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MULTIVARIATE INTEGRATION OF INFINITELY MANY TIMES DIFFERENTIABLE FUNCTIONS IN WEIGHTED KOROBOV SPACES

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Abstract. We study multivariate integration for a weighted Korobov space of periodic infinitely many times differentiable functions for which the Fourier coefficients decay exponentially fast. The weights are defined in terms of two non-decreasing sequences $\mathbf{a} = \{a_i\}$ and $\mathbf{b} = \{b_i\}$ of numbers no less than one and a parameter $\omega \in (0,1)$. Let $e(n,s)$ be the minimal worst-case error of all algorithms that use n function values in the s-variate case. We would like to check conditions on $\mathbf{a}$, $\mathbf{b}$ and ω such that $e(n,s)$ decays exponentially fast, i.e., for some $q \in (0,1)$ and $p > 0$ we have $e(n,s) = \mathcal{O}(q^{n^p})$ as n goes to infinity. The factor in the $\mathcal{O}$ notation may depend on s in an arbitrary way. We prove that exponential convergence holds iff $B := \sum_{i=1}^{\infty} 1/b_i < \infty$ independently of $\mathbf{a}$ and ω. Furthermore, the largest p of exponential convergence is $1/B$. We also study exponential convergence with weak, polynomial and strong polynomial tractability. This means that $e(n,s) \le C(s)\, q^{n^p}$ for all n and s and with $\log C(s) = \exp(o(s))$ for weak tractability, with a polynomial bound on $\log C(s)$ for polynomial tractability, and with uniformly bounded $C(s)$ for strong polynomial tractability. We prove that the notions of weak, polynomial and strong polynomial tractability are equivalent, and hold iff $B < \infty$ and a_i are exponentially growing with i. We also prove that the largest (or the supremum of) p for exponential convergence with strong polynomial tractability belongs to $[1/(2B), 1/B]$.

1. Introduction

Multivariate integration is a popular research subject especially if the number of variables s is large. Such problems occur in many computational applications. We approximate multivariate integrals by algorithms that use n function values. For large s, it seems necessary to study how the errors of algorithms depend not only on n but also on s. The information complexity $n(\varepsilon, s)$ is the minimal number n such that there exists an algorithm using n function values with an error of at most ε in the s-variate case. It is proportional to the minimal cost of computing an ε approximation since linear algorithms are optimal and their cost is proportional to $n(\varepsilon, s)$.

We would like to control how $n(\varepsilon, s)$ depends on ε^{-1} and s and rule out the cases where this dependence is exponential. This is the subject of *tractability*;

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see [5,6]. The lack of exponential dependence is called *weak* tractability. Typically, $n(\varepsilon, s)$ is polynomially dependent on ε^{-1} and s for weighted classes of smooth functions. The notion of weighted classes means that the successive variables and groups of variables are moderated by certain weights. For sufficiently fast decaying weights the information complexity has a bound that is polynomially dependent on s, which corresponds to *polynomial* tractability, or is even independent of s, which corresponds to *strong polynomial* tractability.

In most papers on tractability, smoothness of functions is finite. This means that functions are differentiable only finitely many times. Then the minimal errors of algorithms are bounded by $C(s)\, n^{-\tau}$ for some positive τ which depends on the smoothness of functions. For many classes of such functions we know the largest τ which grows with increasing smoothness and decreasing weights. Then weak tractability holds iff $\log\, C(s) = o(s)$, whereas polynomial tractability holds iff $C(s)$ is polynomially dependent on s, and strong polynomial tractability holds iff $C(s)$ is uniformly bounded in s.

It seems to us that the case of infinitely many times differentiable functions is also of interest. For such classes of functions we would like to have exponential convergence. Let $e(n, s)$ be the minimal worst-case error among all algorithms that use n function values in the s-variate case. We would like to guarantee that

$$e(n, s) \le C(s)\, q^{n^p} \quad \text{for all} \quad n, s \in \mathbb{N}.$$

Here, $q \in (0, 1)$, p is positive, and both are independent of s. A priori it is not obvious what we should require about $C(s)$ although, clearly, the smaller $C(s)$ the better. Obviously, if we do not care about the dependence on s then the mere existence of $C(s)$ is enough. If we do care about the dependence on s, it is reasonable to consider the information complexity which is now bounded by

$$n(\varepsilon, s) \le \left\lceil \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p} \right\rceil \qquad \text{for all} \quad s \in \mathbb{N} \ \text{ and } \ \varepsilon \in (0, 1).$$

(Throughout the paper log means the natural logarithm and $\log^r x$ means $[\log x]^r$.)

Tractability with exponential convergence means that we would like to replace ε^{-1} by $\log\, \varepsilon^{-1}$ and guarantee the same properties on $n(\varepsilon, s)$ as before. Hence, exponential convergence with *weak tractability* holds iff

$$C(s) = \exp\left(\exp\left(o(s)\right)\right) \quad \text{as} \quad s \to \infty.$$

For polynomial or strong polynomial tractability we need to guarantee that $\log\, C(s)$ is polynomially or uniformly bounded in s. Hence, exponential convergence with *polynomial* tractability holds iff there are non-negative A and τ such that

$$\log\, C(s) \le A\, s^{\tau} \quad \text{for all} \quad s \in \mathbb{N}.$$

If $\tau = 0$ then $C(s)$ is uniformly bounded in s and we have exponential convergence with *strong* polynomial tractability.

Exponential convergence with weak, polynomial and strong polynomial tractability was studied in [3]. It was done for multivariate integration for weighted Korobov spaces with exponentially fast decaying Fourier coefficients. In the current paper, which can be viewed as a follow up paper of [3], we also study multivariate integration for weighted Korobov spaces in the worst-case setting. The main difference between these two papers is a different way of defining the decay of Fourier coefficients.

In our case, the decay is defined by two non-decreasing sequences $\boldsymbol{a} = \{a_j\}$ and $\boldsymbol{b} = \{b_j\}$ for which $a_1 = b_1 = 1$, and by a parameter $\omega \in (0,1)$; see Section 2 for more details. We study two questions.

The first question is: For which $(\boldsymbol{a}, \boldsymbol{b}, \omega)$ do we have exponential convergence? We prove that this holds iff

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty,$$

independently of $\boldsymbol{a}$ and ω. Furthermore, the largest p in exponential convergence is $1/B$. This is achieved by the quasi-Monte Carlo algorithm $Q_{n,s}$ that uses sample points from a regular grid with possibly different mesh-sizes m_j, i.e.,

$$Q_{n,s}(f) = \frac{1}{n} \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2-1} \cdots \sum_{j_s=0}^{m_s-1} f\left(\frac{j_1}{m_1}, \frac{j_2}{m_2}, \ldots, \frac{j_s}{m_s}\right),$$

for some positive integers m_j such that $n = \prod_{j=1}^{s} m_j$. Assuming that $B < \infty$, we show how to define m_i such that

$$n(\varepsilon, s) \le n = \mathcal{O}\left(\log^B\left(1 + \frac{1}{\varepsilon}\right)\right)$$

with the factor in the $\mathcal{O}$ notation independent of ε^{-1} but dependent on s. Furthermore, we prove that B is the smallest exponent of $\log(1+\varepsilon^{-1})$ or, equivalently, that the largest p of exponential convergence is $1/B$ and we may take $q = \omega$.

