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V-INTEGRABILITY, ASYMPTOTIC STABILITY AND COMPARISON PROPERTY OF EXPLICIT NUMERICAL SCHEMES FOR NON-LINEAR SDES

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ABSTRACT. Khasminski [*Stochastic Stability of Differential Equations*, Kluwer Academic Publishers, 1980] showed that the asymptotic stability and the integrability of solutions to stochastic differential equations (SDEs) can be obtained via Lyapunov functions. These properties are, however, not necessarily inherited by standard numerical approximations. In this article we introduce a general class of explicit numerical approximations that are amenable to Khasminski's techniques and are particularly suited for non-globally Lipschitz coefficients. We derive general conditions under which these numerical schemes are bounded in expectation with respect to certain Lyapunov functions, and/or inherit the asymptotic stability of the SDEs. Finally we show that by truncating the noise it is possible to recover the comparison theorem for numerical approximations of non-linear scalar SDEs.

1. Introduction

The main goal of this article is to extend the applicability of Lyapunov functions to numerical approximations of SDEs. This is achieved by modifying the standard Euler scheme in such a way that, despite the lack of a discrete version of Itō's formula, one can still analyse asymptotic and qualitative properties of numerical approximations using discrete analogues of classical techniques developed in [14]. In particular, we investigate the integrability and asymptotic stability of numerical approximations of SDEs, paying particular attention to SDEs with non-globally Lipschitz drift and diffusion. This is a relatively new area as the majority of the research on numerical analysis for SDEs is restricted to the global Lipschitz condition [15, 23]. If the global Lipschitz condition does not hold for either of the coefficients, Hutzenthaler, Jentzen and Kloeden [8] showed that the explicit Euler scheme may have unbounded moments, and consequently the classical Euler scheme may fail to converge in the L^p norm, $p \in \mathbb{N}^+$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions, where I is a subset of $[0, \infty)$. Let W_t be an m-dimensional $(\mathcal{F}_t)_{t\in I}$-adapted Wiener process. Consider an SDE:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,\ t \in I, \tag{1.1}$$

where $b : I\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma =: I\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous, and a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}^+ \cup \{0\}$ at least twice continuously differentiable.

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Both integrability and asymptotic stability properties of (1.1) can be deduced by examining the generator

$$\mathcal{L}_t V(x) = \langle \nabla V(x), b(t,x)\rangle + \frac{1}{2}\mathrm{tr}\left[\sigma(t,x)V^{(2)}(x)\sigma(t,x)^\top\right],\ \forall t \in I,\ x \in \mathbb{R}^d,$$

where $V^{(2)}(\cdot)$ is the Hessian matrix of $V(\cdot)$. When $I = [0,T]$ for $T > 0$ fixed, one knows from standard results in stochastic analysis [14] that if there is a constant $\rho > 0$ s.t.

$$\mathcal{L}_t V(x) \leqslant \rho V(x),\ \forall t \in [0,T],\ x \in \mathbb{R}^d, \tag{1.2}$$

then one has a uniform bound:

$$\mathbb{E}V(X_t) \leqslant e^{\rho T}\mathbb{E}V(X_0),\ \forall t \in [0,T]. \tag{1.3}$$

In the context of asymptotic stability, one considers the case where $I = [0,\infty)$ and $b(t,0) \equiv 0,\ \sigma(t,0) \equiv 0,\ \forall t \geqslant 0$ holds[1] (see [17,19]). One also takes a Lyapunov function $V \in \mathcal{C}^2(\mathbb{R}^d)$ that takes value 0 at the origin and is strictly positive elsewhere (e.g. $V(\cdot) = |\cdot|^p$ for some $p \in \mathbb{N}^+$)[2]. Instead of (1.2), a sufficient condition for $X_t \to 0$ a.s. as $t \to \infty$, regardless of the value of X_0, is that

$$\mathcal{L}_t V(\cdot) \leqslant -z(\cdot), \tag{1.4}$$

for some non-negative $z \in \mathcal{C}(\mathbb{R}^d)$ such that $\ker(z) = \{0\}$. Moreover if $z(\cdot) \geqslant \rho V(\cdot)$ for some constant $\rho > 0$, then instead of (1.3) one has

$$\mathbb{E}V(X_t) \leqslant e^{-\rho t}\mathbb{E}V(X_0) \to 0, \tag{1.5}$$

as $t \to \infty$, given that $\mathbb{E}V(X_0) < \infty$. Conditions of the type (1.4) with $z(\cdot) \geqslant \rho V(\cdot)$ also play a crucial role in establishing ergodicity properties of SDEs; see [22].

The majority of the research on integrability or stability of the numerical schemes relies on simple Lyapunov functions such as $V(x) = |x|^p,\ p \geqslant 2$; see, e.g., [9,11,15, 23,24], with the exception of [11,12]. Here we aim at handling more general cases, particularly polynomials of the general form

$$V(x) = \sum_{i=1}^d c_i x_i^{p_i}, \quad c_1, \cdots, c_d \in \mathbb{R}, \tag{1.6}$$

where the p_i's (non-negative) are not necessarily identical. This is necessary if one hopes to analyse many important SDEs in literature; see [11,12][3] and Example 3.19 in the current paper. It turns out that for a special class of Lyapunov functions $V(x) = |x|^p,\ p \geqslant 2$, the drift-implicit Euler scheme admits a discrete-time analogue of (1.3), without the global Lipschitz condition; see [6,20,21]. However, solving an implicit equation at each iteration of the algorithm is usually costly. Furthermore, recently it has been demonstrated that even the explicit Euler scheme, if appropriately modified, can have bounded moments (and hence L^p-convergence). Such modification of explicit schemes is conventionally called *"taming"*; see [9,11,12,24,27]. A tamed Euler scheme is usually of the following form:

$$\bar{X}_{k+1} = \bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1},\ k \in \mathbb{N}, \tag{1.7}$$

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[1] Given the well-posedness of the SDE (1.1) one sees that the system has trivial solution (equilibrium) $X_t \equiv 0,\ \forall t \geqslant 0$ a.s. when $X_0 \equiv 0$ a.s.

[2] Throughout this article we use the notation $|\cdot|$ for modulus of vectors and $\|\cdot\|$ for the 2-norm of matrices.

[3] In [11,12] authors investigated integrability, but not asymptotic stability of explicit schemes allowing Lyapunov functions of the form (1.6).

where $t_k = kh$ with $0 < h \leqslant 1$ being the step size of the uniform partition of I, and $\Delta W_{k+1} = W_{t_{k+1}} - W_{t_k}$. Usually a taming method (b^h, σ^h) is chosen s.t. $b^h(t,x) \to b(t,x)$, $\sigma^h(t,x) \to \sigma(t,x)$ as $h \to 0$ uniformly[4] in $(t,x) \in I \times \mathbb{R}^d$. We also introduce the "tamed" generator:

$$\mathcal{L}_t^h V(x) := \langle \nabla V(x), b^h(t,x) \rangle + \frac{1}{2}\mathrm{tr}\left[\sigma^h(t,x) V^{(2)}(x) \sigma^h(t,x)^\top\right], \ \forall t \in I, \ x \in \mathbb{R}^d.$$

The main challenge is to preserve condition (1.2) or (1.4) for $\mathcal{L}_t^h$ and to benefit from some extra control on the growth of the tamed coefficients. Although integrability results have been established in the literature for some specific explicit schemes of the form (1.7), it is not clear how property (1.3) can be inherited (possibly with a different ρ) under simple assumptions. For example, in [11] the authors showed some criteria for moment bounds (Proposition 2.7) and one can indeed recover (1.3), but an a priori estimate is needed: $\sup_h \max_k \|V(\bar{X}_k)\|_{L^p(\Omega)} h^{(\alpha-1)(1-1/p)} < \infty$ for some $\alpha > 1$. We will show in Section 2 that such a property can be preserved by controlling the generator $\mathcal{L}_t^h$ and the coefficients b^h, σ^h. We will also propose a type of projected schemes (1.8) that preserve the strong convergence rate 1/2 and a uniform bound of the form (1.3), with respect to a larger class of Lyapunov functions.

On the other hand, the problem of asymptotic stability has received less attention in the literature so far and to the best of our knowledge the asymptotic stability of explicit numerical schemes beyond the Lipschitz setting is entirely new. Nonetheless, considerable effort has been made in this direction (mainly for implicit schemes) in [2–5, 7, 20, 22, 28]. We will extend these results in two ways: a) we allow a bigger class of Lyapunov functions; b) we consider explicit Euler-type schemes. The idea seems similar to that of integrability—the main difference, however, lies in the recovery of condition (1.4). The issue here is that for the tamed Euler schemes (1.7) one often can only deduce

$$\mathcal{L}_t V(\cdot) \leqslant -z(\cdot) \ \Rightarrow \ \mathcal{L}_t^h V(\cdot) \leqslant -\rho^h(\cdot) z(\cdot), \ \rho^h(\cdot) \geqslant 0,$$

and finds no strictly positive lower bound for $\rho^h(\cdot)z(\cdot)$. The same problem would occur if one tries to recover the ergodicity of the underlying SDE using scheme (1.7); see [22]. Nevertheless, explicit schemes of type (1.7) can recover the almost-sure stability if $\ker(\rho^h) = \{0\}$, but the exponential stability (1.5) seems not to hold. This, however, can be resolved by the aforementioned projected schemes:

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{1.8}$$

where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ is a projection function that can be customised. The advantage of this method lies in that $\rho^h(\cdot) \geqslant c$ for some $c > 0$.

In the last section we will investigate the preservation of non-negativity and the comparison theorem for explicit numerical schemes. This is aimed at those SDEs whose solutions, for example, only stay in $[0, \infty)$. We will see that $b(t,0) \geqslant 0$, $\sigma(t,0) \equiv 0$ is enough to guarantee $X_t \geqslant 0$ a.s., but not necessarily the case for numerical schemes. We will show that simply by truncating the noise as is done in Section 1.3.4 in [23], one can easily recover non-negativity of the tamed Euler scheme. The same method can readily be used to preserve the comparison theorem for SDEs with non-globally Lipschitz coefficients.

[4] Precise definition of these limits may vary.

To summarise the main contributions and the structure of this paper:

- We give general conditions for a modified (tamed) explicit Euler scheme to be integrable with respect to a rich family (2.2) of Lyapunov functions.
- We establish in Section 3 a result on the asymptotic stability properties for tamed explicit Euler schemes in the non-globally Lipschitz setting, admitting a large class of Lyapunov functions.
- We propose an explicit tamed Euler scheme that is easy to implement, recovers exponential V-stability (1.5), and allows the optimal rate of strong convergence.
- We investigate non-negativity preservation and comparison result of tamed Euler schemes in Section 4. Both properties hold true in the non-globally Lipschitz setting.

2. V-Integrability of tamed Euler schemes

In this section we investigate the integrability of tamed Euler schemes $\{\bar{X}_k\}$, (1.7) or (1.8), for the SDE on a fixed interval $t \in [0,T]$:

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t. \tag{2.1}$$

Following [11], let $p,d \in \mathbb{N}^+$, $\gamma \in (0,1/p]$ and consider the following class of Lyapunov functions $\mathcal{V}^p_\gamma \subset \mathcal{C}^p(\mathbb{R}^d)$, where for $\mathbb{N} \ni p \geqslant 2$ and $0 < \gamma \leqslant \frac{1}{p}$,

$$\begin{aligned}\mathcal{V}^p_\gamma := \{V \geqslant 0 :\ &\ker(V) = \{0\},\ \exists c > 0 \text{ s.t.}\\ &\|V^{(s)}(\cdot)\|_{\mathrm{HS}} \leqslant c(1+V(\cdot))^{1-s\gamma},\ \forall s \in \mathbb{N}\cap[0,p]\Big\}.\end{aligned} \tag{2.2}$$

Here $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm and $V^{(s)}$ denotes the s-th order derivative of V. For example, $V^{(1)} = \nabla V$ and $V^{(2)}$ is the Hessian matrix of V. Note that the set $\mathcal{V}^p_\gamma$ not only covers power functions $|\cdot|^p$, $p > 0$, but also covers polynomials of the form (1.6). Hence it is rich enough for one to choose suitable Lyapunov functions for many of important SDEs (see [11] for more details).

