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AN ENTROPY STABLE, HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. This article proves that a particular space-time, hybridizable discontinuous Galerkin method is entropy stable for the compressible Navier-Stokes equations. In order to facilitate the proof, 'entropy variables' are utilized to rewrite the compressible Navier-Stokes equations in a symmetric form. The resulting form of the equations is discretized with a hybridizable discontinuous finite element approach in space, and a classical discontinuous finite element approach in time. Thereafter, the initial solution is shown to continually bound the solutions at later times.

1. INTRODUCTION

In the last decade, hybridizable, discontinuous finite element methods (FEMs) have emerged as a promising alternative to classical discontinuous FEMs. These methods have several distinct advantages over their classical counterparts: 1) The hybridizable methods maintain high-order accuracy at the individual element level by introducing local degrees of freedom, as opposed to the global degrees of freedom that are introduced for this purpose in classical methods. The local degrees of freedom are in turn explicitly incorporated into local problems that facilitate parallelization, and are only incorporated into the global problem implicitly through the principle of 'static condensation.' 2) In accordance with 1), the number of global degrees of freedom for a particular time level approximately scales with $\mathcal{O}(p_s^{(d-1)})$ for hybridizable methods, while it scales with $\mathcal{O}\left(p_s^d\right)$ for classical methods, where p_s is the spatial polynomial order, as opposed to p_t which is the temporal polynomial order, and d is the number of spatial dimensions. Therefore, for sufficiently high order, i.e., sufficiently large values of p_s, the hybridizable methods will require less global degrees of freedom (cf. [58] for a more precise comparison). 3) The hybridizable methods achieve optimal rates of convergence for the velocity and the velocity gradient: $p_s + 2$ and $p_s + 1$, respectively, for $p_s \geq 1$ and the appropriate post-processing within incompressible, smooth, viscous-dominated flows; alternatively classical methods achieve suboptimal convergence: $p_s + 1$ and p_s, respectively [48].

Broadly speaking, there are two distinct classes of hybridizable discontinuous FEMs: hybridizable discontinuous Galerkin (HDG) methods and hybridizable discontinuous Petrov-Galerkin (HDPG) methods. This article will focus on the former, arguably more popular class of methods, although, the interested reader is referred

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to [41–43] for more information about HDPG methods. For HDG methods, there have been significant efforts to leverage their aforementioned advantages, particularly in the field of computational fluid dynamics [2, 17, 18, 32, 34, 38, 47–50, 53, 54, 57, 59]. (Note that this list of references is by no means exhaustive). In the course of these efforts, there have been attempts to rigorously quantify the stability of HDG methods. In particular, Cockburn et al. ([9, 10, 12, 13]) proved existence and uniqueness of HDG solutions for a large class of elliptic model problems. Nguyen et al. [44, 45] proved that certain HDG schemes satisfy energy inequalities for large classes of linear and nonlinear steady, convection-diffusion model problems. Nguyen et al. [46], and Cockburn and Gopalakrishnan [11], and Egger and Waluga [16] proved the existence and uniqueness of HDG solutions for the steady Stokes equations. Finally, Rhebergen and Cockburn [53] proved the existence and uniqueness of HDG solutions for the Oseen equations.

The primary goal of this article is to extend prior research efforts by rigorously establishing the stability of HDG schemes for a larger class of problems. In particular, this work will focus on proving the stability of a space-time HDG scheme for the unsteady, compressible Navier-Stokes (NS) equations for all $d \geq 2$, $p_s \geq 1$, and $p_t \geq 0$. This scheme technically utilizes an HDG approach in space and a DG approach in time, but it will be henceforth referred to as a space-time HDG scheme for the sake of brevity. To the author's knowledge, this is the first time that such an ambitious proof of stability has been attempted for such a scheme. In what follows, the proof of stability will require a symmetrization of the compressible NS equations via the utilization of so-called 'entropy variables'. The existence of an entropy-based symmetric formulation of the compressible NS equations has been known for some time, and has been discussed in significant detail in [22, 23, 37, 40]. There have been several recent efforts to leverage the symmetric formulation to prove stability of various numerical schemes (for instance to name *just a few* efforts [7, 8, 19–21]). However, overall, there have been relatively few efforts to leverage the symmetric formulation to prove the stability of FEMs. In this regard, the work of Hughes et al. [31], Shakib et al. [55], Barth [4], Jiang et al. [33], and Hou et al. [29] stands out. In [31] and [55], a straightforward symmetrization is introduced, and is successfully employed to prove the entropy stability of a space-time, streamline upwind Petrov-Galerkin (SUPG) scheme for the compressible NS equations. In [4], an alternative symmetrization is introduced and employed to prove the entropy stability of space-time DG and Galerkin least squares (GLS) schemes for the inviscid, compressible NS equations. Finally, in [33] and [29], the entropy stability of a DG method is established for nonlinear systems of hyperbolic conservation laws (a category of laws that contains the inviscid, compressible NS equations). As a result of this work, the entropy stability of classical FEMs is well established. In what follows, this work is expanded to treat hybridizable methods.

The format of this paper is as follows. Section 2 introduces a symmetric formulation of the unsteady, compressible NS equations, and then section 3 discretizes it using an HDG approach in space and a DG approach in time. Section 4 proves the stability of the resulting space-time HDG scheme for the unsteady, compressible NS equations. In this section, stability is first established for the inviscid form of the equations and then the viscous form. Then, section 5 presents some concluding remarks. Finally, Appendix A explains the notation used throughout the article, and Appendix B contains several lemmas that support the proofs that are presented in section 4.

2. The compressible NS equations

Consider the unsteady, compressible NS equations written in conservative form with associated boundary conditions as follows:

$$\boldsymbol{u}_{,t} + \boldsymbol{f}^i_{,\boldsymbol{x}_i} - \boldsymbol{g}^i_{,\boldsymbol{x}_i} = 0 \quad \text{in} \quad \Omega \times T, \tag{2.1}$$

$$L\left(\boldsymbol{u}, \boldsymbol{u}^\partial, \boldsymbol{u}_{,\boldsymbol{x}}, \boldsymbol{u}^\partial_{,\boldsymbol{x}}\right) = 0 \quad \text{on} \quad \partial\Omega \times T, \tag{2.2}$$

where t represents the temporal coordinate in the one-dimensional domain T, $\boldsymbol{x} \in \mathbb{R}^d$ denotes the spatial coordinates in the d-dimensional domain Ω, $\partial\Omega$ denotes the $(d-1)$-dimensional boundary of Ω, $\boldsymbol{u}$ denotes the m-valued solution, $\boldsymbol{f}^i = \boldsymbol{f}^i(\boldsymbol{u})$ denotes the m-valued inviscid fluxes in the $i = 1, \ldots, d$ directions, $\boldsymbol{g}^i = \boldsymbol{g}^i(\boldsymbol{u}, \boldsymbol{u}_{,\boldsymbol{x}})$ denotes the m-valued viscous fluxes also in the $i = 1, \ldots, d$ directions, (where it should be noted that the gradient $\boldsymbol{u}_{,\boldsymbol{x}} = \left\{\boldsymbol{u}_{,\boldsymbol{x}_j}\right\}$ has components in the $j = 1, \ldots, d$ directions as well), and L is a linear function of the solution, its gradient, and the boundary data. The solution, inviscid fluxes, and viscous fluxes take the following precise forms when $d = 3$ and $m = 5$,

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho\mathbf{V}^1 \\ \rho\mathbf{V}^2 \\ \rho\mathbf{V}^3 \\ \rho\left(e + \frac{1}{2}\mathbf{V}^k\mathbf{V}^k\right) \end{bmatrix}, \tag{2.3}$$

$$\boldsymbol{f}^1 = \begin{bmatrix} \rho\mathbf{V}^1 \\ \rho\mathbf{V}^1\mathbf{V}^1 + p \\ \rho\mathbf{V}^1\mathbf{V}^2 \\ \rho\mathbf{V}^1\mathbf{V}^3 \\ \rho\mathbf{V}^1\left(e + \frac{1}{2}\mathbf{V}^k\mathbf{V}^k + \frac{p}{\rho}\right) \end{bmatrix}, \boldsymbol{f}^2 = \begin{bmatrix} \rho\mathbf{V}^2 \\ \rho\mathbf{V}^1\mathbf{V}^2 \\ \rho\mathbf{V}^2\mathbf{V}^2 + p \\ \rho\mathbf{V}^2\mathbf{V}^3 \\ \rho\mathbf{V}^2\left(e + \frac{1}{2}\mathbf{V}^k\mathbf{V}^k + \frac{p}{\rho}\right) \end{bmatrix},$$
$$\boldsymbol{f}^3 = \begin{bmatrix} \rho\mathbf{V}^3 \\ \rho\mathbf{V}^1\mathbf{V}^3 \\ \rho\mathbf{V}^2\mathbf{V}^3 \\ \rho\mathbf{V}^3\mathbf{V}^3 + p \\ \rho\mathbf{V}^3\left(e + \frac{1}{2}\mathbf{V}^k\mathbf{V}^k + \frac{p}{\rho}\right) \end{bmatrix}, \tag{2.4}$$

$$\boldsymbol{g}^1 = \begin{bmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ \tau_{31} \\ \tau_{1j}\mathbf{V}^j + \kappa T_{,\boldsymbol{x}_1} \end{bmatrix}, \boldsymbol{g}^2 = \begin{bmatrix} 0 \\ \tau_{12} \\ \tau_{22} \\ \tau_{32} \\ \tau_{2j}\mathbf{V}^j + \kappa T_{,\boldsymbol{x}_2} \end{bmatrix}, \boldsymbol{g}^3 = \begin{bmatrix} 0 \\ \tau_{13} \\ \tau_{23} \\ \tau_{33} \\ \tau_{3j}\mathbf{V}^j + \kappa T_{,\boldsymbol{x}_3} \end{bmatrix}, \tag{2.5}$$

where ρ is the density, $\mathbf{V} = \left\{\mathbf{V}^i\right\}$ is the velocity vector, e is the internal energy, p is the pressure, κ is the heat conductivity coefficient, T is the temperature, and τ is the viscous stress tensor

$$\tau_{ij} = \mu\left(\mathbf{V}^i_{,\boldsymbol{x}_j} + \mathbf{V}^j_{,\boldsymbol{x}_i}\right) - \frac{2}{3}\mu\delta_{ij}\mathbf{V}^l_{,\boldsymbol{x}_l}, \tag{2.6}$$

and where μ is the shear viscosity coefficient.

In their present form, the compressible NS equations (equation (2.1)) are difficult to analyze or discretize. As a result, it is common practice (cf. for instance [26,27]) to rewrite the compressible NS equations as a first-order system of equations as follows:

$$\boldsymbol{\sigma}^i - \boldsymbol{g}^i = 0, \tag{2.7}$$

$$\boldsymbol{u}_{,t} + \boldsymbol{f}^i_{,\boldsymbol{x}_i} - \boldsymbol{\sigma}^i_{,\boldsymbol{x}_i} = 0, \tag{2.8}$$

where simplification to a first-order system has required the introduction of an auxiliary variable $\boldsymbol{\sigma}^i$. In order to further facilitate analysis, one may 'symmetrize' (2.7) and (2.8) by introducing the m-valued entropy variable $\boldsymbol{v}$, where for $d = 3$ and $m = 5$,

$$\boldsymbol{v} = \frac{1}{e(\gamma - 1)} \begin{bmatrix} -\frac{\mathbf{V}^k \mathbf{V}^k - 2e\left(\gamma - \log\left(e(\gamma-1)\rho^{1-\gamma}\right)\right)}{2e(\gamma-1)} \\ \mathbf{V}^1 \\ \mathbf{V}^2 \\ \mathbf{V}^3 \\ -1 \end{bmatrix}, \tag{2.9}$$

and where γ is the ratio of specific heats. This formulation of $\boldsymbol{v}$ arises from the work of [31] and [55]. Although the precise formulation is not necessary for subsequent analysis, it is provided here for the sake of completeness.

