MATHEMATICS OF COMPUTATION
Volume 87, Number 309, January 2018, Pages 289–308
http://dx.doi.org/10.1090/mcom/3267
Article electronically published on May 31, 2017

SHARP BOUNDS FOR THE MODULUS AND PHASE OF HANKEL FUNCTIONS WITH APPLICATIONS TO JAEGER INTEGRALS

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Abstract. We prove upper and lower bounds for a class of Jaeger integrals $\mathcal{G}_\nu(\tau)$ appearing in axisymmetric diffusive transport related to several physical applications. In particular, we show that these integrals are globaly bounded either from above or from below by the first terms in their corresponding asymptotic expansions in τ, both at zero and infinity. In the case of $\mathcal{G}_0(\tau)$ we show that it is bounded from below by the Ramanujan integral.

These bounds are obtained as a consequence of sharp bounds derived for the modulus and phase of Hankel functions, and for the Ramanujan integral, which we believe to be new and of independent interest, complementing the asymptotic and numerical results in the literature.

1. Introduction

The family of integrals defined by

$$\mathcal{J}(p,q;\tau) = \int_0^\infty \frac{e^{-\tau u^2}}{\left[pJ_1(u)+qJ_0(u)\right]^2+\left[pY_1(u)+qY_0(u)\right]^2}\,\frac{\mathrm{d}u}{u}, \tag{1.1}$$

for positive τ and where J_ν and Y_ν are the Bessel functions of the first and second kind, respectively, was derived in [J] in connection with the flow of heat in domains with cylindrical symmetry. These integrals have received much attention from the numerical analysis community since their introduction, beginning with the work of Jaeger and his collaborators; see [PM] and the references therein.

Recently, some work of a more theoretical nature has been devoted to integrals of the above type and their extensions to general indexes, such as the family

$$\mathcal{G}_\nu(\tau) = \int_0^\infty \frac{e^{-\tau u^2}}{uM_\nu^2(u)}\,\mathrm{d}u \quad (\nu \geq 0) \tag{1.2}$$

studied in [BPPR]. Here M_ν is the modulus of the Hankel function

$$H_\nu^{(1)}(u) = J_\nu(u) + iY_\nu(u) \quad (\nu \in \mathbb{R}),$$

that is,

$$M_\nu(u) = \left[J_\nu^2(u)+Y_\nu^2(u)\right]^{1/2},$$

Received by the editor May 10, 2016 and, in revised form, September 18, 2016.
2010 *Mathematics Subject Classification.* Primary 33C10, 26D15; Secondary 33E20, 35C15.
Key words and phrases. Jaeger integrals, Hankel functions, Ramanujan integral, asymptotic expansions, sharp bounds.
The author was partially supported by FCT (Portugal) through project PTDC/MAT-CAL/4334/2014.

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corresponding to the functions appearing in $\mathcal{J}(1,0;\tau)$ and $\mathcal{J}(0,1;\tau)$ when ν equals 1 or 0, respectively.

The present work, although falling into the latter category, was inspired by Figures 1 and 3 in [PM], where it is seen that the integrals $\mathcal{J}(1,0;\tau)$ and $\mathcal{J}(0,1;\tau)$ ($\mathcal{G}_1$ and $\mathcal{G}_0$, respectively, in our notation) seem to be bounded by the dominant terms in the corresponding asymptotic expansions as τ approaches 0 and $+\infty$. This, together with the fact that a similar behaviour is known for the functions M_ν defined above for large values of the argument ([OM, p. 231] and [Wa]), has prompted us to study the family of integrals (1.2) from this perspective. In particular, and while the majority of the results in the literature address either the asymptotic behaviour of such functions or their numerical approximation [B, BOS, DNT, J, JC, N, L], here we will be mostly concerned with bounding the functions $\mathcal{G}_\nu$ by terms in their asymptotic expansions. These bounds are thus sharp as τ approaches either zero or infinity, complementing both the asymptotic and numerical results given in the literature.

While the existing bounds for M_ν^2 already yield bounds for $\mathcal{G}_\nu(\tau)$, which are sharp for small τ, in order to obtain sharp bounds for large τ we need to obtain first bounds for M_ν^2 (or for the phase function) which are sharp for small argument. To the best of our knowledge the bounds given here are new and, in one case, quite striking, in that they are actually asymptotically sharp both for small and large arguments; see Theorem A.

In some studies such as those in [N, L], the functions J_0 and Y_0 in $\mathcal{G}_0$ are approximated by their expansions for small argument, namely, 1 and $2[\gamma+\log(x/2)]/\pi$, respectively. The resulting integral is known in the literature as Ramanujan's integral [Hd, L], and we shall see that it yields a lower bound to the original integral. Furthermore, this is quite accurate for large values of the argument, as at least the first three terms in the corresponding expansions coincide.

There are some points which should be made at this stage. First it is important to note that these bounds are global and do not hold only for sufficiently small or large values of the argument, although of course they are not necessarily very accurate far away from these limiting values. Second, they are not a direct consequence of an alternating series-type result and, at least in some cases, adding higher-order terms does not necessarily produce further global bounds; see Remark 4 following Corollary B. Finally, the existing bounds for the functions M_ν mentioned above, being for large values of the argument, yield bounds for $\mathcal{G}_\nu$ which are accurate only for small τ, while the most interesting case corresponds precisely to large τ, which requires the new bounds developed here. Note also that in order to be able to replace directly M_ν^2 by a bound in (1.2), it is actually necessary that its asymptotic behaviour at zero is appropriate in order to make the singularity of the integrand integrable; see Remark 2 after Theorem A.

The structure of the paper is as follows. In the next section we gather some basic facts pertaining to Jaeger integrals of the form given by (1.2) and to the modulus and phase of the Hankel function $H_\nu^{(1)}$. This is followed by a section where we state the main results of the paper. The proofs of the results concerning the Hankel functions and the Ramanujan integral may be found in Sections 4 and 5, respectively, while those concerning Jaeger integrals appear in Section 6.

Throughout Section 5 we shall make use of some inequalities that we were not able to find in the literature relating the function $1/\Gamma(1-x)$ to the first terms in its Taylor expansions around 0 and 1 on certain intervals. For completeness, and also because this is largely in the spirit of the paper, we provide the corresponding proofs in Appendix A.

2. Preliminaries and basic facts about $\mathcal{G}_\nu$

We shall first collect some properties of the functions $\mathcal{G}_\nu$ which will be used throughout the paper. The first of these is the monotonicity in both its argument and in the parameter ν, which are a direct consequence of the monotonicity for M_ν. This, in turn, follows easily from Nicholson's integral formula for M_ν^2 (see [Wa, pp. 441ff] or [Wi]). More precisely we have

$$M_\nu^2(x) = \frac{8}{\pi^2}\int_0^{+\infty} K_0(2x\sinh(t))\cosh(2\nu t)\,\mathrm{d}t \tag{2.1}$$

and thus, due to the positivity of all the functions involved and the fact that K_0 and cosh are (strictly) decreasing and increasing in their arguments, respectively, we have that $M_\nu(x)$ is (strictly) decreasing in x and increasing in ν.

From the dependence of $\mathcal{G}_\nu$ on M_ν^2 we then have

$$\mathcal{G}_\nu(\tau) < \mathcal{G}_\mu(\tau), \;\; \forall_{\nu>\mu\geq 0} \text{ and } \forall_{\tau>0}.$$

Since the integrals $\mathcal{G}_\nu$ are known explicitly when ν equals $n-1/2$ for positive integer values of n (see [BPPR, p. 1261]), this immediately provides the simple lower and upper bounds

$$\mathcal{G}_{\lceil \nu+1/2\rceil-1/2}(\tau) \leq \mathcal{G}_\nu(\tau) \leq \mathcal{G}_{\lfloor \nu-1/2\rfloor+1/2}(\tau), \tag{2.2}$$

where, as usual, we have denoted by $\lceil x\rceil$ and $\lfloor x\rfloor$ the smallest integer not less than and the largest integer not greater than x, respectively. The left inequality holds for all non-negative values of ν, while that on the right holds for ν larger than or equal to $1/2$.