The property $\ker(V) = \{0\}$ is in fact not necessary for integrability, but is needed for stability results in Section 3. We introduce this definition here rather than later for the simplicity of presentation: if a non-negative function U only satisfies the growth condition of its derivatives as in (2.2), then $V(x) := U(x) - U(0) \in \mathcal{V}^p_\gamma$ and $U(x)$ is thus equivalent to $1 + V(x)$.

Remark 2.1. The function $|\cdot|^p$ for some even number p is a candidate in the subset $\bar{\mathcal{V}}^p_{1/p} := \mathcal{V}^p_{1/p} \cap \{V^{(p+1)} \equiv 0,\ \exists c > 0 \text{ s.t. } \|V^{(s)}(\cdot)\|_{\mathrm{HS}} \leqslant cV(\cdot)^{1-s/p},\ \forall s \in \mathbb{N}\cap[0,p]\}$.

Once we fix a Lyapunov function $V \in \mathcal{V}^p_\gamma$ it will be useful if the growth conditions of the coefficients of the SDE (2.1) can be expressed in terms of V.

Assumption 2.2. *There exists a Lyapunov function $V \in \mathcal{V}^p_\gamma$ and constants $K, \kappa > 0$, s.t. $\forall t \in [0,T],\ x \in \mathbb{R}^d$,*

$$|b(t,x)| \vee \|\sigma(t,x)\| \leqslant K\left(1 + V(x)^{\kappa\gamma}\right).$$

Take $V(\cdot) = |\cdot|^p \in \bar{\mathcal{V}}^p_{1/p}$, then Assumption 2.2 essentially imposes the polynomial growth condition on the coefficients of the SDE (2.1). Indeed, we may observe that if there exists $L > 0$ such that $\forall t,x,\ |b(t,x)| \leqslant L(1+|x|^{\kappa_1})$, one can find $K > 0$ such that $|b(t,x)| \leqslant K(1+V(x))^{\kappa_1/p}$. The same applies to the diffusion coefficient with polynomial growth of degree κ_2 and let $\kappa = \kappa_1 \vee \kappa_2$. Expressing all estimates in terms of the chosen Lyapunov function[5] makes all calculations convenient.

Definition 2.3. Let V be a non-negative Borel function on $\mathbb{R}^d$. The solution to the SDE (2.1) is said to be integrable with respect to V, or V-integrable, if

$$\sup_{t\in[0,T]} \mathbb{E}V(X_t) < \infty.$$

A time-discretisation $\{\bar{X}_k\}$, with step size $h \in (0,1]$, of the SDE (2.1) is said to be V-integrable, if

$$\sup_{h>0} \max_{0\leqslant k\leqslant \lfloor T/h\rfloor} \mathbb{E}V(\bar{X}_k) < \infty.$$

To clarify the idea of this section without going into too much technical detail let us consider a motivational example.

Example 2.4. Let $(X_t)_{t\in[0,T]}$ be the solution of the 1-d autonomous SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{2.3}$$

with $\mathbb{E}|X_0|^2 < \infty$ and b and σ satisfying Assumption 2.2 and monotonicity condition:

$$2xb(x) + |\sigma(x)|^2 \leqslant \rho(1+|x|^2) \quad \forall x \in \mathbb{R}. \tag{2.4}$$

Note that (2.4) corresponds to the special case of the Lyapunov function $V(x) = |x|^2 \in \bar{\mathcal{V}}^2_{1/2}$, and it immediately follows that for all $t \geqslant 0$, $\mathbb{E}V(X_t) \leqslant e^{\rho t}\mathbb{E}(1+V(X_0))$. We are seeking some condition under which the tamed Euler scheme

$$\bar{X}_{k+1} = \bar{X}_k + b^h(\bar{X}_k)h + \sigma^h(\bar{X}_k)\Delta W_{k+1},$$

is also $|\cdot|^2$-integrable. Let us first square both sides of the scheme to get

$$\mathbb{E}_k|\bar{X}_{k+1}|^2 = |\bar{X}_k|^2 + \big(2\bar{X}_k b^h(\bar{X}_k) + |\sigma^h(\bar{X}_k)|^2\big)h + |b^h(\bar{X}_k)|^2h^2, \tag{2.5}$$

where $\mathbb{E}_k(\cdot) := \mathbb{E}(\cdot|\mathcal{F}_{t_k})$. If a taming method is chosen such that $\exists \mu > 0$,

$$2xb^h(x) + |\sigma^h(x)|^2 \leqslant \rho(1+V(x)) \quad \text{and} \quad |b^h(x)|^2h \leqslant \mu(1+V(x)),\ \forall x \in \mathbb{R}, \tag{2.6}$$

then for all $1 \leqslant k \leqslant \lfloor T/h\rfloor$,

$$\begin{aligned}&\mathbb{E}_k(1+V(\bar{X}_{k+1})) \leqslant 1+V(\bar{X}_k) + (\rho+\mu)(1+V(\bar{X}_k))h\\ \Rightarrow\ &\mathbb{E}V(\bar{X}_{\lfloor T/h\rfloor}) \leqslant e^{(\rho+\mu)T}\mathbb{E}(1+V(X_0)).\end{aligned}$$

One can use taming methods, e.g.,

$$b^h(t,x) := \frac{b(t,x)}{1+G_b(x,h)},\ \sigma^h(t,x) := \frac{\sigma(t,x)}{1+G_\sigma(x,h)},\ \forall t\in[0,T],\ x\in\mathbb{R}^d, \tag{2.7}$$

for some $G_b(\cdot,\cdot), G_\sigma(\cdot,\cdot) \geqslant 0$. Then the first condition in (2.6) holds if $1+G_b(x,h) \leqslant (1+G_\sigma(x,h))^2$. Furthermore for the second condition in (2.6) take $G_\sigma(x,h) = G_b(x,h) := CV(x)^{\kappa_0/2}h^\beta$, with $C = K/\sqrt{\mu}$, $k_0 = (\kappa-1)^+$ and $\beta = 1/2$, so that

$$|b^h(x)|h^{1/2} = \frac{|b(x)|h^{1/2}}{1+CV(x)^{\kappa_0/2}h^{1/2}} \leqslant \frac{KV(x)^{\kappa/2}h^{1/2}}{1+CV(x)^{\kappa_0/2}h^{1/2}} \leqslant \sqrt{\mu}V(x)^{1/2},$$

as required.

[5] This corresponds to the Lyapunov-type functions $\tilde{V}(\cdot) := 1+V(\cdot)$ defined in [11].

2.1. **Taming conditions for V-integrability.** The difficulty in exploiting the integrability of numerical schemes is the lack of a discrete version of Itō's formula. However, if the coefficients are appropriately modified (tamed), one can recover many classical results by controlling the remainder term of the Taylor expansion. This is the result of Theorem 2.5.

In the first part of this section we focus on another subset of $\mathcal{V}_\gamma^p$ denoted by $\hat{\mathcal{V}}_\gamma^p = \mathcal{V}_\gamma^p \cap \{V^{(p+1)} \equiv 0\}$ (this class contains almost all examples of polynomial Lyapunov functions presented in [11]). As an example one may consider a very popular Lypaunov function $V(x) = |x|^p,\ p \geqslant 2$, which allows us to explore the so-called one-sided Lipschitz property of the drift coefficient of the SDE (1.1). Later on we will show that integrability results can be extended to the whole family $\mathcal{V}_\gamma^p$.

Theorem 2.5. *Suppose for the tamed coefficients (b^h, σ^h) as in (1.7) there is a Lyapunov function $V \in \hat{\mathcal{V}}_\gamma^p,\ p \geqslant 2$ s.t. $\mathbb{E}V(X_0) < \infty$ and*

$$\mathcal{L}_t^h V(x) \leqslant \rho(1 + V(x)),\ \forall (t, x) \in [0, T] \times \mathbb{R}^d, \tag{2.8}$$

for some $\rho > 0$. Also assume that there exists $\mu > 0$ s.t.

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu(1 + V(x))^\gamma. \tag{2.9}$$

Then there exists a constant $\tilde{\rho} = O(\mu^2)$ s.t.

$$\mathbb{E}V(\bar{X}_k) \leqslant e^{(\rho+\tilde{\rho})T}\mathbb{E}(1 + V(X_0)) < \infty, \quad \forall 0 \leqslant k \leqslant \lfloor T/h \rfloor .$$

Remark 2.6. V-integrability of numerical schemes has already been studied in [11] (section 2.2), but the results are based on a weaker "semi-stability" condition. Here condition (2.9) ensures full "V-stability" according to their definition.

Proof. Since $V \in \hat{\mathcal{V}}_\gamma^p$, one has the following finite Taylor expansion:

$$\mathbb{E}_k(1 + V(\bar{X}_{k+1})) = 1 + V(\bar{X}_k) + \mathbb{E}_k \sum_{1 \leqslant |\alpha| \leqslant p} \frac{\partial^\alpha V(\bar{X}_k)}{\alpha!}(\bar{X}_{k+1} - \bar{X}_k)^\alpha. \tag{2.10}$$

For the convenience of notation denote $\bar{b}_k := b^h(t_k, \bar{X}_k),\ \bar{\sigma}_k := \sigma^h(t_k, \bar{X}_k)$, and S_s the summation with index $|\alpha| = s,\ s = 1, \cdots, p$. It is easy to see that the conditional expectation of the first two terms of the summation in (2.10) are:

$$\begin{aligned}
\mathbb{E}_k S_1 :=& \mathbb{E}_k \sum_{|\alpha|=1} \frac{\partial^\alpha V(\bar{X}_k)}{\alpha!}(\bar{b}_k h + \bar{\sigma}_k \Delta W_{k+1})^\alpha = \left\langle \bar{b}_k, \nabla V(\bar{X}_k)\right\rangle h,\\
\mathbb{E}_k S_2 =& \frac{1}{2}\sum_{i,j=1}^d \sum_{l=1}^m \frac{\partial^2 V}{\partial x_i \partial x_j}(\bar{X}_k)\bar{\sigma}_k^{(il)}\bar{\sigma}_k^{(jl)} h + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 V}{\partial x_i \partial x_j}(\bar{X}_k)\bar{b}_k^{(i)}\bar{b}_k^{(j)} h^2\\
=& \frac{1}{2}\sum_{l=1}^m \left\langle \bar{\sigma}_k^{(\cdot,l)}, V^{(2)}(\bar{X}_k)\bar{\sigma}_k^{(\cdot,l)}\right\rangle h + \frac{1}{2}\left\langle \bar{b}_k, V^{(2)}(\bar{X}_k)\bar{b}_k\right\rangle h^2\\
\leqslant& \frac{1}{2}\operatorname{tr}\left[V^{(2)}(\bar{X}_k)\bar{\sigma}_k\bar{\sigma}_k^\top\right] h + \frac{1}{2}\left\|V^{(2)}(\bar{X}_k)\right\| \left|\bar{b}_k\right| h^2.
\end{aligned}$$

We can now analyse the rest of the expansion for $|\alpha| = s \geqslant 3$ by rewriting the sum

$$S_s = \frac{1}{s!}\sum_{|\alpha|=s} \binom{s}{\alpha} \left(\bar{X}_{k+1}^{(1)} - \bar{X}_k^{(1)}\right)^{\alpha_1} \cdots \left(\bar{X}_{k+1}^{(d)} - \bar{X}_k^{(d)}\right)^{\alpha_d} \frac{\partial^s}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} V(\bar{X}_k),$$

where for $i = 1, \cdots, d$, each $\left(\bar{X}_{k+1}^{(i)} - \bar{X}_k^{(i)}\right)^{\alpha_i}$ is equal to

$$\left(\bar{b}_k^{(i)} h + \bar{\sigma}_k^{(i,\cdot)} \Delta W_{k+1}\right)^{\alpha_i} = \sum_{r=0}^{\alpha_i} \binom{\alpha_i}{r} \left(\bar{b}_k^{(i)} h\right)^{\alpha_i - r} \left(\bar{\sigma}_k^{(i,\cdot)} \Delta W_{k+1}\right)^r.$$

