h. This completes the proof. □

7. Numerical experiments

In this section, we demonstrate the performance of the error estimator in Theorem 5.4 by some numerical experiments. To this end, we consider the following model problem from [6]. Let $\Omega = (-1.5, 1.5)^2$, $f = -2$, $\chi_h = \chi := 0$ and $u = r^2/2 - \ln(r) - 1/2$ on $\partial\Omega$, where $r^2 = x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$. Then the exact solution is as follows:

$$u := \begin{cases} r^2/2 - \ln(r) - 1/2, & r \ge 1, \\ 0, & \text{otherwise.}\end{cases}$$

Although we have considered the model problem (1.1) with homogeneous boundary condition, the error estimator still provides an upper bound for the error up to some higher order terms involving the non-homogeneous boundary condition. We consider the initial mesh with a four right-angled criss-cross mesh. Then we use the following adaptive algorithm:

SOLVE → ESTIMATE → MARK → REFINE

In the SOLVE step, we use the primal-dual active set strategy [28] to solve the discrete obstacle problem. We then compute the estimator in Theorem 5.4 and use Dörfler's marking strategy [22] with parameter $\theta = 0.3$ for marking the elements for refinement. Using the newest vertex bisection algorithm, we refine the mesh and obtain a new mesh.

In the experiments, we consider three DG formulations: NIPG (6.2), SIPG (6.1) and LDG (6.4). We choose the penalty parameter $\gamma = 1$ for NIPG, $\gamma = 10$ for SIPG and $\gamma = 5$ for LDG. In the LDG formulation (6.4), we take $\beta = (0, 0)$. The convergence history of three methods is depicted in Figures 7.1, 7.2 and 7.3. We observe that all three methods converge optimally, except that both the SIPG and LDG methods exhibit some pre-asymptotic regime. Moreover, the convergence plots show the reliability of the error estimator. The efficiency indices which are reported in Figure 7.4 indicate the efficiency of the error estimator. Finally, Figures 7.5, 7.6 and 7.7 illustrate the mesh refinement in non-contact and free boundary zones.

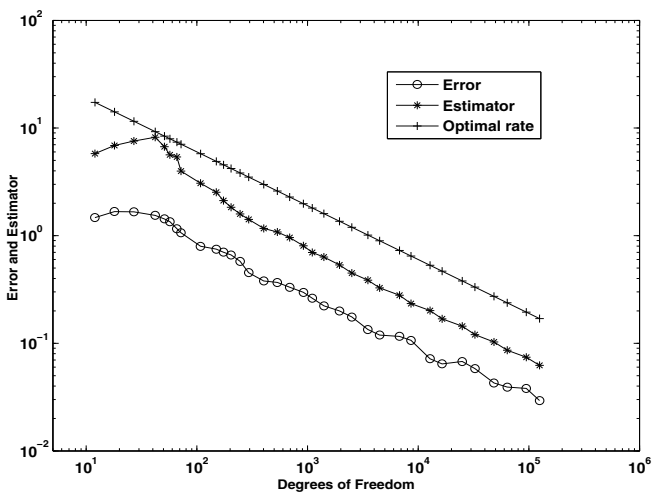

FIGURE 7.1. Error and estimator for NIPG

8. Concluding remarks

In this article, we have derived a computable *a posteriori* error estimator for various discontinuous Galerkin (DG) methods for an elliptic obstacle problem. To this end, we have introduced a smoothing function mapping the DG finite element space to the conforming finite element space which also satisfies the discrete obstacle constraint. Numerical results illustrate the performance of the estimator. Also, it is shown that the non-over-penalized DG methods exhibit uniform stability of the discrete Lagrange multiplier. In the analysis, we have assumed that the underlying mesh is conforming. The extensions to non-conforming meshes introduce additional technical difficulties. However, we hope this case may be handled with more attention and hence leave the subject to the near future.