On writing the solution as a function of the entropy variable ($\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{v})$) and then introducing the result into (2.7) and (2.8), one obtains

$$\boldsymbol{\sigma}^i - \boldsymbol{g}^i_{,\boldsymbol{u}_{,\boldsymbol{x}_j}} \boldsymbol{u}_{,\boldsymbol{v}} \boldsymbol{v}_{,\boldsymbol{x}_j} = 0, \tag{2.10}$$

$$\boldsymbol{u}_{,\boldsymbol{v}} \boldsymbol{v}_{,t} + \boldsymbol{f}^i_{,\boldsymbol{u}} \boldsymbol{u}_{,\boldsymbol{v}} \boldsymbol{v}_{,\boldsymbol{x}_i} - \boldsymbol{\sigma}^i_{,\boldsymbol{x}_i} = 0. \tag{2.11}$$

In line with the approach of Barth [4], one may introduce the following definitions for the quantities that appear in (2.10) and (2.11)

$$M_{ij} = \boldsymbol{g}^i_{,\boldsymbol{u}_{,\boldsymbol{x}_j}}, \quad \widetilde{A}_0 = \boldsymbol{u}_{,\boldsymbol{v}}, \quad \widetilde{M}_{ij} = M_{ij}\widetilde{A}_0, \tag{2.12}$$

$$A_i = \boldsymbol{f}^i_{,\boldsymbol{u}}, \quad \widetilde{A}_i = A_i \widetilde{A}_0, \tag{2.13}$$

where $\widetilde{A}_0$ is symmetric positive-definite (SPD), $\{\widetilde{M}_{ij}\}$ is symmetric positive semi-definite (SPSD), and $\widetilde{A}_i$ is symmetric. These matrices are associated with real-valued, scalar functions $U(\boldsymbol{u})$, $F^i(\boldsymbol{u})$, $\mathcal{U}(\boldsymbol{v})$, and $\mathcal{F}^i(\boldsymbol{v})$ defined such that

$$\mathcal{U}_{,\boldsymbol{v}} = \boldsymbol{u}^T, \qquad \mathcal{U}_{,\boldsymbol{v},\boldsymbol{v}} = \boldsymbol{u}_{,\boldsymbol{v}}, \tag{2.14}$$

$$\mathcal{F}^i_{,\boldsymbol{v}} = \left(\boldsymbol{f}^i\right)^T, \qquad \mathcal{F}^i_{,\boldsymbol{v},\boldsymbol{v}} = \boldsymbol{f}^i_{,\boldsymbol{v}}, \tag{2.15}$$

$$U(\boldsymbol{u}) = \boldsymbol{v}^T(\boldsymbol{u})\,\boldsymbol{u} - \mathcal{U}(\boldsymbol{v}(\boldsymbol{u})), \qquad F^i(\boldsymbol{u}) = \boldsymbol{v}^T(\boldsymbol{u})\,\boldsymbol{f}^i(\boldsymbol{u}) - \mathcal{F}^i(\boldsymbol{v}(\boldsymbol{u})). \tag{2.16}$$

From the definitions of $U(\boldsymbol{u})$, $F^i(\boldsymbol{u})$, $\mathcal{U}(\boldsymbol{v})$, and $\mathcal{F}^i(\boldsymbol{v})$, it immediately follows (cf. [4]) that the following lemma holds.

Lemma 2.1 (Barth [4], p. 199)**.**

$$\boldsymbol{v}^T = U_{,\boldsymbol{u}}, \qquad \boldsymbol{v}^T \boldsymbol{f}^i_{,\boldsymbol{u}} = F^i_{,\boldsymbol{u}}. \tag{2.17}$$

Proof. The proof appears in [4]. □

The identities in (2.13)–(2.17) will be utilized directly and indirectly in the subsequent stability analysis. For now, one may set these aside, and proceed by using the definitions in (2.12) in order to rewrite (2.10) and (2.11) as follows:

$$\boldsymbol{\sigma}^i - \widetilde{M}_{ij}(\boldsymbol{v})\, \boldsymbol{v}_{,\boldsymbol{x}_j} = 0, \tag{2.18}$$

$$\boldsymbol{u}_{,t}(\boldsymbol{v}) + \boldsymbol{f}^i_{,\boldsymbol{x}_i}(\boldsymbol{v}) - \boldsymbol{\sigma}^i_{,\boldsymbol{x}_i} = 0. \tag{2.19}$$

3. A space-time HDG scheme

One can now proceed to begin discretizing (2.18) and (2.19). Towards this end, consider tessellating the space-time domain $(\Omega \times T)$ with space-time elements. Let $I^n = (t^n, t^{n+1})$ denote the nth time interval (where $T = \bigcup_n I^n$) and let $\mathcal{T}_h$ denote the d-triangulation of Ω that contains nonoverlapping, d-dimensional simplex elements T_k. Here, it is assumed that Ω is polygonal, so that Ω and $\mathcal{T}_h$ coincide, i.e., $\Omega = \bigcup_{T_k \in \mathcal{T}_h} T_k$. The full space-time elements $T_k \times I^n$ are shown in Figure 1.

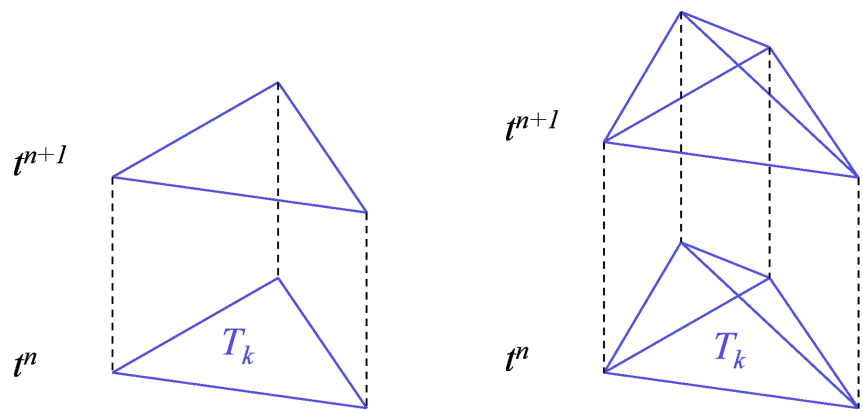

FIGURE 1. Space-time elements for $d = 2$ (left) and $d = 3$ (right).

In addition, one may introduce $\mathcal{E}_h$ in order to denote the set of $(d-1)$-dimensional faces (F_ℓ) of the d-triangulation. For each face, it is possible to arbitrarily label one side as the '$-$' side and the opposite side as the '$+$' side. With this convention, one may define for each face a unit normal $\widehat{\boldsymbol{n}}$ that points from the '$-$' to the '$+$' side of a face.

Now, it is useful to note that the set of d-triangulation faces can be partitioned such that $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$, where $\mathcal{E}_h^0$ and $\mathcal{E}_h^\partial$ are the interior and boundary faces, respectively (cf. Figure 2 that illustrates this for $d = 2$). Also, since Ω is polygonal, its boundary $\partial\Omega$ coincides with $\mathcal{E}_h^\partial$, i.e. $\partial\Omega = \bigcup_{F_{\ell \in \mathcal{E}_h^\partial}} F_\ell$.

In a natural fashion, the $(d-1)$-dimensional faces of the individual elements, $\partial T_k = \{\mathcal{F}_l\}, l = 1, \ldots, d+1$, coincide with the faces of the d-triangulation. For

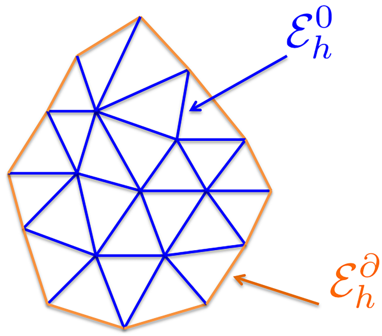

FIGURE 2. Interior and boundary faces for the spatial part of a mesh for $d = 2$.

an element, one can define unit normal vectors $\boldsymbol{n}$ for each face of the element such that they point outward from the interior of the element.

Having defined the d-triangulation $\mathcal{T}_h$ and its faces $\mathcal{E}_h$, one may define the following function spaces:

$$\begin{aligned}
\boldsymbol{\mathcal{W}}^h &= \Big\{ \boldsymbol{w}^h : \boldsymbol{w}^h \in \left(\boldsymbol{L}_2\left(\Omega \times I^n\right)\right)^m, \boldsymbol{w}^h|_{T_k \times I^n} \\
&\qquad \in \left(\boldsymbol{\mathcal{P}}^{(p_t,p_s)}\left(T_k \times I^n\right)\right)^m, \forall T_k \in \mathcal{T}_h \Big\}, \\
\widehat{\boldsymbol{\mathcal{W}}}^h &= \Big\{ \widehat{\boldsymbol{w}}^h : \widehat{\boldsymbol{w}}^h \in \left(\boldsymbol{L}_2\left(\mathcal{E}_h \times I^n\right)\right)^m, \widehat{\boldsymbol{w}}^h|_{F_\ell \times I^n} \\
&\qquad \in \left(\boldsymbol{\mathcal{P}}^{(p_t,p_s)}\left(F_\ell \times I^n\right)\right)^m, \forall F_\ell \in \mathcal{E}_h \Big\}, \\
\boldsymbol{\Phi}^h &= \Big\{ \boldsymbol{\phi}^h : \boldsymbol{\phi}^h \in \left(\left(\boldsymbol{L}_2\left(\Omega \times I^n\right)\right)^m\right)^d, \boldsymbol{\phi}^h|_{T_k \times I^n} \\
&\qquad \in \left(\left(\boldsymbol{\mathcal{P}}^{(p_t,p_s)}\left(T_k \times I^n\right)\right)^m\right)^d, \forall T_k \in \mathcal{T}_h \Big\},
\end{aligned}$$

where $\boldsymbol{\mathcal{P}}^{(p_t,p_s)}\left(T_k \times I^n\right)$ is the function space of polynomials of order $\leq p_t$ and $\leq p_s$ on $T_k \times I^n$.