Due to the independence and the law of the Brownian increments $\Delta W_{k+1}^{(j)}$, the terms with odd r's are zero under $\mathbb{E}_k$. Therefore, with a bit relabelling,

$$\begin{aligned}\mathbb{E}_k S_s \leqslant& \left\| V^{(s)}(\bar{X}_k) \right\| \frac{d^{s-1}}{s!} \sum_{r=0}^{\lfloor s/2 \rfloor} \binom{s}{2r} \left|\bar{b}_k\right|^{s-2r} \left\|\bar{\sigma}_k\right\|^{2r} h^{s-r} \\ \leqslant& \phi_s \left\| V^{(s)}(\bar{X}_k) \right\| \sum_{r=0}^{\lfloor s/2 \rfloor} \left|\bar{b}_k\right|^{s-2r} \left\|\bar{\sigma}_k\right\|^{2r} h^{s-r},\end{aligned}$$

where the positive constants

$$\phi_s := \frac{d^{s-1}}{s!} \max_{r=0,\cdots,s} \binom{s}{r} \leqslant \frac{d^{s-1}}{(\lfloor s/2 \rfloor !)^2}, \tag{2.11}$$

for each s. Returning to (2.10) and using the above estimates, we obtain

$$\mathbb{E}_k(1 + V(\bar{X}_{k+1})) = 1 + V(\bar{X}_k) + \mathcal{L}_{t_k}^h V(\bar{X}_k) h + R^h V(\bar{X}_k), \tag{2.12}$$

where, by relabelling the indices (with $i, j \in \mathbb{N}$) in the summation,

$$\begin{aligned}R^h V(\bar{X}_k) \leqslant& \frac{1}{2} \left\| V^{(2)}(\bar{X}) \right\| \left|\bar{b}_k\right|^2 h^2 \\ &+ \sum_{3 \leqslant i+2j \leqslant p} \phi_{i+2j} \left\| V^{(i+2j)}(\bar{X}_k) \right\| \left|\bar{b}_k\right|^i \left\|\bar{\sigma}_k\right\|^{2j} h^{i+j}.\end{aligned} \tag{2.13}$$

Now given (2.9) and the estimates of $V^{(i+2j)}$ as in (2.2), we have

$$\begin{aligned}R^h V(\bar{X}_k) \leqslant& \frac{1}{2} c\mu^2 (1 + V(\bar{X}_k)) h + \sum_{3 \leqslant i+2j \leqslant p} \phi_{i+2j} c\mu^{i+2j} (1 + V(\bar{X}_k)) h^{\frac{i+j}{2}} \\ =& \left(\frac{1}{2} c\mu^2 + \sum_{s=3}^{p} \sum_{i+2j=s} \phi_s c\mu^s h^{\frac{s}{2}-1} \right) (1 + V(\bar{X}_k)) h \\ \leqslant& \left(\frac{1}{2} c\mu^2 + c \sum_{s=3}^{p} \left\lfloor \frac{s+1}{2} \right\rfloor \phi_s \mu^s \right) (1 + V(\bar{X}_k)) h.\end{aligned}$$

Set $\tilde{\rho} := \frac{1}{2} c\mu^2 + \frac{1}{2} c(p+1) \sum_{s=3}^{p} \phi_s \mu^s h^{s/2-1}$, and from (2.12) we get

$$\begin{aligned}\mathbb{E}(1 + V(\bar{X}_{k+1})) \leqslant& (1 + (\rho + \tilde{\rho})h) \mathbb{E}(1 + V(\bar{X}_k)) \leqslant (1 + (\rho + \tilde{\rho})h)^k \mathbb{E}(1 + V(X_0)) \\ \leqslant& e^{(\rho + \tilde{\rho})T} \mathbb{E}(1 + V(X_0)),\end{aligned}$$

and the result follows by removing 1 from the left-hand side. □

Remark 2.7. For $p = 2$ one only needs to check condition (2.9) for $b^h(\cdot, \cdot)$.

Remark 2.8. In practice one can take $\mu \leqslant 1$ and choose $\tilde{\rho} := c(p^2 - 1) d^{p-1} \mu^2$ since $\sup_{3 \leqslant s \leqslant p} \phi_s \leqslant d^{p-1}$. Therefore $\tilde{\rho}$ can be arbitrarily small by a suitable choice of the parameter μ. For example, the choice $\mu = O(h^\varepsilon)$ for some $\varepsilon > 0$ will lead to the generalisation of Proposition 2.7 in [11], where asymptotically $\tilde{\rho} \to 0$ as $h \to 0$, but the authors proved the result only on a suitable subset of $\mathbb{R}^d$.

In a similar way we extend applicability of tamed Euler schemes to all Lyapunov functions from $\mathcal{V}^p_\gamma$. It turns out that the smoothness of V affects the rate of taming of the diffusion coefficient.

Proposition 2.9. *Let $V \in \mathcal{V}^p_\gamma$, $p \geqslant 3$. Suppose there is a constant $\rho > 0$ s.t. $\mathcal{L}^h V(\cdot) \leqslant \rho V(\cdot)$, and a constant $\mu > 0$ s.t.*

$$\left|b^h(t,x)\right| h^{\beta_1} \vee \left\|\sigma^h(t,x)\right\| h^{\beta_2} \leqslant \mu(1+V(x))^\gamma, \ \forall t, x, \tag{2.14}$$

for some $\beta_1 \leqslant 1/2$ and $\beta_2 \leqslant 1/2 - 1/(p \wedge 4)$. Then there exists $\tilde{\rho} := \tilde{\rho}(\mu)$ s.t.

$$\mathbb{E}V(\bar{X}_k) \leqslant e^{(\rho+\tilde{\rho})T}\mathbb{E}(1+V(X_0)), \ \forall 0 \leqslant k \leqslant \lfloor T/h \rfloor .$$

Proof. The proof is very similar to the proof of Theorem 2.5. We write

$$\begin{aligned}\mathbb{E}_k(1+V(\bar{X}_{k+1})) &= 1+V(\bar{X}_k) + \sum_{1\leqslant|a|\leqslant p-1} \frac{\partial^\alpha V(\bar{X}_k)}{\alpha!}\mathbb{E}_k(\bar{X}_{k+1}-\bar{X}_k)^\alpha \\ &+ p\sum_{|\alpha|=p}\mathbb{E}_k\frac{(\bar{X}_{k+1}-\bar{X}_k)^\alpha}{\alpha!}\int_0^1 (1-t)^{p-1}\partial^\alpha V(\bar{X}_k + t(\bar{X}_{k+1}-\bar{X}_k))\mathrm{d}t.\end{aligned} \tag{2.15}$$

It therefore suffices to look at the remainder term for $p \geqslant 2$:

$$\begin{aligned}\tilde{R}^h &:= p\sum_{|\alpha|=p}\mathbb{E}_k\frac{(\bar{X}_{k+1}-\bar{X}_k)^\alpha}{\alpha!}\int_0^1 (1-t)^{p-1}\partial^\alpha V(\bar{X}_k + t(\bar{X}_{k+1}-\bar{X}_k))\mathrm{d}t \\ &\leqslant p\sum_{|\alpha|=p}\mathbb{E}_k\frac{|(\bar{X}_{k+1}-\bar{X}_k)^\alpha|}{\alpha!}\int_0^1 (1-t)^{p-1}\left\|V^{(p)}(\bar{X}_k + t(\bar{X}_{k+1}-\bar{X}_k))\right\|\mathrm{d}t \\ &\leqslant cp\sum_{|\alpha|=p}\mathbb{E}_k\frac{|(\bar{X}_{k+1}-\bar{X}_k)^\alpha|}{\alpha!}\int_0^1 (1-t)^{p-1}\left(1+V(\bar{X}_k + t(\bar{X}_{k+1}-\bar{X}_k))\right)^{1-p\gamma}\mathrm{d}t.\end{aligned}$$

By Lemma 2.12 in [11] we have

$$1+V(x+y) \leqslant c^{\frac{1}{\gamma}}2^{\frac{1}{\gamma}-1}\Big(1+V(x)+|y|^{\frac{1}{\gamma}}\Big), \ \forall x, y \in \mathbb{R}^d,$$

which leads to

$$\begin{aligned}\Big(1+V(x+y)\Big)^{1-p\gamma} &\leqslant c^{\frac{1}{\gamma}-p}2^{\left(\frac{1}{\gamma}-p\right)(1-\gamma)}\Big(1+V(x)+|y|^{\frac{1}{\gamma}}\Big)^{1-p\gamma} \\ &\leqslant (2c)^{\frac{1}{\gamma}-p}\Big((1+V(x))^{1-p\gamma}+|y|^{\frac{1}{\gamma}-p}\Big),\end{aligned}$$

for $\gamma \in (0, 1/p]$. Consequently,

$$\begin{aligned}\tilde{R}^h &\leqslant cp\sum_{|\alpha|=p}\mathbb{E}_k\frac{|(\bar{X}_{k+1}-\bar{X}_k)^\alpha|}{\alpha!}(2c)^{\frac{1}{\gamma}-p}\Big((1+V(\bar{X}_k))^{1-p\gamma}+\left|\bar{X}_{k+1}-\bar{X}_k\right|^{\frac{1}{\gamma}-p}\Big) \\ &\leqslant p\frac{c^{\frac{1}{\gamma}-p+1}2^{\frac{1}{\gamma}-p}}{p!}\mathbb{E}_k\left(\sum_{i=1}^d|\bar{X}^{(i)}_{k+1}-\bar{X}^{(i)}_k|\right)^p\left((1+V(\bar{X}_k))^{1-p\gamma}+\left|\bar{X}_{k+1}-\bar{X}_k\right|^{\frac{1}{\gamma}-p}\right) \\ &\leqslant \frac{d^{p-1}c^{\frac{1}{\gamma}-p+1}2^{\frac{1}{\gamma}-p}}{(p-1)!}\mathbb{E}_k\left|\bar{X}_{k+1}-\bar{X}_k\right|^p\left((1+V(\bar{X}_k))^{1-p\gamma}+\left|\bar{X}_{k+1}-\bar{X}_k\right|^{\frac{1}{\gamma}-p}\right) \\ &\leqslant c\tilde{\psi}\left(\left|\bar{b}_k\right|^p h^p+\|\bar{\sigma}_k\|^p h^{p/2}\right)(1+V(\bar{X}_k))^{1-p\gamma}+c\tilde{\psi}\big(\left|\bar{b}_k\right|^{\frac{1}{\gamma}}h^{\frac{1}{\gamma}}+\|\bar{\sigma}_k\|^{\frac{1}{\gamma}}h^{\frac{1}{2\gamma}}\big),\end{aligned}$$

where, similar to the proof of Theorem 2.5, $\tilde{\psi} := (d(m+1))^{\frac{1}{\gamma}-1}(2c)^{\frac{1}{\gamma}-p}/(p-1)!$. Now given (2.14), there exists $\tilde{\rho} = \tilde{\rho}(\mu) > 0$ s.t. one has $R^h V(\bar{X}_k) \leqslant \tilde{\rho}(1+V(\bar{X}_k))h$ for R^h defined in (2.12). This is obtained by the following estimate (with $i, j \in \mathbb{N}$):

$$
\begin{aligned}
R^h V(\bar{X}_k) &\leqslant \frac{1}{2}\left\|V^{(2)}(\bar{X})\right\| \left|\bar{b}_k\right|^2 h^2 \\
&\quad + \sum_{3\leqslant i+2j\leqslant p-1} \phi_{i+2j}\left\|V^{(i+2j)}(\bar{X}_k)\right\| \left|\bar{b}_k\right|^i \|\bar{\sigma}_k\|^{2j} h^{i+j} + \tilde{R}^h \\
&\leqslant \left(\frac{1}{2}c\mu^2 h^{1-2\beta_1} + \sum_{3\leqslant i+2j\leqslant p-1} \phi_{i+2j} c\mu^{i+2j} h^{(1/2-\beta_1)i+(1/2-2\beta_2)j}\right)(1+V(\bar{X}_k))h \\
&\quad + c\mu^p\tilde{\psi}(1+V(\bar{X}_k))\left(h^{p(1-\beta_1)-1} + h^{p(1/2-\beta_2)-1}\right)h \\
&\quad + c\mu^{\frac{1}{\gamma}}\tilde{\psi}(1+V(\bar{X}_k))\left(h^{\frac{1-\beta_1}{\gamma}-1} + h^{\frac{1-2\beta_2}{2\gamma}-1}\right)h \\
&\leqslant \tilde{\rho}(1+V(\bar{X}_k))h,
\end{aligned}
$$

for $\beta_1 \leqslant 1/2$ and $\beta_2 \leqslant 1/2 - 1/(p\wedge 4)$, and

$$\tilde{\rho} := \frac{1}{2}c\mu^2 + \frac{1}{2}c(p+1)\sum_{s=3}^{p-1}\mu^s\phi_s + 2c\mu^p\tilde{\psi},$$

where $\{\phi_s\}$ are the same positive constants as in (2.11). □

2.2. **Taming choices.** The results in the previous subsection give us some general integrability conditions for the tamed Euler scheme (1.7). A natural question would be if the assumptions in Theorem 2.5 and Proposition 2.9 can be satisfied for specific taming methods, i.e., for $V \in \mathcal{V}^p_\gamma$ whether for all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\mathcal{L}_t V(x) \leqslant \rho(1+V(x)) \implies \mathcal{L}^h_t V(x) \leqslant \bar{\rho}(1+V(x)), \tag{2.16}$$

for some $\rho, \bar{\rho} > 0$, and for all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$|b^h(t,x)|h^{\beta_1} \vee \|\sigma^h(t,x)\|h^{\beta_2} \leqslant \mu(1+V(x))^\gamma, \tag{2.17}$$

for some $\beta_1 \leqslant 1/2$ and $\beta_2 \leqslant 1/2 - 1/(p\wedge 4)$ hold.

