MATHEMATICS OF COMPUTATION
Volume 84, Number 293, May 2015, Pages 1017–1036
S 0025-5718(2014)02866-4
Article electronically published on August 27, 2014

PIECEWISE $\mathbf{H^1}$ FUNCTIONS AND VECTOR FIELDS ASSOCIATED WITH MESHES GENERATED BY INDEPENDENT REFINEMENTS

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Abstract. We consider piecewise H^1 functions and vector fields associated with a class of meshes generated by independent refinements and show that they can be effectively analyzed in terms of the number of refinement levels and the shape regularity of the subdomains that appear in the meshes. We derive Poincaré-Friedrichs inequalities and Korn's inequalities for such meshes and discuss an application to a discontinuous finite element method.

1. Introduction

In this paper we consider piecewise H^1 functions and vector fields associated with a class of meshes generated by independent refinements. First we recall the concept of a triangulation (conforming mesh) of an open polyhedral domain D in $\mathbb{R}^d$ ($d = 2$ or 3). We say that a collection of open polyhedral subdomains is a triangulation of D if: (i) they form a partition of D, i.e., they are pairwise disjoint and the union of their closures equals the closure of D, (ii) the intersection of the closures of two subdomains are either empty, a common vertex, a common edge, or a common face ($d = 3$).

Let Ω be a polyhedral domain in $\mathbb{R}^d$. For simplicity and concreteness we will first focus on simplicial meshes, but other types of meshes can be considered (cf. the discussion in Section 3.3 below). Let $\mathcal{T}_0 = \{\Omega\}$ be the initial triangulation of Ω consisting of Ω itself. We say a simplicial mesh (or partition) $\mathcal{T}_k$ ($k \geq 1$) is generated from $\mathcal{T}_0$ by k levels of independent refinements if for $1 \leq j \leq k$ the set of simplexes in $\mathcal{T}_j$ is obtained by introducing a simplicial triangulation on each of the subdomains in $\mathcal{T}_{j-1}$ independently. The meshes $\mathcal{T}_k$ for $k \geq 2$ are in general nonconforming, i.e., condition (ii) for a triangulation is not satisfied.

Figure 1.1 illustrates the complexity that can result from just three levels of independent mesh refinements starting with a square. It is also easy to see that many adaptive meshes, nonmatching domain decomposition meshes and standard conforming meshes can be generated by independent refinements (cf. Figure 1.2).

Remark 1.1. Note that the ratios between the diameters of neighboring simplexes in the meshes generated by independent refinements can be arbitrarily large, and the distribution of hanging nodes ($d = 2, 3$) and hanging edges ($d = 3$) can also be

Received by the editor October 30, 2012, and in revised form, August 7, 2013.
2010 *Mathematics Subject Classification.* Primary 65N30.
Key words and phrases. Nonconforming meshes, independent refinements, Poincaré-Friedrichs inequalities, Korn's inequalities, weakly over-penalized symmetric interior penalty method.
This work was supported in part by the National Science Foundation under Grant No. DMS-10-16332.

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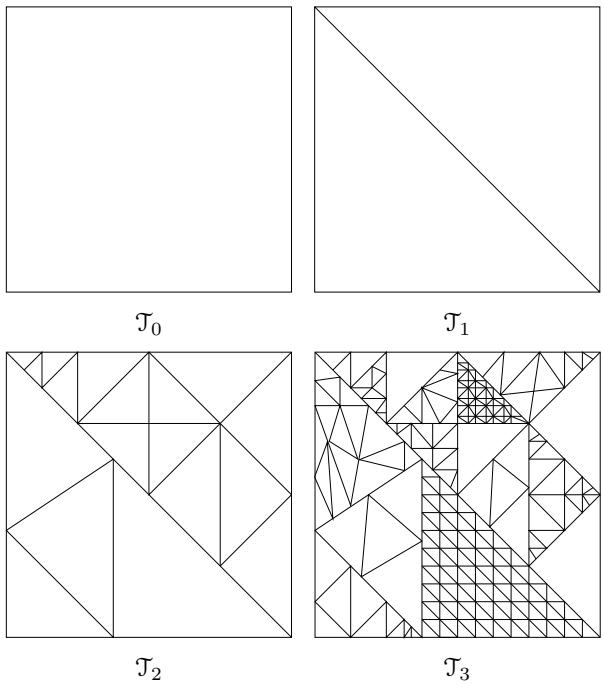


FIGURE 1.1. An example of three levels of independent mesh refinements

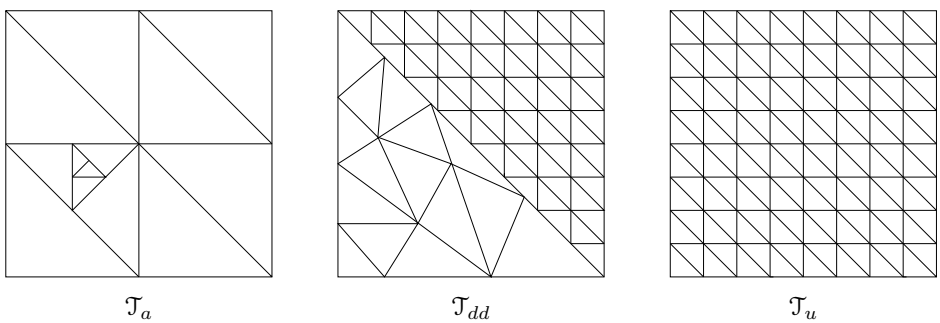


FIGURE 1.2. Adaptive mesh ($\mathcal{T}_a$), nonmatching domain decomposition mesh ($\mathcal{T}_{dd}$) and uniformly refined meshes ($\mathcal{T}_u$)

arbitrary. Therefore, the nonconforming meshes generated by independent refinements in general do not fit the existing theoretical frameworks (cf., for example, [14, 19, 28]).

Our goal is to show that, in spite of the potential high complexity of the meshes generated by independent refinements, piecewise H^1 functions and vector fields associated with such meshes can be analyzed effectively in terms of only the number of refinement levels and the shape regularity of the subdomains in the meshes.

The rest of the paper is organized as follows. We develop the tools for the analysis of piecewise H^1 functions and vector fields in Section 2. As applications we derive Poincaré-Friedrichs inequalities (resp., Korn's inequalities) for piecewise

H^1 functions (resp., H^1 vector fields) associated with independently refined meshes in Section 3, and establish error estimates for a weakly over-penalized symmetric interior penalty method for the Poisson problem in Section 4. We end the paper with some concluding remarks in Section 5.

Throughout the paper we will use standard notation for differential operators and Sobolev spaces that can be found, for example, in [12, 15].

2. Piecewise H^1 functions and vector fields

We will denote the set of the sides (open edges in 2D or open faces in 3D) of a mesh generated by k levels of independent refinements by $\mathcal{S}_k$, where $\mathcal{S}_0$ is the set of all the sides of Ω and $\mathcal{S}_j$ $(j \geq 1)$ is the union of $\mathcal{S}_{j-1}$ and the set of the sides inside the subdomains of $\mathcal{T}_{j-1}$ generated by the j-th refinement. Thus the set $\mathcal{S}_3$ for the mesh $\mathcal{T}_3$ in Figure 1.1 is the union of the four sets of edges depicted in Figure 2.1. We also set $\mathcal{S}_{-1}$ to be the empty set.

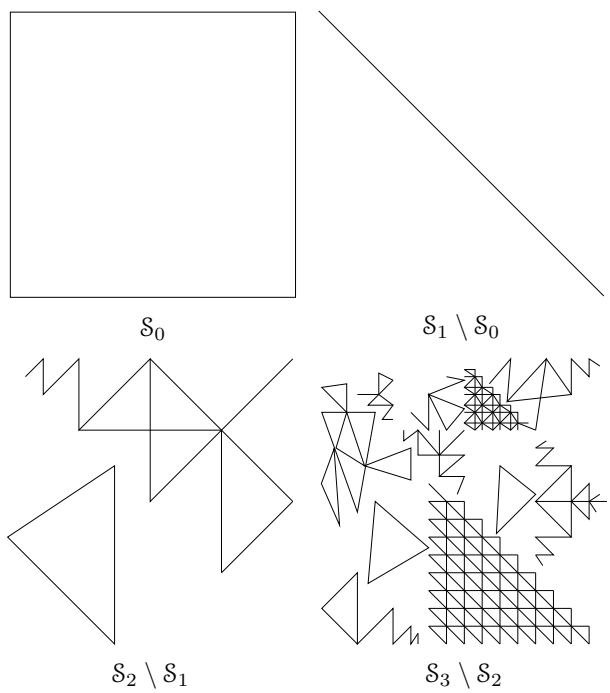


Figure 2.1. The edges in $\mathcal{S}_3$ for the mesh $\mathcal{T}_3$ in Figure 1.1

Remark 2.1. It follows from mathematical induction that for any $\sigma \in \mathcal{S}_k$ the simplexes in $\mathcal{T}_k$ that are from one side of σ induce a partition of σ (cf. Figure 1.1). The two partitions of σ induced by the two sides of σ are in general different, except for $\sigma \in \mathcal{S}_k \setminus \mathcal{S}_{k-1}$ where the partition from either side consists only of σ itself.

