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COMPUTING AUTOMORPHISMS OF MORI DREAM SPACES

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Abstract. We present an algorithm to compute the automorphism group of a Mori dream space. As an example calculation, we determine the automorphism groups of singular cubic surfaces with general parameters. The strategy is to study graded automorphisms of an affine algebra graded by a finitely generated abelian group and apply the results to the Cox ring. Besides the application to Mori dream spaces, our results could be used for symmetry based computing, e.g., for Gröbner bases or tropical varieties.

1. Introduction

We are interested in automorphism groups of Mori dream spaces. Recall that the latter are normal projective varieties X with finitely generated divisor class group $\mathrm{Cl}(X)$ and finitely generated Cox ring

$$\mathcal{R}(X) \;=\; \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

The Cox ring together with an ample class completely encodes a Mori dream space: X can be reconstructed as the GIT quotient of $\operatorname{Spec}\mathcal{R}(X)$ associated to the ample class in $\mathrm{Cl}(X)$; see [1, 11].

The basic idea is to tackle the automorphism group of X via graded automorphisms of its Cox ring. This approach was used in [5] in the case of complete toric varieties X, where root subgroups, dimension and number of connected components of $\mathrm{Aut}(X)$ can be described. In [2], the more general case of complete rational varieties with a torus action of complexity one was considered, where a description of root subgroups is still possible, but general satisfying statements on the dimension or the number of connected components appear to be difficult. The aim of this paper is to provide algorithmic tools for the study automorphism groups for arbitrary Mori dream spaces. The main result in this regard is the following; see Algorithm 4.9.

Algorithm. *Input:* the Cox ring of X in terms of homogeneous generators and relations and an ample class in $\mathrm{Cl}(X)$. *Output:* a presentation of the Hopf algebra of $\mathrm{Aut}(X)$ in terms of generators and relations.

This allows us, in particular, to compute the dimension and the number of connected components of $\mathrm{Aut}(X)$. We also apply the algorithm to detect or exclude symmetries. In Example 5.1 we discuss the blow-up of the projective space in special configuration of six points. As another sample computation, we continue

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the investigation of automorphism groups of cubic surfaces started in [15] by now entering the case with parameters; see Theorem 5.2.

Theorem. *Let $X \subseteq \mathbb{P}^3$ be a singular cubic surface with at most ADE singularities and parameters in its defining equations. Depending on the ADE singularity type $S(X)$, the automorphism group $\mathrm{Aut}(X)$ for a general choice of parameters is the following:*

$S(X)$	A_3	A_2A_1	$2A_2$	$3A_1$	A_2	$2A_1$	A_1
$\mathrm{Aut}(X)$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	$\mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}$	S_3	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$

As a theoretical byproduct of our considerations, we observe a bound on the dimension of the automorphism group of a Mori dream space; see Proposition 4.2. Another result is an extension statement on automorphisms of Mori dream spaces, which we briefly explain here. Any choice of $\mathrm{Cl}(X)$-prime homogeneous generators for the Cox ring of a given Mori dream space X leads to an embedding $X \subseteq Z$ into a toric variety Z such that the embedded X inherits many geometric properties of the ambient Z; see [1]. Our statement concerns the unit component $\mathrm{Aut}(X)^0$ of the automorphism group $\mathrm{Aut}(X)$ and the group $\mathrm{Bir}_2(X)$ of birational automorphisms defined up to codimension two; see Theorem 4.4 for the precise formulation.

Theorem. *Let $X \subseteq Z$ be the toric embedding of a Mori dream space arising from the choice of pairwise $\mathrm{Cl}(X)$-prime homogeneous generators. Then $\mathrm{Aut}(X)^0$ and $\mathrm{Bir}_2(X)$ are induced by the stabilizer subgroups of X in $\mathrm{Aut}(Z)^0$ and $\mathrm{Bir}_2(Z)$, respectively.*

As indicated, our approach goes via graded automorphisms of algebras R graded by finitely generated abelian groups K. Section 2 provides a theoretical study of this setting and Section 3 presents the first algorithmic results: its core is Algorithm 3.9, which represents the graded automorphism group explicitly as a closed subgroup of a general linear group. The final Algorithm 4.9 computing $\mathrm{Aut}(X)$ first determines the graded automorphisms using Algorithm 3.9, then computes the stabilizer of the set of semistable points associated with an ample class of X using Algorithm 4.5 and finally arrives via an invariant ring calculation at $\mathrm{Aut}(X)$.

Note that besides our application to Mori dream spaces, our methods can be used to obtain symmetries of homogeneous ideals. This supports symmetry based algorithms such as the Gröbner basis computations [9, 16] or the computation of tropical varieties [13]. We discuss this effect in Example 5.5. Our algorithms will be made available soon in a suitable software package.

2. Automorphisms of graded algebras

We study algebras graded by finitely generated abelian groups. The aim is to make the graded automorphism group accessible for computations in the case of an effective, pointed grading as introduced below. The precise statements are given in Propositions 2.8 and 2.9.

Let us fix the notation and recall basic background. Our ground field $\mathbb{K}$ is algebraically closed and of characteristic zero. A $\mathbb{K}$-algebra R is *graded* by an abelian group K if it comes with a direct sum decomposition into vector subspaces

$$R \;=\; \bigoplus_K R_w \quad \text{such that} \quad R_w R_{w'} \;\subseteq\; R_{w+w'} \quad \text{for all} \quad w, w' \in K.$$

A *graded homomorphism* of two such algebras $R = \bigoplus_K R_w$ and $S = \bigoplus_L S_u$ is a pair (φ, ψ), where $\varphi\colon R \to S$ is an $\mathbb{K}$-algebra homomorphism and $\psi\colon K \to L$ a homomorphism of the grading groups such that $\varphi(R_w) \subseteq S_{\psi(w)}$ holds for all $w \in K$. We denote by $\mathrm{Aut}_K(R)$ the group of graded automorphisms of R.

If $R = \bigoplus_K R_w$ is integral, then the *weight monoid* is the submonoid $\omega(R) \subseteq K$ of all degrees $w \in K$ with $R_w \neq 0$. We say that the K-grading of R is *effective* if $\omega(R)$ generates K as a group. Moreover, we call the K-grading *pointed* if $R_0 = \mathbb{K}$ holds and the convex cone in $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $\omega(R)$ contains no line.

Let $R = \bigoplus_K R_w$ be integral and the K-grading pointed. Then we have a partial ordering on $\omega(R)$ defined by $w' \leq w$ if $w = w' + w_0$ for some $w_0 \in \omega(R)$. If R is moreover finitely generated, then $\omega(R)$ is also and each $w \in \omega(R)$ admits only finitely many $w' \in \omega(R)$ with $w' \leq w$. With the subalgebras $R_{<w} \subseteq R$ generated by all $R_{w'}$, where $w' < w$, we can then figure out the unique, finite set of *generator degrees*

$$\Omega_R \;:=\; \{w \in \omega(R);\; R_w \not\subseteq R_{<w}\}.$$

Denote by $\mathrm{CAut}_K(R) \subseteq \mathrm{Aut}_K(R)$ the subgroup of all graded $\mathbb{K}$-algebra automorphisms of the form (φ, id) and consider the symmetry group

$$\mathrm{Aut}(\Omega_R) \;:=\; \{\psi \in \mathrm{Aut}(K);\; \psi(\Omega_R) = \Omega_R\} \;\subseteq\; \mathrm{Aut}(K).$$

Proposition 2.1. *Let $R = \bigoplus_K R_w$ be a finitely generated, integral $\mathbb{K}$-algebra with an effective, pointed grading by a finitely generated abelian group K.*

(i) *Every $(\varphi, \psi) \in \mathrm{Aut}_K(R)$ satisfies $\psi(\mathrm{Aut}(\Omega_R)) = \Omega_R$. In particular, there is a well-defined homomorphism of groups:*

$$\pi_\Omega\colon \mathrm{Aut}_K(R) \;\to\; \mathrm{Aut}(\Omega_R), \qquad\qquad (\varphi, \psi) \;\mapsto\; \psi.$$

(ii) *The image $\Gamma := \pi_\Omega(\mathrm{Aut}_K(R)) \subseteq \mathrm{Aut}(\Omega_R)$ is a finite group and we have an exact sequence of affine algebraic groups:*

$$1 \longrightarrow \mathrm{CAut}_K(R) \longrightarrow \mathrm{Aut}_K(R) \xrightarrow{\pi_\Omega} \Gamma \longrightarrow 1\,.$$

Proof. As the set of generator degrees is unique, it is invariant under graded automorphisms. For the fact that all groups of the sequence are affine algebraic see, for example, [2]. The rest is obvious. □

By a *minimal presentation* of a K-graded $\mathbb{K}$-algebra R we mean a K-graded polynomial ring $S := \mathbb{K}[T_1, \ldots, T_r]$ with K-homogeneous variables T_i and a graded epimorphism $(\pi, \kappa)\colon \mathbb{K}[T_1, \ldots, T_r] \to R$ such that $\kappa\colon K \to K$ is an isomorphism of groups and we have

$$\ker(\pi) \;\subseteq\; \langle T_1, \ldots, T_r\rangle^2.$$

We now fix the setting for our study of the group $\mathrm{Aut}_K(R)$ of graded algebra automorphisms: K will always be a finitely generated abelian group, R an integral, finitely generated $\mathbb{K}$-algebra and the K-grading of R will always be effective and pointed. Moreover, we fix a minimal presentation of $(\pi, \mathrm{id})\colon S \to R$, write $I := \ker(\pi)$ for the ideal of relations and we frequently identify R with S/I.

