MATHEMATICS OF COMPUTATION
Volume 87, Number 309, January 2018, Pages 39–67
http://dx.doi.org/10.1090/mcom/3222
Article electronically published on May 11, 2017

AN EXACTLY COMPUTABLE LAGRANGE–GALERKIN SCHEME FOR THE NAVIER–STOKES EQUATIONS AND ITS ERROR ESTIMATES

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ABSTRACT. We present a Lagrange–Galerkin scheme, which is computable exactly, for the Navier–Stokes equations and show its error estimates. In the Lagrange–Galerkin method we have to deal with the integration of composite functions, where it is difficult to get the exact value. In real computations, numerical quadrature is usually applied to the integration to obtain approximate values, that is, the scheme is not computable exactly. It is known that the error caused from the approximation may destroy the stability result that is proved under the exact integration. Here we introduce a locally linearized velocity and the backward Euler method in solving ordinary differential equations in the position of the fluid particle. Then, the scheme becomes computable exactly, and we show the stability and convergence for this scheme. For the $\mathrm{P}_2/\mathrm{P}_1$- and $\mathrm{P}_1+/\mathrm{P}_1$-finite elements optimal error estimates are proved in $\ell^\infty(H^1)\times\ell^2(L^2)$ norm for the velocity and pressure. We present some numerical results, which reflect these estimates and also show robust stability for high Reynolds numbers in the cavity flow problem.

1. Introduction

The purpose of this paper is to present a Lagrange–Galerkin scheme free from numerical quadrature for the Navier–Stokes equations and to prove the convergence. The Lagrange–Galerkin method, which is also called the characteristics finite element method or the Galerkin-characteristics method, is a powerful numerical method for flow problems, having such advantages that it is robust for convection-dominated problems and that the resultant matrix to be solved is symmetric. It has, however, a drawback that it may lose the stability when numerical quadrature is employed with less care to integrate composite function terms that characterize the method. Our scheme presented here overcomes this drawback.

Lagrange–Galerkin schemes for the Navier–Stokes equations have been developed in [1, 4–6, 17, 18, 20, 22, 24]; see also the bibliographies therein. After convergence analysis was done successfully by Pironneau [20] in a suboptimal rate, the optimal convergence result was obtained by Süli [24]. Optimal convergence results by Lagrange–Galerkin schemes were extended to the multi-step method by Boukir et al. [6], to the projection method by Achdou–Guermond [1] and to the pressure-stabilized method by Notsu–Tabata [18]. All these results of the stability and convergence are proved under the condition that the integration of the composite function terms is computed exactly. Since it is difficult to perform the exact

Received by the editor September 4, 2015 and, in revised form, August 2, 2016.
2010 *Mathematics Subject Classification.* Primary 65M12, 65M25, 65M60, 76D05, 76M10.
Key words and phrases. Lagrange–Galerkin scheme, finite element method, Navier–Stokes equations, exact computation.

integration in real problems, numerical quadrature is usually employed. It is, however, reported that instability may occur caused by numerical quadrature error for convection-diffusion problems in [16, 22, 25, 28]. If rough numerical quadrature is used, we observe such instability occurs also for the Navier–Stokes equations by numerical examples in this paper.

Several methods have been studied to avoid the instability; see [4, 5, 16, 21, 22, 28]. There, the map of a fluid particle from the present position to the position a time increment Δt before (the position is often called the foot along the trajectory) is simplified. To find the foot of a particle is nothing but to solve a system of ordinary differential equations (ODEs). Morton et al. [16] solved the ODEs only at the centroids of the elements, and Priestley [22] did only at the vertices of the elements. The map of the other points is approximated by linear interpolation of those values. It becomes possible to perform the exact integration of the composite function terms with the simplified map. Bermejo et al. [4, 5] used the same simplified map as [22] to employ a numerical quadrature of high accuracy to the composite function terms for the Navier–Stokes equations. Tanaka et al. [28] and Tabata–Uchiumi [27] replaced the velocity by a locally linearized velocity and approximated the map by the backward Euler approximation to solve the ODEs for convection-diffusion problems. The approximate map makes possible the exact integration of the composite function terms, whose basic idea is the same as [22].

In this paper we prove the convergence of a Lagrange–Galerkin scheme with the same approximate map as [27, 28] in the $\mathrm{P}_2/\mathrm{P}_1$- and $\mathrm{P}_1+/\mathrm{P}_1$-elements (Taylor–Hood and MINI elements) for the Navier–Stokes equations. Since we neither solve the ODEs nor use numerical quadrature, our scheme is exactly computable to realize the theoretical results. It is, therefore, a genuinely stable Lagrange–Galerkin scheme. Our convergence results are best possible for the velocity and pressure in the $\ell^\infty(H^1) \times \ell^2(L^2)$-norm for both elements as well as for the velocity in the $\ell^\infty(L^2)$-norm in the $\mathrm{P}_1+/\mathrm{P}_1$-finite element.

In order to make clear the argument of convergence in the Lagrange–Galerkin method let us introduce two words, exact and computed finite element solutions. Let (u, p) be an exact solution of the Navier–Stokes equations, (u_h, p_h) an exact finite element solution by a scheme, and (u_h^C, p_h^C) a computed solution of (u_h, p_h). Since the solution got in the real computation is (u_h^C, p_h^C), the desired estimate is, in a norm with positive constants α and β,

$$\|(u - u_h^C, p - p_h^C)\| \le c(\Delta t^\alpha + h^\beta). \tag{1.1}$$

In the standard (not Lagrange–Galerkin) finite element method, we usually do not distinguish (u_h^C, p_h^C) from (u_h, p_h) because all coefficients appearing in the finite element equation are derived from the integrations of polynomials over elements, which are computed exactly except for the errors caused by finite length of digits. Namely, the scheme is exactly computable, and we understand that (u_h^C, p_h^C) is equal to (u_h, p_h). Then, (1.1) is obviously equivalent to the estimate,

$$\|(u - u_h, p - p_h)\| \le c(\Delta t^\alpha + h^\beta). \tag{1.2}$$

In the conventional Lagrange–Galerkin method with quadrature formula of degree m to the integrations of composite function terms, the computed solution $(u_h^C(m), p_h^C(m))$ is not equal to (u_h, p_h). In fact, the difference may cause the instability for low m as shown in Examples 6.1 and 6.2. The theoretical result (1.2) does not imply the convergence result (1.1) of the computed solution. To the best of our knowledge, the error estimate

$$\|(u - u_h^C(m), p - p_h^C(m))\| \le c_m(\Delta t^{\alpha'} + h^{\beta'}) \tag{1.3}$$

has not been obtained. With respect to the present scheme we prove (1.2). Since it is exactly computable without using quadrature, we have $(u_h^C, p_h^C) = (u_h, p_h)$ and (1.1). We note that, in the isoparametric finite element method where numerical quadrature is used, for example, in the Poisson problem in a curved domain, u_h^C is not equal to u_h, but the estimate corresponding to (1.3) is obtained with $\beta' = \beta$ and an appropriate choice m in [7].

As for the efficiency of the computation the present scheme spends about 2.6 times more computation time than the conventional scheme with the quadrature of degree nine in the computation of the cavity flow problem in a square as shown in Example 6.2.

The contents of this paper are as follows. In the next section we describe the Navier–Stokes problem and some preparation. In Section 3, after recalling the conventional Lagrange–Galerkin scheme, we present our Lagrange–Galerkin scheme with a locally linearized velocity. In Section 4 we show convergence results, which are proved in Section 5. In Section 6 we show some numerical results, which reflect the theoretical convergence orders and the robustness of the scheme for high-Reynolds-number problems. In Section 7 we give conclusions.

2. Preliminaries

We state the problem and prepare the notation used throughout this paper.

Let Ω be a polygonal or polyhedral domain of $\mathbb{R}^d$ $(d = 2, 3)$ and $T > 0$ a time. We use the Sobolev spaces $W^{m,p}(\Omega)$ equipped with the norm $\|\cdot\|_{m,p}$ and the semi-norm $|\cdot|_{m,p}$ for $p \in [1, \infty]$ and a non-negative integer m. We denote $W^{0,p}(\Omega)$ by $L^p(\Omega)$. $W_0^{1,p}(\Omega)$ is the subspace of $W^{1,p}(\Omega)$ consisting of functions whose trace vanish on the boundary of Ω. When $p = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the subscript 2 in the corresponding norm and semi-norm. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. For the vector-valued function $w \in W^{1,\infty}(\Omega)^d$ we define the semi-norm $|w|_{1,\infty}$ by

$$\left\| \left\{ \sum_{i,j=1}^{d} \left(\frac{\partial w_i}{\partial x_j} \right)^2 \right\}^{1/2} \right\|_{0,\infty}.$$

The parenthesis $(\cdot, \cdot)$ shows the $L^2(\Omega)^i$-inner product for $i = 1, d$ or $d \times d$. $L_0^2(\Omega)$ is the space of functions $f \in L^2(\Omega)$ satisfying $(f, 1) = 0$. For a Sobolev space $X(\Omega)$ we use the abbreviations $H^m(X) = H^m(0, T; X(\Omega))$ and $C(X) = C([0, T]; X(\Omega))$. We define the function space $Z^m(t_1, t_2)$ by

$$Z^m(t_1, t_2) \equiv \left\{ f \in H^j(t_1, t_2; H^{m-j}(\Omega)^d); j = 0, \ldots, m, \|f\|_{Z^m(t_1,t_2)} < \infty \right\},$$

$$\|f\|_{Z^m(t_1,t_2)} \equiv \left\{ \sum_{j=0}^{m} \|f\|^2_{H^j(t_1,t_2;H^{m-j}(\Omega)^d)} \right\}^{1/2},$$

and denote $Z^m(0, T)$ by Z^m.

We consider the Navier–Stokes equations: find $(u,p) : \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R}$ such that

$$(2.1)\qquad \begin{aligned} \frac{Du}{Dt} - \nu\Delta u + \nabla p &= f && \text{in } \Omega \times (0,T),\\ \nabla\cdot u &= 0 && \text{in } \Omega \times (0,T),\\ u &= 0 && \text{on } \partial\Omega \times (0,T),\\ u &= u^0 && \text{in } \Omega \text{ at } t=0, \end{aligned}$$

where $\partial\Omega$ is the boundary of Ω, $\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + (u\cdot\nabla)u$ is the material derivative and $\nu > 0$ is a viscosity. Functions $f \in C(L^2)$ and $u^0 : \Omega \to \mathbb{R}^d$ are given.

We define the bilinear forms a on $H_0^1(\Omega)^d \times H_0^1(\Omega)^d$ and b on $H_0^1(\Omega)^d \times L_0^2(\Omega)$ by

$$a(u,v) \equiv \nu(\nabla u, \nabla v), \quad b(v,q) \equiv -(\nabla\cdot v, q).$$

Then, we can write the weak form of (2.1) as follows: find $(u,p) : (0,T) \to H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that for $t \in (0,T)$,

$$(2.2a)\qquad \left(\frac{Du}{Dt}(t), v\right) + a(u(t),v) + b(v,p(t)) = (f(t),v), \qquad \forall v \in H_0^1(\Omega)^d,$$

$$(2.2b)\qquad b(u(t),q) = 0, \qquad \forall q \in L_0^2(\Omega),$$

with $u(0) = u^0$.

Let u be smooth. The characteristic curve $X(t;x,s)$ is defined by the solution of the system of the ordinary differential equations,

$$(2.3a)\qquad \frac{dX}{dt}(t;x,s) = u(X(t;x,s),t), \quad t < s,$$

$$(2.3b)\qquad X(s;x,s) = x.$$

Then, we can write the material derivative term $(\frac{\partial}{\partial t} + u\cdot\nabla)u$ as follows:

$$\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)(X(t),t) = \frac{d}{dt}u(X(t),t).$$

Let $\Delta t > 0$ be a time increment. For $w : \Omega \to \mathbb{R}^d$ we define the mapping $X_1(w) : \Omega \to \mathbb{R}^d$ by

$$(2.4)\qquad (X_1(w))(x) \equiv x - w(x)\Delta t.$$

Remark 2.1. The image of x by $X_1(u(\cdot,t))$ is nothing but the approximate value of $X(t-\Delta t; x,t)$ obtained by solving (2.3) by the backward Euler method.