Another straightforward observation is that $\mathcal{G}_\nu$ may be related directly to the argument of the Hankel function $H_\nu^{(1)}$ which is defined by

$$\theta_\nu(x) = \arctan\left[\frac{Y_\nu}{J_\nu}\right].$$

Here θ_ν is the continuous real function with branch defined by

$$\lim_{x\to 0^+}\theta_\nu(x) = -\frac{\pi}{2}.$$

Using the identity [OM, 10.18.8]

$$\theta_\nu'(x) = \frac{2}{\pi x M_\nu^2(x)} \tag{2.3}$$

in the definition (1.2) yields

$$\mathcal{G}_\nu(\tau) \;=\; \frac{\pi}{2}\int_0^\infty e^{-\tau u^2}\theta_\nu'(u)\,\mathrm{d}u \;=\; \frac{\pi^2}{4} + \pi\tau\int_0^\infty u e^{-\tau u^2}\theta_\nu(u)\,\mathrm{d}u$$

upon integration by parts. In fact, by performing the change of variables $u=\sqrt{v}$ in the above integral we are led to

$$\mathcal{G}_\nu(\tau) = \frac{\pi^2}{4} + \frac{\pi\tau}{2}\int_0^\infty \theta_\nu\left(\sqrt{v}\right)e^{-\tau v}\,\mathrm{d}v, \tag{2.4}$$

and we thus see that the function

$$\frac{2}{\pi\tau}\left[\mathcal{G}_\nu(\tau)-\frac{\pi^2}{4}\right]$$

is the Laplace transform of $\theta_\nu\left(\sqrt{v}\right)$.

As we will see below, using bounds for θ_ν in the above relation will, in some cases, allow us to obtain estimates for $\mathcal{G}_\nu$ which are different from those obtainable by using known bounds for M_ν^2 directly.

3. Main results

Our main results are as follows. We first consider the modulus and phase of the Hankel function $H_\nu^{(1)}$, for which we obtain several bounds. These are then used to derive the corresponding estimates for the Jaeger integrals. For the integral $\mathcal{G}_0$ a lower bound is obtained by relating it to the Ramanujan integral, for which we also derive estimates. These allow us to bound $\mathcal{G}_0$ from below by the first three terms in its asymptotic expansion for large argument.

3.1. **Hankel functions.** In [Wa, p. 446] it is shown that $xM_\nu^2(x)$ is increasing (resp. decreasing) for $\nu<1/2$ (resp. $\nu>1/2$). The motivation for studying this is the fact that the first term in the asymptotic expansion of $M_\nu(x)$ for large x is $2/(\pi x)$. Here we consider the analogous result but now for small x. More precisely, we study the monotonicity of $x^{2\nu}M_\nu^2(x)$, from which it is then possible to obtain that the first term in the series expansion of the function M_ν^2 around zero does yield a bound for that function, providing a counterpart to the well-known bounds given by successive terms in the asymptotic expansion of M_ν^2 at infinity [OM, Wa].

Theorem A. *For all positive x,*

$$\frac{d}{dx}\left(x^{2\nu}M_\nu^2(x)\right)\quad\begin{array}{ll}<\ 0, & 0\leq\nu<1/2,\\[2ex] \geq\ 0, & \nu\geq 1/2.\end{array}$$

Furthermore,

$$\begin{array}{rclll} & & M_\nu^2(x) & \leq \dfrac{4^\nu\Gamma^2(\nu)}{\pi^2x^{2\nu}}, & 0<\nu\leq 1/2,\\[2ex] \dfrac{4^\nu\Gamma^2(\nu)}{\pi^2x^{2\nu}} & < & M_\nu^2(x) & \leq \dfrac{4^\nu\Gamma^2(\nu)}{\pi^2x^{2\nu}}+\dfrac{2}{\pi x}, & 1/2<\nu\leq 3/2,\\[2ex] \dfrac{4^\nu\Gamma^2(\nu)}{\pi^2x^{2\nu}}+\dfrac{2}{\pi x} & < & M_\nu^2(x), & & 3/2<\nu,\end{array}$$

with equality if and only if $\nu=1/2$ (first line) and $\nu=3/2$ (second line). For $\nu=0$ and positive x we have

$$M_0^2(x)<1+\frac{4}{\pi^2}\left(\gamma+\log(x/2)\right)^2,$$

where γ denotes Euler's constant.

The above bounds contain the first term in the series expansion of M_ν at zero, and are sharp (asymptotically) in that limit. Those containing the term $2/(\pi x)$ are also sharp as x goes to infinity.

Remark 1. Taking into consideration that $2/(\pi x) = M_{1/2}^2(x)$, it is tempting to consider the possibility that there is a sequence of improved bounds for M_ν^2 on intervals of the form $((2n-1)/2, (2n+1)/2)$ for integer n which would follow the above pattern. However, this will not be the case in general as may be seen from the fact that

$$\frac{16}{\pi^2 x^4} + M_{1/2}^2(x) < M_2^2(x) \not\leq \frac{16}{\pi^2 x^4} + M_{3/2}^2(x),$$

for instance.

Remark 2. There exists another type of bounds for M_ν^2, originally developed by Schafheitlin and which may be found in [Wa, p. 447] or [Ht, Section 4], for instance. One such example is given by

$$xM_\nu^2(x)\left[1-(\nu^2-1/4)x^{-2}\right] < \frac{2}{\pi} \ \ (\nu > 1/2).$$

However, this and similar bounds will not have the appropriate asymptotic behaviour for x close to zero to ensure convergence of the resulting integral, when replaced in (1.2).

We shall now state results concerning the phase of $H_\nu^{(1)}$. The first is a straightforward application of Theorem A above, yielding the following bounds for θ_ν.

Corollary B. *For all positive x,*

$$\theta_\nu(x) \quad \begin{array}{ll} \geq & -\frac{\pi}{2} + \dfrac{\pi}{\nu 4^\nu \Gamma^2(\nu)} x^{2\nu}, \quad 0 < \nu \leq 1/2, \\ \leq & -\frac{\pi}{2} + \dfrac{\pi}{\nu 4^\nu \Gamma^2(\nu)} x^{2\nu}, \quad 1/2 \leq \nu. \end{array}$$

These bounds correspond to the first two terms in the series expansion of θ at the origin and are thus asymptotically sharp as x approaches zero. When $\nu = 0$ we have

$$\theta_0(x) > \arctan\left[\frac{2}{\pi}\left(\gamma + \log\left(\frac{x}{2}\right)\right)\right].$$

Remark 3. These bounds correspond to the inequalities in Theorem A which do not include the term $2/(\pi x)$. Although it is also possible to use these, they will, in general, yield fairly complicated expressions upon integration of the expression obtained after replacing the bound for M_ν^2 in $2/(\pi x M_\nu^2(x))$. Some exceptions will be low-integers and half-integers for which the expressions obtainable are still fairly manageable. As an example, it is possible to derive

$$x - \frac{\pi}{2} + \frac{\log(4) - 2\log(2+\pi x)}{\pi} < \theta_1(x) \ \ (x > 0)$$

which, although not very good for large x, is asymptotically correct up to the second term around zero.