3.1. **Discretizing the auxiliary equation.** In order to begin discretizing the auxiliary equation, consider substituting $\boldsymbol{\sigma}^h \in \boldsymbol{\Phi}^h$ and $\boldsymbol{v}^h \in \boldsymbol{\mathcal{W}}^h$ in place of $\boldsymbol{\sigma}$ and $\boldsymbol{v}$ in (2.18), multiplying the resulting expression by $\boldsymbol{\phi}^h \in \boldsymbol{\Phi}^h$, and integrating over space and time. Upon integrating the resulting expression by parts, replacing $\boldsymbol{v}^h$ in the spatial boundary integral with $\widehat{\boldsymbol{v}}^h$, integrating by parts again, and rearranging the result, one obtains

$$\begin{aligned}
&\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\left(\boldsymbol{\phi}^h\right)^i \right]^T \left(\boldsymbol{\sigma}^h\right)^i dx \right] dt \\
&\qquad = \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\left(\boldsymbol{\phi}^h\right)^i \right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{v}^h_{,\boldsymbol{x}_j} dx \right] dt \\
&\qquad\quad + \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\left(\boldsymbol{\phi}^h\right)^i \right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h \right] \boldsymbol{n}^j d\hat{x} \right] dt,
\end{aligned} \tag{3.1}$$

where $\widehat{\boldsymbol{v}}^h \in \widehat{\boldsymbol{\mathcal{W}}}^h$ and $\boldsymbol{n} \in \mathbb{R}^d$. It is now possible to obtain a definition for $\left(\boldsymbol{\sigma}^h\right)^i$ from (3.1). Towards this end, one may define the following 'element-based lifting operator' $\boldsymbol{R}$ such that

$$\int_{T_k} \vartheta_j^T \, \boldsymbol{R}^j \left(\boldsymbol{q}; \boldsymbol{n}\right) dx = \int_{\partial T_k} \vartheta_j^T \boldsymbol{q} \, \boldsymbol{n}^j d\hat{x}, \tag{3.2}$$

where $\vartheta \in \left(\left(\boldsymbol{L}_2\left(\Omega\right)\right)^m\right)^d$ and $\boldsymbol{q} \in \left(\boldsymbol{\mathcal{W}}^h|_{\partial T_k} + \widehat{\boldsymbol{\mathcal{W}}}^h|_{\partial T_k}\right)$. Note that the existence of $\boldsymbol{R}$ is assured by the Riesz representation theorem. On introducing the definition of $\boldsymbol{R}$ into (3.1), one obtains the following definition of $\left(\boldsymbol{\sigma}^h\right)^i$

$$\left(\boldsymbol{\sigma}^h\right)^i = \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\left[\boldsymbol{v}^h_{,\boldsymbol{x}_j} + \boldsymbol{R}^j\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)\right].$$

In a similar fashion, one can introduce the face-based lifting operator $\boldsymbol{r}_{\mathcal{F}_l}$,

$$\begin{aligned} \int_{T_k} \vartheta_j^T \, \boldsymbol{r}^j_{\mathcal{F}_l}\left(\boldsymbol{q}; \boldsymbol{n}\right) dx &= \int_{\mathcal{F}_l} \vartheta_j^T \boldsymbol{q} \, \boldsymbol{n}^j d\hat{x}, \\ \sum_{\mathcal{F}_l \in \partial T_k} \int_{T_k} \vartheta_j^T \, \boldsymbol{r}^j_{\mathcal{F}_l}\left(\boldsymbol{q}; \boldsymbol{n}\right) dx &= \int_{\partial T_k} \vartheta_j^T \boldsymbol{q} \, \boldsymbol{n}^j d\hat{x}, \end{aligned} \tag{3.3}$$

where it should be noted that the face-based and element-based lifting operators are related as follows:

$$\boldsymbol{R}^j\left(\boldsymbol{q}; \boldsymbol{n}\right) = \sum_{\mathcal{F}_l \in \partial T_k} \boldsymbol{r}^j_{\mathcal{F}_l}\left(\boldsymbol{q}; \boldsymbol{n}\right).$$

In turn, the definition of $\left(\boldsymbol{\sigma}^h\right)^i$ takes the following approximate 'Bassi-Rebay 2' (BR2)-type form (cf. [6]) on the individual faces of an element

$$\left.\left(\boldsymbol{\sigma}^h\right)^i\right|_{\mathcal{F}_l \in \partial T_k} \approx \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\left[\boldsymbol{v}^h_{,\boldsymbol{x}_j} + \eta_{\mathcal{F}_l} \, \boldsymbol{r}^j_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)\right],$$

where $\eta_{\mathcal{F}_l} \in \mathbb{R}$ is a user specified constant in the BR2 approach. Constraints on the value of $\eta_{\mathcal{F}_l}$ that are required in order to ensure stability, will be given in section 4.

3.2. **Discretizing the primary equation.** In order to begin discretizing the primary equation, consider substituting $\boldsymbol{\sigma}^h \in \boldsymbol{\Phi}^h$ and $\boldsymbol{v}^h \in \boldsymbol{\mathcal{W}}^h$ in place of $\boldsymbol{\sigma}$ and $\boldsymbol{v}$ in (2.19), multiplying the resulting expression by $\boldsymbol{w}^h \in \boldsymbol{\mathcal{W}}^h$, and integrating over space and time. Upon integrating the resulting expression by parts, replacing $\boldsymbol{f}^i \boldsymbol{n}^i$ and $\boldsymbol{\sigma}^i \boldsymbol{n}^i$ in the spatial boundary integrals with $\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ and $\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \boldsymbol{n}\right)$, respectively, and adding a stabilizing term, one obtains

$$\begin{aligned} &\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left(-\left[\boldsymbol{w}^h_{,t}\right]^T \boldsymbol{u}\left(\boldsymbol{v}^h\right) - \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i}\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\right) + \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i}\right]^T \left(\boldsymbol{\sigma}^h\right)^i\right) dx\right] dt \\ &+ \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left(\left[\boldsymbol{w}^h\left(t^{n+1}_-\right)\right]^T \boldsymbol{u}\left(\boldsymbol{v}^h\left(t^{n+1}_-\right)\right) - \left[\boldsymbol{w}^h\left(t^n_+\right)\right]^T \boldsymbol{u}\left(\boldsymbol{v}^h\left(t^n_-\right)\right)\right) dx\right] \\ &+ \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\boldsymbol{w}^h\right]^T \left[\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) - \widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \boldsymbol{n}\right)\right] d\hat{x}\right] dt = 0, \end{aligned} \tag{3.4}$$

where one may define

$$\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) = \frac{1}{2}\left[\boldsymbol{f}^i\left(\widehat{\boldsymbol{v}}^h\right) + \boldsymbol{f}^i\left(\boldsymbol{v}^h\right)\right]\boldsymbol{n}^i + \frac{1}{2}\boldsymbol{h}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right), \tag{3.5}$$

$$\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \boldsymbol{n}\right) = \left(\boldsymbol{\sigma}^h\right)^i \boldsymbol{n}^i + \frac{1}{2}\boldsymbol{h}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right). \tag{3.6}$$

Note that $\boldsymbol{h}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ and $\boldsymbol{h}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ will be defined in a subsequent discussion. The aforementioned stabilizing term that was utilized to help form (3.4) takes the precise form

$$\sum_{T_k \in \mathcal{T}_h}\left[\int_{T_k}\left[\boldsymbol{w}^h\left(t^n_+\right)\right]^T\left(\left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t^n_-}^{t^n_+}\right) dx\right],$$

where

$$\left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t^n_-}^{t^n_+} = \boldsymbol{u}\left(\boldsymbol{v}^h\left(t^n_+\right)\right) - \boldsymbol{u}\left(\boldsymbol{v}^h\left(t^n_-\right)\right)$$

is the temporal jump in the solution. This term controls the jumps in the solution that are present due to the DG approach that was utilized for the temporal discretization. It is a necessary term in order to facilitate the subsequent stability proof.

There is one more step required to ensure viability of the scheme. One may introduce an equation that couples the elements and enforces boundary conditions as follows:

$$\begin{aligned}
&\int_{I^n}\sum_{T_k \in \mathcal{T}_h}\left[\int_{\partial T_k}\left[\widehat{\boldsymbol{w}}^h\right]^T\left[\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) - \widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \boldsymbol{n}\right)\right] d\hat{x}\right] dt \\
&+ \int_{I^n}\sum_{F_\ell \in \mathcal{E}^\partial_h}\left[\int_{F_\ell}\left[\widehat{\boldsymbol{w}}^h\right]^T\left[-\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right) + \widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \widehat{\boldsymbol{n}}\right)\right] d\hat{x}\right] dt \\
&+ \int_{I^n}\sum_{F_\ell \in \mathcal{E}^\partial_h}\left[\int_{F_\ell}\left[\widehat{\boldsymbol{w}}^h\right]^T\left[\boldsymbol{b}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^\partial; \widehat{\boldsymbol{n}}\right)\right.\right. \\
&\qquad\qquad \left.\left. + \boldsymbol{b}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^\partial, \boldsymbol{v}^h_{,\boldsymbol{x}}, \boldsymbol{v}^\partial_{,\boldsymbol{x}}; \widehat{\boldsymbol{n}}\right)\right] d\hat{x}\right] dt = 0,
\end{aligned} \tag{3.7}$$

where $\widehat{\boldsymbol{w}}^h \in \widehat{\boldsymbol{\mathcal{W}}}^h$, $\boldsymbol{b}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^\partial; \widehat{\boldsymbol{n}}\right)$ and $\boldsymbol{b}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^\partial, \boldsymbol{v}^h_{,\boldsymbol{x}}, \boldsymbol{v}^\partial_{,\boldsymbol{x}}; \widehat{\boldsymbol{n}}\right)$ are normal fluxes that weakly enforce the boundary conditions for the problem, and $\boldsymbol{v}^\partial$ and $\boldsymbol{v}^\partial_{,\boldsymbol{x}}$ are states defined on the boundary (cf. [49] for details).

One arrives at the full space-time HDG scheme by solving (3.1), (3.4), and (3.7) with $\boldsymbol{w}^h \in \boldsymbol{\mathcal{W}}^h$, $\widehat{\boldsymbol{w}}^h \in \widehat{\boldsymbol{\mathcal{W}}}^h$, and $\boldsymbol{\phi}^h \in \boldsymbol{\Phi}^h$, for unknowns $\boldsymbol{v}^h \in \boldsymbol{\mathcal{W}}^h$, $\widehat{\boldsymbol{v}}^h \in \widehat{\boldsymbol{\mathcal{W}}}^h$, and $\boldsymbol{\sigma}^h \in \boldsymbol{\Phi}^h$.

3.3. **Primal formulation.** A primal formulation of the scheme is obtained by eliminating $\boldsymbol{\sigma}^h$. Towards this end, one may set $\left(\boldsymbol{\phi}^h\right)^i = \boldsymbol{w}^h_{,\boldsymbol{x}_i}$ in (3.1), and then substitute the result into (3.4) in order to obtain

$$
\begin{aligned}
&\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left(- \left[\boldsymbol{w}^h_{,t} \right]^T \boldsymbol{u}\left(\boldsymbol{v}^h\right) - \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i} \right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\right) \right) dx \right] dt \\
&+ \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left(\left[\boldsymbol{w}^h\left(t^{n+1}_{-}\right) \right]^T \boldsymbol{u}\left(\boldsymbol{v}^h\left(t^{n+1}_{-}\right)\right) - \left[\boldsymbol{w}^h\left(t^{n}_{+}\right) \right]^T \boldsymbol{u}\left(\boldsymbol{v}^h\left(t^{n}_{-}\right)\right) \right) dx \right] \\
&+ \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i} \right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{v}^h_{,\boldsymbol{x}_j} dx \right. \\
&\qquad \left. + \int_{\partial T_k} \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i} \right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h \right] \boldsymbol{n}^j d\hat{x} \right] dt \\
&+ \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\boldsymbol{w}^h \right]^T \left[\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) - \widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \boldsymbol{n}\right) \right] d\hat{x} \right] dt = 0.
\end{aligned} \tag{3.8}
$$

For the sake of efficiency, one can solve (3.7) and (3.8) with $\boldsymbol{w}^h \in \boldsymbol{\mathcal{W}}^h$ and $\widehat{\boldsymbol{w}}^h \in \widehat{\boldsymbol{\mathcal{W}}}^h$, for unknowns $\boldsymbol{v}^h \in \boldsymbol{\mathcal{W}}^h$ and $\widehat{\boldsymbol{v}}^h \in \widehat{\boldsymbol{\mathcal{W}}}^h$.

