MATHEMATICS OF COMPUTATION
Volume 86, Number 308, November 2017, Pages 2579–2611
http://dx.doi.org/10.1090/mcom/3173
Article electronically published on March 29, 2017

THE TRIANGULAR SPECTRAL ELEMENT METHOD FOR STOKES EIGENVALUES

WEIKUN SHAN AND HUIYUAN LI

ABSTRACT. A triangular spectral element method is proposed for Stokes eigenvalues, which utilizes the generalized orthogonal Koornwinder polynomials as the local basis functions. The local polynomial projection, which serves as a Fortin interpolation on each triangular element, is defined by the truncated Koornwinder-Fourier series. A sharp estimate on the discrete inf-sup constant of the divergence for our triangular spectral element approximation scheme is then acquired via the stability analysis of the local projection operator. Further, the optimal error estimate of the H^1-orthogonal spectral element projection oriented to Stokes equations is obtained through the globally continuous piecewise polynomial assembled by the union of all local projections. In the sequel, the optimal convergence rate/error estimate theory is eventually established for our triangular spectral element method for both eigenvalue and source problems of the Stokes equations. Finally, numerical experiments are presented to illustrate our theories on both the discrete inf-sup constant of the divergence and the accuracy of the computational eigenvalues.

1. INTRODUCTION

There has been increasing interest in the study of the eigenvalue problem of the Stokes equation, owing to its various applications in fluid mechanics. Abundant literature are contributed to numerical investigations of Stokes eigenvalues by various low order finite element methods [1, 20, 29–31, 38, 39, 44, 59]. In contrast, little attention has been paid to the spectral and the spectral element methods to solve the eigenvalue problem of the Stokes equation, in spite of their popularity in seeking numerical solutions of the source problems (see [18, 19, 40, 46, 50] for example).

Spectral element methods were first introduced by Patera [46] for Chebyshev expansions, then generalized to the Legendre case by Maday and Patera [42]. Analogous to p- and hp-finite element methods, spectral element methods inherit the high accuracy and convergence rate of the traditional spectral methods, while preserving the flexibility of the low order finite element methods. Evidence shows that spectral element methods enjoy some essential priorities over the traditional spectral method and other low order methods for eigenvalue problems [14, 56, 60].

Two types of spectral element methods are commonly used in planar domains. The first type is the quadrilateral element, which is gradually matured both in

Received by the editor April 12, 2015 and, in revised form, April 14, 2015 and February 4, 2016.

2010 *Mathematics Subject Classification.* Primary 65N35, 65N25, 35P15, 35Q30.

Key words and phrases. Stokes eigenvalues, triangular spectral element method, generalized orthogonal Koornwinder polynomials, error analysis.

This work was supported by National Natural Science Foundation of China (No. 91130014, 11471312 and 91430216).

computational practice and in numerical theory owing to its advantages of using tensorial basis functions and naturally diagonal mass matrices [19, 32]. The second is the triangular element, which is proven to be more flexible for complex domains and for adaptivity. Although considerable progress has recently been made in the triangular spectral element method [32, 45, 47, 51, 52, 55], some fundamental issues in numerical analysis remain to be resolved.

The purpose of this paper is to propose a triangular spectral element method and conduct a comprehensive and rigorous (a priori) error analysis for the following Stokes eigenvalue problems,

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \lambda \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ denotes a bounded domain with Lipschitz boundary $\partial\Omega$. Based on the variational formulation of (1.1) and the regular triangulations of Ω, a mixed triangular spectral element approximation scheme is established. To ensure the effectiveness of our approximation scheme, the velocity and the pressure are approximated by globally continuous piecewise polynomials of total degree $\leq M$ and piecewise polynomials of total degree $\leq M-2$, respectively. This yields the so-called $\mathbb{P}_M - \mathbb{P}_{M-2}$ spectral element method [41], which can eliminate the spurious modes on the pressure and guarantee the well-posedness of our scheme [8]. Special kinds of orthogonal Koornwinder polynomials are then utilized in our $\mathbb{P}_M - \mathbb{P}_{M-2}$ spectral element method as the local basis functions on each element such that the corresponding matrix eigenvalue problem can be fast assembled and then be efficiently evaluated by sparse eigensolvers.

For an optimal error estimate of the triangular spectral element method, two ingredients should be highlighted. One is the sharp estimate of the discrete inf-sup constant of divergence in the $\mathbb{P}_M - P_{M-2}$ approximation scheme, the other is the optimal error estimate of the orthogonal spectral element projections.

The discrete inf-sup constant of the divergence with respect to the approximation space $\boldsymbol{V}_\delta \times W_\delta \subseteq H_0^1(\Omega)^2 \times L_0^2(\Omega)$ is defined by

$$\beta_\delta = \inf_{0 \neq q \in W_\delta} \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \boldsymbol{V}_\delta} \frac{-(\operatorname{div} \boldsymbol{v}, q)_\Omega}{\|\boldsymbol{v}\|_{H_0^1(\Omega)^2} \|q\|_{L^2(\Omega)}},$$

where $\delta = \delta(M, h)$ is the discretisation parameter dependent on the polynomial degree M and the mesh size h. The discrete inf-sup condition of the divergence is also referred to as the Ladyzhenskaya-Babuška-Brezzi condition or simply LBB condition [3, 17, 24, 34]. It plays a key role in the discussion of the stability and convergence of the approximation solutions of the equations of hydrodynamics, since it reflects the reliability and effectivity of an approximation scheme. In particular, an optimal convergence analysis of the approximation scheme relies on the sharp estimate of the discrete inf-sup constant. However, one should recognize that the inf-sup condition is a severe requirement, and an analytical estimate on the inf-sup constant is usually out of reach even if the inf-sup condition is satisfied [22, 40, 50].

Based on the orthogonality structure of the tensorial Jacobi polynomial basis on a rectangle or a cuboid, [8] shows that the discrete inf-sup constant behaves asymptotically as $M^{\frac{1-d}{2}}$ for the $\mathbb{Q}_M - \mathbb{Q}_{M-2}$ spectral method in $d = 2, 3$ dimensions, thus the sharp estimate $\beta_\delta \gtrsim M^{\frac{1-d}{2}}$ holds in the corresponding quadrilateral/hexahedral

spectral element method [19]. On the other hand, due to the lack of an orthogonality analysis for the p- and hp-version triangular finite element methods, researchers can only show a theoretical lower bound of $\mathcal{O}(M^{-3})$ up to the present for the discrete inf-sup constant of the $\mathbb{P}_M - \mathbb{P}_{M-2}$ method [49], although numerical tests reveal that $\beta_\delta \gtrsim M^{-1}$ [22]. Now, our first task in numerical analysis is to obtain a sharp estimate on β_δ for the triangular spectral element method, which is compatible with and comparable to that for the quadrilateral spectral element method.

In the optimal error analysis of the spectral element approximation together with the p- or hp-finite element approximation, it is key to construct a globally continuous piecewise polynomial which satisfies the Dirichlet boundary conditions and has the optimal estimate for its approximation error. The construction of such a piecewise polynomial is usually started with a local projection for best approximation on each element [18,26,36,48]. However, a union of all the local projections is not necessarily continuous across the common edge or vertex of two contiguous elements, and usually does not satisfy the Dirichlet boundary conditions. For the global continuity, we have to modify these local projections by resorting to an intricate technique called the polynomial extension or lifting (see [25] and the reference therein). Unfortunately, the above approach may fail for some special purposed approximations such as the best approximation of functions with corner singularities and the best approximation of the divergence free functions by piecewise polynomials in the kernel space $\boldsymbol{X}_\delta = \{\boldsymbol{v} \in \boldsymbol{V}_\delta : (\mathrm{div}\boldsymbol{v}, q)_\Omega = 0,\ \forall q \in W_\delta\}$. So we shall abandon the conventional process and develop a simple new approach in this paper for the optimal error estimate of the triangular spectral element approximations.

For our purposes, we start with the generalized Koornwinder polynomials $\mathcal{J}_l^{-1,-1,-1}$, $l \in \mathbb{N}_0^2$ on an abitrary triangular element T, which are proved to be mutually orthogonal to each other under a properly constructed inner product $\langle\cdot,\cdot\rangle_T$ in $H^2(T)$. The truncated Fourier series $\pi_{M,T}^{-1,-1,-1}u$ in the orthogonal Koornwinder polynomials $\mathcal{J}_l^{-1,-1,-1}$, $l \in \mathbb{N}_0^2$, is then set up as the desired local projection in our numerical analysis. An orthogonality analysis reveals that $\pi_{M,T}^{-1,-1,-1}$ is actually a Fortin interpolation operator in $H_0^1(T)$ whose norm is bounded above by $\sqrt{M}$. Following Fortin's criterion together with the Boland-Nicoliades local-to-global argument, we derive a sharp estimate on the discrete inf-sup constant of the divergence.

Moreover, the truncated Fourier series $\pi_{M,T}^{-1,-1,-1}u$ is an orthogonal projection, hence acts as the best polynomial to u under the norm induced by $\langle\cdot,\cdot\rangle_T$. Hopefully, the projection $\pi_{M,T}^{-1,-1,-1}$ interpolates functions at the vertices of T, and its boundary trace on each side Γ_i of T reduces to the one-dimensional symmetric projection $\pi_{M,\Gamma_i}^{-1,-1}$, $i = 1,2,3$. Thus, the local-to-global mapping can be simply defined as the union of all the local projections $\pi_{M,T}^{-1,-1,-1}u$ such that the resulting global projection is a naturally continuous piecewise polynomial which satisfies the Dirichlet boundary conditions and has the optimal error estimate. Utilizing the global projection as the intermediate piecewise polynomial, we further obtain the optimal error estimate of the H^1-orthogonal spectral element projection for Stokes equations.

Finally, the optimal error estimate of our triangular spectral element method for Stokes eigenvalues follows from the general theory in [44] and [5] on the mixed method for eigenvalue problems. We emphasize that our results are equally applicable to the source problems, since the numerical analysis of the mixed method for

the Stokes eigenvalue problem lies on the convergence theory for the corresponding source problem.

It is worth mentioning that the orthogonal Koornwinder polynomial system on a triangle is not uniquely determined, while the orthogonal projection $\pi_{M,T}^{\alpha_1,\alpha_2,\alpha_3}$ on the triangle T is independent of a particular orthogonal polynomial basis in spite of its concrete representations in truncated Koornwinder-Fourier series. This fundamental theory in approximation ensures the rotational invariance of the local orthogonal projection $\pi_{M,T}^{\alpha_1,\alpha_2,\alpha_3}u$ as shown in Section 4.1, and allows us to perform a numerical analysis direction by direction on an arbitrary triangle in a similar way as on a tensorial rectangle. In return, it simplifies the numerical analysis of the triangular spectral element method.

The remainder of the paper is organized as follows. In Section 2, we introduce notation and conventions, and present orthogonal polynomials on an interval and on a triangle. The triangular spectral element method and its fast implementation for solving Stokes eigenvalues are proposed in Section 3. The main results on the discrete inf-sup constant of divergence and the triangular spectral element approximation are then established in Section 4 and Section 5, respectively. In Section 6 we provide the optimal error estimate on the triangular spectral element method for Stokes eigenvalues. Finally, in Section 7 we present some numerical results which are consistent with our theoretical analysis and we conclude with some remarks.

2. Preliminaries

2.1. **Notation and conventions.** Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain and let w be a generic weight function. Denote by $(u,v)_{w,\Omega} = \int_\Omega u(x)v(x)wdx$ and $\|\cdot\|_{w,\Omega}$ the inner product and the norm of $L^2_w(\Omega)$, respectively. In addition, we denote by $L^2_0(\Omega) = \{u \in L^2(\Omega) : (u,1)_{1,\Omega} = 0\}$ the subspace of L^2-functions with zero mean on Ω. Further, we use $H^s_w(\Omega)$ and $H^s_{0,w}(\Omega)$ to denote the usual weighted Sobolev spaces, whose norms and seminorms are denoted by $\|\cdot\|_{s,w,\Omega}$ and $|\cdot|_{s,w,\Omega}$, respectively. In cases where no confusion could arise, w (if $w \equiv 1$) and Ω may be dropped from the notation.

Let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}^-$ be the sets of the positive integers, nonnegative integers and the negative integers, respectively. For any $k \in \mathbb{N}_0$, we denote by $\mathbb{P}_k(\Omega)$ the space of polynomials of total degree $\leq k$ on Ω.

Denote by c or C a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ (resp. $A \gtrsim B$) to mean that $A \leq cB$ (resp. $A \geq cB$), and $A \asymp B$ to mean $A \lesssim B \lesssim A$.

Let $\hat{T} = \{(\hat{x},\hat{y}) : 0 < \hat{x}, \hat{y}; \hat{x}+\hat{y} < 1\}$ be the reference triangle. We shall abbreviate the differential operators $\frac{\partial}{\partial \hat{x}}$ and $\frac{\partial}{\partial \hat{y}}$ to $\partial_{\hat{x}}$ and $\partial_{\hat{y}}$, respectively. Moreover, we let $\partial_{\hat{z}} = \partial_{\hat{y}} - \partial_{\hat{x}}$. Denote by I the identity operator.

2.2. **Generalized Jacobi polynomials.** Let $I = (-1,1)$. The hypergeometric representation for the classic Jacobi polynomials $J_n^{\alpha_1,\alpha_2}(\zeta)$, $\zeta \in I, n \in \mathbb{N}_0$, with $\alpha_1, \alpha_2 > -1$,

$$J_n^{\alpha_1,\alpha_2}(\zeta) = \binom{n+\alpha_1}{n} {}_2F_1(-n, n+\alpha_1+\alpha_2+1; \alpha_1+1; \frac{1-\zeta}{2}), \quad -n-\alpha_1-\alpha_2 \notin \{1,2,\ldots,n\}, \tag{2.1}$$

furnishes the extension of $J_n^{\alpha_1,\alpha_2}(\zeta)$ to arbitrary α_1 and α_2 [54]. The restriction $-n-\alpha_1-\alpha_2 \notin \{1,2,\ldots,n\}$ is enforced such that the generalized Jacobi polynomial $J_n^{\alpha_1,\alpha_2}(\zeta)$ is exactly of degree n, since a degree reduction occurs in (2.1) if and only if $-n-\alpha_1-\alpha_2 \in \{1,2,\ldots,n\}$ [28, 54].

Denote by $\chi(x)$ a "characteristic" function for negative integers such that $\chi(x) = -x$ if $x \in \mathbb{Z}^-$ and $\chi(x) = 0$ otherwise. The generalized Jacobi polynomials $J_n^{\alpha_1,\alpha_2}(\zeta)$, $n \geq \chi(\alpha_1)+\chi(\alpha_2)$, defined by (2.1) with $\alpha_1 \in \mathbb{Z}^-$ and/or $\alpha_2 \in \mathbb{Z}^-$ are as defined in [36], and also coincide, up to generic constants, with those defined in [27].

For $\alpha_1, \alpha_2 \in \mathbb{Z}^- \cup (-1,\infty)$, the generalized Jacobi polynomials $J_n^{\alpha_1,\alpha_2}$, $n \geq \chi(\alpha_1)+\chi(\alpha_2)$, are mutually orthogonal with respect to the weight function $w^{\alpha_1,\alpha_2} := w^{\alpha_1,\alpha_2}(\zeta) = (1-\zeta)^{\alpha_1}(1+\zeta)^{\alpha_2}$ on I [27, 36], i.e.,

$$
\begin{aligned}
&(J_m^{\alpha_1,\alpha_2}, J_n^{\alpha_1,\alpha_2})_{w^{\alpha_1,\alpha_2},I} = \gamma_n^{\alpha_1,\alpha_2}\delta_{m,n}, \qquad m,n \geq \chi(\alpha_1)+\chi(\alpha_2),\\
&\gamma_n^{\alpha_1,\alpha_2} := \frac{2^{\alpha_1+\alpha_2+1}}{2n+\alpha_1+\alpha_2+1}\,\frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{\Gamma(n+1)\Gamma(n+\alpha_1+\alpha_2+1)}\,\delta_{m,n},
\end{aligned}
\tag{2.2}
$$

where $\delta_{m,n}$ is the Kronecker delta. Moreover, the generalized Jacobi polynomials satisfy the following differential recurrence relation,

$$
\begin{aligned}
\partial_\zeta J_n^{\alpha_1,\alpha_2}(\zeta) = &\frac{n+\alpha_1+\alpha_2+1}{2} J_{n-1}^{\alpha_1+1,\alpha_2+1}(\zeta),\\
&-n-\alpha_1-\alpha_2 \notin \{1,2,\ldots,n\}.
\end{aligned}
\tag{2.3}
$$

Of great interest to us are those polynomials $J_n^{\alpha_1,\alpha_2}, n \in \mathbb{N}_0$, with $\alpha_1 = -1$ and/or $\alpha_2 = -1$. At first, we directly obtain from (2.1) that

$$
J_0^{\alpha_1,-1}(\zeta) = 1, \quad J_n^{\alpha_1,-1}(\zeta) = \frac{n+\alpha_1}{n}\frac{\zeta+1}{2} J_{n-1}^{\alpha_1,1}(\zeta),\ n \geq 1, \quad \alpha_1 > -1, \tag{2.4}
$$

$$
J_0^{-1,\alpha_2}(\zeta) = 1, \quad J_n^{-1,\alpha_2}(\zeta) = \frac{n+\alpha_2}{n}\frac{\zeta-1}{2} J_{n-1}^{1,\alpha_2}(\zeta),\ n \geq 1, \quad \alpha_2 > -1. \tag{2.5}
$$

Meanwhile, an additional definition should be supplemented for $J_1^{-1,-1}$ before we obtain the complete system $\{J_n^{-1,-1}\}$, $n \in \mathbb{N}_0$,

$$
J_0^{-1,-1}(\zeta) = 1, \quad J_1^{-1,-1}(\zeta) = \zeta, \quad J_n^{-1,-1}(\zeta) = \frac{\zeta-1}{2}\frac{\zeta+1}{2} J_{n-2}^{1,1}(\zeta),\ n \geq 2. \tag{2.6}
$$

Such a supplementation preserves the symmetry properties of the classic Jacobi polynomials,

$$
J_n^{\alpha_1,\alpha_2}(\zeta) = (-1)^n J_n^{\alpha_2,\alpha_1}(-\zeta), \qquad n \in \mathbb{N}_0,\ \alpha_1,\alpha_2 \in [-1,\infty). \tag{2.7}
$$

For more about the supplementation of $J_n^{\alpha_1,\alpha_2}$ for $-n-\alpha_1-\alpha_2 \in \{1,2,\ldots,n\}$, please refer to [37].

Define the inner product in $H^1(I)$,

$$
\langle u,v\rangle_I := \underbrace{(\partial_\zeta u, \partial_\zeta v)_I}_{\text{interior}} + \underbrace{u(1)v(1)+u(-1)v(-1)}_{\text{endpoint}}.
$$

Then, a combination of (2.2) and (2.3) gives the orthogonality relation for $J_n^{-1,-1}$, $n \in \mathbb{N}_0$,

$$
\langle J_m^{-1,-1}, J_n^{-1,-1}\rangle_I = \Big[2\delta_{n,0} + 4\delta_{n,1} + \frac{(n-1)^2}{4n-2}(1-\delta_{n,0}-\delta_{n,1})\Big]\delta_{m,n}. \tag{2.8}
$$

Now, any function $u \in H^1(I)$ has a representation in the following Jacobi-Fourier series:

$$u(\zeta) = \sum_{m=0}^{\infty} \hat{u}_m J_m^{-1,-1}(\zeta), \quad \hat{u}_m = \frac{\langle u, J_m^{-1,-1}\rangle_I}{\langle J_m^{-1,-1}, J_m^{-1,-1}\rangle_I}.$$

Further, we define the truncated Jacobi-Fourier series of u:

$$\pi_M^{-1,-1} u(\zeta) = \sum_{m=0}^{M} \hat{u}_m J_m^{-1,-1}(\zeta). \tag{2.9}$$

Thanks to (2.8), $\pi_M^{-1,-1}$ is actually the H^1-orthogonal projection,

$$\langle \pi_M^{-1,-1} u - u, v\rangle_I = 0, \quad v \in \mathbb{P}_M(I),$$

which interpolates u at the endpoints of I, i.e., $\pi_M^{-1,-1} u(\pm 1) = u(\pm 1)$.