Remark 2.2. For the minimal presentation $S \to R$ fixed above, the K-grading on S is effective and pointed as well. Moreover, $\Omega_S = \Omega_R$ consists of the degrees $\deg(T_i) \in K$ of the variables $T_i \in S$.

We now relate the graded automorphism group of R to that of S. For any group action $G \times M \to M$ and subsets $H \subseteq G$ and $N \subseteq M$, the associated *stabilizer* is the subset

$$\mathrm{Stab}_N(H) \;:=\; \{g \in H;\; g \cdot N = N\} \;\subseteq\; G.$$

If the action is algebraic and $H \subseteq \mathrm{GL}(n)$ is closed (a subgroup), then $\mathrm{Stab}_N(H)$ is closed (a subgroup) in G. Here is how the stabilizers occur in our setting.

Proposition 2.3. *Consider a minimally presented K-graded $\mathbb{K}$-algebra $R = S/I$ as before. Then we have a commutative diagram of affine algebraic groups with exact rows and columns*

$$\begin{tikzcd}
 & 1 & 1 & & \\
1 \arrow[r] & \mathrm{CAut}_K(R) \arrow[r] \arrow[u] & \mathrm{Aut}_K(R) \arrow[r, "\pi_\Omega"] \arrow[u] & \Gamma \arrow[r] & 1 \\
 & \mathrm{Stab}_I(\mathrm{CAut}_K(S)) \arrow[r] \arrow[u, "\Phi' : (\varphi,\psi)\mapsto(\varphi_\pi,\psi)"] & \mathrm{Stab}_I(\mathrm{Aut}_K(S)) \arrow[u, "\Phi : (\varphi,\psi)\mapsto(\varphi_\pi,\psi)"'] & & \\
 & \ker(\Phi') \arrow[r] \arrow[u] & \ker(\Phi) \arrow[u] & & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

where the horizontal sequence is as in Proposition 2.1*, the map Φ' is the restriction of Φ and φ_π is the unique homomorphism turning*

$$\begin{tikzcd}
S \arrow[r, "(\varphi,\psi)"] \arrow[d, "(\pi,\mathrm{id})"'] & S \arrow[d, "(\pi,\mathrm{id})"] \\
R \arrow[r, "(\varphi_\pi,\psi)"'] & R
\end{tikzcd}$$

into a commutative diagram of graded homomorphisms. Moreover, the following statements hold.

(i) *We have* $\ker(\Phi) = \ker(\Phi')$*. Setting* $q_i := \deg(T_i)$ *and* $G := \mathrm{Stab}_I(\mathrm{Aut}_K(S))$*, we obtain* $\ker(\Phi)$ *as the subgroup*

$$\ker(\Phi) \;=\; \langle (\varphi, \mathrm{id}) \in G;\; \varphi(T_i) - T_i \in I_{q_i} \text{ for all } i \rangle \;\subseteq\; G.$$

(ii) *If the homogeneous components $I_{q_1}, \ldots, I_{q_r}$ of the degrees $q_i := \deg(T_i)$ are all trivial, then Φ and Φ' are isomorphisms.*

In the proof we need the following two lemmas, where the second one is a certain uniqueness statement on minimal presentations and will also be used later.

Lemma 2.4. *Let $(\varphi, \psi)\colon S \to R$ and $(\pi, \kappa)\colon R' \to R$ be homomorphisms of K-graded $\mathbb{K}$-algebras such that $S = \mathbb{K}[T_1, \ldots, T_r]$ holds, π is surjective and κ an*

isomorphism. Then there is a commutative diagram of homomorphisms of K-graded algebras:

$$\begin{tikzcd} S \arrow[rr, "(\widehat{\varphi},\widehat{\psi})"] \arrow[dr, "(\varphi,\psi)"'] & & R' \arrow[dl, "(\pi,\kappa)"] \\ & R & \end{tikzcd}$$

Proof. Denote by $w_i \in K$ the degree of the variable $T_i \in S$. Each $\varphi(T_i) \in R$ has a preimage $f_i' \in R'_{\kappa^{-1}(\psi(w_i))}$ under (π, κ). Define $\widehat{\varphi}$ by $\widehat{\varphi}(T_i) := f_i'$ and set $\widehat{\psi} := \kappa^{-1} \circ \psi$. □

Lemma 2.5. *Consider two minimal presentations $(\pi_i, \kappa_i)\colon S \to R$ of a $\mathbb{K}$-algebra R with an effective, pointed K-grading. Then there is an isomorphism (φ, ψ) of K-graded algebras fitting into the commutative diagram:*

$$\begin{tikzcd} S \arrow[rr, "(\varphi,\psi)"] \arrow[dr, "(\pi_1,\kappa_1)"'] & & S \arrow[dl, "(\pi_2,\kappa_2)"] \\ & R & \end{tikzcd}$$

Proof. According to Lemma 2.4, there are graded homomorphisms $(\varphi_i, \psi_i)\colon S \to S$ such that the following diagram is commutative:

$$\begin{tikzcd} S \arrow[rr, shift left, "(\varphi_1,\psi_1)"] \arrow[dr, "(\pi_1,\kappa_1)"'] & & S \arrow[ll, shift left, "(\varphi_2,\psi_2)"] \arrow[dl, "(\pi_2,\kappa_2)"] \\ & R & \end{tikzcd}$$

In particular, $\tau := \varphi_2 \circ \varphi_1\colon S \to S$ is a degree preserving graded algebra endomorphism with $\pi_1 \circ \tau = \pi_1$. Using minimality of the presentation $(\pi_1, \kappa_1)\colon S \to R$, we obtain

$$T_i - \tau(T_i) \;\in\; \ker(\pi_1) \;\subseteq\; \langle T_1, \ldots, T_r\rangle^2.$$

We show that τ is surjective. For this, it suffices to show $\tau(S_w) = S_w$ for all $w \in \Omega_S$. Since τ preserves degrees, we have $\tau(S_w) \subseteq S_w$. To verify equality, we proceed by induction on the maximal length $k(w)$ of the possible decompositions

$$w \;=\; w_1 + \ldots + w_k, \qquad w_i \;\in\; \Omega_S \quad \text{indecomposable.}$$

If $k(w) = 1$, then w is indecomposable in Ω_S and S_w is generated as a vector space by some of the T_i. Each $T_i \in S_w$ is fixed by τ which gives $\tau(S_w) = S_w$. Assume $k(w) > 1$. Then we find a basis of the vector space S_w of the form

$$(T_{i_1}, \ldots, T_{i_l}, h_1, \ldots, h_m), \qquad\qquad h_j \;\in\; \langle T_1, \ldots, T_r\rangle^2.$$

Each h_j is a polynomial in variables T_i of degree $w_i \in \Omega_S$ such that $w_i < w$. The latter implies $k(w_i) < k(w)$. By induction hypothesis, these S_{w_i}, and hence h_j, are in the image of τ. This proves surjectivity of τ.

We now immediately obtain bijectivity of τ, because each restriction $\tau\colon S_w \to S_w$ is a surjective linear endomorphism of a finite-dimensional vector space and hence bijective. Thus, $\tau = \varphi_2 \circ \varphi_1$ is an isomorphism. Then the graded homomorphisms φ_i must be isomorphisms as well and we see that $(\varphi, \psi) := (\varphi_1, \psi_1)$ is as wanted. □

Proof of Proposition 2.3. We show that Φ is surjective. Given a graded automorphism, $(\varphi_0, \psi_0) \in \mathrm{Aut}_K(R)$, we have two minimal presentations

$$(\pi, \mathrm{id})\colon S \to R, \qquad (\varphi_0, \psi_0) \circ (\pi, \mathrm{id})\colon S \to R.$$

Lemma 2.5 provides us with an isomorphism (φ, ψ) of graded $\mathbb{K}$-algebras fitting into the commutative diagram:

$$\begin{tikzcd} S \arrow[r, "(\varphi,\psi)"] \arrow[d, "(\pi,\mathrm{id})"'] & S \arrow[d, "(\pi,\mathrm{id})"] \\ R \arrow[r, "(\varphi_0,\psi_0)"'] & R \end{tikzcd}$$

Clearly, $\psi = \psi_0$ and thus (φ, ψ) is the desired preimage of (φ_0, ψ_0) under Φ. This establishes the first part. For the statements on the kernel, one directly verifies

$$\begin{aligned} \ker(\Phi) &= \langle (\varphi, \mathrm{id}) \in \mathrm{Aut}_K(S);\ \varphi_\pi(\overline{T}_i) = \overline{T_i} \in R \text{ for all } i \rangle \\ &= \langle (\varphi, \mathrm{id}) \in \mathrm{Aut}_K(S);\ \varphi(T_i) - T_i \in I \text{ for all } i \rangle \\ &= \langle (\varphi, \mathrm{id}) \in \mathrm{Aut}_K(S);\ \varphi(T_i) - T_i \in I_{q_i} \text{ for all } i \rangle \end{aligned}$$

where $\varphi_\pi\colon R \to R$ is the unique homomorphism induced by (φ, ψ) as introduced in the assertion. □

Similar to the unique set of generator degrees $\Omega_R \subseteq K$ of a K-graded $\mathbb{K}$-algebra R, we have the unique set $\Omega_I \subseteq K$ of *ideal generator degrees* for the homogeneous ideal $I \subseteq S$: let $I_{<w} \subseteq S$ be the ideal generated by all ideal components $I_{w'} \subseteq I$ with $w' < w$ and set

$$\Omega_I := \{w \in K;\ I_w \not\subseteq I_{<w}\}.$$

Proposition 2.6. *Every automorphism* $(\varphi, \psi) \in \mathrm{Aut}_K(R)$ *satisfies* $\psi(\Omega_I) = \Omega_I$.