3.4. **Reformulation for analysis.** In order to prove the entropy stability of the scheme from the previous section, it is necessary to first rewrite (3.7) and (3.8) so that they are more suitable to analysis. Towards this end, one may perform integration by parts in (3.8), and selectively rewrite the resulting expression in terms of summations over faces in the mesh (as opposed to summations over individual element faces). In addition, one may selectively rewrite (3.7) in terms of summations over faces in the mesh. On subtracting the modified form of (3.7) from the modified form of (3.8), one obtains

$$
\begin{aligned}
&\Lambda_{sol}\left(\boldsymbol{w}^h, \boldsymbol{v}^h\right) + \Lambda_{inv}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) + \Lambda_{vis}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) \\
&\qquad - \Lambda_{bc,inv}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) - \Lambda_{bc,vis}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) = 0,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
\Lambda_{sol}\left(\boldsymbol{w}^h, \boldsymbol{v}^h\right) = &\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{w}^h \right]^T \boldsymbol{u}_{,t}\left(\boldsymbol{v}^h\right) dx \right] dt \\
&+ \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{w}^h\left(t^n_{+}\right) \right]^T \left(\left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right) \right]\!\right]_{t^n_{-}}^{t^n_{+}} \right) dx \right],
\end{aligned}
$$

corresponds (at least in part) to terms 1 and 2 in (3.8),

$$\Lambda_{inv}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) = \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{w}^h\right]^T \boldsymbol{f}^i_{,\boldsymbol{x}_i}\left(\boldsymbol{v}^h\right) dx\right] dt$$

$$+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^0} \left[\int_{F_\ell} \left[\boldsymbol{w}^h\left(\boldsymbol{x}_-\right) - \widehat{\boldsymbol{w}}^h\right]^T \widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right), \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right) d\hat{x}\right] dt$$

$$+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^0} \left[\int_{F_\ell} \left[\widehat{\boldsymbol{w}}^h - \boldsymbol{w}^h\left(\boldsymbol{x}_+\right)\right]^T \widehat{\boldsymbol{f}}\left(\widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h\left(\boldsymbol{x}_+\right); \widehat{\boldsymbol{n}}\right) d\hat{x}\right] dt$$

$$+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^0} \left[\int_{F_\ell} \left(-\left[\boldsymbol{w}^h\left(\boldsymbol{x}_-\right)\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right) \widehat{\boldsymbol{n}}^i \right.\right.$$

$$\left.\left. + \left[\boldsymbol{w}^h\left(\boldsymbol{x}_+\right)\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right) \widehat{\boldsymbol{n}}^i\right) d\hat{x}\right] dt,$$

corresponds (at least in part) to terms 1 and 2 in (3.7), and terms 1 and 4 in (3.8),

$$\Lambda_{vis}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) = \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i}\right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{v}^h_{,\boldsymbol{x}_j} dx\right] dt$$

$$- \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\boldsymbol{w}^h_{,\boldsymbol{x}_i}\right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \left[\boldsymbol{v}^h - \widehat{\boldsymbol{v}}^h\right] \ \boldsymbol{n}^j d\hat{x}\right] dt$$

$$- \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\boldsymbol{w}^h - \widehat{\boldsymbol{w}}^h\right]^T \widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \boldsymbol{n}\right) d\hat{x}\right] dt,$$

corresponds (at least in part) to term 1 in (3.7), and terms 3 and 4 in (3.8),

$$\Lambda_{bc,inv}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right)$$

$$= - \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^\partial} \left[\int_{F_\ell} \left[\boldsymbol{w}^h\right]^T \left[\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right) - \boldsymbol{f}^i\left(\boldsymbol{v}^h\right) \widehat{\boldsymbol{n}}^i\right] d\hat{x}\right] dt$$

$$+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^\partial} \left[\int_{F_\ell} \left[\widehat{\boldsymbol{w}}^h\right]^T \boldsymbol{b}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^\partial; \widehat{\boldsymbol{n}}\right) d\hat{x}\right] dt,$$

corresponds (at least in part) to term 3 in (3.7), and terms 1 and 4 in (3.8), and

$$\Lambda_{bc,vis}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right)$$

$$= \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^\partial} \left[\int_{F_\ell} \left[\widehat{\boldsymbol{w}}^h\right]^T \left[\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h_{,\boldsymbol{x}}; \widehat{\boldsymbol{n}}\right) \right.\right.$$

$$\left.\left. + \boldsymbol{b}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h, \boldsymbol{v}^\partial, \boldsymbol{v}^h_{,\boldsymbol{x}}, \boldsymbol{v}^\partial_{,\boldsymbol{x}}; \widehat{\boldsymbol{n}}\right)\right] d\hat{x}\right] dt,$$

corresponds (at least in part) to terms 2 and 3 in (3.7). In addition, the boundary condition terms can be grouped together as follows for the sake of convenience:

$$\Lambda_{bc}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) = \Lambda_{bc,inv}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right) + \Lambda_{bc,vis}\left(\boldsymbol{w}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{w}}^h, \widehat{\boldsymbol{v}}^h\right).$$

4. Entropy stability proof

One first seeks to quantify the 'energy' of the scheme over the time interval $[t_-^0, t_-^N] = \bigcup_{n=0}^{N-1} I^n$. In order to do this, one may set $\boldsymbol{w}^h = \boldsymbol{v}^h$ and $\widehat{\boldsymbol{w}}^h = \widehat{\boldsymbol{v}}^h$ in (3.9), and then sum over the time slabs in the resulting expression in order to obtain

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \left[\Lambda_{sol}\left(\boldsymbol{v}^h, \boldsymbol{v}^h\right) + \Lambda_{inv}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right) + \Lambda_{vis}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right] \\
&= \sum_{n=0}^{N-1} \left[\Lambda_{bc,inv}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right) + \Lambda_{bc,vis}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right].
\end{aligned} \tag{4.1}
$$

In (4.1), one seeks an upper bound on the solution at time t_-^N in terms of the solution at time t_-^0 and the boundary terms $\Lambda_{inv,bc}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)$ and $\Lambda_{vis,bc}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)$. With this in mind, one may consider the subsequent theorems that establish said bounds.

In preparation for viewing the first theorem, one may examine the following definitions.

Definition 4.1 (Mean-value flux $\boldsymbol{h}_{MV}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right)$; [4], p. 216)**.**

$$
\begin{aligned}
&\boldsymbol{h}_{MV}^f\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right), \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right) \\
&\quad \equiv \int_0^1 [1-\theta] \left(\left| \widetilde{A}_i \left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right) \widehat{\boldsymbol{n}}^i \right|_{\widetilde{A}_0} + \left| \widetilde{A}_i \left(\overline{\boldsymbol{v}}^h(\theta)\right) \widehat{\boldsymbol{n}}^i \right|_{\widetilde{A}_0} \right) \left[\boldsymbol{v}^h\left(\boldsymbol{x}_-\right) - \widehat{\boldsymbol{v}}^h\right] d\theta,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
&\boldsymbol{h}_{MV}^f\left(\widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h\left(\boldsymbol{x}_+\right); \widehat{\boldsymbol{n}}\right) \\
&\quad \equiv \int_0^1 [1-\theta] \left(\left| \widetilde{A}_i \left(\underline{\underline{\boldsymbol{v}}}^h(\theta)\right) \widehat{\boldsymbol{n}}^i \right|_{\widetilde{A}_0} + \left| \widetilde{A}_i \left(\underline{\boldsymbol{v}}^h(\theta)\right) \widehat{\boldsymbol{n}}^i \right|_{\widetilde{A}_0} \right) \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right] d\theta,
\end{aligned} \tag{4.3}
$$

where

$$
\overline{\boldsymbol{v}}^h(\theta) = \widehat{\boldsymbol{v}}^h - \theta\left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right], \qquad \overline{\overline{\boldsymbol{v}}}^h(\theta) = \boldsymbol{v}^h\left(\boldsymbol{x}_-\right) + \theta\left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right],
$$

$$
\underline{\boldsymbol{v}}^h(\theta) = \boldsymbol{v}^h\left(\boldsymbol{x}_+\right) - \theta\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right) - \widehat{\boldsymbol{v}}^h\right], \qquad \underline{\underline{\boldsymbol{v}}}^h(\theta) = \widehat{\boldsymbol{v}}^h + \theta\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right) - \widehat{\boldsymbol{v}}^h\right],
$$

$$
\left| \widetilde{A}_i(\cdot)\, \widehat{\boldsymbol{n}}^i \right|_{\widetilde{A}_0} \equiv \left| A_i(\cdot)\, \widehat{\boldsymbol{n}}^i \right| \widetilde{A}_0(\cdot).
$$

Here, the matrix absolute value of $\widetilde{A}_i(\cdot)\,\widehat{\boldsymbol{n}}^i$ is SPSD and has eigenvalues that are equal to the eigenvalues of $\widetilde{A}_i(\cdot)\,\widehat{\boldsymbol{n}}^i$ to within a sign.

Definition 4.2 (Nonnegative function $|||\cdot|||_{|\underline{\widetilde{A}}(\overline{\boldsymbol{v}}^h)|, F_\ell \times \mathcal{I}}$)**.**

$$
\begin{aligned}
|||\cdot|||^2_{|\underline{\widetilde{A}}(\overline{\boldsymbol{v}}^h)|, F_\ell \times \mathcal{I}} \equiv \int_{F_\ell} \int_0^1 [1-\theta] [\cdot]^T \Bigg(&\left(\widetilde{A}_i^+ \left(\overline{\boldsymbol{v}}^h(\theta)\right) \widehat{\boldsymbol{n}}^i \right)_{\widetilde{A}_0} \\
&- \left(\widetilde{A}_i^- \left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right) \widehat{\boldsymbol{n}}^i \right)_{\widetilde{A}_0} \Bigg) [\cdot]\, d\theta\, d\hat{x},
\end{aligned}
$$

where

$$\left(\widetilde{A}_i^{+}\left(\overline{\boldsymbol{v}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right)_{\widetilde{A}_0} = A_i^{+}\left(\overline{\boldsymbol{v}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i \widetilde{A}_0\left(\overline{\boldsymbol{v}}^h(\theta)\right),$$

$$\left(\widetilde{A}_i^{-}\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right)_{\widetilde{A}_0} = A_i^{-}\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i \widetilde{A}_0\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right).$$

Theorem 4.3. *The aforementioned space-time HDG scheme is entropy stable for the Euler equations (the inviscid, compressible NS equations) when the boundary conditions are chosen appropriately,*[1] $p_s \geq 0$, $p_t \geq 0$, *and* $\boldsymbol{h}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right)$ *is chosen to be the mean-value flux. Under these conditions, the solution at time* t_-^N *is governed by the following equation:*

$$\begin{aligned}
&\sum_{T_k \in \mathcal{T}_h}\left[\int_{T_k} U\left(\boldsymbol{v}^h\left(t_-^N\right)\right)dx\right] + \sum_{n=0}^{N-1}\left(\sum_{T_k\in\mathcal{T}_h}\left\|[[\boldsymbol{u}\left(\boldsymbol{v}^h\right)]]_{t_-^n}^{t_+^n}\right\|^2_{\widetilde{A}_0^{-1}, T_k}\right)\\
&+\sum_{n=0}^{N-1}\left(\sum_{F_\ell\in\mathcal{E}_h^0}\left[\left|\left|\left|\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right|\right|\right|^2_{\left|\underline{\widetilde{A}}\left(\overline{\boldsymbol{v}}^h\right)\right|, F_\ell\times\mathcal{I}\times I^n}\right.\right.\\
&\qquad\left.\left.+\left|\left|\left|\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right|\right|\right|^2_{\left|\underline{\widetilde{A}}\left(\underline{\boldsymbol{v}}^h\right)\right|, F_\ell\times\mathcal{I}\times I^n}\right]\right)\\
&= \sum_{T_k\in\mathcal{T}_h}\left[\int_{T_k} U\left(\boldsymbol{v}^h\left(t_-^0\right)\right)dx\right] + \sum_{n=0}^{N-1}\left(-\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^\partial}\left[\int_{F_\ell} F^i\left(\boldsymbol{v}^h\right)\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right.\\
&\qquad\left.+\Lambda_{bc,inv}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right).
\end{aligned} \tag{4.4}$$