Let $N_T \equiv \lfloor T/\Delta t \rfloor$, $t^n \equiv n\Delta t$ and $\psi^n \equiv \psi(\cdot, t^n)$ for a function ψ defined in $\Omega \times (0,T)$. For a set of functions $\psi = \{\psi^n\}_{n=0}^{N_T}$ and a Sobolev space $X(\Omega)$, two norms $\|\cdot\|_{\ell^\infty(X)}$ and $\|\cdot\|_{\ell^2(n_1,n_2;X)}$ are defined by

$$\|\psi\|_{\ell^\infty(X)} \equiv \max\left\{\|\psi^n\|_{X(\Omega)}\,; n = 0,\ldots,N_T\right\},$$

$$\|\psi\|_{\ell^2(n_1,n_2;X)} \equiv \left(\Delta t \sum_{n=n_1}^{n_2} \|\psi^n\|_{X(\Omega)}^2\right)^{1/2},$$

and $\|\psi\|_{\ell^2(1,N_T;X)}$ is denoted by $\|\psi\|_{\ell^2(X)}$. The backward difference operator $\overline{D}_{\Delta t}$ is defined by

$$\overline{D}_{\Delta t}\psi^n \equiv \frac{\psi^n - \psi^{n-1}}{\Delta t}.$$

Let $\mathcal{T}_h$ be a triangulation of $\bar{\Omega}$ and $h \equiv \max_{K\in\mathcal{T}_h} \mathrm{diam}(K)$ the maximum element size. Throughout this paper we consider a regular family of triangulations $\{\mathcal{T}_h\}_{h\downarrow 0}$. Let $V_h \times Q_h \subset H^1_0(\Omega)^d \times L^2_0(\Omega)$ be the $\mathrm{P}_2/\mathrm{P}_1$- or $\mathrm{P}_1+/\mathrm{P}_1$-finite element space, which is called the Hood–Taylor element or the MINI element [2, 10]. Let

$$\Pi_h^{(1)} : C(\bar{\Omega})^d \cap H^1_0(\Omega)^d \to V_h$$

be the Lagrange interpolation operator to the P_1-finite element space. Let $(\widehat{w}_h, \widehat{r}_h) \equiv \Pi_h^S(w, r) \in V_h \times Q_h$ be the Stokes projection of $(w, r) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ defined by

$$a(\widehat{w}_h, v_h) + b(v_h, \widehat{r}_h) = a(w, v_h) + b(v_h, r), \qquad \forall v_h \in V_h, \tag{2.5a}$$

$$b(\widehat{w}_h, q_h) = b(w, q_h), \qquad \forall q_h \in Q_h. \tag{2.5b}$$

We denote by $(\Pi_h^S(w, r))_1$ the first component $\widehat{w}_h$ of $\Pi_h^S(w, r)$.

The symbol $\circ$ stands for the composition of functions, e.g., $(g \circ f)(x) \equiv g(f(x))$.

3. A Lagrange–Galerkin scheme with a locally linearized velocity

The conventional Lagrange–Galerkin scheme, which we call Scheme LG, is described as follows.

Scheme LG. Let $u_h^0 = (\Pi_h^S(u^0, 0))_1$. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(u_h^{n-1})}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = (f^n, v_h), \qquad \forall v_h \in V_h,$$

$$b(u_h^n, q_h) = 0, \qquad \forall q_h \in Q_h,$$

for $n = 1, \ldots, N_T$.

Remark 3.1. Süli [24] used the exact solution X_h^{n-1} of the system of ordinary differential equations,

$$\frac{dX_h^{n-1}}{dt}(t; x, t^n) = u_h^{n-1}(X_h^{n-1}(t; x, t^n), t), \quad t^{n-1} < t < t^n, \tag{3.2a}$$

$$X_h^{n-1}(t^n; x, t^n) = x, \tag{3.2b}$$

instead of $X_1(u_h^{n-1})$.

By a similar way to [24] combined with [6], error estimates

$$\|u_h - u\|_{\ell^\infty(H^1)}, \|p_h - p\|_{\ell^2(L^2)} \le c(h^k + \Delta t), \tag{3.3a}$$

$$\|u_h - u\|_{\ell^\infty(L^2)} \le c(h^{k+1} + \Delta t) \tag{3.3b}$$

can be proved, where $k = 2$ for the $\mathrm{P}_2/\mathrm{P}_1$-element and $k = 1$ for the $\mathrm{P}_1+/\mathrm{P}_1$-element. In the estimate above, the composite function term $(u_h^{n-1} \circ X_1(u_h^{n-1}), v_h)$ is assumed to be exactly integrated.

Although the function u_h^{n-1} is a polynomial on each element K, the composite function $u_h^{n-1} \circ X_1(u_h^{n-1})$ is not a polynomial on K in general since the image $X_1(u_h^{n-1})$ of an element K may spread over plural elements. Hence, it is hard to calculate the composite function term $(u_h^{n-1} \circ X_1(u_h^{n-1}), v_h)$ exactly. In practice, the following numerical quadrature has been used. Let $g : K \to \mathbb{R}$ be a continuous function. A numerical quadrature $I_h[g; K]$ of $\int_K g\, dx$ is defined by

$$I_h[g; K] \equiv \text{meas}(K) \sum_{i=1}^{N_q} w_i g(a_i),$$

where N_q is the number of quadrature points and $(w_i, a_i) \in \mathbb{R} \times K$ is a pair of the weight and the point for $i = 1, \dots, N_q$. We call the practical scheme using numerical quadrature Scheme LG′.

Scheme LG′. Let $u_h^0 = (\Pi_h^S(u^0, 0))_1$. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that

$$\frac{1}{\Delta t}(u_h^n, v_h) - \frac{1}{\Delta t} \sum_{K \in \mathcal{T}_h} I_h[(u_h^{n-1} \circ X_1(u_h^{n-1})) \cdot v_h; K] + a(u_h^n, v_h) + b(v_h, p_h^n) = (f^n, v_h), \qquad \forall v_h \in V_h,$$

$$b(u_h^n, q_h) = 0, \qquad \forall q_h \in Q_h,$$

for $n = 1, \dots, N_T$.

For convection-diffusion equations it has been reported that rough numerical quadrature causes the instability [16, 22, 25–28]. For the Navier–Stokes equations we present numerical results showing the instability of Scheme LG′ in Section 6.

We now present our Lagrange–Galerkin scheme with a locally linearized velocity. It is free from quadrature and exactly computable. We call it Scheme LG-LLV.

Scheme LG-LLV. Let $u_h^0 = (\Pi_h^S(u^0, 0))_1$. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that

(3.4a)
$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(\Pi_h^{(1)} u_h^{n-1})}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = (f^n, v_h), \quad \forall v_h \in V_h,$$

(3.4b)
$$b(u_h^n, q_h) = 0, \qquad \forall q_h \in Q_h,$$

for $n = 1 \dots, N_T$.

In the above scheme the locally linearized velocity $\Pi_h^{(1)} u_h^{n-1}$ is used in place of the original velocity u_h^{n-1}. The error caused by the introduction of the approximate velocity $\Pi_h^{(1)} u_h^{n-1}$ is evaluated properly in Theorems 4.2 and 4.5 in the next section. The following proposition assures that the integration $(u_h^{n-1} \circ X_1(\Pi_h^{(1)} u_h^{n-1}), v_h)$ can be calculated exactly.

Proposition 3.2. *Let u_h, $v_h \in V_h$ and $w \in W_0^{1,\infty}(\Omega)^d$. Suppose $\alpha_{20} \Delta t\, |w|_{1,\infty} < 1$, where α_{20} is the constant defined in* (5.1a) *below. Then, $\int_\Omega (u_h \circ X_1(\Pi_h^{(1)} w)) \cdot v_h\, dx$ is exactly computable.*

Outline of the proof. When u_h and v_h are scalar functions, the result on the exact computability has been proved in [28] and [27, Proposition 1]. Here, we do not repeat the proof but show only the outline. It is necessary that the inclusion $(X_1(\Pi_h^{(1)} w))(\Omega) \subset \Omega$ holds to execute the integration of $u_h \circ X_1(\Pi_h^{(1)} w) \cdot v_h$ over Ω. The condition $\alpha_{20} \Delta t\, |w|_{1,\infty} < 1$ is sufficient for it by virtue of Lemma 5.7-(i) and (5.1a) below. The mapping $X_1(\Pi_h^{(1)} w)$ is linear on each element. When a mapping F is linear, we have the following general result for any two elements K_0 and K_1

and any polynomial ϕ_h of any order k defined on K_1. Proposition 3.2 is proved by applying the following lemma, whose proof is easy; cf. [27, Lemma 1]. □

Lemma 3.3. *Let $K_0, K_1 \in \mathcal{T}_h$ and $F : K_0 \to \mathbb{R}^d$ be linear and one-to-one. Let $E_1 \equiv K_0 \cap F^{-1}(K_1)$ and $\mathrm{meas}(E_1) > 0$. Then, the following hold:*

(i) *E_1 is a polygon ($d = 2$) or a polyhedron ($d = 3$).*

(ii) $\phi_h \circ F_{|E_1} \in \mathrm{P}_k(E_1), \quad \forall \phi_h \in \mathrm{P}_k(K_1)$.

Remark 3.4. (i) In the case of $d = 2$, Priestley [22] approximated $X_h^{n-1}(t^{n-1}; x, t^n)$ in (3.2) by

$$\widetilde{X}_h(x) = B_1\lambda_1(x) + B_2\lambda_2(x) + B_3\lambda_3(x),\ x \in K_0$$

on each $K_0 \in \mathcal{T}_h$, where $B_i = X_h^{n-1}(t^{n-1}; A_i, t^n)$, $\{A_i\}_{i=1}^3$ are vertices of K_0 and $\{\lambda_i\}_{i=1}^3$ are the barycentric coordinates of K_0 with respect to $\{A_i\}_{i=1}^3$. Since $\widetilde{X}_h(x)$ is linear in K_0, the decomposition

$$\int_{K_0} (u_h^{n-1} \circ \widetilde{X}_h) \cdot v_h\ dx = \sum_{l \in \Lambda(K_0)} \int_{E_l} (u_h^{n-1} \circ \widetilde{X}_h) \cdot v_h\ dx,$$

$$\Lambda(K_0) \equiv \left\{ l; K_0 \cap \widetilde{X}_h^{-1}(K_l) \neq \emptyset \right\}, \quad E_l \equiv K_0 \cap \widetilde{X}_h^{-1}(K_l),$$

makes the exact integration possible. The points $B_i = X_h^{n-1}(t^{n-1}; A_i, t^n)$ are the solutions of the system of ordinary differential equations (3.2) and it is not easy to solve it exactly in general since u_h^{n-1} is a piecewise polynomial. In practice, some numerical method, e.g., the Runge–Kutta method, is employed.

(ii) Since our mapping $X_1(\Pi_h^{(1)} u_h^{n-1})$ is also linear in K_0, we can rewrite the mapping as

$$X_1(\Pi_h^{(1)} u_h^{n-1})(x) = C_1\lambda_1(x) + C_2\lambda_2(x) + C_3\lambda_3(x),\ x \in K_0,$$

where $C_i = A_i - (\Pi_h^{(1)} u_h^{n-1})(A_i)\Delta t = A_i - u_h^{n-1}(A_i)\Delta t$ for $i = 1, 2, 3$. Since C_i are the approximate values of $X_h^{n-1}(t^{n-1}; A_i, t^n)$ by the Euler method, our method is also derived from Priestley's method by applying the Euler method to (3.2). In the following we show the convergence for the Navier–Stokes equations, which is not discussed in [22]. As the solver for the ordinary differential equations the Runge–Kutta method is more accurate than the Euler method. However, in order to obtain the best possible convergence order of the velocity and the pressure, at least in the $\ell^\infty(H^1) \times \ell^2(L^2)$-norm for the $\mathrm{P}_2/\mathrm{P}_1$- and $\mathrm{P}_1{+}/\mathrm{P}_1$-finite elements, it is shown to be sufficient to employ the Euler method in (3.2).

4. Main results

We present the main results of error estimates for Scheme LG-LLV, which are proved in the next section. We first state the result when the $\mathrm{P}_2/\mathrm{P}_1$-element is employed.

Hypothesis 1. *The solution of* (2.1) *satisfies*

$$u \in Z^2 \cap H^1(H^3),\ p \in H^1(H^2).$$

Remark 4.1. Hypothesis 1 implies $(u, p) \in C(H^3 \times H^2)$, which yields $\nabla \cdot u^0 = 0$.

Hypothesis 2. *The sequence $\{\mathcal{T}_h\}_{h \downarrow 0}$ satisfies the inverse assumption. In addition, for each h, $\forall K \in \mathcal{T}_h$ has at least one vertex in Ω.*

Theorem 4.2. *Let $V_h \times Q_h$ be the $\mathrm{P}_2/\mathrm{P}_1$-finite element space. Suppose Hypotheses 1 and 2. Then, there exist positive constants c_0 and h_0 such that if $h \in (0, h_0]$ and $\Delta t \le c_0 h^{d/4}$, the solution $(u_h, p_h) \equiv \{(u_h^n, p_h^n)\}_{n=0}^{N_T}$ of Scheme LG-LLV exists and the estimates*

$$\|u_h - u\|_{\ell^\infty(H^1)}, \|p_h - p\|_{\ell^2(L^2)} \le c_1(h^2 + \Delta t)$$

hold, where c_1 is a positive constant independent of h and Δt.

Next, we state the result when the $\mathrm{P}_1+/\mathrm{P}_1$-element is employed.

Hypothesis 1′. *The solution of (2.1) satisfies*

$$u \in Z^2 \cap H^1(H^2), \ p \in H^1(H^1).$$

Remark 4.3. Hypothesis 1′ implies $(u, p) \in C(H^2 \times H^1)$, which yields $\nabla \cdot u^0 = 0$.