The second question is: For which $(\boldsymbol{a}, \boldsymbol{b}, \omega)$ do we have exponential convergence with weak, polynomial or strong polynomial tractability? We prove that the notions of weak, polynomial and strong polynomial tractability are equivalent. This means that the cases of almost doubly exponential $C(s)$, exponentially bounded $C(s)$ and uniformly bounded $C(s)$ are the same. We also prove that exponential convergence with strong polynomial tractability holds iff $B < \infty$ and a_i are exponentially growing with i. If these two conditions hold then again we may use the quasi-Monte Carlo algorithm that uses sample points from a regular grid with different mesh-sizes m_i. The definition of m_i is now different than for exponential convergence and

$$n(\varepsilon, s) \le n = \mathcal{O}\left(\log^t\left(1 + \frac{1}{\varepsilon}\right)\right)$$

where the factor in the $\mathcal{O}$ notation is now independent of ε^{-1} and s. The exponent t can be arbitrarily close to $2B$. We do not know, however, if the smallest exponent (or the infimum of) t is $2B$. We only know that $\inf t \in [B, 2B]$. It would be of interest to close this gap. Equivalently, we know that the largest (or the supremum of) p for exponential convergence with strong polynomial tractability belongs to $[1/(2B), 1/B]$.

In the final section we briefly discuss a relaxed notion of exponential convergence and demand that

$$e(n,s) \le C(s)\, q^{(n/C_1(s))^p} \qquad \text{for all} \qquad n, s \in \mathbb{N}$$

for some function $C_1 : \mathbb{N} \to (0, \infty)$. It turns out that the function C_1 does not help and we have the same results on exponential convergence with and without different notions of tractability as for the special case $C_1(s) = 1$ for all $s \in \mathbb{N}$.

2. Our weighted Korobov space $H(K)$

We consider a weighted Korobov space $H(K)$ of complex-valued periodic functions defined on $[0,1]^s$. The space $H(K)$ is a reproducing kernel Hilbert space with kernel function

$$K(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \omega_{\boldsymbol{h}} \exp(2\pi \mathtt{i} \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})) \quad \text{for all} \quad \boldsymbol{x},\boldsymbol{y}\in[0,1]^s,$$

with the usual dot product $\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y}) = \sum_{j=1}^s h_j(x_j-y_j)$, where h_j, x_j, y_j are the jth components of the vectors $\boldsymbol{h},\boldsymbol{x},\boldsymbol{y}$, respectively. (For general information about reproducing kernel Hilbert spaces we refer to [1].)

We assume that $\omega_{\boldsymbol{0}} = 1$, $\omega_{\boldsymbol{h}} \in [0,1]$ for all $\boldsymbol{h}\in\mathbb{Z}^s$, and that $\omega_{\boldsymbol{h}}$ may also depend on s, i.e., $\omega_{\boldsymbol{h}} = \omega_{s,\boldsymbol{h}}$. The kernel K is well defined if we choose $\omega_{\boldsymbol{h}}$ such that

$$|K(\boldsymbol{x},\boldsymbol{y})| \le K(\boldsymbol{x},\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \omega_{\boldsymbol{h}} < \infty. \tag{1}$$

For $f\in H(K)$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \widehat{f}(\boldsymbol{h}) \exp(2\pi\mathtt{i}\boldsymbol{h}\cdot\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x}\in[0,1]^s,$$

where $\widehat{f}(\boldsymbol{h}) = \int_{[0,1]^s} f(\boldsymbol{x}) \exp(-2\pi\mathtt{i}\boldsymbol{h}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and the norm of f from $H(K)$ is given in terms of its Fourier coefficients $\widehat{f}$ by

$$\|f\|_{H(K)} = \left(\sum_{\boldsymbol{h}\in\mathbb{Z}^s} |\widehat{f}(\boldsymbol{h})|^2 \omega_{\boldsymbol{h}}^{-1}\right)^{1/2} < \infty.$$

The inner product of f and g from $H(K)$ is

$$\langle f,g\rangle = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \widehat{f}(\boldsymbol{h})\,\overline{\widehat{g}(\boldsymbol{h})}\,\omega_{\boldsymbol{h}}^{-1}.$$

In this paper we consider a special choice of the coefficients $\omega_{\boldsymbol{h}}$. Let $\boldsymbol{a} = \{a_j\}$ and $\boldsymbol{b} = \{b_j\}$ be two sequences of real positive weights and let $\omega\in(0,1)$. Then for $\boldsymbol{h}\in\mathbb{Z}^s$ we consider coefficients of the form

$$\omega_{\boldsymbol{h}} := \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}}. \tag{2}$$

The case $a_j = b_j = 1$ for all $j\in\mathbb{N}$ was already considered in [3]. Furthermore, in [3] also a weighted case was considered which is different from that considered here.

It is easy to check that for any $(\boldsymbol{a},\boldsymbol{b},\omega)$ the condition (1) is satisfied. Note that for any positive c the triples $(\boldsymbol{a},\boldsymbol{b},\omega)$ and $(c\,\boldsymbol{a},\boldsymbol{b},\omega^{1/c})$ define the same sequence $\omega_{\boldsymbol{h}}$ of weights. To guarantee a uniqueness we assume that $a_1 = 1$. To omit some technical difficulties we also assume that $b_1 = 1$. We also want to model the situation in which the successive variables have a diminishing role. This can be achieved by assuming

$$1 = a_1 \le a_2 \le \ldots \le a_s \le \ldots, \tag{3}$$

$$1 = b_1 \le b_2 \le \ldots \le b_s \le \ldots. \tag{4}$$

The reproducing kernel K with coefficients of the form (2) depends on $(\boldsymbol{a},\boldsymbol{b},\omega)$. Since ω will be fixed throughout the paper and $\boldsymbol{a}$ and $\boldsymbol{b}$ will be arbitrary sequences satisfying (3) and (4), we denote such a kernel by $K_{s,\boldsymbol{a},\boldsymbol{b}}$ and, accordingly, the

corresponding Korobov space by $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$. Clearly, functions from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ are infinitely many times differentiable.

3. Integration

We are interested in numerical approximation of the values of integrals

$$I_s(f) = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \qquad \text{for all} \quad f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}).$$

Without loss of generality (see, e.g., [8]), we can restrict ourselves to approximating $I_s(f)$ by means of *linear algorithms* $Q_{n,s}$ of the form

$$Q_{n,s}(f) := \sum_{k=1}^{n} q_k f(\boldsymbol{x}_k),$$

where coefficients $q_k \in \mathbb{C}$ and sample points $\boldsymbol{x}_k \in [0,1)^s$. If we choose $q_k = 1/n$ for all $1 \le k \le n$ then we obtain so-called *quasi-Monte Carlo (QMC) algorithms* which are often used in practical applications especially if s is large. We are interested in studying the *worst-case integration error*,

$$e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), Q_{n,s}) = \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \le 1}} |I_s(f) - Q_{n,s}(f)|\,.$$

It is well known (see, for example, [4, Proposition 2.11]) that the worst-case error is equal to

$$e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), Q_{n,s}) = \left(1 - 2\sum_{k=1}^{n} q_k + \sum_{k,l=1}^{n} q_k q_l K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x}_k, \boldsymbol{x}_l)\right)^{1/2}. \tag{5}$$

For QMC algorithms this reduces to

$$e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), Q_{n,s}) = \left(-1 + \frac{1}{n^2}\sum_{k,l=1}^{n} K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x}_k, \boldsymbol{x}_l)\right)^{1/2}. \tag{6}$$

Let $e(n,s)$ be the *nth minimal worst-case error*,

$$e(n,s) = \inf_{q_k, \boldsymbol{x}_k,\ k=1,2,\ldots,n} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \le 1}} \left| I_s(f) - \sum_{k=1}^{n} q_k f(\boldsymbol{x}_k) \right|.$$

For $n = 0$, the best we can do is to approximate $I_s(f)$ simply by zero, and

$$e(0,s) = \|I_s\| = 1 \qquad \text{for all} \qquad s \in \mathbb{N}.$$

Hence, the integration problem is well normalized for all s.