2.3. **Koornwinder polynomials.** The Koornwinder polynomials [33] $\mathcal{J}_l^\alpha(\hat{x}, \hat{y})$, $l \in \mathbb{N}_0^2$, for $\alpha \in [-1, \infty)^3$ on the reference triangle $\hat{T}$ can be defined through the generalized Jacobi polynomials,

$$\mathcal{J}_l^\alpha(\hat{x}, \hat{y}) = (\hat{y} + \hat{x})^{l_1} J_{l_1}^{\alpha_1,\alpha_2}\left(\frac{\hat{y} - \hat{x}}{\hat{y} + \hat{x}}\right) J_{l_2}^{2l_1+\alpha_1+\alpha_2+1,\alpha_3}(1 - 2\hat{x} - 2\hat{y}), \quad (\hat{x}, \hat{y}) \in \hat{T}. \tag{2.10}$$

The definition above further extends the classic/generalized Koornwinder polynomials for $l_1 \geq \chi(\alpha_1) + \chi(\alpha_2)$ and $l_2 \geq \chi(\alpha_3)$ in [36], which are mutually orthogonal with respect to the weight function $w^\alpha(\hat{x}, \hat{y}) = \hat{x}^{\alpha_1} \hat{y}^{\alpha_2} (1 - \hat{x} - \hat{y})^{\alpha_3}$,

$$\begin{aligned} &(\mathcal{J}_l^\alpha, \mathcal{J}_k^\alpha)_{w^\alpha, \hat{T}} = \gamma_l^\alpha \delta_{l,k}, \qquad l_1, k_1 \geq \chi(\alpha_1) + \chi(\alpha_2),\ l_2, k_2 \geq \chi(\alpha_3), \\ &\gamma_l^\alpha := \frac{1}{(2l_1 + \alpha_1 + \alpha_2 + 1)(2|l| + |\alpha| + 2)} \frac{\Gamma(l_1 + \alpha_1 + 1)\Gamma(l_1 + \alpha_2 + 1)}{\Gamma(l_1 + 1)\Gamma(l_1 + \alpha_1 + \alpha_2 + 1)} \\ &\qquad \times \frac{\Gamma(l_2 + \alpha_3 + 1)\Gamma(|l| + l_1 + \alpha_1 + \alpha_2 + 2)}{\Gamma(l_2 + 1)\Gamma(|l| + l_1 + |\alpha| + 2)}. \end{aligned} \tag{2.11}$$

Hereafter, we use the multi-index notations $|l| = l_1 + l_2$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and the convention that $\mathcal{J}_l^\alpha \equiv 0$ for $l_1 < 0$ or $l_2 < 0$. Furthermore, for $-l_1 - \alpha_1 - \alpha_2 \notin \{1, 2, \ldots, l_1\}$ and $-|l| - l_1 - |\alpha| - 1 \notin \{1, 2, \ldots, l_2\}$, the Koornwinder polynomials satisfy the following symmetry relation:

$$\mathcal{J}_l^\alpha(\hat{x}, \hat{y}) = (-1)^{l_1} \mathcal{J}_l^{\alpha_2,\alpha_1,\alpha_3}(\hat{y}, \hat{x}). \tag{2.12}$$

It is worth noting that the classic approximation theory reveals that the multivariate orthogonal polynomials can never be uniquely determined [23]. Let us define

$$\begin{aligned} \mathcal{K}_l^\alpha(\hat{x}, \hat{y}) &= \mathcal{J}_l^{\alpha_2,\alpha_3,\alpha_1}(\hat{y}, 1 - \hat{x} - \hat{y}), \quad l \in \mathbb{N}_0^2, \\ \mathcal{R}_l^\alpha(\hat{x}, \hat{y}) &= \mathcal{J}_l^{\alpha_3,\alpha_1,\alpha_2}(1 - \hat{x} - \hat{y}, \hat{x}), \quad l \in \mathbb{N}_0^2. \end{aligned} \tag{2.13}$$

Then $\mathcal{K}_l^\alpha$ and $\mathcal{R}_l^\alpha$ satisfy the orthogonality relation

$$(\mathcal{K}_l^\alpha, \mathcal{K}_k^\alpha)_{w^\alpha,\hat{T}} = \gamma_l^{\alpha_2,\alpha_3,\alpha_1}\delta_{l,k}, \qquad (\mathcal{R}_l^\alpha, \mathcal{R}_k^\alpha)_{w^\alpha,\hat{T}} = \gamma_l^{\alpha_3,\alpha_1,\alpha_2}\delta_{l,k}, \tag{2.14}$$

for $l_1, k_1 \geq \chi(\alpha_1) + \chi(\alpha_2),\ l_2, k_2 \geq \chi(\alpha_3)$.

The Koornwinder polynomials with $\alpha = (-1,-1,-1)$ are of particular interest. To avoid complicated piecewise expressions for their differentiation relations, we shall adopt the following normalized representation:

$$\begin{cases} \tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y}) = -\frac{2(2l_1-1)(2|l|-1)}{(l_1-1)l_2}\hat{x}\hat{y}(1-\hat{x}-\hat{y})\mathcal{J}^{1,1,1}_{l_1-2,l_2-1}(x,y) \\ \qquad\qquad = \frac{2(2l_1-1)(2|l|-1)}{(l_1-1)(|l|+l_1-1)}\mathcal{J}^{-1,-1,-1}_{l_1,l_2}(\hat{x},\hat{y}), \qquad l_1 \geq 2,\ l_2 \geq 1, \\ \tilde{\mathcal{J}}_{l_1,0}(\hat{x},\hat{y}) = -\frac{2(2l_1-1)}{l_1-1}\hat{x}\hat{y}\mathcal{J}^{1,1,0}_{l_1-2,0}(\hat{x},\hat{y}) \\ \qquad\qquad = \frac{2(2l_1-1)}{l_1-1}\mathcal{J}^{-1,-1,-1}_{l_1,0}(\hat{x},\hat{y}), \qquad l_1 \geq 2, \\ \tilde{\mathcal{J}}_{0,l_2}(\hat{x},\hat{y}) = \frac{2l_2-1}{l_2-1}(\hat{x}+\hat{y})(\hat{x}+\hat{y}-1)\mathcal{J}^{0,0,1}_{0,l_2-2}(\hat{x},\hat{y}) \\ \qquad\qquad = \frac{2l_2-1}{l_2-1}\mathcal{J}^{-1,-1,-1}_{0,l_2}(\hat{x},\hat{y}), \qquad l_2 \geq 2, \\ \tilde{\mathcal{J}}_{1,l_2}(\hat{x},\hat{y}) = \frac{2l_2+1}{l_2}(\hat{x}-\hat{y})(\hat{x}+\hat{y}-1)\mathcal{J}^{0,0,1}_{0,l_2-1}(\hat{x},\hat{y}) \\ \qquad\qquad = \frac{2l_2+1}{l_2+1}\mathcal{J}^{-1,-1,-1}_{1,l_2}(\hat{x},\hat{y}), \qquad l_2 \geq 1, \\ \tilde{\mathcal{J}}_{1,0}(\hat{x},\hat{y}) = \hat{y}-\hat{x}, \quad \tilde{\mathcal{J}}_{0,1}(\hat{x},\hat{y}) = -\hat{y}-\hat{x}, \quad \tilde{\mathcal{J}}_{0,0}(\hat{x},\hat{y}) = 1. \end{cases} \tag{2.15}$$

Correspondingly, we define

$$\tilde{\mathcal{K}}_l(\hat{x},\hat{y}) = \tilde{\mathcal{J}}_l(\hat{y}, 1-\hat{x}-\hat{y}), \quad \tilde{\mathcal{R}}_l(\hat{x},\hat{y}) = \tilde{\mathcal{J}}_l(1-\hat{x}-\hat{y},\hat{x}), \quad l \in \mathbb{N}_0^2. \tag{2.16}$$

Remark 2.1. Although generalized Koornwinder polynomials were originally introduced in [36], similar shape functions using integrated Jacobi polynomials were proposed earlier by Beuchler and Schöberlat in [11] and further studied in their subsequent works [9, 10],

$$\tilde{\phi}_l(\tilde{x},\tilde{y}) = \hat{p}^0_{l_1}\left(\frac{2\tilde{x}}{1-\tilde{y}}\right)\left(\frac{1-\tilde{y}}{2}\right)^{l_1}\hat{p}^{2l_1-1}_{l_2}(\tilde{y}), \qquad |\tilde{x}| < \frac{1-\tilde{y}}{2} < 1,\ l \in \mathbb{N}_0^2,$$

where $\hat{p}^s_i(\zeta) = \int_{-1}^{\zeta} J^{s,0}_{i-1}(\eta)d\eta$ for $i \geq 1$ and $\hat{p}^s_0(\zeta) = 1$. Indeed, one readily finds that for $s > -1$,

$$\begin{aligned} &\hat{p}^s_0(\zeta) = J^{s-1,-1}_0(\zeta), \qquad \hat{p}^0_1(\zeta) = J^{-1,-1}_1(\zeta) + J^{-1,-1}_0(\zeta), \\ &\hat{p}^s_i(\zeta) = \frac{2J^{s-1,-1}_i(\zeta)}{i+s-1} = \frac{2}{2i+s-1}\left[J^{s,-1}_i(\zeta) - \frac{i-1}{i+s-1}J^{s,-1}_{i-1}(\zeta)\right], \\ &\qquad\qquad i \geq 1, i+s-1 \neq 0. \end{aligned}$$

This reveals that, for $l \in \mathbb{N}_0^2$ with $l_1 \neq 1$, there exist two constants $c_l^{(0)}$ and $c_l^{(1)}$ such that

$$\tilde{\phi}_l(\hat{y}-\hat{x}, 1-2\hat{x}-2\hat{y}) = c_l^{(0)}\tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y}) + c_l^{(1)}\tilde{\mathcal{J}}_{l_1,l_2-1}(\hat{x},\hat{y}).$$

We now list some differentiation relations of the Koornwinder polynomials $\tilde{\mathcal{J}}_l$, which are fundamental to our numerical analysis in subsequent sections.

Lemma 2.1. *For $l \in \mathbb{N}_0^2$, it holds that*

$$\partial_{\hat{z}} \tilde{\mathcal{J}}_l(\hat{x},\hat{y}) = \frac{2(2l_1-1)(2|l|-1)}{|l|+l_1-1} \mathcal{J}^{0,0,-1}_{l_1-1,l_2}(\hat{x},\hat{y}), \tag{2.17}$$

$$\begin{aligned} \partial_{\hat{x}} \tilde{\mathcal{J}}_l(\hat{x},\hat{y}) = -(2|l|-1)\Big[&(2-\delta_{l_1,0})\mathcal{J}^{0,-1,0}_{l_1,l_2-1}(\hat{x},\hat{y}) \\ &+ (2-\delta_{l_1,1})\mathcal{J}^{0,-1,0}_{l_1-1,l_2}(\hat{x},\hat{y})\Big], \end{aligned} \tag{2.18}$$

$$\begin{aligned} \partial_{y} \tilde{\mathcal{J}}_l(\hat{x},\hat{y}) = -(2|l|-1)\Big[&(2-\delta_{l_1,0})\mathcal{J}^{-1,0,0}_{l_1,l_2-1}(\hat{x},\hat{y}) \\ &- (2-\delta_{l_1,1})\mathcal{J}^{-1,0,0}_{l_1-1,l_2}(\hat{x},\hat{y})\Big], \end{aligned} \tag{2.19}$$

$$\partial_{\hat{x}}\partial_{\hat{z}} \tilde{\mathcal{J}}_l(x,y) = -2(2|l|-1)\left[l_1 \mathcal{J}^{1,0,0}_{l_1-1,l_2-1}(\hat{x},\hat{y}) + (l_1-1)\mathcal{J}^{1,0,0}_{l_1-2,l_2}(\hat{x},\hat{y})\right], \tag{2.20}$$

$$\partial_{\hat{y}}\partial_{\hat{z}} \tilde{\mathcal{J}}_l(\hat{x},\hat{y}) = -2(2|l|-1)\left[l_1 \mathcal{J}^{0,1,0}_{l_1-1,l_2-1}(\hat{x},\hat{y}) - (l_1-1)\mathcal{J}^{0,1,0}_{l_1-2,l_2}(\hat{x},\hat{y})\right], \tag{2.21}$$

$$\begin{aligned} \partial_{\hat{x}}\partial_{\hat{y}} \tilde{\mathcal{J}}_l(\hat{x},\hat{y}) = (2|l|-1)\Big[&(|l|+l_1)\mathcal{J}^{0,0,1}_{l_1,l_2-2}(\hat{x},\hat{y}) \\ &- (|l|+l_1-2)\mathcal{J}^{0,0,1}_{l_1-2,l_2}(\hat{x},\hat{y})\Big]. \end{aligned} \tag{2.22}$$

Lemma 2.1 can either be excerpted or be deduced from the appendix in [36] for $l_1 \geq 2, l_2 \geq 1$, and then be readily derived for other cases by continuation. We omit the detailed proof here.

Let $\Gamma_1 = \{(0,\hat{y}) : 0 < \hat{y} < 1\}$, $\Gamma_2 = \{(\hat{x},0) : 0 < \hat{x} < 1\}$ and $\Gamma_3 = \{(\hat{x}, 1-\hat{x}) : 0 < \hat{x} < 1\}$ be the sides of $\hat{T}$. Define the inner product in $H^2(\hat{T})$ as

$$\begin{aligned} \langle u, v\rangle_{\hat{T}} =& \underbrace{(\partial_{\hat{x}}\partial_{\hat{z}} u, \partial_{\hat{x}}\partial_{\hat{z}} v)_{w^{1,0,0},\hat{T}} + (\partial_{\hat{y}}\partial_{\hat{z}} u, \partial_{\hat{y}}\partial_{\hat{z}} v)_{w^{0,1,0},\hat{T}}}_{\text{interior}} \\ &+ \underbrace{(\partial_{\hat{y}} u, \partial_{\hat{y}} v)_{\Gamma_1} + (\partial_{\hat{x}} u, \partial_{\hat{x}} v)_{\Gamma_2}}_{\text{edge}} + \underbrace{u(0,0)v(0,0)}_{\text{vertex}}. \end{aligned} \tag{2.23}$$

Then we have the following lemma.

Lemma 2.2. *$\tilde{\mathcal{J}}_l$, $l \in \mathbb{N}_0^2$, are mutually orthogonal polynomials with respect to $\langle \cdot,\cdot\rangle_{\hat{T}}$. More precisely,*

$$\begin{aligned} &\langle \tilde{\mathcal{J}}_l, \tilde{\mathcal{J}}_k\rangle_{\hat{T}} = \gamma_l \delta_{l,k}, \quad l,k \in \mathbb{N}_0^2, \\ &\gamma_l := 4(2|l|-1)[(2-\delta_{l_2,0})l_1 - (1-\delta_{l_1,0})] \\ &\qquad + 2(2|l|-1)(\delta_{l_1,0} + \delta_{l_1,1} - \delta_{l,0}) + \delta_{l,0}. \end{aligned} \tag{2.24}$$

The proof of Lemma 2.2 is postponed to Appendix A.

3. Triangular spectral element scheme and its implementation

3.1. **Variational problem.** Let $\boldsymbol{V} = H_0^1(\Omega)^2$ and $W = L_0^2(\Omega)$. The Stokes eigenvalue problem (1.1) can be written in a variational form as follows: to find $\lambda \in \mathbb{R}$ and $(\boldsymbol{u}, p) \in \boldsymbol{V} \times W$ with $\boldsymbol{u} \neq \boldsymbol{0}$ such that

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \lambda(\boldsymbol{u},\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ b(\boldsymbol{u},q) = 0 & \forall q \in W, \end{cases} \tag{3.1}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(\boldsymbol{u},\boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \qquad b(\boldsymbol{u},q) = -(\operatorname{div} \boldsymbol{u}, q).$$

It is easy to see that both $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms:

$$|a(\boldsymbol{u},\boldsymbol{v})| \le \|\boldsymbol{u}\|_1 \, \|\boldsymbol{v}\|_1, \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}, \tag{3.2}$$

$$|b(\boldsymbol{u},p)| \le \|\boldsymbol{u}\|_1 \, \|p\|, \quad \boldsymbol{u} \in \boldsymbol{V},\, p \in W. \tag{3.3}$$

Furthermore, $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}$, i.e., there exists a constant $\eta > 0$ such that

$$a(\boldsymbol{u},\boldsymbol{u}) \ge \eta \|\boldsymbol{u}\|_1^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{V}, \tag{3.4}$$

and $b(\cdot,\cdot)$ satisfies the following inf-sup condition on $\boldsymbol{V} \times W$ [16, 48],

$$\beta = \beta(\Omega) := \inf_{0 \ne q \in W} \sup_{\boldsymbol{0} \ne \boldsymbol{v} \in \boldsymbol{V}} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_1 \|q\|} > 0. \tag{3.5}$$

Define the kernel space of the divergence operator

$$\boldsymbol{X} = \{\boldsymbol{v} \in \boldsymbol{V} : b(\boldsymbol{v},q) = 0,\ \forall q \in W\}. \tag{3.6}$$

Then the variational problem (3.1) is equivalent to the following one: to find $(\lambda, \boldsymbol{u}) \in \mathbb{R} \times \boldsymbol{X} \setminus \{\boldsymbol{0}\}$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}. \tag{3.7}$$

3.2. **Approximation scheme.** Let $\mathcal{T} = \{T_i\}$ be a triangular partition of Ω. We assume that the partition has the following properties:

- Each element T_i is *spectrally admissible* in the sense that there is a bijective mapping F_i of class C^∞, which maps $\overline{\hat{T}}$ onto $\overline{T_i}$. We shall only consider in the current paper the cases when every T_i has only straight sides such that each F_i is an affine mapping. The cases with curvilinear edges are beyond our scope of discussion here.
- $\mathcal{T}$ is *regular* in the sense that the intersection $\overline{T}_i \cap \overline{T}_j,\ i \ne j$, is either empty or a node or an entire edge of both T_i and T_j.
- $\mathcal{T}$ is *shape regular* which means that there exists a constant κ, independent of each $T \in \mathcal{T}$, such that

$$\sup_{T \in \mathcal{T}} \frac{h_T}{\rho_T} \le \kappa < \infty, \tag{3.8}$$

  where $h_T = \operatorname{diam}(T)$ and ρ_T denotes the diameter of the inner circle of element T.