Hypothesis 3. *The Stokes problem is regular, that is, for all $g \in L^2(\Omega)^d$ the solution $(w, r) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ of the Stokes problem,*

$$-\nu\Delta w + \nabla r = g, \quad x \in \Omega,$$
$$\nabla \cdot w = 0, \quad x \in \Omega,$$

belongs to $H^2(\Omega)^d \times H^1(\Omega)$ and the estimate

$$\|(w, r)\|_{H^2\times H^1} \le c\,\|g\|_0$$

holds, where c is a positive constant independent of g, w and r.

Remark 4.4. Hypothesis 3 holds, for example, if $d = 2$ and Ω is convex [10].

Theorem 4.5. *Let $V_h \times Q_h$ be the $\mathrm{P}_1+/\mathrm{P}_1$-finite element space. Suppose Hypotheses 1′ and 2. Then, there exist positive constants c_0 and h_0 such that if $h \in (0, h_0]$ and $\Delta t \le c_0 h^{d/4}$, the solution $(u_h, p_h) \equiv \{(u_h^n, p_h^n)\}_{n=0}^{N_T}$ of Scheme LG-LLV exists, and the estimates*

$$\|u_h - u\|_{\ell^\infty(H^1)}, \|p_h - p\|_{\ell^2(L^2)} \le c_2(h + \Delta t) \tag{4.1}$$

hold, where c_2 is a positive constant independent of h and Δt. Moreover, under Hypothesis 3, the estimate

$$\|u_h - u\|_{\ell^\infty(L^2)} \le c_3(h^2 + \Delta t) \tag{4.2}$$

holds, where c_3 is a positive constant independent of h and Δt.

Remark 4.6. The convergence proof is easily extended for any pairs satisfying the inf-sup condition. However, the convergence order with respect to the space discretization is bounded by $O(h^2)$ caused by the locally linearized approximation of the velocity. In fact, in the case of the $\mathrm{P}_2/\mathrm{P}_1$-element the estimate (3.3b) with $k = 2$ does not hold in Scheme LG-LLV, cf., Example 6.1 in Section 6.

5. Proofs of the main theorems

The proofs of Theorems 4.2 and 4.5 can be carried out in the way of [6, 24] except for estimates concerning the newly introduced locally linearized velocity. For the paper to be selfcontained, however, we show here the complete proofs. We also give a clear bound on Δt for estimates of the Jacobian of a mapping in Lemma 5.7. In this section and Appendix A the symbol α with a subscript stands for a positive numerical constant, in particular independent of the discretization parameters, h and Δt.

5.1. **Some lemmas.** We recall some results used in proving the main theorems. For proofs of Lemmas 5.1–5.6 we refer to the cited bibliography.

Lemma 5.1 (Poincaré's inequality [7]). *There exists an $\alpha_1(\Omega)$ such that*

$$\|v\|_0 \le \alpha_1 |v|_1, \quad \forall v \in H^1_0(\Omega)^d.$$

Lemma 5.2 (The Lagrange interpolation [7]). *Suppose $\{\mathcal{T}_h\}_{h\downarrow 0}$ is a regular family of triangulations of $\bar\Omega$. Let X_h be the P_2- or P_1+-finite element space and $\Pi_h^{(1)}$ be the Lagrange interpolation operator to the P_1-finite element space. Then, it holds that*

$$\|\Pi_h^{(1)} v\|_{0,\infty} \le \|v\|_{0,\infty}, \quad \forall v \in C(\bar\Omega)^d,$$

and there exist $\alpha_{20} \ge 1$, α_{21} and α_{22} such that

$$|\Pi_h^{(1)} v|_{1,\infty} \le \alpha_{20} |v|_{1,\infty}, \qquad \forall v \in W^{1,\infty}(\Omega)^d, \tag{5.1a}$$

$$\|\Pi_h^{(1)} v - v\|_s \le \alpha_{21} h^{2-s} |v|_2, \qquad s = 0,1,\ \forall v \in H^2(\Omega)^d, \tag{5.1b}$$

$$\|\Pi_h^{(1)} v_h\|_0 \le \alpha_{22} \|v_h\|_0, \qquad \forall v_h \in X_h. \tag{5.1c}$$

Remark 5.3. The inequality (5.1c) holds since X_h is finite-dimensional. If we replace $\Pi_h^{(1)}$ by the Clément interpolation operator [8], this inequality holds for all $v \in L^2(\Omega)^d$.

Lemma 5.4 (The inverse inequality [7]). *Suppose $\{\mathcal{T}_h\}_{h\downarrow 0}$ satisfies the inverse assumption. Let X_h be the P_2- or P_1+-finite element space. Then, there exist α_{30} and α_{31} such that*

$$\|v_h\|_{0,\infty} \le \alpha_{30} h^{-d/6} \|v_h\|_1, \quad \forall v_h \in X_h,$$
$$|v_h|_{1,\infty} \le \alpha_{31} h^{-d/2} |v_h|_1, \quad \forall v_h \in X_h.$$

Lemma 5.5 (The inf-sup condition [2,3,29]). *Suppose Hypothesis* 2*. Let $V_h \times Q_h \subset H^1_0(\Omega)^d \times L^2_0(\Omega)$ be the $\mathrm{P}_2/\mathrm{P}_1$- or $\mathrm{P}_1+/\mathrm{P}_1$-finite element space. Then, there exists an α_4 such that*

$$\inf_{q_h \in Q_h \setminus\{0\}} \sup_{v_h \in V_h\setminus\{0\}} \frac{b(v_h, q_h)}{\|v_h\|_1 \|q_h\|_0} \ge \alpha_4.$$

Lemma 5.6 ([10]). (i) *Suppose Hypothesis* 2 *and that $V_h \times Q_h \subset H^1_0(\Omega)^d \times L^2_0(\Omega)$ is the $\mathrm{P}_2/\mathrm{P}_1$- or $\mathrm{P}_1+/\mathrm{P}_1$-finite element space. Let $(\widehat{w}_h, \widehat{r}_h)$ be the Stokes projection of (w,r) defined in* (2.5)*. Then, there exists an $\alpha_{50}(\nu)$ such that*

$$\|\widehat{w}_h - w\|_1, \|\widehat{r}_h - r\|_0 \le \alpha_{50} h^k \|(w,r)\|_{H^{k+1}\times H^k},$$

where $k=2$ for the $\mathrm{P}_2/\mathrm{P}_1$-element and $k=1$ for the $\mathrm{P}_1+/\mathrm{P}_1$-element.

(ii) *Moreover, suppose Hypothesis* 3*. Then, there exists an $\alpha_{51}(\nu)$ such that*

$$\|\widehat{w}_h - w\|_0 \le \alpha_{51} h^{k+1} \|(w,r)\|_{H^{k+1}\times H^k},$$

where $k=2$ for the $\mathrm{P}_2/\mathrm{P}_1$-element and $k=1$ for the $\mathrm{P}_1+/\mathrm{P}_1$-element.

Lemma 5.7. (i) *Let $w \in W^{1,\infty}_0(\Omega)^d$ and $X_1(w)$ be the mapping defined in* (2.4)*. Then, under the condition $\Delta t\, |w|_{1,\infty} < 1$, $X_1(w): \Omega \to \Omega$ is bijective.*

(ii) *Furthermore, under the condition $\Delta t\, |w|_{1,\infty} \le 1/4$, the estimate*

$$\frac{1}{2} \le \det\left(\frac{\partial X_1(w)}{\partial x}\right) \le \frac{3}{2}$$

holds, where $\det(\partial X_1(w)/\partial x)$ is the Jacobian.

Proof. The former is proved in [23, Proposition 1]. We prove the latter only in the case $d = 3$ since the proof in $d = 2$ is much easier. Let I be the 3×3 identity matrix, $A = (a_{ij})$ and $a_j = (a_{1j}, a_{2j}, a_{3j})^T$, where $a_{ij} = \Delta t\, \partial w_i/\partial x_j$ for $i, j = 1, 2, 3$. The notation $|\cdot|$ stands for the absolute value, or the Euclidean norm in $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$. From the condition

$$|a| = (|a_1|^2 + |a_2|^2 + |a_3|^2)^{1/2} \leq 1/4,$$

we obtain

$$\det(A) \leq |a_1|\,|a_2|\,|a_3| \leq \left(\frac{1}{4\sqrt{3}}\right)^3.$$

Then, we have

$$\begin{aligned}
&\left|\det\left(\frac{\partial X_1(w)}{\partial x}\right) - 1\right| = |\det(I - A) - 1|\\
=&| - (a_{11} + a_{22} + a_{33})\\
&\quad + a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13} - \det(A)|\\
\leq&|a_{11} + a_{22} + a_{33}|\\
&\quad + |a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13}| + |\det(A)|\\
\leq&\sqrt{3}|a| + |a|^2 + |\det(A)| \leq 1/2,
\end{aligned}$$

which implies the result. □

Lemma 5.8. *Let* $1 \leq q < \infty$, $1 \leq p \leq \infty$, $1/p + 1/p' = 1$ *and* $w_i \in W_0^{1,\infty}(\Omega)^d$, $i = 1, 2$. *Under the condition* $\Delta t\, |w_i|_{1,\infty} \leq 1/4$, *it holds that, for* $\psi \in W^{1,qp'}(\Omega)^d$,

$$\|\psi \circ X_1(w_1) - \psi \circ X_1(w_2)\|_{0,q} \leq 2^{1/(qp')}\Delta t\, \|w_1 - w_2\|_{0,pq}\, \|\nabla\psi\|_{0,qp'},$$

where $X_1(\cdot)$ *is defined in* (2.4).

Lemma 5.8 is a direct consequence of [1, Lemma 4.5] and Lemma 5.7-(ii).

Lemma 5.9. *Let* $w \in W_0^{1,\infty}(\Omega)^d$. *Under the condition* $\Delta t\, |w|_{1,\infty} \leq 1/4$, *there exists an* α_6 *such that, for* $\psi \in L^2(\Omega)^d$,

$$\|\psi - \psi \circ X_1(w)\|_{-1} \leq \alpha_6 \Delta t\, \|w\|_{1,\infty}\, \|\psi\|_0,$$

where $X_1(\cdot)$ *is defined in* (2.4).

This is an extension of [9, Lemma 1] obtained in the 1D case. For the self-containedness we give the proof of Lemma 5.9 in Appendix A.

5.2. **Estimates of e_h^n under some assumptions.** Let

$$(e_h^n, \varepsilon_h^n) \equiv (u_h^n - \widehat{u}_h^n, p_h^n - \widehat{p}_h^n), \quad \eta(t) \equiv u(t) - \widehat{u}_h(t), \tag{5.2}$$

where (u, p) is the solution of (2.1), $(\widehat{u}_h(t), \widehat{p}_h(t))$ is the Stokes projection of $(u(t), p(t))$ defined in (2.5) and (u_h^n, p_h^n) is the solution of Scheme LG-LLV at the step n. From (2.2), (2.5) and (3.4) we have the error equations in (e_h^n, ε_h^n):

$$\left(\overline{D}_{\Delta t} e_h^n, v_h\right) + a(e_h^n, v_h) + b(v_h, \varepsilon_h^n) = \sum_{i=1}^{4}(R_i^n, v_h), \qquad \forall v_h \in V_h, \tag{5.3a}$$

$$b(e_h^n, q_h) = 0, \qquad \forall q_h \in Q_h, \tag{5.3b}$$

for $n = 1, \dots, N_T$, where

$$
\begin{aligned}
R_1^n &\equiv \frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1(u^{n-1})}{\Delta t},\\
R_2^n &\equiv \frac{u^{n-1}\circ X_1(\Pi_h^{(1)} u_h^{n-1}) - u^{n-1}\circ X_1(u^{n-1})}{\Delta t},\\
R_3^n &\equiv \frac{\eta^n - \eta^{n-1}\circ X_1(\Pi_h^{(1)} u_h^{n-1})}{\Delta t}, \quad R_4^n \equiv -\frac{e_h^{n-1} - e_h^{n-1}\circ X_1(\Pi_h^{(1)} u_h^{n-1})}{\Delta t}.
\end{aligned} \tag{5.4}
$$

Lemma 5.10. *Suppose Hypotheses* 1 *and* 2. *Under the condition*

$$\Delta t |u^{n-1}|_{1,\infty},\ \Delta t |\Pi_h^{(1)} u_h^{n-1}|_{1,\infty} \le 1/4, \tag{5.5}$$

it holds that

$$
\begin{aligned}
&\|R_1^n\|_0 \le \beta_1 \sqrt{\Delta t}\, \|u\|_{Z^2(t^{n-1},t^n)}, && (5.6a)\\
&\|R_2^n\|_0 \le \beta_2 \left\|e_h^{n-1}\right\|_0 + \beta_3 h^2 (\|(u,p)^{n-1}\|_{H^3\times H^2} + \left|u^{n-1}\right|_2), && (5.6b)\\
&\|R_3^n\|_0 \le \beta_4 \frac{h^2}{\sqrt{\Delta t}} \Big(\|(u,p)\|_{H^1(t^{n-1},t^n;H^3\times H^2)} &&\\
&\qquad + \left\|u_h^{n-1}\right\|_{0,\infty} \|(u,p)\|_{L^2(t^{n-1},t^n;H^3\times H^2)} \Big), && (5.6c)\\
&\|R_4^n\|_0 \le \beta_5 \left\|u_h^{n-1}\right\|_{0,\infty} \left|e_h^{n-1}\right|_1, && (5.6d)
\end{aligned}
$$

for $n = 1, \dots, N_T$, where $\beta_1 = \beta_1(\|u\|_{C(W^{1,\infty})})$, $\beta_2 = \beta_2(|u|_{C(W^{1,\infty})}, \alpha_{22})$, $\beta_3 = \beta_3(|u|_{C(W^{1,\infty})}, \alpha_{21}, \alpha_{22}, \alpha_{50})$, $\beta_4 = \beta_4(\alpha_{50})$, $\beta_5 = \sqrt{2}$ and the notation $\beta_i(A)$ means that a positive constant β_i depends on a set of parameters A.

