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A CLASS OF ORTHOGONAL FUNCTIONS GIVEN BY A THREE TERM RECURRENCE FORMULA

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Abstract. We present a class of functions satisfying a certain orthogonality property for which there also exists a three term recurrence formula. This class of functions, which can be considered as an extension to the class of symmetric orthogonal polynomials on $[-1,1]$, has a complete connection to the orthogonal polynomials on the unit circle. Interpolatory properties, quadrature rules and other properties based on the zeros of these functions are also considered.

1. Introduction

Let $\mathbb{P}_m$ be the linear space of real polynomials of degree at most m and let Ω_m be the linear space of "real" functions on $[-1,1]$ defined as follows: $\Omega_0 \equiv \mathbb{P}_0$ and Ω_m, for $m \geq 1$, is such that if $\mathcal{F} \in \Omega_m$, then $\mathcal{F}(x) = B^{(0)}(x) + \sqrt{1-x^2}\, B^{(1)}(x)$, where $B^{(0)} \in \mathbb{P}_m$ and $B^{(1)} \in \mathbb{P}_{m-1}$ satisfy

$$B^{(0)}(-x) = (-1)^m B^{(0)}(x) \quad \text{and} \quad B^{(1)}(-x) = (-1)^{m-1} B^{(1)}(x).$$

This means, if $\mathcal{F} \in \Omega_{2n}$, then $B^{(0)}$ is an even polynomial of degree at most $2n$ and $B^{(1)}$ is an odd polynomial of degree at most $2n-1$. Likewise, if $\mathcal{F} \in \Omega_{2n+1}$, then $B^{(0)}$ is an odd polynomial of degree at most $2n+1$ and $B^{(1)}$ is an even polynomial of degree at most $2n$.

Note that the dimension of Ω_m is $m+1$. As an example of a basis for Ω_{2n} we have

$$\{1,\ x\sqrt{1-x^2},\ x^2,\ \ldots,\ x^{2n-1}\sqrt{1-x^2},\ x^{2n}\}$$

and as an example of a basis for Ω_{2n+1} we can state

$$\{\sqrt{1-x^2},\ x,\ x^2\sqrt{1-x^2},\ \ldots,\ x^{2n}\sqrt{1-x^2},\ x^{2n+1}\}.$$

It is important to observe that, if $\mathcal{F} \in \Omega_m$, then for all $k \geq 1$,

$$\mathcal{F} \in \Omega_{m+2k} \quad \text{but} \quad \mathcal{F} \notin \Omega_{m+2k-1}.$$

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For $\mathcal{F}(x) = B^{(0)}(x) + \sqrt{1-x^2}\, B^{(1)}(x) \in \Omega_m$, by setting

$$B^{(0)}(x) = \sum_{j=0}^{\lfloor m/2 \rfloor} b_{2j}^{(0)}\, x^{m-2j} \quad \text{and} \quad B^{(1)}(x) = \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} b_{2j}^{(1)}\, x^{m-1-2j},$$

we say that the function $\mathcal{F}$ is of exact degree m if $(b_0^{(0)})^2 + (b_0^{(1)})^2 > 0$. The coefficients $b_0^{(0)}$ and $b_0^{(1)}$ will be referred to as the first and second leading coefficients of $\mathcal{F}$, respectively. The positive number $(b_0^{(0)})^2 + (b_0^{(1)})^2$ may be called the lead factor of $\mathcal{F}$.

Our interest in the study of such functions is broad. Apart from the interesting properties such as the three term recurrence formula, orthogonality and quadrature formulas that can be associated with these functions as shown in this manuscript, the solution of the differential equation

$$(1-x^2)\mathcal{F}''(x) - \big[(2\lambda+1)x - 2\eta\sqrt{1-x^2}\,\big]\mathcal{F}'(x) + m\big[m + 2\lambda + \frac{2\eta x}{\sqrt{1-x^2}}\big]\mathcal{F}(x) = 0,$$

with integer m, is also among these types of functions (see [7]). Note that when $\eta = 0$, the solution of the above differential equation is the ultraspherical polynomial of degree m (see, for example, [4, 19]).

Our studies of these functions were also motivated by the interesting example (Example 2) given in Section 7 of this manuscript. The function $\widehat{\mathcal{W}}_m$ given in this example is exactly a solution of the above differential equation.

Another benefit for studying these types of functions is that with $x = \cos(\theta)$ we can obtain trigonometric polynomials with interesting properties. For example, as a consequence of results obtained in [7], the use of the three term recurrence formula (1.1) below with the choice $\gamma_m = 1$, $m \geq 0$, $\{\beta_m\}_{m=1}^{\infty}$ a real sequence and $\{\alpha_m\}_{m=2}^{\infty}$ a positive chain sequence, permits one to generate a sequence of trigonometric polynomials having complete sets of simple zeros in $(0, \pi)$. Precisely, $W_m(\cos(\theta))$ would be a trigonometric polynomial of degree m with exactly m simple zeros in $(0, \pi)$.

We will also describe in Section 6 of this manuscript the connection that these functions have with orthogonal polynomials on the unit circle (OPUC). In the contexts of OPUC, some properties of functions of a somewhat similar appearance, but without the even odd parity conditions that we have imposed here, have been studied in [1].

The specific aim of the present manuscript is to consider some properties, in particular, the orthogonal properties of the sequence of functions $\{\mathcal{W}_m\}$, where $\mathcal{W}_m \in \Omega_m$, given by

$$\begin{aligned} &\mathcal{W}_0(x) = \gamma_0, \quad \mathcal{W}_1(x) = (\gamma_1 x - \beta_1\sqrt{1-x^2})\gamma_0, \\ &\mathcal{W}_{m+1}(x) = \big[\gamma_{m+1}x - \beta_{m+1}\sqrt{1-x^2}\,\big]\mathcal{W}_m(x) - \alpha_{m+1}\mathcal{W}_{m-1}(x), \quad m \geq 1. \end{aligned} \tag{1.1}$$

Here, $\{\alpha_m\}_{m=2}^{\infty}$, $\{\beta_m\}_{m=1}^{\infty}$ and $\{\gamma_m\}_{m=0}^{\infty}$ are sequences of real numbers.

2. Functions in Ω_m and self-inversive polynomials

It is known that a polynomial Q_m of degree at most m is self-inversive of degree m if $z^m\overline{Q_m(1/\overline{z})} = c_m Q_m(z)$, where $|c_m| = 1$. As one of the earliest references to self-inversive polynomials we cite Bonsall and Marden [2]. The most interesting self-inversive polynomials are those polynomials with all their zeros on the unit

circle. Characterizing self-inversive polynomials with zeros on the unit circle has been of considerable interest (see, for example, [13, 15, 18]).

In this manuscript we adopt the definition that Q_m is a self-inversive polynomial of degree m if Q_m is a polynomial of degree at most m and satisfies

$$Q_m^*(z) = z^m \overline{Q_m(1/\overline{z})} = Q_m(z).$$

Note that we have assumed $c_m = 1$ from the original definition of self-inversive polynomials. We remark that what is considered in the present manuscript as self-inversive polynomials are, as in [18], also known as conjugate reciprocal polynomials.

Functions belonging to Ω_m are connected to self-inversive polynomials of degree m. That is, given $\mathcal{F}_m \in \Omega_m$ then associated with it there exists a unique Q_m which is a self-inversive polynomial of degree m. Precisely, $e^{-im\theta/2} Q_m(e^{i\theta}) = \mathcal{F}_m(x)$, where $x = \cos(\theta/2)$.

The following lemma, which will be of considerable use in this manuscript, gives a more precise statement regarding this connection.

Lemma 2.1. *Let $x = \cos(\theta/2)$. Then the polynomial Q_m is self-inversive of degree m if and only if*

$$e^{-im\theta/2} Q_m(e^{i\theta}) = \mathcal{F}_m(x) = B_m^{(0)}(x) + \sqrt{1-x^2}\, B_m^{(1)}(x) \ \in \ \Omega_m.$$

Thus, $|Q_m(e^{i\theta})|^2 = \left[B_m^{(0)}(x) + \sqrt{1-x^2}\, B_m^{(1)}(x)\right]^2$ and $Q_m(1) = B_m^{(0)}(1)$.

Moreover, Q_m is a self-inversive polynomial with real coefficients if and only if $B_m^{(1)}$ is identically zero.

Proof. This lemma has also been stated and proved in [8]. However, for completeness and for a better understanding of the use of this lemma, we give here a sketch of its proof.

Given any polynomial Q_m of degree at most m, not necessarily self-inversive, by setting $Q_m(z) = \sum_{j=0}^m (c_j^{(m)} + i\, d_j^{(m)}) z^j$ we can write

$$\begin{aligned} z^{-m+1/2} Q_{2m-1}(z) = & \sum_{j=0}^{m-1} (c_{m-1-j}^{(2m-1)} + c_{m+j}^{(2m-1)}) \frac{z^{j+1/2} + z^{-j-1/2}}{2} \\ & - \sum_{j=0}^{m-1} i(d_{m-1-j}^{(2m-1)} - d_{m+j}^{(2m-1)}) \frac{z^{j+1/2} - z^{-j-1/2}}{2} \\ & + \sum_{j=0}^{m-1} i(d_{m-1-j}^{(2m-1)} + d_{m+j}^{(2m-1)}) \frac{z^{j+1/2} + z^{-j-1/2}}{2} \\ & - \sum_{j=0}^{m-1} (c_{m-1-j}^{(2m-1)} - c_{m+j}^{(2m-1)}) \frac{z^{j+1/2} - z^{-j-1/2}}{2} \end{aligned}$$

and

$$\begin{aligned} z^{-m}Q_{2m}(z) = c_m^{(2m)} + \ i\, d_m^{(2m)} \\ + \sum_{j=1}^{m}\left[(c_{m-j}^{(2m)} + c_{m+j}^{(2m)})\frac{z^j + z^{-j}}{2} \ - \ i(d_{m-j}^{(2m)} - d_{m+j}^{(2m)})\frac{z^j - z^{-j}}{2}\right] \\ + \sum_{j=1}^{m}\left[i(d_{m-j}^{(2m)} + d_{m+j}^{(2m)})\frac{z^j + z^{-j}}{2} \ - \ (c_{m-j}^{(2m)} - c_{m+j}^{(2m)})\frac{z^j - z^{-j}}{2}\right]. \end{aligned}$$

Then for $z = e^{i\theta}$, with $x = (z^{1/2} + z^{-1/2})/2 = \cos(\theta/2)$, we obtain

$$e^{-im\theta/2}Q_m(e^{i\theta}) = \mathcal{F}_m(x) + i\,\tilde{\mathcal{F}}_m(x) \quad \text{and} \quad |Q_m(e^{i\theta})|^2 = [\mathcal{F}_m(x)]^2 + [\tilde{\mathcal{F}}_m(x)]^2,$$

where the functions $\mathcal{F}_m$ and $\tilde{\mathcal{F}}_m$, defined for $x \in [-1,1]$, satisfy

$$\begin{aligned} \mathcal{F}_{2m-1}(x) &= \sum_{j=0}^{m-1}(c_{m-1-j}^{(2m-1)} + c_{m+j}^{(2m-1)})T_{2j+1}(x) + (d_{m-1-j}^{(2m-1)} - d_{m+j}^{(2m-1)})\sqrt{1-x^2}\,U_{2j}(x), \\ \tilde{\mathcal{F}}_{2m-1}(x) &= \sum_{j=0}^{m-1}(d_{m-1-j}^{(2m-1)} + d_{m+j}^{(2m-1)})T_{2j+1}(x) - (c_{m-1-j}^{(2m-1)} - c_{m+j}^{(2m-1)})\sqrt{1-x^2}\,U_{2j}(x), \\ \mathcal{F}_{2m}(x) &= c_m^{(2m)} + \sum_{j=1}^{m}(c_{m-j}^{(2m)} + c_{m+j}^{(2m)})T_{2j}(x) + (d_{m-j}^{(2m)} - d_{m+j}^{(2m)})\sqrt{1-x^2}\,U_{2j-1}(x), \\ \tilde{\mathcal{F}}_{2m}(x) &= d_m^{(2m)} + \sum_{j=1}^{m}(d_{m-j}^{(2m)} + d_{m+j}^{(2m)})T_{2j}(x) - (c_{m-j}^{(2m)} - c_{m+j}^{(2m)})\sqrt{1-x^2}\,U_{2j-1}(x). \end{aligned}$$

Here,

$$\begin{aligned} T_j(x) &= \cos(j\theta/2) \\ &= \frac{1}{2}(z^{j/2} + z^{-j/2}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} U_j(x) &= \frac{\sin((j+1)\theta/2)}{\sin(\theta/2)} \\ &= \frac{(z^{(j+1)/2} - z^{-(j+1)/2})}{(z^{1/2} - z^{-1/2})}, \end{aligned}$$

are respectively the Chebyshev polynomials of the first and second kind.