Denote by k_m the polynomial degree for the velocity on the element $T_m \in \mathcal{T}$. Let $\delta = \delta(h, M)$ be the discretization parameter with $h = \max_{T_m \in \mathcal{T}} h_{T_m}$ and $M = \min_{T_m \in \mathcal{T}} k_m$ which tends to zero when h tends to zero and/or M tends to infinity. Generally, $\delta = O(h/M)$. We now define the $\mathbb{P}_M - \mathbb{P}_{M-2}$ approximation spaces $\boldsymbol{V}_\delta \subset \boldsymbol{V}$ and $W_\delta \subset W$ as follows:

$$\boldsymbol{V}_\delta = \{\boldsymbol{v} \in H_0^1(\Omega)^2 :\ \boldsymbol{v}_{|T_m} \circ F_m \in \mathbb{P}_{k_m}(\hat{T})^2,\ T_m \in \mathcal{T}\}, \tag{3.9}$$

$$W_\delta = \{p \in L_0^2(\Omega) :\ p_{|T_m} \circ F_m \in \mathbb{P}_{k_m-2}(\hat{T}),\ T_m \in \mathcal{T}\}. \tag{3.10}$$

For simplicity, we only consider in this paper the case where $k_m = M$ for all $T_m \in \mathcal{T}$.

Now we propose the triangular spectral element approximation scheme for the Stokes eigenvalue problem of (3.1): to find $\lambda_\delta \in \mathbb{R}$ and $(\boldsymbol{u}_\delta, p_\delta) \in \boldsymbol{V}_\delta \times W_\delta$ with $\boldsymbol{u}_\delta \ne \boldsymbol{0}$, such that

$$\begin{cases} a(\boldsymbol{u}_\delta,\boldsymbol{v}) + b(\boldsymbol{v},p_\delta) = \lambda_\delta(\boldsymbol{u}_\delta,\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}_\delta, \\ b(\boldsymbol{u}_\delta,q) = 0 & \forall q \in W_\delta. \end{cases} \tag{3.11}$$

It is known that $b(\cdot,\cdot)$ also satisfies the discrete inf-sup condition [8, 19, 22, 48, 49],

$$\beta_\delta = \inf_{0\neq q\in W_\delta} \sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_\delta} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_1\,\|q\|} > 0. \tag{3.12}$$

Define

$$\boldsymbol{X}_\delta = \{\boldsymbol{v}\in \boldsymbol{V}_\delta : b(\boldsymbol{v},q)=0,\ \forall q\in W_\delta\}. \tag{3.13}$$

Then (3.11) can be rewritten as an equivalent form: to find $(\lambda_\delta, \boldsymbol{u}_\delta)\in \mathbb{R}\times \boldsymbol{X}_\delta\setminus\{\mathbf{0}\}$ such that

$$a(\boldsymbol{u}_\delta,\boldsymbol{v}) = \lambda_\delta(\boldsymbol{u}_\delta,\boldsymbol{v}) \quad \forall \boldsymbol{v}\in \boldsymbol{X}_\delta. \tag{3.14}$$

3.3. **Implementation.** A brief description of the implementation of (3.11) is introduced. We first find a group of (semi)orthogonal basis functions which stem from the Koornwinder polynomials $\tilde{\mathcal{J}}_l$, which are divided into vertex modes, edge modes and interior modes. The vertex mode only has a magnitude at one vertex and is zero at other vertices; the edge mode has magnitudes on one edge and is enforced as zero on other edges and vertices; while the interior mode is identically zero on all edges and vertices.

- Vertex modes:

$$\begin{aligned}
(0,0):&\quad \phi_{0,0}(\hat{x},\hat{y}) = 1-\hat{x}-\hat{y},\\
(1,0):&\quad \phi_{1,0}(\hat{x},\hat{y}) = \hat{x},\\
(0,1):&\quad \phi_{0,1}(\hat{x},\hat{y}) = \hat{y}.
\end{aligned}$$

- Edge modes:

$$\begin{aligned}
\hat{x}+\hat{y}=1:&\quad \phi_{l_1,0}(\hat{x},\hat{y}) = \tilde{\mathcal{J}}_{l_1,0}(\hat{x},\hat{y}), \quad l_1\geq 2,\\
\hat{y}=0:&\quad \phi_{0,l_2}(\hat{x},\hat{y}) = \tilde{\mathcal{J}}_{0,l_2}(\hat{x},\hat{y}) + \tilde{\mathcal{J}}_{1,l_2-1}(\hat{x},\hat{y}), \quad l_2\geq 2,\\
\hat{x}=0:&\quad \phi_{1,l_2-1}(\hat{x},\hat{y}) = \tilde{\mathcal{J}}_{0,l_2}(\hat{x},\hat{y}) - \tilde{\mathcal{J}}_{1,l_2-1}(\hat{x},\hat{y}), \quad l_2\geq 2.
\end{aligned}$$

- Interior modes:

$$\phi_{l_1,l_2}(\hat{x},\hat{y}) = \tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y}),\ l_1\geq 2,\ l_2\geq 1.$$

We list the function expansion and the derivative expansions of $\tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y})$ which can either be excerpted from the appendix of [36] for $l_1\geq 2, l_2\geq 1$ or be readily derived for other cases by continuation.

Lemma 3.1. *For $|l|\geq 2$, it holds that*

$$\begin{aligned}
\tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y}) =& \tfrac{(|l|+l_1)(|l|+l_1+1)}{2|l|(2|l|+1)}\mathcal{J}^{0,0,0}_{l_1,l_2}(\hat{x},\hat{y}) - \tfrac{(l_2+1)(l_2+2)}{2|l|(2|l|+1)}\mathcal{J}^{0,0,0}_{l_1-2,l_2+2}(\hat{x},\hat{y})\\
&- \tfrac{(|l|+l_1)(|l|-3l_1-1)}{(2|l|+1)(2|l|-2)}\mathcal{J}^{0,0,0}_{l_1,l_2-1}(\hat{x},\hat{y}) + \tfrac{(|l|+3l_1-4)(l_2+1)}{(2|l|+1)(2|l|-2)}\mathcal{J}^{0,0,0}_{l_1-2,l_2+1}(\hat{x},\hat{y})\\
&- \tfrac{(l_2-1)(|l|+3l_1)}{2|l|(2|l|-3)}\mathcal{J}^{0,0,0}_{l_1,l_2-2}(\hat{x},\hat{y}) + \tfrac{(|l|+l_1-2)(|l|-3l_1+3)}{2|l|(2|l|-3)}\mathcal{J}^{0,0,0}_{l_1-2,l_2}(\hat{x},\hat{y})\\
&+ \tfrac{(l_2-1)(l_2-2)}{(2|l|-3)(2|l|-2)}\mathcal{J}^{0,0,0}_{l_1,l_2-3}(\hat{x},\hat{y}) - \tfrac{(|l|+l_1-3)(|l|+l_1-2)}{(2|l|-3)(2|l|-2)}\mathcal{J}^{0,0,0}_{l_1-2,l_2-1}(\hat{x},\hat{y}),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\partial_{\hat{x}}\tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y}) =& -(|l|+l_1)\mathcal{J}^{0,0,0}_{l_1,l_2-1} - (2l_1-1)\mathcal{J}^{0,0,0}_{l_1-1,l_2} + (l_2+1)\mathcal{J}^{0,0,0}_{l_1-2,l_2+1}\\
&+ (l_2-1)\mathcal{J}^{0,0,0}_{l_1,l_2-2} - (2l_1-1)\mathcal{J}^{0,0,0}_{l_1-1,l_2-1} - (|l|+l_1-2)\mathcal{J}^{0,0,0}_{l_1-2,l_2},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\partial_{\hat{y}}\tilde{\mathcal{J}}_{l_1,l_2}(\hat{x},\hat{y}) =& -(|l|+l_1)\mathcal{J}^{0,0,0}_{l_1,l_2-1} + (2l_1-1)\mathcal{J}^{0,0,0}_{l_1-1,l_2} + (l_2+1)\mathcal{J}^{0,0,0}_{l_1-2,l_2+1}\\
&+ (l_2-1)\mathcal{J}^{0,0,0}_{l_1,l_2-2} + (2l_1-1)\mathcal{J}^{0,0,0}_{l_1-1,l_2-1} - (|l|+l_1-2)\mathcal{J}^{0,0,0}_{l_1-2,l_2}.
\end{aligned} \tag{3.17}$$

Let Γ be a part of the boundary of $\hat{T}$, which is either the empty set or constituted by one, two or three sides of $\hat{T}$. Define

$$\mathbb{P}_k^{\Gamma}(\hat{T}) = \left\{ v \in \mathbb{P}_k(\hat{T}) : v|_{\Gamma} = 0 \right\} \tag{3.18}$$

and

$$\begin{aligned} V_M(T_m) &= \left\{ v \circ F_m^{-1} : v \in \mathbb{P}_M^{\Gamma}(\hat{T}),\ \Gamma = F_m^{-1}(\partial\Omega \cap T_m) \right\} \\ &= \left\{ \phi_l \circ F_m^{-1} : l \in \Lambda_m \right\}, \end{aligned} \tag{3.19}$$

where the index set $\Lambda_m = \{l \in \mathbb{N}_0^2 : \phi_l \circ F_m^{-1} \in V_M(T_m)\}$. In such a way, the approximation space $\boldsymbol{V}_\delta$ for velocities can be written as

$$\boldsymbol{V}_\delta = V_\delta^2, \qquad V_\delta = \left\{ v \in H^1(\Omega) : v|_{T_m} \in V_M(T_m) \text{ for all } T_m \in \mathcal{T} \right\}. \tag{3.20}$$

We note that the constraint $p \in L_0^2(\Omega)$ is enforced to ensure the uniqueness of p and to avoid the spurious eigenvalue 0. In a practical computation, we can remove this zero mean constraint, and then simply set

$$\begin{aligned} W_\delta &= \{ v : v|_{T_m} \in W_M(T_m) \text{ for all } T_m \in \mathcal{T} \}, \\ W_M(T_m) &= \left\{ \mathcal{J}_k^{0,0,0} \circ F_m^{-1} : |k| \le M-2 \right\}. \end{aligned} \tag{3.21}$$

Then the local stiffness and mass matrices on element T_m can be expressed as

$$\begin{aligned} B^{(m)} &= \left[(\phi_k \circ F_m^{-1}, \phi_l \circ F_m^{-1})_{T_m} \right]_{l,k}, \\ A^{(m)} &= \left[(\nabla(\phi_k \circ F_m^{-1}), \nabla(\phi_l \circ F_m^{-1}))_{T_m} \right]_{l,k}, \\ E_1^{(m)} &= -\left[(\partial_x(\phi_l \circ F_m^{-1}), \mathcal{J}_k^{0,0,0} \circ F_m^{-1})_{T_m} \right]_{l,k}, \\ E_2^{(m)} &= -\left[(\partial_y(\phi_l \circ F_m^{-1}), \mathcal{J}_k^{0,0,0} \circ F_m^{-1})_{T_m} \right]_{l,k}, \end{aligned}$$

where all the matrix entries can be analytically evaluated through Lemma 3.1 without any utilization of quadrature rules.

Finally, assembling the local matrices, we get the global stiffness and mass matrices A, B, E_1, E_2, and then the algebraic eigenvalue system can be expressed by

$$\begin{pmatrix} A & 0 & E_1 \\ 0 & A & E_2 \\ E_1^{\mathsf{T}} & E_2^{\mathsf{T}} & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ P \end{pmatrix} = \lambda_\delta \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ P \end{pmatrix}. \tag{3.22}$$

Note that the matrices on both sides of the algebraic eigenequation (3.22) are singular. To avoid this situation, we just remove one column and row arising from the basis functions $\mathcal{J}_{0,0}^{0,0,0} \circ F_m^{-1}$ on an arbitrary element $T_m \in \mathcal{T}$. The reduced eigen-system (3.22) now has a positive definite matrix on its left side, thus can be efficiently solved by algebraic eigenvalue packages such as ARPACK/PARPACK using the implicitly restarted Arnoldi methods [35].

4. The discrete inf-sup constant of divergence

In this section, we shall prove $\beta_\delta \gtrsim M^{-1/2}$. We shall also show that this lower bound estimate is sharp in the sense that $\beta_\delta \asymp M^{-1/2}$ whenever Ω is a triangle and $\mathcal{T}$ is constituted of the only one element $T = \Omega$. Such an estimate is comparable to and thus compatible with that of the quadrilateral spectral elements. The key to our estimate lies in the truncated Koornwinder-Fourier series, which actually serves as a Fortin interpolation on the reference triangle.

4.1. **Truncated Koornwinder-Fourier series.** For $\alpha \in (-1,\infty)^3$ and $M \in \mathbb{N}_0$, define the weighted L^2-orthogonal projection $\pi_M^\alpha : L^2_{w^\alpha}(\hat{T}) \mapsto \mathbb{P}_M(\hat{T})$ such that

$$(\pi_M^\alpha u - u, v)_{w^\alpha,\hat{T}} = 0 \quad \forall v \in \mathbb{P}_M(\hat{T}). \tag{4.1}$$

Owing to the orthogonality of $\{\mathcal{J}_l^\alpha\}_{l\in\mathbb{N}_0^2}$, $\{\mathcal{K}_l^\alpha\}_{l\in\mathbb{N}_0^2}$ and $\{\mathcal{R}_l^\alpha\}_{l\in\mathbb{N}_0^2}$, $\pi_M^\alpha u$ has explicit representations in truncated Koornwinder-Fourier series,

$$\pi_M^\alpha u = \sum_{|l|\le M} \frac{(u,\mathcal{J}_l^\alpha)_{w^\alpha,\hat{T}}}{\gamma_l^\alpha}\mathcal{J}_l^\alpha = \sum_{|l|\le M} \frac{(u,\mathcal{K}_l^\alpha)_{w^\alpha,\hat{T}}}{\gamma_l^{\alpha_2,\alpha_3,\alpha_1}}\mathcal{K}_l^\alpha = \sum_{|l|\le M} \frac{(u,\mathcal{R}_l^\alpha)_{w^\alpha,\hat{T}}}{\gamma_l^{\alpha_3,\alpha_1,\alpha_2}}\mathcal{R}_l^\alpha,$$

which indicates that $\pi_M^\alpha u$ is independent of a particular system of orthogonal polynomials [23]. Hereafter, we associate any function u defined on $\hat{T}$ with

$$u^{\mathcal{K}}(\hat{x},\hat{y}) = u(\hat{y}, 1-\hat{x}-\hat{y}) \quad \text{and} \quad u^{\mathcal{R}}(\hat{x},\hat{y}) = u(1-\hat{x}-\hat{y},\hat{x}). \tag{4.2}$$

Then one finds from (2.13) that

$$\begin{aligned} \pi_M^\alpha u(\hat{x},\hat{y}) &= \big(\pi_M^{\alpha_3,\alpha_1,\alpha_2} u^{\mathcal{K}}\big)(1-\hat{x}-\hat{y},\hat{x}) \\ &= \big(\pi_M^{\alpha_2,\alpha_3,\alpha_1} u^{\mathcal{R}}\big)(\hat{y},1-\hat{x}-\hat{y}), \quad (\hat{x},\hat{y}) \in \hat{T}. \end{aligned} \tag{4.3}$$

According to Theorem 3.3 in [36], we have the following error estimate on π_M^α for $\alpha_1,\alpha_2,\alpha_3 > -1$.

Lemma 4.1. *Suppose $u \in H^s(\hat{T})$ with $s \ge 0$ and $\alpha_1,\alpha_2,\alpha_3 > -1$, then*

$$\begin{aligned} \|\pi_M^\alpha u - u\|_{w^\alpha,\hat{T}} &\lesssim M^{-s} \sum_{\substack{|k|=s\\ k\in\mathbb{N}_0^3}} \big\|\partial_{\hat{x}}^{k_1}\partial_{\hat{y}}^{k_2}\partial_{\hat{z}}^{k_3} u\big\|_{w^{\alpha_1+k_1+k_3,\alpha_2+k_2+k_3,\alpha_3+k_1+k_2},\hat{T}} \\ &\lesssim M^{-s}|u|_{s,w^\alpha,\hat{T}}. \end{aligned} \tag{4.4}$$

We now turn to the Koornwinder-Fourier series for $\alpha = (-1,-1,-1)$. Define the orthogonal projection $\pi_M^{-1,-1,-1} : H^2(\hat{T}) \mapsto \mathbb{P}_M(\hat{T})$ such that

$$\langle u - \pi_M^{-1,-1,-1} u, v\rangle_{\hat{T}} = 0 \qquad \forall v \in \mathbb{P}_M(\hat{T}). \tag{4.5}$$

The following lemma states that $\pi_M^{-1,-1,-1}$ is independent of any particular system of orthogonal polynomials, thus is rotational invariant under the barycentric coordinates $(\hat{x},\hat{y},1-\hat{x}-\hat{y})$.

Lemma 4.2. *For $M \ge 1$ and $u \in H^2(\hat{T})$, it holds that*

$$\begin{aligned} \pi_M^{-1,-1,-1} u(\hat{x},\hat{y}) &= \big(\pi_M^{-1,-1,-1} u^{\mathcal{R}}\big)^{\mathcal{K}}(\hat{x},\hat{y}) = \big(\pi_M^{-1,-1,-1} u^{\mathcal{K}}\big)^{\mathcal{R}}(\hat{x},\hat{y}) \\ &= \sum_{|l|\le M} \frac{\langle u,\tilde{\mathcal{J}}_l\rangle_{\hat{T}}}{\gamma_l}\tilde{\mathcal{J}}_l(\hat{x},\hat{y}) = \sum_{|l|\le M} \frac{\langle u^{\mathcal{R}},\tilde{\mathcal{J}}_l\rangle_{\hat{T}}}{\gamma_l}\tilde{\mathcal{K}}_l(\hat{x},\hat{y}) \\ &= \sum_{|l|\le M} \frac{\langle u^{\mathcal{K}},\tilde{\mathcal{J}}_l\rangle_{\hat{T}}}{\gamma_l}\tilde{\mathcal{R}}_l(\hat{x},\hat{y}). \end{aligned} \tag{4.6}$$

Lemma 4.2 is an immediate consequence of the general theory in [23, §3.5]. A proof will also be given in Appendix B.

Special attention to the boundary trace of polynomial projections [21, 43] has paid in recent years in spectral element methods and higher order finite element methods. The following lemma states that $\pi_M^{-1,-1,-1}$ interpolates u at the vertices

of $\hat{T}$, and its trace on each side Γ_i of $\hat{T}$ serves as the 1-dimensional orthogonal projection $\pi_{M,\Gamma_i}^{-1,-1}$.