Proof. The inviscid form of (4.1) is as follows:

$$\sum_{n=0}^{N-1}\left[\Lambda_{sol}\left(\boldsymbol{v}^h, \boldsymbol{v}^h\right) + \Lambda_{inv}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right] = \sum_{n=0}^{N-1}\left[\Lambda_{bc,inv}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right]. \tag{4.5}$$

Consider expanding the first term on the LHS of (4.5),

$$\begin{aligned}
&\sum_{n=0}^{N-1}\left[\Lambda_{sol}\left(\boldsymbol{v}^h, \boldsymbol{v}^h\right)\right]\\
&= \sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{T_k} U_{,t}\left(\boldsymbol{v}^h\right)dx\right]dt\right.\\
&\qquad\left.+\sum_{T_k\in\mathcal{T}_h}\left[\int_{T_k}\left[\boldsymbol{v}^h\left(t_+^n\right)\right]^T\left([[\boldsymbol{u}\left(\boldsymbol{v}^h\right)]]_{t_-^n}^{t_+^n}\right)dx\right]\right),
\end{aligned} \tag{4.6}$$

[1]For instance, if periodic boundary conditions are enforced on all boundaries such that the boundary terms vanish. Or the contributions of the boundary terms are designed such that they are nonpositive in line with the approach of [1], although precise details are beyond the scope of this work.

where the fact that $\left[\boldsymbol{v}^h\right]^T \boldsymbol{u}_{,t}\left(\boldsymbol{v}^h\right) = U_{,t}\left(\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right) = U_{,t}\left(\boldsymbol{v}^h\right)$ has been used. (This identity follows from (2.17) in Lemma 2.1.) Next, one may note that

(4.7)
$$\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} U_{,t}\left(\boldsymbol{v}^h\right) dx\right] dt = \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left([\![U\left(\boldsymbol{v}^h\right)]\!]_{t_-^n}^{t_-^{n+1}} - [\![U\left(\boldsymbol{v}^h\right)]\!]_{t_-^n}^{t_+^n} \right) dx\right].$$

Equation (4.7) can be immediately substituted into (4.6). Thereafter, one may rewrite (B.1) from Lemma B.1 with t_-^n and t_+^n in place of t_a and t_b, and substitute this expression into the result from the previous step. This sequence of actions is carried out below.

$$\begin{aligned}
&\sum_{n=0}^{N-1} \left[\Lambda_{sol}\left(\boldsymbol{v}^h, \boldsymbol{v}^h\right)\right] \\
&= \sum_{n=0}^{N-1} \left(\sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left([\![U\left(\boldsymbol{v}^h\right)]\!]_{t_-^n}^{t_-^{n+1}} - [\![U\left(\boldsymbol{v}^h\right)]\!]_{t_-^n}^{t_+^n} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \left[\boldsymbol{v}^h\left(t_+^n\right)\right]^T \left([\![\boldsymbol{u}\left(\boldsymbol{v}^h\right)]\!]_{t_-^n}^{t_+^n} \right) \right) dx \right] \right) \\
&= \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[U\left(\boldsymbol{v}^h\left(t_-^N\right)\right) - U\left(\boldsymbol{v}^h\left(t_-^0\right)\right)\right] dx \right] \\
&\quad + \sum_{n=0}^{N-1} \left(\sum_{T_k \in \mathcal{T}_h} \left\| [\![\boldsymbol{u}\left(\boldsymbol{v}^h\right)]\!]_{t_-^n}^{t_+^n} \right\|_{\widetilde{A}_0^{-1}, T_k}^2 \right).
\end{aligned} \tag{4.8}$$

The Λ_{sol} term in (4.8) now relates the initial condition at t_-^0 and the current condition at t_-^N to nonnegative functions of the solution, as required by the main result (equation (4.4)) of Theorem 4.3.

Now, it is useful to examine the second term on the LHS of (4.5):

$$\begin{aligned}
&\sum_{n=0}^{N-1} \left[\Lambda_{inv}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right] = \sum_{n=0}^{N-1} \left(\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{v}^h\right]^T \boldsymbol{f}^i_{,\boldsymbol{x}_i}\left(\boldsymbol{v}^h\right) dx \right] dt \right. \\
&+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^0} \left[\int_{F_\ell} \left[\boldsymbol{v}^h\left(\boldsymbol{x}_-\right) - \widehat{\boldsymbol{v}}^h\right]^T \widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right), \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right) d\hat{x} \right] dt \\
&+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^0} \left[\int_{F_\ell} \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right]^T \widehat{\boldsymbol{f}}\left(\widehat{\boldsymbol{v}}^h, \boldsymbol{v}^h\left(\boldsymbol{x}_+\right); \widehat{\boldsymbol{n}}\right) d\hat{x} \right] dt \\
&+ \int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^0} \left[\int_{F_\ell} \left(- \left[\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right) \widehat{\boldsymbol{n}}^i \right.\right. \\
&\qquad\qquad \left.\left.\left. + \left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right) \widehat{\boldsymbol{n}}^i \right) d\hat{x} \right] dt \right).
\end{aligned} \tag{4.9}$$

The first term on the RHS of (4.9) can be rewritten as

$$\begin{aligned}
&\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{T_k}\left[\boldsymbol{v}^h\right]^T\boldsymbol{f}^i_{,\boldsymbol{x}_i}\left(\boldsymbol{v}^h\right)dx\right]dt\right)\\
&=\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\left[F^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right)-F^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right)\right]\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right.\\
&\qquad\left.+\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^\partial}\left[\int_{F_\ell}F^i\left(\boldsymbol{v}^h\right)\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right),
\end{aligned}\tag{4.10}$$

where the divergence theorem and the fact that

$$\left[\boldsymbol{v}^h\right]^T\boldsymbol{f}^i_{,\boldsymbol{x}_i}\left(\boldsymbol{v}^h\right)=\left[\boldsymbol{v}^h\right]^T\boldsymbol{f}^i_{,\boldsymbol{x}_i}\left(\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right)=F^i_{,\boldsymbol{x}_i}\left(\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right)=F^i_{,\boldsymbol{x}_i}\left(\boldsymbol{v}^h\right)$$

have been used. (The identity follows from (2.17) in Lemma 2.1.) The remaining terms on the RHS of (4.9) can be rewritten in terms of the definition of $\widehat{\boldsymbol{f}}\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h;\boldsymbol{n}\right)$ in (3.5). On introducing this definition along with (4.10) into (4.9), and simplifying the results, one obtains

$$\begin{aligned}
&\sum_{n=0}^{N-1}\left[\Lambda_{inv}\left(\boldsymbol{v}^h,\boldsymbol{v}^h;\widehat{\boldsymbol{v}}^h,\widehat{\boldsymbol{v}}^h\right)\right]\\
&=\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\left[F^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right)-F^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right)\right]\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right.\\
&+\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^\partial}\left[\int_{F_\ell}F^i\left(\boldsymbol{v}^h\right)\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\\
&+\frac{1}{2}\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\left[\widehat{\boldsymbol{v}}^h+\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]^T\left[\boldsymbol{f}^i\left(\widehat{\boldsymbol{v}}^h\right)-\boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right)\right]\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\\
&+\frac{1}{2}\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)+\widehat{\boldsymbol{v}}^h\right]^T\left[\boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right)-\boldsymbol{f}^i\left(\widehat{\boldsymbol{v}}^h\right)\right]\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\\
&+\frac{1}{2}\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\left[\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)-\widehat{\boldsymbol{v}}^h\right]^T\boldsymbol{h}^f\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right),\widehat{\boldsymbol{v}}^h;\widehat{\boldsymbol{n}}\right)d\hat{x}\right]dt\\
&\left.+\frac{1}{2}\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right]^T\boldsymbol{h}^f\left(\widehat{\boldsymbol{v}}^h,\boldsymbol{v}^h\left(\boldsymbol{x}_+\right);\widehat{\boldsymbol{n}}\right)d\hat{x}\right]dt\right).
\end{aligned}\tag{4.11}$$

Setting (4.11) aside for the moment, consider the relations below that follow directly from Lemma B.2, (B.2):

(4.12)
$$F^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right)-F^i\left(\widehat{\boldsymbol{v}}^h\right)+\frac{1}{2}\left[\widehat{\boldsymbol{v}}^h+\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]^T\left[\boldsymbol{f}^i\left(\widehat{\boldsymbol{v}}^h\right)-\boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right)\right]$$
$$=\frac{1}{2}\int_0^1[1-\theta]\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]^T\left[\widetilde{A}_i\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\right]\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]d\theta,$$

(4.13)
$$F^i\left(\widehat{\boldsymbol{v}}^h\right)-F^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right)+\frac{1}{2}\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)+\widehat{\boldsymbol{v}}^h\right]^T\left[\boldsymbol{f}^i\left(\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right)-\boldsymbol{f}^i\left(\widehat{\boldsymbol{v}}^h\right)\right]$$
$$=\frac{1}{2}\int_0^1[1-\theta]\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)-\widehat{\boldsymbol{v}}^h\right]^T\left[\widetilde{A}_i\left(\underline{\underline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\underline{\boldsymbol{v}}^h(\theta)\right)\right]\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)-\widehat{\boldsymbol{v}}^h\right]d\theta.$$

On substituting (4.12) and (4.13) into (4.11), one obtains

$$\begin{aligned}
\sum_{n=0}^{N-1}\left[\Lambda_{inv}\left(\boldsymbol{v}^h,\boldsymbol{v}^h;\widehat{\boldsymbol{v}}^h,\widehat{\boldsymbol{v}}^h\right)\right] &= \sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^\partial}\left[\int_{F_\ell}F^i\left(\boldsymbol{v}^h\right)\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right.\\
&+\frac{1}{2}\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\int_0^1[1-\theta]\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]^T\right.\\
&\cdot\left[\widetilde{A}_i\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\right]\widehat{\boldsymbol{n}}^i\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]d\theta\, d\hat{x}\\
&+\int_{F_\ell}\int_0^1[1-\theta]\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)-\widehat{\boldsymbol{v}}^h\right]^T\\
&\cdot\left[\widetilde{A}_i\left(\underline{\underline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\underline{\boldsymbol{v}}^h(\theta)\right)\right]\widehat{\boldsymbol{n}}^i\left[\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)-\widehat{\boldsymbol{v}}^h\right]d\theta\, d\hat{x}\\
&+\int_{F_\ell}\left[\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)-\widehat{\boldsymbol{v}}^h\right]^T\boldsymbol{h}^f\left(\boldsymbol{v}^h\left(\boldsymbol{x}_-\right),\widehat{\boldsymbol{v}}^h;\widehat{\boldsymbol{n}}\right)d\hat{x}\\
&\left.\left.+\int_{F_\ell}\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_+\right)\right]^T\boldsymbol{h}^f\left(\widehat{\boldsymbol{v}}^h,\boldsymbol{v}^h\left(\boldsymbol{x}_+\right);\widehat{\boldsymbol{n}}\right)d\hat{x}\right]dt\right).
\end{aligned}\tag{4.14}$$