2.2.1. *Balanced schemes.* Let us first look at the balanced schemes proposed in [9, 25, 27], which in general are of the form

$$b^h(t,x) := \frac{b(t,x)}{1+G_b(x,h)},\ \sigma^h(t,x) := \frac{\sigma(t,x)}{1+G_\sigma(x,h)},\ \forall t,x, \tag{2.18}$$

where $G_b, G_\sigma \geqslant 0$ and $G_b(\cdot,h), G_\sigma(\cdot,h) \to 0$ as $h\to 0$. In this case requirement (2.16) is interpreted as

$$
\begin{aligned}
\mathcal{L}^h_t V(x) &:= \nabla V(x)\cdot b^h(t,x) + \frac{1}{2}\mathrm{tr}\left[V^{(2)}(x)\sigma^h(\sigma^h)^\top(t,x)\right] \\
&= \frac{\nabla V(x)\cdot b(t,x)}{1+G_b(x,h)} + \frac{1}{2}\frac{\mathrm{tr}\left[V^{(2)}(x)\sigma\sigma^\top(t,x)\right]}{(1+G_\sigma(x,h))^2} \leqslant \rho(1+V(x)).
\end{aligned}
$$

Hence, condition (2.16) holds if either of the following conditions is satisfied:

(i) $1+G_b(x,h) = (1+G_\sigma(x,h))^2,\ \forall x,h$;
(ii) $1+G_b(x,h) \leqslant (1+G_\sigma(x,h))^2,\ \forall x,h$, if $\mathrm{tr}\left[V^{(2)}(x)\sigma\sigma^\top(t,x)\right] > 0,\ \forall x\in\mathbb{R}^d$ (this is the case for most Lyapunov functions).

One may consider case (i) and let, e.g.,

$$G_b(x,h) := 2CV(x)^{\kappa^*\gamma}h^{\beta_2} + C^2V(x)^{2\kappa^*\gamma}h^{2\beta_2} \quad \text{and} \quad G_\sigma(x,h) := CV(x)^{\kappa^*\gamma}h^{\beta_2}.$$

In order for (2.17) to hold we take $\beta_1 = 2\beta_2$, $C \geqslant K/\mu$ and $k^* \geqslant \kappa - 1$ so that

$$\|\sigma^h(t,x)\|h^{\beta_2} = \frac{\|\sigma(t,x)\|h^{\beta_2}}{1+CV(x)^{\kappa^*\gamma}h^{\beta_2}} \leqslant \frac{K(1+V(x))^{\kappa\gamma}h^{\beta_2}}{1+CV(x)^{\kappa^*\gamma}h^{\beta_2}} \leqslant \mu(1+V(x))^\gamma,$$

by Assumption 2.2. We also need to choose $C^2 \geqslant K/\mu$ so that

$$|b^h(t,x)|h^{\beta_1} \leqslant \frac{K(1+V(x))^{\kappa\gamma}h^{2\beta_2}}{1+2CV(x)^{\kappa^*\gamma}h^{\beta_2}+C^2V(x)^{2\kappa^*\gamma}h^{2\beta_2}} \leqslant \mu(1+V(x))^\gamma,$$

as $2\kappa^* \geqslant \kappa - 1$. Therefore we choose $\kappa^* \geqslant \kappa - 1$ and $C \geqslant (K/\mu) \vee 1$, which gives a reasonable taming method for the scheme to be bounded with respect to V.

2.2.2. *Projected schemes.* Motivated by a different type of projected scheme introduced in [1], where the authors considered 1-d SDEs with strong solutions on $[0,\infty)$, we propose a new type of Euler schemes:

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b(t_k,\bar{X}_k)h + \sigma(t_k,\bar{X}_k)\Delta W_{k+1}\right), \tag{2.19}$$

where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ defined s.t. $|\Pi(x)| = |x| \wedge h^{-r}$ with $r > 0$ to be chosen. For example, one can define $\Pi(x) = (\Pi_i(x_i))_{i=1}^d$ as a truncation, where $\Pi_i(x_i) = (-h^{-r} \vee x_i \wedge h^{-r})/\sqrt{d}$, or as a scaling: $\Pi(x) = \min\{1, h^{-r}|x|^{-1}\}x$. In order to ensure $|\bar{X}_k| \leqslant h^{-r}$ for all $k \geqslant 0$ we may assume $|X_0| \leqslant h^{-r}$, otherwise send in $\Pi(X_0)$ for the first iteration. Integrability of this scheme becomes straightforward for Lyapunov functions V satisfying $V \circ \Pi(\cdot) \leqslant V(\cdot)$. This additional condition does not significantly narrow the set $\mathcal{V}^p_\gamma$ of choices; in particular, it is usually satisfied for polynomials of the general form (1.6). In Section 3 we will show that these schemes preserve the exponential stability, which balanced schemes may fail to achieve.

Theorem 2.10. *Consider a projected scheme $\{\bar{X}_k\}$ defined by* (2.19)*. Let Assumption* 2.2 *hold and $V \in \mathcal{V}^p_\gamma$ s.t. for all $x \in \mathbb{R}^d$, $V(\Pi(x)) \leqslant V(x) \leqslant \nu(1+|x|^q)$ for some constants $\nu > 0$, $q \geqslant 1$. If there exists $\rho > 0$ s.t.*

$$\mathcal{L}_tV(x) \leqslant \rho(1+V(x)),\ \forall (t,x) \in [0,T]\times\mathbb{R}^d,$$

and $\mathbb{E}V(X_0) < \infty$, then $\{\bar{X}_k\}$ is V-integrable for $r \leqslant (1/2 - 1/(p\wedge 4))/((\kappa-1)q\gamma)$.

Proof. The same arguments in the proofs of Theorem 2.5 and Proposition 2.9 imply

$$\begin{aligned}\mathbb{E}_kV(\bar{X}_{k+1}) =& V\left(\Pi(\bar{X}_k + b(t_k,\bar{X}_k)h + \sigma(t_k,\bar{X}_k)\Delta W_{k+1})\right)\\ \leqslant& V(\bar{X}_k + b(t_k,\bar{X}_k)h + \sigma(t_k,\bar{X}_k)\Delta W_{k+1})\\ =& V(\bar{X}_k) + \mathcal{L}_{t_k}V(\bar{X}_k)h + R^hV(\bar{X}_k) + M_{k+1},\end{aligned} \tag{2.20}$$

where M_{k+1} is a local martingale, as the expression given in (2.12). This immediately shows that one need only work with $\mathcal{L}_tV(x)$, $b(t,x)$ and $\sigma(t,x)$ directly for $|x| \leqslant h^{-r}$. Thus (2.16) is redundant and we have

$$\begin{aligned}|b(t,x)|h^{\frac{1}{2}} \vee \|\sigma(t,x)\|h^{\frac{1}{2}-\frac{1}{p\wedge 4}} \leqslant& K(1+V(x))^{\kappa\gamma}h^{\frac{1}{2}-\frac{1}{p\wedge 4}}\\ \leqslant& 2K\nu\left(1+|x|^{q(\kappa-1)\gamma}\right)(1+V(x))^\gamma h^{\frac{1}{2}-\frac{1}{p\wedge 4}}\\ \leqslant& 4K\nu h^{\frac{1}{2}-\frac{1}{p\wedge 4}-r(\kappa-1)q\gamma}(1+V(x))^\gamma =: \mu(1+V(x))^\gamma,\end{aligned} \tag{2.21}$$

by choosing $r \leqslant (1/2 - 1/(p \wedge 4))/((\kappa - 1)q\gamma)$, which achieves (2.17). The result thus follows by Theorem 2.9. $\square$

2.2.3. *Strong convergence.* Now given the integrability (in particular, bounded moments) of the scheme we can explain how in general one may establish the strong convergence of (1.7) based on the results in [11] (Definition 3.1 and Corollary 3.12) and [27] (the proof of Lemma 3.2 and Theorem 2.1). Roughly speaking, both results state that provided that appropriate moment bounds ($V(\cdot) = |\cdot|^p$) for the tamed Euler scheme (1.7) are achieved, and that the strong and weak one-step differences against the standard Euler scheme are given by appropriate rates, then the tamed Euler scheme (1.7) converges to the solution of the SDE (1.1) in L^p. Precise statements are made in Appendix A.

Proposition 2.11. *Under appropriate assumptions (more precisely, let Assumption* A.1 *in Appendix* A *hold for $p = 2$ and some even number $p_0 > 2$ sufficiently large), the projected schemes* (2.19) *converge to the solution to the SDE* (2.1) *in L^2 with rate* $1/2$ *for $r < 1/(2(\kappa - 1))$.*

Corollary 2.12. *If a tamed Euler scheme* (1.7) *already satisfies the conditions for L^2-convergence (see Theorem* A.2 *in Appendix* A*), then the composed scheme*

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{2.22}$$

with an appropriate value of r chosen, also converges in L^2 with the same rate.

The proofs of both claims above can be found in Appendix B.

3. Asymptotic stability of equilibrium

Suppose for all $\mathcal{F}_0$-measurable X_0, there exists a unique (strong or weak) solution to the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,\ t \geqslant 0, \tag{3.1}$$

with drift and diffusion satisfying $b(t, x^*) \equiv 0,\ \sigma(t, x^*) \equiv 0,\ \forall t \geqslant 0$ for some $x^* \in \mathbb{R}^d$. When almost surely $X_0 = x^*$, the SDE has trivial solution $X_t = x^*$ a.s. Analogous to the concept of equilibria of ODEs, one can rewrite the SDE as

$$Y_t := X_t - x^* = \int_0^t b(s, Y_s + x^*)\mathrm{d}s + \sigma(s, Y_s + x^*)\mathrm{d}W_s =: \int_0^t \tilde{b}(s, Y_s)\mathrm{d}s + \tilde{\sigma}(s, Y_s)\mathrm{d}W_s,$$

and therefore assume, without loss of generality, the equilibrium $x^* = 0$ and

$$b(t, 0) \equiv 0,\ \sigma(t, 0) \equiv 0,\ \forall t \geqslant 0. \tag{3.2}$$

In the context of stability one still needs to model the growth of b and of σ in terms of the selected Lyapunov function in the class V_γ^p. But instead of $1 + V$ as in the integrability discussion before, we need a different assumption than Assumption 2.2 to model the growth conditions of b and σ, due to (3.2) and the possibility of V taking the form (1.6). More precisely,

Assumption 3.1. *There is a $V \in \mathcal{V}_\gamma^p$ and a non-negative function $U \in \mathcal{C}(\mathbb{R}^d)$, $\ker(U) = \{0\}$, s.t. $V(\cdot) \leqslant U(\cdot)$, and constants $K > 0,\ \kappa_{1,2} \geqslant 1$ s.t.*

$$|b(t, x)| \leqslant KU(x)^{\kappa_1\gamma},\ \|\sigma(t, x)\| \leqslant KU(x)^{\kappa_2\gamma},\ \forall t \geqslant 0,\ x \in \mathbb{R}^d.$$

In most cases the function U can be reasonably assumed to have polynomial growth in the sense

$$U(\cdot) \lesssim |\cdot|^{q_1} + |\cdot|^{q_2},$$

with $0 < q_1 \leqslant q_2$, which gives polynomial growth for b and σ; see Example 3.18.