Lemma 4.3. *Denote by G_i the linear mapping from Γ_i onto $I = (-1,1)$. Then for any $u \in H^2(\hat{T})$ and $M \geq 1$,*

$$(\pi_M^{-1,-1,-1}u)\big|_{\Gamma_i} = \pi_{M,\Gamma_i}^{-1,-1}(u|_{\Gamma_i}) := \big[\pi_M^{-1,-1}(u|_{\Gamma_i} \circ G_i^{-1})\big] \circ G_i, \quad i = 1,2,3, \tag{4.7}$$

where $\pi_M^{-1,-1}$ is defined in (2.9). *In particular, $(\pi_M^{-1,-1,-1}u)\big|_{\Gamma_i} = 0$ if $u|_{\Gamma_i} = 0$, and $u - \pi_M^{-1,-1,-1}u$ vanishes at all the vertices of $\hat{T}$.*

Proof. Since $u \in H^2(\hat{T})$, both u and $\pi_M^{-1,-1,-1}u$ can be expanded into the Koornwinder-Fourier series,

$$u = \sum_{l \in \mathbb{N}_0^2} \hat{u}_l \tilde{\mathcal{J}}_l, \quad \pi_M^{-1,-1,-1}u = \sum_{|l| \leq M} \hat{u}_l \tilde{\mathcal{J}}_l, \qquad \hat{u}_l = \frac{1}{\gamma_l}\langle u, \tilde{\mathcal{J}}_l\rangle_{\hat{T}}.$$

On the edge Γ_1, by (2.15),

$$\begin{aligned} u|_{\Gamma_1} = u(0,\hat{y}) &= \hat{u}_{0,0} + (\hat{u}_{1,0} - \hat{u}_{0,1})\hat{y} \\ &\quad + \sum_{l_2 \geq 2} \frac{2l_2 - 1}{l_2 - 1}(\hat{u}_{1,l_2-1} - \hat{u}_{0,l_2})\hat{y}(1-\hat{y})J_{l_2-2}^{1,1}(1-2\hat{y}), \\ \pi_M^{-1,-1,-1}u|_{\Gamma_1} &= \hat{u}_{0,0} + (\hat{u}_{1,0} - \hat{u}_{0,1})\hat{y} \\ &\quad + \sum_{2 \leq l_2 \leq M} \frac{2l_2 - 1}{l_2 - 1}(\hat{u}_{1,l_2-1} - \hat{u}_{0,l_2})\hat{y}(1-\hat{y})J_{l_2-2}^{1,1}(1-2\hat{y}). \end{aligned}$$

Therefore

$$\big[\pi_M^{-1,-1}(u|_{\Gamma_1} \circ G_1^{-1})\big] \circ G_1 = (\pi_M^{-1,-1,-1}u)|_{\Gamma_1},$$

which gives (4.7) for $i = 1$. In analogy, one also proves (4.7) for $i = 2,3$.

Both the vanishing of $u - \pi_M^{-1,-1,-1}u$ at three vertices and the vanishing of $(\pi_M^{-1,-1,-1}u)|_{\Gamma_i}$ for a vanishing $u|_{\Gamma_i}$ are immediate consequences of (4.7). □

Theorem 4.1. *For any $u \in H^2(\hat{T}) \cap H_0^1(\hat{T})$, it holds that*

$$\big|\pi_M^{-1,-1,-1}u\big|_{1,\hat{T}} \leq \sqrt{\frac{M+1}{2}}|u|_{1,\hat{T}}. \tag{4.8}$$

Proof. In view of Lemma 4.2, we first expand $u \in H^2(\hat{T})$ and $\pi_M^{-1,-1,-1}u$ into the Koornwinder-Fourier series in $\tilde{\mathcal{J}}_l$,

$$u = \sum_{l \in \mathbb{N}_0^2} \hat{u}_l \tilde{\mathcal{J}}_l, \quad \pi_M^{-1,-1,-1}u = \sum_{|l| \leq M} \hat{u}_l \tilde{\mathcal{J}}_l, \qquad \hat{u}_l = \frac{1}{\gamma_l}\langle u, \tilde{\mathcal{J}}_l\rangle_{\hat{T}}.$$

According to (3.16) and (3.17), it holds that

$$\begin{aligned}\partial_{\hat z} u &= \sum_{l\in\mathbb{N}_0^2} \hat u_l \partial_{\hat z}\tilde{\mathcal{J}}_l = \sum_{l\in\mathbb{N}_0^2}(4l_1-2)\hat u_l[\mathcal{J}^{0,0,0}_{l_1-1,l_2}+\mathcal{J}^{0,0,0}_{l_1-1,l_2-1}]\\ &= \sum_{l\in\mathbb{N}_0^2}(4l_1+2)(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})\mathcal{J}^{0,0,0}_{l_1,l_2},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\partial_{\hat z}\pi_M^{-1,-1,-1}u &= \sum_{|l|\le M}\hat u_l\partial_{\hat z}\tilde{\mathcal{J}}_l = \sum_{|l|\le M}(4l_1-2)\hat u_l[\mathcal{J}^{0,0,0}_{l_1-1,l_2}+\mathcal{J}^{0,0,0}_{l_1-1,l_2-1}]\\ &= \sum_{|l|\le M-2}(4l_1+2)\big[\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1}\big]\mathcal{J}^{0,0,0}_{l_1,l_2}\\ &\quad+\sum_{|l|=M-1}(4l_1+2)\hat u_{l_1+1,l_2}\mathcal{J}^{0,0,0}_{l_1,l_2}.\end{aligned} \tag{4.10}$$

Using the orthogonality relation (2.11) with $\alpha=(0,0,0)$, we further get that

$$\big\|\partial_{\hat z}u\big\|^2_{\hat T} = \sum_{l\in\mathbb{N}_0^2}(4l_1+2)^2(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})^2\gamma_l^{0,0,0}, \tag{4.11}$$

$$\begin{aligned}\big\|\partial_{\hat z}\pi_M^{-1,-1,-1}u\big\|^2_{\hat T} &= \sum_{|l|\le M-2}(4l_1+2)^2(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})^2\gamma_l^{0,0,0}\\ &\quad+\sum_{|l|=M-1}(4l_1+2)^2\hat u^2_{l_1+1,l_2}\gamma_l^{0,0,0} := I_1+I_2.\end{aligned} \tag{4.12}$$

Since $u\in H^2(\hat T)\cap H_0^1(\hat T)$, Lemma 4.3 indicates that $\pi_M^{-1,-1,-1}u\in\mathbb{P}_M(\hat T)\cap H_0^1(\hat T)$ thus $\hat u_{l_1,l_2}=0$ for $l_1=0,1$ or $l_2=0$. We now derive from the Cauchy-Schwarz inequality that

$$\begin{aligned}|\hat u_{l_1+1,M-1-l_1}|^2 &= \Big|\sum_{l_2=0}^{M-2-l_1}(-1)^{M-2-l_1-l_2}(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})\Big|^2\\ &\le \sum_{l_2=0}^{M-2-l_1}(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})^2\gamma^{0,0,0}_{l_1,l_2}\sum_{l_2=0}^{M-2-l_1}\frac{1}{\gamma^{0,0,0}_{l_1,l_2}}\\ &\overset{(2.11)}{=}(2l_1+1)(M-1-l_1)(M+l_1)\sum_{l_2=0}^{M-2-l_1}(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})^2\gamma^{0,0,0}_{l_1,l_2}.\end{aligned}$$

As an immediate consequence,

$$\begin{aligned}I_2 &\le \frac{1}{2M}\sum_{l_1=0}^{M-2}(M-1-l_1)(M+l_1)(4l_1+2)^2\sum_{l_2=0}^{M-2-l_1}(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})^2\gamma^{0,0,0}_{l_1,l_2}\\ &\le \frac{M^2-M}{2M}\sum_{|l|\le M-2}(4l_1+2)^2(\hat u_{l_1+1,l_2}+\hat u_{l_1+1,l_2+1})^2\gamma_l^{0,0,0} = \frac{M-1}{2}I_1.\end{aligned}$$

In return, we obtain from (4.12) and (4.11) that

$$\big\|\partial_{\hat z}\pi_M^{-1,-1,-1}u\big\|^2_{\hat T}\le\frac{M+1}{2}I_1\le\frac{M+1}{2}\|\partial_{\hat z}u\|^2_{\hat T}.$$

By the rotational invariance property (4.6) of $\pi_M^{-1,-1,-1}$, we also deduce that

$$\left\|\partial_{\hat{x}}\pi_M^{-1,-1,-1}u\right\|_{\hat{T}}^2 = \left\|\partial_{\hat{z}}\pi_M^{-1,-1,-1}u^{\mathcal{K}}\right\|_{\hat{T}}^2 \le \frac{M+1}{2}\left\|\partial_{\hat{z}}u^{\mathcal{K}}\right\|_{\hat{T}}^2 = \frac{M+1}{2}\left\|\partial_{\hat{x}}u\right\|_{\hat{T}}^2,$$
$$\left\|\partial_{\hat{y}}\pi_M^{-1,-1,-1}u\right\|_{\hat{T}}^2 = \left\|\partial_{\hat{z}}\pi_M^{-1,-1,-1}u^{\mathcal{R}}\right\|_{\hat{T}}^2 \le \frac{M+1}{2}\left\|\partial_{\hat{z}}u^{\mathcal{R}}\right\|_{\hat{T}}^2 = \frac{M+1}{2}\left\|\partial_{\hat{y}}u\right\|_{\hat{T}}^2.$$

This finally completes the proof. □

4.2. **Estimate on the discrete inf-sup constant.** Suppose $u \in H^2(\hat{T}) \cap H_0^1(\hat{T})$. Then for $l_1 \ge 2$ and $l_2 \ge 1$, by integration by parts, combined with (A.10) in [36], (2.15), (2.11) and (2.24), it holds that

$$\begin{aligned}\hat{u}_l =& \frac{\langle u, \tilde{\mathcal{J}}_l\rangle_{\hat{T}}}{\gamma_l} = \frac{1}{\gamma_l}\left[(\partial_{\hat{x}}\partial_{\hat{z}}u, \partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l)_{w^{1,0,0},\hat{T}} + (\partial_{\hat{y}}\partial_{\hat{z}}u, \partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l)_{w^{0,1,0},\hat{T}}\right]\\ =& \frac{1}{\gamma_l}(u, \left[\partial_{\hat{x}}\partial_{\hat{z}}\left(w^{1,0,0}\partial_{\hat{x}}\partial_{\hat{z}}\right) + \partial_{\hat{y}}\partial_{\hat{z}}\left(w^{0,1,0}\partial_{\hat{y}}\partial_{\hat{z}}\right)\right]\tilde{\mathcal{J}}_l)_{\hat{T}}\\ =& \frac{l_1(l_1-1)l_2(2l_1+l_2-1)}{\gamma_l}(u, w^{-1,-1,-1}\tilde{\mathcal{J}}_l)_{\hat{T}} = \frac{(u, \mathcal{J}_l^{-1,-1,-1})_{w^{-1,-1,-1},\hat{T}}}{\frac{2(2l_1-1)(2|l|-1)}{(l_1-1)(|l|+l_1-1)}\gamma_l^{-1,-1,-1}}.\end{aligned}$$

This allows us to extend $\pi_M^{-1,-1,-1}$ as a projection from $H_0^1(\hat{T})$ onto $\mathbb{P}_M(\hat{T}) \cap H_0^1(\hat{T})$:

$$\pi_M^{-1,-1,-1}u = \sum_{|l|\le M-3} \frac{(u, \mathcal{J}_{l_1+2,l_2+1}^{-1,-1,-1})_{w^{-1,-1,-1},\hat{T}}}{\gamma_{l_1+2,l_2+1}^{-1,-1,-1}}\mathcal{J}_{l_1+2,l_2+1}^{-1,-1,-1}. \tag{4.13}$$

Lemma 4.4. *$\pi_M^{-1,-1,-1} : H_0^1(\hat{T}) \mapsto \mathbb{P}_M(\hat{T}) \cap H_0^1(\hat{T})$ is a Fortin interpolation operator in the sense that*

$$(\mathrm{div}(\boldsymbol{v} - \pi_M^{-1,-1,-1}\boldsymbol{v}), q)_{\hat{T}} = 0 \qquad \forall q \in \mathbb{P}_{M-2}(\hat{T}),\ \boldsymbol{v} \in H_0^1(\hat{T})^2, \tag{4.14}$$

which satisfies

$$\left|\pi_M^{-1,-1,-1}u\right|_{1,\hat{T}} \le \sqrt{\frac{M+1}{2}}|u|_{1,\hat{T}} \qquad \forall u \in H_0^1(\hat{T}). \tag{4.15}$$

Proof. Since $H^2(\hat{T}) \cap H_0^1(\hat{T})$ is dense in $H_0^1(\hat{T})$, (4.15) is then derived from Theorem 4.1 by a standard density argument. To prove (4.14), it suffices to prove that

$$(v - \pi_M^{-1,-1,-1}v, q)_{\hat{T}} = 0 \qquad \forall q \in \mathbb{P}_{M-3}(\hat{T}),\ v \in H_0^1(\hat{T}).$$

Actually, for any $v \in H_0^1(\hat{T})$ and any $\mathcal{J}_{l_1+2,l_2+1}^{-1,-1,-1} = -\frac{2l_1+l_2+4}{l_2+1}w^{1,1,1}\mathcal{J}_l^{1,1,1}$, $|l| \le M-3$, according to (4.13) and (2.11), it holds that

$$(\pi_M^{-1,-1,-1}v, w^{-1,-1,-1}\mathcal{J}_{l_1+2,l_2+1}^{-1,-1,-1})_{\hat{T}} = (v, \mathcal{J}_{l_1+2,l_2+1}^{-1,-1,-1})_{w^{-1,-1,-1},\hat{T}},$$

which finally completes the proof. □

Making use of Fortin's criterion [15], we arrive now at the sharp estimate on the discrete inf-sup constant on the reference triangle $\hat{T}$.

Theorem 4.2. *Let $W_M := \mathbb{P}_{M-2}(\hat{T}) \cap L_0^2(\hat{T})$ and $\boldsymbol{V}_M := \mathbb{P}_M^{\partial\hat{T}}(\hat{T})^2 = \mathrm{span}\big\{\tilde{\mathcal{J}}_{l_1,l_2} : l_1 \ge 2, l_2 \ge 1, l_1 + l_2 \le M\big\}^2$. It holds that*

$$\inf_{0\ne q\in W_M}\sup_{\boldsymbol{0}\ne\boldsymbol{v}\in\boldsymbol{V}_M}\frac{(\mathrm{div}\ \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\hat{T}}\|q\|_{\hat{T}}} \asymp \frac{1}{\sqrt{M}}. \tag{4.16}$$

Proof. For any $q \in W_M$, according to (4.14) and (4.15), it holds that

$$\sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_M} \frac{(\text{div } \boldsymbol{v}, q)_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}} = \sup_{\mathbf{0}\neq \boldsymbol{v}\in H_0^1(\hat{T})^2} \frac{(\text{div } \pi_M^{-1,-1,-1}\boldsymbol{v}, q)_{\hat{T}}}{\|\pi_M^{-1,-1,-1}\boldsymbol{v}\|_{1,\hat{T}}}$$
$$\gtrsim \frac{1}{\sqrt{M}} \sup_{\mathbf{0}\neq \boldsymbol{v}\in H_0^1(\hat{T})^2} \frac{(\text{div } \boldsymbol{v}, q)_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}} \gtrsim \frac{\beta(\hat{T})}{\sqrt{M}}\|q\|_{\hat{T}},$$

which gives the lower bound of the discrete inf-sup constant.

On the other hand, we set

$$q^*(\hat{x},\hat{y}) = \sum_{k=1}^{M-2} (k+1)^2 \mathcal{J}_{0,k}^{0,0,0}(\hat{x},\hat{y})$$

such that $q^* \in W_M$ and

$$\|q^*\|_{\hat{T}}^2 = \sum_{k=1}^{M-2} \frac{(k+1)^3}{2} = \frac{M^2(M-1)^2}{8} - \frac{1}{2}.$$

Then by (3.16), (3.17) together with the orthogonality relation (2.11) with $\alpha = (0,0,0)$, one readily checks that

$$(\text{div } \boldsymbol{v}, q^* + M^2 \mathcal{J}_{0,M-1}^{0,0,0}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_M.$$

Thus, for any $\boldsymbol{v} \in \boldsymbol{V}_M$,

$$\frac{(\text{div } \boldsymbol{v}, q^*)_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}\|q^*\|_{\hat{T}}} = \frac{-M^2(\text{div } \boldsymbol{v}, \mathcal{J}_{0,M-1}^{0,0,0})_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}\|q^*\|_{\hat{T}}} \leq \frac{M^2\|\text{div } \boldsymbol{v}\|_{\hat{T}}\, \|\mathcal{J}_{0,M-1}^{0,0,0}\|_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}\|q^*\|_{\hat{T}}}$$
$$\lesssim \frac{M^2\|\mathcal{J}_{0,M-1}^{0,0,0}\|_{\hat{T}}}{\|q^*\|_{\hat{T}}} \lesssim \frac{1}{\sqrt{M}}.$$

As a result,

$$\inf_{0\neq q\in W_M} \sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_M} \frac{(\text{div } \boldsymbol{v}, q)_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}\|q\|_{\hat{T}}} \leq \sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_M} \frac{(\text{div } \boldsymbol{v}, q^*)_{\hat{T}}}{\|\boldsymbol{v}\|_{1,\hat{T}}\|q^*\|_{\hat{T}}} \lesssim \frac{1}{\sqrt{M}},$$

which asserts the upper bound of (4.16). The proof is now completed. □

By the Boland-Nicoliades local-to-global argument [13, 19, 49, 53], one obtains the main theorem for the discrete inf-sup constant on Ω from Theorem 4.2.

Theorem 4.3. *Let $\boldsymbol{V}_\delta, W_\delta$ be defined as in* (3.9) *and* (3.10). *Then it holds for $M \geq 2$ that*

$$\beta_\delta = \inf_{0\neq q\in W_\delta} \sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_\delta} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1\|q\|} \gtrsim \frac{1}{\sqrt{M}}. \tag{4.17}$$

5. Triangular spectral element approximations

We devote this section to the error estimate of the orthogonal spectral element projections. To this end, we shall first define the local projection on each element, and assemble all local projections into a globally continuous piecewise polynomial which has an optimal estimate for the approximation error.

5.1. **Fortin interpolation on Ω.** For each $T_m \in \mathcal{T}$, the local projection $\pi_{M,T_m}^{-1,-1,-1}$: $H^2(T_m) \mapsto \mathbb{P}_M(T_m)$ is defined by

$$\pi_{M,T_m}^{-1,-1,-1} u = \left[\pi_M^{-1,-1,-1}(u \circ F_m)\right] \circ F_m^{-1}.$$

By Lemma 4.3, we can simply use the union of the local projections as the local-to-global mapping and set up a global projection $\pi_{M,h}^{-1,-1,-1} : V_{\mathcal{T}}^2(\Omega) \mapsto V_\delta$ such that

$$(\pi_{M,h}^{-1,-1,-1} u)\big|_{T_m} = \pi_{M,T_m}^{-1,-1,-1}(u|_{T_m}), \qquad T_m \in \mathcal{T}, \tag{5.1}$$

where we define $V_{\mathcal{T}}^s(\Omega) = \{u \in H^1(\Omega) : u|_{T_m} \in H^s(T_m) \text{ for any } T_m \in \mathcal{T}\}$ for $s \geq 2$.

Since $F_m : \overline{\hat{T}} \mapsto \overline{T}_m$ is an affine mapping, one gets from Lemma 4.2, (3.16) and (3.17) that

$$(\text{div } (\pi_{M,T_m}^{-1,-1,-1}\boldsymbol{u} - \boldsymbol{u}), q)_{T_m} = 0 \quad \forall q \in \mathbb{P}_{M-2}(T_m), \boldsymbol{u} \in H^2(T_m)^2,$$

which reveals that $\pi_{M,h}^{-1,-1,-1}\boldsymbol{u}$ is a Fortin interpolation of $\boldsymbol{u}$ on Ω,

$$b(\pi_{M,h}^{-1,-1,-1}\boldsymbol{u} - \boldsymbol{u}, q) = 0 \quad \forall q \in W_\delta, \boldsymbol{u} \in V_{\mathcal{T}}^2(\Omega)^2. \tag{5.2}$$

Recalling that $h_{T_m} = \text{diam}(T_m)$ and $h = \max_{T_m \in \mathcal{T}} h_{T_m}$, we are now in the position to state the error estimate of $\pi_{M,h}^{-1,-1,-1}$.

Theorem 5.1. *Suppose $u \in V_{\mathcal{T}}^s(\Omega)$ with $s \geq 2$. Then*

$$\left\|\nabla(\pi_{M,h}^{-1,-1,-1} u - u)\right\| \lesssim h^{s-1} M^{1-s} |u|_s. \tag{5.3}$$

Theorem 5.1 is a direct consequence of the following lemma.