Proof. We prove (5.6a). We decompose R_1^n as follows:

$$
\begin{aligned}
R_1^n(x) =& \left\{ \frac{\partial u^n}{\partial t}(x) + (u^{n-1}(x)\cdot\nabla)u^n(x) - \frac{u^n - u^{n-1}\circ X_1(u^{n-1})}{\Delta t}(x) \right\}\\
&+ (u^n(x) - u^{n-1}(x))\cdot\nabla u^n(x) \equiv R_{11}^n(x) + R_{12}^n(x).
\end{aligned}
$$

Setting

$$y(x,s) = x + (s-1)\Delta t\ u^{n-1}(x), \quad t(s) = t^{n-1} + s\Delta t,$$

we have

$$\frac{u^n - u^{n-1}\circ X_1(u^{n-1})}{\Delta t} = \frac{1}{\Delta t}\big[u(y(\cdot,s), t(s))\big]_{s=0}^1,$$

which implies that

$$
\begin{aligned}
R_{11}^n &= \frac{\partial u^n}{\partial t} + (u^{n-1}\cdot\nabla)u^n - \int_0^1 \left\{ (u^{n-1}(\cdot)\cdot\nabla)u + \frac{\partial u}{\partial t} \right\} (y(\cdot,s), t(s))ds\\
&= \Delta t \int_0^1 ds \int_s^1 \left\{ \left(u^{n-1}(\cdot)\cdot\nabla + \frac{\partial}{\partial t}\right)^2 u \right\} (y(\cdot,s_1), t(s_1))ds_1\\
&= \Delta t \int_0^1 s_1 \left\{ \left(u^{n-1}(\cdot)\cdot\nabla + \frac{\partial}{\partial t}\right)^2 u \right\} (y(\cdot,s_1), t(s_1))ds_1.
\end{aligned}
$$

Hence, we have

$$\|R_{11}^n\|_0 \le \Delta t \int_0^1 s_1 \left\| \left\{ \left(u^{n-1}(\cdot)\cdot\nabla + \frac{\partial}{\partial t}\right)^2 u \right\} (y(\cdot,s_1),t(s_1)) \right\|_0 ds_1$$
$$\le \beta_1'(\|u\|_{C(L^\infty)})\sqrt{\Delta t}\,\|u\|_{Z^2(t^{n-1},t^n)},$$

where we have used the transformation of independent variables from x to y and s_1 to t and the estimate $|\det(\partial x/\partial y)| \le 2$ by virtue of Lemma 5.7-(ii). It is easy to show

$$\|R_{12}^n\|_0 \le \sqrt{\Delta t}\,|u^n|_{1,\infty} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(t^{n-1},t^n;L^2)}.$$

From the triangle inequality we get (5.6a).

We prove (5.6b). Using Lemma 5.8 with $q=2$, $p=1$, $p'=\infty$, $w_1=\Pi_h^{(1)}u_h^{n-1}$, $w_2=u^{n-1}$ and $\psi=u^{n-1}$, we have

$$\|R_2^n\|_0 \le \left|u^{n-1}\right|_{1,\infty} \|\Pi_h^{(1)}u_h^{n-1}-u^{n-1}\|_0$$
$$\le \left|u^{n-1}\right|_{1,\infty} (\|\Pi_h^{(1)}u_h^{n-1}-\Pi_h^{(1)}u^{n-1}\|_0 + \|\Pi_h^{(1)}u^{n-1}-u^{n-1}\|_0).$$

From Lemmas 5.2 and 5.6-(i) we evaluate the first term as follows:

$$\begin{aligned}
&\|\Pi_h^{(1)}u_h^{n-1}-\Pi_h^{(1)}u^{n-1}\|_0 = \|\Pi_h^{(1)}(u_h^{n-1}-\Pi_h^{(1)}u^{n-1})\|_0\\
&\le\alpha_{22}\|u_h^{n-1}-\Pi_h^{(1)}u^{n-1}\|_0\\
&\le\alpha_{22}(\left\|u_h^{n-1}-\widehat{u}_h^{n-1}\right\|_0+\left\|\widehat{u}_h^{n-1}-u^{n-1}\right\|_0+\|u^{n-1}-\Pi_h^{(1)}u^{n-1}\|_0)\\
&\le\alpha_{22}(\|e_h^{n-1}\|_0+\alpha_{50}h^2\|(u,p)^{n-1}\|_{H^3\times H^2}+\alpha_{21}h^2\left|u^{n-1}\right|_2).
\end{aligned} \tag{5.7}$$

The second term is evaluated as follows:

$$\|\Pi_h^{(1)}u^{n-1}-u^{n-1}\|_0 \le \alpha_{21}h^2\left|u^{n-1}\right|_2.$$

Thus, we have

$$\|R_2^n\|_0 \le \left|u^{n-1}\right|_{1,\infty}\Big\{\alpha_{22}(\left\|e_h^{n-1}\right\|_0+\alpha_{50}h^2\|(u,p)^{n-1}\|_{H^3\times H^2})$$
$$+\alpha_{21}(1+\alpha_{22})h^2\left|u^{n-1}\right|_2\Big\},$$

which implies (5.6b).

We prove (5.6c). Let

$$y(x)=x+(s-1)\Delta t\Pi_h^{(1)}u_h^{n-1}(x), \quad t(s)=t^{n-1}+s\Delta t.$$

Since R_3^n is rewritten as

$$R_3^n=\int_0^1\left\{\frac{\partial\eta}{\partial t}+(\Pi_h^{(1)}u_h^{n-1}(\cdot)\cdot\nabla)\eta\right\}(y(\cdot,s),t(s))ds,$$

we have, by using the change of the variable and Lemma 5.7-(ii),

$$
\begin{aligned}
\|R_3^n\|_0 \le & \left\| \int_0^1 \left| \frac{\partial \eta}{\partial t} \right| (y(\cdot,s),t(s)) ds \right\|_0 + \left\| \int_0^1 |(\Pi_h^{(1)} u_h^{n-1}(\cdot)\cdot\nabla)\eta|(y(\cdot,s),t(s)) ds \right\|_0 \\
\le & \frac{\sqrt{2}}{\sqrt{\Delta t}} \left(\left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(t^{n-1},t^n;L^2)} + \|\Pi_h^{(1)} u_h^{n-1}\|_{0,\infty} \|\nabla \eta\|_{L^2(t^{n-1},t^n;L^2)} \right) \\
\le & \frac{\sqrt{2}\alpha_{50} h^2}{\sqrt{\Delta t}} \big(\|(u,p)\|_{H^1(t^{n-1},t^n;H^3\times H^2)} \\
& \qquad + \left\|u_h^{n-1}\right\|_{0,\infty} \|(u,p)\|_{L^2(t^{n-1},t^n;H^3\times H^2)} \big),
\end{aligned}
$$

which implies (5.6c).

The inequality (5.6d) is obtained from Lemma 5.8 with $q=2$, $p=\infty$, $p'=1$, $w_1=0$, $w_2=\Pi_h^{(1)} u_h^{n-1}$ and $\psi = e_h^{n-1}$. □

Lemma 5.11. *Suppose Hypotheses* 1 *and* 2. *Let $n \in \{1,\cdots,N_T\}$ be any integer and let $u_h^{n-1} \in V_h$ be known. Suppose that u_h^{n-1} satisfies*

$$
b(u_h^{n-1}, q_h) = 0, \qquad \forall q_h \in Q_h. \tag{5.8}
$$

Under the condition (5.5), *there exists a solution (u_h^n, p_h^n) of* (3.4) *and it holds that*

$$
\begin{aligned}
& \left\|\overline{D}_{\Delta t} e_h^n\right\|_0^2 + \overline{D}_{\Delta t}(\nu\, |e_h^n|_1^2) \\
\le & \beta_{21}(\left\|u_h^{n-1}\right\|_{0,\infty}) \nu \left|e_h^{n-1}\right|_1^2 + \beta_{22}(\left\|u_h^{n-1}\right\|_{0,\infty}) \Big\{ \Delta t\, \|u\|_{Z^2(t^{n-1},t^n)}^2 \\
& + \frac{h^4}{\Delta t} \|(u,p)\|_{H^1(t^{n-1},t^n;H^3\times H^2)}^2 + h^4 \left(\|(u,p)^{n-1}\|_{H^3\times H^2}^2 + \left|u^{n-1}\right|_2^2 \right) \Big\},
\end{aligned}
$$

where e_h^n is defined in (5.2), *and $\beta_{21}(\xi)$ and $\beta_{22}(\xi)$ are the functions defined in* (5.10) *below.*

Proof. Since it holds that $\Delta t |\Pi_h^{(1)} u_h^{n-1}|_{1,\infty} \le 1/4$, the mapping $X_1(\Pi_h^{(1)} u_h^{n-1}) : \Omega \to \Omega$ is bijective from Lemma 5.7-(i). Hence, there exists a solution (u_h^n, p_h^n) of (3.4). Substituting $v_h = \overline{D}_{\Delta t} e_h^n$ in (5.3a), we have

$$
\left\|\overline{D}_{\Delta t} e_h^n\right\|_0^2 + \overline{D}_{\Delta t}\Big(\frac{\nu}{2} \|\nabla e_h^n\|_0^2\Big) + b(\overline{D}_{\Delta t} e_h^n, \varepsilon_h^n) \le \sum_{i=1}^4 (R_i^n, \overline{D}_{\Delta t} e_h^n). \tag{5.9}
$$

From (5.8) and (3.4) the term $b(\overline{D}_{\Delta t} e_h^n, \varepsilon_h^n)$ of the left-hand side vanishes. Using Schwarz' and Young's inequalities and Lemma 5.10, we have

$$
\begin{aligned}
& \left\|\overline{D}_{\Delta t} e_h^n\right\|_0^2 + \overline{D}_{\Delta t}\Big(\frac{\nu}{2} |e_h^n|_1^2\Big) \le 2\Big\{ \beta_1^2 \Delta t\, \|u\|_{Z^2(t^{n-1},t^n)}^2 \\
& + \big(\beta_2 \left\|e_h^{n-1}\right\|_0 + \beta_3 h^2 (\|(u,p)^{n-1}\|_{H^3\times H^2} + \left|u^{n-1}\right|_2)\big)^2 \\
& + \beta_4^2 \frac{h^4}{\Delta t} \left(\|(u,p)\|_{H^1(t^{n-1},t^n;H^3\times H^2)} + \left\|u_h^{n-1}\right\|_{0,\infty} \|(u,p)\|_{L^2(t^{n-1},t^n;H^3\times H^2)} \right)^2 \\
& + \beta_5^2 \left\|u_h^{n-1}\right\|_{0,\infty}^2 \left|e_h^{n-1}\right|_1^2 \Big\} + \frac{1}{2} \left\|\overline{D}_{\Delta t} e_h^n\right\|_0^2,
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\overline{D}_{\Delta t}(\nu\,|e_h^n|_1^2) + \left\|\overline{D}_{\Delta t} e_h^n\right\|_0^2 \\
\le&\beta_{11}\Big(\left\|e_h^{n-1}\right\|_0^2 + \left\|u_h^{n-1}\right\|_{0,\infty}^2 \left|e_h^{n-1}\right|_1^2\Big) + \beta_{12}\Big\{\Delta t\,\|u\|_{Z^2(t^{n-1},t^n)}^2 \\
&+ \frac{h^4}{\Delta t}(\|(u,p)\|_{H^1(t^{n-1},t^n;H^3\times H^2)}^2 + \left\|u_h^{n-1}\right\|_{0,\infty}^2 \|(u,p)\|_{L^2(t^{n-1},t^n;H^3\times H^2)}^2) \\
&+ h^4\left(\|(u,p)^{n-1}\|_{H^3\times H^2}^2 + \left|u^{n-1}\right|_2^2\right)\Big\},
\end{aligned}$$

where β_{11} and β_{12} are constants depending only on $\beta_1,\ldots,\beta_5$. Using Poincaré's inequality $\|e_h^{n-1}\|_0 \le \alpha_1 |e_h^{n-1}|_1$ and defining the functions β_{21} and β_{22} by

$$\beta_{21}(\xi) = \frac{\beta_{11}}{\nu}(\alpha_1^2+\xi^2),\ \beta_{22}(\xi) = \beta_{12}(1+\xi^2), \tag{5.10}$$

we have the conclusion. □

5.3. **Definitions of constants c_*, c_0 and h_0.** We first define constants β_{21}^* and β_{22}^* by

$$\beta_{21}^* \equiv \beta_{21}(\|u\|_{C(L^\infty)}+1),\ \beta_{22}^* \equiv \beta_{22}(\|u\|_{C(L^\infty)}+1).$$

We define two positive constants c_* and c_0 by

$$\begin{aligned}
c_* \equiv&\big\{\nu^{-1}(1+\alpha_1^2)\exp(\beta_{21}^* T)\beta_{22}^*\big\}^{1/2} \max\Big\{\|u\|_{Z^2},\Big(\|(u,p)\|_{H^1(H^3\times H^2)}^2 \\
&+ T\left(\|(u,p)\|_{C(H^3\times H^2)}^2 + |u|_{C(H^2)}^2\right) + \nu\alpha_{50}^2|p^0|_2^2\Big)^{1/2}\Big\}
\end{aligned}$$

and

$$c_0 \equiv \frac{1}{4}\sqrt{\frac{1}{\alpha_{20}\alpha_{31}c_*}}. \tag{5.11}$$

Let a positive constant h_0 be small enough to satisfy that

$$\begin{aligned}
&\alpha_{30}h_0^{1-d/6}\left(c_*h_0 + \alpha_{50}h_0\|(u,p)\|_{C(H^3\times H^2)} + \alpha_{21}\,|u|_{C(H^2)}\right) \\
&\qquad\qquad + \alpha_{30}c_*c_0h_0^{d/12} \le 1,
\end{aligned} \tag{5.12a}$$

$$\begin{aligned}
&c_0\Big\{\alpha_{31}h_0^{1-d/4}\left(c_*h_0 + \alpha_{50}h_0\|(u,p)\|_{C(H^3\times H^2)} + \alpha_{21}\,|u|_{C(H^2)}\right) \\
&\qquad\qquad + \alpha_{20}h_0^{d/4}\,|u|_{C(W^{1,\infty})}\Big\} \le \frac{3}{16\alpha_{20}},
\end{aligned} \tag{5.12b}$$

which are possible since all the powers of h_0 are positive.