Proof. According to Lemma 2.5, the graded automorphism (φ, ψ) admits a lifting $(\widehat{\varphi}, \psi)$ with respect to the minimal presentation $(\pi, \mathrm{id})\colon S \to R$. □

Construction 2.7. Consider $S = \mathbb{K}[T_1, \ldots, T_r]$ and $G := \mathrm{Aut}_K(S)$. Then we have finite-dimensional vector subspaces

$$V := \bigoplus_{w \in \Omega_S} S_w \subseteq S, \qquad W := \bigoplus_{u \in \Omega_I} S_u \subseteq S.$$

Moreover, V is invariant under the (linear) G-action on S and the induced representation $G \to \mathrm{GL}(V)$ is faithful.

Proof. Since the grading is pointed and Ω_S as well as Ω_I are finite, V and W are of finite dimension. Proposition 2.1 (i) guarantees that V is G-invariant and the induced representation is faithful, because V generates S as a $\mathbb{K}$-algebra. □

The following observation will allow us to compute the stabilizer $\mathrm{Stab}_I(G)$, which is an essential step in the computation of $\mathrm{Aut}_K(R)$:

Proposition 2.8. *Notation as in Construction* 2.7. *Set* $I_W := I \cap W$. *Then* I_W *generates the ideal* I *and the stabilizer* $\mathrm{Stab}_I(G)$ *is given as*

$$\mathrm{Stab}_I(G) = \mathrm{Stab}_{I_W}(G) = \{g \in G;\ g \cdot I_W = I_W\} \subseteq G.$$

Finally, we arrive at a finite-dimensional faithful representation of $\mathrm{Aut}_K(R)$ used in our computations.

Proposition 2.9. *Notation is as in Construction* 2.7. *Set* $I_V := I \cap V$. *Then the minimal presentation* $(\pi, \mathrm{id})\colon S \to R$ *induces an isomorphism of vector spaces*

$$V/I_V \;\to\; \bigoplus_{w \in \Omega_R} R_w.$$

Moreover, I_V *is invariant under* $\mathrm{Stab}_I(G)$, *we have an induced representation* $\varrho\colon \mathrm{Stab}_I(G) \to \mathrm{GL}(V/I_V)$ *and an isomorphism*

$$\varrho(\mathrm{Stab}_I(G)) \;\cong\; \mathrm{Aut}_K(R).$$

Proof. Proposition 2.3 gives us the desired isomorphism. □

Corollary 2.10. *Let* K *be a finitely generated abelian group and* $R = \bigoplus_{w \in K} R_w$ *an integral, finitely generated* $\mathbb{K}$*-algebra with an effective, pointed* K*-grading. Then the dimension of the group of graded automorphisms is bounded by*

$$\dim(\mathrm{Aut}_K(R)) \;\leq\; \sum_{w \in \Omega_R} \dim(R_w)^2.$$

3. Basic algorithms

We present the algorithms for computing the automorphism group $\mathrm{Aut}_K(R)$ of a $\mathbb{K}$-algebra R graded by a finitely generated abelian group K. Our algorithms are formulated in a manner allowing, up to standard considerations in linear algebra, a direct implementation.

We work in the setting of Section 2. In particular, the K-grading of R is effective and pointed. Moreover, we fix a minimal presentation $(\pi, \mathrm{id})\colon S \to R$ with a polynomial ring $S := \mathbb{K}[T_1, \ldots, T_r]$ and denote $I := \ker(\pi)$; such a minimal presentation is directly obtained by removing successively redundant generators from any presentation by generators and relations.

Recall that we have $\Omega_S = \Omega_R$ for the respective sets of generator weights of S and R. A first basic step is to determine the automorphism group $\mathrm{Aut}(\Omega_S)$.

Remark 3.1 (Computing $\mathrm{Aut}(\Omega_S)$). Represent the grading group K as a direct sum of a free part and its p-torsion parts, i.e.,

$$K \;=\; \mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} K_{p_i}, \qquad\qquad K_{p_i} \;=\; \bigoplus_{j=1}^{n_i} \mathbb{Z}/p_i^{k_{ij}}\mathbb{Z}$$

with pairwise different primes p_i and $k_{ij} \leq k_{ij+1}$. Set $n := k + n_1 + \ldots + n_l$. Then the automorphisms $\psi\colon K \to K$ are induced by integral block matrices:

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n \\ \downarrow & & \downarrow \\ K & \xrightarrow[\psi]{} & K \end{array}, \qquad\qquad A \;=\; \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

with $B \in \mathrm{GL}(k, \mathbb{Z})$ and D suitably block diagonal; see [4, 17] for the precise conditions on D. The entries of D and C are nonnegative and bounded by the smallest annihilator $a > 0$ of the torsion part of K. The subgroup $\mathrm{Aut}(\Omega_S) \subseteq \mathrm{Aut}(K)$ is now obtained as follows.

- Determine the (finitely many) $B \in \mathrm{GL}(k, \mathbb{Z})$ stabilizing the set of the $\mathbb{Z}^k$-parts of the generator degrees Ω_S.

- Determine the (finitely many) matrices A with a B-block from the previous step and figure out those defining an automorphism of K stabilizing Ω_S.

As a second step, we make Construction 2.7 explicit and realize $\mathrm{Aut}_K(S)$ via concrete equations as a subgroup of a general linear group $\mathrm{GL}(n)$. For later applications, we include the treatment of subgroups of the following type in $\mathrm{Aut}_K(S)$: given a subgroup $\Sigma \subseteq \mathrm{Aut}(\Omega_S)$, set

$$\mathrm{Aut}_{K,\Sigma}(S) \ := \ \{(\varphi,\psi) \in \mathrm{Aut}_K(S);\ \psi \in \Sigma\} \ \subseteq \ \mathrm{Aut}_K(S).$$

Construction 3.2. Consider the K-graded ring $S = \mathbb{K}[T_1,\ldots,T_r]$ and write $\Omega_S = \{w_1,\ldots,w_s\}$ with pairwise different w_j. For each $i = 1,\ldots,s$, set $d_i := \dim_{\mathbb{K}}(S_{w_i})$ and fix a basis $\mathcal{B}_i$ consisting of monomials for the homogeneous component S_{w_i}. The concatenation

$$\mathcal{B} \ := \ (\mathcal{B}_1,\ldots,\mathcal{B}_s)$$

is a basis for the direct sum $V := \bigoplus_i S_{w_i}$ of all S_{w_i}. Now, every graded automorphism $(\varphi,\psi) \in \mathrm{Aut}_K(S)$ defines a linear automorphism

$$V \to \ V, \qquad S_{w_i} \ni f \ \mapsto \ \varphi(f) \in S_{\psi(w_i)}.$$

We denote by $A(\varphi,\psi) \in \mathrm{GL}(n)$, where $n = d_1 + \ldots + d_s$, the matrix representing this map with respect to $\mathcal{B}$. This gives rise to a faithful matrix representation

$$\varrho\colon \mathrm{Aut}_K(S) \ \to \ \mathrm{GL}(n), \qquad (\varphi,\psi) \ \mapsto \ A(\varphi,\psi).$$

Construction 3.3. Notation as in Construction 3.2. Via the basis $\mathcal{B}$, every matrix $A \in \mathrm{GL}(n)$ defines a linear endomorphism $\varphi_A\colon V \to V$. Since the variables $T_1,\ldots,T_r$ of S belong to V we can define for every $f \in S$ the polynomial

$$A \cdot f \ := \ f(\varphi_A(T_1),\ldots,\varphi_A(T_r)) \ \in \ S.$$

Definition 3.4. Notation as in Constructions 3.2 and 3.3. Let $\Sigma \subseteq \mathrm{Aut}(\Omega_S)$ be a subgroup. We introduce three types of matrices:

(i) a matrix $A \in \mathrm{GL}(n)$ is *S-admissible* if for any two monomials T^{ν_1}, T^{ν_2} such that the degrees of T^{ν_1}, T^{ν_2} and $T^{\nu_1+\nu_2}$ belong to $\Omega_S = \{w_1,\ldots,w_s\}$ we have

$$A \cdot (T^{\nu_1}T^{\nu_2}) \ = \ (A \cdot T^{\nu_1})(A \cdot T^{\nu_2});$$

(ii) a matrix $A \in \mathrm{GL}(n)$ is *Ω_S-diagonal* if it is block diagonal with blocks $A_1,\ldots,A_s$, where $A_i \in \mathrm{GL}(d_i)$;

(iii) a matrix $B \in \mathrm{GL}(n)$ is *Σ-permuting* if there is a $\sigma \in \Sigma$ such that B sends each $\mathcal{B}_i$ bijectively to $\mathcal{B}_j$, where $w_j = \sigma(w_i)$.