As in [3], we say that we achieve exponential convergence for $e(n,s)$ if there exist numbers $q \in (0,1)$, $p > 0$ and a function $C : \mathbb{N} \to (0,\infty)$ such that

$$e(n,s) \le C(s)\, q^{n^p} \quad \text{for all} \quad s, n \in \mathbb{N}. \tag{7}$$

We stress that q and p are independent of s.

For $\varepsilon \in (0,1)$, we define the *information complexity* of integration

$$n(\varepsilon, s) = \min\{n \,:\, e(n,s) \le \varepsilon\}$$

as the minimal number of function values needed to obtain an ε-approximation.

Without loss of generality we may assume that the nth minimal worst-case error $e(n,s)$ is attained for some linear algorithms, i.e., for some q_k and $\boldsymbol{x}_k$ for $k = 1,2,\ldots,n$, where $n = n(\varepsilon, s)$.

As in [3], we note that if (7) holds, then

$$n(\varepsilon,s) \le \left\lceil \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p} \right\rceil \qquad \text{for all} \quad s \in \mathbb{N} \ \text{ and } \ \varepsilon \in (0,1). \tag{8}$$

Furthermore, if (8) holds, then

$$e(n+1,s) \le C(s)\, q^{n^p} \qquad \text{for all} \quad s,n \in \mathbb{N}.$$

This means that (7) and (8) are practically equivalent. Note that the parameter p determines the power of $\log \varepsilon^{-1}$ in the information complexity, whereas the parameter q effects only the multiplier of $\log^{1/p} \varepsilon^{-1}$. From this point of view, the parameter p is more important than the parameter q. That is why we would like to have (7) with the largest possible p. We shall see how to find or estimate such p for the parameters $(\boldsymbol{a}, \boldsymbol{b}, \omega)$ of the weighted Korobov space.

Exponential convergence implies that asymptotically with respect to ε tending to zero, we need $\mathcal{O}(\log^{1/p} \varepsilon^{-1})$ function values to compute an ε-approximation to multivariate integrals. However, it is not clear how long we have to wait to see this nice asymptotic behaviour especially for large s. This, of course, depends on $C(s)$ and is the subject of tractability. Tractability means that we control the behaviour of $C(s)$ and rule out the cases for which $n(\varepsilon,s)$ depends exponentially on s. Since there are many ways of controlling the lack of exponential dependence, we have many notions of tractability. We consider the notions of weak, polynomial and strong polynomial tractability in this paper; for more on tractability we refer to [5,6].

We say that we have *exponential convergence with weak tractability* iff

$$C(s) = \exp\left(\exp\left(o(s)\right)\right) \qquad \text{as} \qquad s \to \infty.$$

Hence, even for almost doubly exponential functions $C(s)$ of s we still have weak tractability.

We say that we have *exponential convergence with polynomial tractability* iff there exist non-negative numbers A, p_1, p_2 such that

$$n(\varepsilon,s) \le A\left(s^{p_1} + \log^{p_2} \varepsilon^{-1}\right) \quad \text{for all} \quad s \in \mathbb{N},\ \varepsilon \in (0,1).$$

If $p_1 = 0$, we say that we have *exponential convergence with strong polynomial tractability.*

Assume that (7) is satisfied. Then polynomial tractability holds if

$$\sup_{s \in \mathbb{N}} s^{-\tau} \log(1 + C(s)) < \infty \quad \text{for some} \quad \tau \ge 0,$$

and strong polynomial tractability if $\tau = 0$ in the formula above. If so, then we have (strong) polynomial tractability with $p_1 = \tau/p$ and $p_2 = 1/p$.

Hence, polynomial tractability holds if there exist non-negative numbers A and τ such that

$$C(s) \le \exp\left(A s^{\tau}\right) \quad \text{for all} \quad s \in \mathbb{N}.$$

The condition on $C(s)$ seems to be quite weak since even for singly exponential $C(s)$ we have polynomial tractability. Strong polynomial tractability holds if $C(s)$ is uniformly bounded in s.

4. Main result

We prove that QMC algorithms based on regular grids with different mesh-sizes for successive variables are optimal. These algorithms have already been studied in [3]. We recall their definition.

Definition 1. For $s \in \mathbb{N}$, a regular grid with mesh-sizes $m_1, \ldots, m_s \in \mathbb{N}$ is defined as the point set $\mathcal{G}_{n,s}$ given by

$$\left(\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}\right)$$

for $k_j = 0, 1, \ldots, m_j - 1$ and $j = 1, 2, \ldots, s$, where $n = \prod_{j=1}^{s} m_j$ is the cardinality of $\mathcal{G}_{n,s}$.

The QMC algorithm is now of the form

$$Q_{n,s}(f) = \frac{1}{n} \sum_{\boldsymbol{x} \in \mathcal{G}_{n,s}} f(\boldsymbol{x}). \tag{9}$$

Note that for dimension $s = 1$ this is the trapezoidal rule applied to a one-periodic function f. For the trapezoidal rule it is well known that exponential convergence rates can be achieved for periodic integrands; see, for example, [2, Chapter 8] or [9].

We are ready to formulate the main result of this paper.

Theorem 1.

- *Exponential convergence holds iff*

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty$$

independently of $\boldsymbol{a}$ and ω. Furthermore, the largest exponent p in (7) is $1/B$ or, equivalently, the smallest exponent $1/p$ in (8) is B. It is achieved by the QMC algorithm using the regular grid with mesh-sizes given by

$$m_j := \left\lfloor m^{1/(B \cdot b_j)} \right\rfloor \qquad \textit{for} \quad j = 1, 2, \ldots, s$$

with

$$m = \max_{j=1,2,\ldots,s} \left\lceil \left(\frac{2^{b_j}}{a_j} \frac{\log\left(1 + \frac{2s}{\log(1+\varepsilon^2)}\right)}{\log \omega^{-1}} \right)^B \right\rceil.$$

Then $e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), Q_{n,s}) \le \varepsilon$ and for all s there is a positive $C(s)$ such that

$$n(\varepsilon, s) \le n = \prod_{j=1}^{s} m_j \le C(s) \log^B\left(1 + \frac{1}{\varepsilon}\right) \qquad \textit{for all} \quad \varepsilon \in (0,1).$$