Remark 2.2. From mathematical induction we also see that each edge ($d = 2$) or face ($d = 3$) of a simplex in $\mathcal{T}_k$ is a subset of exactly one $\sigma \in \mathcal{S}_k$ (cf. Figure 1.1 and Figure 2.1).

Remark 2.3. Since $\mathcal{T}_k$ is a refinement of $\mathcal{T}_j$ for $1 \le j \le k-1$, the shape regularity of the simplexes in $\mathcal{T}_1, \ldots, \mathcal{T}_{k-1}$ are determined by the shape regularity of the simplexes in $\mathcal{T}_k$.

2.1. **Piecewise H^1 functions.** The space of piecewise H^1 functions associated with $\mathcal{T}_k$ is denoted by $H^1(\mathcal{T}_k)$, i.e.,

$$H^1(\mathcal{T}_k) = \{v \in L_2(\Omega) : v_T = v\big|_T \in H^1(T) \quad \forall T \in \mathcal{T}_k\}. \tag{2.1}$$

Note that these spaces are nested, i.e.,

$$H^1(\Omega) = H^1(\mathcal{T}_0) \subseteq H^1(\mathcal{T}_1) \subseteq H^1(\mathcal{T}_2) \subseteq \cdots \subseteq H^1(\mathcal{T}_k).$$

Let $\sigma \in \mathcal{S}_k$ be an edge ($d = 2$) or face ($d = 3$) in Ω and let n_σ be a unit normal of σ. The side of σ where n_σ is pointing will be designated as the positive side and the opposite side will be the negative side. We define the jump of $v \in H^1(\mathcal{T}_k)$ over σ to be the vector

$$[\![v]\!]_\sigma = \big[\big(v_-\big|_\sigma\big) - \big(v_+\big|_\sigma\big)\big] n_\sigma, \tag{2.2}$$

where $v_+\big|_\sigma$ (resp., $v_-\big|_\sigma$) is the trace of v from the positive (resp., negative) side of σ. If $\sigma \in \mathcal{S}$ is a subset of $\partial\Omega$, then we define $[\![v]\!]_\sigma$ to be the vector vn_σ on σ, where n_σ is the unit normal of σ pointing towards the outside of Ω.

Remark 2.4. The definition of $[\![v]\!]_\sigma$ in (2.2) is independent of the choice of n_σ.

The following result concerning piecewise H^1 functions with respect to a triangulation of a single polyhedral domain is crucial for understanding $H^1(\mathcal{T}_k)$.

Lemma 2.5. *Let D be a polyhedral domain in $\mathbb{R}^d$, $\mathcal{T}_D$ a simplicial triangulation of D, $\mathcal{S}_D^I$ the set of the edges ($d = 2$) or faces ($d = 3$) of the simplexes in $\mathcal{T}_D$ that are interior to D, and let v be a piecewise H^1 function on D with respect to $\mathcal{T}_D$. There exists a function $w \in H^1(D)$ such that*

$$\begin{aligned} \sum_{T \in \mathcal{T}_D} \big(h_T^{-1}\|v - w\|^2_{L_2(\partial T)} + h_T^{-2}\|v - w\|^2_{L_2(T)} + \|\nabla(v - w)\|^2_{L_2(T)}\big) \\ \le C\Big[\sum_{T \in \mathcal{T}_D} \|\nabla v\|^2_{L_2(T)} + \sum_{\sigma \in \mathcal{S}_D^I} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big], \end{aligned} \tag{2.3}$$

where the positive constant C in (2.3) *depends only on the shape regularity of the simplexes in $\mathcal{T}_D$.*

Proof. Let v be an arbitrary piecewise H^1 function on D with respect to $\mathcal{T}_D$ and $\bar{v}$ the piecewise constant function that takes the mean value of v on each $T \in \mathcal{T}_D$. We have a standard estimate [12, 15]

$$h_T^{-1}\|v - \bar{v}\|^2_{L_2(\partial T)} + h_T^{-2}\|v - \bar{v}\|^2_{L_2(T)} \le C\|\nabla v\|^2_{L_2(T)} \qquad \forall T \in \mathcal{T}_D. \tag{2.4}$$

Let $w \in H^1(D)$ be the P_1 finite element function whose value at any vertex p of $\mathcal{T}_D$ is the average of $\bar{v}$ at the simplexes that share p as a common vertex. A simple calculation (cf. [12, Section 10.6]) shows that

$$\begin{aligned} \sum_{T \in \mathcal{T}_D} \big(h_T^{-1}\|\bar{v} - w\|^2_{L_2(\partial T)} + h_T^{-2}\|\bar{v} - w\|^2_{L_2(T)} + \|\nabla w\|^2_{L_2(T)}\big) \\ \le C \sum_{\sigma \in \mathcal{S}_D^I} h_\sigma^{-1}\|[\![\bar{v}]\!]_\sigma\|^2_{L_2(\sigma)} = C \sum_{\sigma \in \mathcal{S}_D^I} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![\bar{v}]\!]_\sigma\|^2_{L_2(\sigma)}. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) we find

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_D}\big(h_T^{-1}\|v-w\|^2_{L_2(\partial T)}+h_T^{-2}\|v-w\|^2_{L_2(T)}+\|\nabla(v-w)\|^2_{L_2(T)}\big)\\
&\quad\le 2\sum_{T\in\mathcal{T}_D}\big(h_T^{-1}\|v-\bar v\|^2_{L_2(\partial T)}+h_T^{-2}\|v-\bar v\|^2_{L_2(T)}+\|\nabla v\|^2_{L_2(T)}\big)\\
&\qquad+2\sum_{T\in\mathcal{T}_D}\big(h_T^{-1}\|\bar v-w\|^2_{L_2(\partial T)}+h_T^{-2}\|\bar v-w\|^2_{L_2(T)}+\|\nabla w\|^2_{L_2(T)}\big)\\
&\quad\le C\Big(\sum_{T\in\mathcal{T}_D}\|\nabla v\|^2_{L_2(T)}+\sum_{\sigma\in\mathcal{S}_D^I}h_\sigma^{-1}\|\Pi_{0,\sigma}[\![\bar v]\!]_\sigma\|^2_{L_2(\sigma)}\Big)\\
&\quad\le C\Big(\sum_{T\in\mathcal{T}_D}\|\nabla v\|^2_{L_2(T)}+\sum_{\sigma\in\mathcal{S}_D^I}h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}+\sum_{T\in\mathcal{T}_D}h_\sigma^{-1}\|v-\bar v\|^2_{L_2(\partial T)}\Big)\\
&\quad\le C\Big(\sum_{T\in\mathcal{T}_D}\|\nabla v\|^2_{L_2(T)}+\sum_{\sigma\in\mathcal{S}_D^I}h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big).
\end{aligned}$$

□

Let the piecewise gradient of $v\in H^1(\mathcal{T}_j)$ be defined by

$$(\nabla_j v)\big|_T=\nabla v_T\quad\forall\,T\in\mathcal{T}_j. \tag{2.6}$$

The next result provides a link between $H^1(\mathcal{T}_j)$ and $H^1(\mathcal{T}_{j-1})$.

Lemma 2.6. *Given any $v\in H^1(\mathcal{T}_j)$, there exists $w\in H^1(\mathcal{T}_{j-1})$ such that*

$$\begin{aligned}
\|\nabla_{j-1}w\|^2_{L_2(\Omega)}&+\sum_{T\in\mathcal{T}_j}\big(h_T^{-1}\|v-w\|^2_{L_2(\partial T)}+h_T^{-2}\|v-w\|^2_{L_2(T)}\big)\\
&\le C\Big(\|\nabla_j v\|^2_{L_2(\Omega)}+\sum_{\sigma\in\mathcal{S}_j\setminus\mathcal{S}_{j-1}}h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big),
\end{aligned} \tag{2.7}$$

where the positive constant C depends only on the shape regularity of the simplexes in $\mathcal{T}_j$.

Proof. Let $v\in H^1(\mathcal{T}_j)$ be arbitrary, and for each $D\in\mathcal{T}_{j-1}$, let $\mathcal{T}_D$ be the triangulation of D generated by the j-th level refinement. There exists, by Lemma 2.5, a function $w_D\in H^1(D)$ such that

$$\begin{aligned}
\sum_{T\in\mathcal{T}_D}&\big(h_T^{-1}\|v-w_D\|^2_{L_2(\partial T)}+h_T^{-2}\|v-w_D\|^2_{L_2(T)}+\|\nabla(v-w_D)\|^2_{L_2(T)}\big)\\
&\le C\Big(\sum_{T\in\mathcal{T}_D}\|\nabla v\|^2_{L_2(T)}+\sum_{\sigma\in\mathcal{S}_D^I}h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big).
\end{aligned} \tag{2.8}$$

Let $w \in H^1(\mathcal{T}_{j-1})$ be defined by $w\big|_D = w_D$ for all $D \in H^1(\mathcal{T}_{j-1})$. It follows from (2.8) that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_j} \big(h_T^{-1}\|v-w\|^2_{L_2(\partial T)} + h_T^{-2}\|v-w\|^2_{L_2(T)} + \|\nabla(v-w)\|^2_{L_2(T)}\big) \\
&\quad = \sum_{D\in\mathcal{T}_{j-1}} \sum_{T\in\mathcal{T}_D} \big(h_T^{-1}\|v-w\|^2_{L_2(\partial T)} + h_T^{-2}\|v-w_D\|^2_{L_2(T)} \\
&\qquad\qquad + \|\nabla(v-w_D)\|^2_{L_2(T)}\big) \\
&\quad \le C \sum_{D\in\mathcal{T}_{j-1}} \Big(\sum_{T\in\mathcal{T}_D} \|\nabla v\|^2_{L_2(T)} + \sum_{\sigma\in\mathcal{S}_D^I} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big).
\end{aligned} \tag{2.9}$$

The estimate (2.9) implies (2.7) immediately. □

We can use Lemma 2.6 to build links between $H^1(\mathcal{T}_k)$ and $H^1(\mathcal{T}_j)$ for $0 \le j \le k-1$.