Now, consider substituting $\boldsymbol{h}_{MV}^f\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h;\widehat{\boldsymbol{n}}\right)$ (equations (4.2) and (4.3)) in place of $\boldsymbol{h}^f\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h;\widehat{\boldsymbol{n}}\right)$ in (4.14), so that one obtains

(4.15)
$$\begin{aligned}
\sum_{n=0}^{N-1}\left[\Lambda_{inv}\left(\boldsymbol{v}^h,\boldsymbol{v}^h;\widehat{\boldsymbol{v}}^h,\widehat{\boldsymbol{v}}^h\right)\right] &= \sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^\partial}\left[\int_{F_\ell}F^i\left(\boldsymbol{v}^h\right)\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right.\\
&+\frac{1}{2}\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\int_{F_\ell}\int_0^1[1-\theta]\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]^T\right.\\
&\cdot\left[\widetilde{A}_i\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\right]\widehat{\boldsymbol{n}}^i\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\left(\boldsymbol{x}_-\right)\right]d\theta\, d\hat{x}
\end{aligned}$$

$$+\int_{F_\ell}\int_0^1 [1-\theta]\left[\boldsymbol{v}^h(\boldsymbol{x}_+)-\widehat{\boldsymbol{v}}^h\right]^T \left[\widetilde{A}_i\left(\underline{\underline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\underline{\boldsymbol{v}}^h(\theta)\right)\right]\widehat{\boldsymbol{n}}^i$$

$$\cdot\left[\boldsymbol{v}^h(\boldsymbol{x}_+)-\widehat{\boldsymbol{v}}^h\right]d\theta\, d\hat{x}$$

$$+\int_{F_\ell}\int_0^1 [1-\theta]\left[\boldsymbol{v}^h(\boldsymbol{x}_-)-\widehat{\boldsymbol{v}}^h\right]^T \left(\left|\widetilde{A}_i\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}+\left|\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}\right)$$

$$\cdot\left[\boldsymbol{v}^h(\boldsymbol{x}_-)-\widehat{\boldsymbol{v}}^h\right]d\theta\, d\hat{x}$$

$$+\int_{F_\ell}\int_0^1 [1-\theta]\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_+)\right]^T \left(\left|\widetilde{A}_i\left(\underline{\underline{\boldsymbol{v}}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}+\left|\widetilde{A}_i\left(\underline{\boldsymbol{v}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}\right)$$

$$\cdot\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_+)\right]d\theta\, d\hat{x}\Bigg]dt\Bigg).$$

The central part of the fourth term on the RHS of (4.15) can be simplified with the right change of variables (i.e., setting $\theta = 1-\theta'$, rearranging terms, and rewriting everything in terms of θ again) as follows:

$$\begin{aligned}&\int_0^1 [1-\theta]\left[\boldsymbol{v}^h(\boldsymbol{x}_-)-\widehat{\boldsymbol{v}}^h\right]^T \left(\left|\widetilde{A}_i\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}\right.\\ &\qquad\qquad\left.+\left|\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}\right)\left[\boldsymbol{v}^h(\boldsymbol{x}_-)-\widehat{\boldsymbol{v}}^h\right]d\theta\\ &=\int_0^1 \left[\boldsymbol{v}^h(\boldsymbol{x}_-)-\widehat{\boldsymbol{v}}^h\right]^T\left|\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\widehat{\boldsymbol{n}}^i\right|_{\widetilde{A}_0}\left[\boldsymbol{v}^h(\boldsymbol{x}_-)-\widehat{\boldsymbol{v}}^h\right]d\theta.\end{aligned}\tag{4.16}$$

In addition, the central part of the second term on the RHS of (4.15) can be simplified as follows:

$$\begin{aligned}&\int_0^1 [1-\theta]\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_-)\right]^T\left[\widetilde{A}_i\left(\overline{\overline{\boldsymbol{v}}}^h(\theta)\right)-\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\right]\widehat{\boldsymbol{n}}^i\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_-)\right]d\theta\\ &=\int_0^1 \left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_-)\right]^T[1-2\theta]\left[\widetilde{A}_i\left(\overline{\boldsymbol{v}}^h(\theta)\right)\right]\widehat{\boldsymbol{n}}^i\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_-)\right]d\theta.\end{aligned}\tag{4.17}$$

Similar steps can be taken to simplify the third and fifth terms on the RHS of (4.15). On performing these steps, combining (4.15)–(4.17), splitting the matrices into plus and minus parts, performing one more change of variables, and rewriting everything in terms of θ, one obtains the following:

$$\sum_{n=0}^{N-1}\left[\Lambda_{inv}\left(\boldsymbol{v}^h,\boldsymbol{v}^h;\widehat{\boldsymbol{v}}^h,\widehat{\boldsymbol{v}}^h\right)\right]=\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^\partial}\left[\int_{F_\ell}F^i\left(\boldsymbol{v}^h\right)\widehat{\boldsymbol{n}}^i d\hat{x}\right]dt\right.$$

$$\left.+\int_{I^n}\sum_{F_\ell\in\mathcal{E}_h^0}\left[\left|\left|\left|\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_-)\right|\right|\right|^2_{|\overline{A}(\overline{\boldsymbol{v}}^h)|,F_\ell\times\mathcal{I}}+\left|\left|\left|\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h(\boldsymbol{x}_+)\right|\right|\right|^2_{|\underline{A}(\underline{\boldsymbol{v}}^h)|,F_\ell\times\mathcal{I}}\right]dt\right),$$

where the nonnegative function from Definition 4.2 has been used. This concludes the proof of Theorem 4.3. $\square$

Remark 4.4. Entropy stability is implied by Theorem 4.3 because it states that a measure of the entropy function of the solution at some later time $U\left(\boldsymbol{v}^h\left(t_-^N\right)\right)$ is equal to a measure of the entropy function at the initial time $U\left(\boldsymbol{v}^h\left(t_-^0\right)\right)$ plus some additional terms, i.e.,

$$\sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} U\left(\boldsymbol{v}^h\left(t_-^N\right)\right) dx \right] = \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} U\left(\boldsymbol{v}^h\left(t_-^0\right)\right) dx \right] \\ + \text{non-positive terms} + \text{B.C. terms}.$$

It is clear from the expression above that if the boundary condition terms vanish (i.e., we have periodic boundary conditions) or are nonpositive, then the measure of the entropy function cannot grow to be larger than its value at the initial time.

Remark 4.5. One should note that the aforementioned HDG scheme is also stable for any $\boldsymbol{h}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ such that

$$\left[\boldsymbol{v}^h - \widehat{\boldsymbol{v}}^h\right]^T \boldsymbol{h}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) \geq \left[\boldsymbol{v}^h - \widehat{\boldsymbol{v}}^h\right]^T \boldsymbol{h}_{MV}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right).$$

There are a number of suitable numerical fluxes that satisfy this condition, including a particular form of the Lax-Friedrichs flux (cf. [4], pp. 229–230).

Remark 4.6. Entropy stability is a valuable property for a scheme to possess. It effectively ensures that the solution remains bounded, and while this does not ensure pointwise convergence, it facilitates the convergence of boundary functionals of engineering interest [28, 39]. Furthermore, in flows with strong shocks, an entropy stable method will still remain stable. However, it will exhibit spurious oscillations (referred to as Gibbs phenomenon) at the location of the discontinuities. It is usually necessary to damp these oscillations in order to ensure that the scheme produces reasonable results (i.e., accurately 'captures' the shocks). There are a number of approaches for shock-capturing, including limiting, artificial diffusion, and ENO (Essentially Non-Oscillatory) or WENO (Weighted ENO) schemes. The reader may consult [15] for a detailed overview of these shock-capturing methodologies. Some of the more popular approaches for shock-capturing for high-order methods on unstructured grids are documented in [3, 5, 24, 25, 28, 30, 35, 36, 51, 52, 56]. (Note that this list of references is by no means exhaustive). Of particular interest is the recent approach of [28], which is entropy stable.

Remark 4.7. In addition to the mathematical proof given above, there is numerical evidence that the aforementioned HDG scheme is stable. The HDG scheme is very similar to a scheme formulated by Fidkowski in [18]. The primary difference is that Fidkowski does not employ entropy variables, but rather utilizes the standard conservative variables. In addition, a different stabilization strategy is utilized, which is effectively a Roe-like flux. The Roe flux can be shown to not succeed in preserving entropy stability [4]. Nevertheless, stable results are obtained for an Euler vortex problem, and a pitching and plunging NACA 0012 airfoil [18]. It is expected that the stability properties of the scheme proposed in this article, will be greater than or equal to those of the scheme in [18].

In preparation for viewing the final theorem, one may examine the following definitions.

Definition 4.8 (Interior Penalty (IP)-type flux $\boldsymbol{h}_{IP}^{\sigma}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ [14])**.**

$$\boldsymbol{h}_{IP}^{\sigma}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) \equiv \beta_{\mathcal{F}_l}\, \boldsymbol{n}^i \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{n}^j \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right], \tag{4.18}$$

where $\beta_{\mathcal{F}_l} \geq 0$.

Definition 4.9 (Function $\Theta\left(\boldsymbol{v}_{,\boldsymbol{x}}^h, \cdot\right)_{\widetilde{M},T_k}$ and the nonnegative functions $\Theta_{abs}\left(\boldsymbol{v}_{,\boldsymbol{x}}^h, \cdot\right)_{\widetilde{M},T_k}$ and $|||\cdot|||_{\widetilde{M},T_k}$)**.**

$$\Theta\left(\boldsymbol{v}_{,\boldsymbol{x}}^h, \cdot\right)_{\widetilde{M},T_k} \equiv \int_{T_k} \left[\boldsymbol{v}_{,\boldsymbol{x}_i}^h\right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) [\cdot]\, dx,$$

$$\Theta_{abs}\left(\boldsymbol{v}_{,\boldsymbol{x}}^h, \cdot\right)_{\widetilde{M},T_k} \equiv \int_{T_k} \left|\left[\boldsymbol{v}_{,\boldsymbol{x}_i}^h\right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) [\cdot]\right| dx,$$

$$|||\cdot|||_{\widetilde{M},T_k}^2 \equiv \int_{T_k} [\cdot]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) [\cdot]\, dx.$$