Remark 4. The bounds above are what may be obtained in full generality with respect to successive terms in the series expansion of θ_ν, as the case of $\nu = 1$ shows. In that instance

$$\theta_1(x) = -\frac{\pi}{2} + \frac{\pi x^2}{4} + \frac{1}{32}\left[4\pi\log(x) + 4\gamma\pi - 3\pi - 4\pi\log(2)\right]x^4 + \cdots$$

and while $\theta_1(x)$ will be smaller than the first two terms as indicated by the theorem, although adding the term in $\log(x)x^4$ may possibly improve the bound for small x, it definitely makes it worse for x larger than 1. Furthermore, if one adds the full term shown above, $\theta_1(x)$ will be larger than the resulting expression for small x, but again smaller for larger values.

The next result is a consequence of the more general but less explicit Theorems 4.1 and 4.2 in Section 4.5.

Theorem C. *For all positive x,*

$$
\begin{aligned}
&x - \frac{\pi}{2} \le \theta_\nu(x) \le x - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi, \quad 0 \le \nu \le 1/2,\\
&x - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi \le \theta_\nu(x) \le x - \left(\frac{\nu}{2} - \frac{1}{4}\right)\pi - \arctan(x), \quad 1/2 \le \nu \le 3/2,\\
&x - \left(\frac{\nu}{2} - \frac{1}{4}\right)\pi - \arctan(x) \le \theta_\nu(x) \le x - \frac{\pi}{2} - \arctan(x), \quad 3/2 \le \nu.
\end{aligned}
$$

Remark 5. All the above bounds are sharp asymptotically to the second (constant) term as x goes to $+\infty$, with the exception of the lower bound for ν in $[0, 1/2)$ and the upper bound for ν larger than $3/2$, which are only sharp in the first term.

3.2. **Jaeger integrals.** By taking advantage of the monotonicity properties of $\mathcal{G}_\nu(\tau)$ mentioned in Section 2 and the fact that $\mathcal{G}_{n+1/2}(\tau)$ may be computed explicitly for integer n (see [BPPR, p. 1261]), it is possible to derive a set of general bounds for $\mathcal{G}_\nu$.

Theorem D. *Let $n = \lfloor \nu + 1/2 \rfloor$. For all positive τ we have*

$$
\begin{aligned}
&\mathcal{G}_{1/2}(\tau) \le \mathcal{G}_\nu(\tau) \le \mathcal{G}_{1/2}(\tau) + \frac{\pi^2}{8}(1-2\nu), \quad 0 \le \nu < 1/2,\\
&\mathcal{G}_{n-1/2}(\tau) + \frac{\pi^2}{8}\left[2(n-\nu)-1\right] \le \mathcal{G}_\nu(\tau) \le \mathcal{G}_{n+1/2}(\tau) + \frac{\pi^2}{8}\left[2(n-\nu)+1\right], \quad 1/2 \le \nu.
\end{aligned}
$$

The lower bound above for $0 \le \nu < 1/2$ is already in [BPPR]. As an application of this result we obtain, for instance, the following bounds. For $0 \le \nu < 1/2$,

$$
\mathcal{G}_\nu(\tau) \le \frac{\pi^{3/2}}{4\sqrt{\tau}} + (1-2\nu)\frac{\pi^2}{8},
$$

while for $1/2 \le \nu < 3/2$ Theorem D yields

$$
\frac{\pi^{3/2}}{4\sqrt{\tau}} + (1-2\nu)\frac{\pi^2}{8} \le \mathcal{G}_\nu(\tau) \le \frac{\pi^{3/2}}{4\sqrt{\tau}} - \frac{\pi^2}{4}e^\tau \operatorname{erfc}(\sqrt{\tau}) + \frac{\pi^2}{8}(3-2\nu).
$$

Here erfc denotes the complementary error function defined by

$$
\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{+\infty} e^{-t^2}\,\mathrm{d}t.
$$

Both bounds are sharp, in the sense that their series expansions as τ approaches zero coincide with those of $\mathcal{G}_\nu(\tau)$ up to the second term and we thus see that in this range $\mathcal{G}_\nu$ is bounded from below by the first two terms in its expansion at zero.

Clearly for large values of τ these bounds become worse than those given by (2.2), which in this instance are

$$\frac{\pi^{3/2}}{4\sqrt{\tau}} - \frac{\pi^2}{4} e^{\tau} \operatorname{erfc}(\sqrt{\tau}) \le \mathcal{G}_\nu(\tau) \le \frac{\pi^{3/2}}{4\sqrt{\tau}} \qquad (1/2 \le \nu \le 3/2).$$

The bounds for M_ν^2 given in Theorem A may again be used directly, now to derive corresponding bounds for $\mathcal{G}_\nu$, which are sharp in the asymptotic limit when τ approaches infinity.

Theorem E. *The integrals $\mathcal{G}_\nu$ satisfy*

$$\mathcal{G}_\nu(\tau) \begin{array}{ll} > & \dfrac{\pi^2}{2^{2\nu+1}\Gamma(\nu)}\tau^{-\nu}, \quad 0 < \nu \le 1/2, \\ < & \dfrac{\pi^2}{2^{2\nu+1}\Gamma(\nu)}\tau^{-\nu}, \quad 1/2 < \nu, \end{array}$$

with $\mathcal{G}_1$ further satisfying

$$\frac{\pi^2}{8\tau} - \frac{\pi^{7/2}}{32\tau^{3/2}} < \frac{\pi^2}{2}\int_0^{+\infty} \frac{u e^{-\tau u^2}}{2+\pi u}\,\mathrm{d}u < \mathcal{G}_1(\tau) < \frac{\pi^2}{8\tau}.$$

All bounds are asymptotically correct to the first term as τ goes to $+\infty$.

From Theorem A and Corollary B above, we expect the results for ν equal to zero to be of a different form and this is indeed the case. We will proceed in two steps, first by relating $\mathcal{G}_0$ to the Ramanujan integral and then by obtaining explicit sharp bounds for the latter.

Theorem F. *For ν equal to zero we have*

$$\frac{\pi^2}{2} N\left(4e^{-2\gamma}\tau\right) < \mathcal{G}_0(\tau),$$

where N is the Ramanujan integral defined by

$$N(x) = \int_0^\infty \frac{e^{-xu}}{u\left[\pi^2 + \log^2(u)\right]}\,\mathrm{d}u. \tag{3.1}$$

This integral is mentioned by Hardy in [Hd, p. 25 and Chapter XI]. As a particular case of what is referred to in [Hd] as formula (C), we have

$$N(x) = e^x - \int_0^{+\infty} \frac{x^u}{\Gamma(u+1)}\,\mathrm{d}u.$$

It also appears in connection with diffusion and neutron transport problems and was considered in [B, DNT, N] and, more recently, in [L]. It arises precisely from an approximation of an integral of Jaeger type which may already be found in [N], for instance, where a heuristic argument is used to replace $J_0(u)$ and $Y_0(u)$ by 1 and $2(\log(u/2) + \gamma)$, respectively.

The asymptotic behaviour for $N(x)$ as x becomes large was studied in some of these papers and is given by

$$N(x) = \frac{1}{\log(x)} - \frac{\gamma}{\log^2(x)} + \frac{\gamma^2 - \zeta(2)}{\log^3(x)} + \mathrm{O}\left(\log^{-4}(x)\right),$$

where ζ is Riemann's zeta function. Comparing this with the asymptotic behaviour of $\mathcal{G}_0$ for the large argument given in [J, p. 228], namely,

$$\frac{2}{\pi^2}\mathcal{G}_0(\tau) = \frac{1}{y} - \frac{\gamma}{y^2} + \frac{\gamma^2 - \zeta(2)}{y^3} + \mathrm{O}\left(y^{-4}\right),$$

where $y = \log(4\tau) - 2\gamma$, we see that the functions

$$\frac{2}{\pi^2}\mathcal{G}_0(\tau) \text{ and } N\left(4e^{-2\gamma}\tau\right)$$

do not differ in the first three terms of their asymptotic expansions for large τ.