These relations enable one to establish a connection between any polynomial $Q_m(z)$ and the functions $\mathcal{F}_m(x)$ and $\tilde{\mathcal{F}}_m(x)$ both in Ω_m.

However, if $Q_m(z) = \sum_{j=0}^{m}(c_j^{(m)} + i\,d_j^{(m)})z^j$ is self-inversive, that is, $Q_m^*(z) = Q_m(z)$, then $c_j^{(m)} = c_{m-j}^{(m)}$ and $d_j^{(m)} = -d_{m-j}^{(m)}$ and hence $\tilde{\mathcal{F}}_m(x)$ is identically zero. This proves the first part of the lemma. To obtain the latter part of the lemma, we use $c_j^{(m)} = c_{m-j}^{(m)}$ and $d_j^{(m)} = -d_{m-j}^{(m)} = 0$. □

Within the contexts of the latter part of Lemma 2.1, the transformation $x = (z^{1/2} + z^{-1/2})/2 = \cos(\theta/2)$, sometimes known as the DG-transformation, has been used by many, including [3, 6, 21], to explore the connection between real OPUC and symmetric orthogonal polynomials in $[-1,1]$.

Since a polynomial of degree at most m cannot have more than m zeros on the unit circle, an immediate consequence of Lemma 2.1 is the following.

Theorem 2.2. *Let $\mathcal{F} \in \Omega_m$. Then the number of zeros of $\mathcal{F}$ in $(-1,1)$ cannot exceed m. Moreover, if 1 is a zero of $\mathcal{F}$, then -1 is also a zero of $\mathcal{F}$, and in this case the number of zeros of $\mathcal{F}$ in $[-1,1]$ cannot exceed $m+1$.*

Another interesting result regarding functions in Ω_m is their interpolation property, which is virtually the interpolation property of self-inversive polynomials on the unit circle $|z| = 1$. Results regarding interpolation by polynomials, including the idea behind the proof of the theorem below, are well known and can be found in any numerical analysis texts. For a recent reference of such a text we cite [14].

Theorem 2.3. *Given the $m+1$ pairs of real numbers (x_j, f_j), $j = 1, 2, \ldots, m+1$, where $-1 < x_1 < x_2 < \ldots < x_{m+1} < 1$, then there exists a unique $\mathcal{F} \in \Omega_m$ such that*

$$\mathcal{F}(x_j) = f_j, \quad j = 1, 2, \ldots, m+1.$$

Moreover, this interpolation function can be given by

$$\mathcal{F}(x) = \sum_{k=1}^{m+1} \mathcal{L}_k(x)\, f_k, \tag{2.1}$$

where

$$\mathcal{L}_k(x) = z^{-m/2}\, z_k^{m/2} \prod_{\substack{l=1\\ l\neq k}}^{m+1} \frac{z - z_l}{z_k - z_l}, \qquad k = 1, 2, \ldots, m+1,$$

with $z = e^{i\theta}$, $\theta = 2\arccos(x)$, $z_k = e^{i\theta_k}$ and $\theta_k = 2\arccos(x_k)$.

Proof. Uniqueness follows from Theorem 2.2. That is, if $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are two different functions in Ω_m such that

$$\mathcal{F}(x_j) = f_j \quad \text{and} \quad \tilde{\mathcal{F}}(x_j) = f_j, \quad j = 1, 2, \ldots, m+1,$$

then $\mathcal{G}$, where $\mathcal{G}(x) = \mathcal{F}(x) - \tilde{\mathcal{F}}(x) \in \Omega_m$ and $\mathcal{G}_m(x) \neq 0$, has $m+1$ zeros in $(-1, 1)$ contradicting Theorem 2.2.

To show the existence we construct the required function as follows. It is easily seen that the scaled Lagrange polynomials

$$z_k^{m/2} \prod_{\substack{l=1\\ l\neq k}}^{m+1} \frac{z - z_l}{z_k - z_l}, \qquad k = 1, 2, \ldots, m+1,$$

defined on the set $\{z_1, z_2, \ldots, z_m\}$ are self-inversive. Consequently, $\mathcal{L}_k$ are in Ω_m and they satisfy

$$\mathcal{L}_k(x_j) = \delta_{jk}, \quad j = 1, 2, \ldots, m+1.$$

Hence, the formula (2.1) immediately leads to the required interpolation function. □

3. Some basic properties

Theorem 3.1. *Let the sequence functions $\{\mathcal{F}_m\}$, where $\mathcal{F}_0(x) = b_{0,0}^{(0)}$,*

$$\mathcal{F}_m(x) = \sum_{j=0}^{\lfloor m/2 \rfloor} b_{m,2j}^{(0)}\, x^{m-2j} \; + \; \sqrt{1-x^2} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} b_{m,2j}^{(1)}\, x^{m-1-2j}, \quad m \geq 1,$$

be such that $b_{m+1,0}^{(0)} b_{m,0}^{(0)} + b_{m+1,0}^{(1)} b_{m,0}^{(1)} \neq 0$, $m \geq 0$. Here, $b_{0,0}^{(1)} = 0$. Then the following hold:

1. *A basis for Ω_{2m} is*

$$\{\mathcal{F}_{2m}(x), \sqrt{1-x^2}\mathcal{F}_{2m-1}(x), \mathcal{F}_{2m-2}(x), \ldots, \mathcal{F}_2(x), \sqrt{1-x^2}\mathcal{F}_1(x), \mathcal{F}_0(x)\};$$

2. *A basis for* Ω_{2m+1} *is*

$$\{\mathcal{F}_{2m+1}(x), \sqrt{1-x^2}\mathcal{F}_{2m}(x), \mathcal{F}_{2m-1}(x), \ldots, \sqrt{1-x^2}\mathcal{F}_2(x), \mathcal{F}_1(x), \sqrt{1-x^2}\mathcal{F}_0(x)\}.$$

Proof. Since $b^{(0)}_{m+1,0}b^{(0)}_{m,0} + b^{(1)}_{m+1,0}b^{(1)}_{m,0} \neq 0$ for $m \geq 0$ implies $(b^{(0)}_{m,0})^2 + (b^{(1)}_{m,0})^2 > 0$ for $m \geq 0$, the function $\mathcal{F}_m$ is of exact degree m for $m \geq 0$.

Now to prove the theorem, by observing that the dimension of Ω_m is $m+1$, all we have to do is to verify that the above sets are linearly independent. We prove this for the even indices and the proof for the odd indices is similar.

Clearly, $\mathcal{F}_0(x) = b^{(0)}_{0,0} \neq 0$ is a basis for Ω_0. Now we verify that

$$\{\mathcal{F}_2(x), \sqrt{1-x^2}\mathcal{F}_1(x), \mathcal{F}_0(x)\}$$

is a basis for Ω_2.

Let c_0, c_1, c_2 be such that $c_0\,\mathcal{F}_2(x) + c_1\,\sqrt{1-x^2}\,\mathcal{F}_1(x) + c_2\,\mathcal{F}_0(x) = 0$. Since the dimension of Ω_2 is 3, we need to verify that this is possible only if $c_0 = c_1 = c_2 = 0$. By considering the coefficients of x^2 and $x\sqrt{1-x^2}$, we have

$$c_0\,b^{(0)}_{2,0} - c_1\,b^{(1)}_{1,0} = 0,$$
$$c_0\,b^{(1)}_{2,0} + c_1\,b^{(0)}_{1,0} = 0.$$

The determinant of this system is $b^{(0)}_{2,0}b^{(0)}_{1,0} + b^{(1)}_{2,0}b^{(1)}_{1,0}$, which is different from zero. We must therefore have $c_0 = c_1 = 0$. This reduces our verification to finding c_2 such that $c_2\,\mathcal{F}_0(x) = 0$. Clearly, $c_2 = 0$, which follows from $\mathcal{F}_0(x) = b^{(0)}_{0,0} \neq 0$.

Now assuming

$$\{\mathcal{F}_{2m}(x), \sqrt{1-x^2}\mathcal{F}_{2m-1}(x), \mathcal{F}_{2m-2}(x), \ldots, \mathcal{F}_2(x), \sqrt{1-x^2}\mathcal{F}_1(x), \mathcal{F}_0(x)\} \tag{3.1}$$

is a basis for Ω_{2m}, we show that

$$\{\mathcal{F}_{2m+2}(x), \sqrt{1-x^2}\mathcal{F}_{2m+1}(x), \mathcal{F}_{2m}(x), \ldots, \mathcal{F}_2(x), \sqrt{1-x^2}\mathcal{F}_1(x), \mathcal{F}_0(x)\}$$

is a basis for Ω_{2m+2}. Let $c_0, c_1, c_2, \ldots, c_{2m+1}, c_{2m+2}$ be such that

$$c_0\,\mathcal{F}_{2m+2}(x) + c_1\,\sqrt{1-x^2}\mathcal{F}_{2m+1}(x) + \ldots + c_{2m+1}\sqrt{1-x^2}\mathcal{F}_1(x) + c_{2m+2}\,\mathcal{F}_0(x) = 0.$$

By considering the coefficients of x^{2m+2} and $x^{2m+1}\sqrt{1-x^2}$, we have

$$c_0\,b^{(0)}_{2m+2,0} - c_1\,b^{(1)}_{2m+1,0} = 0,$$
$$c_0\,b^{(1)}_{2m+2,0} + c_1\,b^{(0)}_{2m+1,0} = 0.$$

The determinant of this system is $b^{(0)}_{2m+2,0}b^{(0)}_{2m+1,0} + b^{(1)}_{2m+2,0}b^{(1)}_{2m+1,0}$. Since this determinant is different from zero, we must have $c_0 = c_1 = 0$. This reduces our verification to finding $c_2, c_3, \ldots, c_{2m+2}$, such that

$$c_2\,\mathcal{F}_{2m}(x) + c_3\,\sqrt{1-x^2}\mathcal{F}_{2m-1}(x) + \ldots + c_{2m+1}\sqrt{1-x^2}\mathcal{F}_1(x) + c_{2m+2}\,\mathcal{F}_0(x) = 0.$$

Clearly, $c_2 = c_3 = \ldots = c_{2m+1} = c_{2m+2} = 0$, which follow from the assumption in (3.1). Thus, the results of the theorem for even indices follow by induction. □

Now with the assumptions $\gamma_0 \neq 0$, $\gamma_1 \neq 0$ and $\gamma_m^2 + \beta_m^2 > 0$, $m \geq 2$, we consider the functions $\mathcal{W}_m$ given by the recurrence formula (1.1). It is easily seen that $\mathcal{W}_m$ takes the form

$$\mathcal{W}_m(x) = A^{(0)}_m(x) + \sqrt{1-x^2}A^{(1)}_m(x), \tag{3.2}$$

where $A_m^{(0)}$ and $A_m^{(1)}$ are, respectively, polynomials of degree at most m and $m-1$, such that

$$A_m^{(0)}(-x) = (-1)^m A_m^{(0)}(x) \quad \text{and} \quad A_m^{(1)}(-x) = (-1)^{m-1} A_m^{(1)}(x).$$

Hence, $\mathcal{W}_m \in \Omega_m$, $m \geq 0$. Setting

$$A_m^{(0)}(x) = \sum_{j=0}^{\lfloor m/2 \rfloor} a_{m,\,2j}^{(0)}\, x^{m-2j} \quad \text{and} \quad A_m^{(1)}(x) = \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} a_{m,\,2j}^{(1)}\, x^{m-1-2j}, \tag{3.3}$$

we have the following.