5.4. **Induction.** For $n = 0,\ldots,N_T$ we define the property $\mathrm{P}(n)$ by

(a) $\nu\,|e_h^n|_1^2 + \left\|\overline{D}_{\Delta t}e_h\right\|_{\ell^2(1,n;L^2)}^2 \le \exp(\beta_{21}^* n\Delta t)\beta_{22}^*\Big\{\Delta t^2\,\|u\|_{Z^2(t^0,t^n)}^2$
$+ h^4(\|(u,p)\|_{H^1(t^0,t^n;H^3\times H^2)}^2 + \|(u,p)\|_{\ell^2(0,n-1;H^3\times H^2)}^2 + |u|_{\ell^2(0,n-1;H^2)}^2)$
$+ \nu\left|e_h^0\right|_1^2\Big\}$.

(b) $\|u_h^n\|_{0,\infty} \le \|u\|_{C(L^\infty)} + 1$.

(c) $\Delta t|\Pi_h^{(1)}u_h^n|_{1,\infty} \le 1/4$.

Proof of Theorem 4.2. We first prove that P(n) holds for $n = 0, \ldots, N_T$ by induction. When $n = 0$, the property P(0)-(a) obviously holds with the equality. The properties P(0)-(b) and (c) are proved in similar ways to and easier than P(n)-(b) and (c) below. We omit the proofs.

Let $n \in \{1, \cdots, N_T\}$ be any integer. Supposing that P(k), $k = 1, \ldots, n-1$, holds true, we prove that P(n) holds. We now apply Lemma 5.11. The condition (5.8) is satisfied trivially when $n \geq 2$. When $n = 1$, from the choice of u_h^0, (2.5) and Remark 4.1 we have

$$b(e_h^0, q_h) = b(u_h^0, q_h) - b(\widehat{u}_h^0, q_h) = 0 - 0 = 0, \quad \forall q_h \in Q_h. \tag{5.13}$$

We consider the condition (5.5). The former condition follows from $\Delta t \leq c_0 h^{d/4}$ and (5.12b) by the inequality

$$\Delta t |u^{n-1}|_{1,\infty} \leq c_0 h_0^{d/4} |u|_{C(W^{1,\infty})} \leq \frac{3}{16\alpha_{20}} \leq \frac{1}{4},$$

and the latter condition $\Delta t |\Pi_h^{(1)} u_h^{n-1}|_{1,\infty} \leq 1/4$ follows from P($n-1$)-(c). Hence, there exists a solution (u_h^n, p_h^n) at the step n.

We begin the proof of P(n)-(a). By putting

$$\begin{aligned} x_n \equiv & \nu \, |e_h^n|_1^2, \quad y_n \equiv \left\| \overline{D}_{\Delta t} e_h^n \right\|_0^2, \\ b_n \equiv & \Delta t \, \|u\|_{Z^2(t^{n-1}, t^n)}^2 \\ & + h^4 \left(\tfrac{1}{\Delta t} \|(u,p)\|_{H^1(t^{n-1}, t^n; H^3 \times H^2)}^2 + \|(u,p)^{n-1}\|_{H^3 \times H^2}^2 + \left| u^{n-1} \right|_2^2 \right), \end{aligned}$$

P(n)-(a) is rewritten as

$$x_n + \Delta t \sum_{i=1}^{n} y_i \leq \exp(\beta_{21}^* n \Delta t) \beta_{22}^* \left(x_0 + \Delta t \sum_{i=1}^{n} b_i \right). \tag{5.14}$$

On the other hand, Lemma 5.11 implies that

$$x_n + \Delta t y_n \leq (1 + \beta_{21}^* \Delta t) x_{n-1} + \beta_{22}^* \Delta t b_n,$$

where we have used the inequalities $\beta_{2i}(\left\| u_h^{n-1} \right\|_{0,\infty}) \leq \beta_{2i}^*$, $i = 1, 2$, obtained from P($n-1$)-(b). Using the inequalities $1 \leq 1 + x \leq \exp(x)$ for $x \geq 0$ and P($n-1$)-(a) rewritten by (5.14), we have

$$\begin{aligned} & x_n + \Delta t \sum_{i=1}^{n} y_i = x_n + \Delta t y_n + \Delta t \sum_{i=1}^{n-1} y_i \\ \leq & (1 + \beta_{21}^* \Delta t) x_{n-1} + \beta_{22}^* \Delta t b_n + \Delta t \sum_{i=1}^{n-1} y_i \\ \leq & (1 + \beta_{21}^* \Delta t) \exp(\beta_{21}^* (n-1) \Delta t) \beta_{22}^* \left(x_0 + \Delta t \sum_{i=1}^{n-1} b_i \right) + \beta_{22}^* \Delta t b_n \\ \leq & \exp(\beta_{21}^* n \Delta t) \beta_{22}^* \left(x_0 + \Delta t \sum_{i=1}^{n} b_i \right), \end{aligned}$$

which is nothing but P(n)-(a).

Since u_h^0 is the first component of $\Pi_h^S(u^0, 0)$, we have

$$e_h^0 = u_h^0 - \widehat{u}_h^0 = (\Pi_h^S(0, -p^0))_1 = ((\Pi_h^S - I)(0, -p^0))_1,$$

which implies $\left|e_h^0\right|_1 \le \alpha_{50} h^2 \left|p^0\right|_2$. From P(0)-(a) and the definition of c_*, we have

$$\|e_h^n\|_1 \le c_*(h^2 + \Delta t). \tag{5.15}$$

P(n)-(b) is proved as follows:

$$\begin{aligned}
&\|u_h^n\|_{0,\infty}\\
\le&\|u_h^n - \Pi_h^{(1)} u^n\|_{0,\infty} + \|\Pi_h^{(1)} u^n\|_{0,\infty}\\
\le&\alpha_{30} h^{-d/6} \|u_h^n - \Pi_h^{(1)} u^n\|_1 + \|u^n\|_{0,\infty} \quad \text{(by Lemmas 5.4 and 5.2)}\\
\le&\alpha_{30} h^{-d/6} (\|u_h^n - \widehat{u}_h^n\|_1 + \|\widehat{u}_h^n - u^n\|_1 + \|u^n - \Pi_h^{(1)} u^n\|_1) + \|u^n\|_{0,\infty}\\
\le&\alpha_{30} h^{-d/6} \left(c_*(h^2 + \Delta t) + \alpha_{50} h^2 \|(u,p)^n\|_{H^3\times H^2} + \alpha_{21} h \left|u^n\right|_2\right) + \|u^n\|_{0,\infty}\\
&\qquad \text{(by (5.15), Lemma 5.6-(i) and Lemma 5.2)}\\
\le&\alpha_{30} h^{1-d/6} \left(c_* h + \alpha_{50} h \|(u,p)^n\|_{H^3\times H^2} + \alpha_{21} \left|u^n\right|_2\right) + \alpha_{30} c_* c_0 h^{d/12} + \|u^n\|_{0,\infty}\\
&\qquad \text{(since } \Delta t \le c_0 h^{d/4})\\
\le&1 + \|u\|_{C(L^\infty)} \qquad \text{(since } h \le h_0 \text{ and by (5.12a)).}
\end{aligned}$$

We prove P(n)-(c). We can estimate $|u_h^n|_{1,\infty} \Delta t$ as follows:

$$\begin{aligned}
&|u_h^n|_{1,\infty} \Delta t\\
\le&(|u_h^n - \Pi_h^{(1)} u^n|_{1,\infty} + |\Pi_h^{(1)} u^n|_{1,\infty})\Delta t\\
\le&\{\alpha_{31} h^{-d/2} (|u_h^n - \Pi_h^{(1)} u^n|_1) + \alpha_{20} \left|u^n\right|_{1,\infty}\}\Delta t \qquad \text{(by Lemmas 5.4 and 5.2)}\\
\le&\{\alpha_{31} h^{-d/2} (|u_h^n - \widehat{u}_h^n|_1 + |\widehat{u}_h^n - u^n|_1 + |u^n - \Pi_h^{(1)} u^n|_1) + \alpha_{20} \left|u^n\right|_{1,\infty}\}\Delta t\\
\le&\Big\{\alpha_{31} h^{-d/2} \left(c_*(h^2 + \Delta t) + \alpha_{50} h^2 \|(u,p)^n\|_{H^3\times H^2} + \alpha_{21} h \left|u^n\right|_2\right)\\
&+ \alpha_{20} \left|u^n\right|_{1,\infty}\Big\}\Delta t \qquad \text{(by (5.15), Lemma 5.6-(i) and Lemma 5.2)}\\
\le&c_0 \left\{\alpha_{31} h^{1-d/4} \left(c_* h + \alpha_{50} h \|(u,p)^n\|_{H^3\times H^2} + \alpha_{21} \left|u^n\right|_2\right) + \alpha_{20} h^{d/4} \left|u^n\right|_{1,\infty}\right\}\\
&+ \alpha_{31} c_* c_0^2 \qquad \text{(since } \Delta t \le c_0 h^{d/4})\\
\le&\frac{3}{16\alpha_{20}} + \frac{1}{16\alpha_{20}} = \frac{1}{4\alpha_{20}} \quad \text{(since } h \le h_0\text{, and by (5.12b) and (5.11)).}
\end{aligned}$$

From this estimate and the definition of α_{20}, we have $\Delta t|\Pi_h^{(1)} u_h^n|_{1,\infty} \le 1/4$.

Thus, we have proved that P(n) holds for $n = 0, \cdots, N_T$.

From P(n)-(a), $n = 0, \ldots, N_T$, we obtain

$$\|e_h\|_{\ell^\infty(H^1)}, \left\|\overline{D}_{\Delta t} e_h\right\|_{\ell(1,N_T;L^2)} \le c_*(h^2 + \Delta t). \tag{5.16}$$

Using the triangle inequality $\|u_h - u\|_{\ell^\infty(H^1)} \le \|e_h\|_{\ell^\infty(H^1)} + \|\eta\|_{\ell^\infty(H^1)}$, we get

$$\|u_h - u\|_{\ell^\infty(H^1)} \le c_1(h^2 + \Delta t).$$

We now prove the estimate on the pressure. We can evaluate ε_h^n as follows:

$$
\begin{aligned}
\|\varepsilon_h^n\|_0 \le & \frac{1}{\alpha_4} \sup_{v_h\in V_h} \frac{b(v_h,\varepsilon_h^n)}{\|v_h\|_1} \qquad \text{(by Lemma 5.5)} \\
= & \frac{1}{\alpha_4} \sup_{v_h\in V_h} \frac{1}{\|v_h\|_1}\left(\sum_{i=1}^4 (R_i^n, v_h) - (\overline{D}_{\Delta t} e_h^n, v_h) - a(e_h^n, v_h)\right) \qquad \text{(by (5.3a))} \\
\le & \frac{1}{\alpha_4}\left(\sum_{i=1}^4 \|R_i^n\|_0 + \left\|\overline{D}_{\Delta t} e_h^n\right\|_0 + \nu\, |e_h^n|_1\right) \\
\le & c\Bigg(\left\|\overline{D}_{\Delta t} e_h^n\right\|_0 + \nu\,|e_h^n|_1 + \|e_h^{n-1}\|_1 + \sqrt{\Delta t}\,\|u\|_{Z^2(t^{n-1},t^n)} \\
& + h^2\|(u,p)^{n-1}\|_{H^3\times H^2} + \frac{h^2}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^3\times H^2)} + h^2\left|u^{n-1}\right|_2\Bigg) \\
& \text{(by Lemma 5.10 and P}(n-1)\text{-(b)),}
\end{aligned}
$$

which implies that, from (5.16)

$$
\|\varepsilon_h\|_{\ell^2(L^2)} \le c(\left\|\overline{D}_{\Delta t} e_h\right\|_{\ell^2(L^2)} + h^2 + \Delta t) \le c(h^2+\Delta t),
$$

where c is a positive constant independent of h and Δt. Using the triangle inequality

$$
\|p_h - p\|_{\ell^2(L^2)} \le \|\varepsilon_h\|_{\ell^2(L^2)} + \|p - \widehat{p}_h\|_{\ell^2(L^2)}\,,
$$

we obtain $\|p_h - p\|_{\ell^2(L^2)} \le c_1(h^2+\Delta t)$. □

5.5. **Proof of Theorem 4.5.** In this subsection we prove the result on the P_1+/P_1-element. At first we replace the estimates of R_2^n and R_3^n in Lemma 5.10.