As in the case of the Hankel functions, we also present sharp bounds for the Ramanujan integral together with an identity for N based on Bouwkamp's approach [B].

Theorem G. *The function N satisfies the identity*

$$N(x) = -\int_0^{+\infty} \frac{e^{-xu}}{\pi^2 + \log^2(u)}\,\mathrm{d}u + \int_0^1 \frac{1}{\Gamma(1-u)}x^{-u}\,\mathrm{d}u \tag{3.2}$$

for all positive values of x. Furthermore,

$$\frac{1}{\log x} - \frac{\gamma}{\log^2 x} + \frac{\gamma - \zeta(2)}{\log^3 x} < N(x) < \frac{1}{\log x} - \frac{\gamma}{\log^2 x} - \frac{1}{x\log x}\left(1 - \gamma - \frac{\gamma}{\log x}\right),$$

where the left-hand side inequality holds for x larger than one and the right-hand side inequality for all positive x.

Remark 6. From this we see that N is bounded from below by the first three terms in its asymptotic expansion at infinity and, since the last term on the right becomes negative for sufficiently large x (in fact, for x larger than approximately 3.92), N is then smaller than the first two terms in the same expansion. Using the same method as in the proof of this inequality, it is possible to show that N is also smaller than the first term in the expansion minus $1/(x\log(x))$, showing that the results above may not be obtained as a consequence of an alternating series type result.

Remark 7. The infinite integral in the above identity is in fact equal to $N'(x)$ and thus this may also be written as

$$N'(x) - N(x) = -\int_0^1 \frac{1}{\Gamma(1-u)}x^{-u}\,\mathrm{d}u,$$

which we will use in the proof of the theorem.

As an immediate corollary, we obtain a lower bound for $\mathcal{G}_0$ from the last two results.

Corollary H. *For $\tau > e^{2\gamma}/4 \approx 0.793$ we have*

$$\frac{\pi^2}{2}\left[\frac{1}{y} - \frac{\gamma}{y^2} + \frac{\gamma^2 - \zeta(2)}{y^3}\right] < \mathcal{G}_0(\tau),$$

where $y = \log(4\tau) - 2\gamma$.

Finally, and since to the best of our knowledge only in the cases of ν equal to zero and one are the asymptotic expansions of $\mathcal{G}_\nu$ both as τ approaches zero and plus infinity known (see [J, pp. 227–228], [JC, p. 229] and also [PM]), we complement these with the first few terms in both expansions for general ν.

Theorem I. *For positive values of ν we have the following asymptotic expansions:*

$$\mathcal{G}_\nu(\tau) = \frac{\pi^{3/2}}{4\sqrt{\tau}} + (1-2\nu)\frac{\pi^2}{8} + \left(4\nu^2-1\right)\frac{\pi^{3/2}}{16}\tau^{1/2} + \mathrm{O}(\tau) \ \textit{as}\ \tau \to 0$$

and

$$\mathcal{G}_\nu(\tau) = \frac{\pi^2}{2^{2\nu+1}\Gamma(\nu)}\tau^{-\nu} + \mathrm{o}(\tau^{-\nu}) \ \textit{as}\ \tau \to +\infty.$$

4. Bounds for the modulus and phase of Hankel functions

4.1. **The modulus of Hankel functions: Proof of Theorem A.** By using standard properties of the Bessel functions J_ν and Y_ν it is possible to obtain the following expression for the derivative of $x^{2\nu}M_\nu^2(x)$:

$$\frac{d}{dx}\left[x^{2\nu}M_\nu^2(x)\right] = 2x^{2\nu}\left[J_{\nu-1}(x)J_\nu(x) + Y_{\nu-1}(x)Y_\nu(x)\right].$$

We shall now make use of a Nicholson-type integral representation formula for the above combination of Bessel functions. More precisely, it is stated in [Wa, p. 445] that

$$J_\mu(x)J_\nu(x) + Y_\mu(x)Y_\nu(x) = \frac{4}{\pi^2}\int_0^{+\infty} K_{\nu-\mu}(2x\sinh(t)) \times \left[e^{(\mu+\nu)t} + \cos\left[(\nu-\mu)\pi\right]e^{-(\mu+\nu)t}\right]\,\mathrm{d}t,$$

for $|\operatorname{Re}(\mu-\nu)| < 1$. The case $\mu = \nu - 1$ which interests us does not satisfy this last condition. However, it is not difficult to see that, since the function $e^{(\mu+\nu)t} + \cos\left[(\nu-\mu)\pi\right]e^{-(\mu+\nu)t}$ becomes $2\sinh\left[(2\nu-1)\right]$ in this limit, the function in the integral will be integrable on $(0,+\infty)$ and we have

$$J_{\nu-1}(x)J_\nu(x) + Y_{\nu-1}(x)Y_\nu(x) = \frac{8}{\pi^2}\int_0^{+\infty} K_1(2x\sinh(t))\sinh\left[(2\nu-1)t\right]\,\mathrm{d}t.$$

The monotonicity of $x^{2\nu}M_\nu^2(x)$ then follows immediately from the sign of $2\nu-1$.

To prove the upper and lower bounds for ν less than $1/2$ and ν between $1/2$ and $3/2$, respectively, it remains to integrate this differential inequalty between y and x with $0 < y < x$ and then let y converge to zero while taking into account the behaviour of M_ν^2 at zero.

4.2. $1/2 < \nu \leq 3/2$**.** From the expressions above we obtain

$$\frac{d}{dx}\left[x^{2\nu}M_\nu^2(x)\right] = \frac{16x^{2\nu}}{\pi^2}\int_0^{+\infty} K_1\left(2x\sinh(t)\right)\sinh\left[(2\nu-1)t\right]\,\mathrm{d}t.$$

This last integral may be written as

$$\begin{aligned}
&\int_0^{+\infty} K_1\left(2x\sinh(t)\right)\sinh\left[(2\nu-1)t\right]\,\mathrm{d}t\\
&\quad= -\int_0^{+\infty} \frac{dK_0\left(2x\sinh(t)\right)}{dt}\frac{\sinh\left[(2\nu-1)t\right]}{2x\cosh(t)}\,\mathrm{d}t\\
&\quad= -K_0\left(2x\sinh(t)\right)\frac{\sinh\left[(2\nu-1)t\right]}{2x\cosh(t)}\Big|_0^{+\infty}\\
&\qquad+\frac{1}{2x}\int_0^{+\infty} K_0\left(2x\sinh(t)\right)\frac{g_\nu(t)}{\cosh^2(t)}\,\mathrm{d}t\\
&\quad= \frac{1}{2x}\int_0^{+\infty} K_0\left(2x\sinh(t)\right)\frac{g_\nu(t)}{\cosh^2(t)}\,\mathrm{d}t,
\end{aligned}$$

where

$$g_\nu(t) = (2\nu-1)\cosh(t)\cosh\left[(2\nu-1)t\right] - \sinh(t)\sinh\left[(2\nu-1)t\right].$$

Note that $g_\nu(0) = 2\nu-1$ and that its derivative with respect to t may be written as

$$g_\nu'(t) = 4\nu(\nu-1)\cosh(t)\sinh\left[(2\nu-1)t\right].$$

We shall now first consider $1/2 < \nu \leq 1$. In this range we have $g_\nu'(t) < 0$, $g_\nu(x) < 2\nu-1$ for positive x and

$$\begin{aligned}
\frac{d}{dx}\left[x^{2\nu}M_\nu^2(x)\right] &< \frac{8(2\nu-1)x^{2\nu-1}}{\pi^2}\int_0^{+\infty}\frac{K_0\left(2x\sinh(t)\right)}{\cosh^2(t)}\,\mathrm{d}t\\
&< \frac{8(2\nu-1)x^{2\nu-1}}{\pi^2}\int_0^{+\infty} K_0\left(2xt\right)\,\mathrm{d}t\\
&= \frac{2(2\nu-1)x^{2\nu-2}}{\pi^2}.
\end{aligned}$$

Integrating both sides between 0 and x yields the bound for $1/2 < \nu < 1$.