Theorem 3.2. *With $\gamma_0 \neq 0$, $\gamma_1 \neq 0$ and $\gamma_m^2 + \beta_m^2 > 0$, $m \geq 2$, let $\{\mathcal{W}_m\}$ be the sequence of functions obtained from the three term recurrence formula* (1.1)*. Then with respect to the leading coefficients $a_{m,0}^{(0)}$ and $a_{m,0}^{(1)}$ and the lead factor $\lambda_m = (a_{m,0}^{(0)})^2 + (a_{m,0}^{(1)})^2$ of $\mathcal{W}_m$ given by* (3.2) *and* (3.3)*, the following hold:*

$$\begin{bmatrix} a_{m,0}^{(0)} \\ a_{m,0}^{(1)} \end{bmatrix} = \begin{bmatrix} \gamma_m & \beta_m \\ -\beta_m & \gamma_m \end{bmatrix} \begin{bmatrix} a_{m-1,0}^{(0)} \\ a_{m-1,0}^{(1)} \end{bmatrix}, \quad m \geq 1,$$

with $a_{0,0}^{(0)} = \gamma_0$ and $a_{0,0}^{(1)} = 0$. Consequently,

$$\lambda_0 = \gamma_0^2, \qquad \lambda_m = (\gamma_m^2 + \beta_m^2)\lambda_{m-1}, \quad m \geq 1,$$

$$\lambda_{m,1} = a_{m+1,0}^{(0)}\, a_{m,0}^{(0)} + a_{m+1,0}^{(1)}\, a_{m,0}^{(1)} = \gamma_{m+1}\lambda_m, \quad m \geq 0,$$

$$a_{m+1,0}^{(0)}\, a_{m,0}^{(1)} - a_{m+1,0}^{(1)}\, a_{m,0}^{(0)} = \beta_{m+1}\lambda_m, \quad m \geq 0,$$

$$a_{m+1,0}^{(0)}\, a_{m-1,0}^{(0)} + a_{m+1,0}^{(1)}\, a_{m-1,0}^{(1)} = (\gamma_m\gamma_{m+1} - \beta_m\beta_{m+1})\lambda_{m-1}, \quad m \geq 1$$

and

$$a_{m+1,0}^{(0)}\, a_{m-1,0}^{(1)} - a_{m+1,0}^{(1)}\, a_{m-1,0}^{(0)} = (\gamma_{m+1}\beta_m + \gamma_m\beta_{m+1})\lambda_{m-1}, \quad m \geq 1.$$

Proof. From (1.1), (3.2) and (3.3), by equating the coefficients of x^{m+1} and $x^m\sqrt{1-x^2}$,

$$\begin{aligned} a_{m+1,0}^{(0)} &= \gamma_{m+1} a_{m,0}^{(0)} + \beta_{m+1}\, a_{m,0}^{(1)}, \\ a_{m+1,0}^{(1)} &= -\beta_{m+1}\, a_{m,0}^{(0)} + \gamma_{m+1} a_{m,0}^{(1)}. \end{aligned} \tag{3.4}$$

From these the matrix formula in the theorem follows. The equalities and recurrence can be obtained as follows.

From the matrix formula,

$$\begin{aligned} (a_{m,0}^{(0)})^2 + (a_{m,0}^{(1)})^2 &= \begin{bmatrix} a_{m,0}^{(0)} & a_{m,0}^{(1)} \end{bmatrix} \begin{bmatrix} a_{m,0}^{(0)} \\ a_{m,0}^{(1)} \end{bmatrix} \\ &= \begin{bmatrix} a_{m-1,0}^{(0)} & a_{m-1,0}^{(1)} \end{bmatrix} \begin{bmatrix} \gamma_m & -\beta_m \\ \beta_m & \gamma_m \end{bmatrix} \begin{bmatrix} \gamma_m & \beta_m \\ -\beta_m & \gamma_m \end{bmatrix} \begin{bmatrix} a_{m-1,0}^{(0)} \\ a_{m-1,0}^{(1)} \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} (a_{m,0}^{(0)})^2 + (a_{m,0}^{(1)})^2 &= \begin{bmatrix} a_{m-1,0}^{(0)} & a_{m-1,0}^{(1)} \end{bmatrix} \begin{bmatrix} \gamma_m^2 + \beta_m^2 & 0 \\ 0 & \gamma_m^2 + \beta_m^2 \end{bmatrix} \begin{bmatrix} a_{m-1,0}^{(0)} \\ a_{m-1,0}^{(1)} \end{bmatrix} \\ &= (\gamma_m^2 + \beta_m^2)\left[(a_{m-1,0}^{(0)})^2 + (a_{m-1,0}^{(1)})^2\right] = (\gamma_m^2 + \beta_m^2)\,\lambda_{m-1}, \end{aligned}$$

which gives the recurrence formula for λ_m. Similarly,

$$\begin{aligned} a^{(0)}_{m+1,0}\, a^{(0)}_{m,0} + a^{(1)}_{m+1,0}\, a^{(1)}_{m,0} &= \begin{bmatrix} a^{(0)}_{m,0} & a^{(1)}_{m,0} \end{bmatrix} \begin{bmatrix} a^{(0)}_{m+1,0} \\ a^{(1)}_{m+1,0} \end{bmatrix} \\ &= \begin{bmatrix} a^{(0)}_{m,0} & a^{(1)}_{m,0} \end{bmatrix} \begin{bmatrix} \gamma_{m+1} & \beta_{m+1} \\ -\beta_{m+1} & \gamma_{m+1} \end{bmatrix} \begin{bmatrix} a^{(0)}_{m,0} \\ a^{(1)}_{m,0} \end{bmatrix} \\ &= \gamma_{m+1}\big[(a^{(0)}_{m,0})^2 + (a^{(1)}_{m,0})^2\big] = \gamma_{m+1}\lambda_m. \end{aligned}$$

From (3.4),

$$\begin{aligned} &a^{(0)}_{m+1,0}\, a^{(1)}_{m,0} - a^{(1)}_{m+1,0}\, a^{(0)}_{m,0} \\ &\qquad = [\gamma_{m+1}a^{(0)}_{m,0} + \beta_{m+1}a^{(1)}_{m,0}]a^{(1)}_{m,0} - [-\beta_{m+1}a^{(0)}_{m,0} + \gamma_{m+1}a^{(1)}_{m,0}]a^{(0)}_{m,0} \\ &\qquad = \beta_{m+1}\big[(a^{(0)}_{m,0})^2 + (a^{(1)}_{m,0})^2\big] = \beta_{m+1}\lambda_m. \end{aligned}$$

Again from (3.4),

$$\begin{aligned} &a^{(0)}_{m+1,0}\, a^{(0)}_{m-1,0} + a^{(1)}_{m+1,0}\, a^{(1)}_{m-1,0} \\ &\qquad = [\gamma_{m+1}a^{(0)}_{m,0} + \beta_{m+1}a^{(1)}_{m,0}]a^{(0)}_{m-1,0} + [-\beta_{m+1}a^{(0)}_{m,0} + \gamma_{m+1}a^{(1)}_{m,0}]a^{(1)}_{m-1,0} \\ &\qquad = \gamma_{m+1}\big[a^{(0)}_{m,0}\, a^{(0)}_{m-1,0} + a^{(1)}_{m,0}\, a^{(1)}_{m-1,0}\big] - \beta_{m+1}\big[a^{(0)}_{m,0}\, a^{(1)}_{m-1,0} - a^{(1)}_{m,0}\, a^{(0)}_{m-1,0}\big] \\ &\qquad = (\gamma_m\gamma_{m+1} - \beta_m\beta_{m+1})\lambda_{m-1}. \end{aligned}$$

Similarly, the value associated with $a^{(0)}_{m+1,0}\, a^{(1)}_{m-1,0} - a^{(1)}_{m+1,0}\, a^{(0)}_{m-1,0}$ is also obtained. □

Clearly, with the assumptions $\gamma_0 \neq 0$, $\gamma_1 \neq 0$ and $\beta_m^2 + \gamma_m^2 > 0$, $m \geq 2$, it holds that

$$\lambda_m = (a^{(0)}_{m,0})^2 + (a^{(1)}_{m,0})^2 > 0, \quad m \geq 0,$$

which means the leading coefficients of $A^{(0)}_m$ and $A^{(1)}_m$ cannot be zero simultaneously and $\mathcal{W}_m \in \Omega_m$ is of exact degree m.

With a more restrictive condition than $\gamma_0 \neq 0$, $\gamma_1 \neq 0$ and $\beta_m^2 + \gamma_m^2 > 0$, $m \geq 2$, we can state the following theorem.

Theorem 3.3. *Let $\gamma_m \neq 0$, $m \geq 0$ and let $\{\mathcal{W}_m\}$ be the sequence of functions given by the recurrence formula* (1.1). *Then for any $m \geq 0$,*

1. *A basis for Ω_{2m} is*

$$\{\mathcal{W}_{2m}(x), \sqrt{1-x^2}\mathcal{W}_{2m-1}(x), \mathcal{W}_{2m-2}(x), \ldots, \mathcal{W}_2(x), \sqrt{1-x^2}\mathcal{W}_1(x), \mathcal{W}_0(x)\}.$$

2. *A basis for Ω_{2m+1} is*

$$\{\mathcal{W}_{2m+1}(x), \sqrt{1-x^2}\mathcal{W}_{2m}(x), \ldots, \sqrt{1-x^2}\mathcal{W}_2(x), \mathcal{W}_1(x), \sqrt{1-x^2}\mathcal{W}_0(x)\}.$$

Proof. From Theorem 3.2 we observe that, with $\gamma_m \neq 0$, $m \geq 0$, the leading coefficients of $\mathcal{W}_m$, $m \geq 0$ are such that $\lambda_{m,1} = a^{(0)}_{m+1,0}\, a^{(0)}_{m,0} + a^{(1)}_{m+1,0}\, a^{(1)}_{m,0} \neq 0$, $m \geq 0$. Hence, the present theorem follows from Theorem 3.1. □

Finally, by denoting the self-inversive polynomial associated with $\mathcal{W}_m$ by K_m, we have the following.