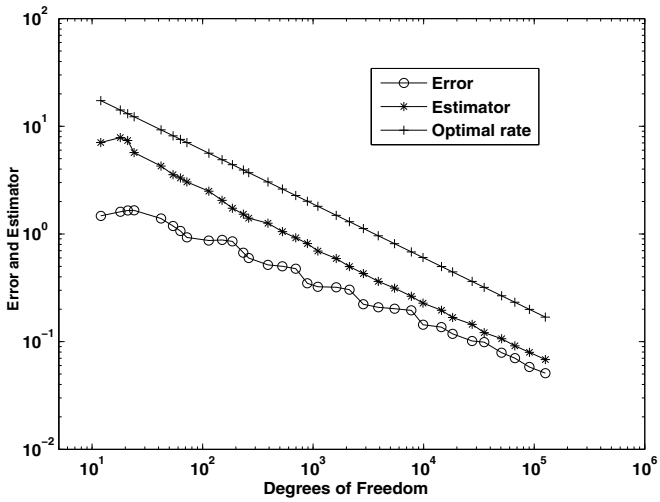

FIGURE 7.2. Error and estimator for SIPG

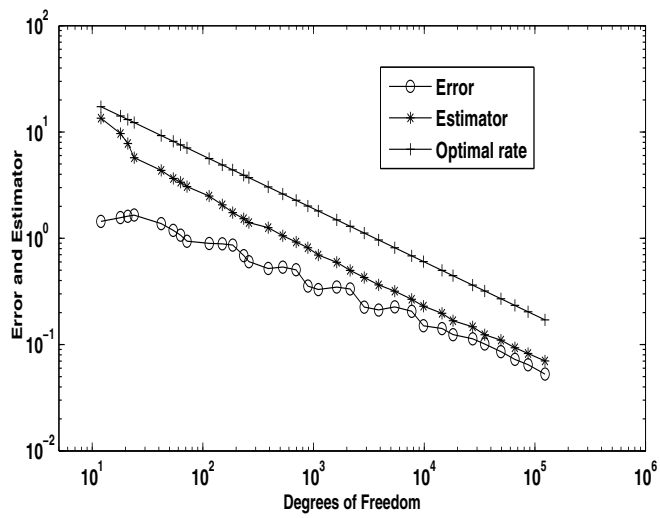

FIGURE 7.3. Error and estimator for LDG

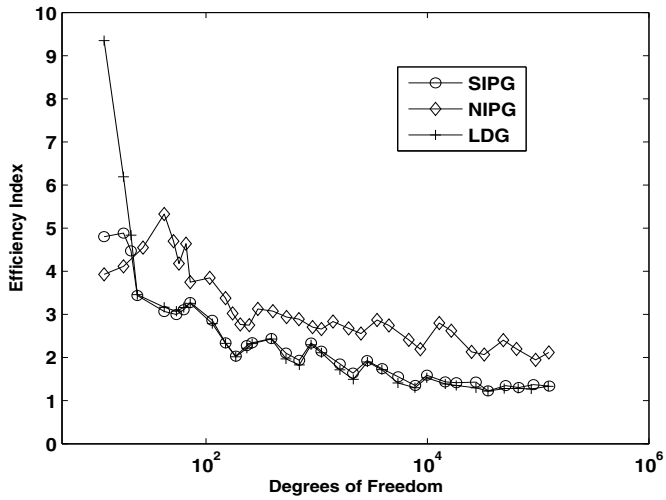

FIGURE 7.4. Efficiency indices

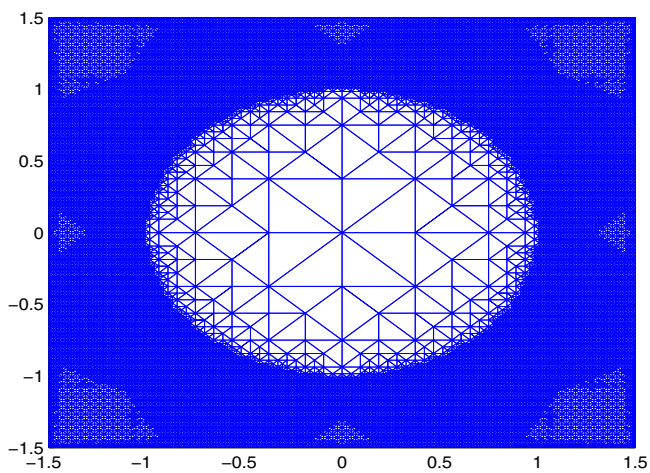

FIGURE 7.5. Adaptive mesh for NIPG

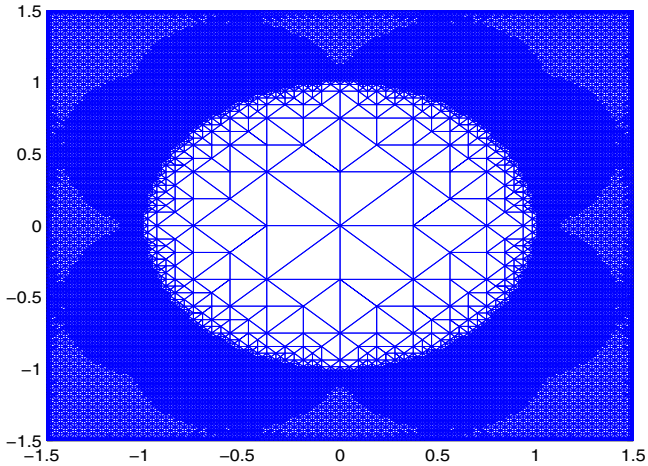

FIGURE 7.6. Adaptive mesh for SIPG

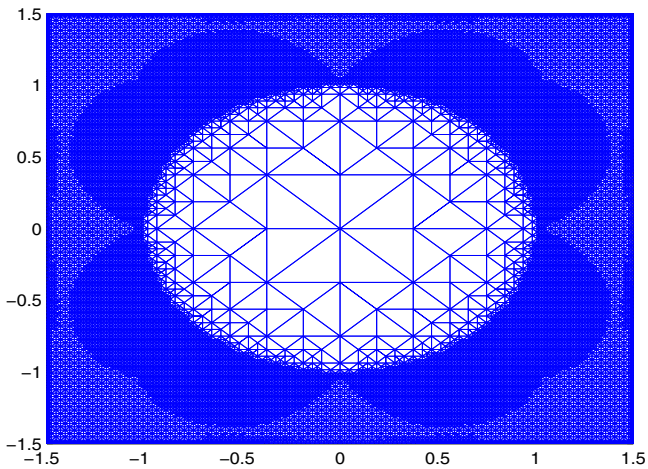

FIGURE 7.7. Adaptive mesh for LDG

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Department of Mathematics, Indian Institute of Science, Bangalore - 560012, India
E-mail address: gudi@math.iisc.ernet.in

Department of Mathematics, Indian Institute of Science, Bangalore - 560012, India
E-mail address: kamana@math.iisc.ernet.in