To proceed in the case of $1 < \nu < 3/2$ we write

$$\frac{d}{dx}\left[x^{2\nu}M_\nu^2(x)\right] = \frac{8x^{2\nu-1}}{\pi^2}\int_0^{+\infty} K_0\left(2x\sinh(t)\right)\cosh(t)h_\nu(t)\,\mathrm{d}t, \tag{4.1}$$

where

$$h_\nu(t) = \frac{2\nu-1}{\cosh^3(t)}\left[\cosh(t)\cosh\left[(2\nu-1)t\right] - \frac{\sinh(t)\sinh\left[(2\nu-1)t\right]}{2\nu-1}\right].$$

In this range of values we have $0 < 2\nu-1 < 2$ and so

$$\begin{aligned}
h_\nu(t) &< \frac{2\nu-1}{\cosh^3(t)}\left[\cosh(t)\cosh\left[(2\nu-1)t\right] - \tfrac{1}{2}\sinh(t)\sinh\left[(2\nu-1)t\right]\right]\\
&= \frac{2\nu-1}{\cosh^3(t)}\left[\cosh\left[2(\nu-1)t\right] + \tfrac{1}{2}\sinh(t)\sinh\left[(2\nu-1)t\right]\right]\\
&< \frac{2\nu-1}{\cosh^3(t)}\left[\cosh(t) + \tfrac{1}{2}\sinh(t)\sinh(2t)\right]\\
&= 2\nu-1.
\end{aligned}$$

Replacing this back in (4.1) and proceeding as in the previous case yields the result in this range of the parameter.

4.3. $3/2 < \nu$. This case follows in the same way as when $1 < \nu < 3/2$, except that now $2 < 2\nu - 1$ so that the first inequality is reversed. In the case of the second inequality, $\nu = 3/2$ is now the smallest possible value of ν, which implies that the second inequality is also reversed. The rest of the proof follows in the same way.

4.4. **The case $\nu = 0$.** We now write

$$h_0(x) = M_0^2(x) - \left[1 + \frac{4}{\pi^2}\left(\gamma + \log\left(\frac{x}{2}\right)\right)^2\right]$$

and begin by noting that, from the expansion of M_0^2 around zero, for instance, we know that $h_0(0)$ vanishes. Writing h_0 by means of Nicholson's integral and differentiating with respect to x yields

$$\begin{aligned} h_0'(x) &= \frac{8}{\pi^2}\int_0^{+\infty} K_0'\left(2x\sinh(t)\right) 2\sinh(t)\,\mathrm{d}t - \frac{8}{\pi^2 x}\left[\gamma + \log\left(\frac{x}{2}\right)\right] \\ &= -\frac{8}{\pi^2 x}\left[\int_0^{+\infty} \frac{K_0\left(2x\sinh(t)\right)}{\cosh^2(t)}\,\mathrm{d}t + \gamma + \log\left(\frac{x}{2}\right)\right], \end{aligned} \tag{4.2}$$

where the last step follows by integrating by parts.

We now observe that from the series expansion of K_0 around zero (see, for instance, [OM, p. 252, 10.31.2]) we have

$$K_0(z) \geq -\left[\gamma + \log\left(\frac{z}{2}\right)\right] I_0(z).$$

Clearly if the term multiplying I_0 is negative we always have

$$K_0(z) \geq -\left[\gamma + \log\left(\frac{z}{2}\right)\right] \tag{4.3}$$

while if it is positive, using $I_0(z) > 1$ for positive z also yields the same inequality. Thus

$$\begin{aligned} \int_0^{+\infty} \frac{K_0\left(2x\sinh(t)\right)}{\cosh^2(t)}\,\mathrm{d}t &> -\int_0^{+\infty} \frac{\gamma + \log(x) + \log\left[\sinh(t)\right]}{\cosh^2(t)}\,\mathrm{d}t \\ &= -\left[\gamma + \log(x)\right]\int_0^{+\infty} \frac{1}{\cosh^2(t)}\,\mathrm{d}t - \int_0^{+\infty} \frac{\log\left[\sinh(t)\right]}{\cosh^2(t)}\,\mathrm{d}t \\ &= -\gamma - \log(x) + \log(2). \end{aligned}$$

Replacing this in identity (4.2) yields $h_0'(x) < 0$ for all positive x. Combining this with the fact that $h_0(0)$ vanishes yields the result and completes the proof of Theorem A.

Remark 8. The above bound is of interest only for small values of x, since we know that M_0^2 is decreasing in x. Using the monotonicity of $M_\nu^2(x)$ we know that

$$M_0^2(x) \leq \frac{2}{\pi x},$$

which is better than the above for x larger than or equal to approximately 0.5065.

4.5. **The phase of Hankel functions: Proof of Theorem C.**

Theorem 4.1. *Given ν greater than or equal to $1/2$, let $n = \lfloor \nu + 1/2 \rfloor$. Then, for all positive x,*

$$-\frac{2}{\pi}\int_x^{+\infty}\left[\frac{1}{tM_{n-1/2}^2(t)}-\frac{\pi}{2}\right]\mathrm{d}t \le \theta_\nu(x) - x + \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi$$
$$\le -\frac{2}{\pi}\int_x^{+\infty}\left[\frac{1}{tM_{n+1/2}^2(t)}-\frac{\pi}{2}\right]\mathrm{d}t.$$

The upper bound holds for any non-negative value of ν and equality holds for the lower bound when ν is of the form $m - 1/2$ for integer m.

Remark 9. All the above bounds are sharp asymptotically to the second (constant) term as x goes to $+\infty$.

Proof. Due to the monotonicity of M_ν with respect to ν we have

$$M_{n-1/2}^2(x) \le M_\nu^2(x) \le M_{n+1/2}^2(x), \text{ for } n = \lfloor \nu + 1/2 \rfloor \tag{4.4}$$

and ν larger than or equal to $1/2$; note that the upper bound is valid for all non-negative ν, and thus the proof also holds in that case.

From (2.3) and using the above monotonicity of M_ν^2 we obtain

$$\frac{2}{\pi x M_{n+1/2}^2} \le \theta_\nu'(x) \le \frac{2}{\pi x M_{n-1/2}^2}.$$

Integrating between x and y $(0 < x < y)$ yields

$$\frac{2}{\pi}\int_x^y \frac{1}{tM_{n+1/2}^2(t)}dt \le \theta_\nu(y) - \theta_\nu(x) \le \frac{2}{\pi}\int_x^y \frac{1}{tM_{n-1/2}^2(t)}dt \tag{4.5}$$

and

$$\frac{2}{\pi}\int_x^y\left[\frac{1}{tM_{n+1/2}^2(t)}-\frac{\pi}{2}\right]dt \le \theta_\nu(y) - y - \theta_\nu(x) + x \le \frac{2}{\pi}\int_x^y\left[\frac{1}{tM_{n-1/2}^2(t)}-\frac{\pi}{2}\right]dt.$$