Proposition 3.5. *Notation is as in Constructions* 3.2 *and* 3.3. *Then* $\varrho(\mathrm{Aut}_{K,\Sigma}(S))$ *consists exactly of the S-admissible matrices $AB \in \mathrm{GL}(n)$, where $A \in \mathrm{GL}(n)$ is Ω_S-diagonal and $B \in \mathrm{GL}(n)$ is Σ-permuting.*

Proof. First let $(\varphi,\psi) \in \mathrm{Aut}_{K,\Sigma}(S)$. Then, for each $i = 1,\ldots,s$, the linear isomorphism $\varphi\colon V \to V$ restricts to a linear isomorphism $\varphi_i\colon V_{w_i} \to V_{\psi(w_i)}$. Denote by $A_i \in \mathrm{GL}(d_i)$ the representing matrix of φ_i with respect to the bases $\mathcal{B}_i$ and $\mathcal{B}_{j(i)}$, where $j(i)$ is defined via $\psi(w_i) = w_{j(i)}$. Moreover, let $B \in \mathrm{GL}(n)$ denote the permutation matrix sending the basis vectors v_{ik} of $\mathcal{B}_i = (v_{i1},\ldots,v_{id_i})$ to the basis vectors $v_{j(i)k}$ of $\mathcal{B}_{j(i)} = (v_{j(i)1},\ldots,v_{j(i)d_i})$. Then $A := \mathrm{diag}(A_{j(1)},\ldots,A_{j(s)})$ is Ω_S-diagonal, B is Σ-permuting and we have $A(\varphi,\psi) = AB$. As the matrix

AB represents the restriction of the algebra homomorphism φ to the vector subspace $V \subseteq S$, it is compatible with the multiplication of polynomials and thus S-admissible.

Conversely, let $A \in \mathrm{GL}(n)$ be Ω_S-diagonal and let $B \in \mathrm{GL}(n)$ be Σ-permuting with respect to $\psi \in \Sigma$ such that AB is S-admissible. Via the basis $\mathcal{B}$, the matrix AB defines the linear automorphism $\varphi_{AB}\colon V \to V$. Moreover, $f \mapsto AB \cdot f$ defines a graded algebra automorphism $(\varphi, \psi)\colon S \to S$, where $\psi \in \Sigma$ belongs to B as fixed above. Since AB is S-admissible, we see that φ and φ_{AB} coincide on the monomials lying in V. Being linear maps, φ and φ_{AB} then coincide on V and we conclude $AB = A(\varphi, \psi)$. □

Corollary 3.6. *In the special cases* $\Sigma = \{\mathrm{id}_K\}$ *and* $\Sigma = \mathrm{Aut}(\Omega_S)$*, Proposition* 3.5 *gives the following.*

(i) *The image* $\varrho(\mathrm{CAut}_K(S))$ *consists exactly of the matrices* $A \in \mathrm{GL}(n)$ *which are* Ω_S*-diagonal and* S*-admissible.*

(ii) *The image* $\varrho(\mathrm{Aut}_K(S))$ *consists exactly of the* S*-admissible matrices* $AB \in \mathrm{GL}(n)$*, where* $A \in \mathrm{GL}(n)$ *is* Ω_S*-diagonal and* $B \in \mathrm{GL}(n)$ *is* $\mathrm{Aut}(\Omega_S)$*-permuting.*

In the subsequent algorithms, we say that an ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_n]$ *describes* a subset $X \subseteq \mathbb{K}^n$ if X equals the zero set of $\mathfrak{a}$ and we say that $\mathfrak{a}$ *defines* a closed subset $X \subseteq \mathbb{K}^n$ if $\mathfrak{a}$ equals the vanishing ideal of X.

Algorithm 3.7 (Computing $\mathrm{Aut}_{K,\Sigma}(S)$)**.** *Input:* the K-graded polynomial ring S and a subgroup $\Sigma \subseteq \mathrm{Aut}(\Omega_S)$.

- Compute a basis $\mathcal{B} = (b_1, \ldots, b_n)$ for $V = \bigoplus_i S_{w_i}$ as in Construction 3.2.
- Set $S' := \mathbb{K}[T_{ij};\ 1 \le i, j \le n]$.
- Let $I' \subseteq S'$ be an ideal describing the S-admissible matrices in $\mathrm{GL}(n)$.
- Let $J \subseteq S'$ be an ideal describing the Ω_S-diagonal matrices in $\mathrm{GL}(n)$.
- Compute the (finite) set $\mathfrak{B}$ of Σ-permuting matrices.
- For each $B \in \mathfrak{B}$ form the product $J := J \cdot (B^* J)$.

Output: the ideal $I' + J \subseteq \mathbb{K}[T_{ij};\ 1 \le i, j \le n]$. It describes the subgroup $\mathrm{Aut}_{K,\Sigma}(S) \subseteq \mathrm{GL}(n)$.

We proceed with computing equations for the stabilizer of the ideal of relations $I \subseteq S$ of R in $\mathrm{Aut}_{K,\Sigma}(S) \subseteq \mathrm{GL}(n)$, where we follow the lines of Proposition 2.8.

Algorithm 3.8 (Computing $\mathrm{Stab}_I(\mathrm{Aut}_{K,\Sigma}(S))$)**.** *Input:* the K-graded polynomial ring S and the defining ideal $I \subseteq S$ of R and a subgroup $\Sigma \subseteq \mathrm{Aut}(\Omega_S)$.

- Let $I' + J \subseteq S' := \mathbb{K}[T_{ij};\ 1 \le i, j \le n]$ be the output of Algorithm 3.7.
- Determine Ω_I and the vector space $W := \bigoplus_{\Omega_I} S_u$.
- Determine a basis $(h_1, \ldots, h_l)$ for the vector subspace $I_W = I \cap W \subseteq W$.
- Compute linear forms $\ell_1, \ldots, \ell_m \in W^*$ with I_W as common zero set.
- With $T = (T_{ij})$ and the $\mathrm{GL}(n)$-action on S from Construction 3.3 define the ideal

$$J' \ := \ \langle \ell_i\,(T \cdot h_j)\,;\ 1 \le i \le m,\ 1 \le j \le l \rangle \ \subseteq \ S'.$$

Output: the ideal $I' + J + J' \subseteq S'$. It describes the subgroup $\mathrm{Stab}_I(\mathrm{Aut}_{K,\Sigma}(S)) \subseteq \mathrm{GL}(n)$.

Finally, we implement Proposition 2.9 to compute, in particular, $\mathrm{Aut}_K(R)$ as a subgroup of a suitable general linear group $\mathrm{GL}(k)$. We still consider subgroups $\Sigma \subseteq \mathrm{Aut}(\Omega_S) = \mathrm{Aut}(\Omega_R)$ and set

$$\mathrm{Aut}_{K,\Sigma}(R) \;:=\; \{(\varphi,\psi) \in \mathrm{Aut}_K(R);\ \psi \in \Sigma\} \;\subseteq\; \mathrm{Aut}_K(R).$$

Algorithm 3.9 (Computing $\mathrm{Aut}_{K,\Sigma}(R)$)**.** *Input:* the K-graded polynomial ring S and the defining ideal $I \subseteq S$ of R and a subgroup $\Sigma \subseteq \mathrm{Aut}(\Omega_S)$.

- Let $I' + J + J' \subseteq S' := \mathbb{K}[T_{ij};\ 1 \le i,j \le n]$ be the output of Algorithm 3.8.
- Determine the vector subspace $I_V = I \cap V$ of $V = \bigoplus_{\Omega_S} S_w$.
- Determine a basis $u_1, \ldots, u_k$ for V/I_V and set $S'' := \mathbb{K}[T_{ij};\ 1 \le i,j \le k]$.
- Compute a defining ideal $J'' \subseteq S''$ for the image of the homomorphism

$$\mathrm{GL}(n) \supseteq \mathrm{Stab}_I(\mathrm{Aut}_{K,\Sigma}(S)) \;\to\; \mathrm{GL}(k) = \mathrm{Aut}(V/I_V)$$

  of algebraic groups arising from the projection $\mathbb{K}^n = V \to V/I_V = \mathbb{K}^k$.

Output: the ideal $J'' \subseteq \mathbb{K}[T_{ij};\ 1 \le i,j \le k]$. It defines the subgroup $\mathrm{Aut}_{K,\Sigma}(R) \subseteq \mathrm{GL}(k)$.