Theorem 4.10. *The aforementioned space-time HDG scheme is entropy stable for the compressible NS equations when the boundary conditions are chosen appropriately,*[2] $p_s \geq 1$, $p_t \geq 0$, $\boldsymbol{h}^f\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \widehat{\boldsymbol{n}}\right)$ *is chosen to be the mean-value flux,* $\boldsymbol{h}^{\sigma}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ *is chosen to be an IP-type flux,* $\eta_{\mathcal{F}_l} \geq N_{\mathcal{F}}$, *and* $N_{\mathcal{F}} = d+1$. *Under these conditions, the solution at time* t_-^N *is governed by the following equation:*

$$\begin{aligned}
&\sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} U\left(\boldsymbol{v}^h\left(t_-^N\right)\right) dx\right] + \sum_{n=0}^{N-1} \left(\sum_{T_k \in \mathcal{T}_h} \left\| [[\boldsymbol{u}\left(\boldsymbol{v}^h\right)]]_{t_-^n}^{t_+^n} \right\|_{\widetilde{A}_0^{-1},T_k}^2\right) \\
&+ \sum_{n=0}^{N-1} \left(\sum_{F_\ell \in \mathcal{E}_h^0} \left[|||\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_-\right)|||_{|\underline{\widetilde{A}}\left(\overline{\boldsymbol{v}}^h\right)|,F_\ell \times \mathcal{I} \times I^n}^2 \right.\right. \\
&\qquad \left.\left. + |||\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\left(\boldsymbol{x}_+\right)|||_{|\underline{\widetilde{A}}\left(\underline{\boldsymbol{v}}^h\right)|,F_\ell \times \mathcal{I} \times I^n}^2 \right]\right) \\
&+ \sum_{n=0}^{N-1} \left(\sum_{T_k \in \mathcal{T}_h} \left[2\,\Theta\left(\boldsymbol{v}_{,\boldsymbol{x}}^h, \boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)\right)_{\widetilde{M},T_k \times I^n} + |||\boldsymbol{v}_{,\boldsymbol{x}}^h|||_{\widetilde{M},T_k \times I^n}^2 \right.\right. \\
&+ \sum_{\mathcal{F}_l \in \partial T_k} \left(\eta_{\mathcal{F}_l} |||\boldsymbol{r}_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)|||_{\widetilde{M},T_k \times I^n}^2 \right. \\
&\qquad \left.\left.\left. + \frac{1}{2}\beta_{\mathcal{F}_l} |||\boldsymbol{n}\left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right]|||_{\widetilde{M},\mathcal{F}_l \times I^n}^2 \right)\right]\right) \\
&= \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} U\left(\boldsymbol{v}^h\left(t_-^0\right)\right) dx\right] \\
&+ \sum_{n=0}^{N-1} \left(-\int_{I^n} \sum_{F_\ell \in \mathcal{E}_h^\partial} \left[\int_{F_\ell} F^i\left(\boldsymbol{v}^h\right) \widehat{\boldsymbol{n}}^i d\hat{x}\right] dt + \Lambda_{bc}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right) \right),
\end{aligned} \tag{4.19}$$

[2]See footnote 1.

where

$$
\begin{aligned}
&2\sum_{n=0}^{N-1}\left(\sum_{T_k\in\mathcal{T}_h}\left[\Theta_{abs}\left(\boldsymbol{v}^h_{,\boldsymbol{x}},\boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right)_{\widetilde{M},T_k\times I^n}\right]\right)\\
&\leq\sum_{n=0}^{N-1}\left(\sum_{T_k\in\mathcal{T}_h}\left[\left|\!\left|\!\left|\boldsymbol{v}^h_{,\boldsymbol{x}}\right|\!\right|\!\right|^2_{\widetilde{M},T_k\times I^n}\right.\right.\\
&\qquad\left.\left.+\sum_{\mathcal{F}_l\in\partial T_k}\eta_{\mathcal{F}_l}\left|\!\left|\!\left|\boldsymbol{r}_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right|\!\right|\!\right|^2_{\widetilde{M},T_k\times I^n}\right]\right).
\end{aligned}
\tag{4.20}
$$

Proof. The proof of Theorem 4.10 follows from the proof of Theorem 4.3 and appropriate manipulations of the viscous terms in (4.1). One may begin by considering the third (and final) term on the LHS of (4.1)

$$
\begin{aligned}
&\sum_{n=0}^{N-1}\left[\Lambda_{vis}\left(\boldsymbol{v}^h,\boldsymbol{v}^h;\widehat{\boldsymbol{v}}^h,\widehat{\boldsymbol{v}}^h\right)\right]=\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{T_k}\left[\boldsymbol{v}^h_{,\boldsymbol{x}_i}\right]^T\widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\boldsymbol{v}^h_{,\boldsymbol{x}_j}dx\right]dt\right.\\
&-\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{\partial T_k}\left[\boldsymbol{v}^h_{,\boldsymbol{x}_i}\right]^T\widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\left[\boldsymbol{v}^h-\widehat{\boldsymbol{v}}^h\right]\,\boldsymbol{n}^j d\hat{x}\right]dt\\
&\left.-\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{\partial T_k}\left[\boldsymbol{v}^h-\widehat{\boldsymbol{v}}^h\right]^T\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h,\boldsymbol{v}^h_{,\boldsymbol{x}};\boldsymbol{n}\right)d\hat{x}\right]dt\right).
\end{aligned}
\tag{4.21}
$$

The second term on the RHS of (4.21) can be simplified by utilizing the definition of the element-based lifting operator (equation (3.2)) as follows:

$$
\begin{aligned}
&-\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{\partial T_k}\left[\boldsymbol{v}^h_{,\boldsymbol{x}_i}\right]^T\widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\left[\boldsymbol{v}^h-\widehat{\boldsymbol{v}}^h\right]\,\boldsymbol{n}^j d\hat{x}\right]dt\\
&=\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{T_k}\left[\boldsymbol{v}^h_{,\boldsymbol{x}_i}\right]^T\widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\boldsymbol{R}^j\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)dx\right]dt.
\end{aligned}
\tag{4.22}
$$

Now, consider the third term on the RHS of (4.21). One can express this term utilizing the definition of $\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h,\boldsymbol{v}^h_{,\boldsymbol{x}};\boldsymbol{n}\right)$ (equation (3.6)) as follows:

$$
\begin{aligned}
&-\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{\partial T_k}\left[\boldsymbol{v}^h-\widehat{\boldsymbol{v}}^h\right]^T\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h,\boldsymbol{v}^h_{,\boldsymbol{x}};\boldsymbol{n}\right)d\hat{x}\right]dt\\
&=\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{\partial T_k}\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\right]^T\boldsymbol{n}^i\,\widetilde{M}_{ij}\left(\boldsymbol{v}^h\right)\left[\boldsymbol{v}^h_{,\boldsymbol{x}_j}+\eta_{\mathcal{F}_l}\boldsymbol{r}^j_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right]d\hat{x}\right]dt\\
&+\frac{1}{2}\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\int_{\partial T_k}\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\right]^T\boldsymbol{h}^\sigma\left(\boldsymbol{v}^h,\widehat{\boldsymbol{v}}^h;\boldsymbol{n}\right)d\hat{x}\right]dt.
\end{aligned}
\tag{4.23}
$$

The first term on the RHS of (4.23) can be simplified by utilizing the expressions for the element-based and face-based lifting operators (equations (3.2) and (3.3)) as follows:

(4.24)
$$\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right]^T \boldsymbol{n}^i \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \left[\boldsymbol{v}^h_{,\boldsymbol{x}_j} + \eta_{\mathcal{F}_l} \boldsymbol{r}^j_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)\right] d\hat{x} \right] dt$$
$$= \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \boldsymbol{R}^i \left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{v}^h_{,\boldsymbol{x}_j} dx \right] dt$$
$$+ \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\sum_{\mathcal{F}_l \in \partial T_k} \eta_{\mathcal{F}_l} \int_{T_k} \boldsymbol{r}^i_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{r}^j_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right) dx \right] dt.$$

The second term on the RHS of (4.23) can be expanded by setting $\boldsymbol{h}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) = \boldsymbol{h}^\sigma_{IP}\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right)$ (from (4.18)), as follows:

(4.25)
$$\frac{1}{2} \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{\partial T_k} \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right]^T \boldsymbol{h}^\sigma\left(\boldsymbol{v}^h, \widehat{\boldsymbol{v}}^h; \boldsymbol{n}\right) d\hat{x} \right] dt$$
$$= \frac{1}{2} \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\sum_{\mathcal{F}_l \in \partial T_k} \beta_{\mathcal{F}_l} \int_{\mathcal{F}_l} \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right]^T \boldsymbol{n}^i \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{n}^j \left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right] d\hat{x} \right] dt.$$

Upon combining (4.21), (4.22), (4.24), and (4.25), one obtains

$$\begin{aligned}
\sum_{n=0}^{N-1} \left[\Lambda_{vis}\left(\boldsymbol{v}^h, \boldsymbol{v}^h; \widehat{\boldsymbol{v}}^h, \widehat{\boldsymbol{v}}^h\right)\right] &= \sum_{n=0}^{N-1} \Bigg(\int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[|||\boldsymbol{v}^h_{,\boldsymbol{x}}|||^2_{\widetilde{M}, T_k} \right] dt \\
&+ 2 \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[\boldsymbol{v}^h_{,\boldsymbol{x}_i}\right]^T \widetilde{M}_{ij}\left(\boldsymbol{v}^h\right) \boldsymbol{R}^j \left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right) dx \right] dt \\
&+ \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\sum_{\mathcal{F}_l \in \partial T_k} \eta_{\mathcal{F}_l} |||\boldsymbol{r}_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)|||^2_{\widetilde{M}, T_k} \right] dt \\
&+ \frac{1}{2} \int_{I^n} \sum_{T_k \in \mathcal{T}_h} \left[\sum_{\mathcal{F}_l \in \partial T_k} \beta_{\mathcal{F}_l} |||\boldsymbol{n}\left[\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h\right]|||^2_{\widetilde{M}, \mathcal{F}_l} \right] dt \Bigg),
\end{aligned} \tag{4.26}$$

where the second nonnegative function from Definition 4.9 has been used to simplify the result.

Now, for stability purposes, the nonnegative terms on the RHS of (4.26) need to dominate the element-based lifting operator term on the RHS. The element-based lifting operator term itself is bounded as follows:

$$\begin{aligned}
&2 \sum_{T_k \in \mathcal{T}_h} \left[\Theta_{abs}\left(\boldsymbol{v}^h_{,\boldsymbol{x}}, \boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)\right)_{\widetilde{M}, T_k} \right] \\
&\leq \sum_{T_k \in \mathcal{T}_h} \left[\varepsilon |||\boldsymbol{v}^h_{,\boldsymbol{x}}|||^2_{\widetilde{M}, T_k} + \left(\frac{1}{\varepsilon}\right) |||\boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h - \boldsymbol{v}^h; \boldsymbol{n}\right)|||^2_{\widetilde{M}, T_k} \right],
\end{aligned} \tag{4.27}$$

where the Cauchy-Schwarz inequality and Young's inequality have been used (on the integrand), and the first nonnegative function in Definition 4.9 has been used. Therefore, for stability, one requires that the nonnegative terms on the RHS of (4.26) bound the RHS of (4.27) as follows:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\varepsilon\left|\left|\left|\boldsymbol{v}^h_{,\boldsymbol{x}}\right|\right|\right|^2_{\widetilde{M},T_k}+\left(\frac{1}{\varepsilon}\right)\left|\left|\left|\boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right|\right|\right|^2_{\widetilde{M},T_k}\right]dt\right)\\
&\leq\sum_{n=0}^{N-1}\left(\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\left|\left|\left|\boldsymbol{v}^h_{,\boldsymbol{x}}\right|\right|\right|^2_{\widetilde{M},T_k}+\sum_{\mathcal{F}_l\in\partial T_k}\eta_{\mathcal{F}_l}\left|\left|\left|\boldsymbol{r}_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right|\right|\right|^2_{\widetilde{M},T_k}\right]dt\right.\\
&\qquad\left.+\frac{1}{2}\int_{I^n}\sum_{T_k\in\mathcal{T}_h}\left[\sum_{\mathcal{F}_l\in\partial T_k}\beta_{\mathcal{F}_l}\left|\left|\left|\boldsymbol{n}\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\right]\right|\right|\right|^2_{\widetilde{M},\mathcal{F}_l}\right]dt\right).
\end{aligned}\tag{4.28}$$