Letting y go to infinity and taking into account the asymptotic expansion of $\theta_\nu(y)$ at infinity [OM, p. 231] we have

$$\frac{2}{\pi}\int_x^{+\infty}\left[\frac{1}{tM_{n+1/2}^2(t)}-\frac{\pi}{2}\right]dt \le x - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi - \theta_\nu(x)$$
$$\le \frac{2}{\pi}\int_x^{+\infty}\left[\frac{1}{tM_{n-1/2}^2(t)}-\frac{\pi}{2}\right]dt$$

yielding the bounds in the theorem; note that convergence of the above integral on $[x, +\infty)$ for any positive x follows from the asymptotic expansion of M_ν^2 at infinity. Equality follows directly from the fact that, when ν is of the form $m-1/2$, we have equality in the lower bound in (4.4) □

For values of ν in intervals of the form $(m - 1/2, m + 1/2)$, m integer, the above theorem provides different bounds for θ_ν, with the resulting expressions becoming increasingly more complex as m gets larger. For ν in $[0, 1/2)$ only the upper bound holds and we have

$$M_{1/2}^2 = \frac{2}{\pi x},$$

yielding the corresponding upper bound in Theorem C. For ν in $[1/2, 3/2)$ we have

$$M_{3/2}^2 = \frac{2}{\pi x}\left(1 + \frac{1}{x^2}\right)$$

and then

$$x - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi \le \theta_\nu(x) \le x - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi + \frac{1}{2}\left[\pi - 2\arctan(x)\right],$$

for $1/2 \le \nu \le 3/2$, yielding the corresponding lower and upper bounds in Theorem C.

If instead of letting the upper limit in the integral in (4.5) go to infinity we let the lower limit converge to zero, we obtain the following result.

Theorem 4.2. *Given ν greater than or equal to $1/2$, let $n = \lfloor \nu + 1/2 \rfloor$. Then, for all positive x,*

$$\frac{2}{\pi}\int_0^x \frac{1}{tM_{n+1/2}^2(t)}\,\mathrm{d}t \le \theta_\nu(x) + \frac{\pi}{2} \le \frac{2}{\pi}\int_0^x \frac{1}{tM_{n-1/2}^2(t)}\,\mathrm{d}t.$$

The lower bound holds for any non-negative value of ν and equality holds for the upper bound when ν is of the form $m - 1/2$ for integer m.

Remark 10. These bounds are now asymptotically sharp as x approaches zero.

As above, by picking ν in different ranges we obtain different specific bounds. Thus for ν in $[0, 1/2)$ we get the lower bound given in Theorem C for this range, while for ν in $[1/2, 3/2)$ we obtain

$$x - \arctan(x) - \frac{\pi}{2} \le \theta_\nu(x) \le x - \frac{\pi}{2}.$$

To obtain the bounds in Theorem C which hold for ν larger than $3/2$ we proceed as above using the inequality $M_{3/2}^2 \le M_\nu^2$ for ν in that range, and let y go to infinity to obtain the lower bound and x go to zero to obtain the upper bound.

5. Ramanujan integral: Proof of Theorem G

We begin by proving identity (3.2). In order to do this, we proceed as in [B] and define the function

$$g(s) = -\frac{e^{i\pi s}}{\pi}\int_0^{+\infty} \frac{u^{s+\sigma-1}e^{-xu}}{\pi i + \log u}\,\mathrm{d}u$$

for positive σ and s. Then

$$\mathrm{Im}[\mathrm{g}(0)] = \int_0^{+\infty} \frac{\mathrm{u}^{\sigma-1}\mathrm{e}^{-\mathrm{xu}}}{\pi^2 + \log^2 \mathrm{u}}\,\mathrm{du}$$

and by differentiating g with respect to s we have

$$g'(s) = -\frac{e^{i\pi s}}{\pi}\int_0^{+\infty} u^{s+\sigma-1}e^{-xu}\,\mathrm{d}u = -\frac{e^{i\pi s}}{\pi}\frac{\Gamma(s+\sigma)}{x^{s+\sigma}}.$$

Integrating this between 0 and 1 yields

$$g(1) - g(0) = -\int_0^1 \frac{e^{i\pi s}}{\pi}\frac{\Gamma(s+\sigma)}{x^{s+\sigma}}\,\mathrm{d}s$$

which, upon taking the imaginary parts, yields

$$(5.1) \qquad \int_0^{+\infty} \frac{u^\sigma e^{-xu}}{\pi^2+\log^2 u}\,\mathrm{d}u + \int_0^{+\infty} \frac{u^{\sigma-1} e^{-xu}}{\pi^2+\log^2 u}\,\mathrm{d}u = \int_0^1 \frac{\sin(\pi s)}{\pi}\frac{\Gamma(s+\sigma)}{x^{s+\sigma}}\,\mathrm{d}s.$$

Identity (3.2) now follows by taking limits as $\sigma \to 0^+$ and using the identity

$$\frac{\sin(\pi s)}{\pi}\Gamma(s) = \frac{1}{\Gamma(1-s)}.$$

We will now prove the upper bound for N. As pointed out in Remark 7, identity (3.2) is equivalent to

$$(5.2) \qquad N'(x) - N(x) = -\int_0^1 \frac{x^{-s}}{\Gamma(1-s)}\,\mathrm{d}s.$$

From Lemma A.2 we have that on $(0,1)$,

$$0 < \frac{1}{\Gamma(1-s)} < 1-\gamma x$$

and we obtain, by replacing this in (5.2),

$$N'(x) - N(x) > -\int_0^1 (1-\gamma s)x^{-s}\,\mathrm{d}s = -\frac{1}{\log x} + \frac{1}{\log^2 x} + \frac{1-\gamma}{x\log x} - \frac{\gamma}{x\log^2 x}.$$

Multiplying now by e^{-x} and integrating between x and $+\infty$ yields

$$-e^{-x}N(x) > \int_x^{+\infty} e^{-s}\left(-\frac{1}{\log s} + \frac{1}{\log^2 s} + \frac{1-\gamma}{s\log s} - \frac{\gamma}{s\log^2 s}\right)\mathrm{d}s$$

and

$$N(x) < e^x \int_x^{+\infty} e^{-s}\left(\frac{1}{\log s} - \frac{1}{\log^2 s} - \frac{1-\gamma}{s\log s} + \frac{\gamma}{s\log^2 s}\right)\mathrm{d}s.$$

From the fact that the function multiplying e^{-s} inside the integral equals

$$\int_0^1 (1-\gamma s)x^{-s}\,\mathrm{d}s,$$

which is easily seen to be positive and decreasing in x, it follows that

$$N(x) < e^x \int_x^{+\infty} e^{-s}\,\mathrm{d}s \left(\frac{1}{\log x} - \frac{1}{\log^2 x} - \frac{1-\gamma}{x\log x} + \frac{\gamma}{x\log^2 x}\right),$$

proving the upper bound in Theorem G.