Theorem 3.4. *Let* $K_m(z) = \sum_{j=0}^{m} k_j^{(m)} z^j$ *be such that*

$$e^{-im\theta/2} K_m(e^{i\theta}) = \mathcal{W}_m(x),$$

where $x = \cos(\theta/2)$. *Then*

$$k_0^{(m)} = \overline{k_m^{(m)}} = 2^{-m}[a_{m,0}^{(0)} + i\, a_{m,0}^{(1)}] \quad \textit{and} \quad |k_0^{(m)}|^2 = |k_m^{(m)}|^2 = 2^{-2m}\lambda_m.$$

4. Orthogonal properties associated with $\mathcal{W}_m$

Let ψ be a positive measure on $[-1,1]$. We consider the sequence of functions $\{\mathcal{W}_m\}$, where $\mathcal{W}_m \in \Omega_m$ is of exact degree m, is such that

$$\begin{aligned} &\int_{-1}^{1} \mathcal{W}_{2n}(x)\, \mathcal{W}_{2m}(x)\, \sqrt{1-x^2}\, d\psi(x) = \rho_{2m}\, \delta_{n,m}, \\ &\int_{-1}^{1} \mathcal{W}_{2n+1}(x)\, \mathcal{W}_{2m+1}(x)\, \sqrt{1-x^2}\, d\psi(x) = \rho_{2m+1}\, \delta_{n,m}, \\ &\int_{-1}^{1} \mathcal{W}_{2n+1}(x)\, \mathcal{W}_{2m}(x)\, d\psi(x) = 0, \end{aligned} \tag{4.1}$$

for $n, m = 0, 1, 2, \dots$.

We use the notation $\mathcal{W}_0(x) = a_{0,0}^{(0)}$,

$$\begin{aligned} \mathcal{W}_m(x) &= \sum_{j=0}^{\lfloor m/2 \rfloor} a_{m,\,2j}^{(0)}\, x^{m-2j} \quad + \sqrt{1-x^2} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} a_{m,\,2j}^{(1)}\, x^{m-1-2j}, \quad m \geq 1, \\ \lambda_m &= (a_{m,0}^{(0)})^2 + (a_{m,0}^{(1)})^2, \end{aligned}$$

and

$$\lambda_{m,1} = a_{m+1,0}^{(0)} a_{m,0}^{(0)} + a_{m+1,0}^{(1)} a_{m,0}^{(1)}, \quad m \geq 0,$$

with $a_{0,0}^{(1)} = 0$.

Observe that, in the case $a_{m,2j}^{(1)} = 0$, $j = 0, 1, \dots, \lfloor (m-1)/2 \rfloor$, $m \geq 1$, then $\{\mathcal{W}_m\}$ are symmetric polynomials and (4.1) reduces to the orthogonality of symmetric polynomials with respect to $\sqrt{1-x^2}\, d\psi(x)$ in $[-1,1]$. For references to some of the classical texts on orthogonal polynomials on the real line we cite [4, 12, 19].

Theorem 4.1. *The sequence of functions* $\{\mathcal{W}_m\}$ *that satisfies the orthogonality property* (4.1), *where* $\mathcal{W}_m \in \Omega_m$ *and is of exact degree* m, *always exists. Moreover, the orthogonal sequence* $\{\mathcal{W}_m\}$ *can always be given by the three term recurrence formula*

$$\begin{aligned} &\mathcal{W}_0(x) = \gamma_0, \qquad \mathcal{W}_1(x) = \big[\gamma_1 x - \beta_1 \sqrt{1-x^2}\big]\gamma_0, \\ &\mathcal{W}_{m+1}(x) = \big[\gamma_{m+1} x - \beta_{m+1}\sqrt{1-x^2}\big]\mathcal{W}_m(x) - \alpha_{m+1}\mathcal{W}_{m-1}(x), \quad m \geq 1, \end{aligned} \tag{4.2}$$

where $\gamma_m \neq 0$, $m \geq 0$, $\beta_1 = \gamma_1\, \rho_0^{-1} \int_{-1}^{1} x\, \mathcal{W}_0^2(x)\, d\psi(x)$,

(4.3)

$$\beta_{m+1} = \gamma_{m+1}\, \frac{1}{\rho_m} \int_{-1}^{1} x\, \mathcal{W}_m^2(x)\, d\psi(x), \qquad \textit{and}$$

$$\alpha_{m+1} = \frac{1}{\rho_{m-1}} \int_{-1}^{1} \left[\gamma_{m+1} x - \beta_{m+1}\sqrt{1-x^2}\right] \mathcal{W}_{m-1}(x)\, \mathcal{W}_m(x) \sqrt{1-x^2}\, d\psi(x),$$

for $m \geq 1$. *Here,* $\rho_m = \int_{-1}^{1} \mathcal{W}_m^2(x)\, \sqrt{1-x^2}\, d\psi(x)$, $m \geq 0$.

Proof. It follows from Theorem 3.2 that, with $\gamma_m \neq 0$, $m \geq 0$, the function $\mathcal{W}_m$ obtained from the three term recurrence formula (4.2) is such that $\mathcal{W}_m \in \Omega_m$ and is of exact degree m. Hence, we first show that with α_m and β_m successively chosen as in (4.3) the sequence of functions $\{\mathcal{W}_m\}$ generated by the above three term recurrence formula satisfies the orthogonality (4.1).

Since $\beta_1 = \gamma_1\, \rho_0^{-1} \int_{-1}^{1} x\, \mathcal{W}_0^2(x)\, d\psi(x)$, with $\mathcal{W}_1(x) = \left[\gamma_1 x - \beta_1\sqrt{1-x^2}\right]\gamma_0$ it follows that $\int_{-1}^{1} \mathcal{W}_0(x)\, \mathcal{W}_1(x)\, d\psi(x) = 0$.

Since β_2 and α_2 are as in (4.3), with

$$\mathcal{W}_2(x) = \left[\gamma_2 x - \beta_2\sqrt{1-x^2}\right]\mathcal{W}_1(x) - \alpha_2 \mathcal{W}_0(x),$$

we have

$$\int_{-1}^{1} \mathcal{W}_1(x)\, \mathcal{W}_2(x)\, d\psi(x) = 0 \quad \text{and} \quad \int_{-1}^{1} \mathcal{W}_0(x)\, \mathcal{W}_2(x)\, \sqrt{1-x^2}\, d\psi(x) = 0.$$

Now suppose that for $N \geq 2$ the sequence of functions $\{\mathcal{W}_m\}_{m=0}^{N}$ obtained from the three term recurrence formula (4.2) satisfies (4.1).

Since $x\mathcal{W}_{N-2k}(x) \in \Omega_{N+1-2k}$, by Theorem 3.3 there exists $c_0, c_1, \ldots, c_{N+1-2k}$ such that

$$x\mathcal{W}_{N-2k}(x) = c_0 \mathcal{W}_{N+1-2k}(x) + c_1\sqrt{1-x^2}\mathcal{W}_{N-2k}(x) + c_2 \mathcal{W}_{N-1-2k}(x) + \ldots\ .$$

Hence, we have

$$\int_{-1}^{1} x\mathcal{W}_{N-2k}(x)\, \mathcal{W}_N(x)\, d\psi(x) = 0, \quad k = 1, 2, \ldots \lfloor N/2 \rfloor.$$

Likewise, since $x\sqrt{1-x^2}\mathcal{W}_{N-1-2k}(x) \in \Omega_{N+1-2k}$, we have

$$\int_{-1}^{1} x\sqrt{1-x^2}\mathcal{W}_{N-1-2k}(x)\, \mathcal{W}_N(x)\, d\psi(x) = 0, \quad k = 1, 2, \ldots \lfloor (N-1)/2 \rfloor \tag{4.4}$$

and, since $(1-x^2)\mathcal{W}_{N-1-2k}(x) \in \Omega_{N+1-2k}$, we also have

$$\int_{-1}^{1} (1-x^2)\mathcal{W}_{N-1-2k}(x)\, \mathcal{W}_N(x)\, d\psi(x) = 0, \quad k = 1, 2, \ldots \lfloor (N-1)/2 \rfloor. \tag{4.5}$$

Note that in (4.4) and (4.5) the value of N is assumed to be ≥ 3.

Hence, from

$$\mathcal{W}_{N+1}(x) = \left[\gamma_{N+1} x - \beta_{N+1}\sqrt{1-x^2}\right]\mathcal{W}_N(x) - \alpha_{N+1}\mathcal{W}_{N-1}(x),$$

it follows that

$$\int_{-1}^{1} \mathcal{W}_{N-2k}(x)\,\mathcal{W}_{N+1}(x)\,d\psi(x) = 0, \quad k = 1, 2, \ldots, \lfloor N/2 \rfloor,$$

$$\int_{-1}^{1} \mathcal{W}_{N-1-2k}(x)\,\mathcal{W}_{N+1}(x)\,\sqrt{1-x^2}d\psi(x) = 0, \quad k = 1, 2, \ldots, \lfloor (N-1)/2 \rfloor.$$

Moreover, since α_{N+1} and β_{N+1} are as in (4.3), we also have

$$\int_{-1}^{1} \mathcal{W}_{N}(x)\,\mathcal{W}_{N+1}(x)\,d\psi(x) = 0,$$

$$\int_{-1}^{1} \mathcal{W}_{N-1}(x)\,\mathcal{W}_{N+1}(x)\,\sqrt{1-x^2}d\psi(x) = 0.$$

Thus, by induction we conclude that the sequence of functions given by the three term recurrence formula satisfies (4.1).

On the other hand, to show that any sequence of functions $\{\mathcal{W}_m\}$, where $\mathcal{W}_m \in \Omega_m$ is of exact degree m, for which (4.1) holds, must also satisfy the three term recurrence formula (4.2), we proceed as follows.

Clearly we can write $\mathcal{W}_0(x) = \gamma_0 \neq 0$ and $\mathcal{W}_1(x) = a_{1,0}^{(0)}x + a_{1,0}^{(1)}\sqrt{1-x^2} = [\gamma_1 x - \beta_1\sqrt{1-x^2}]\gamma_0$. Then for $\int_{-1}^{1} \mathcal{W}_0(x)\,\mathcal{W}_1(x)\,d\psi(x) = 0$ to hold such that $\mathcal{W}_1(x)$ is of exact degree 1, one must have $\gamma_1 \neq 0$ and $\beta_1 = \gamma_1\,\rho_0^{-1}\int_{-1}^{1} x\,\mathcal{W}_0^2(x)\,d\psi(x)$.