Remark 3.10. Algorithm 3.8 makes no use of Gröbner bases, whereas the image computation in the last step of Algorithm 3.9 involves determining the kernel of a ring map, which usually is performed via Gröbner basis computations. However, if all homogeneous components $I_w \subseteq I$ with $w \in \Omega_S$ are trivial, then Proposition 2.3 provides a describing ideal for $\mathrm{Aut}_K(R) \cong \mathrm{Stab}_I(\mathrm{Aut}_K(S))$ without any Gröbner basis computation.

Remark 3.11. The output of Algorithm 3.9 allows us to compute the dimension as well as the number of connected components of $\mathrm{Aut}_K(R)$ by standard algorithms using Gröbner bases.

Remark 3.12 (Computing Γ from Proposition 2.1). Compute the groups $\mathrm{CAut}_K(R)$ and $\mathrm{Aut}_K(R)$ with Algorithm 3.7. Then the factor group Γ consists of those $\sigma \in \mathrm{Aut}(\Omega_S)$ that admit an $\mathrm{Aut}(\Omega_S)$-permuting matrix B with

$$B^*\mathrm{Stab}_I(\mathrm{Aut}_K(S)) \cap \mathrm{Stab}_I(\mathrm{Aut}_K(S)) \;\neq\; \emptyset.$$

The $\mathrm{Aut}(\Omega_S)$-permuting matrices are computed in Algorithm 3.7 and the displayed condition is computable in terms of the describing ideal of the stabilizer, which in turn is provided by Algorithm 3.8.

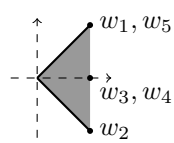
Remark 3.13. Proceeding similarly as in Algorithm 3.8, one can compute the *transporter* of two homogeneous ideals $I_1, I_2 \subseteq S$, that is, the closed subset $\{g \in \mathrm{Aut}_K(S);\ g \cdot I_1 \subseteq I_2\}$.

Example 3.14 ($A_3 2A_1$-singular Gorenstein log del Pezzo $\mathbb{K}^*$-surface)**.** Recall, for example from [1, 12], that the Cox ring R of the singular Gorenstein log del Pezzo $\mathbb{K}^*$-surface X with singularity type $A_3 2A_1$ is

$$R \;=\; S/I, \qquad S \;:=\; \mathbb{K}[T_1, \ldots, T_5], \qquad I \;:=\; \langle T_1T_2 + T_3^2 + T_4^2 \rangle,$$

where R is effectively and pointedly graded by $K := \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ with generator degrees $w_i = \deg(T_i)$ given by

$$[w_1, \ldots, w_5] \;:=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0} \end{bmatrix},$$



$\subseteq K \otimes \mathbb{Q}.$

Observe that we have $\Omega_S = \{w_1, \ldots, w_5\}$ and Remark 3.1 yields the symmetries of generator weights

$$\mathrm{Aut}(\Omega_S) \;\cong\; \{\mathrm{id},\, \psi_1\} \;\cong\; \mathbb{Z}/2\mathbb{Z}, \qquad \psi_1 \colon K \to K, \quad g \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot g.$$

Algorithm 3.7 first produces the bases $\mathcal{B}_i = (T_i)$ for the components S_{w_i} from Construction 3.2 and then returns the description of $G := \mathrm{Aut}_K(S)$:

$$\begin{aligned} G \;\cong\; & \{\mathrm{diag}(a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, a_{5,5});\; a_{i,j} \in \mathbb{K}^*\} \\ & \cup \left\{ \begin{bmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{3,4} & 0 \\ 0 & 0 & a_{4,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} \end{bmatrix};\; a_{i,j} \in \mathbb{K}^* \right\} \;\subseteq\; \mathrm{GL}(5). \end{aligned}$$

Note that in this case, we did not need equations for the S-admissible property. These matrices correspond to explicit automorphisms as noted in Construction 3.2. For instance, the second type of matrices in the above description of G belongs to elements $(\varphi, \psi) \in \mathrm{Aut}_K(S)$ with $\psi = \psi_1$ and φ given by

$$T_1 \mapsto a_{1,1}T_1, \quad T_2 \mapsto a_{2,2}T_2, \quad T_3 \mapsto a_{3,4}T_4, \quad T_4 \mapsto a_{4,3}T_3, \quad T_5 \mapsto a_{5,5}T_5.$$

Since all $I_{\deg(T_i)}$ are trivial, we can use Algorithm 3.8 and see that $\mathrm{Aut}_K(R)$, as a subgroup of $\mathrm{GL}(5)$, is given as a union of two sets, each of which consists of two connected components:

$$\begin{aligned} \mathrm{Aut}_K(R) \;=\; & \left\{ \mathrm{diag}(a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, a_{5,5}) \in \mathrm{GL}(5);\; \begin{matrix} a_{3,3}^2 = a_{4,4}^2, \\ a_{1,1}a_{2,2} = a_{3,3}^2 \end{matrix} \right\} \\ & \cup \left\{ \begin{bmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{3,4} & 0 \\ 0 & 0 & a_{4,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} \end{bmatrix} \in \mathrm{GL}(5);\; \begin{matrix} a_{3,4}^2 = a_{4,3}^2, \\ a_{1,1}a_{2,2} = a_{4,3}^2 \end{matrix} \right\}. \end{aligned}$$

In particular, we obtain $\dim(\mathrm{Aut}_K(R)) = 3$ and we see that the unit component $\mathrm{Aut}_K(R)^0$ is of index four in $\mathrm{Aut}_K(R)$. Moreover, we directly verify that $\mathrm{Aut}_K(R)$ is isomorphic to the semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times (\mathbb{K}^*)^3)$.

4. Automorphism groups of Mori dream spaces

Here we apply the results of the preceding sections to study automorphism groups of Mori dream spaces. Recall that a normal projective variety X is a *Mori dream space* if it has finitely generated divisor class group $\mathrm{Cl}(X)$ and finitely generated *Cox ring*

$$\mathcal{R}(X) \;=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

where the case of torsion in $\mathrm{Cl}(X)$ requires a little care in this definition; see [1, Sec. 1.4.2] for details. Any Mori dream space X has a *total coordinate space* $\overline{X} = \mathrm{Spec}\, \mathcal{R}(X)$. The $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$ defines an action of the *characteristic*

quasitorus $H := \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$ on $\overline{X}$ and we recover X as a GIT quotient $X = \widehat{X}/\!\!/H$ of the set of semistable points $\widehat{X} \subseteq \overline{X}$ associated with any ample class $[D] \in \operatorname{Cl}(X) = \mathbb{X}(H)$ of X; see [1, Sec 3.2.1] for details.

In the study of the automorphism group $\operatorname{Aut}(X)$, several related groups are important. The *H-equivariant automorphisms* of $\overline{X}$ are pairs $(\varphi, \tilde{\varphi})$, where $\varphi\colon \overline{X} \to \overline{X}$ is an automorphism of varieties and $\tilde{\varphi}\colon H \to H$ is an automorphism of linear algebraic groups such that for all $x \in \overline{X}$ and $h \in H$ we have

$$\varphi(t \cdot x) \;=\; \tilde{\varphi}(t) \cdot \varphi(x).$$

Write $\operatorname{Aut}_H(\overline{X})$ for the group of all such automorphisms of $\overline{X}$ and, analogously, $\operatorname{Aut}_H(\widehat{X})$ for those of $\widehat{X}$. Also, the group $\operatorname{Bir}_2(X)$ of birational automorphisms of X defined on an open subset of X having complement of codimension at least two plays a role. All these groups are affine algebraic and are related to each other via the following commutative diagram where the rows are exact sequences and the upward inclusions are of finite index:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & \operatorname{Aut}_H(\overline{X}) & \longrightarrow & \operatorname{Bir}_2(X) & \longrightarrow & 1 \\
 & & \| & & \cup\!\mid & & \cup\!\mid & & \\
1 & \longrightarrow & H & \longrightarrow & \operatorname{Aut}_H(\widehat{X}) & \longrightarrow & \operatorname{Aut}(X) & \longrightarrow & 1;
\end{array}$$

see Theorem [2, Thm. 2.1]. Moreover, the subgroup $\operatorname{CAut}_H(\overline{X}) \subseteq \operatorname{Aut}_H(\overline{X})$, consisting of all pairs of the form $(\varphi, \mathrm{id}) \in \operatorname{Aut}_H(\overline{X})$, occurs according to [2, Cor. 2.8] in the exact sequence

$$1 \longrightarrow H \longrightarrow \operatorname{CAut}_H(\overline{X})^0 \longrightarrow \operatorname{Aut}(X)^0 \longrightarrow 1.$$

Remark 4.1. The link to the results of the preceding sections is the following. Set $R := \mathcal{R}(X)$ and $K := \operatorname{Cl}(X)$. Then $\operatorname{CAut}_H(\overline{X})$ is isomorphic to $\operatorname{CAut}_K(R)$ and $\operatorname{Aut}_H(\overline{X})$ is isomorphic to $\operatorname{Aut}_K(R)$; see [2, Cor. 2.3].