Finally, to obtain the lower bound we shall again start from equation (5.2) but we will now first obtain a direct bound on N' in order to bound N. To this end, we consider equation (5.1) but now with σ equal to one, that is,

$$\int_0^{+\infty} \frac{u e^{-xu}}{\pi^2+\log^2 u}\,\mathrm{d}u + \int_0^{+\infty} \frac{e^{-xu}}{\pi^2+\log^2 u}\,\mathrm{d}u = \int_0^1 \frac{\sin(\pi s)}{\pi}\frac{\Gamma(s+1)}{x^{s+1}}\,\mathrm{d}s.$$

Writing

$$P(x) = -N'(x) = \int_0^{+\infty} \frac{e^{-xu}}{\pi^2+\log^2 u}\,\mathrm{d}u$$

and making the change of variables $s \to s+1$ in the integral on the right-hand side yields the equation

$$P'(x) - P(x) = \int_1^2 \frac{x^{-s}}{\Gamma(1-s)}\,\mathrm{d}s.$$

For our purposes now it is enough to consider the inequality

$$\frac{1}{\Gamma(1-x)} > 1-x$$

for x in $(1,2)$ given by Lemma A.1. Then

$$P'(x)-P(x) > \int_1^2 (1-s)x^{-s}\,\mathrm{d}s = -\frac{1}{x\log^2 x}+\frac{1}{x^2\log x}+\frac{1}{x^2\log^2 x}.$$

Again multiplying by e^{-x} on both sides, integrating between x and $+\infty$ and using the monotonicity of the function in the integrand multiplying the exponential yields

$$P(x) < \frac{1}{x\log^2 x}-\frac{1}{x^2\log x}-\frac{1}{x^2\log^2 x}.$$

Recalling equation (5.2) with $P=-N'$ we have

$$N(x) = \int_0^1 \frac{x^{-s}}{\Gamma(1-s)}\,\mathrm{d}s - P(x).$$

Using the above bound for P and the lower bound from A.2 on $(0,1)$,

$$\frac{1}{\Gamma(1-x)} > 1-\gamma x+\frac{\gamma^2-\zeta(2)}{2}x^2,$$

we obtain

$$\begin{aligned} N(x) &> \int_0^1 \left[1-\gamma s+\frac{\gamma^2-\zeta(2)}{2}s^2\right]x^{-s}\,\mathrm{d}s - \frac{1}{x\log^2 x}+\frac{1}{x^2\log x}+\frac{1}{x^2\log^2 x} \\ &= \frac{1}{\log x}-\frac{\gamma}{\log^2 x}+\frac{\gamma-\zeta(2)}{\log^3 x} \\ &\quad +\frac{\gamma-\gamma^2/2+\zeta(2)/2-1}{x\log x}+\frac{\gamma-\gamma^2+\zeta(2)-1}{x\log^2 x}+\frac{\zeta(2)-\gamma^2}{x\log^3 x} \\ &\quad +\frac{1}{x^2\log x}+\frac{1}{x^2\log^2 x}. \end{aligned}$$

Since the coefficients of all the terms other than the three $\log^{-k} x$ $(k=1,2,3)$ terms are positive, we see that N must become larger than those first three terms for x larger than one, concluding the proof of Theorem G.

6. Jaeger integrals: Proof of Theorems D, E, F and I

Proof of Theorem D. For $\nu<1/2$ it is only necessary to prove the upper bound, as the lower bound is known. As derived in Section 2 we have

$$\mathcal{G}_\nu(\tau) = \frac{\pi^2}{4}+\pi\tau\int_0^\infty ue^{-\tau u^2}\theta_\nu(u)\,\mathrm{d}u. \tag{6.1}$$

Replacing θ by the upper bound for $\nu<1/2$ given in Theorem C yields

$$\mathcal{G}_\nu(\tau) \le \frac{\pi^2}{4}+\pi\tau\int_0^\infty u\left[u-\left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right]e^{-\tau u^2}\,\mathrm{d}u,$$

providing the upper bound for $\mathcal{G}_\nu$ in this range upon integration.

For ν larger than $1/2$ we replace the upper bound for θ_ν in Theorem 4.1 in the expression for $\mathcal{G}_\nu$ given by (6.1) to obtain

$$\begin{aligned}
\mathcal{G}_\nu(\tau) &\leq \frac{\pi^2}{4} + \pi\tau \int_0^{+\infty} ue^{-\tau u^2} \left[u - \left(\frac{\nu}{2} + \frac{1}{4}\pi\right) \right.\\
&\qquad\qquad \left. -\frac{2}{\pi}\int_u^{+\infty} \left[\frac{1}{tM_{n+1/2}^2(t)} - \frac{\pi}{2} \right] \mathrm{d}t \right] \mathrm{d}u\\
&= \frac{\pi^{3/2}}{4\sqrt{\tau}} + \frac{\pi^2}{8}(1-2\nu) - 2\tau \int_0^{+\infty} \left[\frac{1}{tM_{n+1/2}^2(t)} - \frac{\pi}{2} \right] \int_0^t ue^{-\tau u^2}\, \mathrm{d}u\, \mathrm{d}t\\
&= \frac{\pi^{3/2}}{4\sqrt{\tau}} + \frac{\pi^2}{8}(1-2\nu) - \int_0^{+\infty} \left[\frac{1}{tM_{n+1/2}^2(t)} - \frac{\pi}{2} \right] \left(1 - e^{-\tau t^2}\right) \mathrm{d}t\\
&= \frac{\pi^{3/2}}{4\sqrt{\tau}} + \frac{\pi^2}{8}(1-2\nu) - \int_0^{+\infty} \left[\frac{1}{tM_{n+1/2}^2(t)} - \frac{\pi}{2} \right] \mathrm{d}t\\
&\quad + \mathcal{G}_{n+1/2}(\tau) - \frac{\pi}{2}\int_0^{+\infty} e^{-\tau t^2}\, \mathrm{d}t\\
&= \mathcal{G}_{n+1/2}(\tau) + \frac{\pi^2}{8}(2n+1-2\nu),
\end{aligned}$$

yielding the upper bound for $1/2 \leq \nu$. The lower bound in this range may be obtained in the same way, but using the lower bound for θ_ν in Theorem 4.1 instead. □

Proof of Theorem E. The proof of the one-term bounds is a direct application of the corresponding bounds in Theorem A, where for ν larger than $3/2$ we use the weaker bound without the $2/(\pi x)$ term. To prove the lower bound for $\mathcal{G}_1$ we first use the upper bound for M_1^2 in Theorem A to obtain

$$\frac{\pi^2}{2}\int_0^{+\infty} \frac{ue^{-\tau u^2}}{2+\pi u}\, \mathrm{d}u < \mathcal{G}_1(\tau).$$

This integral may be evaluated in terms of other special integrals, yielding a bound that is sharp asymptotically for both small and large arguments. We shall, however, simplify things in order to obtain a simple bound, while losing the sharpness for small τ in the process. To do this, we just use the fact that

$$\frac{1}{2+\pi u} \geq \frac{1}{2} - \frac{\pi}{4}u,$$

which upon replacement in the above integral yields the desired bound. □

Proof of Theorem F. Using the lower bound for M_0 given by Theorem A we obtain

$$\mathcal{G}_0(\tau) \geq \pi^2 \int_0^{+\infty} \frac{e^{-\tau u^2}}{u\left[\pi^2 + (2\gamma + 2\log(u/2))^2\right]}\, \mathrm{d}u,$$

and the result follows by making the change of variables $v = e^{2\gamma}u^2/4$. □

Proof of Theorem I. We shall first consider the case when τ approaches zero. From equation (6.1) we have

$$\begin{aligned}\mathcal{G}_\nu(\tau) &= \frac{\pi^2}{4} + \pi\tau \int_0^\infty u e^{-\tau u^2} \theta_\nu(u)\, \mathrm{d}u \\ &= \frac{\pi^2}{4} + \pi\tau \int_0^\infty u e^{-\tau u^2} \left[\theta_\nu(u) - \left(u - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi + \frac{4\nu^2-1}{8u}\right)\right] \mathrm{d}u \\ &\quad + \pi\tau \int_0^\infty u e^{-\tau u^2} \left[u - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi + \frac{4\nu^2-1}{8u}\right] \mathrm{d}u.\end{aligned}$$