- *The notions of exponential convergence with weak, polynomial and strong polynomial tractability are equivalent.*
- *Exponential convergence with strong polynomial tractability holds iff $B < \infty$ and for any $p_1 \in (0, 1/B)$ there is a positive β_1 such that*

$$a_j \ge \beta_1 \, 2^{j\, p_1} \qquad \textit{for all} \quad j \in \mathbb{N}.$$

Furthermore, it is achieved by the QMC algorithm using the regular grid with mesh-sizes given by

$$m_j = \left\lceil \left(\frac{\log\left(1 + \frac{\pi^2}{3} \frac{j^2}{\log(1+\varepsilon^2)}\right)}{a_j \, \log \omega^{-1}} \right)^{1/b_j} \right\rceil .$$

Then $e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), Q_{n,s}) \le \varepsilon$ *and for any positive* δ *there exists a positive number* C_δ *such that*

$$n(\varepsilon, s) \le n = \prod_{j=1}^{s} m_j \le C_\delta \left(\log^{2B+\delta} \left(1 + \frac{1}{\varepsilon}\right)\right) \qquad \textit{for all} \quad \varepsilon \in (0,1).$$

Theorem 1 states that a necessary and sufficient condition for exponential convergence of integration is that the coefficients b_i go to infinity so fast that $B = \sum_{j=1}^{\infty} b_j^{-1} < \infty$, with no extra conditions on a_i and ω. Furthermore, the largest exponent p of exponential convergence is $1/B$. Note that p is always less than 1 since $p < b_1 = 1$, and for large B the exponent p is small. The smallest exponent in (8) is B. For example, if we take $b_j = j^2$ then $B = \pi^2/6 = 1.64449\ldots$ and the largest exponent p of exponential convergence is $1/B = 0.60792\ldots$.

Assume that $B < \infty$. Then Theorem 1 states that the notions of weak, polynomial and strong polynomial tractability are equivalent. Hence, it does not matter if we assume that $C(s)$ is almost doubly exponential in s or $C(s)$ is exponential or $C(s)$ is uniformly bounded. Furthermore, strong polynomial tractability holds iff coefficients a_i are exponentially large in i.

The proof of this theorem will be done as follows. In Section 5 we prove necessary conditions. More precisely, we prove that exponential convergence implies $B < \infty$ and $p \le 1/B$, and that exponential convergence with weak tractability implies $a_j \ge \beta_1 2^{j\,p_1}$ for all j. In Section 6 we prove sufficient conditions. More precisely, we prove that the QMC algorithms of the form (9) with m_j given as in Theorem 1 have the worst-case error at most ε and the corresponding numbers n of points used by the QMC algorithms satisfy bounds presented in Theorem 1. The equivalence of the notions of weak, polynomial and strong polynomial tractability now easily follows. Of course, it is enough to show that weak tractability implies strong polynomial tractability. It is indeed so since weak tractability implies that $a_j \ge \beta_1 2^{j\,p_1}$, whereas the last bound on a_j implies strong polynomial tractability.

5. Lower bounds

We present a lower bound on the nth minimal worst-case error for integration based on [3, Theorem 1] whose proof is adopted from [7].

Lemma 1. *Let* A_s *be a finite subset of* $\mathbb{Z}^s$. *Then*

$$e(n,s) \ge \left[\max_{\boldsymbol{h}^* \in A_s} \sum_{\boldsymbol{h} \in A_s} \frac{1}{\omega_{\boldsymbol{h}-\boldsymbol{h}^*}} \right]^{-1/2} \qquad \textit{for all} \quad n < |A_s|.$$

Proof. It is enough to repeat the proof of [3, Theorem 1]. □

For $\boldsymbol{t} = (t_1, t_2, \ldots, t_s)$ with non-negative integers t_j, we now define

$$A_s = A_{s,\boldsymbol{t}} = \{\boldsymbol{h} \in \mathbb{Z}^s \ : \ h_j \in \{0, 1, \ldots, t_j\} \quad \text{for all} \quad j = 1, 2, \ldots, s\}.$$

Clearly, $|A_{s,\boldsymbol{t}}| = \prod_{j=1}^{s}(1+t_j)$. For $\boldsymbol{h}, \boldsymbol{h}^* \in A_{s,\boldsymbol{t}}$ we have

$$\omega_{\boldsymbol{h}-\boldsymbol{h}^*}^{-1} = \omega^{-\sum_{j=1}^{s} a_j |h_j - h_j^*|^{b_j}} \le \omega^{-\sum_{j=1}^{s} a_j t_j^{b_j}}.$$

Therefore,

$$\max_{\boldsymbol{h}^* \in A_{s,\boldsymbol{t}}} \sum_{\boldsymbol{h} \in A_{s,\boldsymbol{t}}} \frac{1}{\omega_{\boldsymbol{h}-\boldsymbol{h}^*}} \le \omega^{-\sum_{j=1}^{s} a_j t_j^{b_j}} \; |A_{s,\boldsymbol{t}}|.$$

This leads to the following corollary.

Corollary 1. *For all non-negative integers t_j,*

$$e(n,s) \ge \omega^{2^{-1} \sum_{j=1}^{s} a_j t_j^{b_j}} \prod_{j=1}^{s} (1+t_j)^{-1/2} \qquad \text{for all} \quad n < \prod_{j=1}^{s}(1+t_j).$$

We are ready to prove necessary conditions in Theorem 1. More precisely, we prove the following theorem.

Theorem 2.

- *Assume that we have exponential convergence, i.e., there exist numbers $q \in (0,1)$, $p > 0$ and a function $C : \mathbb{N} \to (0,\infty)$ such that*

$$e(n,s) \le C(s)\, q^{n^p} \qquad \text{for all} \quad s, n \in \mathbb{N}. \tag{10}$$

  Then for arbitrary $s \in \mathbb{N}$ and for all $\boldsymbol{t} = (t_1, t_2, \ldots, t_s) \in \mathbb{N}_0$ with $\|\boldsymbol{t}\|_{s,\infty} := \max_{j=1,\ldots,s} t_j$ tending to infinity we have

$$\liminf_{\|\boldsymbol{t}\|_{s,\infty} \to \infty} \frac{\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s}(1+t_j)^p} \ge \frac{2 \log q^{-1}}{\log \omega^{-1}} > 0.$$

  In particular, this implies that

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \qquad \text{and} \qquad p \le \frac{1}{B}$$

  independently of positive $\boldsymbol{a}$ and ω.