Definition 3.2. The solution to the SDE (3.1) is said to be almost surely stable, if $X_t \to 0$ a.s. as $t \to \infty$, regardless of the value of X_0.

A time-discretisation $\{\bar{X}_k\}$, with step size $h \in (0, 1]$, of the solution to the SDE (3.1) is said to be almost surely stable, if for fixed step size $h > 0$, $\bar{X}_k \to 0$ a.s. as $k \to \infty$, regardless of the value of X_0.

Definition 3.3. Let $V \in \mathcal{V}^p_\gamma$. The solution to the SDE (3.1) is said to be exponentially stable with respect to V, or V-exponentially stable, with rate ρ, if $\mathbb{E}V(X_0) < \infty$ and there exists $\rho > 0$ s.t.

$$\mathbb{E}V(X_t) \leqslant e^{-\rho t}\mathbb{E}V(X_0),\ \forall t \geqslant 0.$$

A time-discretisation $\{\bar{X}_k\}$, with step size $h \in (0, 1]$, of the solution to the SDE (3.1) is said to be V-exponentially stable with rate $\tilde{\rho}$, if for fixed time-step $h > 0$ there exists $\tilde{\rho} > 0$ s.t.

$$\mathbb{E}V(\bar{X}_k) \leqslant e^{-\tilde{\rho} kh}\mathbb{E}V(X_0),\ \forall k \geqslant 0.$$

Remark 3.4. By the Borel-Cantelli lemma, V-exponential stability implies almost-sure stability.

First we check the conditions for stability of equilibrium on the SDE level. We first quote a simplified version of stochastic LaSalle theorem regarding the almost-sure stability of SDE (3.1) from [18, 21, 26]:

Theorem 3.5. *Let b and σ be locally Lipschitz in x and $V \in \mathcal{C}^2(\mathbb{R}^d)$ be non-negative. If $V(X_0) < \infty$ a.s. and there is a non-negative $z \in \mathcal{C}(\mathbb{R}^d)$ s.t.*

$$\mathcal{L}_t V(x) \leqslant -z(x),\ \forall (t, x) \in [0, \infty) \times \mathbb{R}^d, \tag{3.3}$$

then almost surely we have

$$\overline{\lim_{t\to\infty}} V(X_t) < \infty,\ \lim_{t\to\infty} z(X_t) = 0,$$

regardless of the value of X_0. In addition, if $\ker(z) = \{0\}$, then $X_t \to 0$ a.s. as $t \to \infty$.

Moreover, when $z(\cdot) \geqslant \rho V(\cdot)$ for some constant $\rho > 0$, then the solution X_t is V-exponentially stable.

One can use Theorem 3.5 to determine whether a system is almost surely stable. In particular, mean-square stability, i.e., $V(\cdot) = |\cdot|^2$, is the most popular choice. Before introducing stability results for tamed Euler schemes let us consider the following simple case.

Example 3.6. The solution to

$$\mathrm{d}X_t = -|X_t|^2 X_t \mathrm{d}t + |X_t|^2 \mathrm{d}W_t,\ |X_0|^2 < \infty \text{ a.s.}$$

is almost surely stable at 0.

Indeed one finds $\mathcal{L}|x|^2 = -2|x|^4 + |x|^4 = -|x|^4 =: -z(x)$, where $z(x) \geqslant 0$ and $z(x) = 0 \Leftrightarrow x = 0$. Note that in this case the solution is not necessarily mean-square exponentially stable, but Theorem 3.5 still holds.

Nevertheless, the stability property of numerical schemes is not immediate. One may, for example, consider the following balanced scheme:

$$b^h(x) = \frac{b(x)}{1+G(x)h^\alpha}, \ \sigma^h(x) = \frac{\sigma(x)}{1+G(x)h^\alpha}, \ 0 < \alpha \leqslant 1. \tag{3.4}$$

This is a simple version of (2.18). One first notices that by (2.5),

$$|\bar{X}_{k+1}|^2 = |\bar{X}_k|^2 + \mathcal{L}^h|\bar{X}_k|^2 h + |b^h(\bar{X}_k)|^2 h^2 + M_{k+1},$$

where M_{k+1} denotes a local martingale. We then calculate

$$\mathcal{L}^h|x|^2 = 2\frac{x \cdot b(x)}{1+G(x)h^\alpha} + \frac{\|\sigma(x)\|^2}{(1+G(x)h^\alpha)^2} \leqslant \frac{1}{1+G(x)h^\alpha}\mathcal{L}|x|^2 = -\frac{z(x)}{1+G(x)h^\alpha}.$$

One can choose $\alpha \leqslant 1$ and $G(x) := 2|x|^2$, s.t.

$$\begin{aligned} A^h(x) :=& \frac{z(x)}{1+G(x)h^\alpha} - \frac{|b(x)|^2 h}{(1+G(x)h^\alpha)^2} = \frac{|x|^4}{1+2|x|^2h^\alpha} - \frac{|x|^6 h}{(1+2|x|^2h^\alpha)^2} \\ \geqslant& \frac{2|x|^6h^\alpha - |x|^6 h}{(1+2|x|^2h^\alpha)^2} \geqslant \frac{|x|^6 h}{(1+2|x|^2h^\alpha)^2} \geqslant 0, \end{aligned}$$

and $A^h(x) = 0 \Leftrightarrow x = 0$. Thus we have

$$|\bar{X}_{k+1}|^2 \leqslant |\bar{X}_k|^2 - A^h(\bar{X}_k)h + M_{k+1} \leqslant |\bar{X}_0|^2 - \sum_{l=0}^{k} A^h(\bar{X}_l)h + \sum_{l=0}^{k} M_{l+1},$$

from which we deduce $\lim_{l\to\infty} A^h(\bar{X}_l) = 0$ a.s. and $\lim_{k\to\infty} \bar{X}_k = 0$ a.s. since $\ker(A^h) = \{0\}$. This can be seen by applying the following lemma, quoted from [19] (Theorem 1.3.9), to the non-negative process $V_k := V_0 - \sum_{l=0}^{k} A^h(\bar{X}_l)h + \sum_{l=0}^{k} M_{l+1}$ with $V_0 := |X_0|^2$.

Lemma 3.7. *Consider a non-negative stochastic process $\{V_k\}$ with representation*

$$V_k = V_0 + A_k^1 - A_k^2 + M_k,$$

where $\{A_k^1\}$ and $\{A_k^2\}$ are almost surely non-decreasing, predictable processes with $A_0^1 = A_0^2 = 0$, and $\{M_k\}$ is a local martingale adapted to $\{\mathcal{F}_{t_k}\}$ with $M_0 = 0$. Then

$$\{\lim_{k\to\infty} A_k^1 < \infty\} \subset \{\lim_{k\to\infty} A_k^2 < \infty\} \cap \{\lim_{k\to\infty} V_k < \infty \text{ exists}\} \text{ a.s.} \tag{3.5}$$

This is in fact a discrete version of Theorem 2.6.7 in [16] for special semi-martingales.

Now we investigate, for a general tamed explicit Euler scheme

$$\bar{X}_{k+1} = \bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1} \tag{3.6}$$

to be almost surely stable, what conditions one should impose on the tamed coefficients (b^h, σ^h). We first remark that a result on the preservation of almost-sure stability for the drift-implicit Euler scheme has been studied in [21], where the authors considered only $V = |\cdot|^2$.

Theorem 3.8. *Let $V \in \hat{\mathcal{V}}^p_\gamma := \mathcal{V}^p_\gamma \cap \{V^{(p+1)} \equiv 0\}$ be dominated by a non-negative function U and $\mathbb{E}V(X_0) < \infty$. Suppose there is a non-negative function $z^h \in \mathcal{C}(\mathbb{R}^d)$, s.t. for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$,*

$$\mathcal{L}^h_t V(x) \leqslant -z^h(x), \tag{3.7}$$

and a constant $0 < \mu \leqslant 1$ s.t.

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu \frac{(1+U(x))^\gamma z^h(x)}{1+U(x)+z^h(x)}. \tag{3.8}$$

Then for $\mu < \sqrt{2}/\sqrt{c + cd^{p-1}(p^2-1)}$, the scheme (3.6) satisfies:

$$\overline{\lim_{k\to\infty}} V(\bar{X}_k) < \infty, \ \lim_{k\to\infty} z^h(\bar{X}_k) = 0, \ a.s.,$$

and hence if $\ker(z^h) = \{0\}$, then $\bar{X}_k \to 0$ a.s. as $k \to \infty$.

Moreover, in the particular case where $z^h(\cdot) \geqslant \rho V(\cdot)$ for some $\rho > 0$, if there exists $\mu > 0$ s.t. for all $t \geqslant 0$ and $x \in \mathbb{R}^d$,

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu V(x)^\gamma, \tag{3.9}$$

then the scheme (3.6), with $\mu < \sqrt{2\rho}/\sqrt{c + cd^{p-1}(p^2-1)}$, admits V-exponential stability with a rate $\hat{\rho} \in (0,\rho)$, $\rho - \hat{\rho} = O(\mu^2)$.

Proof. The proof is almost identical to that of Theorem 2.5. However, by the estimate for the remainder (2.13), instead of (2.12) we have the following estimate:

$$\begin{aligned} V(\bar{X}_{k+1}) &= V(\bar{X}_k) + \mathcal{L}^h_{t_k} V(\bar{X}_k)h + R^h V(\bar{X}_k) + M_{k+1} \\ &\leqslant V(\bar{X}_k) - \mathcal{L}^h_{t_k} V(\bar{X}_k)h + \frac{1}{2}\left\|V^{(2)}(\bar{X}_k)\right\| \left|\bar{b}_k\right|^2 h^2 \\ &\quad + \sum_{3\leqslant i+2j\leqslant p} \phi_{i+2j} \left\|V^{(i+2j)}(\bar{X}_k)\right\| \left|\bar{b}_k\right|^i \left\|\bar{\sigma}_k\right\|^{2j} h^{i+2j} + M_{k+1}, \end{aligned} \tag{3.10}$$

where M_{k+1} is a local martingale and again $i,j \in \mathbb{N}$. Notice that all derivatives of V have upper bounds as defined in (2.2). Now apply (3.7) and (3.8) and we get (recall that $\gamma \leqslant 1/p$):

$$\begin{aligned} V(\bar{X}_{k+1}) &\leqslant V(\bar{X}_k) - z^h(\bar{X}_k)h + \frac{1}{2}c\mu^2(1+V(\bar{X}_k))^{1-2\gamma}\left(\frac{(1+U(\bar{X}_k))^\gamma z^h(\bar{X}_k)}{1+U(\bar{X}_k)+z^h(\bar{X}_k)}\right)^2 h \\ &\quad + \sum_{3\leqslant i+2j\leqslant p} \phi_{i+2j} c\mu^{i+2j}(1+V(\bar{X}_k))^{1-(i+2j)\gamma}\left(\frac{(1+U(\bar{X}_k))^\gamma z^h(\bar{X}_k)}{1+U(\bar{X}_k)+z^h(\bar{X}_k)}\right)^{i+2j} h^{\frac{i+j}{2}} \\ &\quad + M_{k+1} \\ &\leqslant V(\bar{X}_k) - z^h(\bar{X}_k)h + \frac{1}{2}c\mu^2 \frac{1+U(\bar{X}_k)}{\left(1+\frac{1+U(\bar{X}_k)}{z^h(\bar{X}_k)}\right)^2} h \\ &\quad + \sum_{3\leqslant i+2j\leqslant p} \phi_{i+2j} c\mu^{i+2j} \frac{1+U(\bar{X}_k)}{\left(1+\frac{1+U(\bar{X}_k)}{z^h(\bar{X}_k)}\right)^{i+2j}} h^{\frac{i+j}{2}} + M_{k+1}. \end{aligned}$$

By the fact that $1 + (1 + U(\bar{X}_k)/z^h(\bar{X}_k)) > 1$, the above expression reduces to

$$\begin{aligned} V(\bar{X}_{k+1}) \leqslant & V(\bar{X}_k) - z^h(\bar{X}_k)h + \frac{1}{2}c\mu^2 z^h(\bar{X}_k)h + \sum_{s=3}^{p} \sum_{i+2j=s} \phi_s c\mu^s z^h(\bar{X}_k)h^{\frac{s}{2}} \\ & + M_{k+1} \\ \leqslant & V(\bar{X}_k) - z^h(\bar{X}_k)h + \frac{1}{2}c\mu^2 z^h(\bar{X}_k)h + \sum_{s=3}^{p} \left\lfloor \frac{s+1}{2} \right\rfloor \phi_s c\mu^s z^h(\bar{X}_k)h^{\frac{s}{2}} \\ & + M_{k+1}. \end{aligned}$$