Writing

$$g_\nu(u) = \theta_\nu(u) - \left[u - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi + \frac{4\nu^2-1}{8u}\right]$$

we have

$$g_\nu(u) = \mathrm{O}(u^{-3}), \qquad \text{as } u \to +\infty,$$

from the asymptotic behaviour of $\theta_\nu(u)$ as u goes to $+\infty$. Hence

$$\begin{aligned}\left|\int_0^\infty u e^{-\tau u^2} g_\nu(u)\, \mathrm{d}u\right| &\le \int_0^1 u e^{-\tau u^2} |g_\nu(u)|\, \mathrm{d}u + \int_1^\infty u e^{-\tau u^2} |g_\nu(u)|\, \mathrm{d}u \\ &\le \int_0^1 u\, |g_\nu(u)|\, \mathrm{d}u + \int_1^\infty u\, |g_\nu(u)|\, \mathrm{d}u \\ &= C\end{aligned}$$

for some constant C, yielding that

$$\pi\tau \int_0^\infty u e^{-\tau u^2} \left[\theta_\nu(u) - \left(u - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi + \frac{4\nu^2-1}{8u}\right)\right] \mathrm{d}u = \mathrm{O}(\tau)$$

as τ goes to zero. On the other hand,

$$\begin{aligned}&\pi\tau \int_0^\infty u e^{-\tau u^2} \left[u - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi + \frac{4\nu^2-1}{8u}\right] \mathrm{d}u \\ &\qquad = \frac{\pi^{3/2}}{4\sqrt{\tau}} - \frac{(2\nu+1)\pi^2}{8} + \frac{4\nu^2-1}{16}\pi^{3/2}\sqrt{\tau},\end{aligned}$$

yielding the desired asymptotics.

For large τ and strictly positive ν, the first term in the asymptotic expansion of $\mathcal{G}_\nu$ follows directly by combining expression (2.4) with the asymptotic expansion of $\theta_\nu(u)$ for small v given by

$$\theta_\nu(\sqrt{v}) = -\frac{\pi}{2} + \frac{\pi}{\nu 4^\nu \Gamma^2(\nu)} v^\nu + \mathrm{o}(v^\nu), \text{ as } \nu \to 0,$$

and then using Watson's lemma. □

Appendix A. Bounds for the Γ function

Here we prove the three estimates for the Γ function which were used in the paper. In all cases we prove that this function is bounded by the first one, two, or three terms in a certain Taylor series. The first and simplest case corresponds to the expansion around 1, namely,

$$\frac{1}{\Gamma(1-x)} = -(x-1) + \gamma(x-1)^2 + \mathrm{O}((x-1)^3),$$

for which we have the following result.

Lemma A.1. *For all* $x \in (1,2)$

$$1 - x < \frac{1}{\Gamma(1-x)} < 0.$$

Proof. It is enough to notice that, since $\Gamma(1-x)$ is negative on $(1,2)$, the result is equivalent to showing $1 < (1-x)\Gamma(1-x) = \Gamma(2-x) = \Gamma(y)$ for $y \in (0,1)$, which holds, since Γ is decreasing on $(0,1)$ and $\Gamma(1) = 1$. □

The second result is related to the expansion around 0 which now reads

$$\frac{1}{\Gamma(1-x)} = 1 - \gamma x + \frac{\gamma^2 - \zeta(2)}{2}x^2 + \mathrm{O}(x^3),$$

and its proof is more involved.

Lemma A.2. *For all* $x \in (0,1)$

$$1 - \gamma x + \frac{\gamma^2 - \zeta(2)}{2}x^2 < \frac{1}{\Gamma(1-x)} < 1 - \gamma x.$$

Proof. In both cases we shall make use of the expansion

$$\log\Gamma(1-x) = -\log(1-x) - (1-\gamma)x + \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k}x^k, \ |x| < 2, \tag{A.1}$$

which may be found, for instance, in [AR, p. 139]. The upper bound in the lemma is equivalent to showing that $(1-\gamma x)\Gamma(1-x) > 1$. Upon taking logarithms on both sides this becomes equivalent to $\log(1-\gamma x) + \log\Gamma(1-x) > 0$ which, using (A.1), then becomes

$$\log(1-\gamma x) - \log(1-x) - (1-\gamma)x + \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k}x^k > 0.$$

To prove this, it will be sufficient to look at the function $g(x) = \log(1-\gamma x) - \log(1-x) - (1-\gamma)x$, since the remaining term is positive and we have

$$g'(x) = \frac{\gamma}{\gamma x - 1} + \frac{1}{1-x} + \gamma - 1 = \frac{x}{(1-x)(1-\gamma x)}\left[(\gamma-1)\gamma x + 1 - \gamma^2\right].$$

Hence, since the term $(\gamma-1)\gamma x + 1 - \gamma^2$ is decreasing in x and equals $1-\gamma$ when x is one, we see that $g'(x)$ is always positive. Since $g(0)$ vanishes, the result follows.

To prove the lower bound we shall use a similar strategy. We first observe that

$$1 - \gamma x + \frac{\gamma^2 - \zeta(2)}{2}x^2$$

vanishes at two points, one negative that needs not concern us, and the other which we shall denote by x_0 given by

$$x_0 = \frac{-\gamma + \sqrt{2\zeta(2) - \gamma^2}}{\zeta(2) - \gamma^2} \approx 0.87.$$

Since $\Gamma(1-x)$ is positive on $(0,1)$, the result holds for $x \in (x_0, 1)$. From now on we shall thus assume $x \in (0, x_0)$. On this interval the lower bound is positive and we may thus take logarithms on both sides of the left-hand inequality in the lemma to obtain that this is equivalent to showing that

$$\log\left[1-\gamma x+\frac{\gamma^2-\zeta(2)}{2}x^2\right]+\log\Gamma(1-x)<0.$$

Again using the series for $\log\Gamma(1-x)$, we see that the left-hand side, which we shall now denote by h, equals

$$h(x)=\log\left[1-\gamma x+\frac{\gamma^2-\zeta(2)}{2}x^2\right]-\log(1-x)-(1-\gamma)x+\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}x^k.$$

Since we now which to show that h is negative on $(0, x_0)$, it is no longer possible to ignore the series and we shall begin by bounding this. We have

$$\begin{aligned}
\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}x^k &= \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}x^k-\sum_{k=2}^{\infty}\frac{x^k}{k}\\
&< \frac{\zeta(2)}{2}x^2+[\zeta(3)-1]\sum_{k=3}^{\infty}\frac{x^k}{k}-\frac{x^2}{2}\\
&= \frac{\zeta(2)}{2}x^2+\frac{\zeta(3)-1}{2}\left[-x^2-2x-2\log(1-x)\right]-\frac{x^2}{2}\\
&= \frac{\zeta(2)-\zeta(3)}{2}x^2+[1-\zeta(3)]\left[x+\log(1-x)\right],
\end{aligned}$$

and thus,

$$\begin{aligned}
h(x)<h_0(x)=\log&\left[1-\gamma x+\frac{\gamma^2-\zeta(2)}{2}x^2\right]\\
&-\zeta(3)\log(1-x)+[\gamma-\zeta(3)]\,x+\frac{\zeta(2)-\zeta(3)}{2}x^2
\end{aligned}$$

It is easily seen that $h_0(0)=h_0'(0)=h_0''(0)=0$ and

$$h_0'''(0)=-3\gamma\zeta(2)+2\zeta(3)+\gamma^3\approx-0.25<0.$$

Since

$$h_0'(x)=x^2\left[\frac{(\gamma^2-\zeta(2))\zeta(2)x-3\gamma\zeta(2)+\gamma^3}{(\gamma^2-\zeta(2))x^2-2\gamma x+2}-\frac{\zeta(3)}{x-1}\right],$$

we see that h_0 will have a double zero root and two other roots which are situated approximately at -0.24 and 1.78 and thus outside the range of interest; note also that the singularities inside the interval $[0,1]$ are at x_0 and 1. We thus have that $h_0'(x)$ must remain negative on $(0, x_0)$, proving the lemma. $\square$

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