With the next element $\mathcal{W}_2$ of the given orthogonality sequence, let γ_2 and β_2 be such that $\mathcal{W}_2(x) - \left[\gamma_2 x - \beta_2\sqrt{1-x^2}\right]\mathcal{W}_1(x) \in \Omega_0$. With respect to the leading coefficients of $\mathcal{W}_2$ and $\mathcal{W}_1$, the elements γ_2 and β_2 must satisfy

$$\begin{bmatrix} a_{2,0}^{(0)} \\ a_{2,0}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{1,0}^{(0)} & a_{1,0}^{(1)} \\ a_{1,0}^{(1)} & -a_{1,0}^{(0)} \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \beta_2 \end{bmatrix}.$$

The determinant of this system is $-[(a_{1,0}^{(0)})^2 + (a_{1,0}^{(1)})^2]$. Since $\mathcal{W}_1$ is of exact degree 1 this determinant is different from zero. Hence, the values of γ_2 and β_2 are uniquely found. Writing

$$\mathcal{W}_2(x) = \left[\gamma_2 x - \beta_2\sqrt{1-x^2}\right]\mathcal{W}_1(x) - \alpha_2\mathcal{W}_0(x),$$

we find, with the orthogonality and with the additional observation that $\mathcal{W}_2$ is of exact degree 2, that $\gamma_2 \neq 0$,

$$\beta_2 = \gamma_2\,\frac{1}{\rho_1}\int_{-1}^{1} x\,\mathcal{W}_1^2(x)\,d\psi(x) \qquad \text{and}$$

$$\alpha_2 = \frac{1}{\rho_0}\int_{-1}^{1} \left[\gamma_2 x - \beta_2\sqrt{1-x^2}\right]\mathcal{W}_0(x)\,\mathcal{W}_1(x)\sqrt{1-x^2}\,d\psi(x).$$

Now for $N \geq 2$ assume that the orthogonal functions $\mathcal{W}_m$, $m = 0, 1, \ldots, N$ satisfy the three term recurrence formula (4.2), with $\gamma_m \neq 0$, $m = 0, 1, \ldots, N$. Let γ_{N+1} and β_{N+1} be such that $\mathcal{W}_{N+1}(x) - \left[\gamma_{N+1}x - \beta_{N+1}\sqrt{1-x^2}\right]\mathcal{W}_N(x) \in \Omega_{N-1}$. With respect to the leading coefficients of $\mathcal{W}_{N+1}$ and $\mathcal{W}_N$, the elements γ_{N+1} and β_{N+1} must satisfy

$$\begin{bmatrix} a_{N+1,0}^{(0)} \\ a_{N+1,0}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{N,0}^{(0)} & a_{N,0}^{(1)} \\ a_{N,0}^{(1)} & -a_{N,0}^{(0)} \end{bmatrix} \begin{bmatrix} \gamma_{N+1} \\ \beta_{N+1} \end{bmatrix}.$$

The determinant of this system is $-[(a_{N,0}^{(0)})^2 + (a_{N,0}^{(1)})^2]$, which is different from zero because $\mathcal{W}_N$ is of exact degree N. Hence, the values for γ_{N+1} and β_{N+1} are uniquely found.

Now using Theorem 3.3, there exist $c_0, c_1, \ldots, c_{N-1}$ such that

$$\mathcal{W}_{N+1}(x) = \big[\gamma_{N+1}x - \beta_{N+1}\sqrt{1-x^2}\big]\mathcal{W}_N(x) + c_0\mathcal{W}_{N-1}(x) + c_1\sqrt{1-x^2}\mathcal{W}_{N-2}(x) + c_2\mathcal{W}_{N-3}(x) + \ldots .$$

Applications of the orthogonality properties (4.1) of the sequence $\{\mathcal{W}_m\}_{m=0}^{N+1}$, together with the observation that $\mathcal{W}_{N+1}$ is of exact degree $N+1$, lead to $c_1 = c_2 = \ldots = c_{N-1} = 0$, $\gamma_{N+1} \neq 0$,

$$\beta_{N+1} = \gamma_{N+1}\frac{1}{\rho_N}\int_{-1}^{1} x\,\mathcal{W}_N^2(x)\,d\psi(x) \qquad \text{and}$$

$$\alpha_{N+1} = -c_0 = \frac{1}{\rho_{N-1}}\int_{-1}^{1}\big[\gamma_{N+1}x - \beta_{N+1}\sqrt{1-x^2}\big]\,\mathcal{W}_{N-1}(x)\,\mathcal{W}_N(x)\sqrt{1-x^2}\,d\psi(x).$$

This concludes the proof of the theorem. □

Following as in Theorem 3.2 we have $\gamma_{m+1} = \lambda_{m,1}/\lambda_m$. Thus, the orthogonal functions $\mathcal{W}_m$, $m \geq 1$, are such that

$$\lambda_{m,1} = a_{m+1,0}^{(0)}\,a_{m,0}^{(0)} + a_{m+1,0}^{(1)}\,a_{m,0}^{(1)} \neq 0, \quad m \geq 0.$$

Corollary 4.1.1. *The sequence of functions $\{\mathcal{W}_m\}$, where $\mathcal{W}_m \in \Omega_m$ and is of exact degree m, satisfies the orthogonality property* (4.1) *if and only if, for $m \geq 1$,*

$$\int_{-1}^{1}\mathcal{F}(x)\,\mathcal{W}_m(x)\,d\psi(x) = 0 \quad \textit{whenever} \quad \mathcal{F} \in \Omega_{m-1}. \tag{4.6}$$

Proof. First we assume that (4.6) holds. Observe that, if $\mathcal{F} \in \Omega_{m+1-2k}$ for $k = 1, 2, \ldots, \lfloor (m+1)/2 \rfloor$, then $\mathcal{F} \in \Omega_{m-1}$. Hence, $\mathcal{W}_{m+1-2k} \in \Omega_{m+1-2k}$ and $\mathcal{W}_{m-2k}\sqrt{1-x^2} \in \Omega_{m+1-2k}$ leads immediately to (4.1).

On the other hand, if (4.1) holds, then from Theorem 4.1 and from Theorem 3.3 we can write

$$\mathcal{F}_{m+1-2k}(x) = c_0\mathcal{W}_{m+1-2k}(x) + c_1\sqrt{1-x^2}\mathcal{W}_{m-2k}(x) + c_2\mathcal{W}_{m-1-2k}(x) + \ldots ,$$

for $k = 1, 2, \ldots, \lfloor (m+1)/2 \rfloor$. Hence, (4.6) is immediate. □

It is quite straightforward that another way to present the above corollary is the following.

Corollary 4.1.2. *The sequence of functions $\{\mathcal{W}_m\}$, where $\mathcal{W}_m \in \Omega_m$ and is of exact degree m, satisfies the orthogonality property* (4.1) *if and only if, for $m \geq 1$,*

$$\int_{-1}^{1} B^{(0)}(x)\,\mathcal{W}_m(x)\,d\psi(x) = 0 \quad \textit{and} \quad \int_{-1}^{1} B^{(1)}(x)\,\mathcal{W}_m(x)\sqrt{1-x^2}\,d\psi(x) = 0,$$

where $B^{(0)} \in \mathbb{P}_{m-1}$ and $B^{(1)} \in \mathbb{P}_{m-2}$ satisfy

$$B^{(0)}(-x) = (-1)^{m-1}B^{(0)}(x) \quad \textit{and} \quad B^{(1)}(-x) = (-1)^{m-2}B^{(1)}(x).$$

The following corollary provides one other way to express the orthogonality (4.1) of the sequence $\{\mathcal{W}_n\}$.

Corollary 4.1.3. *The sequence of functions $\{\mathcal{W}_m\}$, where $\mathcal{W}_m \in \Omega_m$ and is of exact degree m, satisfies the orthogonality property* (4.1) *if and only if, for $m \geq 1$,*

$$\int_{-1}^{1} (x + i\sqrt{1-x^2}\,)^{-m+1+2s}\, \mathcal{W}_m(x)\, d\psi(x) = 0, \quad s = 0, 1, \ldots, m-1. \tag{4.7}$$

Proof. We just give the proof for $m = 2n$ and the proof for $m = 2n+1$ is similar.

Since $(x + i\sqrt{1-x^2}\,)(x - i\sqrt{1-x^2}\,) = 1$ the orthogonality (4.7) for $m = 2n$ can be written as

$$\int_{-1}^{1} (x \pm i\sqrt{1-x^2}\,)^{2l+1}\, \mathcal{W}_{2n}(x)\, d\psi(x) = 0, \quad l = 0, 1, \ldots, n-1.$$

Observe that

$$\begin{aligned}&(x \pm i\sqrt{1-x^2}\,)^{2l+1}\\ &\quad = \sum_{k=0}^{l} \binom{2l+1}{2k+1} x^{2k+1}(x^2-1)^{l-k} \pm i\sqrt{1-x^2} \sum_{k=0}^{l} \binom{2l+1}{2k} x^{2k}(x^2-1)^{l-k}.\end{aligned}$$

Since the polynomials represented by the above sums are, respectively, odd and even polynomials of exact degrees $2l+1$ and $2l$, the required result follows from Corollary 4.1.2. □

The following theorem, in addition to showing some further orthogonality properties of the functions $\{\mathcal{W}_m\}$ given by Theorem 4.1, also gives another expression for the coefficients α_n given in Theorem 4.1.

Theorem 4.2.

$$\int_{-1}^{1} x\sqrt{1-x^2}\, \mathcal{W}_1(x)\, \mathcal{W}_2(x)\, d\psi(x) = \frac{\gamma_2\gamma_3 - \beta_2\beta_3}{(\gamma_2^2 + \beta_2^2)\gamma_3}\, \rho_2,$$

$$\int_{-1}^{1} (1-x^2)\, \mathcal{W}_1(x)\, \mathcal{W}_2(x)\, d\psi(x) = -\frac{\gamma_3\beta_2 + \gamma_2\beta_3}{(\gamma_2^2 + \beta_2^2)\gamma_3}\, \rho_2,$$

and for $m \geq 2$,

$$\begin{aligned}&\int_{-1}^{1} x\sqrt{1-x^2}\, \mathcal{W}_{m-1-2k}(x)\, \mathcal{W}_m(x)\, d\psi(x)\\ &\qquad = \begin{cases} \dfrac{\gamma_m\gamma_{m+1} - \beta_m\beta_{m+1}}{(\gamma_m^2 + \beta_m^2)\gamma_{m+1}}\, \rho_m, & k = 0,\\ 0, & 1 \leq k \leq \lfloor (m-1)/2 \rfloor,\end{cases}\end{aligned}$$

$$\begin{aligned}&\int_{-1}^{1} (1-x^2)\, \mathcal{W}_{m-1-2k}(x)\, \mathcal{W}_m(x)\, d\psi(x)\\ &\qquad = \begin{cases} -\dfrac{\gamma_{m+1}\beta_m + \gamma_m\beta_{m+1}}{(\gamma_m^2 + \beta_m^2)\gamma_{m+1}}\, \rho_m, & k = 0,\\ 0, & 1 \leq k \leq \lfloor (m-1)/2 \rfloor.\end{cases}\end{aligned}$$

Consequently,

$$\alpha_{m+1} = \frac{\gamma_{m+1}^2 + \beta_{m+1}^2}{\gamma_m^2 + \beta_m^2}\, \frac{\gamma_m}{\gamma_{m+1}}\, \frac{\rho_m}{\rho_{m-1}}, \quad m \geq 1.$$

Proof. Since $x\sqrt{1-x^2}\,\mathcal{W}_{m-1-2k}(x)$ and $(1-x^2)\,\mathcal{W}_{m-1-2k}(x)$ are in Ω_{m+1-2k}, the results corresponding to $1 \le k \le \lfloor (m-1)/2 \rfloor$ follow from Corollary 4.1.1.

We now prove the result associated with $\int_{-1}^{1} x\sqrt{1-x^2}\,\mathcal{W}_{m-1}(x)\,\mathcal{W}_m(x)\,d\psi(x)$.