A very first observation is a bound on the dimension of the automorphism group of a Mori dream space in terms of its Cox ring and the rank its divisor class group.

Proposition 4.2. *Let X be a Mori dream space with Cox ring $\mathcal{R}(X)$ and let $\Omega_X \subseteq \operatorname{Cl}(X)$ denote the set of generator degrees. Then we have*

$$\dim(\operatorname{Aut}(X)) \;\leq\; \left(\sum_{w \in \Omega_X} \dim(\mathcal{R}(X)_w)^2 \right) - \dim(\operatorname{Cl}(X)_{\mathbb{Q}}).$$

Proof. The divisor class group $\operatorname{Cl}(X)$ is the character group of the characteristic quasitorus H and thus the dimension of H equals the dimension of the rational vector space $\operatorname{Cl}(X)_{\mathbb{Q}}$. The assertion is then an immediate consequence the above sequences, Remark 4.1 and Corollary 2.10. □

Remark 4.3. As the example of the projective spaces $\mathbb{P}^n$ shows, the bound of Proposition 4.2 is sharp.

Another application relates the automorphism group of X to that of a certain ambient toric variety. The choice of a minimal system of K-prime generators of the Cox ring R gives rise to a minimal presentation $\mathbb{K}[T_1, \ldots, T_r] \to R$. Fixing an ample class $[D] \in \mathrm{Cl}(X)$ of X leads to the commutative diagram

$$\begin{array}{ccc} \overline{X} & \longrightarrow & \overline{Z} \\ \cup\!\mid & & \cup\!\mid \\ \widehat{X} & \longrightarrow & \widehat{Z} \\ {\scriptstyle /\!/H}\downarrow & & \downarrow{\scriptstyle /\!/H} \\ X & \longrightarrow & Z \end{array}$$

where $\overline{Z} = \mathbb{K}^r$, the subsets $\widehat{X} \subseteq \overline{X}$ and $\widehat{Z} \subseteq \overline{Z}$ are the sets of semistable points defined by $[D]$ and the horizontal arrows are closed embeddings. The quotient variety $Z = \widehat{Z}/\!/H$ is a projective toric variety. We refer to [1, Sec. 3.2.5] for the details of this construction.

Theorem 4.4. *Consider the closed embedding $X \subseteq Z$ into the toric variety Z given above.*

(i) *There is an open subset $U \subseteq Z$ with $U \cap X \neq \emptyset$ such that all maps of $\mathrm{Bir}_2(Z)$ induce automorphisms of U.*

(ii) *According to* (i), *the stabilizer $\mathrm{Stab}_X(\mathrm{Bir}_2(Z))$ is well defined and, with the subgroup $\mathrm{Cent}_X(\mathrm{Bir}_2(Z))$ of elements defining the identity on X, we have*

$$\mathrm{Bir}_2(X) \;\cong\; \mathrm{Stab}_X(\mathrm{Bir}_2(Z))/\mathrm{Cent}_X(\mathrm{Bir}_2(Z)).$$

(iii) *With the stabilizer $\mathrm{Stab}_X(\mathrm{Aut}(Z))$ of X and the subgroup $\mathrm{Cent}_X(\mathrm{Aut}(Z))$ of all elements leaving all points of X fixed, we have*

$$\mathrm{Aut}(X)^0 \;\cong\; (\mathrm{Stab}_X(\mathrm{Aut}(Z))/\mathrm{Cent}_X(\mathrm{Aut}(Z)))^0.$$

(iv) *Let $w_i := \deg(T_i) \in \mathrm{Cl}(X)$ be the degrees of the generators $T_i \in S = \mathbb{K}[T_1, \ldots, T_r]$. If $S_{w_1} \cup \ldots \cup S_{w_r}$ contains no relations of $\mathcal{R}(X)$, then we have*

$$\mathrm{Bir}_2(X) \;\cong\; \mathrm{Stab}_X(\mathrm{Bir}_2(Z)), \qquad \mathrm{Aut}(X)^0 \;\cong\; \mathrm{Stab}_X(\mathrm{Aut}(Z))^0.$$

Proof. For (i) recall from the proof of [2, Thm. 2.1] that the image $U \subseteq Z$ under $\widehat{Z} \to Z$ of the intersection over all possible nonempty sets of semistable points of the H-action on Z is a nonempty open set, where all elements of $\mathrm{Bir}_2(Z)$ define automorphisms. By this construction, we have $U \cap X \neq \emptyset$.

Assertion (ii) is proved by relating the groups $\mathrm{Bir}_2(Z)$ and $\mathrm{Bir}_2(X)$ to each other via the following commutative diagram, where all vertical and horizontal sequences

are exact:

$$\begin{tikzcd}
1 \arrow[d] & & 1 \arrow[d] & & 1 \arrow[d] & \\
H \arrow[d] & = & H \arrow[d] & = & H \arrow[d] & \\
\mathrm{Aut}_H(\overline{Z}) \arrow[d] & \supseteq & \mathrm{Stab}_{\overline{X}}(\mathrm{Aut}_H(\overline{Z})) \arrow[r] \arrow[d] & \mathrm{Aut}_H(\overline{X}) \arrow[r] \arrow[d] & 1 & \\
\mathrm{Bir}_2(Z) \arrow[d] & \supseteq & \mathrm{Stab}_X(\mathrm{Bir}_2(Z)) \arrow[r] \arrow[d] & \mathrm{Bir}_2(X) \arrow[r] \arrow[d] & 1 & \\
1 & & 1 & 1 & &
\end{tikzcd}$$

To obtain the diagram, we combined Proposition 2.3 via Remark 4.1 with the upper horizontal sequence of [2, Thm. 2.1] stated before.

Assertion (iii) is proved in the same manner as (ii), one just uses the vertical sequence involving $\mathrm{CAut}_K(R)$ of Proposition 2.3 and the sequence of [2, Cor. 2.8] given above. Assertion (iv) is then a direct application of Proposition 2.3 (ii). $\square$

It seems that details on the full automorphism group $\mathrm{Aut}(X)$ are not directly visible from the picture developed above. Even in the more accessible special case of X having a torus action of complexity one, where for example the root subgroups can be determined explicitly [2], we do not know how to recognize instantaneously the dimension or the number of its connected components.

However, we may study $\mathrm{Aut}(X)$ by means of the algorithms developed in the preceding sections. As in Section 2, we fix a minimal presentation $R = S/I$ with $S = \mathbb{K}[T_1, \ldots, T_r]$ for the K-graded Cox ring R of X. As mentioned before, $\widehat{X} \subseteq \overline{X}$ is the set of semistable points of an ample class $w = [D]$ in $K = \mathrm{Cl}(X)$. In fact, the ample cone of X is the relative interior of the *GIT-cone* $\lambda \subseteq K_{\mathbb{Q}}$ determined by w; see, e.g., [1,3].

Algorithm 4.5 (Computing $\mathrm{Aut}_H(\widehat{X})$)**.** *Input:* the K-graded Cox ring $R = S/I$ and an ample class $w \in K$ of a Mori dream space X.

- Compute the GIT-cone $\lambda \subseteq K_{\mathbb{Q}}$ of w, e.g., using [14].
- Compute the subgroup $\Sigma \subseteq \mathrm{Aut}(\Omega_S)$ stabilizing $\lambda \subseteq K_{\mathbb{Q}}$.
- Run Algorithm 3.9 with input S, I and Σ.

Output: the ideal $J'' \subseteq \mathbb{K}[T_{ij}; 1 \leq i, j \leq k]$ computed by Algorithm 3.9. It describes $\mathrm{Aut}_H(\widehat{X}) \subseteq \mathrm{GL}(k)$.

Proof. The subgroup $\mathrm{Aut}_H(\widehat{X}) \subseteq \mathrm{Aut}_H(\overline{X})$ is the stabilizer of the subset $\widehat{X} \subseteq \overline{X}$. By definition of the set $\widehat{X}$ of semistable points associated with the ample class w, the graded algebra automorphisms defining elements of $\mathrm{Aut}_H(\widehat{X})$ are precisely the $(\varphi, \psi) \in \mathrm{Aut}_K(R)$ such that ψ stabilizes the GIT-cone λ. $\square$

Passing from $\mathrm{Aut}_H(\widehat{X})$ to $\mathrm{Aut}(X)$ involves an invariant ring computation, which in our setting amounts to computing a degree zero *Veronese subalgebra*. Recall

that for a K-graded $\mathbb{K}$-algebra R and a subgroup $K' \subseteq K$, the associated Veronese subalgebra is

$$R(K') \;:=\; \bigoplus_{w \in K'} R_w \;\subseteq\; \bigoplus_{w \in K} R_w \;=\; R.$$

For any graded presentation $R = S/I$ with $S = \mathbb{K}[T_1, \ldots, T_r]$, the *degree map* is the linear homomorphism $\delta\colon \mathbb{Z}^r \to K$ with $\delta(e_i) = \deg(T_i) \in K$.