Proposition 2.7. *Given any $v \in H^1(\mathcal{T}_k)$, there exist $v_j \in H^1(\mathcal{T}_j)$ for $0 \le j \le k-1$ such that*

$$\begin{aligned}
&\|\nabla_j v_j\|^2_{L_2(\Omega)} + \sum_{T\in\mathcal{T}_j} \big(h_T^{-1}\|v-v_j\|^2_{L_2(\partial T)} + h_T^{-2}\|v-v_j\|^2_{L_2(T)}\big) \\
&\qquad\qquad \le C\Big(\|\nabla_k v\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_j} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big),
\end{aligned} \tag{2.10}$$

where the positive constant C depends only on k and the shape regularity of the simplexes in $\mathcal{T}_k$.

Proof. By Lemma 2.6, there exists $v_j \in H^1(\mathcal{T}_j)$ for $0 \le j \le k$ such that $v_k = v$ and

$$\begin{aligned}
&\|\nabla_{j-1} v_{j-1}\|^2_{L_2(\Omega)} + \sum_{T\in\mathcal{T}_j} \big(h_T^{-1}\|v_j-v_{j-1}\|^2_{L_2(\partial T)} + h_T^{-2}\|v_j-v_{j-1}\|^2_{L_2(T)}\big) \\
&\qquad \le C\Big(\|\nabla_j v_j\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_j\setminus\mathcal{S}_{j-1}} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v_j]\!]_\sigma\|^2_{L_2(\sigma)}\Big) \qquad \text{for } 1 \le j \le k,
\end{aligned} \tag{2.11}$$

where the constant C depends only on the shape regularity of the simplexes in $\mathcal{T}_k$ (cf. Remark 2.3).

In view of Remark 2.1 and Remark 2.2, the estimate (2.11) implies

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_j\setminus\mathcal{S}_{j-1}} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v_j]\!]_\sigma\|^2_{L_2(\sigma)} \\
&\quad \le C \sum_{\sigma\in\mathcal{S}_j\setminus\mathcal{S}_{j-1}} h_\sigma^{-1}\Big(\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)} + \sum_{\ell=j+1}^{k} \|\Pi_{0,\sigma}[\![v_\ell - v_{\ell-1}]\!]_\sigma\|^2_{L_2(\sigma)}\Big) \\
&\quad \le C\Big[\sum_{\sigma\in\mathcal{S}_j\setminus\mathcal{S}_{j-1}} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)} \\
&\qquad + \sum_{\ell=j+1}^{k} \Big(\|\nabla_\ell v_\ell\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_\ell\setminus\mathcal{S}_{\ell-1}} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v_\ell]\!]_\sigma\|^2_{L_2(\sigma)}\Big)\Big]
\end{aligned} \tag{2.12}$$

for $1 \le j \le k$.

Combining (2.11) and (2.12), we find

$$\|\nabla_{j-1} v_{j-1}\|^2_{L_2(\Omega)} \le C\Big(\|\nabla_k v\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_{j-1}} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big) \tag{2.13}$$

for $1 \le j \le k$.

From (2.11) we also have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_j}\big(h_T^{-1}\|v-v_j\|^2_{L_2(\partial T)} + h_T^{-2}\|v-v_j\|^2_{L_2(T)}\big)\\
&\qquad\le \sum_{T\in\mathcal{T}_j}(k-j)\sum_{\ell=j+1}^{k}\big(h_T^{-1}\|v_\ell - v_{\ell-1}\|^2_{L_2(\partial T)} + h_T^{-2}\|v_\ell - v_{\ell-1}\|^2_{L_2(T)}\big)\\
&\qquad\le C\sum_{\ell=j+1}^{k}\sum_{T\in\mathcal{T}_\ell}\big(h_T^{-1}\|v_\ell - v_{\ell-1}\|^2_{L_2(\partial T)} + h_T^{-2}\|v_\ell - v_{\ell-1}\|^2_{L_2(T)}\big)\\
&\qquad\le C\sum_{\ell=j+1}^{k}\Big(\|\nabla_\ell v_\ell\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_\ell\setminus\mathcal{S}_{\ell-1}} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v_\ell]\!]_\sigma\|^2_{L_2(\sigma)}\Big),
\end{aligned}$$

which together with (2.12) and (2.13) implies

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_j}\big(h_T^{-1}\|v-v_j\|^2_{L_2(\partial T)} + h_T^{-2}\|v-v_j\|^2_{L_2(T)}\big)\\
&\qquad\le C\Big(\|\nabla_k v\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_j} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)}\Big) \qquad \text{for } 0\le j\le k-1.
\end{aligned}$$

□

2.2. **Piecewise H^1 vector fields.** The space of piecewise H^1 vector fields associated with $\mathcal{T}_k$ is denoted by $[H^1(\mathcal{T}_k)]^d$. Let $\sigma \in \mathcal{S}_k$ be an edge ($d = 2$) or face ($d = 3$) in Ω and let n_σ be a unit normal of σ. We define the jump of $\boldsymbol{v} \in [H^1(\mathcal{T}_k)]^d$ over σ to be the $d \times d$ tensor

$$[\![\boldsymbol{v}]\!]_\sigma = \big[\big(\boldsymbol{v}_-\big|_\sigma\big) - \big(\boldsymbol{v}_+\big|_\sigma\big)\big] \otimes n_\sigma. \tag{2.14}$$

If $\sigma \in \mathcal{S}_k$ is a subset of $\partial\Omega$, then we define $[\![\boldsymbol{v}]\!]_\sigma$ to be $\boldsymbol{v} \otimes n_\sigma$ on σ, where n_σ is the unit normal of σ pointing towards the outside of Ω.

Let the piecewise gradient tensor (of the first order derivatives) of $\boldsymbol{v}$ be defined by

$$(\nabla_k \otimes \boldsymbol{v})\big|_T = \nabla \otimes \boldsymbol{v}_T \qquad \forall\, T \in \mathcal{T}_k.$$

For any $\boldsymbol{v} \in [H^1(\mathcal{T}_k)]^d$, the piecewise strain tensor $\epsilon_k(\boldsymbol{v})$ is defined by

$$\epsilon_k(\boldsymbol{v})\big|_T = \epsilon(\boldsymbol{v}_T) = \frac{1}{2}\big[(\nabla\otimes\boldsymbol{v}_T) + (\nabla\otimes\boldsymbol{v}_T)^t\big] \qquad \forall\, T \in \mathcal{T}_k.$$

Let RM be the space of rigid motions defined by

$$\mathrm{RM} = \{\boldsymbol{a} + \boldsymbol{\eta}\boldsymbol{x} : \boldsymbol{a} \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{\eta} \in \mathfrak{so}(d)\},$$

where $\boldsymbol{x} = [x_1, \ldots, x_d]^t$ and $\mathfrak{so}(d)$ is the Lie algebra of anti-symmetric $d\times d$ matrices. Then $\|\epsilon_k(\boldsymbol{v})\|_{L_2(\Omega)} = 0$ if and only if $\boldsymbol{v}_T$ belongs to the space $\mathrm{RM}(T)$ (the restriction of RM to T) for all $T \in \mathcal{T}_k$.

For any simplex T, let Π_T be the interpolation operator from $[H^1(T)]^d$ to the space $\mathrm{RM}(T)$ be defined by

$$\Big|\int_T (\boldsymbol{\zeta} - \Pi_T\boldsymbol{\zeta})dx\Big| = \Big|\int_T \nabla\times(\boldsymbol{\zeta} - \Pi_T\boldsymbol{\zeta})dx\Big| = 0 \qquad \forall\,\boldsymbol{\zeta}\in[H^1(T)]^d.$$

Then we have [7, Corollary A.3]

$$\|\nabla\otimes(\boldsymbol{\zeta} - \Pi_T\boldsymbol{\zeta})\|_{L_2(T)} \le C\|\epsilon(\boldsymbol{\zeta})\|_{L_2(T)} \qquad \forall\,\boldsymbol{\zeta}\in[H^1(T)]^d. \tag{2.15}$$

It follows from (2.15), a classical Poincaré-Friedrichs inequality, and the trace theorem with scaling that

$$\begin{aligned} h_T^{-1}\|\boldsymbol{\zeta} - \Pi_T\boldsymbol{\zeta}\|^2_{L_2(\partial T)} + h_T^{-2}\|\boldsymbol{\zeta} - \Pi_T\boldsymbol{\zeta}\|^2_{L_2(T)} \\ \le C\|\epsilon(\boldsymbol{\zeta})\|^2_{L_2(T)} \qquad \forall\,\boldsymbol{\zeta}\in[H^1(T)]^d. \end{aligned} \tag{2.16}$$

The following lemma is an analog of Lemma 2.5.