Lemma 5.10′. *Suppose Hypotheses* 1′ *and* 2*. Under the condition* (5.5) *it holds that*

$$
\begin{aligned}
\|R_2^n\|_0 \le & \beta_2\left\|e_h^{n-1}\right\|_0 + \beta_3(h\|(u,p)^{n-1}\|_{H^2\times H^1} + h^2\left|u^{n-1}\right|_2), \\
\|R_3^n\|_0 \le & \beta_4\frac{h}{\sqrt{\Delta t}}\big(\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} \\
& + \left\|u_h^{n-1}\right\|_{0,\infty}\|(u,p)\|_{L^2(t^{n-1},t^n;H^2\times H^1)}\big)
\end{aligned}
$$

for $n = 1,\dots,N_T$.

The proof is similar to Lemma 5.10 by replacing the order $k=2$ by $k=1$ in Lemma 5.6-(i).

Proof of Theorem 4.5. We only show the outline of the proof for the existence of (u_h, p_h) and the inequality (4.1) since the proof is similar to that of Theorem 4.2. We replace the definition of c_* by

$$
\begin{aligned}
c_* \equiv & \big\{\nu^{-1}(1+\alpha_1^2)\exp(\beta_{21}^* T)\beta_{22}^*\big\}^{1/2}\max\Big\{\|u\|_{Z^2}, \Big(\|(u,p)\|^2_{H^1(H^2\times H^1)} \\
& + T\left(\|(u,p)\|^2_{C(H^2\times H^1)} + |u|^2_{C(H^2)}\right) + \nu\alpha_{50}^2|p^0|_1^2\Big)^{1/2}\Big\},
\end{aligned}
$$

redefine c_0 by (5.11) with the new c_*, and replace the condition (5.12) on h_0 by

$$\alpha_{30}h_0^{1-d/6}\Big(c_* + \alpha_{50}\|(u,p)\|_{C(H^2\times H^1)} + \alpha_{21}\,|u|_{C(H^2)}\Big) + \alpha_{30}c_*c_0h_0^{d/12} \le 1, \tag{5.17a}$$

$$c_0\Big\{\alpha_{31}h_0^{1-d/4}\Big(c_* + \alpha_{50}\|(u,p)\|_{C(H^2\times H^1)} + \alpha_{21}\,|u|_{C(H^2)}\Big) + \alpha_{20}h_0^{d/4}\,|u|_{C(W^{1,\infty})}\Big\} \le \frac{3}{16\alpha_{20}}. \tag{5.17b}$$

We also replace P(n)-(a) by

$$\begin{aligned}&\nu\,|e_h^n|_1^2 + \big\|\overline{D}_{\Delta t}e_h\big\|_{\ell^2(1,n;L^2)}^2 \le \exp(\beta_{21}^*n\Delta t)\beta_{22}^*\Big\{\Delta t^2\,\|u\|_{Z^2(t^0,t^n)}^2\\&+ h^2\left(\|(u,p)\|_{H^1(t^0,t^n;H^2\times H^1)}^2 + \|(u,p)\|_{\ell^2(0,n-1;H^2\times H^1)}^2 + |u|_{\ell^2(0,n-1;H^2)}^2\right)\\&+\nu\,\big|e_h^0\big|_1^2\Big\}.\end{aligned}$$

P(n)-(a) implies the estimate

$$\|e_h^n\|_1 \le c_*(h+\Delta t). \tag{5.18}$$

The choice (5.17) is sufficient to derive P(n)-(b) and (c). Hence, the existence of the solution and the estimate (4.1) are obtained similarly.

We now prove the estimate (4.2), following [24] except for the introduction of $X_1(\Pi_h^{(1)}u_h^{n-1})$. Substituting $(v_h, q_h) = (e_h^n, \varepsilon_h^n)$ in (5.3), we have

$$\frac{1}{2}\overline{D}_{\Delta t}\,\|e_h^n\|_0^2 + \frac{1}{2\Delta t}\,\big\|e_h^n - e_h^{n-1}\big\|_0^2 + \nu\,|e_h^n|_1^2 = \sum_{i=1}^4 (R_i^n, e_h^n), \tag{5.19}$$

where R_i, $i = 1,\cdots,4$, are defined in (5.4). The term (R_1^n, e_h^n) is evaluated by (5.6a). From Lemma 5.6-(ii) we have

$$\big\|\widehat{u}_h^{n-1} - u^{n-1}\big\|_0 \le \alpha_{51}h^2\|(u,p)^{n-1}\|_{H^2\times H^1}.$$

Using this estimate in the last line in (5.7), we have

$$(R_2^n, e_h^n) \le \|R_2^n\|_0\,\|e_h^n\|_0 \le \{\beta_2\,\big\|e_h^{n-1}\big\|_0 + \beta_3' h^2(\|(u,p)^{n-1}\|_{H^2\times H^1} + \big|u^{n-1}\big|_2)\}\,\|e_h^n\|_0\,.$$

We divide the term (R_3^n, e_h^n) as follows:

$$\begin{aligned}(R_3^n, e_h^n) =& \frac{1}{\Delta t}(\eta^n - \eta^{n-1}, e_h^n) + \frac{1}{\Delta t}(\eta^{n-1} - \eta^{n-1}\circ X_1(\Pi_h^{(1)}u^{n-1}), e_h^n)\\&+\frac{1}{\Delta t}(\eta^{n-1}\circ X_1(\Pi_h^{(1)}u^{n-1}) - \eta^{n-1}\circ X_1(\Pi_h^{(1)}u_h^{n-1}), e_h^n)\\&\equiv I_1 + I_2 + I_3.\end{aligned}$$

The first term I_1 is evaluated as

$$I_1 \le \frac{1}{\sqrt{\Delta t}}\left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)}\|e_h^n\|_0 \le \frac{\alpha_{51}h^2}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}\,\|e_h^n\|_0\,.$$

By Lemma 5.9 the second term I_2 is evaluated as

$$\begin{aligned}I_2 \le&\alpha_6\alpha_{20}\,\|u\|_{C(W^{1,\infty})}\,\big\|\eta^{n-1}\big\|_0\,\|e_h^n\|_1\\ \le&\alpha_6\alpha_{20}\,\|u\|_{C(W^{1,\infty})}\,\alpha_{51}h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\,\|e_h^n\|_1\,.\end{aligned}$$

In order to evaluate I_3 we prepare the estimate

$$\alpha_{30} h^{-d/6} \left|\eta^{n-1}\right|_1 \le \alpha_{30} h^{1-d/6} \alpha_{50} \|(u,p)^{n-1}\|_{H^2\times H^1} \le 1,$$

where we have used Lemma 5.6-(i) and (5.17a).

Using Lemma 5.8 with $q=1$, $p=p'=2$, $w_1=\Pi_h^{(1)}u^{n-1}$, $w_2=\Pi_h^{(1)}u_h^{n-1}$ and $\psi=\eta^{n-1}$, Lemma 5.4, the above estimate and (5.7), we can evaluate I_3 as follows:

$$\begin{aligned} I_3 \le& \frac{1}{\Delta t}\|\eta^{n-1}\circ X_1(\Pi_h^{(1)}u^{n-1}) - \eta^{n-1}\circ X_1(\Pi_h^{(1)}u_h^{n-1})\|_{0,1}\, \|e_h^n\|_{0,\infty} \\ \le& \sqrt{2}\alpha_{30} h^{-d/6} \left|\eta^{n-1}\right|_1 \|\Pi_h^{(1)}u^{n-1} - \Pi_h^{(1)}u_h^{n-1}\|_0 \|e_h^n\|_1 \\ \le& \sqrt{2}\|\Pi_h^{(1)}u^{n-1} - \Pi_h^{(1)}u_h^{n-1}\|_0 \|e_h^n\|_1 \\ \le& \sqrt{2}\alpha_{22}(\|e_h^{n-1}\|_0 + \alpha_{51}h^2\|(u,p)^{n-1}\|_{H^2\times H^1} + \alpha_{21}h^2\left|u^{n-1}\right|_2)\,\|e_h^n\|_1 . \end{aligned}$$

In order to evaluate (R_4^n, e_h^n) we prepare the estimate

$$\alpha_{30} h^{-d/6} \left|e_h^{n-1}\right|_1 \le \alpha_{30} h^{-d/6} c_*(h+\Delta t) \le \alpha_{30} c_* h^{-d/6}(h + c_0 h^{d/4}) \le 1,$$

where we have used (5.18) and (5.17a). Using Lemma 5.9, the above inequality and a similar estimate to I_3 in (R_3^n, e_h^n), we can evaluate (R_4^n, e_h^n) as follows:

$$\begin{aligned} (R_4^n, e_h^n) =& -\frac{1}{\Delta t}(e_h^{n-1} - e_h^{n-1}\circ X_1(\Pi_h^{(1)}u^{n-1}), e_h^n) \\ &- \frac{1}{\Delta t}(e_h^{n-1}\circ X_1(\Pi_h^{(1)}u^{n-1}) - e_h^{n-1}\circ X_1(\Pi_h^{(1)}u_h^{n-1}), e_h^n) \\ \le& \alpha_6\alpha_{20}\, \|u\|_{C(W^{1,\infty})}\, \|e_h^{n-1}\|_0\, \|e_h^n\|_1 \\ &+ \sqrt{2}\alpha_{22}(\|e_h^{n-1}\|_0 + \alpha_{51}h^2\|(u,p)^{n-1}\|_{H^2\times H^1} + \alpha_{21}h^2\left|u^{n-1}\right|_2)\,\|e_h^n\|_1 . \end{aligned}$$

Combining (5.19) with these estimates and using Young's inequality and Poincaré's inequality $\|e_h^n\|_0 \le \alpha_1 |e_h^n|_1$, we have,

$$\begin{aligned} &\frac{1}{2}\overline{D}_{\Delta t}\, \|e_h^n\|_0^2 + \nu\, |e_h^n|_1^2 \le \beta_{31}\Big(\left\|e_h^{n-1}\right\|_0 + \sqrt{\Delta t}\, \|u\|_{Z^2(t^{n-1},t^n)} \\ &+ \frac{h^2}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + h^2\|(u,p)^{n-1}\|_{H^2\times H^1} + h^2\left|u^{n-1}\right|_2\Big)\, \|e_h^n\|_1 \\ \le& \nu\, |e_h^n|_1^2 + \beta_{32}\left\|e_h^{n-1}\right\|_0^2 + \beta_{33}\Big\{\Delta t\, \|u\|_{Z^2(t^{n-1},t^n)}^2 + \frac{h^4}{\Delta t}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} \\ &+ h^4\left(\|(u,p)^{n-1}\|_{H^2\times H^1}^2 + \left|u^{n-1}\right|_2^2\right)\Big\}, \end{aligned}$$

where β_{31}, β_{32} and β_{33} are positive constants independent of h and Δt. Applying discrete Gronwall's inequality, we obtain (4.2). □

6. Numerical results

We show numerical results in $d=2$ for the P_2/P_1-element. We compare the present Scheme LG-LLV with the conventional Scheme LG′.

In order to divide an element K_0 in Lemma 3.3 into the union of sub-triangles we use the algorithm by Priestley [22] and Jack [13], where the edges and vertices are traced to identify the polygon E_1 and it is divided into the union of sub-triangles. For the quadrature in Scheme LG′ we employ the seven-point formula of degree five [11] and the 21-point formula of degree nine [14]. We denote the schemes with these formulae by LG′(5) and LG′(9).

TABLE 1. Symbols used in Figures 2, 3 and 4, and Tables 2 and 3.

ϕ	u	p	u	u
X	$\ell^\infty(H_0^1)$	$\ell^2(L^2)$	$\ell^\infty(L^2)$	$\ell^\infty(L^2)$
Δt	h^2	h^2	h^2	h^3
Scheme LG′	●	■	▲	▼
Scheme LG-LLV	○	□	△	▽

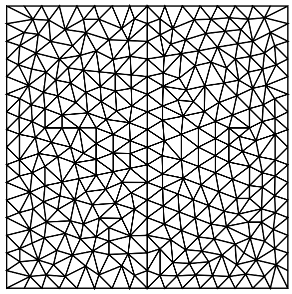

FIGURE 1. The triangulation of $\bar{\Omega}$ for $N = 16$.