Since $x\sqrt{1-x^2}\,\mathcal{W}_{m-1}(x) \in \Omega_{m+1}$, there exist $c_0, c_1, \ldots, c_{m+1}$ such that

$$x\sqrt{1-x^2}\,\mathcal{W}_{m-1}(x) = c_0\mathcal{W}_{m+1}(x) + c_1\sqrt{1-x^2}\mathcal{W}_m(x) + c_2\mathcal{W}_{m-1}(x) + \ldots$$

and that $\int_{-1}^{1} x\sqrt{1-x^2}\,\mathcal{W}_{m-1}(x)\,\mathcal{W}_m(x)\,d\psi(x) = c_1\rho_m$. Moreover, comparing the leading coefficients on both sides

$$\begin{aligned} -a^{(1)}_{m-1,0} &= c_0\, a^{(0)}_{m+1,0} - c_1\, a^{(1)}_{m,0}, \\ a^{(0)}_{m-1,0} &= c_0\, a^{(1)}_{m+1,0} + c_1\, a^{(0)}_{m,0}. \end{aligned}$$

This gives $c_1 = \left[a^{(0)}_{m+1,0}a^{(0)}_{m-1,0} + a^{(1)}_{m+1,0}a^{(1)}_{m-1,0}\right] / \left[a^{(0)}_{m+1,0}a^{(0)}_{m,0} + a^{(1)}_{m+1,0}a^{(1)}_{m,0}\right]$. Thus, from Theorem 3.2 we have

$$c_1 = \frac{\gamma_m\gamma_{m+1} - \beta_m\beta_{m+1}}{(\gamma_m^2 + \beta_m^2)\gamma_{m+1}}$$

and the required result for $\int_{-1}^{1} x\sqrt{1-x^2}\,\mathcal{W}_{m-1}(x)\,\mathcal{W}_m(x)\,d\psi(x)$.

The result associated with $\int_{-1}^{1}(1-x^2)\,\mathcal{W}_{m-1}(x)\,\mathcal{W}_m(x)\,d\psi(x)$ is obtained similarly, and the latter result of the theorem then follows from (4.3). □

Now we can state the following theorem with respect to the zeros of $\mathcal{W}_m$.

Theorem 4.3. *Let $\{\mathcal{W}_m\}$ be the sequence of functions given as in Theorem 4.1. Then for $m \ge 1$, the function $\mathcal{W}_m$ has exactly m distinct zeros in $(-1,1)$.*

Proof. From (4.1) since at least one of the integrals

$$\int_{-1}^{1} \mathcal{W}_0(x)\mathcal{W}_m(x)d\psi(x) \quad \text{and} \quad \int_{-1}^{1} \mathcal{W}_0(x)\mathcal{W}_m(x)\sqrt{1-x^2}d\psi(x)$$

is zero, we can say that $\mathcal{W}_m$ changes sign at least once in $(-1,1)$. According to Theorem 2.2 the number of sign changes of $\mathcal{W}_m$ also cannot exceed m.

Suppose that $\mathcal{W}_m(x)$ changes sign k $(1 \le k \le m)$ times in $(-1,1)$, namely at the points $y_1, y_2, \ldots, y_k$. Let $\theta_j = 2\arccos(y_j)$, $j = 1, 2, \ldots, k$ and consider the self-inversive polynomial (of degree k) defined by

$$q_k(z) = e^{-i(k\pi+\theta_1+\theta_2+\ldots+\theta_k)/2}(z - e^{i\theta_1})(z - e^{i\theta_2})\cdots(z - e^{i\theta_k}).$$

Now, with $x = \cos(\theta/2) = (z^{1/2} + z^{-1/2})/2$, if we consider the function $\mathcal{F}_k$ given by

$$\mathcal{F}_k(x) = e^{-ik\theta/2}q_k(e^{i\theta}),$$

then by Lemma 2.1 $\mathcal{F}_k \in \Omega_k$ and further $\mathcal{F}_k$ has exactly the k zeros $y_1, y_2, \ldots, y_k$, which are the points of sign changes in $\mathcal{W}_m$. Hence, the function $\mathcal{F}_k(x)\mathcal{W}_m(x)$ does not change sign in $(-1,1)$ which then leads to the conclusion

$$\int_{-1}^{1} \mathcal{F}_k(x)\mathcal{W}_m(x)d\psi(x) \neq 0 \quad \text{and} \quad \int_{-1}^{1} \mathcal{F}_k(x)\mathcal{W}_m(x)\sqrt{1-x^2}d\psi(x) \neq 0.$$

On the other hand, if $k < m$, then from Corollary 4.1.1 at least one of the integrals

$$\int_{-1}^{1} \mathcal{F}_k(x)\mathcal{W}_m(x)d\psi(x) \quad \text{and} \quad \int_{-1}^{1} \mathcal{F}_k(x)\mathcal{W}_m(x)\sqrt{1-x^2}d\psi(x)$$

must be equal to zero, contradicting the earlier conclusion. Hence, the only possibility is $k = m$ and that $\mathcal{W}_m$ has m sign changes in $(-1, 1)$. That is, $\mathcal{W}_m$ has exactly m zeros in $(-1, 1)$. □

Now we look at the sequence of functions $\{\widehat{\mathcal{W}}_m\}$, obtained from the sequence of orthogonal functions $\{\mathcal{W}_m\}$ by the scaling

$$\widehat{\mathcal{W}}_0(x) = \frac{1}{\gamma_0}\,\mathcal{W}_0(x), \qquad \widehat{\mathcal{W}}_m(x) = \frac{1}{\gamma_0 \cdots \gamma_m}\,\mathcal{W}_m(x), \quad m \geq 0.$$

By considering the properties of $\{\widehat{\mathcal{W}}_m\}$ we can state the following.

Theorem 4.4. *Let the sequence of functions $\{\widehat{\mathcal{W}}_m\}$ be given by*

$$\widehat{\mathcal{W}}_0(x) = 1, \qquad \widehat{\mathcal{W}}_1(x) = x - \widehat{\beta}_1\sqrt{1-x^2},$$

$$\widehat{\mathcal{W}}_{m+1}(x) = \left[x - \widehat{\beta}_{m+1}\sqrt{1-x^2}\right]\widehat{\mathcal{W}}_m(x) - \widehat{\alpha}_{m+1}\widehat{\mathcal{W}}_{m-1}(x), \quad m \geq 1,$$

where $\widehat{\beta}_1 = \frac{1}{\widehat{\rho}_0}\int_{-1}^{1} x\,\widehat{\mathcal{W}}_0^2(x)\,d\psi(x)$ *and for* $m \geq 1$,

$$\widehat{\beta}_{m+1} = \frac{1}{\widehat{\rho}_m}\int_{-1}^{1} x\,\widehat{\mathcal{W}}_m^2(x)\,d\psi(x) \qquad \textit{and}$$

$$\widehat{\alpha}_{m+1} = \frac{1}{\widehat{\rho}_{m-1}}\int_{-1}^{1}\left[x - \widehat{\beta}_{m+1}\sqrt{1-x^2}\right]\widehat{\mathcal{W}}_{m-1}(x)\,\widehat{\mathcal{W}}_m(x)\sqrt{1-x^2}\,d\psi(x)$$

$$= \frac{1+\widehat{\beta}_{m+1}^2}{1+\widehat{\beta}_m^2}\,\frac{\widehat{\rho}_m}{\widehat{\rho}_{m-1}}.$$

Here, $\widehat{\rho}_m = \int_{-1}^{1}\widehat{\mathcal{W}}_m^2(x)\sqrt{1-x^2}\,d\psi(x)$, $m \geq 0$. *Then* $\{\widehat{\mathcal{W}}_m\}$, *where* $\widehat{\mathcal{W}}_m \in \Omega_m$ *and is of exact degree* m, *satisfies the orthogonality* (4.1).

Observe that $\widehat{\alpha}_{m+1} > 0$, $m \geq 1$. In fact (see Theorem 6.2), more can be said about these coefficients if the measure ψ is such that $\int_{-1}^{1}(1-x^2)^{-1/2}d\psi(x)$ exists.

5. Quadrature rules associated with $\mathcal{W}_m$

In order to be able to obtain the quadrature rules based on the zeros of $\mathcal{W}_m$ we first present the following theorem.

Theorem 5.1. *Let $\{\mathcal{W}_m\}$ be the sequence of orthogonal functions as defined before.*

Given any function $\mathcal{E} \in \Omega_{4m-1}$, $m \geq 1$, then associated with it there exists a unique function $\mathcal{F} \in \Omega_{2m-1}$ and a unique function $\mathcal{G} \in \Omega_{2m-1}$ such that

$$\mathcal{E}(x) = \mathcal{F}(x)\,\mathcal{W}_{2m}(x) + \mathcal{G}(x).$$

Likewise, given any $\mathcal{E} \in \Omega_{4m}$, $m \geq 1$, then associated with it there exists a unique function $\mathcal{F} \in \Omega_{2m-1}$ and a unique function $\mathcal{G} \in \Omega_{2m}$ such that

$$\mathcal{E}(x) = \mathcal{F}(x)\,\mathcal{W}_{2m+1}(x) + \mathcal{G}(x).$$

Proof. We give a proof of the latter formula. If the formula holds then the uniqueness of $\mathcal{F}$ and $\mathcal{G}$ can be easily verified with the use of Theorem 2.3. Now to prove the existence, let P, Q, R and K_{2m+1} be the respective self-inversive polynomials associated with the functions $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{W}_{2m+1}$. Then one needs to prove that given the self inversive-polynomial P of degree at most $4m$ there exists a self-inversive

polynomial Q of degree at most $2m-1$ and a self-inversive polynomial R of degree at most $2m$ such that

$$P(z) = Q(z)\, K_{2m+1}(z) + z^m R(z). \tag{5.1}$$

With the self-inversive property, we can write

$$\begin{aligned}
P(z) &= \sum_{j=0}^{4m} p_j\, z^j \;=\; \sum_{j=0}^{2m-1} p_j\, z^j \;+ p_{2m}\, z^{2m} + \sum_{j=2m+1}^{4m} \overline{p}_{4m-j}\, z^j,\\
Q(z) &= \sum_{j=0}^{2m-1} q_j\, z^j \;=\; \sum_{j=0}^{m-1} q_j\, z^j \;+ \sum_{j=m}^{2m-1} \overline{q}_{2m-1-j}\, z^j,\\
R(z) &= \sum_{j=0}^{2m} r_j\, z^j \;=\; \sum_{j=0}^{m-1} r_j\, z^j \;+ r_m\, z^m + \sum_{j=m+1}^{2m} \overline{r}_{2m-j}\, z^j,
\end{aligned}$$

and

$$K_{2m+1}(z) = \sum_{j=0}^{2m+1} k_j^{(2m+1)}\, z^j \;=\; \sum_{j=0}^{m} k_j^{(2m+1)}\, z^j \;+ \sum_{j=m+1}^{2m+1} \overline{k_{2m+1-j}^{(2m+1)}}\, z^j.$$

Note that p_{2m} and r_m are both real.

Comparing the coefficients of the potentials $1, z, z^2, \ldots, z^{2m}$ on both sides of (5.1) we have

$$\begin{aligned}
q_0 &= p_0/k_0^{(2m+1)}\\
q_j &= \big[p_j - \sum_{l=0}^{j-1} q_l\, k_{j-l}^{(2m+1)}\big]/k_0^{(2m+1)}, \quad j = 1, 2, \ldots, m-1,\\
r_j &= p_{m+j} - \sum_{l=0}^{m+j} q_l\, k_{m+j-l}^{(2m+1)}, \quad j = 0, 1, \ldots, m-1,
\end{aligned}$$

and

$$r_m = p_{2m} - \sum_{l=0}^{2m-1} q_l\, k_{2m-l}^{(2m+1)} = p_{2m} - \sum_{l=0}^{m-1} [q_l\, k_{2m-l}^{(2m+1)} + \overline{q_l\, k_{2m-l}^{(2m+1)}}].$$

From Theorem 3.4 since $k_0^{(2m+1)} \neq 0$ the above formulas are well defined. Thus, with the further observation that equalities in the higher potentials lead to the same results, the existence of Q and R in (5.1) is verified.

The proof of the first formula of the theorem is similar. □

We now look at the interpolatory type quadrature rule at the zeros of $\mathcal{W}_m$. For the notion of polynomial interpolatory quadrature rules we cite [10].