Lemma 2.8. *Let D be a polyhedral domain in $\mathbb{R}^d$, $\mathcal{T}_D$ a simplicial triangulation of D, $\mathcal{S}_D^I$ the set of the edges ($d=2$) or the faces ($d=3$) of the simplexes in $\mathcal{T}_D$ that are interior to D, and let $\boldsymbol{v}$ be a piecewise H^1 vector field on D with respect to $\mathcal{T}_D$. There exists a vector field $\boldsymbol{w}\in[H^1(D)]^d$ such that*

$$\begin{aligned} \sum_{T\in\mathcal{T}_D}\big(h_T^{-1}\|\boldsymbol{v}-\boldsymbol{w}\|^2_{L_2(\partial T)} + h_T^{-2}\|\boldsymbol{v}-\boldsymbol{w}\|^2_{L_2(T)} + \|\nabla\otimes(\boldsymbol{v}-\boldsymbol{w})\|^2_{L_2(T)}\big) \\ \le C\Big[\sum_{T\in\mathcal{T}_D}|\epsilon(\boldsymbol{v})|^2_{L_2(T)} + \sum_{\sigma\in\mathcal{S}_D^I} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\boldsymbol{v}]\!]_\sigma\|^2_{L_2(\sigma)}\Big], \end{aligned} \tag{2.17}$$

where the positive constant C depends only on the shape regularity of the simplexes in $\mathcal{T}_D$.

Proof. Let $\boldsymbol{v}$ be an arbitrary piecewise H^1 vector field on Ω with respect to $\mathcal{T}_D$ and $\tilde{\boldsymbol{v}}$ the piecewise linear vector field defined by

$$\tilde{\boldsymbol{v}}\big|_T = \Pi_T\boldsymbol{v}_T \qquad \forall\,T\in\mathcal{T}_k. \tag{2.18}$$

It follows from (2.15), (2.16) and (2.18) that

$$\begin{aligned} h_T^{-1}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L_2(\partial T)} + h_T^{-2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L_2(T)} + \|\nabla\otimes(\boldsymbol{v}-\tilde{\boldsymbol{v}})\|^2_{L_2(T)} \\ \le C\|\epsilon(\boldsymbol{v})\|^2_{L_2(T)} \qquad \forall\,T\in\mathcal{T}_D. \end{aligned} \tag{2.19}$$

Let $\boldsymbol{w}\in[H^1(D)]^d$ be the P_1 vector finite element function whose value at any vertex p of $\mathcal{T}_D$ is the average of $\tilde{\boldsymbol{v}}$ at p from the simplexes that share p as a common vertex. We have, by a direct calculation,

$$\begin{aligned} &\sum_{T\in\mathcal{T}_D}\big(h_T^{-1}\|\tilde{\boldsymbol{v}}-\boldsymbol{w}\|^2_{L_2(\partial T)} + h_T^{-2}\|\tilde{\boldsymbol{v}}-\boldsymbol{w}\|^2_{L_2(T)} + \|\nabla\otimes(\tilde{\boldsymbol{v}}-\boldsymbol{w})\|^2_{L_2(T)}\big) \\ &\qquad \le C\sum_{\sigma\in\mathcal{S}_D^I} h_\sigma^{-1}\|[\![\tilde{\boldsymbol{v}}]\!]_\sigma\|^2_{L_2(\sigma)} \\ &\qquad = C\sum_{\sigma\in\mathcal{S}_D^I} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\tilde{\boldsymbol{v}}]\!]_\sigma\|^2_{L_2(\sigma)} \\ &\qquad \le C\Big(\sum_{\sigma\in\mathcal{S}_D^I} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\boldsymbol{v}]\!]_\sigma\|^2_{L_2(\sigma)} + \sum_{T\in\mathcal{T}_D} h_\sigma^{-1}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L_2(\partial T)}\Big). \end{aligned} \tag{2.20}$$

The estimate (2.17) follows from (2.19) and (2.20). □

The following lemma is an analog of Lemma 2.6 and we omit its proof since it can be derived from Lemma 2.8 in the same way that Lemma 2.6 is derived from Lemma 2.5.

Lemma 2.9. *Given any $\boldsymbol{v} \in [H^1(\mathcal{T}_j)]^d$, there exists $\boldsymbol{w} \in [H^1(\mathcal{T}_{j-1})]^d$ such that*

$$\|\nabla \otimes \boldsymbol{w}\|^2_{L_2(\Omega)} + \sum_{T\in\mathcal{T}_j} \big(h_T^{-1}\|\boldsymbol{v}-\boldsymbol{w}\|^2_{L_2(\partial T)} + h_T^{-2}\|\boldsymbol{v}-\boldsymbol{w}\|^2_{L_2(T)}\big)$$
$$\leq C\Big(\|\epsilon_j(\boldsymbol{v})\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_j\setminus\mathcal{S}_{j-1}} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\boldsymbol{v}]\!]_\sigma\|^2_{L_2(\sigma)}\Big),$$

where the positive constant C depends only on the shape regularity of the simplexes in $\mathcal{T}_j$.

The following proposition is an analog of Proposition 2.7. Again we omit its proof since it follows from Lemma 2.9 in exactly the same way that Proposition 2.7 follows from Lemma 2.6.

Proposition 2.10. *Given any $\boldsymbol{v} \in [H^1(\mathcal{T}_k)]^d$, there exist $\boldsymbol{v}_j \in [H^1(\mathcal{T}_j)]^d$ for $0 \leq j \leq k-1$ such that*

$$\|\nabla \otimes \boldsymbol{v}_j\|^2_{L_2(\Omega)} + \sum_{T\in\mathcal{T}_j} \big(h_T^{-1}\|\boldsymbol{v}-\boldsymbol{v}_j\|^2_{L_2(\partial T)} + h_T^{-2}\|\boldsymbol{v}-\boldsymbol{v}_j\|^2_{L_2(T)}\big)$$
$$\leq C\Big(\|\epsilon_k(\boldsymbol{v})\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_j} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\boldsymbol{v}]\!]_\sigma\|^2_{L_2(\sigma)}\Big), \tag{2.21}$$

where the positive constant C depends only on k and the shape regularity of the simplexes in $\mathcal{T}_k$.

3. Poincaré-Friedrichs inequalities and Korn's inequalities

In this section we will establish Poincaré-Friedrichs inequalities and Korn's inequalities for functions and vector fields that are piecewise H^1 with respect to a simplicial mesh generated by independent refinements. These inequalities complement existing results on Poincaré-Friedrichs inequalities and Korn's inequalities for piecewise H^1 functions and vector fields (cf. [6, 7] and the references therein) since the meshes generated by independent refinements can have very high complexity (cf. Remark 1.1).

3.1. Poincaré-Friedrichs inequalities.

We can easily derive Poincaré-Friedrichs inequalities by using Proposition 2.7.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain, Γ a measurable subset of $\partial\Omega$ with a positive $(d-1)$-dimensional measure, and let the simplicial mesh $\mathcal{T}_k$ on Ω be generated from $\mathcal{T}_0 = \{\Omega\}$ through k levels of independent refinements. Then there exists a positive constant C depending only on k and the shape regularity of the simplexes in $\mathcal{T}_k$ such that for all $v \in H^1(\mathcal{T}_k)$ we have*

$$\|v\|^2_{L_2(\Omega)} \leq C\Big[\|\nabla_k v\|^2_{L_2(\Omega)} + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|^2_{L_2(\sigma)} + \Big(\int_\Gamma v\,ds\Big)^2\Big], \tag{3.1}$$

where ds denotes infinitesimal arclength ($d = 2$) *or surface area* ($d = 3$), *and*

$$\|v\|_{L_2(\Omega)}^2 \le C\Big[\|\nabla_k v\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|_{L_2(\sigma)}^2 + \Big(\int_\Omega v\,dx\Big)^2\Big]. \tag{3.2}$$

Proof. First we recall the following classical Poincaré-Friedrichs inequalities [25]:

$$\|w\|_{L_2(\Omega)} \le C_\Omega\Big(\|\nabla w\|_{L_2(\Omega)} + \Big|\int_\Gamma w\,ds\Big|\Big) \qquad \forall\, w \in H^1(\Omega), \tag{3.3}$$

$$\|w\|_{L_2(\Omega)} \le C_\Omega\Big(\|\nabla w\|_{L_2(\Omega)} + \Big|\int_\Omega w\,dx\Big|\Big) \qquad \forall\, w \in H^1(\Omega). \tag{3.4}$$