Algorithm 4.6 (Computing Veronese subalgebras)**.** *Input:* a K-graded $\mathbb{K}$-algebra $R = S/I$ where $S = \mathbb{K}[T_1, \ldots, T_r]$ and a subgroup $K' \subseteq K$.

- Compute generators $\mu_1, \ldots, \mu_m$ for the monoid $\delta^{-1}(K') \cap \mathbb{Z}^r_{\geq 0}$.
- Compute the preimage $\Psi^{-1}(I)$ under $\Psi\colon \mathbb{K}[Y_1, \ldots, Y_m] \to S$, $Y_j \mapsto T^{\mu_j}$.

Output: the ideal $\Psi^{-1}(I) \subseteq \mathbb{K}[Y_1, \ldots, Y_m]$. Then $R(K')$ is isomorphic to the K'-graded $\mathbb{K}$-algebra $\mathbb{K}[Y_1, \ldots, Y_m]/\Psi^{-1}(I)$.

Remark 4.7. The first step of Algorithm 4.6 can be carried out by a Hilbert basis computation for a pointed, convex, rational cone.

In order to apply Algorithm 4.6, we have to identify the characteristic quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ of X inside $\operatorname{Aut}_H(\widehat{X})$ in terms of the description as a subgroup of $\mathrm{GL}(k)$ as provided by Algorithm 4.5. The key observation for this is the following.

Remark 4.8. Consider the monomial basis $\mathcal{B} = (b_1, \ldots, b_n)$ of the vector subspace $V \subseteq R$ as in Construction 3.2. Setting $\deg(T_{ij}) := u_j := \deg(b_j) \in K$, we define a K-grading on

$$\mathcal{O}(\mathrm{GL}(n)) \;=\; \mathbb{K}[T_{ij};\ 1 \leq i, j \leq n]_{\det}.$$

Observe that det is K-homogeneous. The action of $h \in H$ on $A \in \mathrm{GL}(n)$ is then given by multiplying from the right to A the diagonal matrix

$$\operatorname{diag}(\chi^{u_1}(h), \ldots, \chi^{u_n}(h)).$$

The ideals provided by Algorithms 3.7 and 3.8 are K-homogeneous and, finally, the K-grading of $\mathcal{O}(\mathrm{GL}(n))$ induces a K-grading of $\mathcal{O}(\mathrm{GL}(k))$, where $\mathrm{GL}(k)$ is the ambient group of $\operatorname{Aut}_{K,\Sigma}(R)$ provided by Algorithm 3.9, which in turn represents H as a subgroup of $\operatorname{Aut}_H(\widehat{X})$.

Algorithm 4.9 (Computing the Hopf algebra $\mathcal{O}(\operatorname{Aut}(X))$)**.** *Input:* the K-graded Cox ring $R = S/I$ and an ample class $w \in K$ of a Mori dream space X.

- Compute with Algorithm 4.5 the ideal $J'' \subseteq S' := \mathbb{K}[T_{ij};\ 1 \leq i, j \leq k]$ of $\operatorname{Aut}_H(\widehat{X}) \subseteq \mathrm{GL}(k)$.
- Install the K-grading from Remark 4.8 on S' and compute with Algorithm 4.6 a presentation $(S'/J'')(0) = \mathbb{K}[Y_1, \ldots, Y_m]/J$.

Output: $\mathbb{K}[Y_1, \ldots, Y_m]/J$. This is the Hopf algebra $\mathcal{O}(\operatorname{Aut}(X))$ of $\operatorname{Aut}(X)$.

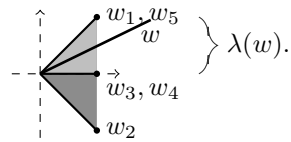
Proof. From the previous discussion, we have $\operatorname{Aut}(X) \cong \operatorname{Aut}_H(\widehat{X})/H$. To mod out the subgroup H, one passes to the degree zero Veronese subalgebra of the K-grading of $\mathcal{O}(\operatorname{Aut}_H(\widehat{X}))$ established in Remark 4.8. $\square$

Remark 4.10. In Algorithm 4.9, the Hopf algebra structure on $\mathcal{O}(\operatorname{Aut}(X))$ is inherited from that of $\mathrm{GL}(k)$ via inclusion and restriction:

$$\mathcal{O}(\mathrm{GL}(k)) \;\to\; \mathcal{O}\left(\operatorname{Aut}_H(\widehat{X})\right) \;\supseteq\; \mathcal{O}(\operatorname{Aut}(X)).$$

Example 4.11 ($A_3 2A_1$-singular Gorenstein log del Pezzo $\mathbb{K}^*$-surface II)**.** We continue Example 3.14. Since X is a surface, there is exactly one GIT-cone $\lambda(w)$ within $\mathrm{Mov}(X)$; we can choose $w := (2,1) \in K_{\mathbb{Q}}$ and obtain

$$\begin{aligned} \mathrm{Aut}_H\big(\widehat{X}\big) &= \mathrm{Aut}_H(\overline{X}) \\ &\cong \mathrm{Aut}_K(R) \\ &= \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times (\mathbb{K}^*)^3), \end{aligned}$$



We can then compute $\mathrm{Aut}(X)$ with Algorithm 4.9. As an affine variety, it is given by

$$\begin{aligned} \mathrm{Aut}(X) \;\cong\; V(\mathbb{K}^{16};\; & T_{16}, T_{14}+T_{15}-1, T_{11}, T_{10}, T_7, T_5, T_4-T_6+T_{12}, T_3-T_8-T_9, \\ & T_2-T_9, T_1-T_8, T_{15}^2-T_{15}, T_{12}T_{15}-T_{12}, T_9T_{15}, T_8T_{15}-T_8, \\ & T_6T_{15}-T_{12}, T_8T_{12}-T_{15}, T_6T_9+T_{15}-1, T_9^2T_{13}-T_6^2+T_{12}^2, \\ & T_8^2T_{13}-T_{12}^2). \end{aligned}$$

As a group, one verifies that $\mathrm{Aut}(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{K}^*$. The four components of $\mathrm{Aut}_H(\widehat{X})$ are mapped onto the two components of $\mathrm{Aut}(X)$, which in turn are given by

$$\begin{aligned} V(\mathbb{K}^{16};\; & T_{16}, T_{15}-1, T_{14}, T_{11}, \\ & T_{10}, T_9, T_7, T_6-T_{12}, T_5, \\ & T_4, T_3-T_8, T_2, T_1-T_8, \\ & T_8T_{12}-1, T_8^2T_{13}-T_{12}^2), \end{aligned} \qquad \begin{aligned} V(\mathbb{K}^{16};\; & T_{16}, T_{15}, T_{14}-1, T_{12}, T_{11}, \\ & T_{10}, T_8, T_7, T_5, T_4-T_6, \\ & T_3-T_9, T_2-T_9, T_1, \\ & T_6T_9-1, T_9^2T_{13}-T_6^2). \end{aligned}$$

5. Applications

We discuss three applications. In the first one, we consider the blow-up of a projective space $\mathbb{P}^3$ in a special configuration of six points; recall that the two-dimensional analogue leads to generalized del Pezzo surfaces. We check if our variety admits an effective two-torus action, as a naive glimpse at the defining equations of its Cox ring could suggest.

Example 5.1. Consider the blow-up $X \to \mathbb{P}^3$ along the points $[1,0,0,1]$, $[0,1,1,0]$ and the four standard toric fixed points. According to [10, Thm. 7.1(iv)], the Cox ring R of X is given by

$$\begin{array}{l} R = \mathbb{K}[T_1, \ldots, T_{12}]/I \\ \text{where } I \text{ is generated by} \\ T_3T_8 - T_5T_{12} - T_2T_9, \\ T_4T_7 - T_6T_{11} - T_1T_{10}, \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the columns of the matrix are the degrees $\deg(T_i) \in K := \mathbb{Z}^7$ of the variables T_i. Using Algorithm 3.8, we obtain that $\mathrm{Aut}_K(R)$ is of dimension 8. Thus, $\mathrm{Aut}(X)$ is one-dimensional and cannot contain a two-dimensional torus.

The second application concerns singular cubic surfaces $X \subseteq \mathbb{P}^3$ with at most ADE-singularities. There are 20 singularity types of such surfaces; in some types, there occur infinite families due to the existence of parameters in the equation describing the surface in $\mathbb{P}^3$; we refer to [8, Sec. 8, 9] for details. The Cox rings of singular cubic surfaces with at most ADE singularities have been determined in [6,7]. So far, the automorphism groups have been computed in the cases without parameters [15].