In order to find an element including a given point we use the efficient local search algorithm by Löhner and Ambrosiano [15]. For the triangulation of the domain the FreeFem++ [12] is used.

The relative error E_X is defined by

$$E_X(\phi) \equiv \frac{\|\Pi_h\phi - \phi_h\|_X}{\|\Pi_h\phi\|_X},$$

for $\phi = u$ in $X = \ell^\infty(H_0^1)$ and $\ell^\infty(L^2)$, and for $\phi = p$ in $X = \ell^2(L^2)$. Table 1 shows the symbols used in the graphs and tables. Since every graph of the relative error E_X versus h is depicted in the logarithmic scale, the slope corresponds to the convergence order.

Example 6.1. In (2.1), let $\Omega \equiv (0,1)^2$, $T = 1$. We consider the two cases, $\nu = 10^{-2}$ and 10^{-4}. The functions f and u^0 are defined so that the exact solution is

$$\begin{aligned} u_1(x,t) &= \phi(x_1, x_2, t), \\ u_2(x,t) &= -\phi(x_2, x_1, t), \\ p(x,t) &= \sin(\pi(x_1 + 2x_2) + 1 + t), \end{aligned}$$

where $\phi(a,b,t) \equiv -\sin(\pi a)^2 \sin(\pi b)\{\sin(\pi(a+t)) + 3\sin(\pi(a+2b+t))\}$.

Let N be the division number of each side of Ω. We set $h \equiv 1/N$. Figure 1 shows the triangulation of $\bar{\Omega}$ for $N = 16$. The time increment Δt is set to be $\Delta t = h^2$ ($N = 16, 23, 32, 45$ and 64) or $\Delta t = h^3$ ($N = 16, 19, 23, 27$ and 32) so that we can observe the convergence behavior of order h^2 or h^3. The purpose of the choice $\Delta t = O(h^2)$ or $O(h^3)$ is to examine the theoretical convergence order, but it is not based on the stability condition, which is much weaker as shown in Theorem 4.2.

At first we consider the case of $\nu = 10^{-2}$. We use the quadrature formula of degree five in Scheme LG′. Figure 2 shows the graphs of $E_{\ell^\infty(H_0^1)}(u)$, $E_{\ell^2(L^2)}(p)$ and $E_{\ell^\infty(L^2)}(u)$ versus h. Their values and convergence orders are listed in Table 2.

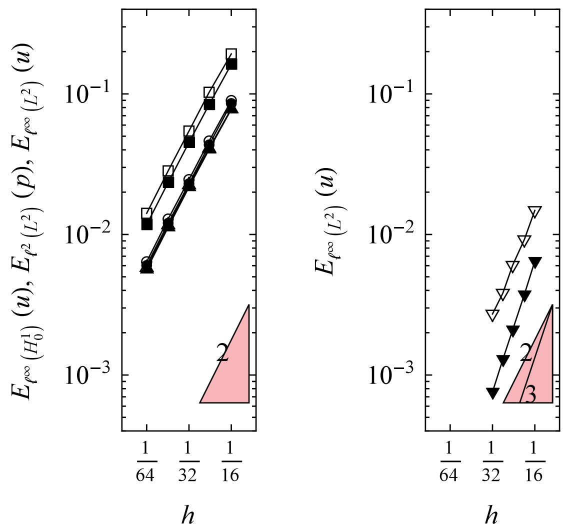

FIGURE 2. Relative errors $E_{\ell^\infty(H_0^1)}(u)$, $E_{\ell^2(L^2)}(p)$, $E_{\ell^\infty(L^2)}(u)$ with $\Delta t = h^2$ (left) and $E_{\ell^\infty(L^2)}(u)$ with $\Delta t = h^3$ (right) of Scheme LG′(5) and Scheme LG-LLV in the case of $\nu = 10^{-2}$ in Example 6.1.

TABLE 2. The values of relative errors and orders in Figure 2 by Scheme LG′(5) (top) and Scheme LG-LLV (bottom).

N	●	order	■	order	▲	order
16	8.55e-2		1.63e-1		7.77e-2	
23	4.34e-2	1.87	8.40e-2	1.82	4.03e-2	1.81
32	2.30e-2	1.93	4.52e-2	1.88	2.17e-2	1.87
45	1.20e-2	1.90	2.34e-2	1.92	1.13e-2	1.93
64	6.02e-3	1.97	1.18e-2	1.96	5.64e-3	1.96

N	▼	order
16	6.45e-3	
19	3.73e-3	3.19
23	2.10e-3	3.02
27	1.29e-3	3.02
32	7.57e-4	3.15

N	○	order	□	order	△	order
16	8.97e-2		1.93e-1		7.84e-2	
23	4.62e-2	1.83	1.03e-1	1.73	4.10e-2	1.78
32	2.46e-2	1.92	5.44e-2	1.92	2.25e-2	1.82
45	1.29e-2	1.90	2.84e-2	1.91	1.17e-2	1.93
64	6.39e-3	1.99	1.41e-2	1.97	5.81e-3	1.98

N	▽	order
16	1.48e-2	
19	9.19e-3	2.78
23	6.04e-3	2.19
27	3.83e-3	2.85
32	2.72e-3	2.01

When $\Delta t = h^2$, the convergence orders of $E_{\ell^\infty(H_0^1)}(u)$ (●, ○), $E_{\ell^2(L^2)}(p)$ (■, □) and $E_{\ell^\infty(L^2)}(u)$ (▲, △) are almost 2 in both schemes. When $\Delta t = h^3$, the order of $E_{\ell^\infty(L^2)}(u)$ is almost 3 in Scheme LG′(5) (▼) and 2 in Scheme LG-LLV (▽). They reflect the theoretical results.

Next, we consider a higher-Reynolds-number case of $\nu = 10^{-4}$. We use two quadrature formulae of degree five and nine in Scheme LG′. Figure 3 shows the graphs by LG′(5) and LG-LLV and Figure 4 shows the graphs by LG′(9) and

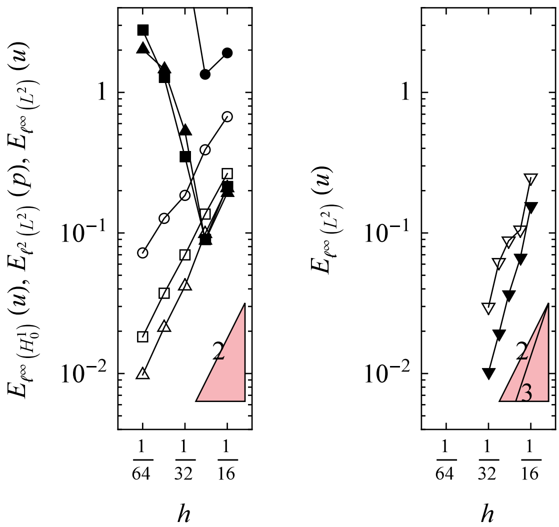

FIGURE 3. Relative errors $E_{\ell^\infty(H_0^1)}(u)$, $E_{\ell^2(L^2)}(p)$, $E_{\ell^\infty(L^2)}(u)$ with $\Delta t = h^2$ (left) and $E_{\ell^\infty(L^2)}(u)$ with $\Delta t = h^3$ (right) of Scheme LG$'$(5) and Scheme LG-LLV in the case of $\nu = 10^{-4}$ in Example 6.1.

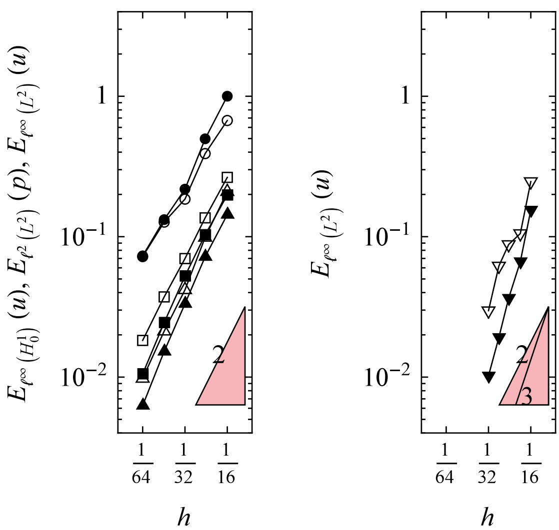

FIGURE 4. Relative errors $E_{\ell^\infty(H_0^1)}(u)$, $E_{\ell^2(L^2)}(p)$, $E_{\ell^\infty(L^2)}(u)$ with $\Delta t = h^2$ (left) and $E_{\ell^\infty(L^2)}(u)$ with $\Delta t = h^3$ (right) of Scheme LG$'$(9) and Scheme LG-LLV in the case of $\nu = 10^{-4}$ in Example 6.1.

LG-LLV. Their values are listed in Table 3. When $\Delta t = h^2$, all errors increase abnormally at $N = 32, 45$ and 64 in Scheme LG$'$(5) (●, ■, ▲) in Figure 3 while the convergence is observed in LG$'$(9) (●, ■, ▲) in Figure 4 and Scheme LG-LLV (○, □, △), but the orders of $E_{\ell^\infty(H_0^1)}(u)$ (●, ○) are less than 2. In order to obtain

TABLE 3. The values of relative errors and orders in Figures 3 and 4 by Scheme LG′(5) (top), Scheme LG′(9) (middle) and Scheme LG-LLV (bottom).

N	●	order	■	order	▲	order
16	1.91e+0		2.14e-1		1.93e-1	
23	1.34e+0	0.97	8.97e-2	2.39	8.81e-2	2.16
32	9.42e+0	-5.90	3.48e-1	-4.11	5.28e-1	-5.43
45	4.10e+1	-4.31	1.28e+0	-3.81	1.46e+0	-2.98
64	8.82e+1	-2.18	2.77e+0	-2.20	2.02e+0	-0.93

N	▼	order
16	1.55e-1	
19	6.64e-2	4.92
23	3.65e-2	3.14
27	1.92e-2	4.01
32	1.02e-2	3.71

N	●	order	■	order	▲	order
16	1.00e+0		1.98e-1		1.44e-1	
23	4.98e-1	1.93	1.03e-1	1.80	7.19e-2	1.91
32	2.18e-1	2.50	5.24e-2	2.05	3.33e-2	2.33
45	1.32e-1	1.47	2.44e-2	2.24	1.52e-2	2.29
64	7.31e-2	1.69	1.05e-2	2.39	6.27e-3	2.52

N	▼	order
16	1.55e-1	
19	6.64e-2	4.92
23	3.65e-2	3.14
27	1.92e-2	4.01
32	1.02e-2	3.71

N	○	order	□	order	△	order
16	6.72e-1		2.65e-1		2.09e-1	
23	3.91e-1	1.50	1.36e-1	1.83	9.88e-2	2.07
32	1.85e-1	2.26	6.98e-2	2.02	4.18e-2	2.60
45	1.27e-1	1.10	3.73e-2	1.84	2.12e-2	1.99
64	7.21e-2	1.61	1.83e-2	2.03	9.78e-3	2.20

N	▽	order
16	2.47e-1	
19	1.05e-1	4.96
23	8.80e-2	0.94
27	6.18e-2	2.20
32	2.97e-2	3.29

the theoretical convergence order $O(h^2)$ in Scheme LG-LLV, it seems that finer meshes will be necessary. When $\Delta t = h^3$, the orders of $E_{\ell^\infty(L^2)}(u)$ are more than 3 in Schemes LG′(5) and LG′(9) (▼) while it is less than 3 between $N = 19$ and 23, and $N = 23$ and 27 in Scheme LG-LLV (▽), cf. Remark 4.6. We can observe that convergence is recovered in Scheme LG′(5) for much smaller time increments $\Delta t = h^3$. The reason is explained in Appendix B. In Table 3 (▼) the differences of the results of LG′(5) and LG′(9) are too small to be distinguished in three significant figures.

The above numerical results show that, for the stable computation by Scheme LG′, quadrature formula of degree five is sufficient in the case of $\nu = 10^{-2}$ but not so in a higher-Reynolds-number case of $\nu = 10^{-4}$. Quadrature formula of degree nine gives the stable computation for $\nu = 10^{-4}$. Scheme LG-LLV is always stable, as proved in Theorem 4.2.

We now consider a cavity problem to see that Scheme LG-LLV is robust for high Reynolds numbers while Scheme LG′ is not. This problem is subject to an inhomogeneous Dirichlet boundary condition, but we can solve it by Scheme LG′ and Scheme LG-LLV. In order to assure the existence of the solution we deal with a regularized cavity problem, where the prescribed velocity is continuous on the boundary.

Example 6.2. Let $\Omega \equiv (0,1)^2$, $f = 0$, $u^0 = 0$. We consider the two cases, $\nu = 10^{-4}$ and 10^{-5}. The boundary condition is described in Figure 5 (left), where $g_1 = 4x_1(1 - x_1)$.