Let $v \in H^1(\mathcal{T}_k)$ be arbitrary and let $v_0 \in H^1(\mathcal{T}_0) = H^1(\Omega)$ be the function provided by Proposition 2.7. It follows from (2.10) that

$$\begin{aligned}\|\nabla v_0\|_{L_2(\Omega)}^2 + \|v - v_0\|_{L_2(\partial\Omega)}^2 + \|v - v_0\|_{L_2(\Omega)}^2 \\ \le C\Big(\|\nabla_k v\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|_{L_2(\sigma)}^2\Big).\end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), we arrive at

$$\begin{aligned}\|v\|_{L_2(\Omega)}^2 &\le 2\|v - v_0\|_{L_2(\Omega)}^2 + 2\|v_0\|_{L_2(\Omega)}^2 \\ &\le C\Big[\|\nabla_k v\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|_{L_2(\sigma)}^2 + \|\nabla v_0\|_{L_2(\Omega)}^2 + \Big(\int_\Gamma v_0\,ds\Big)^2\Big] \\ &\le C\Big[\|\nabla_k v\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|_{L_2(\sigma)}^2 + \|v - v_0\|_{L_2(\partial\Omega)}^2 + \Big(\int_\Gamma v\,ds\Big)^2\Big] \\ &\le C\Big[\|\nabla_k v\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{0,\sigma}[\![v]\!]_\sigma\|_{L_2(\sigma)}^2 + \Big(\int_\Gamma v\,ds\Big)^2\Big].\end{aligned}$$

The derivation of (3.2) from (3.4) and (3.5) is completely analogous. □

3.2. **Korn's inequalities.** We can also easily derive Korn's inequalities by using Proposition 2.10.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain, Γ a measurable subset of $\partial\Omega$ with a positive $(d-1)$-dimensional measure, and let the simplicial mesh $\mathcal{T}_k$ on Ω be generated from $\mathcal{T}_0 = \{\Omega\}$ through k levels of independent refinements. Then there exists a positive constant C depending only on k and the shape regularity of the simplexes in $\mathcal{T}_k$ such that for all $\boldsymbol{v} \in [H^1(\mathcal{T}_k)]^d$ we have*

$$\|\nabla_k \otimes \boldsymbol{v}\|_{L_2(\Omega)}^2 \le C\Big[\|\epsilon_k(\boldsymbol{v})\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\boldsymbol{v}]\!]_\sigma\|_{L_2(\sigma)}^2 + \|\boldsymbol{v}\|_{L_2(\Gamma)}^2\Big], \tag{3.6}$$

where ds denotes infinitesimal arclength ($d = 2$) *or surface area* ($d = 3$), *and*

$$\|\nabla_k \otimes \boldsymbol{v}\|_{L_2(\Omega)}^2 \le C\Big[\|\epsilon_k(\boldsymbol{v})\|_{L_2(\Omega)}^2 + \sum_{\sigma\in\mathcal{S}_k\setminus\mathcal{S}_0} h_\sigma^{-1}\|\Pi_{1,\sigma}[\![\boldsymbol{v}]\!]_\sigma\|_{L_2(\sigma)}^2 + \|\boldsymbol{v}\|_{L_2(\Omega)}^2\Big]. \tag{3.7}$$

The proof of Theorem 3.2, which is based on Proposition 2.10 and the classical Korn's inequalities [7, 26]

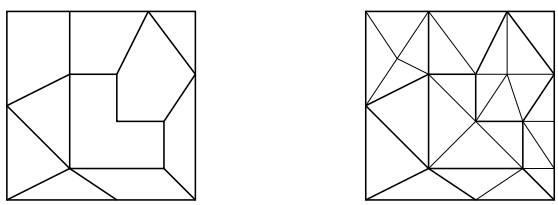
$$\|\nabla \otimes \boldsymbol{v}\|_{L_2(\Omega)} \le C\Big(\|\epsilon(\boldsymbol{v})\|_{L_2(\Omega)} + \|\boldsymbol{v}\|_{L_2(\Gamma)}\Big) \qquad \forall\, \boldsymbol{v} \in [H^1(\Omega)]^d, \tag{3.8}$$

$$\|\nabla \otimes \boldsymbol{v}\|_{L_2(\Omega)} \le C\big(\|\epsilon(\boldsymbol{v})\|_{L_2(\Omega)} + \|\boldsymbol{v}\|_{L_2(\Omega)}\big) \qquad \forall\, \boldsymbol{v} \in [H^1(\Omega)]^d, \tag{3.9}$$

is similar to the proof of Theorem 3.1 and hence omitted.

3.3. **Extensions.** The simplexes in $\mathcal{T}_1, \ldots, \mathcal{T}_k$ can be replaced by other polyhedral subdomains such as quadrilaterals ($d=2$), parallelepipeds ($d=3$), prisms ($d=3$) and pyramids ($d = 3$), and we can generalize the results in Section 3.1 and Section 3.2 to these meshes. The key is to extend Lemma 2.5 and Lemma 2.8 to a triangulation $\mathcal{T}_D$ consisting of arbitrary polyhedral subdomains.

We can accomplish this by introducing a simplicial triangulation $\tilde{\mathcal{T}}_D$ of D such that each subdomain in $\mathcal{T}_D$ is the union of a collection of simplexes in $\tilde{\mathcal{T}}_D$ whose diameters are comparable to the diameter of that subdomain. An example of such a construction for a general triangulation $\mathcal{T}_D$ of a square D is given in Figure 3.1.



$\mathcal{T}_D$ $\tilde{\mathcal{T}}_D$

FIGURE 3.1. An example of $\mathcal{T}_D$ and $\tilde{\mathcal{T}}_D$

By applying Lemma 2.5 (resp., Lemma 2.8) for piecewise H^1 functions (resp., vector fields) with respect to $\tilde{\mathcal{T}}_D$, we see that the estimate (2.3) (resp., (2.17)) remains valid for $\mathcal{T}_D$, but now the constant C will depend on the shapes of the simplexes in $\tilde{\mathcal{T}}_D$, which in some sense provide a measure of the shape regularity of the subdomains in $\mathcal{T}_D$.

Proposition 2.7 and Proposition 2.10 for general independently refined meshes follow from Lemma 2.5 and Lemma 2.8 for general $\mathcal{T}_D$ in the same way that Proposition 2.7 and Proposition 2.10 for independently refined simplicial meshes follow from Lemma 2.5 and Lemma 2.8 for simplicial $\mathcal{T}_D$. In turn they imply Theorem 3.1 and Theorem 3.2 for general independently refined meshes, where the constant C will also depend on the shape regularity of the simplexes in the additional refinements that turn $\mathcal{T}_k$ into a simplicial mesh $\tilde{\mathcal{T}}_k$.

For concreteness we have only presented the most popular forms of the Poincaré-Friedrichs inequalities and Korn's inequalities. The more general forms of these inequalities in [6, 7, 23] can also be extended to independently refined meshes by similar techniques.

4. A WEAKLY OVER-PENALIZED SYMMETRIC INTERIOR PENALTY METHOD

In this section we apply the results in Section 2 to the analysis of a weakly over-penalized symmetric interior penalty (WOPSIP) method [10, 27] on nonconforming

meshes. For simplicity, we will focus on the following model problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \int_\Omega f v\, dx \qquad \forall\, v \in H_0^1(\Omega), \tag{4.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$, $f \in L_2(\Omega)$ and

$$a(w,v) = \int_\Omega \nabla w \cdot \nabla v\, dx \qquad \forall\, v, w \in H^1(\Omega).$$

4.1. **Elliptic regularity.** Let $\omega_1, \ldots, \omega_L$ be the interior angles at the corners $c_1, \ldots, c_L$ of Ω, and let the parameters $\mu_1, \ldots, \mu_L$ be chosen according to the rule

$$\begin{cases} \mu_\ell = 1 & \text{if} \quad \omega_\ell < \pi, \\ \frac{1}{2} < \mu_\ell < \frac{\pi}{\omega_\ell} & \text{if} \quad \omega_\ell > \pi. \end{cases} \tag{4.2}$$

We use the weight function ϕ_μ given by $\phi_\mu(x) = \prod_{\ell=1}^L |x - c_\ell|^{1-\mu_\ell}$ to define the weighted L_2 space

$$L_{2,\mu}(\Omega) = \{g \in L_{2,\mathrm{loc}}(\Omega) : \|g\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi_\mu^2(x) g^2(x)\, dx < \infty\}. \tag{4.3}$$

By the elliptic regularity theory [17,21,22,24], the solution u of the model problem (4.1) belongs to the weighted Sobolev space $H^2_\mu(\Omega)$, i.e.,

$$\phi_0^{|\alpha|-2}\Big(\frac{\partial^\alpha u}{\partial x^\alpha}\Big) \in L_{2,\mu}(\Omega) \qquad \forall\, |\alpha| \le 2,$$

where $\phi_0(x) = \prod_{\ell=1}^L |x - c_\ell|$, and we have the following elliptic regularity estimate

$$\|u\|_{H^2_\mu(\Omega)} = \Big(\sum_{|\alpha|\le 2} \|\phi_0^{|\alpha|-2}(\partial^\alpha u/\partial x^\alpha)\|^2_{L_{2,\mu}(\Omega)}\Big)^{\frac{1}{2}} \le C_\Omega \|f\|_{L_2(\Omega)}. \tag{4.4}$$

The weighted Sobolev norm $\|u\|_{H^2_\mu(\Omega)}$ is equivalent to the standard Sobolev norm $\|u\|_{H^2(\Omega)}$ if Ω is convex. If Ω is nonconvex, then the norms $\|\cdot\|_{H^2_\mu}$ and $\|\cdot\|_{H^2}$ are still equivalent away from the reentrant corners. In a (small) neighborhood Ω_ℓ of a reentrant corner c_ℓ, $u \in H^2_\mu(\Omega)$ implies that $u \in H^{1+\mu_\ell}(\Omega_\ell)$ and

$$\|u\|_{H^{1+\mu_\ell}(\Omega_\ell)} \le C_{\Omega_\ell} \|u\|_{H^2_\mu(\Omega)}. \tag{4.5}$$

Therefore, $u \in H^s(\Omega)$ for a nonconvex Ω where $s = \min_{1\le\ell\le L}(1+\mu_\ell)$ and the parameters μ_ℓ are chosen according to (4.2).