Lemma 5.1. *Suppose $T_m \in \mathcal{T}$ and $u \in H^s(T_m)$ with $s \geq 2$. Then*

$$\|\nabla(\pi_{M,T_m}^{-1,-1,-1} u - u)\|_{T_m} \lesssim h_{T_m}^{s-1} M^{1-s} |u|_{s,T_m}. \tag{5.4}$$

Proof. It suffices to prove (5.4) with $T_m = \hat{T}$. We now proceed with the Koornwinder-Fourier series of u in $\tilde{\mathcal{J}}_l$, $l \in \mathbb{N}_0^2$. By (4.9), (4.10) and (2.11),

$$\partial_{\hat{z}}(\mathrm{I} - \pi_M^{-1,-1,-1})u = \sum_{|l| \geq M-1} (4l_1 + 2)\left[(1 - \delta_{|l|,M-1})\hat{u}_{l_1+1,l_2} + \hat{u}_{l_1+1,l_2+1}\right]\mathcal{J}_l^{0,0,0},$$

$$\begin{aligned}
\left\|\partial_{\hat{z}}(\mathrm{I} - \pi_M^{-1,-1,-1})u\right\|_{\hat{T}}^2 &= \sum_{|l| \geq M-1} \frac{4l_1 + 2}{|l| + 1}\left[(1 - \delta_{|l|,M-1})\hat{u}_{l_1+1,l_2} + \hat{u}_{l_1+1,l_2+1}\right]^2 \\
&\leq \sum_{|l| \geq M-1} \frac{4(2l_1 + 1)}{|l| + 1}\left[(1 - \delta_{|l|,M-1})\hat{u}_{l_1+1,l_2}^2 + \hat{u}_{l_1+1,l_2+1}^2\right] \\
&= \sum_{|l| \geq M} 4(2l_1 + 1)\left[\frac{1}{|l| + 1} + \frac{1 - \delta_{l_2,0}}{|l|}\right]\hat{u}_{l_1+1,l_2}^2.
\end{aligned} \tag{5.5}$$

On the other hand, (2.20) together with the definition of $\pi_{M-2}^{1,0,0}$ yields

$$\begin{aligned}\partial_{\hat{x}}\partial_{\hat{z}}u &= \sum_{l\in\mathbb{N}_0^2}\hat{u}_l\partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l = -2\sum_{l\in\mathbb{N}_0^2}\hat{u}_l(2|l|-1)\big[l_1\mathcal{J}_{l_1-1,l_2-1}^{1,0,0}+(l_1-1)\mathcal{J}_{l_1-2,l_2}^{1,0,0}\big]\\ &= -2\sum_{l\in\mathbb{N}_0^2}(2|l|+3)(l_1+1)\big[\hat{u}_{l_1+1,l_2+1}+\hat{u}_{l_1+2,l_2}\big]\mathcal{J}_l^{1,0,0},\\ \pi_{M-2}^{1,0,0}\partial_{\hat{x}}\partial_{\hat{z}}u &= -2\sum_{|l|\le M-2}(2|l|+3)(l_1+1)\big[\hat{u}_{l_1+1,l_2+1}+\hat{u}_{l_1+2,l_2}\big]\mathcal{J}_l^{1,0,0}.\end{aligned}$$

Analogously, (2.21) and the definition of $\pi_{M-2}^{0,1,0}$ give that

$$\begin{aligned}\partial_{\hat{y}}\partial_{\hat{z}}u &= -2\sum_{l\in\mathbb{N}_0^2}(2|l|+3)(l_1+1)\big[\hat{u}_{l_1+1,l_2+1}-\hat{u}_{l_1+2,l_2}\big]\mathcal{J}_l^{0,1,0},\\ \pi_{M-2}^{0,1,0}\partial_{\hat{y}}\partial_{\hat{z}}u &= -2\sum_{|l|\le M-2}(2|l|+3)(l_1+1)\big[\hat{u}_{l_1+1,l_2+1}-\hat{u}_{l_1+2,l_2}\big]\mathcal{J}_l^{0,1,0}.\end{aligned}$$

Thus, (2.11) yields

(5.6)
$$\begin{aligned}&\big\|(\pi_{M-2}^{1,0,0}-\mathrm{I})\partial_{\hat{x}}\partial_{\hat{z}}u\big\|_{w^{1,0,0},\hat{T}}^2+\big\|(\pi_{M-2}^{0,1,0}-\mathrm{I})\partial_{\hat{y}}\partial_{\hat{z}}u\big\|_{w^{0,1,0},\hat{T}}^2\\ &\quad=\sum_{|l|\ge M-1}2(2|l|+3)(l_1+1)\big[(\hat{u}_{l_1+1,l_2+1}+\hat{u}_{l_1+2,l_2})^2+(\hat{u}_{l_1+1,l_2+1}-\hat{u}_{l_1+2,l_2})^2\big]\\ &\quad=\sum_{|l|\ge M}4(2|l|+1)[(2-\delta_{l_1,0}-\delta_{l_2,0})l_1+(1-\delta_{l_2,0})]\hat{u}_{l_1+1,l_2}^2.\end{aligned}$$

A combination of (5.5) and (5.6) gives

$$\begin{aligned}\big\|\partial_{\hat{z}}(\pi_M^{-1,-1,-1}-\mathrm{I})u\big\|_{\hat{T}}^2 \le{}& M^{-2}\big\|(\pi_{M-2}^{1,0,0}-\mathrm{I})\partial_{\hat{x}}\partial_{\hat{z}}u\big\|_{w^{1,0,0},\hat{T}}^2\\ &+M^{-2}\big\|(\pi_{M-2}^{0,1,0}-\mathrm{I})\partial_{\hat{y}}\partial_{\hat{z}}u\big\|_{w^{0,1,0},\hat{T}}^2.\end{aligned}$$

By the rotational property (4.6) and (4.3), we further deduce that

$$\begin{aligned}&\big\|\partial_{\hat{y}}(\pi_M^{-1,-1,-1}-\mathrm{I})u\big\|_{\hat{T}}^2=\big\|\partial_{\hat{z}}(\pi_M^{-1,-1,-1}-\mathrm{I})u^{\mathcal{R}}\big\|_{\hat{T}}^2\\ &\le M^{-2}\left(\big\|(\pi_{M-2}^{1,0,0}-\mathrm{I})\partial_{\hat{x}}\partial_{\hat{z}}u^{\mathcal{R}}\big\|_{w^{1,0,0},\hat{T}}^2+\big\|(\pi_{M-2}^{0,1,0}-\mathrm{I})\partial_{\hat{y}}\partial_{\hat{z}}u^{\mathcal{R}}\big\|_{w^{0,1,0},\hat{T}}^2\right)\\ &=M^{-2}\left(\big\|(\pi_{M-2}^{1,0,0}-\mathrm{I})(\partial_{\hat{y}}\partial_{\hat{z}}u)^{\mathcal{R}}\big\|_{w^{1,0,0},\hat{T}}^2+\big\|(\pi_{M-2}^{0,1,0}-\mathrm{I})(\partial_{\hat{x}}\partial_{\hat{y}}u)^{\mathcal{R}}\big\|_{w^{0,1,0},\hat{T}}^2\right)\\ &=M^{-2}\left(\big\|(\pi_{M-2}^{0,1,0}-\mathrm{I})\partial_{\hat{y}}\partial_{\hat{z}}u\big\|_{w^{0,1,0},\hat{T}}^2+\big\|(\pi_{M-2}^{0,0,1}-\mathrm{I})\partial_{\hat{y}}\partial_{\hat{x}}u\big\|_{w^{0,0,1},\hat{T}}^2\right)\end{aligned}$$

and

$$\begin{aligned}\big\|\partial_{\hat{x}}(\pi_M^{-1,-1,-1}-\mathrm{I})u\big\|_{\hat{T}}^2 \le{}& M^{-2}\big\|(\pi_{M-2}^{0,0,1}-\mathrm{I})\partial_{\hat{y}}\partial_{\hat{x}}u\big\|_{w^{0,0,1},\hat{T}}^2\\ &+M^{-2}\big\|(\pi_{M-2}^{1,0,0}-\mathrm{I})\partial_{\hat{x}}\partial_{\hat{z}}u\big\|_{w^{1,0,0},\hat{T}}^2.\end{aligned}$$

Then, resorting to Lemma 4.1 we finally derive that

$$\begin{aligned}
\|\nabla(\pi_M^{-1,-1,-1}u-u)\|_{\hat T}^2 &= \big\|\partial_{\hat x}\pi_M^{-1,-1,-1}u-\partial_{\hat x}u\big\|_{\hat T}^2+\big\|\partial_{\hat y}\pi_M^{-1,-1,-1}u-\partial_{\hat y}u\big\|_{\hat T}^2\\
&\le M^{-2}\left(\big\|(\pi_{M-2}^{0,0,1}-\mathrm{I})\partial_{\hat y}\partial_{\hat x}u\big\|_{w^{0,0,1},\hat T}^2+\big\|(\pi_{M-2}^{1,0,0}-\mathrm{I})\partial_{\hat x}\partial_{\hat z}u\big\|_{w^{1,0,0},\hat T}^2\right)\\
&\quad+M^{-2}\left(\big\|(\pi_{M-2}^{0,1,0}-\mathrm{I})\partial_{\hat y}\partial_{\hat z}u\big\|_{w^{0,1,0},\hat T}^2+\big\|(\pi_{M-2}^{0,0,1}-\mathrm{I})\partial_{\hat y}\partial_{\hat x}u\big\|_{w^{0,0,1},\hat T}^2\right)\\
&\lesssim M^{2-2s}\Big[|\partial_{\hat y}\partial_{\hat x}u|_{s-2,\hat T}^2+|\partial_{\hat y}\partial_{\hat z}u|_{s-2,\hat T}^2+|\partial_{\hat x}\partial_{\hat z}u|_{s-2,\hat T}^2\Big]\\
&\lesssim M^{2-2s}|u|_{s,\hat T}^2,
\end{aligned}$$

which completes the proof. □

5.2. **Orthogonal spectral element projections.** We first introduce the L^2-orthogonal projection $\pi_\delta : L_0^2(\Omega)\mapsto W_\delta$ such that

$$(\pi_\delta u-u,v)=0,\quad v\in W_\delta. \tag{5.7}$$

It is readily verified that

$$(\pi_\delta u)|_{T_m}=\big[\pi_{M-2}^{0,0,0}(u|_{T_m}\circ F_m)\big]\circ F_m^{-1},\quad T_m\in\mathcal{T}.$$

As an immediate consequence of Lemma 4.1, we have the following corollary.

Corollary 5.1. *Suppose $u\in L_0^2(\Omega)\cap H^s(\Omega)$ for $s\ge 0$. Then*

$$\inf_{v\in W_\delta}\|u-v\|=\big\|\pi_\delta u-u\big\|\lesssim h^sM^{-s}\sum_{T_m\in\mathcal{T}}|u|_{s,T_m}\lesssim h^sM^{-s}|u|_s.$$

Further define the orthogonal projection $\Pi_\delta : \boldsymbol{V}\mapsto \boldsymbol{X}_\delta$ (defined in (3.13)), such that for $\boldsymbol{u}\in\boldsymbol{V}$,

$$a(\boldsymbol{u}-\Pi_\delta\boldsymbol{u},\boldsymbol{v})=(\nabla(\boldsymbol{u}-\Pi_\delta\boldsymbol{u}),\nabla\boldsymbol{v})=0\qquad\forall\boldsymbol{v}\in\boldsymbol{X}_\delta. \tag{5.8}$$

Theorem 5.2. *Suppose $\boldsymbol{u}\in\boldsymbol{X}\cap H^s(\Omega)^2$ for $s\ge 1$. Then*

$$\inf_{\boldsymbol{v}\in\boldsymbol{X}_\delta}|\boldsymbol{u}-\boldsymbol{v}|_1=|\boldsymbol{u}-\Pi_\delta\boldsymbol{u}|_1\lesssim h^{s-1}M^{1-s}\sum_{T_m\in\mathcal{T}}|\boldsymbol{u}|_{s,T_m}\lesssim h^{s-1}M^{1-s}|\boldsymbol{u}|_s. \tag{5.9}$$

Proof. By the projection theorem, it holds that

$$\big\|\nabla(\boldsymbol{u}-\Pi_\delta\boldsymbol{u})\big\|=\inf_{\boldsymbol{v}\in\boldsymbol{X}_\delta}\big\|\nabla(\boldsymbol{u}-\boldsymbol{v})\big\|\le\|\nabla(\boldsymbol{u}-\boldsymbol{v}_\delta)\|\qquad\forall\boldsymbol{v}_\delta\in\boldsymbol{X}_\delta. \tag{5.10}$$

Taking $\boldsymbol{v}_\delta=0$, we have

$$|\boldsymbol{u}-\Pi_\delta\boldsymbol{u}|_1\le|\boldsymbol{u}|_1. \tag{5.11}$$

Meanwhile, (5.2) states that $b(\pi_{M,h}^{-1,-1,-1}\boldsymbol{u},q)=0$ for all $q\in W_\delta$ and $\boldsymbol{u}\in\boldsymbol{X}\cap H^2(\Omega)^2$. In the sequel, $\pi_{M,h}^{-1,-1,-1}\boldsymbol{u}\in\boldsymbol{X}_\delta$. For $s\ge 2$, we take $\boldsymbol{v}_\delta=\pi_{M,h}^{-1,-1,-1}\boldsymbol{u}$ in (5.10), and derive from Theorem 5.1 that

$$|\boldsymbol{u}-\Pi_\delta\boldsymbol{u}|_1\le|\boldsymbol{u}-\boldsymbol{v}_\delta|_1\lesssim h^{s-1}M^{1-s}|\boldsymbol{u}|_s,\quad s\ge 2, \tag{5.12}$$

which finally completes the proof of (5.9). □

6. Convergence analysis

In this section, we resort to the general theory in [5,44] and [7], and then perform the convergence analysis for the Stokes eigenvalue problem (3.1). For this purpose, we first turn to the analysis of the triangular spectral element method for the source problem of the Stokes equation.

6.1. **Source problem of Stokes equations.** Let $\boldsymbol{H} = L^2(\Omega)^2$. Then $\boldsymbol{V}$ is compactly embedded into $\boldsymbol{H}$. For any $\boldsymbol{g} \in \boldsymbol{H}$, we consider the associated source problem: to find $A\boldsymbol{g} \in \boldsymbol{V}$ and $B\boldsymbol{g} \in W$ such that

$$\begin{cases} a(A\boldsymbol{g}, \boldsymbol{v}) + b(\boldsymbol{v}, B\boldsymbol{g}) = (\boldsymbol{g}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ b(A\boldsymbol{g}, q) = 0 & \forall q \in W; \end{cases} \tag{6.1}$$

and the approximate source problem: to find $A_\delta \boldsymbol{g} \in \boldsymbol{V}_\delta$ and $B_\delta \boldsymbol{g} \in W_\delta$ such that

$$\begin{cases} a(A_\delta \boldsymbol{g}, \boldsymbol{v}) + b(\boldsymbol{v}, B_\delta \boldsymbol{g}) = (\boldsymbol{g}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}_\delta, \\ b(A_\delta \boldsymbol{g}, q) = 0 & \forall q \in W_\delta. \end{cases} \tag{6.2}$$

Owing to (3.4), (3.5) and (4.17), combined with Corollary 2.1 and Corollary 2.2 in [7], one can conclude that both the problems above are uniquely solvable. Moreover, the solutions satisfy

$$\|A\boldsymbol{g}\|_1 + \beta\|B\boldsymbol{g}\| \le (1 + \frac{1}{\eta})\|\boldsymbol{g}\|, \qquad \|A_\delta \boldsymbol{g}\|_1 + \beta_\delta\|B_\delta \boldsymbol{g}\| \le (1 + \frac{1}{\eta})\|\boldsymbol{g}\|,$$

where $\eta > 0$ as in (3.4).

The following lemma measures the distance between the solutions of (6.1) and (6.2).

Lemma 6.1. *For any $\boldsymbol{g}, \boldsymbol{g}^\delta \in \boldsymbol{H}$, it holds that*

$$\begin{aligned} &\|A\boldsymbol{g} - A_\delta \boldsymbol{g}^\delta\|_1 + \beta_\delta\|B\boldsymbol{g} - B_\delta \boldsymbol{g}^\delta\| \\ &\qquad \lesssim \inf_{\boldsymbol{v} \in \boldsymbol{X}_\delta} \|A\boldsymbol{g} - \boldsymbol{v}\|_1 + \inf_{q \in W_\delta} \big\|B\boldsymbol{g} - q\big\| + \sup_{\boldsymbol{v} \in \boldsymbol{V}_\delta} \frac{(\boldsymbol{g} - \boldsymbol{g}^\delta, \boldsymbol{v})}{\|\boldsymbol{v}\|_1}. \end{aligned} \tag{6.3}$$

Further suppose the following shift theorem holds for equation (6.1),

$$\|A\boldsymbol{g}\|_2 + \|B\boldsymbol{g}\|_1 \lesssim \|\boldsymbol{g}\|. \tag{6.4}$$

Then

$$\|(A - A_\delta)\boldsymbol{g}\| \lesssim hM^{-1}\big[\inf_{\boldsymbol{v} \in \boldsymbol{X}_\delta} \|A\boldsymbol{g} - \boldsymbol{v}\|_1 + \inf_{q \in W_\delta} \big\|B\boldsymbol{g} - q\big\|\big]. \tag{6.5}$$

Proof. The estimate (6.3) can be derived directly from or be obtained in a way similar to [7, Theorem 2.2-2.3] or [8, Theorem 23.9].