- *Assume that we have exponential convergence with weak tractability, i.e., (10) holds with $C(s) = \exp(\exp(o(s)))$. Then for all $s \in \mathbb{N}$ and for all $\boldsymbol{t} = (t_1, t_2, \ldots, t_s) \in \mathbb{N}$ with $\prod_{j=1}^{s}(1+t_j)$ tending to infinity we have*

$$\liminf_{\prod_{j=1}^{s}(1+t_j) \to \infty} \frac{\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s}(1+t_j)^p} \ge \frac{2 \log q^{-1}}{\log \omega^{-1}} > 0.$$

  In particular, this means that there exists a positive number β for which

$$\sum_{j=1}^{s} a_j \ge \beta\, 2^{s\,p} \qquad \text{for all} \qquad s \in \mathbb{N},$$

  and for any $p_1 \in (0,p)$ there is a positive β_1 such that

$$a_j \ge \beta_1\, 2^{j\,p_1} \qquad \text{for all} \qquad j \in \mathbb{N}.$$

Before we prove this theorem we explain how the limit inferiors are taken in the successive points. For exponential convergence, the dimension s is arbitrary but fixed and we take the vectors $\boldsymbol{t}$ with at least one component going to infinity. It is important to remember that now we do not yet control the behaviour of $C(s)$. When we consider exponential convergence with weak tractability then we

control the behaviour of $C(s)$. We now may take varying s and vectors $\boldsymbol{t}$ such that $\prod_{j=1}^{s}(1+t_j)$ goes to infinity. For example all t_j can now be the same and positive, $t_j = t \in \mathbb{N}$ so that the norm $\|\boldsymbol{t}\|_{s,\infty}$ does not go to infinity as was required in the previous case. However, the product $\prod_{j=1}^{s}(1+t) = (1+t)^s$ goes to infinity with s. Note that now all $t_j \geq 1$ and $\prod_{j=1}^{s}(1+t_j) \geq 2^s$.

Proof. From (10) and Corollary 1 with $n = -1 + \prod_{j=1}^{s}(1+t_j)$ we have

$$C(s) \geq \exp\left(-\frac{1}{2}\sum_{j=1}^{s} a_j t_j^{b_j} \log\frac{1}{\omega} \;-\; \frac{1}{2}\sum_{j=1}^{s}\log(1+t_j) \;+ n^p \log\frac{1}{q}\right).$$

This implies that

$$\begin{aligned} &\frac{\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s}(1+t_j)^p} \;+\; \frac{2\log C(s) \;+\; \sum_{j=1}^{s}\log(1+t_j)}{[\prod_{j=1}^{s}(1+t_j)^p]\,\log\omega^{-1}} \\ &\qquad \geq\; 2\left(1-\frac{1}{\prod_{j=1}^{s}(1+t_j)}\right)^p \frac{\log q^{-1}}{\log\omega^{-1}}. \end{aligned} \tag{11}$$

For fixed s, when $\|\boldsymbol{t}\|_{s,\infty}$ goes to infinity then the second term of the left-hand side goes to zero, and the right-hand side goes to $2\log q^{-1}/\log\omega^{-1} > 0$. Thus, the first necessary condition holds.

For a positive t now take

$$t_j = \lceil t^{1/b_j}\rceil \qquad \text{for all} \qquad j = 1,2,\ldots,s.$$

Clearly, $\lim_{t\to\infty}\lceil t^{1/b_j}\rceil/t^{1/b_j} = 1$. Then for t tending to infinity we have

$$\begin{aligned} \frac{\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s}(1+t_j)^p} &= t^{1-p\sum_{j=1}^{s} b_j^{-1}} \frac{\sum_{j=1}^{s} a_j(\lceil t^{1/b_j}\rceil/t^{1/b_j})^{b_j}}{\prod_{j=1}^{s}\left(\lceil t^{1/b_j}\rceil/t^{1/b_j} + t^{-1/b_j}\right)^p} \\ &= t^{1-p\sum_{j=1}^{s} b_j^{-1}}\,(1+o(1))\sum_{j=1}^{s} a_j. \end{aligned}$$

Since this expression is positive when t goes to infinity, we must have $p\sum_{j=1}^{s}\frac{1}{b_j} \leq 1$. This holds for all s. Hence for s tending to infinity we conclude that

$$p\sum_{j=1}^{\infty}\frac{1}{b_j} = p\,B \leq 1,$$

which completes this part of the proof.

We now assume that we also have weak tractability, then $\log C(s) = \exp(o(s))$. This means that for any $c_1 > 0$ there exists $C_1 > 0$ such that

$$\log C(s) \leq C_1\,\exp(c_1 s) = C_1\,2^{c_1 s/\log 2} \qquad \text{for all} \qquad s \in \mathbb{N}.$$

Without loss of generality, we can now assume that $C(s) \geq 1$, i.e., $\log C(s) \in \left[0, C_1\,2^{c_1 s/\log 2}\right]$. We take a positive c_1 such that $c_1/\log 2 < p$.

We now consider $t_j \geq 1$ and therefore

$$\prod_{j=1}^{s}(1+t_j)^p \geq 2^{(s-1)p}\,(1+\|\boldsymbol{t}\|_{s,\infty})^p\,.$$

We estimate the second term in (11) from above by

$$\frac{2C_1\, 2^{c_1 s/\log 2} + s\,\log(1+\|\boldsymbol{t}\|_{s,\infty})}{2^{(s-1)p}(1+\|\boldsymbol{t}\|_{s,\infty})^p\,\log\omega^{-1}} = \frac{2C_1\, +\, s2^{-c_1 s/\log 2}\,\log(1+\|\boldsymbol{t}\|_{s,\infty})}{2^{s(p-c_1/\log 2)-p}\,(1+\|\boldsymbol{t}\|_{s,\infty})^p\,\log\omega^{-1}}.$$

Note that $\prod_{j=1}^s(1+t_j)\to\infty$ iff $s\to\infty$ or $\|\boldsymbol{t}\|_{s,\infty}\to\infty$. In either case, the expression above as well as the second term in (11) goes to zero. The rest is unchanged.

To prove the last point take all $t_j=1$ and let s tend to infinity. Then the limit inferior of $\sum_{j=1}^s a_j/2^{sp}$ is positive. This implies the existence of a positive number β such that

$$\sum_{j=1}^s a_j \ge \beta\, 2^{s\,p} \qquad \text{for all} \qquad s\in\mathbb{N},$$

as claimed. Finally, since a_j's are ordered we have $sa_s\ge\sum_{j=1}^s a_j$ and

$$a_s \ge \frac{\beta}{s}\,2^{s\,p} = \frac{\beta\, 2^{s(p-p_1)}}{s}\,2^{s\,p_1} \ge \beta_1\, 2^{s\,p_1}$$

with $\beta_1=\beta\,\inf_{s\in\mathbb{N}} 2^{s(p-p_1)}/s>0$. This completes the proof. □

6. Upper bounds

Upper bounds in Theorem 1 are obtained by the QMC algorithms that use regular grids with different mesh-sizes. We first present the worst-case error of the QMC algorithm that uses mesh-sizes with arbitrary $m_i\in\mathbb{N}$, i.e., it uses the points from the regular grid $\mathcal{G}_{n,s}$. The worst-case error of the QMC algorithm $Q_{n,s}$ is now denoted by $e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}_{n,s})$ to stress the dependence on $\mathcal{G}_{n,s}$.