In particular, u belongs to $H^{1+\alpha}(\Omega)$ and we have

$$\|u\|_{H^{1+\alpha}(\Omega)} \le C_{\Omega,\alpha} \|f\|_{L_2(\Omega)}, \tag{4.6}$$

where $\alpha = 1$ if Ω is convex and $(1/2) < \alpha < \pi/(\text{maximum reentrant angle})$ if Ω is nonconvex.

The solution u of (4.1) also has a singular function representation [17,21,24] that can be used to justify the following integration by parts formula [10, Lemma 2.1]:

$$\int_T \nabla u \cdot \nabla v\, dx = \int_{\partial T} \Big(\frac{\partial u}{\partial n}\Big) v\, ds + \int_T f v\, dx \qquad \forall\, v \in H^1(T), \tag{4.7}$$

where T is any triangle in Ω.

4.2. **The WOPSIP method.** Let $\mathcal{T}_h = \mathcal{T}_k$ $(k \geq 1)$ be a simplicial mesh generated by k level(s) of independent simplicial refinements from $\mathcal{T}_0 = \{\Omega\}$, where $h = \max_{T\in\mathcal{T}_k} h_T$ denotes the mesh parameter. We assume $\mathcal{T}_h$ satisfies the additional condition that

(4.8) if an edge of a triangle $T \in \mathcal{T}_h$ has hanging nodes, then it is the common edge of T and a triangle in $\mathcal{T}_j$ for some $j \in \{1, \ldots, k-1\}$.

Note that (4.8) implies, in particular,

(4.9) any edge of $T \in \mathcal{T}_h$ with hanging nodes must be the union of the edges of other triangles in $\mathcal{T}_h$.

An example of a nonconforming mesh generated by two independent refinements that satisfies (4.8) is depicted in Figure 4.1.

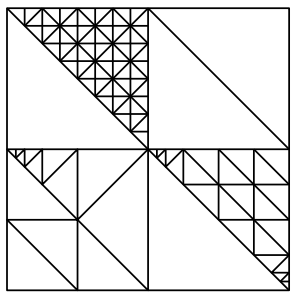


FIGURE 4.1. A mesh generated by two independent refinements that satisfies the condition (4.8)

The set $\mathcal{E}_h$ that will be used in the formulation of the WOPSIP method is defined as follows. An edge of a triangle in $\mathcal{T}_h$ belongs to $\mathcal{E}_h$ if and only if: (i) it is a subset of $\partial\Omega$, (ii) it is the common edge of two triangles in $\mathcal{T}_h$, or (iii) it contains at least one hanging node. In other words, we only include the "long" edges in $\mathcal{E}_h$ when hanging nodes are present. For any $e \in \mathcal{E}_h$,

(4.10) $\mathcal{T}_e$ is the set of the triangle(s) in $\mathcal{T}_h$ such that e is an edge of T.

Note that $\mathcal{T}_e$ consists of one triangle under conditions (i) and (iii) for the membership of $\mathcal{E}_h$ and two triangles under condition (ii).

Let V_h be the discontinuous P_1 finite element space associated with $\mathcal{T}_h$, i.e.,

$$V_h = \{v \in L_2(\Omega) : \, v_T = v\big|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

The WOPSIP method computes $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{4.11}$$

where

$$\begin{aligned} a_h(w, v) = &\sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx \\ &+ \sum_{e\in\mathcal{E}_h} \frac{1}{|e|^3} \int_e \Pi_{0,e}[\![w]\!]_e \cdot \Pi_{0,e}[\![v]\!]_e \, ds \qquad \forall\, v, w \in H^1(\mathcal{T}_k), \end{aligned} \tag{4.12}$$

$|e|$ is the length of e, and $\Pi_{0,e}$ is the projection from $L_2(e)$ to the space of constants on e. Note that

$$a_h(\zeta, v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla\zeta \cdot \nabla v \, dx \qquad \forall\, \zeta \in H_0^1(\Omega),\ v \in H^1(\mathcal{T}_k). \tag{4.13}$$

Remark 4.1. Implementation issues for WOPSIP were discussed in [8] and higher order WOPSIP methods can be found in [11]. An adaptive implementation using nonconforming meshes was carried out in [9].

Remark 4.2. The analysis for the WOPSIP method was carried out in [10] for conforming meshes and it was also observed there that the WOPSIP method can be implemented on nonconforming meshes with similar performance. Below we will provide an analysis of the WOPSIP method on independently refined meshes under condition (4.8). Note that one can impose (4.8) on many meshes at the expense of one additional independent refinement.

An important ingredient for the analysis is the Crouzeix-Raviart interpolation operator (cf. [16]) $\Pi_T^{CR} : H^1(T) \longrightarrow P_1(T)$ defined by

$$\int_e \Pi_T^{CR}\zeta \, ds = \int_e \zeta \, ds \qquad \text{for all the edges } e \text{ of } T. \tag{4.14}$$

We have the following well-known estimate [16] that follows from the Bramble-Hilbert Lemma [5, 20]:

$$\|\zeta - \Pi_T^{CR}\zeta\|_{L_2(T)} + h_T\|\nabla(\zeta - \Pi_T^{CR}\zeta)\|_{L_2(T)} \le C h_T^{1+\tau}|\zeta|_{H^{1+\tau}(T)} \tag{4.15}$$

for all $\zeta \in H^{1+\tau}(T)$ and $\tau \in [0, 1]$, where the constant C depends only on the shape regularity of T. The global interpolation operator $\Pi_h^{CR} : H^1(\Omega) \longrightarrow V_h$ is then defined by

$$(\Pi_h^{CR}\zeta)\big|_T = \Pi_T^{CR}(\zeta\big|_T) \qquad \forall\, T \in \mathcal{T}_h.$$

A key observation is that, because of (4.9) and (4.14), we have

$$\Pi_{0,e}[\![\Pi_h^{CR}\zeta]\!]_e = 0 \qquad \forall\, \zeta \in H^1(\Omega),\ e \in \mathcal{E}_h, \tag{4.16}$$

and hence

$$\|u - \Pi_h^{CR}u\|_h = \|\nabla_h(u - \Pi_h^{CR}u)\|_{L_2(\Omega)}. \tag{4.17}$$

Remark 4.3. The relation (4.17), where the interpolation operator is defined locally on each element, is possible because in the WOPSIP approach there is no need to distinguish the two sides of an edge with hanging nodes as a master side and a slave side. Note that a relation like (4.17) is not available under condition (4.9) for the discontinuous Galerkin methods discussed in [2, 18, 29] because the standard jumps across the edges with hanging nodes would not necessarily vanish for any interpolant of u that is defined locally.

Proposition 2.7 will also play an important role in the error analysis below (cf. Lemma 4.4). Here we note that condition (4.9) implies every $\sigma \in \mathcal{S}_k$ is the union of edges from $\mathcal{E}_h$ and hence

$$\sum_{\sigma\in\mathcal{S}_k} \frac{1}{|\sigma|}\|\Pi_{0,e}[\![v]\!]_e\|_{L_2(\sigma)}^2 \le \sum_{\sigma\in\mathcal{S}_k} \frac{1}{|\sigma|}\|[\![v]\!]_e\|_{L_2(\sigma)}^2 \le \sum_{e\in\mathcal{E}_h} \frac{1}{|e|}\|[\![v]\!]_e\|_{L_2(e)}^2. \tag{4.18}$$

4.3. **Error estimates.** We take the mesh-dependent energy norm $\|\cdot\|_h$ to be

$$\|v\|_h^2 = a_h(v,v) = \|\nabla_h v\|_{L_2(\Omega)}^2 + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|^3}\|\Pi_{0,e}[\![v]\!]_e\|_{L_2(e)}^2, \tag{4.19}$$

where ∇_h is the operator ∇_k defined in (2.6). It follows from (4.13), (4.17) and (4.19) that

$$\begin{aligned}\|u-u_h\|_h &\le \|\nabla_h(u-\Pi_h^{CR}u)\|_{L_2(\Omega)} + \sup_{v\in V_h\setminus\{0\}} \frac{a_h(\Pi_h^{CR}u-u_h,v)}{\|v\|_h} \\ &\le 2\|\nabla_h(u-\Pi_h^{CR}u)\|_{L_2(\Omega)} + \sup_{v\in V_h\setminus\{0\}} \frac{a_h(u-u_h,v)}{\|v\|_h}\end{aligned} \tag{4.20}$$

and it only remains to analyze the term $a_h(u-u_h,v)$.