We now turn to the proof of (6.5) and let $\boldsymbol{v} \in \boldsymbol{H}$. Using (6.1) and (6.2) repeatedly yields

$$\begin{aligned} (\boldsymbol{v}, A\boldsymbol{g} - A_\delta \boldsymbol{g}) &= a(A\boldsymbol{v}, A\boldsymbol{g} - A_\delta \boldsymbol{g}) + b(A\boldsymbol{g} - A_\delta \boldsymbol{g}, B\boldsymbol{v}) \\ &= \big[a(A\boldsymbol{v} - \Pi_\delta A\boldsymbol{v}, A\boldsymbol{g} - A_\delta \boldsymbol{g}) - b(\Pi_\delta A\boldsymbol{v}, B\boldsymbol{g} - \pi_\delta B\boldsymbol{g})\big] + b(A\boldsymbol{g} - A_\delta \boldsymbol{g}, B\boldsymbol{v}) \\ &= \big[a(A\boldsymbol{v} - \Pi_\delta A\boldsymbol{v}, A\boldsymbol{g} - A_\delta \boldsymbol{g}) - b(\Pi_\delta A\boldsymbol{v} - A\boldsymbol{v}, B\boldsymbol{g} - \pi_\delta B\boldsymbol{g})\big] \\ &\quad + b(A\boldsymbol{g} - A_\delta \boldsymbol{g}, B\boldsymbol{v} - \pi_\delta B\boldsymbol{v}). \end{aligned}$$

Thus by (3.2), (3.3) together with the estimates in Corollary 5.1 and Theorem 5.2, it holds that

$$\begin{aligned} |(\boldsymbol{v}, A\boldsymbol{g} - A_\delta \boldsymbol{g})| &\le \|A\boldsymbol{v} - \Pi_\delta A\boldsymbol{v}\|_1(\|A\boldsymbol{g} - A_\delta \boldsymbol{g}\|_1 + \|B\boldsymbol{g} - \pi_\delta B\boldsymbol{g}\|) \\ &\quad + \|A\boldsymbol{g} - A_\delta \boldsymbol{g}\|_1 \, \|B\boldsymbol{v} - \pi_\delta B\boldsymbol{v}\| \\ &\le (\|A\boldsymbol{v} - \Pi_\delta A\boldsymbol{v}\|_1 + \|B\boldsymbol{v} - \pi_\delta B\boldsymbol{v}\|)(\|A\boldsymbol{g} - A_\delta \boldsymbol{g}\|_1 + \|B\boldsymbol{g} - \pi_\delta B\boldsymbol{g}\|) \\ &\lesssim hM^{-1}(\|A\boldsymbol{v}\|_2 + \|B\boldsymbol{v}\|_1)(\|A\boldsymbol{g} - A_\delta \boldsymbol{g}\|_1 + \|B\boldsymbol{g} - \pi_\delta B\boldsymbol{g}\|), \end{aligned}$$

which implies by (6.4) and (6.3) that

$$\|A\boldsymbol{g} - A_\delta \boldsymbol{g}\| = \sup_{\boldsymbol{v}\in\boldsymbol{H}} \frac{|(A\boldsymbol{g} - A_\delta\boldsymbol{g}, \boldsymbol{v})|}{\|\boldsymbol{v}\|} \lesssim hM^{-1}(\|A\boldsymbol{g} - A_\delta\boldsymbol{g}\|_1 + \|B\boldsymbol{g} - \pi_\delta B\boldsymbol{g}\|)$$
$$\lesssim hM^{-1}(\|A\boldsymbol{g} - \Pi_\delta A\boldsymbol{g}\|_1 + \|B\boldsymbol{g} - \pi_\delta B\boldsymbol{g}\|).$$

Now the proof is completed. □

6.2. **Error analysis of Stokes eigenvalues.** Lemma 6.1 indicates that

$$\lim_{\delta\to 0} \|A - A_\delta\|_{\mathscr{L}(\boldsymbol{H},\boldsymbol{H})} = \lim_{\delta\to 0} \|A - A_\delta\|_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})} = 0, \tag{6.6}$$

which implies A is compact since A_δ is compact. Hereafter, for $A \in \mathscr{L}(H, G)$, we denote by $\|A\|_{\mathscr{L}(H,G)} = \sup_{u\in H, \|u\|_H=1} \|Au\|_G$ the norm of A.

It is well known that the eigenvalues of (3.1)–(3.7) (resp. (3.11)–(3.14)) are reciprocals of the eigenvalues of A (resp. A_δ) and the eigenfunctions of (3.7) (resp. (3.14)) are the same as the eigenfunctions of A (resp. A_δ).

Let λ be an eigenvalue of (3.1) with the algebraic/geometric multiplicity m. Define the eigenspace corresponding to λ:

$$E = E(\lambda) = \mathrm{Ker}(\lambda^{-1} - A).$$

There are exactly m eigenvalues $\lambda_{1,\delta}, \ldots, \lambda_{m,\delta}$ (counted according to the algebraic/geometric multiplicity) of (3.11) converge to λ. We denote by $E_\delta = E_\delta(\lambda)$ the direct sum of the eigenspaces corresponding to $\lambda_{1,\delta}, \ldots, \lambda_{m,\delta}$.

We now state the spectral theory on the approximate eigenvalue problem (3.11) [44, Theorems 5.1 and 5.2].

Lemma 6.2. *There is a positive constant δ^* such that for $0 < \delta \le \delta^*$ and $1 \le i \le m$,*

$$\begin{aligned} |\lambda - \lambda_{i,\delta}| \lesssim & \big\|(A - A_\delta)|_E\big\|_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})} \big\|(B - \pi_\delta B)|_E\big\|_{\mathscr{L}(\boldsymbol{H},W)} \\ & + \big\|(A - A_\delta)|_E\big\|^2_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})}, \end{aligned} \tag{6.7}$$

$$\hat{\delta}(E, E_\delta) \lesssim \big\|(A - A_\delta)|_E\big\|_{\mathscr{L}(\boldsymbol{H},\boldsymbol{H})}, \tag{6.8}$$

where $A|_E$ denotes the restriction of A to E, and $\hat{\delta}(E, E_\delta)$ is the gap of two subspaces $E \subseteq \boldsymbol{H}$ and $E_\delta \subseteq \boldsymbol{H}$ evaluated in the $\boldsymbol{H}$-norm:

$$\hat{\delta}(E, E_\delta) = \max(\delta(E, E_\delta), \delta(E_\delta, E)), \quad \delta(E, E_\delta) = \sup_{\boldsymbol{w}\in E, \|\boldsymbol{w}\|_{\boldsymbol{H}}=1} \inf_{\boldsymbol{v}\in E_\delta} \|\boldsymbol{w} - \boldsymbol{v}\|_{\boldsymbol{H}}.$$

Proof. (6.8) follows directly from [44, Theorem 5.2] and [5, Theorem 7.1].

By Theorem 4.1 in [44], we have

$$|\lambda - \lambda_{i,\delta}| \lesssim \sup_{\boldsymbol{u}\in E, \|\boldsymbol{u}\|=1} \sup_{\boldsymbol{v}\in E, \|\boldsymbol{v}\|=1} ((A - A_\delta)\boldsymbol{u}, \boldsymbol{v}) + \big\|(A - A_\delta)|_E\big\|^2_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})}.$$

It remains to consider the first term in the right side above. By (6.1), (6.2) and (5.7), we have

$$\begin{aligned} ((A - A_\delta)\boldsymbol{u}, \boldsymbol{v}) &= a((A - A_\delta)\boldsymbol{u}, A\boldsymbol{v}) + b((A - A_\delta)\boldsymbol{u}, B\boldsymbol{v}) \\ &= a((A - A_\delta)\boldsymbol{u}, A\boldsymbol{v}) + b((A - A_\delta)\boldsymbol{u}, (B - \pi_\delta B)\boldsymbol{v}), \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}, \\ (\boldsymbol{u} - \boldsymbol{u}, A_\delta\boldsymbol{v}) &= a((A - A_\delta)\boldsymbol{u}, A_\delta\boldsymbol{v}) + b(A_\delta\boldsymbol{v}, (B - B_\delta)\boldsymbol{u}) \\ &= a((A - A_\delta)\boldsymbol{u}, A_\delta\boldsymbol{v}) + b((A_\delta - A)\boldsymbol{v}, (B - \pi_\delta B)\boldsymbol{u}), \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}. \end{aligned}$$

As a result, for any $\boldsymbol{u}, \boldsymbol{v} \in E$ with $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$, one gets from (3.2) and (3.3) that

$$\begin{aligned}((A-A_\delta)\boldsymbol{u}, \boldsymbol{v}) &= a((A-A_\delta)\boldsymbol{u}, (A-A_\delta)\boldsymbol{v}) \\ &\quad + b((A-A_\delta)\boldsymbol{u}, (B-\pi_\delta B)\boldsymbol{v}) + b((A-A_\delta)\boldsymbol{v}, (B-\pi_\delta B)\boldsymbol{u}) \\ &\le \|(A-A_\delta)\boldsymbol{u}\|_1 \|(A-A_\delta)\boldsymbol{v}\|_1 + \|(A-A_\delta)\boldsymbol{u}\|_1 \|(B-\pi_\delta B)\boldsymbol{v}\| \\ &\quad + \|(B-\pi_\delta B)\boldsymbol{u}\| \|(A-A_\delta)\boldsymbol{v}\|_1 \\ &\le \|(A-A_\delta)|_E\|^2_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})} + 2\|(A-A_\delta)|_E\|_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})} \|(B-\pi_\delta B)|_E\|_{\mathscr{L}(\boldsymbol{H},W)},\end{aligned}$$

which completes the proof. □

Finally, we arrive at the convergence theorem of the triangular spectral element method for Stokes eigenvalues.

Theorem 6.1. *Let $E^* = E^*(\lambda) = \{(\boldsymbol{u}, p) = (\lambda A\boldsymbol{u}, \lambda B\boldsymbol{u}) : \|\boldsymbol{u}\| = 1\}$ be the collection of normalized eigenfunctions of* (3.1) *corresponding to λ. Suppose that $(\boldsymbol{u}, p) \in H^{1+s}(\Omega)^2 \times H^s(\Omega)\,(s \ge 0)$ for any $(\boldsymbol{u}, p) \in E^*$. It holds that*

$$|\lambda - \lambda_{i,\delta}| \lesssim h^{2s} M^{-2s} \sup_{(\boldsymbol{u},p)\in E^*(\lambda)} (\|\boldsymbol{u}\|_{s+1} + \|p\|_s)^2, \quad 1 \le i \le m. \tag{6.9}$$

Let $(\boldsymbol{u}_{i,\delta}, p_{i,\delta}) = (\lambda_{i,\delta} A_\delta \boldsymbol{u}, \lambda_{i,\delta} B_\delta \boldsymbol{u})$ with $\|\boldsymbol{u}_{i,\delta}\| = 1$ be an eigenfunction pair of (3.11) *corresponding to $\lambda_{i,\delta}$ for $1 \le i \le m$. There exists $(\boldsymbol{u}, p) \in E^*$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_{i,\delta}\|_1 + M^{-1/2}\|p - p_{i,\delta}\| \lesssim h^s M^{-s} \sup_{(\boldsymbol{u},p)\in E^*} (\|\boldsymbol{u}\|_{s+1} + \|p\|_s); \tag{6.10}$$

moreover, it also holds under the hypothesis (6.4) *that*

$$\|\boldsymbol{u} - \boldsymbol{u}_{i,\delta}\| \lesssim h^{1+s} M^{-1-s} \sup_{(\boldsymbol{u},p)\in E^*} (\|\boldsymbol{u}\|_{s+1} + \|p\|_s). \tag{6.11}$$

Proof. Both (6.9) and (6.11) follow directly from Lemma 6.1, Corollary 5.1 and Theorem 5.2.

Also, a general estimate for $\|\boldsymbol{u} - \boldsymbol{u}_{i,\delta}\|$ without the hypothesis (6.4) can be deduced from (6.8), Corollary 5.1 and Theorem 5.2,

$$\|\boldsymbol{u} - \boldsymbol{u}_{i,\delta}\| \lesssim \|(A - A_\delta)|_E\|_{\mathscr{L}(\boldsymbol{H},\boldsymbol{V})} \lesssim h^s M^{-s} \sup_{(\boldsymbol{u},p)\in E^*} (\|\boldsymbol{u}\|_{s+1} + \|p\|_s). \tag{6.12}$$

Thus by (6.3),

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_{i,\delta}\|_1 + M^{-1/2}\|p - p_{i,\delta}\| &= \|\lambda A\boldsymbol{u} - \lambda_{i,\delta} A_\delta \boldsymbol{u}_{i,\delta}\|_1 + M^{-1/2}\|\lambda B\boldsymbol{u} - \lambda_{i,\delta} B_\delta \boldsymbol{u}_{i,\delta}\| \\ &\lesssim \|\boldsymbol{u} - \Pi_\delta \boldsymbol{u}\|_1 + \big\|p - \pi_\delta p\big\| + \sup_{\boldsymbol{v}\in \boldsymbol{V}_\delta} \frac{(\lambda - \lambda_{i,\delta})(\boldsymbol{u}, \boldsymbol{v}) + \lambda_{i,\delta}(\boldsymbol{u} - \boldsymbol{u}_{i,\delta}, \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \\ &\lesssim \|\boldsymbol{u} - \Pi_\delta \boldsymbol{u}\|_1 + \big\|p - \pi_\delta p\big\| + |\lambda - \lambda_{i,\delta}| + \lambda_{i,\delta}\|\boldsymbol{u} - \boldsymbol{u}_{i,\delta}\|.\end{aligned}$$

Now, a combination of (6.12), (6.9), Corollary 5.1 and Theorem 5.2 gives (6.10). □

7. Numerical results

To illustrate the validation of our theory, in this section we carry out some numerical computation for the discrete inf-sup constant of divergence, and implement the triangular spectral element approximation for the Stokes eigenvalues on both the square and the L-shape domain.

7.1. **Square domain.** We first consider the Stokes eigenvalue problem (1.1) on the square $\Omega = [-1,1] \times [-1,1]$ by the triangular spectral element method (TSEM).

The 5 smallest Stokes eigenvalues by TSEM are tabulated in Table 7.1, Table 7.2 and Table 7.3, which are solved on a special pattern of the uniform triangular meshes shown in Figure 7.2 with $h = 2$, $h = 1$ and $h = 1/2$, respectively. Reference eigenvalues, excerpted from [39], are also given in five decimal digits in the last column of each table.

As indicated in Tables 7.1, 7.2 and 7.3, the discrete eigenvalues converge asymptotically from below to the exact ones as M/h tends to infinity. Moreover, for a fixed mesh size h, the computational eigenvalues using a small M can even attain the same precision as the reference ones.

We now take 13.0861727921039 as the reference value of the first eigenvalue λ_1 of (1.1) on $\Omega = [-1,1]^2$, which, together with the reference value of its eigenfunction $(\boldsymbol{u}_1, p_1)$, is obtained by the classic $\mathbb{Q}_M - \mathbb{Q}_{M-2}$ rectangular spectral method with the separate polynomial degree $M = 512$. This number is believed to be 5-digit more accurate than the best one 13.086172775 [12,57,58] available so far in the literature. The relative errors $|\lambda_1 - \lambda_{1,\delta}|/\lambda_1$ versus M/h are then plotted in Figure 7.1 in semi-log scale for $h = 2$, $h = 1$ and $h = 1/2$, respectively. Since the Stokes eigenfunctions are sufficiently smooth on $\Omega = [-1,1]^2$, nearly exponential convergence rates are observed in all three cases. Moreover, the relative errors $|\lambda_1 - \lambda_{1,\delta}|/\lambda_1$ versus h with a fixed polynomial degree are shown in Figure 7.3 in log-log scale from which the algebraic convergence with respect to mesh size h is observed.

Let $(\boldsymbol{u}_{1,\delta}, p_{1,\delta})$ be the eigenfunction pair corresponding to $\lambda_{1,\delta}$. Then the convergence behaviour of the velocity, more precisely, the L^2 error $\|\boldsymbol{u}_1 - \boldsymbol{u}_{1,\delta}\|$ and H^1 error $|\boldsymbol{u}_1 - \boldsymbol{u}_{1,\delta}|_1$ versus M, is depicted in Figure 7.4. Exponential convergence rates are also observed.

TABLE 7.1. Numerical eigenvalues λ_δ on the mesh of size $h = 2$ in Figure 7.2 (left).

Mode/M	4	8	16	32	ref
1	12.366543	13.086109	13.086173	13.086173	13.086
2	22.682809	23.029943	23.031098	23.031098	23.031
3	22.682809	23.029943	23.031098	23.031098	23.031
4	33.779867	32.045151	32.052396	32.052396	32.053
5	38.118378	38.524360	38.531366	38.531366	38.532

TABLE 7.2. Numerical eigenvalues λ_δ on the mesh of size $h = 1$ in Figure 7.2 (center).

Mode/M	4	8	16	32	ref
1	13.073134	13.086173	13.086173	13.086173	13.086
2	23.022465	23.031098	23.031098	23.031098	23.031
3	23.022465	23.031098	23.031098	23.031098	23.031
4	31.912263	32.052381	32.052396	32.052396	32.053
5	38.580621	38.531362	38.531366	38.531366	38.532

TABLE 7.3. Numerical eigenvalues λ_δ on the mesh of size $h = 1/2$ in Figure 7.2 (right).

Mode/M	4	6	8	12	16	ref
1	13.085414	13.086173	13.086173	13.086173	13.086173	13.086
2	23.029124	23.031096	23.031098	23.031098	23.031098	23.031
3	23.029124	23.031096	23.031098	23.031098	23.031098	23.031
4	32.045437	32.052375	32.052396	32.052396	32.052396	32.053
5	38.530045	38.531365	38.531366	38.531366	38.531366	38.532

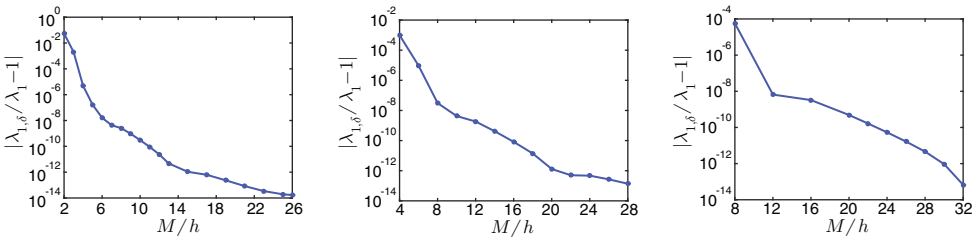

FIGURE 7.1. Relative errors $|1 - \lambda_{1,\delta}/\lambda_1|$ versus M/h in semi-log scale for TSEM on $[-1,1]^2$ with the meshes corresponding to Figure 7.2 from left to right.

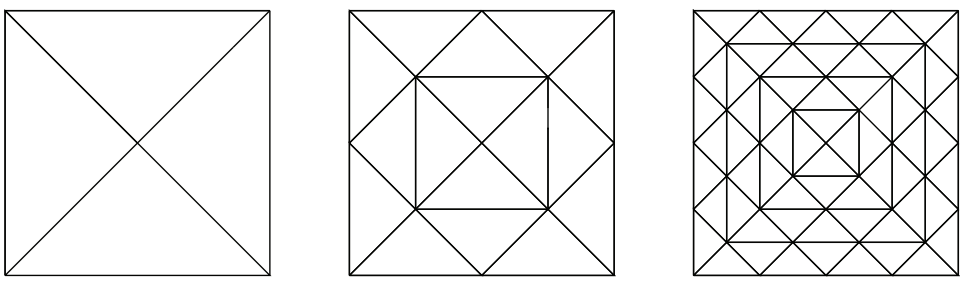

FIGURE 7.2. Uniform triangulations on $\Omega = [-1,1]^2$ with mesh size $h = 2$ (left), $h = 1$ (center) and $h = 1/2$ (right).

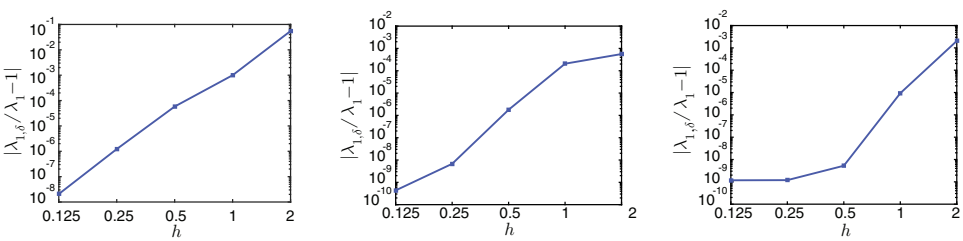

FIGURE 7.3. Relative errors $|1 - \lambda_{1,\delta}/\lambda_1|$ versus h in log-log scale for TSEM on $[-1,1]^2$ with $M = 4$ (left), $M = 5$ (center) and $M = 6$ (right).

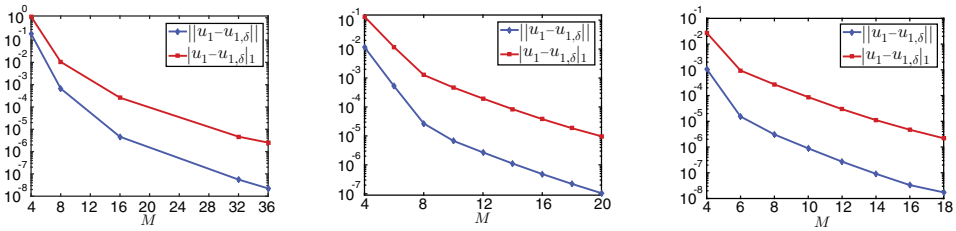

FIGURE 7.4. L^2- and H^1-errors of the velocity versus M in semi-log scale for TSEM on $[-1,1]^2$ with the meshes corresponding to Figure 7.2 from left to right.