Here, it is clear that one requires that ε is chosen such that $\varepsilon\leq 1$ (in addition to requiring that $\varepsilon>0$), so that the term containing $\left|\left|\left|\boldsymbol{v}^h_{,\boldsymbol{x}}\right|\right|\right|^2_{\widetilde{M},T_k}$ on the RHS bounds the corresponding term on the LHS. In addition, one requires that $\eta_{\mathcal{F}_l}$ is chosen such that the face-based lifting operator term on the RHS bounds the element-based lifting operator term on the LHS. Towards this end, one should note that by the Cauchy-Schwarz inequality (applied to the integrand of the element-based lifting operator term), one obtains

$$\begin{aligned}
&\sum_{T_k\in\mathcal{T}_h}\left[\left|\left|\left|\boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right|\right|\right|^2_{\widetilde{M},T_k}\right]\\
&\leq N_{\mathcal{F}}\sum_{T_k\in\mathcal{T}_h}\left[\sum_{\mathcal{F}_l\in\partial T_k}\left|\left|\left|\boldsymbol{r}_{\mathcal{F}_l}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right|\right|\right|^2_{\widetilde{M},T_k}\right],
\end{aligned}\tag{4.29}$$

where $N_{\mathcal{F}}$ is the number of faces for each simplex element (i.e., $N_{\mathcal{F}}=d+1$). In light of (4.29), it follows that in order for the face-based lifting operator term on the RHS of (4.28) to bound the element-based lifting operator term on the LHS, one requires that each $\eta_{\mathcal{F}_l}\geq\frac{N_{\mathcal{F}}}{\varepsilon}$. It is convenient to choose $\varepsilon=1$ so that the requirement becomes that $\eta_{\mathcal{F}_l}\geq N_{\mathcal{F}}$ instead of $\eta_{\mathcal{F}_l}\gg N_{\mathcal{F}}$ as would be required for smaller values of ε. With these choices for $\eta_{\mathcal{F}_l}$ and ε, (4.20) holds. This, in turn, completes the proof of Theorem 4.10. □

Remark 4.11. There is an alternative approach to stabilizing the aforementioned space-time HDG scheme. In particular, one may require that $\beta_{\mathcal{F}_l}$ is chosen such that the jump term on the RHS of (4.28) bounds the element-based lifting operator term on the LHS. The resulting scheme is stable because the following holds true:

$$\begin{aligned}
&2\sum_{n=0}^{N-1}\left(\sum_{T_k\in\mathcal{T}_h}\left[\Theta_{abs}\left(\boldsymbol{v}^h_{,\boldsymbol{x}},\boldsymbol{R}\left(\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h;\boldsymbol{n}\right)\right)_{\widetilde{M},T_k\times I^n}\right]\right)\\
&\leq\sum_{n=0}^{N-1}\left(\sum_{T_k\in\mathcal{T}_h}\left[\left|\left|\left|\boldsymbol{v}^h_{,\boldsymbol{x}}\right|\right|\right|^2_{\widetilde{M},T_k\times I^n}+\frac{1}{2}\sum_{\mathcal{F}_l\in\partial T_k}\beta_{\mathcal{F}_l}\left|\left|\left|\boldsymbol{n}\left[\widehat{\boldsymbol{v}}^h-\boldsymbol{v}^h\right]\right|\right|\right|^2_{\widetilde{M},\mathcal{F}_l\times I^n}\right]\right)
\end{aligned}\tag{4.30}$$

for large enough values of $\beta_{\mathcal{F}_l}$ (although the necessary minimum value of $\beta_{\mathcal{F}_l}$ is difficult to determine *a priori*). As a result of (4.30), (4.19) holds.

This analysis demonstrates that if $\beta_{\mathcal{F}_l}$ is chosen appropriately, $\eta_{\mathcal{F}_l} = 0$ is acceptable, and there is no need to form the face-based lifting operators.

5. Conclusion

This work has succeeded in constructing a mathematical proof for the entropy stability of a space-time HDG scheme for the compressible NS equations. In particular, stability has been shown for the compressible NS equations when the boundary conditions are chosen appropriately, the inviscid numerical flux is chosen to be a mean-value flux or the Lax-Friedrichs flux, and the viscous numerical flux is chosen to be a BR2-type flux (optionally accompanied by an IP-type flux). It is hoped that the entropy stable HDG scheme along with its variants will serve as useful tools in the toolbox of fluid dynamicists. This work is merely a preliminary step in constructing these tools, and it is anticipated that future work will consist of applying the schemes to practical problems, and extending the schemes to treat complex boundary conditions, shock waves, and other phenomena that are likely to arise in such problems.

Appendix A. Notation

This paper utilizes a mixture of index notation and vector notation. The notation is best explained via an example. Consider the term $\left[\boldsymbol{w}^h_{,\boldsymbol{x}_i}\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\right)$. This is shorthand for the following when $d = 3$:

$$\left[\boldsymbol{w}^h_{,\boldsymbol{x}_i}\right]^T \boldsymbol{f}^i\left(\boldsymbol{v}^h\right) = \left[\frac{\partial \boldsymbol{w}^h}{\partial \boldsymbol{x}_1}\right]^T \boldsymbol{f}^1\left(\boldsymbol{v}^h\right) + \left[\frac{\partial \boldsymbol{w}^h}{\partial \boldsymbol{x}_2}\right]^T \boldsymbol{f}^2\left(\boldsymbol{v}^h\right) + \left[\frac{\partial \boldsymbol{w}^h}{\partial \boldsymbol{x}_3}\right]^T \boldsymbol{f}^3\left(\boldsymbol{v}^h\right),$$

where the transpose allows for the standard dot product between two m-vectors (say $\left[\boldsymbol{w}^h_{,\boldsymbol{x}_1}\right]$ and $\boldsymbol{f}^1\left(\boldsymbol{v}^h\right)$), and the repeated index i allows for the standard Einstein summation over d dimensions.

Appendix B. Supporting lemmas

Lemma B.1 (Trivial generalization of a result from Barth; [4], p. 221)**.** *The temporal jumps in the solution are governed by the relation*

$$\begin{aligned}&\sum_{T_k \in \mathcal{T}_h} \left[\int_{T_k} \left[-\left[\!\left[U\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b} + \left[\boldsymbol{v}^h\left(t_b\right)\right]^T \left(\left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b}\right)\right] dx\right] \\ &\qquad = \sum_{T_k \in \mathcal{T}_h} \left\| \left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b} \right\|^2_{\widetilde{A}_0^{-1}, T_k},\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}&\left\| \left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b} \right\|^2_{\widetilde{A}_0^{-1}, T_k} \\ &\quad \equiv \int_{T_k} \left(\int_0^1 \left[1-\theta\right] \left(\left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b}\right)^T \widetilde{A}_0^{-1}\left(\overline{\boldsymbol{u}}\left(\theta\right)\right) \left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b} d\theta\right) dx \geq 0,\end{aligned}$$

is a nonnegative function, and where

$$\overline{\boldsymbol{u}}\left(\theta\right) \equiv \boldsymbol{u}\left(\boldsymbol{v}^h\left(t_b\right)\right) - \theta \left[\!\left[\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right]\!\right]_{t_a}^{t_b}, \qquad \left(\boldsymbol{v}^h\right)^T = U_{,\boldsymbol{u}}\left(\boldsymbol{u}\left(\boldsymbol{v}^h\right)\right), \qquad \widetilde{A}_0^{-1} = U_{,\boldsymbol{u},\boldsymbol{u}}.$$

Proof. The proof appears in [4] with t^n_- and t^n_+ in place of t_a and t_b. A similar argument is also presented in [55]. ∎

Lemma B.2 (Trivial generalization of a result from Barth; [4], pp. 221–222)**.** *The spatial jumps in the entropy flux are governed by the relation*

$$
\begin{aligned}
&F^i(\boldsymbol{v}_a) - F^i(\boldsymbol{v}_b) + \frac{1}{2}\left[\boldsymbol{v}_b + \boldsymbol{v}_a\right]^T \left[\boldsymbol{f}^i(\boldsymbol{v}_b) - \boldsymbol{f}^i(\boldsymbol{v}_a)\right] \\
&= \frac{1}{2}\int_0^1 \left[1-\theta\right]\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right]^T \left[\widetilde{A}_i\left(\check{\check{\boldsymbol{v}}}(\theta)\right) - \widetilde{A}_i\left(\check{\boldsymbol{v}}(\theta)\right)\right]\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] d\theta,
\end{aligned}
\tag{B.2}
$$

where

$$
\check{\boldsymbol{v}}(\theta) = \boldsymbol{v}_b - \theta\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right], \qquad \check{\check{\boldsymbol{v}}}(\theta) = \boldsymbol{v}_a + \theta\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right].
$$

Proof. Recall that $\mathcal{F}^i = \mathcal{F}^i(\boldsymbol{v})$. On utilizing this fact in conjunction with Taylor's theorem, one can obtain the following:

$$
\begin{aligned}
&\mathcal{F}^i(\boldsymbol{v}_b) - \mathcal{F}^i(\boldsymbol{v}_a) - \mathcal{F}^i_{,\boldsymbol{v}}(\boldsymbol{v}_b)\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] \\
&+ \int_0^1 \left[1-\theta\right]\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right]^T \mathcal{F}^i_{,\boldsymbol{v},\boldsymbol{v}}\left(\check{\boldsymbol{v}}(\theta)\right)\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] d\theta = 0,
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
&\mathcal{F}^i(\boldsymbol{v}_b) - \mathcal{F}^i(\boldsymbol{v}_a) - \mathcal{F}^i_{,\boldsymbol{v}}(\boldsymbol{v}_a)\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] \\
&- \int_0^1 \left[1-\theta\right]\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right]^T \mathcal{F}^i_{,\boldsymbol{v},\boldsymbol{v}}\left(\check{\check{\boldsymbol{v}}}(\theta)\right)\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] d\theta = 0.
\end{aligned}
\tag{B.4}
$$

Upon multiplying (B.3) and (B.4) by (1/2) and summing the results, one obtains

$$
\begin{aligned}
&\mathcal{F}^i(\boldsymbol{v}_b) - \mathcal{F}^i(\boldsymbol{v}_a) - \frac{1}{2}\left[\boldsymbol{f}^i(\boldsymbol{v}_b) + \boldsymbol{f}^i(\boldsymbol{v}_a)\right]^T \left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] \\
&= \frac{1}{2}\int_0^1 \left[1-\theta\right]\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right]^T \left[\widetilde{A}_i\left(\check{\check{\boldsymbol{v}}}(\theta)\right) - \widetilde{A}_i\left(\check{\boldsymbol{v}}(\theta)\right)\right]\left[\boldsymbol{v}_b - \boldsymbol{v}_a\right] d\theta,
\end{aligned}
\tag{B.5}
$$

where the fact that $\left[\boldsymbol{f}^i\right]^T = \mathcal{F}^i_{,\boldsymbol{v}}$ and $\widetilde{A}_i = \mathcal{F}^i_{,\boldsymbol{v},\boldsymbol{v}}$ has been used. Setting (B.5) aside for the moment, consider the following jump identity that derives from (2.16)

$$
\begin{aligned}
&F^i(\boldsymbol{v}_b) - F^i(\boldsymbol{v}_a) + \mathcal{F}^i(\boldsymbol{v}_b) - \mathcal{F}^i(\boldsymbol{v}_a) \\
&= \frac{1}{2}\left[\boldsymbol{v}_b + \boldsymbol{v}_a\right]^T \left[\boldsymbol{f}^i(\boldsymbol{v}_b) - \boldsymbol{f}^i(\boldsymbol{v}_a)\right] + \frac{1}{2}\left[\boldsymbol{f}^i(\boldsymbol{v}_b) + \boldsymbol{f}^i(\boldsymbol{v}_a)\right]^T \left[\boldsymbol{v}_b - \boldsymbol{v}_a\right].
\end{aligned}
\tag{B.6}
$$

Substituting (B.5) into (B.6) completes the proof of (B.2). This in turn, completes the proof of Lemma B.2. ◻

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