From (4.7), (4.8) and (4.13), we have

$$a_h(u,v) = \sum_{e\in\mathcal{E}_h}\int_e \nabla u\cdot[\![v]\!]_e\,ds + \int_\Omega fv\,dx \qquad \forall\, v\in H^1(\mathcal{T}_k), \tag{4.21}$$

which together with (4.11) implies that

$$a_h(u-u_h,v) = \sum_{e\in\mathcal{E}_h}\int_e \nabla u\cdot[\![v]\!]_e\,ds \qquad \forall\, v\in V_h\,(\subset H^1(\mathcal{T}_k)). \tag{4.22}$$

Next we will estimate the edge integrals that appear on the right-hand side of (4.22).

Let $e\in\mathcal{E}_h$ and T_e be a triangle in $\mathcal{T}_e$ (cf. (4.10)). First we write

$$\int_e \nabla u\cdot[\![v]\!]_e\,ds = \int_e \nabla(u-\Pi_{T_e}^{CR}u)\cdot[\![v]\!]_e\,ds + \int_e \nabla(\Pi_{T_e}^{CR}u)\cdot\Pi_{0,e}[\![v]\!]_e\,ds, \tag{4.23}$$

and we estimate the second term on the right-hand side of (4.23) by

$$\begin{aligned}\int_e \nabla(\Pi_{T_e}^{CR}u)\cdot\Pi_{0,e}[\![v]\!]_e\,ds &\le \big(|e|^{3/2}\|\nabla(\Pi_{T_e}^{CR}u)\|_{L_2(e)}\big)\big(|e|^{-3/2}\|\Pi_{0,e}[\![v]\!]_e\|_{L_2(e)}\big) \\ &\le C\big(|e|\|\nabla(\Pi_{T_e}^{CR}u)\|_{L_2(T_e)}\big)\big(|e|^{-3/2}\|\Pi_{0,e}[\![v]\!]_e\|_{L_2(e)}\big),\end{aligned}$$

where we have used a standard inverse estimate [12, 15]. Summing up this estimate over all the edges in $\mathcal{E}_h$, we obtain, by (4.15), (4.19) and the Cauchy-Schwarz inequality,

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h}\int_e \nabla(\Pi_{T_e}^{CR}u)\cdot\Pi_{0,e}[\![v]\!]_e\,ds \\ &\qquad\le C\sum_{e\in\mathcal{E}_h}\big(|e|^{-3/2}\|\Pi_{0,e}[\![v]\!]_e\|_{L_2(e)}\big)\sum_{T\in\mathcal{T}_e}|e|\|\nabla(\Pi_{T_e}^{CR}u)\|_{L_2(T)} \\ &\qquad\le C\Big(\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}|e|^2\|\nabla u\|_{L_2(T)}^2\Big)^{\frac12}\|v\|_h.\end{aligned} \tag{4.24}$$

The following lemma is useful for the analysis of the first term on the right-hand side of (4.23).

Lemma 4.4. *There exists a positive constant C depending on the refinement level k and the shape regularity of the triangles in $\mathcal{T}_k$ $(=\mathcal{T}_h)$ such that, for any $v\in H^1(\mathcal{T}_k)$,*

$$\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|[\![v]\!]_e\|_{L_2(e)}^2 \le C\Big(\|\nabla_h v\|_{L_2(\Omega)}^2 + \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|\Pi_{0,e}[\![v]\!]_e\|_{L_2(e)}^2\Big) \le C\|v\|_h^2. \tag{4.25}$$

Proof. We can represent $\mathcal{E}_h$ as the union of $\mathcal{E}_h^{\mathtt{C}} = \{e \in \mathcal{E}_h : e$ does not contain any hanging node$\}$ (the set of conforming edges) and $\mathcal{E}_h^{\mathtt{N}} = \mathcal{E}_h \setminus \mathcal{E}_h^{\mathtt{C}}$ (the set of nonconforming edges).

If e belongs to $\mathcal{E}_h^{\mathtt{C}}$, then e is either the common edge of two triangles in $\mathcal{T}_h$ or a subset of $\partial\Omega$. Suppose e is the common edge of $T_\pm \in \mathcal{T}_h$ (i.e., $\mathcal{T}_e = \{T_-, T_+\}$) and $v_\pm = v\big|_{T_\pm}$. We have, by the Cauchy-Schwarz inequality and the Bramble-Hilbert lemma,

$$\begin{aligned}\frac{1}{|e|}\|[\![v]\!]_e\|^2_{L_2(e)} &\le \frac{3}{|e|}\big(\|v_- - \Pi_{0,e}v_-\|^2_{L_2(e)} + \|v_+ - \Pi_{0,e}v_+\|^2_{L_2(e)} + \|\Pi_{0,e}[\![v]\!]_e\|^2_{L_2(e)}\big)\\ &\le C\big(\sum_{T\in\mathcal{T}_e}\|\nabla v\|^2_{L_2(T)} + \frac{1}{|e|}\|\Pi_{0,e}[\![v]\!]_e\|^2_{L_2(e)}\big),\end{aligned}$$

where the positive constant C only depends on the shape regularity of the triangles in $\mathcal{T}_h$. By a similar argument, this estimate also holds when e is a subset of $\partial\Omega$. It follows that

$$\sum_{e\in\mathcal{E}_h^{\mathtt{C}}}\frac{1}{|e|}\|[\![v]\!]_e\|^2_{L_2(e)} \le C\Big(\|\nabla_h v\|^2_{L_2(\Omega)} + \sum_{e\in\mathcal{E}_h^{\mathtt{C}}}\frac{1}{|e|}\|\Pi_{0,e}[\![v]\!]_e\|^2_{L_2(e)}\Big). \tag{4.26}$$

We now consider the edges in $\mathcal{E}_h^{\mathtt{N}}$. According to condition (4.8), for any $e \in \mathcal{E}_h^{\mathtt{N}}$ there exists a triangle $D_e \in \mathcal{T}_j$ for some $j \in \{1, \dots, k-1\}$ such that e is the common edge of T_e and D_e. We can therefore represent $\mathcal{E}_h^{\mathtt{N}}$ as the union of $\mathcal{E}_{h,1}^{\mathtt{N}}, \dots, \mathcal{E}_{h,k-1}^{\mathtt{N}}$, where $e \in \mathcal{E}_{h,j}^{\mathtt{N}}$ if and only if $D_e \in \mathcal{T}_j \setminus \mathcal{T}_{j-1}$.

Let $e \in \mathcal{E}_{h,j}^{\mathtt{N}}$, $v_- = v\big|_{T_e}$ and $v_+ = v\big|_{D_e}$. There exists, by Proposition 2.7 (applied to D_e) and the relation (4.18), a function $\phi \in H^1(D_e)$ such that

$$\begin{aligned}\frac{1}{|e|}\|[\![v]\!]_e\|^2_{L_2(e)} &\le \frac{2}{|e|}\Big(\|v_- - \phi\|^2_{L_2(e)} + \|\phi - v_+\|^2_{L_2(e)}\Big)\\ &\le C\Big[\big(\|\nabla v\|^2_{L_2(T_e)} + \|\nabla\phi\|^2_{L_2(D_e)} + \frac{1}{|e|}\|\Pi_{0,e}(v_- - \phi)\|^2_{L_2(e)}\big) + \frac{1}{|e|}\|\phi - v_+\|^2_{L_2(e)}\Big]\\ &\le C\Big[\|\nabla v\|^2_{L_2(T_e)} + \|\nabla\phi\|^2_{L_2(D_e)} + \frac{1}{|e|}\|\Pi_{0,e}[\![v]\!]_e\|^2_{L_2(e)} + \frac{1}{|e|}\|\phi - v_+\|^2_{L_2(e)}\Big]\\ &\le C\Big[\|\nabla v\|^2_{L_2(T_e)} + \sum_{T\in\mathcal{T}_h(D_e)}\|\nabla v\|^2_{L_2(T)} + \frac{1}{|e|}\|\Pi_{0,e}[\![v]\!]_e\|^2_{L_2(e)}\\ &\qquad + \sum_{e'\in\mathcal{E}_h^I(D_e)}\frac{1}{|e'|}\|[\![v]\!]_{e'}\|^2_{L_2(e')}\Big],\end{aligned}$$

where $\mathcal{T}_h(D_e)$ is the partition of D_e induced by $\mathcal{T}_h$ and $\mathcal{E}_h^I(D_e)$ is the edges in $\mathcal{E}_h$ that are interior to D_e. Note that the edges in $\mathcal{E}_h^I(D_e)$ belong to either $\mathcal{E}_h^{\mathtt{C}}$ or $\mathcal{E}_{h,\ell}^{\mathtt{N}}$ for $j+1 \le \ell \le k-1$. Hence we have, for $1 \le j \le k-1$,

$$\begin{aligned}\sum_{e\in\mathcal{E}_{h,j}^{\mathtt{N}}}\frac{1}{|e|}\|[\![v]\!]_e\|^2_{L_2(e)} \le C\Big(&\|\nabla_h v\|^2_{L_2(\Omega)} + \sum_{e\in\mathcal{E}_{h,j}^{\mathtt{N}}}\frac{1}{|e|}\|\Pi_{0,e}[\![v]\!]_e\|^2_{L_2(e)}\\ &+ \sum_{\ell=j+1}^{k-1}\sum_{e\in\mathcal{E}_{h,\ell}^{\mathtt{N}}}\frac{1}{|e|}\|[\![v]\!]_e\|^2_{L_2(e)} + \sum_{e\in\mathcal{E}_h^{\mathtt{C}}}\frac{1}{|e|}\|[\![v]\!]_e\|^2_{L_2(e)}\Big),\end{aligned} \tag{4.27}$$

where the positive constant C only depends on the refinement level k and the shape regularity of the triangles in $\mathcal{T}_h$.