This implies that for all k,

$$\begin{aligned} V(\bar{X}_{k+1}) \leqslant & V(X_0) - \sum_{l=0}^{k} \left(1 - \frac{1}{2}c\mu^2 - \frac{1}{2}c(p+1)\sum_{s=3}^{p} \phi_s \mu^s h^{\frac{s}{2}-1}\right) z^h(\bar{X}_l)h \\ & + \sum_{l=0}^{k} M_{l+1}. \end{aligned} \tag{3.11}$$

One should then find a taming method with μ and h sufficiently small s.t.

$$a_s(\mu, h) := 1 - \frac{1}{2}c\mu^2 - \frac{1}{2}c(p+1)\sum_{s=3}^{p} \phi_s \mu^s h^{\frac{s}{2}-1} > 0,$$

so that $\sum_{l=0}^{k} a_s(\mu, h)z^h(\bar{X}_l)h$ is increasing in k. Now the same argument used at the end of Example 3.6 applies, according to Lemma 3.7, $\lim_{k\to\infty} V(\bar{X}_k) < \infty$ and $\sum_{l=0}^{\infty} a_s(\mu, h)z^h(\bar{X}_l)h < \infty$ a.s., implying that $\lim_{k\to\infty} z^h(\bar{X}_k) = 0$ a.s. Moreover when $z^h(x) = 0$ iff $x = 0$ one concludes that $\lim_{k\to\infty} \bar{X}_k = 0$ a.s. In fact, assuming $\mu, h \leqslant 1$, by Remark 2.8 one just needs to choose $\mu < 1/\sqrt{c/2 + cd^{p-1}(p^2-1)/2}$.

If $z^h(\cdot) \geqslant \rho V(\cdot)$ for some $\rho > 0$, then instead of (3.11) one runs the same calculation to get

$$\begin{aligned} V(\bar{X}_{k+1}) \leqslant & V(\bar{X}_k) - \left(\rho - \frac{1}{2}c\mu^2 - \frac{1}{2}c(p+1)\sum_{s=3}^{p} \phi_s \mu^s h^{\frac{s}{2}-1}\right) V(\bar{X}_k)h + M_{k+1} \\ =: & V(\bar{X}_k) - \tilde{\rho}V(\bar{X}_k)h + M_{k+1}. \end{aligned}$$

Choose μ and h sufficiently small s.t. $\tilde{\rho} > 0$. Taking expectation on both sides,

$$\mathbb{E}V(\bar{X}_{k+1}) \leqslant (1 - \hat{\rho}h)\mathbb{E}V(\bar{X}_k) \leqslant (1 - \hat{\rho}h)^{k+1}\mathbb{E}V(X_0) \leqslant e^{-\hat{\rho}(k+1)h}\mathbb{E}V(X_0) \to 0,$$

as $k \to \infty$. This can be done by choosing $\mu < \sqrt{\rho}/\sqrt{c/2 + cd^{p-1}(p^2-1)/2}$. □

Remark 3.9. In analogy to Proposition 2.9, Theorem 3.8 also holds for $V \in \mathcal{V}_\gamma^p$.

Remark 3.10. By (3.10), condition (3.8) can be weakened to

$$\left\|V^{(i+2j)}(x)\right\| \left|b^h(t,x)\right|^i \left\|\sigma^h(t,x)\right\|^{2j} h^{\frac{i+j}{2}} \leqslant \mu z^h(x), \ \forall t \geqslant 0, \ x \in \mathbb{R}^d, \tag{3.12}$$

for $i = 2, j = 0$ and all $i, j \in \mathbb{N}$ s.t. $3 \leqslant i + 2j \leqslant p$.

Remark 3.11. For $V \in \bar{\mathcal{V}}_\gamma^p$ condition (3.8) can be simplified to

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu \frac{U(x)^\gamma z^h(x)}{U(x) + z^h(x)}, \ \forall t \geqslant 0, \ x \in \mathbb{R}^d, \tag{3.13}$$

which also implies (3.12) for $0 < \mu \leqslant 1$.

Notice that (3.13) is reasonable since from (3.7) we have

$$\begin{aligned} z^h(x) \leqslant& \|\nabla V(x)\| \left|b^h(t,x)\right| + \frac{1}{2}\left\|V^{(2)}(x)\right\| \left\|\sigma^h(t,x)\right\|^2 \\ \leqslant& KU(x)^{1+(\kappa_1-1)\gamma} + KU(x)^{1+2(\kappa_2-1)\gamma}, \end{aligned} \tag{3.14}$$

which ensures no singularity in the right-hand term in (3.13).

3.1. **Balanced schemes.** Now with Theorem 3.8 one can determine whether a certain type of taming methods can preserve stability. For this we may derive some general conditions with respect to Lyapunov functions in $\mathcal{V}_\gamma^p$. Although most practically relevant Lyapunov functions can be found in the subset $\bar{\mathcal{V}}_\gamma^p$ defined in Remark 2.1, we may treat them as a special case.

Let us first investigate the following type of tamed schemes adopted by [9,25,27]:

$$b^h(t,x) = \frac{b(t,x)}{1+G(x)h^\alpha},\ \sigma^h(t,x) = \frac{\sigma(t,x)}{1+G(x)h^\alpha}, \tag{3.15}$$

for some $G(\cdot) \geqslant 0 < \alpha \leqslant 1$. Given the growth condition (3.14), which also holds for $z(\cdot)$, it turns out that by imposing some lower bounds on z one can recover almost-sure stability for (3.15).

Proposition 3.12. *Let Assumption* 3.1 *hold for $V \in \mathcal{V}_\gamma^p$ s.t. the coefficients of the SDE* (3.1) *satisfy*

$$\mathcal{L}_t V(x) \leqslant -z(x),\ \forall (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{3.16}$$

for some $0 \leqslant z \in \mathcal{C}(\mathbb{R}^d)$ satisfying

$$z(x) \geqslant \lambda(1+U(x))^{1-\gamma}\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right),\ \forall x \in \mathbb{R}^d, \tag{3.17}$$

for some $\lambda > 0$. Then by choosing $h < (\mu\lambda/K)^4$, $G(x) = C(U(x)^{(\kappa_1-1)\gamma} \vee U(x)^{(\kappa_2-1)\gamma})$, $C \geqslant 1/\left(\mu/K - h^{1/4}/\lambda\right)$, and $\alpha \leqslant 1/4$, the Euler scheme (3.6) *with tamed coefficients* (3.15) *preserves almost-sure stability for the trivial solution, where μ satisfies the requirement in Theorem* 3.8.

Proof. First one calculates

$$\begin{aligned} \mathcal{L}_t^h V(x) =& \nabla V(x) \cdot \frac{b(t,x)}{1+G(x)h^\alpha} + \frac{1}{2(1+G(x)h^\alpha)^2} \operatorname{tr}\left[\nabla^2 V(x) \sigma\sigma^\top(t,x)\right] \\ \leqslant& \frac{1}{1+G(x)h^\alpha}\mathcal{L}|x|^2 \leqslant -\frac{z(x)}{1+G(x)h^\alpha} =: -z^h(x), \end{aligned} \tag{3.18}$$

which satisfies $z^h(x) = 0 \Leftrightarrow x = 0$. Now one only needs to select appropriate $G(\cdot)$ and α s.t. condition (3.8) is satisfied, i.e.,

$$\begin{aligned} \frac{|b(t,x)|h^{\frac{1}{2}} \vee \|\sigma(t,x)\|h^{\frac{1}{4}}}{1+G(x)h^\alpha} \leqslant& \mu \frac{(1+U(x))^\gamma}{1+U(x)+\frac{z(x)}{1+G(x)h^\alpha}} \frac{z(x)}{1+G(x)h^\alpha} \\ \Leftrightarrow\ |b(t,x)|h^{\frac{1}{2}} \vee \|\sigma(t,x)\|h^{\frac{1}{4}} \leqslant& \frac{\mu(1+U(x))^\gamma}{\frac{1+U(x)}{z(x)} + \frac{1}{1+G(x)h^\alpha}}. \end{aligned}$$

One has an upper bound for the left-hand side above by Assumption 3.1 and a lower bound for the right-hand side by (3.17). Hence for the above inequality to hold, one can require

$$K\left(U(x)^{\kappa_1\gamma}\vee U(x)^{\kappa_2\gamma}\right)h^{1/4}\leqslant\frac{\mu(1+U(x))^\gamma}{\left(\frac{(1+U(x))^\gamma}{\lambda(U(x)^{\kappa_1\gamma}\vee U(x)^{\kappa_2\gamma})}+\frac{1}{1+G(x)h^\alpha}\right)}$$
$$\Leftrightarrow\ \mu(1+U(x))^\gamma\geqslant\frac{K}{\lambda}h^{1/4}(1+U(x))^\gamma+\frac{K\left(U(x)^{\kappa_1\gamma}\vee U(x)^{\kappa_2\gamma}\right)}{1+G(x)h^\alpha}h^{1/4}$$
$$\Leftrightarrow\ 1+G(x)h^\alpha\geqslant\frac{K\left(U(x)^{\kappa_1\gamma}\vee U(x)^{\kappa_2\gamma}\right)}{(\mu-Kh^{1/4}/\lambda)(1+U(x))^\gamma}h^{1/4},$$

where for fixed $\mu\leqslant 1$ we choose $h\leqslant h_0<(\mu\lambda/K)^4$. Thus by choosing $\alpha=1/4$ and $G(x):=C\left(U(x)^{(\kappa_1-1)\gamma}\vee U(x)^{(\kappa_2-1)\gamma}\right)$, the taming condition (3.13) is satisfied for $\mu\geqslant K\left(1/C+h^{1/4}/\lambda\right)$. Hence by Remark 3.11 and Theorem 3.8, the scheme (3.15) is almost surely stable when C and h are chosen sufficiently large and small, respectively. □

When $U(\cdot)=|\cdot|^{q_1}+|\cdot|^{q_2}$, $0<q_1\leqslant q_2$, one sees $U(\cdot)^{\kappa_1\gamma}\vee U(\cdot)^{\kappa_2\gamma}=|\cdot|^{(\kappa_1\wedge\kappa_2)q_1\gamma}+|\cdot|^{(\kappa_1\vee\kappa_2)q_2\gamma}$.

Corollary 3.13. *In the special case where $V(\cdot)=|\cdot|^p$ and $z(x)\gtrsim|x|^{\kappa_1+p-1}+|x|^{\kappa_2+p-1}$, one just needs to choose $\alpha=1/4$ and $G(x):=C(|x|^{\kappa_1-1}+|x|^{\kappa_2-1})$ with C sufficiently large.*

3.2. **Projected schemes.** In general there is no evident clue that the balanced scheme (3.15) can preserve moment-exponential stability, since the factor $1/(1+G(x)h^\alpha)$ has no positive lower bound. However, this can be resolved if at every step the scheme is projected onto a bounded range:

$$\bar X_{k+1}=\Pi\left(\bar X_k+b^h(t_k,\bar X_k)h+\sigma^h(t_k,\bar X_k)\Delta W_{k+1}\right),\tag{3.19}$$

where $\Pi:\mathbb{R}^d\to\mathbb{R}^d$ is a function such that $|\Pi(x)|=|x|\wedge h^{-r}$ for some $r>0$, $\forall x\in\mathbb{R}^d$, and b^h,σ^h are as in (3.15). By adopting this scheme one can immediately have z^h in (3.18) replaced by just z itself (with scaling):

$$z^h(x)=\frac{z(x)}{1+G(x)h^\alpha}=\frac{z(x)}{1+C|x|^{\kappa^*}h^\alpha}\geqslant\frac{z(x)}{1+Ch^{\alpha-rq\kappa^*}}\geqslant\frac{1}{1+C}z(x),\ \forall x\in\mathbb{R}^d,$$

by choosing $r<\alpha/(q\kappa^*)$, where $G(\cdot)$ is, for instance, as in Example 3.6, chosen to be $C|\cdot|^{\kappa^*}$ for some $C,\kappa^*>0$. This motivates the idea that (3.19) can remedy the shortcoming of the balanced scheme (3.15). Indeed, when $z(\cdot)\geqslant\rho V(\cdot)$, for the balance schemes one has

$$\mathcal{L}_t^hV(x)\leqslant-\rho\frac{V(x)}{1+G(x)h^\alpha},$$

where one sees that $z^h(\cdot)\gtrsim V(\cdot)$ is violated due to the unboundedness of $G(\cdot)$. However, this can be avoided by using projection (3.19).