Theorem 5.2. *Let $X \subseteq \mathbb{P}^3$ be a singular cubic surface with at most ADE singularities and parameters in its defining equations. Depending on the ADE singularity type $S(X)$, the automorphism group $\mathrm{Aut}(X)$ for a general choice of parameters is the following:*

$S(X)$	A_3	A_2A_1	$2A_2$	$3A_1$	A_2	$2A_1$	A_1
$\mathrm{Aut}(X)$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	$\mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}$	S_3	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$

Remark 5.3. Consider the output of Algorithm 3.9. The ideal describing $H = \operatorname{Spec} \mathbb{K}[K]$ inside $\mathrm{GL}(k)$ is obtained as follows. Let Q denote the degree matrix of the induced K-grading of $\mathcal{O}(\mathrm{GL}(k))$ from Remark 4.8 and compute a Gale dual for Q, i.e., a matrix P fitting into

$$0 \longleftarrow K \xleftarrow[Q]{} \mathbb{Z}^{k^2} \xleftarrow[P^*]{} \ker(Q) \longleftarrow 0 \,.$$

Next, compute the lattice ideal $I(P) \subseteq \mathbb{K}[T_{ij};\ 1 \le i,j \le k]$ associated with P through $E_k \in \mathrm{GL}(k)$; see e.g. [10, Rem. 5.5]. Then $I(P)$ describes $H \subseteq \mathrm{CAut}_K(R)$ inside $\mathrm{GL}(k)$.

Proof of Theorem 5.2. The families of Cox rings of the resolutions of cubic surfaces have been determined in [7]. The Cox rings of the cubic surfaces are obtained by contracting all (-2)-curves according to [10, Prop. 2.1], respectively. The cases involving parameters have the singularity types listed in the table. Write $R := \mathcal{R}(X)$ and $K := \mathrm{Cl}(X)$ for the respective cubic surface X.

Cases A_3, A_2A_1, $3A_1$, A_2, $2A_1$ and A_1: In each case, we use Algorithm 3.9 to compute the ideal describing $\mathrm{CAut}_K(R) \subseteq \mathrm{GL}(k)$. Remark 5.3 delivers the ideal describing the inclusion $H \subseteq \mathrm{GL}(k)$ of the characteristic quasitorus H of X. Comparing ideals, we verify that $H = \mathrm{CAut}_K(R)$ holds. Therefore, $\mathrm{Aut}(X)$ is isomorphic to $\Gamma = \mathrm{Aut}_K(R)/H$. The group Γ is obtained using Remark 3.12.

Case $2A_2$: The characteristic quasitorus H is isomorphic to $(\mathbb{K}^*)^3$ and we have $R = S/I$ with $S = \mathbb{K}[T_1, \ldots, T_7]$ and an ideal $I \subseteq S$. In the setting of Construction 3.2, we have $\mathcal{B} = (T_1, \ldots, T_7)$ and $\Omega_S = \{w_1, \ldots, w_7\}$ with $w_i := \deg(T_i)$. An application of Algorithm 3.7 shows that $\mathrm{Aut}_K(S)$ consists of the matrices $A, AB \in \mathrm{GL}(7)$, where A is a Ω_S-diagonal matrix and B a $\mathrm{Aut}(\Omega_S)$-permuting matrix as follows:

$$A = \mathrm{diag}(*, \ldots, *) \in \mathrm{GL}(7),$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}(7). \tag{1}$$

Using Algorithm 3.8, we compute an ideal $J \subseteq \mathbb{K}[T_{i,j};\, 1 \le i \le 7]$ describing $\mathrm{Aut}_K(R) \subseteq \mathrm{GL}(7)$; it is given by

$$J \;=\; J' \cdot B^* J' \quad \text{with}$$
$$J' \;=\; \left\langle \begin{matrix} T_{2,2}T_{3,3} - T_{1,1}T_{4,4}, & -aT_{5,5}^2 + aT_{6,6}T_{7,7}, \\ (a-1)T_{1,1}T_{4,4} + (-a+1)T_{5,5}^2, & T_{2,2}T_{3,3} - T_{6,6}T_{7,7}, \end{matrix} \right\rangle + \langle T_{i,j};\, i \ne j \rangle,$$

where $a \in \mathbb{K}^* \setminus \{1\}$ is a parameter coming from the ideal of $X \subseteq \mathbb{P}^3$. Since B is $\mathrm{Aut}(\Omega_S)$-permuting, there is $\sigma_B \in \mathrm{Aut}(\Omega_S)$ such that $\deg(T_{j,j}) = \sigma_B(\deg(T_{i,i}))$. By the previous decomposition of $\mathrm{Aut}_K(S)$, we obtain a homomorphism

$$\alpha\colon \mathrm{Aut}_K(R) \;\to\; \Gamma \;\cong\; \mathbb{Z}/2\mathbb{Z}, \qquad B'A' \;\mapsto\; \sigma_{B'}.$$

Moreover, we have a homomorphism $\beta\colon \mathbb{K}^* \times H \to \mathrm{Aut}_K(R)$ that maps a tuple (v,s,t,u) to the product $A_v \cdot B_s \cdot C_t \cdot D_u$ where

$$A_v := \mathrm{diag}(v^{-1},1,1,v,1,1,1), \qquad B_s := \mathrm{diag}(s^{-1},s,s^{-1},s,1,1,1),$$
$$C_t := \mathrm{diag}(t^{-2},t,t^{-3},1,t^{-1},t^{-2},1), \qquad D_u := \mathrm{diag}(1,u,u^{-1},1,1,u^{-1},u).$$

One directly checks that β is well defined. Comparing entries in $A_v \cdot B_s \cdot C_t \cdot D_u$, we see that β is injective. We thus have a split exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{K}^* \times H \arrow[r, "\beta"] & \mathrm{Aut}_K(R) \arrow[r, "\alpha"] & \Gamma \arrow[r] \arrow[l, bend left, "\gamma"] & 1 \end{tikzcd},$$

where we define the homomorphism $\gamma\colon \Gamma \to \mathrm{Aut}_K(R)$ by mapping id to the 7×7-unit matrix and σ_B with B as in (1) to the element $BA' \in \mathrm{Aut}_K(R)$, where $A' := \mathrm{diag}(1,\dots,1)$. Thus, we have a semidirect product representation for $\mathrm{Aut}_K(R)$ which in turn descends to

$$\mathrm{Aut}(X) \;\cong\; \mathrm{Aut}_K(R)/H \;\cong\; \mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}. \qquad \square$$

Remark 5.4. Note that passing to special parameters for the surfaces of Theorem 5.2 may lead to larger automorphism groups. Whereas Algorithms 3.8 and 4.9 return equations including parameters, the description of the group needs to be done anew. As an example of this effect, consider the $\mathbb{Z}^7$-graded rings

$$R_\lambda \;:=\; \mathbb{K}[T_1,\dots,T_{10}]/\langle T_2T_5^2T_8 + T_3T_6^2T_9 + T_4T_7^2T_{10} - \lambda T_1T_2T_3T_4T_5T_6T_7 \rangle,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

where $\lambda \in \{0,1\}$ and the columns of the matrix are the degrees $\deg(T_i) \in \mathbb{Z}^7$. The rings R_λ are the Cox rings $\mathcal{R}(X_\lambda)$ of the minimal resolutions $X_\lambda \to X'_\lambda$ of cubic surfaces X'_λ with singularity type D_4; see [6, case D_4, p. 28]. One has $X_\lambda \cong X_1$ for $\lambda \ne 0$ and from [15, Thm. 2], we infer

$$\mathrm{Aut}(X_1) \;\cong\; S_3, \qquad \mathrm{Aut}(X_0) \;\cong\; \mathbb{K}^* \rtimes S_3.$$

Observe that Proposition 4.2 gives the bound $\dim(\mathrm{Aut}(X_\lambda)) \le 3$, which is not sharp in the present case.

Our third application is in computer algebra: Algorithm 3.8 is a linear algebra-based, Gröbner basis-free way to detect symmetries of graded ideals. These symmetries can then be used to speed up computations on ideals.

Example 5.5. The affine cone over the Grassmannian $G(2,5)$ has the coordinate ring $R = S/I$ where $S = \mathbb{K}[T_1, \dots, T_{10}]$ and I is generated by the Plücker relations

$$T_5T_{10} - T_6T_9 + T_7T_8, \qquad T_1T_9 - T_2T_7 + T_4T_5, \qquad T_1T_8 - T_2T_6 + T_3T_5,$$
$$T_1T_{10} - T_3T_7 + T_4T_6, \qquad T_2T_{10} - T_3T_9 + T_4T_8.$$

We use that R is effectively and pointedly graded by $\mathbb{Z}^5$: the degrees of the variables T_i are the columns of

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Applying Algorithm 3.8 to R, we obtain $\mathrm{Stab}_I(\mathrm{Aut}_K(S)) \cong S_5$. Ten of the 120 elements are just permutations of variables: written as elements of S_{10}, besides the identity we have

$$(10,9,7,4,8,6,3,5,2,1), \qquad (5,6,7,1,8,9,2,10,3,4), \qquad (10,3,6,8,4,7,9,1,2,5),$$
$$(8,9,2,5,10,3,6,4,7,1), \qquad (8,6,3,10,5,2,9,1,7,4), \qquad (4,3,2,1,10,9,7,8,6,5),$$
$$(1,7,6,5,4,3,2,10,9,8), \qquad (5,2,9,8,1,7,6,4,3,10), \qquad (4,7,9,10,1,2,3,5,6,8).$$

Then the computation of the Gröbner fan of I with `gfan` [13] using the `symmetries` option and the ten listed symmetries was computed instantly on a 5 year old machine (core2duo), whereas the case of no given symmetries took about two seconds.

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