The estimate (4.25) follows from (4.26) and (4.27). $\square$

It follows from Lemma 4.4 and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e \nabla(u-\Pi_{T_e}^{CR}u)\cdot[\![v]\!]_e\,ds \\
&\qquad\le \Big(\sum_{e\in\mathcal{E}_h}|e|\|\nabla(u-\Pi_{T_e}^{CR})\|_{L_2(e)}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![v]\!]_e\|_{L_2(e)}^2\Big)^{\frac12} \\
&\qquad\le C\Big(\sum_{e\in\mathcal{E}_h}|e|\sum_{T\in\mathcal{T}_e}\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(e)}^2\Big)^{1/2}\|v\|_h.
\end{aligned}\tag{4.28}$$

Putting (4.20)–(4.24) and (4.28) together, we arrive at the following theorem.

Theorem 4.5. *Let $\mathcal{T}_h$ be a simplicial mesh generated from $\mathcal{T}_0$ by k levels of independent refinements that satisfy condition* (4.8)*. There exists a positive constant C that depends only on k and the shape regularity of the triangles in $\mathcal{T}_h$ such that*

$$\begin{aligned}
\|u-u_h\|_h \le C\Big[&\|\nabla_h(u-\Pi_h^{CR}u)\|_{L_2(\Omega)}+\Big(\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}|e|^2\|\nabla u\|_{L_2(T)}^2\Big)^{\frac12} \\
&+\Big(\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}|e|\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(e)}^2\Big)^{1/2}\Big],
\end{aligned}\tag{4.29}$$

where $\mathcal{T}_e$ is defined in (4.10)*.*

In order to derive concrete error estimates from Theorem 4.5, we need to estimate the three terms on the right-hand side of (4.29).

Since $\|\nabla u\|_{L_2(\Omega)}\le C_\Omega\|f\|_{L_2(\Omega)}$, it is clear that

$$\Big(\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}|e|^2\|\nabla u\|_{L_2(T)}^2\Big)^{\frac12}\le Ch\|f\|_{L_2(\Omega)}.\tag{4.30}$$

The estimates for the other two terms depend on the mesh $\mathcal{T}_h$. Without any additional information on $\mathcal{T}_h$, we have the following estimate that follows from (4.6), (4.15), and the trace theorem with scaling:

$$\|\nabla_h(u-\Pi_h^{CR}u)\|_{L_2(\Omega)}+\Big(\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}|e|\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(e)}^2\Big)^{1/2}\le Ch^\alpha\|f\|_{L_2(\Omega)}.$$

We can use additional information on $\mathcal{T}_h$ to refine this estimate. Let γ_T be the center of $T\in\mathcal{T}_h$. For a triangle $T\in\mathcal{T}_h$ away from the reentrant corners, we have

$$\begin{aligned}
&h_T\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(\partial T)}^2+\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(T)}^2 \\
&\qquad\le Ch_T^2\|\nabla^2u\|_{L^2(T)}^2\le Ch_T^2(\phi_\mu(\gamma_T))^{-2}\|\phi_\mu\nabla^2u\|_{L^2(T)}^2,
\end{aligned}\tag{4.31}$$

where ∇^2u is the Hessian matrix of the second order derivatives of u. If one of the vertices of T is a reentrant corner with interior angle $\omega_\ell>\pi$, then we have

$$h_T\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(\partial T)}^2+\|\nabla(u-\Pi_T^{CR}u)\|_{L_2(T)}^2\le Ch_T^{2\mu_\ell}|u|_{H^{1+\mu_\ell}(T)}^2,\tag{4.32}$$

where μ_ℓ satisfies (4.2).

Under the assumption that there exists a positive constant C_* such that

$$\begin{aligned} &(i)\ h_T \le C_* h \phi_\mu(\gamma_T) \text{ for any triangle } T \in \mathcal{T}_h \text{ away from the reentrant} \\ &\qquad \text{corners, and} \\ &(ii)\ h_T^{\mu_\ell} \le C_* h \text{ for any triangle } T \in \mathcal{T}_h \text{ that touches a reentrant corner } c_\ell, \end{aligned} \tag{4.33}$$

the elliptic regularity estimates (4.4)–(4.5) together with (4.31)–(4.32) imply

$$\|\nabla_h(u - \Pi_h^{CR} u)\|_{L_2(\Omega)} + \Big(\sum_{e \in \mathcal{E}_h} \sum_{T \in \mathcal{T}_e} |e| \|\nabla(u - \Pi_T^{CR} u)\|_{L_2(e)}^2 \Big)^{1/2} \le Ch \|f\|_{L_2(\Omega)}.$$

By taking β to be α for a general mesh $\mathcal{T}_h$ and 1 for a mesh $\mathcal{T}_h$ that satisfies the additional conditions in (4.33) and recalling (4.17), we can unify the interpolation error estimates as

$$\|u - \Pi_h^{CR} u\|_h + \Big(\sum_{e \in \mathcal{E}_h} \sum_{T \in \mathcal{T}_e} |e| \|\nabla(u - \Pi_T^{CR} u)\|_{L_2(e)}^2 \Big)^{1/2} \le Ch^\beta \|f\|_{L_2(\Omega)}. \tag{4.34}$$

The following theorem is an immediate consequence of Theorem 4.5, (4.30) and (4.34).

Theorem 4.6. *Let $\mathcal{T}_h$ be a simplicial mesh generated from $\mathcal{T}_0$ by k levels of independent refinements that satisfy condition* (4.8)*. There exists a positive constant C that depends only on k and the shape regularity of the triangles in $\mathcal{T}_h$ such that*

$$\|u - u_h\|_h \le Ch^\alpha \|f\|_{L_2(\Omega)},$$

where α is the index of elliptic regularity in (4.6)*.*

If the mesh satisfies the additional conditions (*i*) *and* (*ii*) *in* (4.33)*, we have*

$$\|u - u_h\|_h \le Ch \|f\|_{L_2(\Omega)},$$

where the constant C also depends on C_*.

Remark 4.7. The construction of graded meshes that satisfy (4.33) is discussed, for example, in [1, 3].

Using Lemma 4.4, (4.34) and Theorem 4.6, we can derive the following L_2 error estimates by a duality argument similar to the one in [10, Theorem 4.1].

Theorem 4.8. *Let $\mathcal{T}_h$ be a simplicial mesh generated from $\mathcal{T}_0$ by k levels of independent refinements that satisfy condition* (4.8)*. There exists a positive constant C that depends only on k and the shape regularity of the triangles in $\mathcal{T}_h$ such that*

$$\|u - u_h\|_{L_2(\Omega)} \le Ch^{2\alpha} \|f\|_{L_2(\Omega)},$$

where α is the index of elliptic regularity in (4.6)*.*

If the mesh satisfies the additional conditions (*i*) *and* (*ii*) *in* (4.33)*, we have*

$$\|u - u_h\|_{L_2(\Omega)} \le Ch^2 \|f\|_{L_2(\Omega)},$$

where the constant C also depends on C_*.

5. Concluding remarks

We have only considered piecewise H^1 Sobolev spaces in this paper because they provide the tools for the analysis of the WOPSIP method. It is of course possible to consider piecewise W_p^1 Sobolev spaces (cf. [12, Section 10.5]) and higher order piecewise Sobolev spaces (cf. [13]) associated with meshes generated by independent refinements. These topics will be treated elsewhere.

The WOPSIP method can be extended to three dimensions in a straightforward manner and the analog of Theorem 4.5 remains valid. However, the derivations of concrete error estimates on graded meshes would be more complicated. The WOPSIP approach can also be applied to the Stokes problem [4] within the framework of mixed finite element methods, and the resulting mixed finite element method on meshes generated by independent refinements can be analyzed by the tools developed in this paper.

Finally we would like to mention that it is possible to use the techniques developed in Section 2 to show that, on independently refined meshes, the choices of the penalty parameters for standard discontinuous Galerkin methods (for the sake of ensuring stability) only depend on k and the shape regularity of the simplexes in $\mathcal{T}_k$.

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