7.2. **L-shaped domain.** In our second example, we consider the Stokes eigenvalue (1.1) on the L-shaped domain $\Omega = [-1,1]^2 \backslash [0,1]^2$, which is obtained by removing the top right quadrant of the square. The L-shaped domain has one reentrant corner, which may induce the corner singularity of the eigenfunctions and thus admits only a limited convergence rate for some eigenvalues. In particular, the eigenfunction $\boldsymbol{u}_1$ corresponding to λ_1 exhibits a singularity of type $|x|^\eta$, where $\eta \approx 0.5445$ is the smallest positive imaginary part of the solutions $z \neq +\mathrm{i}$, of the equation $\mathrm{sh}^2(3z/2) = z^2$ [8, Remark 23.3 and 2.1]. As a result, $(\boldsymbol{u}_1, p_1) \in H^{\eta+1-\varepsilon}(\Omega)^2 \times H^{\eta-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ [8, Remark 23.3 and 2.1], while $(\boldsymbol{u}_1, p_1) \in B^{2\eta+1}(\Omega)^2 \times B^{2\eta}(\Omega)$ [6] with $B^s(\Omega)$ being certain s-order Besov space. This indicates that the convergence rate of $\lambda_{1,\delta}$ for the h-version FEM does not exceed order 2η, while the spectral element method and hp-FEM will exactly attain the order 4η.

Reference values of the Stokes eigenvalues are obtained by the TSEM using 15 levels of geometric meshes, just as indicated in Figure 7.6. Specifically, $\lambda_1 \approx 32.132693, \lambda_2 \approx 37.018334, \lambda_3 \approx 41.939832, \lambda_4 \approx 48.983584$ and $\lambda_5 \approx 55.415426$. All the (rounded) digits are confirmed by the hp-FEM on the specific geometric meshes [4], which are gradually refined around the origin with an optimal mesh size ratio $q \approx 0.1716$. The local polynomial degree of each element increases linearly away from the origin such that this specific hp-FEM has an exponential convergence rate [4] and offers the best approach so far for handling such a reentrant corner singularity.

A low accuracy is shown in Figure 7.7 for the relative errors of the 5 smallest eigenvalues, among which an asymptotic convergence rate around order $4\eta \approx 2.2$ is observed for the first, third and fifth eigenvalues and an order around 7 to 8 for the second and fourth eigenvalues. Hence, our numerical results are in good agreement with the theoretical estimates.

7.3. **Discrete inf-sup constant.** Making use of the basis functions defined in Section 3, the algebraic equations corresponding to the discrete Stokes eigenvalue problem (3.11) can be written under a matrix form:

$$\boldsymbol{A}\underline{u}_\delta + \boldsymbol{E}\underline{p}_\delta = \lambda_\delta \boldsymbol{B}\underline{u}_\delta, \tag{7.1}$$

$$\boldsymbol{E}^\mathsf{T}\underline{u}_\delta = 0, \tag{7.2}$$

where $\boldsymbol{A}$ and $\boldsymbol{E}$ are matrices corresponding to discrete Laplace operator and discrete divergence operator respectively, and $\boldsymbol{B}$ is the mass matrix. Moreover, let $\tilde{\boldsymbol{B}}$ be

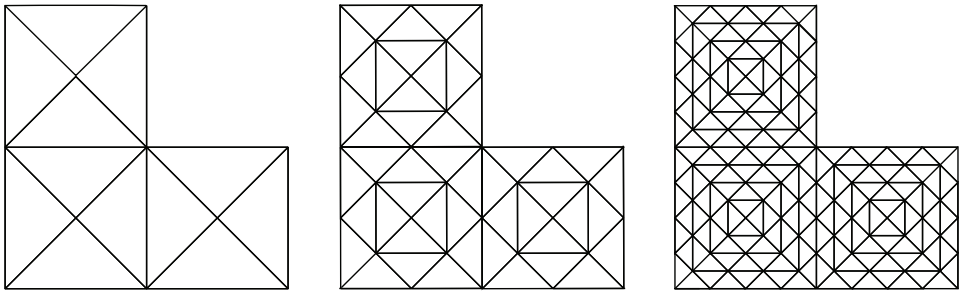

FIGURE 7.5. Uniform triangulations on L-shaped domain with mesh size $h = 1$ (left), $h = 1/2$ (center), $h = 1/4$ (right).

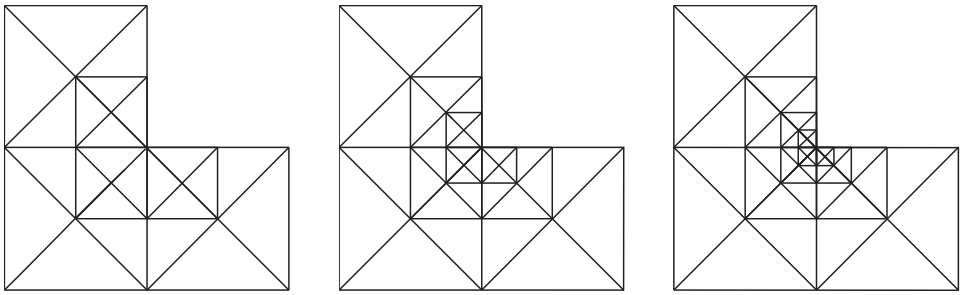

FIGURE 7.6. Geometric meshes of level 2, 3 and 4 from left to right.

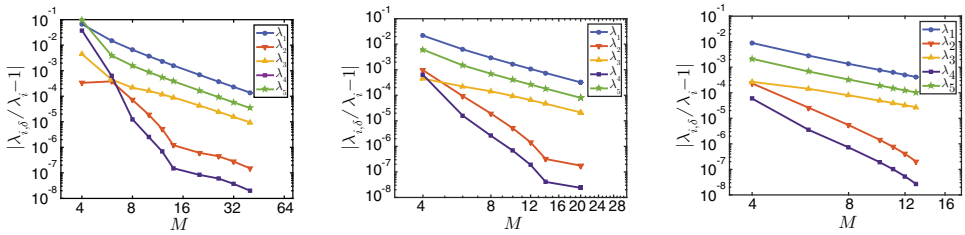

FIGURE 7.7. Relative errors of the five smallest eigenvalues on L-shaped domain by the TSEM on the meshes corresponding to Figure 7.5 from left to right. M denotes the polynomial degree of the discrete velocity space on each element.

the mass matrix associated with the pressure subspace, i.e.,

$$\tilde{\boldsymbol{B}} = [(\mathcal{J}_k^{0,0,0} \circ F_m^{-1}, \mathcal{J}_l^{0,0,0} \circ F_m^{-1})_{T_m}]$$

for all admissible k, l and $T_m \in \mathcal{T}$. Define

$$\boldsymbol{S} := \boldsymbol{E}^{\mathsf{T}} \boldsymbol{A}^{-1} \boldsymbol{E} \tag{7.3}$$

and let $\lambda_{min}^S, \lambda_{max}^S$ be the minimum and maximum eigenvalues of the algebraic eigenvalue system $\boldsymbol{S} = \lambda \tilde{\boldsymbol{B}}$. Then referring to [40], [2] and [50], we know that

$$\lambda_{max}^S = 1, \quad \lambda_{min}^S = \beta_\delta^2. \tag{7.4}$$

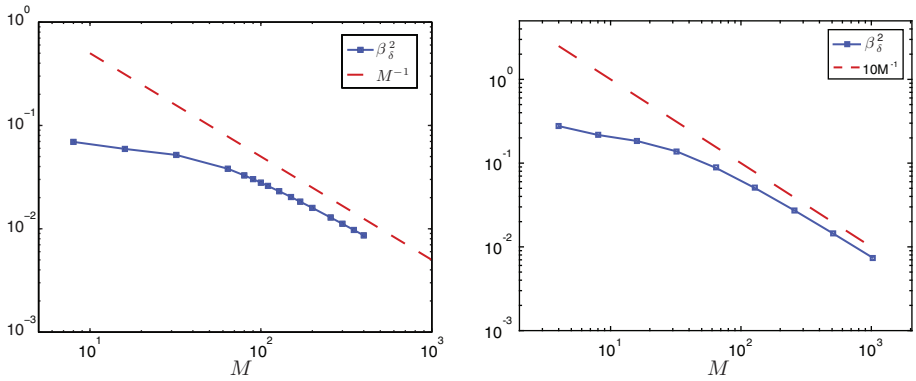

FIGURE 7.8. $\lambda_{min}^S = \beta_\delta^2$ versus M in log-log scale. Left: the mono-domain TSEM on $\Omega = \hat{T}$; Right: the rectangular spectral method on $\Omega = [-1,1]^2$.

Numerical results are then reported for the mono-domain case. In Figure 7.8, we plot β_δ^2 on $\Omega = \hat{T}$ versus M in log-log scale which states a slope tending to M^{-1}. As a comparison, the square of the discrete inf-sup constant β_δ^2 versus the separate polynomial degree M is also plotted for the $\mathbb{Q}_M - \mathbb{Q}_{M-2}$ spectral method on $\Omega = [-1,1]^2$. In both cases, the discrete inf-sup constant β_δ decays as $M^{-1/2}$ asymptotically for large M. The result coincides with Theorem 4.3, which, in turn, shows the sharpness of our estimate. Futhermore, a pre-asymptotic phenomena is observed for $M \leq 100$, which reflects a transition process to the *hp* behaviour.

CONCLUDING REMARKS

In summary, we have proposed the triangular spectral element method for the Stokes eigenvalue problems with Dirichlet boundary conditions based on the generalized Koornwinder polynomials, then emphasis has been set on the comprehensive rigorous numerical analysis on the triangular spectral element method.

The fundamental tool in our numerical analysis is the truncated Fourier series $\pi_M^{-1,-1,-1}u$ in the orthogonal Koornwinder polynomials $\mathcal{J}_l^{-1,-1,-1}$, $l \in \mathbb{N}_0^2$, which serves as an orthogonal projection of u on the reference triangle $\hat{T}$. An orthogonality analysis further reveals that $\pi_M^{-1,-1,-1}u$ is actually a Fortin interpolation in $H_0^1(\hat{T})$ whose norm $\lesssim \sqrt{M}$. Following the Fortin's criterion together with the Boland-Nicoliades local-to-global argument, we finally derive a sharp estimate on the discrete inf-sup constant of the divergence.

Since the local projection on each triangular element interpolates the function u at its vertices and its trace on each side of the triangle reduces to a common type of the one-dimensional orthogonal projection. The local-to-global mapping is then simply defined as the union of all the local projections such that the resulting global projection is a naturally continuous piecewise polynomial which satisfies the Dirichlet boundary conditions and has the optimal error estimate. Taking the global projection as the intermediate piecewise polynomial, we have further obtained the optimal error estimate of the H^1-orthogonal spectral element projection from the well-known projection theorem.

As a result, the optimal error estimate of our triangular spectral element method for Stokes eigenvalues finally follows from the spectral theory on the mixed methods for variationally posed eigenvalue problems. More importantly, inspired by the classic approximation theory that the orthogonal projection is independent of a particular orthogonal polynomial basis, we have proved that the local orthogonal projection $\pi_M^{-1,-1,-1}u$ is rotationally invariant. In some sense, this allows us to perform a numerical analysis direction by direction on an arbitrary triangle in a similar way as on a tensorial domain, which, in return, largely simplifies our work on the numerical analysis of the triangular spectral element method. Heuristically, one can develop the tetrahedral spectral element method in analogue and extend the theory in the current paper into three or even higher dimensions.

Appendix A. The proof of Lemma 2.2

Proof of Lemma 2.2. By (2.20) and (2.21),

$$\begin{aligned}
&(\partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l, \partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_k)_{w^{1,0,0},\hat{T}} + (\partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l, \partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{J}}_k)_{w^{0,1,0},\hat{T}} \\
=&4(2|l|-1)(2|k|-1) \\
&\quad\times((l_1-1)\mathcal{J}^{1,0,0}_{l_1-2,l_2} + l_1\mathcal{J}^{1,0,0}_{l_1-1,l_2-1}, (k_1-1)\mathcal{J}^{1,0,0}_{k_1-2,k_2} + k_1\mathcal{J}^{1,0,0}_{k_1-1,k_2-1})_{w^{1,0,0},\hat{T}} \\
&+4(2|l|-1)(2|k|-1) \\
&\quad\times((l_1-1)\mathcal{J}^{0,1,0}_{l_1-2,l_2} - l_1\mathcal{J}^{0,1,0}_{l_1-1,l_2-1}, (k_1-1)\mathcal{J}^{0,1,0}_{k_1-2,k_2} - k_1\mathcal{J}^{0,1,0}_{k_1-1,k_2-1})_{w^{0,1,0},\hat{T}}.
\end{aligned}$$

Using the symmetry property (2.12) of $\mathcal{J}_l^{0,1,0}$ together with exchange of the order of integration, we further get that

$$\begin{aligned}
&(\partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l, \partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_k)_{w^{1,0,0},\hat{T}} + (\partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{J}}_l, \partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{J}}_k)_{w^{0,1,0},\hat{T}} \\
=&4(2|l|-1)(2|k|-1)(1+(-1)^{l_1+k_1})\times((l_1-1)\mathcal{J}^{1,0,0}_{l_1-2,l_2} + l_1\mathcal{J}^{1,0,0}_{l_1-1,k_2-1}, \\
&\quad (k_1-1)\mathcal{J}^{1,0,0}_{k_1-2,k_2} + k_1\mathcal{J}^{1,0,0}_{k_1-1,k_2-1})_{w^{1,0,0},\hat{T}} \\
=&8(2|l|-1)^2\left[(l_1-1)^2\gamma^{1,0,0}_{l_1-2,l_2} + l_1^2\gamma^{1,0,0}_{l_1-1,l_2-1}\right]\delta_{l,k},
\end{aligned} \tag{A.1}$$

where the last equality sign is derived from the orthogonality relation (2.11).

Further, by (2.18), (2.10) and (2.4), we have

$$\begin{aligned}
\partial_{\hat{x}}\tilde{\mathcal{J}}_l(\hat{x},0) =& -(2|l|-1)\left[(2-\delta_{l_1,1})\mathcal{J}^{0,-1,0}_{l_1-1,l_2}(\hat{x},0) + (2-\delta_{l_1,0})\mathcal{J}^{0,-1,0}_{l_1,l_2-1}(\hat{x},0)\right] \\
=& -(2|l|-1)\left[\delta_{l_1,1}J^{0,0}_{l_2}(1-2\hat{x}) + \delta_{l_1,0}J^{0,0}_{l_2-1}(1-2\hat{x})\right] \\
=& -(2|l|-1)(\delta_{l_1,1}+\delta_{l_1,0})J^{0,0}_{|l|-1}(1-2\hat{x}).
\end{aligned} \tag{A.2}$$

In analogy,

$$\partial_{\hat{y}}\tilde{\mathcal{J}}_l(0,\hat{y}) = (-1)^{l_1+1}(2|l|-1)(\delta_{l_1,1}+\delta_{l_1,0})J^{0,0}_{|l|-1}(1-2\hat{y}). \tag{A.3}$$

Thus we obtain that

(A.4)

$$
\begin{aligned}
&(\partial_{\hat{x}}\tilde{\mathcal{J}}_l, \partial_{\hat{x}}\tilde{\mathcal{J}}_k)_{\Gamma_2} + (\partial_{\hat{y}}\tilde{\mathcal{J}}_l, \partial_{\hat{y}}\tilde{\mathcal{J}}_k)_{\Gamma_1} \\
&=\frac{1}{2}(2|l|-1)(2|k|-1)(\delta_{l_1,0}+\delta_{l_1,1})(\delta_{k_1,0}+\delta_{k_1,1})(J^{0,0}_{|l|-1}, J^{0,0}_{|k|-1})_I \\
&\quad + \frac{1}{2}(2|l|-1)(2|k|-1)(\delta_{l_1,0}+\delta_{l_1,1})(\delta_{k_1,0}+\delta_{k_1,1})(-1)^{l_1+k_1}(J^{0,0}_{|l|-1}, J^{0,0}_{|k|-1})_I \\
&=\frac{1}{2}(2|l|-1)(2|k|-1)(\delta_{l_1,0}+\delta_{l_1,1})(\delta_{k_1,0}+\delta_{k_1,1}) \times (1+(-1)^{l_1+k_1})\gamma^{0,0}_{|l|-1}\delta_{|l|,|k|} \\
&=(2|l|-1)^2\gamma^{0,0}_{|l|-1}(\delta_{l_1,0}+\delta_{l_1,1})\delta_{l,k}.
\end{aligned}
$$

Moreover, it can be easily seen from the definition (2.15) of $\tilde{\mathcal{J}}_l$ that

$$
\tilde{\mathcal{J}}_l(0,0)\tilde{\mathcal{J}}_k(0,0) = \delta_{l,0}\delta_{k,0}. \tag{A.5}
$$

Combining (A.1), (A.4) and (A.5), we finally get that

$$
\begin{aligned}
&\langle \tilde{\mathcal{J}}_l, \tilde{\mathcal{J}}_k\rangle_{\hat{T}} = \gamma_l\delta_{l,k}, \\
&\gamma_l := 8(2|l|-1)^2\Big[(l_1-1)^2\gamma^{1,0,0}_{l_1-2,l_2} + l_1^2\gamma^{1,0,0}_{l_1-1,l_2-1}\Big] \\
&\qquad + (2|l|-1)^2\gamma^{0,0}_{|l|-1}(\delta_{l_1,0}+\delta_{l_1,1}) + \delta_{l,0} > 0,
\end{aligned}
$$

where the last inequality sign is derived from the fact that, for $d \in \mathbb{N}$ and $\alpha \in (-1,\infty)^{d+1}$, $\gamma^\alpha_l > 0$ if $l \in \mathbb{N}_0^d$ and $\gamma^\alpha_l = 0$ otherwise. This completes the proof. □

Appendix B. The proof of Lemma 4.2

Proof of Lemma 4.2. Owing to the orthogonality of $\{\tilde{\mathcal{J}}_l\}_{l\in\mathbb{N}_0^2}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\hat{T}}$ in $H^2(\hat{T})$, the validation of the third equality sign of (4.6) is obvious:

$$
u = \sum_{l\in\mathbb{N}_0^2}\hat{u}_l\tilde{\mathcal{J}}_l, \quad \pi_M^{-1,-1,-1}u = \sum_{|l|\le M}\hat{u}_l\tilde{\mathcal{J}}_l, \qquad \hat{u}_l = \frac{\langle u, \tilde{\mathcal{J}}_l\rangle_{\hat{T}}}{\gamma_l}.
$$

As a result,

$$
\begin{aligned}
&u(\hat{x},\hat{y}) = u^{\mathcal{R}}(\hat{y}, 1-\hat{x}-\hat{y}) = \sum_{l\in\mathbb{N}_0^2}\widehat{u_l^{\mathcal{R}}}\tilde{\mathcal{J}}_l(\hat{y}, 1-\hat{x}-\hat{y}) = \sum_{l\in\mathbb{N}_0^2}\widehat{u_l^{\mathcal{R}}}\tilde{\mathcal{K}}_l(\hat{x},\hat{y}), \\
&\widehat{u_l^{\mathcal{R}}} = \frac{\langle u^{\mathcal{R}}, \tilde{\mathcal{J}}_l\rangle_{\hat{T}}}{\gamma_l},
\end{aligned}
$$

which, in return, implies that

$$
\big(\pi_M^{-1,-1,-1}u^{\mathcal{R}}\big)^{\mathcal{K}}(\hat{x},\hat{y}) = \big(\pi_M^{-1,-1,-1}u^{\mathcal{R}}\big)(\hat{y}, 1-\hat{x}-\hat{y}) = \sum_{|l|\le M}\widehat{u_l^{\mathcal{R}}}\tilde{\mathcal{K}}_l(\hat{x},\hat{y}).
$$

According to the definition (4.5) of $\pi_M^{-1,-1,-1}$, an equivalent statement for the first and the fourth equality signs in (4.6) then reads

$$
\langle u - \big(\pi_M^{-1,-1,-1}u^{\mathcal{R}}\big)^{\mathcal{K}}, \tilde{\mathcal{J}}_k\rangle_{\hat{T}} = 0, \quad k\in\mathbb{N}_0^2,\ |k|\le M.
$$

Thus it suffices to prove that $\tilde{\mathcal{K}}_l$, $|l|\ge 2$ is a linear combination of $\tilde{\mathcal{J}}_k, |k| = |l|$.