Lemma 2. *Let $\mathcal{G}_{n,s}$ denote the regular grid with possibly different mesh-sizes $m_1,m_2,\ldots,m_s\in\mathbb{N}$ and let $n=m_1m_2\cdots m_s$. Then*

$$e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}_{n,s}) = -1+\prod_{j=1}^s\left(1+2\sum_{h=1}^\infty \omega^{a_j(m_jh)^{b_j}}\right).$$

Proof. From (6) and the definition of the kernel $K_{s,\boldsymbol{a},\boldsymbol{b}}$ we obtain

$$\begin{aligned}
&e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}_{n,s})\\
&\quad= -1+\frac{1}{n^2}\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{G}_{n,s}} K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y})\\
&\quad= \frac{1}{n^2}\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{G}_{n,s}}\sum_{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\}} \omega^{\sum_{j=1}^s a_j|h_j|^{b_j}}\exp(2\pi\mathtt{i}\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y}))\\
&\quad= \frac{1}{n^2}\sum_{k_1,l_1=0}^{m_1-1}\cdots\sum_{k_s,l_s=0}^{m_s-1}\sum_{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\}} \omega^{\sum_{j=1}^s a_j|h_j|^{b_j}}\exp\left(2\pi\mathtt{i}\sum_{j=1}^s\frac{h_j(k_j-l_j)}{m_j}\right)\\
&\quad= \frac{1}{n^2}\sum_{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\}} \omega^{\sum_{j=1}^s a_j|h_j|^{b_j}}\prod_{j=1}^s\sum_{k,l=0}^{m_j-1}\exp\left(2\pi\mathtt{i}\frac{h_j(k-l)}{m_j}\right)\\
&\quad= \frac{1}{n^2}\sum_{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\}} \omega^{\sum_{j=1}^s a_j|h_j|^{b_j}}\prod_{j=1}^s\left|\sum_{k=0}^{m_j-1}\exp\left(2\pi\mathtt{i}\frac{h_jk}{m_j}\right)\right|^2.
\end{aligned}$$

We now use the well-known fact that for any $m \in \mathbb{N}$ and $h \in \mathbb{Z}$ we have

$$\sum_{k=0}^{m-1} \exp\left(2\pi \mathtt{i} \frac{hk}{m}\right) = \begin{cases} m & \text{if } h \equiv 0 \,(\mathrm{mod}\, m), \\ 0 & \text{if } h \not\equiv 0 \,(\mathrm{mod}\, m). \end{cases}$$

Then

$$\begin{aligned} e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}_{n,s}) &= \sum_{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}} \omega^{\sum_{j=1}^s a_j (m_j |h_j|)^{b_j}} \\ &= -1 + \prod_{j=1}^s \left(1 + 2 \sum_{h=1}^\infty \omega^{a_j (m_j h)^{b_j}}\right), \end{aligned}$$

as claimed. □

We now define m_j as in the first point of Theorem 1 and show that the corresponding QMC algorithm has the worst-case error at most ε and achieves exponential convergence.

Theorem 3. *Assume that*

$$B := \sum_{j=1}^\infty \frac{1}{b_j} < \infty.$$

For $s \in \mathbb{N}$ and $\varepsilon \in (0,1)$ define

$$m = \max_{j=1,2,\ldots,s} \left\lceil \left(\frac{2^{b_j}}{a_j} \frac{\log\left(1 + \frac{2s}{\log(1+\varepsilon^2)}\right)}{\log \omega^{-1}} \right)^B \right\rceil.$$

Let $\mathcal{G}^*_{n,s}$ be a regular grid with mesh-sizes $m_1, m_2, \ldots, m_s$ given by

$$m_j := \left\lfloor m^{1/(B \cdot b_i)} \right\rfloor \qquad \textit{for} \quad j = 1, 2, \ldots, s \quad \textit{and} \quad n = \prod_{j=1}^s m_j.$$

Then

$$e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le \varepsilon, \qquad \textit{and} \qquad n(\varepsilon, s) \le n = \mathcal{O}\left(\log^B\left(1 + \frac{1}{\varepsilon}\right)\right)$$

with the factor in the $\mathcal{O}$ notation independent of ε^{-1} but dependent on s.

Proof. First note that

$$n = \prod_{j=1}^s m_j = \prod_{j=1}^s \left\lfloor m^{1/(B \cdot b_j)} \right\rfloor \le m^{\frac{1}{B} \sum_{j=1}^s 1/b_j} \le m = \mathcal{O}\left(\log^B\left(1 + \frac{1}{\varepsilon}\right)\right).$$

Since $\lfloor x \rfloor \ge x/2$ for all $x \ge 1$, we have

$$a_j |m_j h_j|^{b_j} \ge a_j (|h_j|/2)^{b_j} \, m^{1/B}$$

for every $j = 1, 2, \ldots, s$. From Lemma 2 we obtain

$$e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le -1 + \prod_{j=1}^s \left(1 + 2 \sum_{h=1}^\infty \omega^{m^{1/B} a_j 2^{-b_j} \, h^{b_j}}\right).$$

Since $b_j \ge 1$ we further estimate

$$\sum_{h=1}^\infty \omega^{m^{1/B} a_j 2^{-b_j} \, h^{b_j}} \le \sum_{h=1}^\infty \omega^{m^{1/B} a_j 2^{-b_j} \, h} = \frac{\omega^{m^{1/B} a_j 2^{-b_j}}}{1 - \omega^{m^{1/B} a_j 2^{-b_j}}}.$$

From the definition of m we have for all $j = 1, 2, \ldots, s$,

$$\frac{\omega^{m^{1/B} a_j 2^{-b_j}}}{1 - \omega^{m^{1/B} a_j 2^{-b_j}}} \le \frac{\log(1+\varepsilon^2)}{2s}.$$

This proves

$$\begin{aligned} e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) &\le \left[-1 + \left(1 + \frac{\log(1+\varepsilon^2)}{s}\right)^s \right]^{1/2} \\ &\le \left[-1 + \exp(\log(1+\varepsilon^2)) \right]^{1/2} = \varepsilon, \end{aligned}$$

and completes the proof. □

We now define m_i as in the last point of Theorem 1 and show that the corresponding QMC algorithm has the worst-case error at most ε and achieves exponential convergence with strong polynomial tractability.

Theorem 4. *Assume that*

$$B = \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty$$

and for any $p_1 \in (0, 1/B)$ there is a positive β_1 such that

$$a_j \ge \beta_1 \, 2^{j\, p_1} \qquad \text{for all} \qquad j \in \mathbb{N}.$$

Let $\mathcal{G}^*_{n,s}$ be a regular grid with mesh-sizes given by

$$m_j = \left\lceil \left(\frac{\log\left(1 + \frac{\pi^2}{3} \frac{j^2}{\log(1+\varepsilon^2)}\right)}{a_j \, \log \omega^{-1}} \right)^{1/b_j} \right\rceil.$$

Then $e(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le \varepsilon$ and for any positive δ there exists a positive number C_δ such that

$$n(\varepsilon, s) \le n = \prod_{j=1}^{s} m_j \le C_\delta \, \log^{2B+\delta}\left(1 + \frac{1}{\varepsilon}\right) \qquad \text{for all} \qquad \varepsilon \in (0,1), \ s \in \mathbb{N}.$$

Proof. We first prove that $e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le \varepsilon^2$. As before for $h \ge 1$ we can estimate $(m_j h)^{b_j} \ge m_j^{b_j} \, h$ since $b_j \ge 1$. Then Lemma 2 yields

$$e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le -1 + \prod_{j=1}^{s} \left(1 + 2 \frac{\omega^{a_j m_j^{b_j}}}{1 - \omega^{a_j m_j^{b_j}}} \right).$$

Note that m_j is defined such that

$$\frac{\omega^{a_j m_j^{b_j}}}{1 - \omega^{a_j m_j^{b_j}}} \le \frac{3}{\pi^2} \, \frac{\log(1+\varepsilon^2)}{j^2}.$$

Therefore,

$$\begin{aligned} e^2(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) &\le -1 + \prod_{j=1}^{s} \left(1 + \frac{6}{\pi^2} \, \frac{\log(1+\varepsilon^2)}{j^2} \right) \\ &= -1 + \exp\left(\sum_{j=1}^{s} \log\left(1 + \frac{6}{\pi^2} \, \frac{\log(1+\varepsilon^2)}{j^2} \right) \right) \end{aligned}$$

$$\begin{aligned} &\le -1 + \exp\left(\frac{6}{\pi^2}\,\log(1+\varepsilon^2)\,\sum_{j=1}^{s} j^{-2}\right) \\ &\le -1 + \exp\big(\log(1+\varepsilon^2)\big) = \varepsilon^2, \end{aligned}$$

as claimed.