First we denote the zeros of $\mathcal{W}_m$ by $x_k^{(m)}$, $k = 1, 2, \ldots, m$ and let $z_k^{(m)} = e^{i2\arccos(x_k^{(m)})}$, $k = 1, 2, \ldots, m$. Hence, if $\mathcal{G} \in \Omega_{m-1}$, then by Theorem 2.3 we can write

$$\mathcal{G}(x) = \sum_{k=1}^{m} \mathcal{L}_{m,k}(x)\, \mathcal{G}(x_k^{(m)}),$$

where $\mathcal{L}_{m,k} \in \Omega_{m-1}$ are functions associated with $\mathcal{W}_m$, given by

$$\mathcal{L}_{m,k}(x) = z^{-(m-1)/2}\ (z_k^{(m)})^{(m-1)/2} \prod_{\substack{l=1\\ l\neq k}}^{m} \frac{z - z_l^{(m)}}{z_k^{(m)} - z_l^{(m)}}, \qquad k = 1, 2, \ldots, m,$$

with $x = \cos(\theta/2)$ and $z = e^{i\theta}$. Consequently, if

$$\widehat{\lambda}_k^{(m)} = \int_{-1}^{1} \mathcal{L}_{m,k}(x)\, d\psi(x) \quad \text{and} \quad \widetilde{\lambda}_k^{(m)} = \int_{-1}^{1} \mathcal{L}_{m,k}(x)\, \sqrt{1-x^2}\, d\psi(x),$$

for $k = 1, 2, \ldots, m$ then, for any $\mathcal{G} \in \Omega_{m-1}$, we have the following interpolatory quadrature rules

$$\begin{aligned} \int_{-1}^{1} \mathcal{G}(x)\, d\psi(x) &= \sum_{k=1}^{m} \widehat{\lambda}_k^{(m)}\, \mathcal{G}(x_k^{(m)}), \\ \int_{-1}^{1} \mathcal{G}(x)\, \sqrt{1-x^2}\, d\psi(x) &= \sum_{k=1}^{m} \widetilde{\lambda}_k^{(m)}\, \mathcal{G}(x_k^{(m)}). \end{aligned} \tag{5.2}$$

Clearly, these quadrature rules hold for any distinct set of points $x_k^{(m)}$, $k=1, 2, \ldots, m$. However, since $x_k^{(m)}$ are the zeros of the functions $\mathcal{W}_m$ we can say more.

Theorem 5.2. *Let $x_k^{(m)}$, $k = 1, 2, \ldots, m$ be the zeros of $\mathcal{W}_m$ and let*

$$\lambda_k^{(m)} = \int_{-1}^{1} [\mathcal{L}_{m,k}(x)]^2\ \sqrt{1-x^2}\, d\psi(x), \quad k = 1, 2, \ldots, m.$$

If $\mathcal{E} \in \Omega_{4m-1}$, $m \geq 1$, then

$$\int_{-1}^{1} \mathcal{E}(x)\, d\psi(x) = \sum_{k=1}^{2m} \frac{1}{\sqrt{1-(x_k^{(2m)})^2}}\, \lambda_k^{(2m)}\, \mathcal{E}(x_k^{(2m)})$$

and if $\mathcal{E} \in \Omega_{4m}$, $m \geq 0$, then

$$\int_{-1}^{1} \mathcal{E}(x)\, \sqrt{1-x^2}\, d\psi(x) = \sum_{k=1}^{2m+1} \lambda_k^{(2m+1)}\, \mathcal{E}(x_k^{(2m+1)}).$$

Proof. To obtain the first quadrature, with $\mathcal{E} \in \Omega_{4m-1}$ we have from Theorem 5.1 that there exist $\mathcal{F} \in \Omega_{2m-1}$ and $\mathcal{G} \in \Omega_{2m-1}$ such that $\mathcal{E}(x) = \mathcal{F}(x)\, \mathcal{W}_{2m}(x) + \mathcal{G}(x)$. Hence, from the orthogonality given by Corollary 4.1.1 it follows that

$$\int_{-1}^{1} \mathcal{E}(x)\, d\psi(x) = \int_{-1}^{1} \mathcal{G}(x)\, d\psi(x).$$

Therefore, from $\mathcal{E}(x_k^{(2m)}) = \mathcal{G}(x_k^{(2m)})$, $k = 1, 2, \ldots, 2m$ and from (5.2), we have

$$\int_{-1}^{1} \mathcal{E}(x)\, d\psi(x) = \sum_{k=1}^{2m} \widehat{\lambda}_k^{(2m)}\, \mathcal{E}(x_k^{(2m)}),$$

which holds for $\mathcal{E} \in \Omega_{4m-1}$. With the choice $\mathcal{E}(x) = \sqrt{1-x^2}\, [\mathcal{L}_{2m,j}(x)]^2 \in \Omega_{4m-1}$ we then obtain that

$$\widehat{\lambda}_j^{(2m)} = \frac{1}{\sqrt{1-(x_j^{(2m)})^2}}\, \lambda_j^{(2m)}, \quad j = 1, 2, \ldots, 2m.$$

Now to obtain the latter quadrature rule, with $\mathcal{E} \in \Omega_{4m}$ it follows from Theorem 5.1 that there exist $\mathcal{F} \in \Omega_{2m-1}$ and $\mathcal{G} \in \Omega_{2m}$ such that $\mathcal{E}(x) = \mathcal{F}(x)\,\mathcal{W}_{2m+1}(x) + \mathcal{G}(x)$. This leads to the interpolatory quadrature rule

$$\int_{-1}^{1} \mathcal{E}(x)\,\sqrt{1-x^2}\,d\psi(x) = \sum_{k=1}^{2m+1} \widetilde{\lambda}_k^{(2m+1)}\,\mathcal{E}(x_k^{(2m+1)}),$$

which holds for $\mathcal{E} \in \Omega_{4m}$. With the choice $\mathcal{E}(x) = [\mathcal{L}_{2m+1,j}(x)]^2 \in \Omega_{4m}$ we then obtain that

$$\widetilde{\lambda}_j^{(2m+1)} = \lambda_j^{(2m+1)}, \quad j = 1, 2, \ldots, 2m+1.$$

This completes the proof of the theorem. □

Since $\lambda_k^{(m)}$ in Theorem 5.2 are all positive, we can now state the following theorem.

Theorem 5.3. *For $m \geq 1$, let $x_0^{(m)} = -1$, $x_{m+1}^{(m)} = 1$ and let $x_j^{(m)}$, $j = 1, 2, \ldots, m$ be the zeros of $\mathcal{W}_m(x)$ arranged such that $x_j^{(m)} < x_{j+1}^{(m)}$, $j = 1, 2, \ldots, m-1$. Then in any of the intervals $(x_j^{(m)}, x_{j+1}^{(m)})$, $j = 0, 1, \ldots, m$, there is a zero of $\mathcal{W}_{2m_r}$, where $m_r = \lfloor (m+2r)/2 \rfloor$ with $r \geq 1$. In particular, there holds the interlacing property*

$$x_1^{(2n)} < x_1^{(2n-1)} < x_2^{(2n)} < \ldots < x_{2n-1}^{(2n)} < x_{2n-1}^{(2n-1)} < x_{2n}^{(2n)}, \quad n \geq 1.$$

Proof. From Corollary 4.1.1 and from the first quadrature rule in Theorem 5.2,

$$\begin{aligned} 0 &= \int_{-1}^{1} \mathcal{W}_m(x)\mathcal{F}_{m-1}(x)\,d\psi(x) \\ &= \sum_{k=1}^{2m_r} \frac{\lambda_k^{(2m_r)}}{\sqrt{1-(x_k^{(2m_r)})^2}}\,\mathcal{W}_m(x_k^{(2m_r)})\mathcal{F}_{m-1}(x_k^{(2m_r)}), \end{aligned} \tag{5.3}$$

where $\mathcal{F}_{m-1} \in \Omega_{m-1}$. Now suppose that the sequence $\{\mathcal{W}_m(x_1^{(2m_r)}), \mathcal{W}_m(x_2^{(2m_r)}), \ldots, \mathcal{W}_m(x_{2m_r}^{(2m_r)})\}$, obtained from the evaluations of $\mathcal{W}_m$ at the points $x_1^{(2m_r)}, x_2^{(2m_r)}, \ldots, x_{2m_r}^{(2m_r)}$, has ℓ $(< m)$ sign changes. Let $y_1, y_2, \ldots, y_\ell$ be points chosen within each of the intervals where the sign changes occur and let $z_j = e^{i2\arccos(y_j)}$, $j = 1, 2, \ldots, \ell$. Hence, if we choose $\mathcal{F}_{m-1}$ such that

$$\mathcal{F}_{m-1}(x) = e^{-i(m-1)\theta/2} e^{i(m-1)\pi/2} (z_1 z_2 \cdots z_\ell)^{-1/2} (e^{i\theta}-1)^{(m-1)-\ell} \prod_{j=1}^{\ell} (e^{i\theta} - z_j),$$

where $x = \cos(\theta/2)$, then the right-hand side of the quadrature sum in (5.3) cannot be zero (because all the terms are of the same sign). Thus $\ell < m$ leads to a contradiction in (5.3). Hence, considering also that $\mathcal{W}_m(x)$ has exactly m zeros in $(-1, 1)$, we must have exactly m sign changes in the sequence $\{\mathcal{W}_m(x_1^{(2m_r)}), \mathcal{W}_m(x_2^{(2m_r)}), \ldots, \mathcal{W}_m(x_{2m_r}^{(2m_r)})\}$. The first part of the above theorem is a consequence of this.

To arrive at the latter part of the theorem, we take $m = 2n-1$ and $r = 1$. □

6. Connection with orthogonal polynomials on the unit circle

From now on let ψ be a positive measure on $[-1,1]$ such that $\int_{-1}^{1}(1-x^2)^{-1/2}d\psi(x)$ exists. Let μ be a positive measure on the unit circle that satisfies

$$-\sin^2(\theta/2)\,d\mu(e^{i\theta}) = d\psi(x), \tag{6.1}$$

where $x = (z^{1/2}+z^{-1/2})/2 = \cos(\theta/2)$.

Observe that the measure μ is not unique. If $\widetilde{\mu}$ is a positive measure on the unit circle such that

$$\int_{\mathcal{C}} f(z)\,d\widetilde{\mu}(z) = \int_{\mathcal{C}} f(z)\,d\mu(z) + \delta\, f(1),$$

where δ is some nonzero constant, then (6.1) also holds for $\widetilde{\mu}$.

Now consider the sequence of self-inversive polynomials $\{\widehat{K}_m\}$ given by

$$e^{-im\theta/2}\widehat{K}_m(e^{i\theta}) = 2^m\widehat{\mathcal{W}}_m(x), \quad m\geq 0,$$

where $\{\widehat{\mathcal{W}}_m\}$ are the normalized orthogonal functions given in Theorem 4.4. Hence, one easily obtains from Corollary 4.1.3 and Theorem 4.4 the following.

Theorem 6.1. *Elements of the sequence of polynomials $\{\widehat{K}_m\}$ satisfy*

$$\int_{\mathcal{C}} z^{-m+s}\widehat{K}_m(z)\,(1-z)d\mu(z) = 0, \quad s=0,1,\dots,m-1, \quad m\geq 1.$$

Moreover,

$$\begin{aligned} &\widehat{K}_0(z) = 1, \qquad \widehat{K}_1(z) = (1+i\widehat{\beta}_1)z + (1-i\widehat{\beta}_1),\\ &\widehat{K}_{m+1}(z) = \big[(1+i\widehat{\beta}_{m+1})z + (1-i\widehat{\beta}_{m+1})\big]\widehat{K}_m(z)\\ &\qquad\qquad - 4\widehat{\alpha}_{m+1}z\,\widehat{K}_{m-1}(z), \quad m\geq 1, \end{aligned} \tag{6.2}$$

where $\widehat{\beta}_m$, $\widehat{\alpha}_{m+1}$, $m\geq 1$ are as in Theorem 4.4.