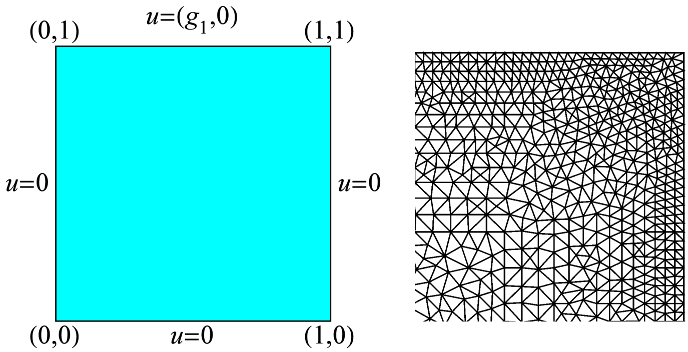

FIGURE 5. The domain Ω and the boundary condition (left) and the local mesh in $[0.7, 1] \times [0.7, 1]$ (right) in Example 6.2.

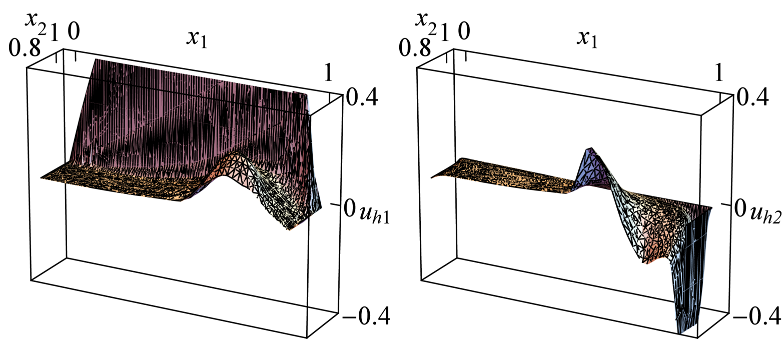

FIGURE 6. Stereographs of u_{h1}^n (left) and u_{h2}^n (right) at $t^n = 8$ by Scheme LG′(5) in Example 6.2 when $\nu = 10^{-4}$.

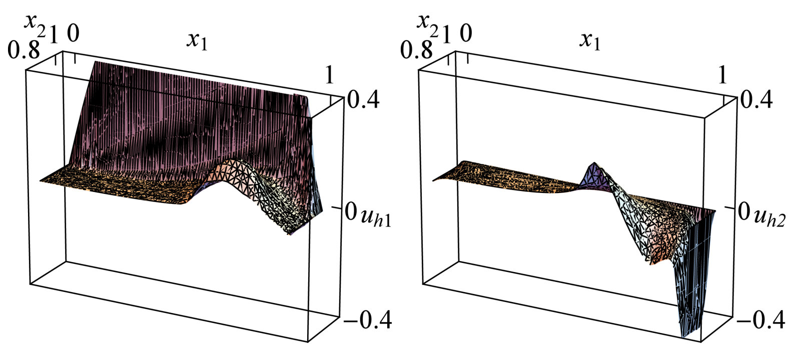

FIGURE 7. Stereographs of u_{h1}^n (left) and u_{h2}^n (right) at $t^n = 8$ by Scheme LG-LLV in Example 6.2 when $\nu = 10^{-4}$.

Figure 5 (right) shows the local mesh in $[0.7, 1] \times [0.7, 1]$ of the triangulation of $\bar{\Omega}$, where each side is divided into 100 segments and smaller triangles are employed near the boundary. The total element number is 9,774. We take $\Delta t = 0.01$. We use the formulae of degree five and nine in Scheme LG′. Figures 6 and 7 show the stereographs of the solution u_h^n at $t^n = 8$ in the subdomain $[0, 1] \times [0.8, 1.0]$

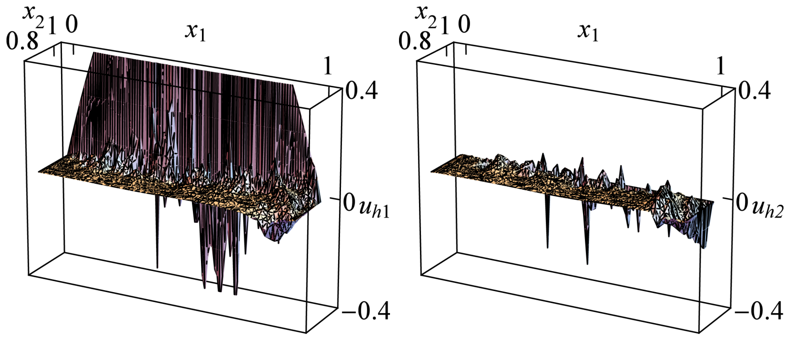

FIGURE 8. Stereographs of u_{h1}^n (left) and u_{h2}^n (right) at $t^n = 8$ by Scheme LG$'$(9) in Example 6.2 when $\nu = 10^{-5}$.

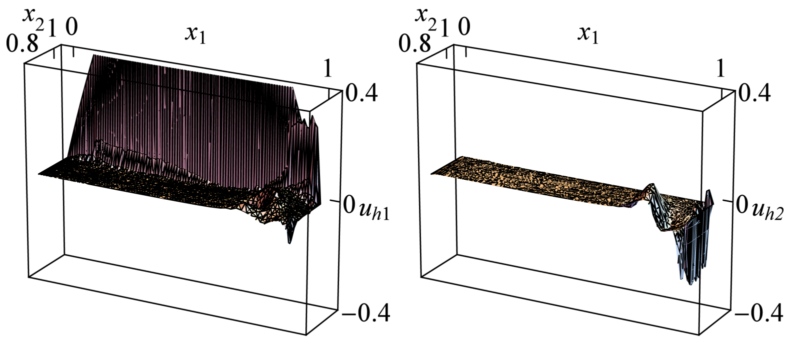

FIGURE 9. Stereographs of u_{h1}^n (left) and u_{h2}^n (right) at $t^n = 8$ by Scheme LG-LLV in Example 6.2 when $\nu = 10^{-5}$.

by Scheme LG$'$(5) and Scheme LG-LLV, respectively, when $\nu = 10^{-4}$. They are depicted in the range $[-0.4, 0.4]$. Neither solution is oscillating though u_{h2} by Scheme LG$'$(5) takes larger values than that by Scheme LG-LLV as seen clearly around $x_1 = 0.6$. Figures 8 and 9 show the stereographs of the solution u_h^n at $t^n = 8$ in the subdomain $[0, 1] \times [0.8, 1.0]$ by Scheme LG$'$(9) and Scheme LG-LLV, respectively, when $\nu = 10^{-5}$. They are depicted in the range $[-0.4, 0.4]$. Oscillation is clearly observed for both components of the solution by Scheme LG$'$(9) in Figure 8. Oscillation is also observed when the formula of degree five is used though the figure is not shown here. This shows that the degree of the formula is not enough. On the other hand, we can see that the solution by Scheme LG-LLV is solved well only with few oscillation at the boundary layer near the corner $(1, 1)$ in Figure 9. In order to eliminate the oscillation completely smaller elements will be required near the corner. Thus, Scheme LG-LLV is more robust than Scheme LG$'$(9) for high Reynolds numbers. Table 4 shows the CPU times in computing the composite function term until $t^n = 2$ when $\nu = 10^{-4}$. Scheme LG-LLV requires about 2.6 times longer CPU time than LG$'$(9). The computations were carried out on a machine with Intel Xeon Processor E3-1245 (3.30GHz) and Windows 7. The code is compiled by Microsoft Visual C++ 2010 with O2 option.

Table 4. CPU times (s) until $t^n = 2$ when $\nu = 10^{-4}$ in Example 6.2.

LG′(5)	LG′(9)	LG-LLV
1.387	3.672	9.448

7. Conclusions

We have presented a Lagrange–Galerkin scheme free from numerical quadrature for the Navier–Stokes equations. The scheme has a close relation with Priestley's as described in Remark 3.4. By virtue of the introduction of a locally linearized velocity, the scheme is exactly computable and the theoretical stability and convergence results are assured for numerical solutions. We have shown optimal error estimates in $\ell^\infty(H^1) \times \ell^2(L^2)$-norm for the velocity and pressure in the case of $\mathrm{P}_2/\mathrm{P}_1$- and $\mathrm{P}_1+/\mathrm{P}_1$-finite elements. Numerical results have reflected these estimates and the robustness of the scheme for high-Reynolds-number problems. The scheme presented here is of first-order in time. The extension to second-order in time can be done as in [6, 19]. The extension of the subdivision algorithm of the triangle by Priestley [22] and Jack [13] to the tetrahedron is not straightforward. Numerical results in 3D will be reported in the future.

Appendix A. Proof of Lemma 5.9

Proof. We simply denote $X_1(w)$ by F. From the definition of the norm $\|\cdot\|_{-1}$, we have

(A.1)
$$\|\psi - \psi \circ F\|_{-1} = \sup_{v \in H_0^1(\Omega)^d} \|v\|_1^{-1} \int_\Omega \{\psi(x) - \psi(F(x))\} \cdot v(x) dx \equiv \sup_{v \in H_0^1(\Omega)^d} \|v\|_1^{-1} I_1.$$

The change of variables from x to $y = F(x)$ yields

$$\begin{aligned} I_1 &= \int_\Omega \psi(x) \cdot v(x) dx - \int_\Omega \psi(y) \cdot v(F^{-1}(y)) \left|\det\left(\frac{\partial x}{\partial y}\right)\right| dy \\ &\le \|\psi\|_0 \left\| v - v \circ F^{-1} \left|\det\left(\frac{\partial x}{\partial y}\right)\right| \right\|_0 \equiv \|\psi\|_0 I_2, \end{aligned} \tag{A.2}$$

where $\det(\frac{\partial x}{\partial y})$ is the Jacobian. We evaluate I_2 by dividing

$$I_2 \le \left\| v - v \circ F^{-1} \right\|_0 + \left\| v \circ F^{-1} \left(1 - \left|\det\left(\frac{\partial x}{\partial y}\right)\right|\right) \right\|_0 \equiv I_{21} + I_{22}. \tag{A.3}$$

Using the estimates of the Jacobian (Lemma 5.7) and the change of variable from x to $z(x, s) \equiv sF(x) + (1 - s)x = x - sw(x)\Delta t$, we have

$$\begin{aligned} I_{21}^2 &= \int_\Omega |v(y) - v(F^{-1}(y))|^2 dy \le \alpha_{61} \int_\Omega |v(F(x)) - v(x)|^2 dx \\ &\le \alpha_{61} \Delta t^2 \|w\|_{0,\infty}^2 \int_\Omega \int_0^1 |\nabla v(z(x, s))|^2 ds dx \le \alpha_{62} \Delta t^2 \|w\|_{0,\infty}^2 \|\nabla v\|_0^2. \end{aligned} \tag{A.4}$$

From a similar estimate to the proof of Lemma 5.7, we get

$$\left| 1 - \left|\det\left(\frac{\partial x}{\partial y}\right)\right| \right| \le \left| 1 - \det\left(\frac{\partial x}{\partial y}\right) \right| = \left|\det\left(\frac{\partial x}{\partial y}\right)\right| \left|\det\left(\frac{\partial y}{\partial x}\right) - 1\right| \le \alpha_{63} \Delta t \, |w|_{1,\infty},$$

which implies

$$I_{22} \le \left\| v \circ F^{-1} \right\|_0 \left\| 1 - \left| \det \left(\frac{\partial x}{\partial y} \right) \right| \right\|_{0,\infty} \le \alpha_{64} \Delta t \, |w|_{1,\infty} \, \|v\|_0 \, . \tag{A.5}$$

Combining the estimates (A.1)–(A.5), we have the conclusion. □

Appendix B. A scheme equivalent to Scheme LG′ for small time increments

First we recall that the P_2-element is employed for the velocity. In the numerical quadrature formulae used in Section 6 all quadrature points a_j are in the interior of the element. If Δt is sufficiently small, the image $[X_1(u_h^{n-1})](a_j)$ and a_j are in the same element. In such a case the first equation of Scheme LG′ can be rewritten as

$$\begin{aligned} &\big(\bar{D}_{\Delta t} u_h^n, v_h\big) + \big((u_h^{n-1} \cdot \nabla) u_h^{n-1}, v_h\big) + a(u_h^n, v_h) + b(v_h, p_h^n) \\ &- \frac{\Delta t}{2} \sum_{K \in \mathcal{T}_h} I_h \left[\left\{ (u_h^{n-1})^T (\nabla^2 u_h^{n-1}) u_h^{n-1} \right\} \cdot v_h ; K \right] = (f^n, v_h), \qquad \forall v_h \in V_h, \end{aligned} \tag{B.1}$$

where $\nabla^2 u_h^{n-1}$ is the Hesse matrix. Here, we have used the Taylor expansion and the condition that the quadrature is of order $m \ge 5$. Hence Scheme LG′ is equivalent to a scheme consisting of (B.1) and the second equation of Scheme LG′. For this scheme we can show the stability and convergence. For the details we refer to a forthcoming paper.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments, which helped to improve the quality of the paper. The first author was supported by JSPS (Japan Society for the Promotion of Science) under Grants-in-Aid for Scientific Research (C), No. 25400212 and (S), No. 24224004 and under the Japanese-German Graduate Externship (Mathematical Fluid Dynamics) and by Waseda University under Project research, Spectral analysis and its application to the stability theory of the Navier-Stokes equations of Research Institute for Science and Engineering. The second author was supported by JSPS under Grant-in-Aid for JSPS Fellows, No. 26·964.

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