We now estimate m_j and then $n = \prod_{j=1}^{s} m_j$. Clearly, $m_j \ge 1$ for all $j \in \mathbb{N}$. We prove that $m_j = 1$ for large j. Indeed, $m_j = 1$ if

$$a_j \log \frac{1}{\omega} \ge \log\left(1 + \frac{\pi^2}{3}\,\frac{j^2}{\log(1+\varepsilon^2)}\right).$$

We take $p_1 = 1/(B+\delta)$. Since $a_j \ge \beta_1 2^{j\,p_1}$ then the last inequality holds for all $j \ge j^*$, where j^* is the smallest positive integer for which

$$j^* \ge \frac{1}{p_1\,\log\,2}\,\log\left(\frac{1}{\beta_1 \log \omega^{-1}}\,\log\left(1 + \frac{\pi^2}{3}\,\frac{[j^*]^2}{\log(1+\varepsilon^2)}\right)\right).$$

Clearly,

$$j^* = \frac{1}{p_1\,\log\,2}\log\,\log\,\varepsilon^{-1} + \mathcal{O}(1) \quad \text{as} \quad \varepsilon \to 0.$$

Without loss of generality we can restrict ourselves to $\varepsilon \le \mathrm{e}^{-\mathrm{e}}$, where $\mathrm{e} = \exp(1)$, so that $\log\log\varepsilon^{-1} \ge 1$. Then there exists a number $C_0 \ge 1$, independent of ε and s, such that

$$m_j = 1 \quad \text{for all} \quad j > \left\lfloor C_0 + \frac{1}{p_1 \log\,2}\,\log\,\log\,\varepsilon^{-1}\right\rfloor.$$

We now estimate m_j for $j \le \left\lfloor C_0 + \frac{1}{p_1\,\log\,2}\,\log\,\log\,\varepsilon^{-1}\right\rfloor$. Note that

$$\log(1+xy) \le \log(1+x) + \log(1+y) \quad \text{for all} \quad x, y \in [0,\infty).$$

Hence,

$$\log\left(1 + \frac{\pi^2}{3}\,\frac{j^2}{\log(1+\varepsilon^2)}\right) \le \log\left(1 + \frac{\pi^2}{3\,\log(1+\varepsilon^2)}\right) + \log(1+j^2).$$

Then $a_j \ge \beta_1 2^{j\,p_1}$ also implies that

$$C(\boldsymbol{a}) := \sup_{j\in\mathbb{N}} \frac{\log(1+j^2)}{a_j} < \infty.$$

Furthermore, there exists a number $C_1 \ge 1$, independent of ε and s such that

$$\log\left(1 + \frac{\pi^2}{3\,\log(1+\varepsilon^2)}\right) \le C_1 + 2\log\frac{1}{\varepsilon} \quad \text{for all} \quad \varepsilon \in (0,1).$$

This yields

$$m_j \le 1 + \left(\frac{C(\boldsymbol{a}) + C_1 + 2\log\varepsilon^{-1}}{\log\omega^{-1}}\right)^{1/b_j}$$

for all

$$j \le \left\lfloor C_0 + \frac{1}{p_1 \log\,2}\,\log\,\log\frac{1}{\varepsilon}\right\rfloor.$$

Let

$$k = \min\left(s, \left\lfloor C_0 + \frac{1}{p_1 \log\,2}\,\log\,\log\frac{1}{\varepsilon}\right\rfloor\right).$$

Then for $C = \max\left(C(\boldsymbol{a}) + C_1, -2\mathrm{e} + \log\omega^{-1}\right)$ we have

$$\max\left(1, \frac{C(\boldsymbol{a}) + C_1 + 2\log\varepsilon^{-1}}{\log\omega^{-1}}\right) \le \frac{C + 2\log\varepsilon^{-1}}{\log\omega^{-1}}$$

and

$$\begin{aligned} n &= \prod_{j=1}^{s} m_j = \prod_{j=1}^{k} m_j \le \prod_{j=1}^{k}\left(1 + \left(\frac{C + 2\log\varepsilon^{-1}}{\log\omega^{-1}}\right)^{1/b_j}\right) \\ &= \left(\frac{C + 2\log\varepsilon^{-1}}{\log\omega^{-1}}\right)^{\sum_{j=1}^{k} 1/b_j} \prod_{j=1}^{k}\left(1 + \left(\frac{\log\omega^{-1}}{C + 2\log\varepsilon^{-1}}\right)^{1/b_j}\right) \\ &\le \left(\frac{C + 2\log\varepsilon^{-1}}{\log\omega^{-1}}\right)^{B} 2^k. \end{aligned}$$

Note that

$$2^k \le 2^{C_0} \exp\left(\frac{1}{p_1}\log\log\frac{1}{\varepsilon}\right) = 2^{C_0} \log^{B+\delta}\frac{1}{\varepsilon}.$$

Therefore there is a positive number C_δ independent of ε^{-1} and s such that

$$n \le C_\delta \log^{2B+\delta}\left(1 + \frac{1}{\varepsilon}\right),$$

as claimed. This completes the proof. □

We briefly add that the extra exponent $B + \delta$ of $\log\varepsilon^{-1}$ is due to the ceiling function occurring in the definition of m_i. We do not know if a more careful analysis of

$$\prod_{j=1}^{k}\left(1 + \left(\frac{\log\omega^{-1}}{C + 2\log\varepsilon^{-1}}\right)^{1/b_j}\right)$$

will allow us to improve the exponent of $\log\varepsilon^{-1}$.

7. Relaxed exponential convergence

One may hope to get less restrictive conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ if the notion of exponential convergence is relaxed. So far we wanted to have $e(n,s) \le C(s)\, q^{n^p}$ for all $n, s \in \mathbb{N}$. Suppose we relax this inequality by demanding that

$$e(n,s) \le C(s)\, q^{(n/C_1(s))^p} \qquad \text{for all} \qquad n, s \in \mathbb{N}. \tag{12}$$

As before $q \in (0,1)$, $p > 0$ and both are independent of n and s, whereas

$$C, C_1 : \mathbb{N} \to (0,\infty)$$

are some functions of s. The case studied before was for $C_1(s) = 1$. We want to check what happens for general functions C_1.