Proposition 3.14. *Let Assumption 3.1 hold with $U(\cdot)=V(\cdot)\leqslant\nu(1+|\cdot|^q)$ for some $\nu,q>0$, and*

$$V(\Pi(x))\leqslant V(x),\ \forall x\in\mathbb{R}^d,\tag{3.20}$$

for a chosen projection Π. Suppose there exists $\rho>0$ s.t. for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$,

$$\mathcal{L}_tV(x)\leqslant-\rho V(x).$$

Then, with $G(x) := C(1+|x|^{(\check{\kappa}-1)q\gamma})$, $C \geqslant K\nu^{(\check{\kappa}-1)\gamma}/\mu$, $\alpha \leqslant 1/4$, $r < \alpha/((\check{\kappa}-1)q\gamma)$, *the scheme* (3.19) *is* V*-exponentially stable, where* $\check{\kappa} = \kappa_1 \vee \kappa_2$ *and* μ *satisfies the requirement in Theorem* 3.8.

Proof. Notice that by the same argument as in the proof of Theorem 2.10, we treat $\mathcal{L}_t^h(b^h, \sigma^h)$ as $\mathcal{L}_t(b^h, \sigma^h)$ restricted on $\{|x| \leqslant h^{-r}\}$, and b^h, σ^h in Theorem 3.8 are just as in (3.15). We first verify condition (3.9) by finding a sufficient condition:

$$\begin{aligned} &\frac{|b(t,x)|\, h^{1/2} \vee \|\sigma(t,x)\|\, h^{1/4}}{1+G(x)h^\alpha} \leqslant \mu V(x)^\gamma \\ \Leftarrow\ & K(V(x)^{\kappa_1\gamma} \vee V(x)^{\kappa_2\gamma})h^{1/4} \leqslant \mu V(x)^\gamma G(x) h^\alpha, \end{aligned}$$

which is achieved by choosing $\alpha \leqslant 1/4$, $G(x) := C(1+|x|^{(\check{\kappa}-1)q\gamma})$, $C \geqslant K\nu^{(\check{\kappa}-1)\gamma}/\mu$, assuming $\nu \geqslant 1$ without loss of generality. Also for $x \in \{|x| \leqslant h^{-r}\}$, we have $G(x) \leqslant C + Ch^{-r(\check{\kappa}-1)q\gamma}$, and thus

$$\mathcal{L}_t^h V(x) \leqslant -\frac{\rho}{1+G(x)h^\alpha} V(x) \leqslant -\frac{1}{1+Ch^\alpha + Ch^{\alpha - r(\check{\kappa}-1)q\gamma}} V(x) =: -\tilde{\rho} V(x),$$

for $\tilde{\rho} > 0$ if we choose $r < \alpha/((\check{\kappa}-1)q\gamma)$. Note that there is no restriction on the step size h. ☐

In fact, one can show that projecting the standard Euler scheme, with the original drift and diffusion,

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b(t_k, \bar{X}_k)h + \sigma(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{3.21}$$

is enough to inherit V-exponential stability under suitable conditions. This has been introduced earlier in (2.19), which by Proposition 2.11 is well-defined.

Proposition 3.15. *Let Assumption* 3.1 *hold with* $U = V$ *satisfying* (3.20) *for a chosen projection* Π *and* $V(\cdot) \leqslant \nu(1+|\cdot|^q)$ *for some* $\nu, q > 0$. *If there exists* $\rho > 0$ *s.t. for all* $(t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\mathcal{L}_t V(x) \leqslant -\rho V(x),$$

then with $r < 1/(4(\check{\kappa}-1)q\gamma)$, $h < (\mu/(2K\nu^{(\check{\kappa}-1)\gamma}))^\beta$, *the scheme* (3.21) *preserves* V*-exponential stability, where* $\beta = 1/4 - r(\check{\kappa}-1)q\gamma$ *and* μ *satisfies the requirement in Theorem* 3.8.

Proof. As shown in (2.20) condition (3.7) is redundant and one only needs to verify condition (3.9) for b and σ, i.e.,

$$|b(t,x)|\, h^{1/2} \vee \|\sigma(t,x)\|\, h^{1/4} \leqslant \mu V(x)^\gamma, \ \forall t, x. \tag{3.22}$$

The left-hand term has upper bound $K\left(V(x)^{\kappa_1\gamma}h^{1/2}\right) \vee \left(V(x)^{\kappa_2\gamma}h^{1/4}\right)$, and for scheme (3.21) we know $|\bar{X}_k| \leqslant h^{-r}$. Since $V(\cdot) \leqslant \nu\,(1+|\cdot|^q)$, one can require

$$\begin{aligned} &\mu V(x)^\gamma \geqslant KV(x)^\gamma \left(V(x)^{(\kappa_1-1)\gamma}h^{1/2}\right) \vee \left(V(x)^{(\kappa_2-1)\gamma}h^{1/4}\right) \\ &\Leftarrow \mu \geqslant K\nu^{(\check{\kappa}-1)\gamma}\left(1+|x|^{(\kappa_1-1)q\gamma}\right)h^{1/2} \vee \left(1+|x|^{(\kappa_2-1)q\gamma}\right)h^{1/4} \\ &\Leftarrow \mu \geqslant 2K\nu^{(\check{\kappa}-1)\gamma}\left(h^{1/2-r(\kappa_1-1)q\gamma} \vee h^{1/4-r(\kappa_2-1)q\gamma}\right) \end{aligned}$$

$$\Leftarrow \mu \geqslant 2K\nu^{(\check{\kappa}-1)\gamma}h^\beta. \tag{3.23}$$

Note that one can immediately let inequality (3.23) hold by choosing

$$r < \frac{1}{2(\kappa_1 - 1)q\gamma} \wedge \frac{1}{4(\kappa_2 - 1)q\gamma},\ h < h_0 \leqslant \Big(\frac{\mu}{2K\nu^{(\breve{\kappa}-1)\gamma}}\Big)^{1/\beta}, \tag{3.24}$$

for fixed μ. Therefore, the scheme (3.21) preserves V-exponential stability when such r is chosen and h is sufficiently small. $\square$

Moment-exponential stability immediately follows when $V(\cdot) = U(\cdot) = |\cdot|^p$, $q = p = 1/\gamma$.

On the other hand, scheme (3.21), as expected, also admits almost-sure stability given the same conditions as for scheme (3.15).

Proposition 3.16. *Let Assumption* 3.1 *hold with V satisfying* (3.20) *for a chosen projection Π. Suppose there exists $0 \leqslant z \in \mathcal{C}(\mathbb{R}^d)$ satisfying* (3.17)*, s.t. for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$, $\mathcal{L}_t V(x) \leqslant -z(x)$. If there exist $\nu, q > 0$ s.t. $U(\cdot) \leqslant \nu(1 + |\cdot|^q)$, then, with $r < (4(\breve{\kappa} - 1)q\gamma)^{-1}$, $h < \big(\mu\lambda/(K + 2\lambda K\nu^{(\breve{\kappa}-1)\gamma})\big)^{1/\beta}$, the scheme* (3.21) *is almost-surely stable, where $\beta = 1/4 - r(\breve{\kappa} - 1)q\gamma$ and μ satisfies the requirement in Theorem* 3.8.

Proof. Again one only needs to check condition (3.8) for b and σ for scheme (3.21), which satisfies $|\bar{X}_k| \leqslant h^{-r}$, $\forall k \geqslant 1$, with $z^h(\cdot) = z(\cdot)$. Indeed for all x (regardless of X_0 since we are only interested in the long-term behaviour),

$$|b(t,x)|h^{1/2} \vee \|\sigma(t,x)\|h^{1/4} \leqslant \mu \frac{(1 + U(x))^\gamma z(x)}{1 + U(x) + z(x)},$$

where, the left-hand term above has upper bound $Kh^{1/4}\big(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2)\gamma}\big)$, and the right-hand term minimizes when $z(x)$ reaches its lower bound in (3.17). Thus, due to $|x| \leqslant h^{-r}$, one can require

$$\begin{aligned}
&Kh^{1/4}\Big(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2)\gamma}\Big) \leqslant \mu \frac{\lambda(1 + U(x))^\gamma \big(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2)\gamma}\big)}{(1 + U(x))^\gamma + \lambda\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right)} \\
\Leftrightarrow\ &Kh^{1/4}\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right) \leqslant \left(\mu - \frac{K}{\lambda}h^{1/4}\right)(1 + U(x))^\gamma \\
\Leftarrow\ &\nu^{(\breve{\kappa}-1)\gamma} K h^{1/4}(1 + |x|^{(\breve{\kappa}-1)q\gamma}) \leqslant \mu - \frac{K}{\lambda}h^{1/4} \\
\Leftarrow\ &\left(\frac{K}{\lambda} + \nu^{(\breve{\kappa}-1)\gamma}K\right)h^{1/4} + \nu^{(\breve{\kappa}-1)\gamma}Kh^{1/4 - r(\breve{\kappa}-1)q\gamma} \leqslant \mu.
\end{aligned}$$

Set $r < (4(\breve{\kappa} - 1)q\gamma)^{-1}$ s.t. $\beta = 1/4 - r\breve{\kappa}q\gamma > 0$. One can then choose $h < \big(\mu\lambda/(K + 2\lambda K\nu^{(\breve{\kappa}-1)\gamma})\big)^{1/\beta}$, and hence almost-sure stability is achieved. $\square$

In most cases $V(\cdot) = U(\cdot) = |\cdot|^p$ is chosen, then $q = p = 1/\gamma$ and the conditions become much simpler.

Corollary 3.17. *In the special case where $V(\cdot) = |\cdot|^p$ and $z(x) \gtrsim |x|^{\kappa_1 + p - 1} + |x|^{\kappa_2 + p - 1}$, one just needs to choose r and h sufficiently small.*

3.3. **Other examples.**

Example 3.18. Consider the stochastic Lorenz equation [11] in $\mathbb{R}^3$ driven by a 3-d Wiener process:

$$b(x) = \begin{pmatrix} \alpha_1(x_2 - x_1) \\ -\alpha_1 x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \alpha_2 x_3 \end{pmatrix}, \ \sigma(x) = \begin{pmatrix} \beta_1 x_1 & 0 & 0 \\ 0 & \beta_2 x_2 & 0 \\ 0 & 0 & \beta_3 x_3 \end{pmatrix}, \tag{3.25}$$

where $2\alpha_1 > \beta_1^2,\ \beta_2^2 < 2,\ 2\alpha_2 > \beta_3^2$.

One can immediately check for the Lyapunov function $V(\cdot) = |\cdot|^2 \in \bar{\mathcal{V}}^2_{1/2}$:

$$\mathcal{L}|x|^2 = -(2\alpha_1 - \beta_1^2)x_1^2 - (2 - \beta_2^2)x_2^2 - (2\alpha_2 - \beta_3^2)x_3^2 \leqslant -\rho|x|^2,$$

where $\rho := (2\alpha_1 - \beta_1^2) \wedge (2 - \beta_2^2) \wedge (2\alpha_2 - \beta_3^2)$. According to Theorem 3.5 the system (3.18) is mean-square stable for the equilibrium. One can thus choose taming method (3.21) to preserve mean-square stability for the tamed Euler scheme. One observes

$$\begin{aligned} |b(x)| &= \sqrt{\alpha_1^2(x_2 - x_1)^2 + (\alpha_1 x_1 + x_2 + x_3)^2 + (x_1 x_2 - \alpha_2 x_3)^2} \leqslant K(|x| + |x|^2), \\ \|\sigma(x)\| &= \sqrt{\beta_1^2 x_1^2 + \beta_2^2 x_2^2 + \beta_3 x_3^2} \leqslant K|x|, \end{aligned}$$

where $K = \sqrt{5\alpha_1^2 + 4\alpha_1 + \alpha_2^2 + 4} \vee \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$. Then one can choose $U(x) = |x| + |x|^2,\ \kappa_1 = 2,\ \kappa_2 = 1$ for Assumption 3.1 to hold. Note that due to $p = 2$ in this case, one only needs the requirement on $b(t, x)$ as in (3.22). Hence according to Proposition 3.15, one needs to choose $r < 1/2$ and $h < (2K)^{-1/(1/2-r)}$ sufficiently small.