The remaining results in this section, stated mainly without any proofs, follows from recent results obtained in [5].

The polynomials $\widehat{K}_m$, $m\geq 0$, are constant multiples of the CD kernels

$$\mathcal{K}_m(z,1) = \frac{\overline{s^*_{m+1}(1)}\,s^*_{m+1}(z) - \overline{s_{m+1}(1)}\,s_{m+1}(z)}{1-z} = \sum_{j=0}^{m}\overline{s_j(1)}\,s_j(z), \quad m\geq 0,$$

where s_m, $m\geq 0$, are the orthonormal polynomials on the unit circle with respect to the positive measure μ. Orthogonal polynomials on the unit circle (OPUC) were introduced by Gábor Szegő in the first half of the 20th century. Many details regarding the earlier research on these polynomials can be found, for example, in the texts [9], [11], [19] and [20]. For recent and more up to date texts on this subject we refer to the two volumes of Simon [16] and [17]. There is also a nice chapter about these polynomials in Ismail [12].

Now the following result can be stated for the coefficients $\{\widehat{\alpha}_{m+1}\}_{m=1}^{\infty}$ that appear in the three term recurrence formula given in Theorem 4.4 (and also in the three term recurrence formula given in Theorem 6.1).

Theorem 6.2. *Let the positive measure ψ on $[-1,1]$ be such that the integral $\int_{-1}^{1}(1-x^2)^{-1/2}d\psi(x)$ exists. Then the sequence of positive numbers $\{\widehat{\alpha}_{m+1}\}_{m=1}^{\infty}$ in Theorem 4.4 is a positive chain sequence. Moreover, this positive chain sequence*

is such that its maximal parameter sequence does not coincide with its minimal parameter sequence.

In (see [7]) the knowledge of $\{\widehat{\alpha}_{m+1}\}_{m=1}^{\infty}$ being a positive chain sequence is used, together with (6.2), to prove the interlacing of the zeros of $\mathcal{W}_m$ and $\mathcal{W}_{m+1}$ (see [7]). Observe that Theorem 5.3 in the present paper gives only a partial interlacing property.

The results of the above theorem may be true even without the assumed condition for the measure ψ. However, we have not been able to verify this.

Even though the polynomials $\widehat{K}_m$ are uniquely defined in terms of the measure ψ, we have already observed that the measure μ that satisfies (6.1) is not unique (varying according to the size of the jump at $z = 1$). Hence, with such distinct measures there exist distinct sets of OPUC. However, with $0 \le t < 1$, if one defines the probability measure $\mu^{(t)}$ such that

$$\begin{aligned} &-\sin^2(\theta/2)\, d\mu^{(t)}(e^{i\theta}) = c(t)\ d\psi(x), \\ &\mu^{(t)} \text{ has a jump } t \text{ at } z = 1 \text{ (i.e. } \mu^{(t)} \text{ has a pure point of size } t \text{ at } z = 1), \end{aligned} \tag{6.3}$$

where $x = (z^{1/2} + z^{-1/2})/2 = \cos(\theta/2)$ and $c(t)$ is the normalizing constant so that $\int_{\mathcal{C}} d\mu^{(t)}(z) = 1$, then we can say more about the associated monic OPUC $S_n^{(t)}$ and hence, also the orthonormal polynomials $s_n^{(t)}$.

Let $\{M_m\}_{m=1}^{\infty}$ be the maximal parameter sequence of the positive chain sequence $\{\widehat{\alpha}_m\}_{m=2}^{\infty}$. Using the value of t and the sequence $\{M_m\}_{m=1}^{\infty}$, we now consider the new positive chain sequence $\{\widetilde{\alpha}_m\}_{m=1}^{\infty}$ given by

$$\widetilde{\alpha}_1 = (1-t)M_1, \qquad \widetilde{\alpha}_{m+1} = \widehat{\alpha}_{m+1} = (1-M_m)M_{m+1}, \quad m \ge 1.$$

It is easily verified (see [4]) that the maximal parameter sequence of the positive chain sequence $\{\widetilde{\alpha}_m\}_{m=1}^{\infty}$ is precisely $\{M_m\}_{m=0}^{\infty}$, with $M_0 = t$. Let $\{\mathfrak{m}_m^{(t)}\}_{m=0}^{\infty}$ be the minimal parameter sequence of $\{\widetilde{\alpha}_m\}_{m=1}^{\infty}$. That is,

$$\mathfrak{m}_0^{(t)} = 0, \qquad \mathfrak{m}_1^{(t)} = \widetilde{\alpha}_1 = (1-t)M_1, \qquad \mathfrak{m}_{m+1}^{(t)} = \widetilde{\alpha}_{m+1}/(1-\mathfrak{m}_m^{(t)}), \quad m \ge 1.$$

Then we can state the following.

Theorem 6.3. *Let $S_m^{(t)}$, $m \ge 0$, be the monic OPUC with respect to the measure $\mu^{(t)}$ given by* (6.3)*. Then the associated Verblunsky coefficients $a_{m-1}^{(t)} = -\overline{S_m^{(t)}(0)}$, $m \ge 1$, satisfy*

$$a_{m-1}^{(t)} = \frac{1}{\tau_{m-1}} \frac{1 - 2\mathfrak{m}_m^{(t)} - i\widehat{\beta}_m}{1 - i\widehat{\beta}_m} \quad m \ge 1,$$

where $\tau_0 = 1$ *and* $\tau_m = \dfrac{1 - i\widehat{\beta}_m}{1 + i\widehat{\beta}_m}\, \tau_{m-1}, \ m \ge 1.$

7. Examples

Given any measure ψ on $[-1, 1]$, one can easily obtain by numerical computation the coefficients in the three term recurrence formulas in Theorem 4.1, hence, also information about the required orthogonal functions $\mathcal{W}_m$. However, for two good reasons we like to consider the normalized orthogonal functions $\widehat{\mathcal{W}}_m$ given by Theorem 4.4. One of these reasons is that when the measure is symmetric then

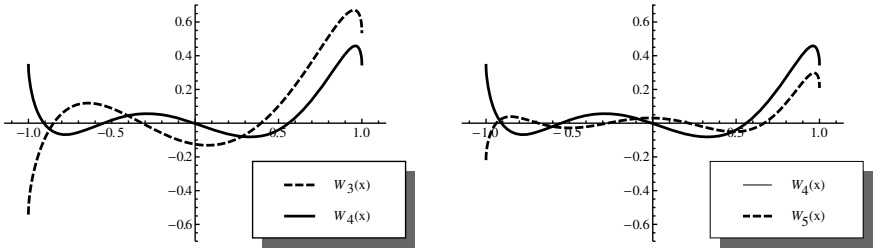

FIGURE 1. Plots of the functions $\widehat{\mathcal{W}}_3$, $\widehat{\mathcal{W}}_4$ and $\widehat{\mathcal{W}}_5$ when $d\psi(x) = (1-x)dx$.

these functions turn out to be monic polynomials. The other reason is because of the chain sequence property given by Theorem 6.2.

Example 1. Let $d\psi(x) = (1-x)dx$. We obtain by numerical computation the following values for the first few $\widehat{\alpha}_m$'s and $\widehat{\beta}_m$'s in the three term recurrence formula in Theorem 4.4:

m	1	2	3	4	5	6
$\widehat{\beta}_m$	-0.424413	-0.302998	-0.239816	-0.200358	-0.173083	-0.152964
$\widehat{\alpha}_m$		0.222958	0.240821	0.245531	0.247399	0.248315

In the two graphs in Figure 1 we give, respectively, plots of the functions $\widehat{\mathcal{W}}_3$ and $\widehat{\mathcal{W}}_4$ and plots of the functions $\widehat{\mathcal{W}}_4$ and $\widehat{\mathcal{W}}_5$, separated in this way to be able see clearly the interlacing of the zeros as pointed out after Theorem 6.2.

Glancing at the plot of $\widehat{\mathcal{W}}_4$ it appears as though this function has a zero at the origin. To be precise, the zero is only very near to the origin and the value of this zero is roughly equal to -0.0055075.

The second example we give here is interesting from the point of view of knowing many things explicitly.

Example 2. Let $d\psi(x) = [e^{-\arccos(x)}]^{2\eta}\,[1-x^2]^{\lambda-1}dx$, where $\eta, \lambda \in \mathbb{R}$ and $\lambda > 1/2$. Here, we assume $\arccos(x)$ between 0 and π. From results given in [8] we have

$$\widehat{\mathcal{W}}_m(x) = 2^{-m}\,\frac{(2\lambda)_m}{(\lambda)_m}\,e^{-im\theta/2}\,{}_2F_1(-m, b;\, b+\bar{b};\, 1-e^{i\theta}),$$

where $x = \cos(\theta/2)$, $b = \lambda + i\eta$ and the hypergeometric polynomial ${}_2F_1(-m, b;\, b+\bar{b};\, 1-z)$ is self-inversive. The orthogonality of $\{\widehat{\mathcal{W}}_m\}$ can be explicitly written as

$$\int_{-1}^{1} \widehat{\mathcal{W}}_{2n}(x)\,\widehat{\mathcal{W}}_{2m}(x)\,[e^{-\arccos(x)}]^{2\eta}\,[1-x^2]^{\lambda-1/2}dx = \widehat{\rho}_{2m}\,\delta_{n,m},$$

$$\int_{-1}^{1} \widehat{\mathcal{W}}_{2n+1}(x)\,\widehat{\mathcal{W}}_{2m+1}(x)\,[e^{-\arccos(x)}]^{2\eta}\,[1-x^2]^{\lambda-1/2}dx = \widehat{\rho}_{2m+1}\,\delta_{n,m},$$

$$\int_{-1}^{1} \widehat{\mathcal{W}}_{2n+1}(x)\,\widehat{\mathcal{W}}_{2m}(x)\,[e^{-\arccos(x)}]^{2\eta}\,[1-x^2]^{\lambda-1}dx = 0,$$

for $n, m = 0, 1, 2, \ldots$, where

$$\widehat{\rho}_m = \frac{\pi\, m!\,(\lambda+m)\,\Gamma(2\lambda+m)}{2^{2\lambda+2m-1}e^{\eta\pi}\,|\Gamma(b+m+1)|^2}\,\frac{1}{[(\lambda)_m]^2}\,\big[\big(\mathcal{R}e[(b)_m]\big)^2 + \big(\mathcal{I}m[(b)_m]\big)^2\big].$$

Here, Γ represents the gamma function and that $(b)_0 = 1$ and $(b)_m = b(b+1)\cdots(b+m-1)$ for $m \geq 1$ are the Pochhammer symbols.

Moreover, in the three term recurrence formula (see Theorem 4.4) for $\{\widehat{\mathcal{W}}_m\}$,

$$\widehat{\beta}_m = \frac{\eta}{m+\lambda-1} \quad \text{and} \quad \widehat{\alpha}_{m+1} = \frac{1}{4}\frac{m(m+2\lambda-1)}{(m+\lambda-1)(m+\lambda)}, \quad m \geq 1.$$

Observe that when $\eta = 0$, the functions $\widehat{\mathcal{W}}_m$ reduce to the monic ultraspherical polynomials $C_m^{(\lambda-1/2)}$.

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