Unfortunately, the function C_1 does not help and we have exactly the same conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ to achieve exponential convergence and exponential convergence with weak or polynomial or strong polynomial tractability.

For exponential convergence without any notion of tractability this may not be surprising since if we do not care about the dependence on s there is no real difference between n and $n/C_1(s)$. This may explain why we have the same condition on $\boldsymbol{b}$. However, the case with tractability, especially with weak tractability, may look a little surprising. In fact, we hoped that exponential convergence with weak

tractability would require a more relaxed condition on $\boldsymbol{a}$. We now show why this is not the case.

From (12) the information complexity now has a bound

$$n(\varepsilon, s) \le \left\lceil C_1(s) \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p} \right\rceil$$

for all $s \in \mathbb{N}$ and $\varepsilon \in (0,1)$.

For exponential convergence, we do not require anything about the functions C and C_1. For weak tractability we require that

$$C(s) = \exp(\exp(o(s))) \quad \text{and} \quad C_1(s) = \exp(o(s)) \qquad \text{as} \quad s \to \infty,$$

for polynomial tractability we require that there are positive A and τ such that

$$\log\, C(s) \le A\, s^{\tau} \quad \text{and} \quad C_1(s) \le A\, s^{\tau} \qquad \text{for all} \qquad s \in \mathbb{N},$$

and for strong polynomial tractability both $C(s)$ and $C_1(s)$ are uniformly bounded in s.

We show how Theorem 2 and its proof are modified when the function C_1 is present.

Theorem 5.

- *Assume that we have exponential convergence in the sense of* (12). *Then for arbitrary $s \in \mathbb{N}$ and for all $\boldsymbol{t} = (t_1, t_2, \ldots, t_s) \in \mathbb{N}_0$ with $\|\boldsymbol{t}\|_{s,\infty} := \max_{j=1,\ldots,s} t_j$ tending to infinity we have*

$$\liminf_{\|\boldsymbol{t}\|_{s,\infty} \to \infty} \frac{\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s} (1+t_j)^p} \ge \frac{2 \log q^{-1}}{\log \omega^{-1}}\, C_1(s)^{-p} > 0.$$

  In particular, this implies that

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \qquad \textit{and} \qquad p \le \frac{1}{B}$$

  independently of positive $\boldsymbol{a}$ and ω.

- *Assume that we have exponential convergence with weak tractability, i.e.,* (12) *holds with $C(s) = \exp(\exp(o(s))$ and $C_1(s) = \exp(o(s))$. Then for all $s \in \mathbb{N}$ and for all $\boldsymbol{t} = (t_1, t_2, \ldots, t_s) \in \mathbb{N}$ with $\prod_{j=1}^{s}(1+t_j)$ tending to infinity we have*

$$\liminf_{\prod_{j=1}^{s}(1+t_j) \to \infty} \frac{C_1^p(s) \sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s} (1+t_j)^p} \ge \frac{2 \log q^{-1}}{\log \omega^{-1}} > 0.$$

  In particular, this means that there exists a positive number β for which

$$\sum_{j=1}^{s} a_j \ge \beta\, 2^{s\,p} / C_1^p(s) \qquad \textit{for all} \qquad s \in \mathbb{N},$$

  and for any $p_1 \in (0,p)$ there is a positive β_1 such that

$$a_j \ge \beta_1\, 2^{j\,p_1} \qquad \textit{for all} \qquad j \in \mathbb{N}.$$

,

Proof. From (12) and Corollary 1 with $n = -1 + \prod_{j=1}^{s}(1+t_j)$ we obtain after some algebra

$$C(s) \geq \exp\left(-\frac{1}{2}\sum_{j=1}^{s} a_j t_j^{b_j} \log\frac{1}{\omega} \; - \; \frac{1}{2}\sum_{j=1}^{s}\log(1+t_j) \; + \left(\frac{n}{C_1(s)}\right)^p \log\frac{1}{q}\right).$$

This implies that

$$\begin{aligned} \frac{\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s}(1+t_j)^p} \quad &+ \quad \frac{2\log\, C(s) \,+\, \sum_{j=1}^{s}\log(1+t_j)}{[\prod_{j=1}^{s}(1+t_j)^p]\,\log\omega^{-1}} \\ &\geq \quad 2\left(1 - \frac{1}{\prod_{j=1}^{s}(1+t_j)}\right)^p \frac{\log q^{-1}}{\log\omega^{-1}}\, C_1(s)^{-p}. \end{aligned}$$

For fixed s, when $\|\boldsymbol{t}\|_{s,\infty}$ goes to infinity then the second term of the left-hand side goes to zero, and the first necessary condition holds. The condition on $\boldsymbol{b}$ is then shown as in the proof of Theorem 2.

Assume now exponential convergence and weak tractability. Then we rewrite the last displayed formula as

$$\begin{aligned} \frac{C_1^p(s)\,\sum_{j=1}^{s} a_j t_j^{b_j}}{\prod_{j=1}^{s}(1+t_j)^p} \quad &+ \quad \frac{2\,C_1^p(s)\,\log\, C(s) \,+\, C_1^p(s)\,\sum_{j=1}^{s}\log(1+t_j)}{[\prod_{j=1}^{s}(1+t_j)^p]\,\log\omega^{-1}} \\ &\geq \quad 2\left(1 - \frac{1}{\prod_{j=1}^{s}(1+t_j)}\right)^p \frac{\log q^{-1}}{\log\omega^{-1}}. \end{aligned}$$

For any c_1 there exists $C_2 > 0$ such that

$$\max(\log\, C(s), C_1(s)) \leq C_2 \exp(c_1 s) = C_2\, 2^{c_1 s/\log\, 2} \qquad \text{for all} \qquad s \in \mathbb{N}.$$

We now take a positive c_1 such that $c_1/\log\, 2 < p/(p+1)$. Then the corresponding part of the proof of Theorem 2 shows that the second term of the left-hand side goes to zero. This shows the successive point of the theorem.

To prove the last point take all $t_j = 1$ and let s tend to infinity. Then the limit inferior of $C_1^p(s)\,\sum_{j=1}^{s} a_j/2^{sp}$ is positive. This implies the existence of a positive number β such that

$$\sum_{j=1}^{s} a_j \geq \beta\, 2^{\,s\,p}/C_1^p(s) \qquad \text{for all} \qquad s \in \mathbb{N},$$

as claimed. Finally, since a_j's are ordered we have $s a_s \geq \sum_{j=1}^{s} a_j$ and

$$a_s \geq \frac{\beta}{s\, C_1^p(s)}\, 2^{\,s\,p} = \frac{\beta\, 2^{s(p-p_1)}}{s\, C_1^p(s)}\, 2^{\,s\,p_1} \geq \beta_1\, 2^{\,s\,p_1}$$

with $\beta_1 = \beta\, \inf_{s\in\mathbb{N}} \frac{2^{s(p-p_1)}}{s\, C_1^p(s)}$, which is positive since $C_1^p(s) = 2^{o(s)}$. This completes the proof. □

Comparing Theorems 2 and 5 we see that we have the same conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ for $C_1(s) = 1$ and for an arbitrary function C_1 if we want to guarantee exponential convergence with or without tractability. This shows that exponential convergence (7) and (12) are equivalent.

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