By (2.16), (2.22) and (2.13),

$$\begin{aligned}\partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{K}}_l(\hat{x},\hat{y}) &= \partial_{\hat{x}}\partial_{\hat{z}}\big[\tilde{\mathcal{J}}_l(\hat{y},1-\hat{x}-\hat{y})\big]\\ &= -(2|l|-1)\big[(|l|+l_1)\mathcal{J}^{0,0,1}_{l_1,l_2-2}(\hat{y},1-\hat{x}-\hat{y}) - (|l|+l_1-2)\mathcal{J}^{0,0,1}_{l_1-2,l_2}(\hat{y},1-\hat{x}-\hat{y})\big]\\ &= -(2|l|-1)\big[(|l|+l_1)\mathcal{K}^{1,0,0}_{l_1,l_2-2}(\hat{x},\hat{y}) - (|l|+l_1-2)\mathcal{K}^{1,0,0}_{l_1-2,l_2}(\hat{x},\hat{y})\big].\end{aligned}$$

Similarly,

$$\partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{K}}_l(\hat{x},\hat{y}) = 2(2|l|-1)\big[l_1\mathcal{K}^{0,1,0}_{l_1-1,l_2-1}(\hat{x},\hat{y}) + (l_1-1)\mathcal{K}^{0,1,0}_{l_1-2,l_2}(\hat{x},\hat{y})\big].$$

The two equations above together with the orthogonality of both $\{\mathcal{J}_l^\alpha\}$ and $\{\mathcal{K}_l^\alpha\}$ in $L^2_{w^\alpha}(\hat{T})$ for $\alpha\in(-1,\infty)^3$ imply that

$$(\partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{K}}_l,\partial_{\hat{x}}\partial_{\hat{z}}\tilde{\mathcal{J}}_k)_{w^{1,0,0},\hat{T}} + (\partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{K}}_l,\partial_{\hat{y}}\partial_{\hat{z}}\tilde{\mathcal{J}}_k)_{w^{0,1,0},\hat{T}} = 0,\quad |l|\neq|k|. \tag{B.1}$$

Further, by (2.16), (2.19), (2.10), (2.5) and (2.7) successively,

$$\begin{aligned}\partial_{\hat{x}}\tilde{\mathcal{K}}_l(\hat{x},0) &= \partial_{\hat{x}}\tilde{\mathcal{J}}_l(0,1-\hat{x})\\ &=(2|l|-1)\left[(2-\delta_{l_1,0})\mathcal{J}^{-1,0,0}_{l_1,l_2-1}(0,1-\hat{x}) - (2-\delta_{l_1,1})\mathcal{J}^{-1,0,0}_{l_1-1,l_2}(0,1-\hat{x})\right]\\ &=(2|l|-1)\left[\delta_{l_1,0}J^{0,0}_{l_2-1}(2\hat{x}-1) - \delta_{l_1,1}J^{0,0}_{l_2}(2\hat{x}-1)\right]\\ &=(-1)^{l_2+1}(2|l|-1)(\delta_{l_1,1}+\delta_{l_1,0})J^{0,0}_{|l|-1}(1-2\hat{x}).\end{aligned} \tag{B.2}$$

In analogy,

$$\partial_{\hat{y}}\tilde{\mathcal{K}}_l(0,\hat{y}) = -\frac{2(2l_1-1)(2|l|-1)}{|l|+l_1-1}\delta_{l_2,0}J^{0,0}_{|l|-1}(1-2\hat{y}). \tag{B.3}$$

Owing to the orthogonality of the Jacobi polynomials $\{J^{0,0}_n,\, n\in\mathbb{N}_0\}$, we derive from (A.2), (B.2), (A.3) and (B.3) that

$$(\partial_{\hat{x}}\tilde{\mathcal{K}}_l,\partial_{\hat{x}}\tilde{\mathcal{J}}_k)_{\Gamma_2} + (\partial_{\hat{y}}\tilde{\mathcal{K}}_l,\partial_{\hat{y}}\tilde{\mathcal{J}}_k)_{\Gamma_1} = 0,\quad |l|\neq|k|. \tag{B.4}$$

Moreover, it can be easily seen from (2.15) and (2.16) that

$$\tilde{\mathcal{K}}_l(0,0)\tilde{\mathcal{J}}_k(0,0) = 0,\quad |l|\geq 2. \tag{B.5}$$

Finally, a combination of (B.1), (B.4), (B.5) yields

$$\langle\tilde{\mathcal{K}}_l,\tilde{\mathcal{J}}_k\rangle_{\hat{T}} = 0,\quad |l|\neq|k|,$$

which implies $\tilde{\mathcal{K}}_l, |l|\geq 2$, is a linear combination of $\tilde{\mathcal{J}}_k, |k|=|l|$.

In analogy, we prove that $\tilde{\mathcal{R}}_l, |l|\geq 2$, is also a linear combination of $\tilde{\mathcal{J}}_k, |k|=|l|$, which leads to the second and the fifth equality sign in (4.6). The proof is now completed. □

References

[1] M. G. Armentano and V. Moreno, *A posteriori error estimates of stabilized low-order mixed finite elements for the Stokes eigenvalue problem*, J. Comput. Appl. Math. **269** (2014), 132–149, DOI 10.1016/j.cam.2014.03.027. MR3197278

[2] M. Azaiez, J. Shen, C. Xu, and Q. Zhuang, *A Laguerre-Legendre spectral method for the Stokes problem in a semi-infinite channel*, SIAM J. Numer. Anal. **47** (2008/09), no. 1, 271–292, DOI 10.1137/070698269. MR2475939

[3] I. Babuška, *The finite element method with Lagrangian multipliers*, Numer. Math. **20** (1972/73), 179–192. MR0359352

[4] I. Babuška and B. Guo, *Approximation properties of the h-p version of the finite element method*, Computer Methods in Applied Mechanics and Engineering **1133** (1996), no. 3-4, 319–346.

[5] I. Babuška and J. Osborn, *Eigenvalue Problems*, Handbook of Numerical Analysis, vol. 2, Elsevier, 1991, pp. 641–787.

[6] I. Babuška and B. Guo, *Direct and inverse approximation theorems for the p-version of the finite element method in the framework of weighted Besov spaces. II. Optimal rate of convergence of the p-version finite element solutions*, Math. Models Methods Appl. Sci. **12** (2002), no. 5, 689–719, DOI 10.1142/S0218202502001854. MR1909423

[7] C. Bernardi, C. Canuto, and Y. Maday, *Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem*, SIAM J. Numer. Anal. **25** (1988), no. 6, 1237–1271, DOI 10.1137/0725070. MR972452

[8] C. Bernardi and Y. Maday, *Spectral methods*, Handb. Numer. Anal., V, North-Holland, Amsterdam, 1997, pp. 209–485, DOI 10.1016/S1570-8659(97)80003-8. MR1470226

[9] S. Beuchler and V. Pillwein, *Completions to sparse shape functions for triangular and tetrahedral p-FEM*, Domain decomposition methods in science and engineering XVII, Lect. Notes Comput. Sci. Eng., vol. 60, Springer, Berlin, 2008, pp. 435–442, DOI 10.1007/978-3-540-75199-1_55. MR2436111

[10] S. Beuchler, V. Pillwein, J. Schöberl, and S. Zaglmayr, *Sparsity optimized high order finite element functions on simplices*, Numerical and Symbolic Scientific Computing, Texts Monogr. Symbol. Comput., Springer, New York and Vienna, 2012, pp. 21–44, DOI 10.1007/978-3-7091-0794-2_2. MR3060506

[11] S. Beuchler and J. Schöberl, *New shape functions for triangular p-FEM using integrated Jacobi polynomials*, Numer. Math. **103** (2006), no. 3, 339–366, DOI 10.1007/s00211-006-0681-2. MR2221053

[12] P. E. Bjørstad and B. P. Tjøstheim, *High precision solutions of two fourth order eigenvalue problems*, Computing **63** (1999), no. 2, 97–107, DOI 10.1007/s006070050053. MR1736662

[13] J. M. Boland and R. A. Nicolaides, *Stability of finite elements under divergence constraints*, SIAM J. Numer. Anal. **20** (1983), no. 4, 722–731, DOI 10.1137/0720048. MR708453

[14] J. P. Boyd, *Chebyshev and Fourier spectral methods*, 2nd ed., Dover Publications, Inc., Mineola, NY, 2001. MR1874071

[15] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, 2007.

[16] J. H. Bramble, *A proof of the inf-sup condition for the Stokes equations on Lipschitz domains*, Math. Models Methods Appl. Sci. **13** (2003), no. 3, 361–371, DOI 10.1142/S0218202503002544. MR1977631

[17] F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers* (English, with loose French summary), Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8** (1974), no. R-2, 129–151. MR0365287

[18] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Scientific Computation, Springer-Verlag, Berlin, 2006. MR2223552

[19] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Evolution to Complex Geometries and Applications to Fluid Dynamics*, Scientific Computation, Springer, Berlin, 2007. MR2340254

[20] W. Chen and Q. Lin, *Approximation of an eigenvalue problem associated with the Stokes problem by the stream function-vorticity-pressure method*, Appl. Math. **51** (2006), no. 1, 73–88, DOI 10.1007/s10492-006-0006-x. MR2197324

[21] A. Chernov, *Optimal convergence estimates for the trace of the polynomial L^2-projection operator on a simplex*, Math. Comp. **81** (2012), no. 278, 765–787, DOI 10.1090/S0025-5718-2011-02513-5. MR2869036

[22] L. K. Chilton, *Locking-free mixed hp finite element methods for linear and geometrically nonlinear elasticity*, ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)–University of Maryland, Baltimore County. MR2696632

[23] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001. MR1827871

[24] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Series in Computational Mathematics, vol. 15, Springer-Verlag, New York, 1991. MR1115205

[25] B. Guo and I. Babuška, *Local Jacobi operators and applications to the p-version of finite element method in two dimensions*, SIAM J. Numer. Anal. **48** (2010), no. 1, 147–163, DOI 10.1137/090747208. MR2608363

[26] B.-Y. Guo and L.-L. Wang, *Error analysis of spectral method on a triangle*, Adv. Comput. Math. **26** (2007), no. 4, 473–496, DOI 10.1007/s10444-005-7471-8. MR2291668

[27] B.-Y. Guo, J. Shen, and L.-L. Wang, *Optimal spectral-Galerkin methods using generalized Jacobi polynomials*, J. Sci. Comput. **27** (2006), no. 1-3, 305–322, DOI 10.1007/s10915-005-9055-7. MR2285783

[28] B.-Y. Guo, J. Shen, and L.-L. Wang, *Generalized Jacobi polynomials/functions and their applications*, Appl. Numer. Math. **59** (2009), no. 5, 1011–1028, DOI 10.1016/j.apnum.2008.04.003. MR2495135

[29] J. Hu and Y. Huang, *Lower bounds for eigenvalues of the Stokes operator*, Adv. Appl. Math. Mech. **5** (2013), no. 1, 1–18. MR3021142

[30] P. Huang, *Lower and upper bounds of Stokes eigenvalue problem based on stabilized finite element methods*, Calcolo **52** (2015), no. 1, 109–121, DOI 10.1007/s10092-014-0110-3. MR3313590

[31] S. Jia, H. Chen, and H. Xie, *A posteriori error estimator for eigenvalue problems by mixed finite element method*, Sci. China Math. **56** (2013), no. 5, 887–900, DOI 10.1007/s11425-013-4614-0. MR3047040

[32] G. E. Karniadakis and S. J. Sherwin, *Spectral/hp Element Methods for Computational Fluid Dynamics*, 2nd ed., Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2005. MR2165335

[33] T. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and application of special functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975), Academic Press, New York, 1975, pp. 435–495. MR0402146

[34] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969. MR0254401

[35] R. B. Lehoucq, *Implicitly restarted Arnoldi methods and subspace iteration*, SIAM J. Matrix Anal. Appl. **23** (2001), no. 2, 551–562 (electronic), DOI 10.1137/S0895479899358595. MR1871329

[36] H. Li and J. Shen, *Optimal error estimates in Jacobi-weighted Sobolev spaces for polynomial approximations on the triangle*, Math. Comp. **79** (2010), no. 271, 1621–1646, DOI 10.1090/S0025-5718-09-02308-4. MR2630005

[37] H. Li and Y. Xu, *Spectral approximation on the unit ball*, SIAM J. Numer. Anal. **52** (2014), no. 6, 2647–2675, DOI 10.1137/130940591. MR3276427

[38] H. Liu, W. Gong, S. Wang, and N. Yan, *Superconvergence and a posteriori error estimates for the Stokes eigenvalue problems*, BIT **53** (2013), no. 3, 665–687, DOI 10.1007/s10543-013-0422-8. MR3095260

[39] C. Lovadina, M. Lyly, and R. Stenberg, *A posteriori estimates for the Stokes eigenvalue problem*, Numer. Methods Partial Differential Equations **25** (2009), no. 1, 244–257, DOI 10.1002/num.20342. MR2473688

[40] Y. Maday, D. Meiron, A. T. Patera, and E. M. Rønquist, *Analysis of iterative methods for the steady and unsteady Stokes problem: application to spectral element discretizations*, SIAM J. Sci. Comput. **14** (1993), no. 2, 310–337, DOI 10.1137/0914020. MR1204233

[41] Y. Maday, A. Patera, and E. Rønquist, *A well-posed optimal Legendre spectral element approximation of the Stokes semi-periodic problem*, ICASE Report (1987), no. 87-48.

[42] Y. Maday and A. Patera, *Spectral element methods for the incompressible Navier-Stokes equations*, IN: State-of-the-art surveys on computational mechanics (A90-47176 21-64). New York, American Society of Mechanical Engineers, 1989, p. 71-143. Research supported by DARPA., vol. 1, 1989, pp. 71–143.

[43] J. M. Melenk and T. Wurzer, *On the stability of the boundary trace of the polynomial L^2-projection on triangles and tetrahedra*, Comput. Math. Appl. **67** (2014), no. 4, 944–965, DOI 10.1016/j.camwa.2013.12.016. MR3163888

[44] B. Mercier, J. Osborn, J. Rappaz, and P.-A. Raviart, *Eigenvalue approximation by mixed and hybrid methods*, Math. Comp. **36** (1981), no. 154, 427–453, DOI 10.2307/2007651. MR606505

[45] Richard Pasquetti and Francesca Rapetti, *Spectral element methods on triangles and quadrilaterals: comparisons and applications*, Journal of Computational Physics **198** (2004), 349–362.

[46] Anthony T Patera, *A spectral element method for fluid dynamics: laminar flow in a channel expansion*, Journal of computational Physics **54** (1984), no. 3, 468–488.

[47] M. D. Samson, H. Li, and L.-L. Wang, *A new triangular spectral element method I: implementation and analysis on a triangle*, Numer. Algorithms **64** (2013), no. 3, 519–547, DOI 10.1007/s11075-012-9677-4. MR3120004

[48] Ch. Schwab, *p- and hp-Finite Element Methods*: *Theory and Applications in Solid and Fluid Mechanics*, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 1998. MR1695813

[49] C. Schwab and M. Suri, *Mixed hp finite element methods for Stokes and non-Newtonian flow*, Comput. Methods Appl. Mech. Engrg. **175** (1999), no. 3-4, 217–241, DOI 10.1016/S0045-7825(98)00355-7. MR1702217

[50] J. Shen, *On fast direct Poisson solver,* inf-sup *constant and iterative Stokes solver by Legendre-Galerkin method*, J. Comput. Phys. **116** (1995), no. 1, 184–188, DOI 10.1006/jcph.1995.1017. MR1315216

[51] J. Shen, L.-L. Wang, and H. Li, *A triangular spectral element method using fully tensorial rational basis functions*, SIAM J. Numer. Anal. **47** (2009), no. 3, 1619–1650, DOI 10.1137/070702023. MR2505867

[52] S. J. Sherwin and G. Em. Karniadakis, *A new triangular and tetrahedral basis for high-order* (*hp*) *finite element methods*, Internat. J. Numer. Methods Engrg. **38** (1995), no. 22, 3775–3802, DOI 10.1002/nme.1620382204. MR1362003

[53] R. Stenberg and M. Suri, *Mixed hp finite element methods for problems in elasticity and Stokes flow*, Numer. Math. **72** (1996), no. 3, 367–389, DOI 10.1007/s002110050174. MR1367655

[54] Gabor Szegö, *Orthogonal polynomials*, vol. 23, American Mathematical Society, 1939.

[55] T. C. Warburton, S. J. Sherwin, and G. E. Karniadakis, *Basis functions for triangular and quadrilateral high-order elements*, SIAM J. Sci. Comput. **20** (1999), no. 5, 1671–1695, DOI 10.1137/S1064827597315716. MR1694678

[56] J. A. C. Weideman and L. N. Trefethen, *The eigenvalues of second-order spectral differentiation matrices*, SIAM J. Numer. Anal. **25** (1988), no. 6, 1279–1298, DOI 10.1137/0725072. MR972454

[57] C. Wieners, *A numerical existence proof of nodal lines for the first eigenfunction of the plate equation*, Arch. Math. (Basel) **66** (1996), no. 5, 420–427, DOI 10.1007/BF01781561. MR1383907

[58] C. Wieners, *Bounds for the N lowest eigenvalues of fourth-order boundary value problems*, Computing **59** (1997), no. 1, 29–41, DOI 10.1007/BF02684402. MR1465309

[59] Y. Yang and W. Jiang, *Upper spectral bounds and a posteriori error analysis of several mixed finite element approximations for the Stokes eigenvalue problem*, Sci. China Math. **56** (2013), no. 6, 1313–1330, DOI 10.1007/s11425-013-4582-4. MR3063973

[60] Z. Zhang, *How many numerical eigenvalues can we trust?*, J. Sci. Comput. **65** (2015), no. 2, 455–466, DOI 10.1007/s10915-014-9971-5. MR3411273

Laboratory of Parallel Computing, Institute of Software, Chinese Academy of Sciences, Beijing 100190, People's Republic of China – and – University of Chinese Academy of Sciences, Beijing 100190, People's Republic of China

E-mail address: `shanweikun11@mails.ucas.ac.cn`

State Key Laboratory of Computer Science/Laboratory of Parallel Computing, Institute of Software, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

E-mail address: `huiyuan@iscas.ac.cn`