Example 3.19. Consider the following 2-d SDE with drift and diffusion similar to the stochastic Duffing-van der Pol oscillator [11]:

$$b(x) = \begin{pmatrix} x_2 - \alpha_1 x_1 \\ -\alpha_2 x_2 - x_1^3 \end{pmatrix}, \ \sigma(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta x_2 & 0 \end{pmatrix}, \tag{3.26}$$

where $\alpha_1 > 0,\ 2\alpha_2 > \beta^2$.

In this case one can set the Lyapunov function to be

$$V(x) = x_1^4 + 2x_2^2,$$

which is from a broader class $\hat{\mathcal{V}}^4_{1/4}$. Then one observes that

$$\mathcal{L}V(x) = -4\alpha_1 x_1^4 - (4\alpha_2 - 2\beta^2)x_2^2 \leqslant -\rho V(x),$$

where $\rho := 4 \wedge (4\alpha_2 - 2\beta^2)$. According to Theorem 3.5, the trivial solution of (3.26) is V-exponentially stable. Therefore we consider using the projected scheme (3.21), for which all conditions regarding (b^h, σ^h, z^h) are reduced to those of (b, σ, z) on the set $\{x : \ |x| \leqslant h^{-r}\}$. In this 2-d case one can, for example, define

$$\Pi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -h^{-r} \vee x_1 \wedge h^{-r} \\ -h^{-r} \vee x_2 \wedge h^{-r} \end{pmatrix},$$

s.t. $|\Pi x| \leqslant h^{-r}$ and (3.20) is satisfied. Hence in order to verify condition (3.9), one only needs to check for the points (x_1, x_2) satisfying $|x_1| \vee |x_2| \leqslant h^{-r}/\sqrt{2}$:

$$\begin{aligned}
|b(x)|h^{1/2} =& \left((\alpha_2+1)|x_2| + \alpha_1|x_1| + |x_1|^3\right) h^{1/2} \\
\leqslant& \frac{\alpha_2+1}{\sqrt[4]{2}}|x_2|^{1/2}h^{1/2-r/2} + \frac{\alpha_1+1}{2}|x_1|h^{1/2-2r} \\
\leqslant& \frac{\alpha_1 \vee \alpha_2 + 1}{2} h^{1/2-2r}(|x_1| + 2|x_2|^{1/2}) \leqslant \mu V(x)^{1/4}, \\
\|\sigma(x)\|h^{1/4} =& |\beta||x_2|h^{1/4} \leqslant \frac{|\beta|}{\sqrt[4]{2}} h^{1/4-r/2}|x_2|^{1/2} \leqslant \mu V(x)^{1/4},
\end{aligned}$$

where we choose $r < 1/4$ and $\mu := \max\{4(\alpha_1 \vee \alpha_2 + 1)h^{1/2-2r}/2, |\beta|h^{1/4-r/2}/\sqrt[4]{2}\} \leqslant 1$. Thus according to Theorem 3.8, the projected scheme (3.21) is exponentially stable with respect to V when h is chosen sufficiently small.

4. Preservation of non-negativity and comparison property

Apart from integrability and stability, there are some other properties on the SDE level that can be preserved via taming. For example, some SDEs have solution only in a bounded region, and especially in 1-d case two SDEs with the same diffusion can be compared, subject to some conditions.

4.1. **Non-negativity.** The issue of non-negativity preservation can be seen from the following 1-d linear SDE with non-zero constants μ and σ:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t. \tag{4.1}$$

The solution $X_t = X_0 \exp\left\{\left(\mu - \sigma^2/2\right)t + \sigma W_t\right\} \geqslant 0$ a.s. if $X_0 \geqslant 0$ a.s. However, this may not be the case for the standard Euler scheme

$$\bar{X}_{k+1} = (1+\mu h)\bar{X}_k + \sigma \bar{X}_k \Delta W_{k+1}.$$

More precisely, suppose that $\bar{X}_k \geqslant 0$ a.s., then for $\sigma > 0$,

$$\mathbb{P}(\bar{X}_{k+1} < 0) = \mathbb{P}\left(\Delta W_{k+1} < -\frac{1+\mu h}{\sigma}\right) > 0;$$

the same applies for $\sigma < 0$ due to the symmetry of the Gaussian distribution. However, one can avoid this situation by simply truncating the Wiener process. For SDEs with super-linear growth coefficients a little bit more work is needed to preserve non-negativity.

Non-negativity of the SDE can be regarded as a corollary of the comparison theorem to be mentioned later (Theorem 4.3). However, it turns out that for non-negativity the requirement on the drift is slightly weaker than that for the comparison theorem.

Lemma 4.1. *Given a* 1*-d SDE*

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \tag{4.2}$$

with $X_0 \geqslant 0$ *a.s. and* $\mathbb{E}X_0 < \infty$ *Suppose*

(i) *there exists a unique,* $|\cdot|^\kappa$*-integrable, strong solution of* (4.2) *for some* $\kappa \geqslant 1$*;*

(ii) $|b(t,x)| \vee |\sigma(t,x)|^2 \lesssim 1 + |x|^\kappa$, $\forall (t,x) \in [0,\infty) \times \mathbb{R}$, *and* b *satisfies the one-sided Lipschitz condition:*

$$(x-y)(b(t,x)-b(t,y)) \leqslant K|x-y|^2, \ \forall x,y \in \mathbb{R}, \ \forall t \geqslant 0; \tag{4.3}$$

(iii) $b(t,0) \geqslant 0, \ \sigma(t,0) = 0, \ \forall t \geqslant 0.$

Then $X_t \geqslant 0$ *a.s. for all* t.

This has been mentioned and heuristically explained in [2]. We give a proof of it in Appendix C.

Now consider a tamed Euler scheme for (4.2):

$$\hat{X}_{k+1} = \hat{X}_k + b^h(t_k, \hat{X}_k)h + \sigma^h(t_k, \hat{X}_k)\sqrt{h}\xi, \tag{4.4}$$

where $\xi \sim N(0,1)$. Non-negativity generally does not hold any more for $\hat{X}_k$, but one can recover this property by truncating the noise:

$$\zeta_h = (-A_h) \vee \xi \wedge A_h, \tag{4.5}$$

where one takes $A_h = \sqrt{2|\log h|}$. This idea is introduced in Section 1.3.4 in [23] for mean-square convergence of the implicit Euler scheme. We would like to point out that such a truncation can be used to preserve non-negativity.

Theorem 4.2. *Let the assumptions in Lemma* 4.1 *hold. If one can find a taming method* (b^h, σ^h) *such that* $b^h(\cdot, 0) \geqslant 0$ *and there exist* $\mu, \alpha > 0$*,*

$$|b^h(t,x) - b^h(t,0)|h^\alpha \vee |\sigma^h(t,x)|h^{\alpha/2} \leqslant \mu|x|, \ \forall (t,x) \in [0,\infty) \times \mathbb{R}, \tag{4.6}$$

then the tamed Euler scheme

$$\bar{X}_{k+1} = \bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\sqrt{h}\zeta_h, \tag{4.7}$$

is almost surely non-negative for $\alpha < 1$ *and* h, μ *sufficiently small.*

Proof. Rewrite the scheme (4.7) and inductively assume $\bar{X}_k \geqslant 0$ a.s.,

$$\begin{aligned} \bar{X}_{k+1} =& \bar{X}_k + b^h(t_k,0)h + (b^h(t_k,\bar{X}_k) - b^h(t_k,0))h + \sigma^h(t_k,\bar{X}_k))\sqrt{h}\zeta_h \\ \geqslant& \bar{X}_k \left(1 - \mu h^{1-\alpha} - \mu h^{1/2-\alpha/2} A_h\right), \end{aligned} \tag{4.8}$$

as $b^h(t,0) \geqslant 0$. In order for (4.8) to stay non-negative, we set $\alpha < 1$ and $h^{1-\alpha} + h^{1/2-\alpha/2}A_h \leqslant 1/\mu$. □

If $|b(\cdot,x) - b(\cdot,0)| \lesssim |x| + |x|^m$ for some $m \geqslant 1$, then (4.6) can be realised by a suitable balanced scheme as discussed in Subsection 2.2, for which the constant μ can be arbitrarily small.

Under the same assumption, condition (4.6) can also be realised by the projected scheme (3.21) by choosing an appropriate r. In fact, in this case one need not truncate the noise via (4.5). Instead one need only define a reasonable projection:

$$\Pi(x) = \left(0 \vee x_i \wedge h^{-r}\right)_{i=1,\cdots,d}, \tag{4.9}$$

where r is chosen s.t. Proposition 2.11 holds. This is similar to what is suggested in [1], where the authors ensure the approximation stay strictly positive. For that one just replaces the 0 above with h^r.

4.2. **Comparison result.** As an extension of non-negativity preservation, one can preserve comparison result for SDEs by applying taming techniques. It is known that two SDEs with the same diffusion and noise can be compared by the comparison theorem:

Theorem 4.3. *Consider two* 1*-d SDEs:*

$$\begin{aligned} dX_t =&\nu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ dY_t =&\lambda(t, Y_t)\,dt + \sigma(t, Y_t)\,dW_t, \end{aligned}$$

with $X_0 \leqslant Y_0$ *a.s. and* $\mathbb{E}|Y_0| \vee \mathbb{E}|X_0| < \infty$*. Assume the following conditions:*

(i) *each SDE has a unique,* $|\cdot|^\kappa$*-integrable, strong solution for some* $\kappa \geqslant 1$*;*
(ii) $|\nu(t,x)| \vee |\lambda(t,x)| \vee |\sigma(t,x)|^2 \lesssim 1 + |x|^\kappa,\ \forall (t,x) \in [0,\infty) \times \mathbb{R}$*;*
(iii) σ *is locally Hölder in* x *with exponent* $\alpha \geqslant 1/2$*;*
(iv) $\nu(t,x) \leqslant \lambda(t,x),\ \forall (t,x) \in [0,\infty) \times \mathbb{R}$*;*
(v) *either* λ *or* ν *satisfies one-sided Lipschitz condition* (4.3)*.*

Then $X_t \leqslant Y_t$ *a.s., for all* $t \geqslant 0$*.*

Although condition (v) is weaker than usually stated in the literature, e.g., Proposition 5.2.18 in [13], one still applies Itō's formula to the process $(Y_t - X_t)^-$ via smooth approximation (for which (iii) is needed), and the result follows from the same arguments adopted in Appendix C.

Now consider the Euler scheme for each equation:

$$\begin{aligned} \hat{X}_{k+1} =&\hat{X}_k + \nu(t_k, \hat{X}_k)h + \sigma(t_k, \hat{X}_k)\sqrt{h}\xi, \\ \hat{Y}_{k+1} =&\hat{Y}_k + \lambda(t_k, \hat{Y}_k)h + \sigma(t_k, \hat{Y}_k)\sqrt{h}\xi, \end{aligned}$$

where $\xi \sim N(0,1)$. In general the comparison property does not necessarily hold for $\hat{X}_k$ and $\hat{Y}_k$, but by truncating the noise using (4.5) it can be recovered.

Theorem 4.4. *Let the assumptions in Theorem* 4.3 *hold with* λ *satisfying one-sided Lipschitz condition* (4.3)*. If there is a taming method* (λ^h, σ^h) *s.t. there exist* $\mu, \alpha > 0$*, for all* $x, y \in \mathbb{R},\ t \geqslant 0$*,*

$$|\lambda^h(t,x) - \lambda^h(t,y)|h^\alpha \vee |\sigma^h(t,x) - \sigma^h(t,y)|h^{\alpha/2} \leqslant \mu|x-y|, \tag{4.10}$$

and $\nu^h(t,x) \leqslant \lambda^h(t,x)$*, then, for* $\alpha < 1$*,* ζ_h *defined as in* (4.5) *and* h, μ *sufficiently small, the tamed Euler schemes*

$$\begin{aligned} \bar{X}_{k+1} =&\bar{X}_k + \nu^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\sqrt{h}\zeta_h, \\ \bar{Y}_{k+1} =&\bar{Y}_k + \lambda^h(t_k, \bar{Y}_k)h + \sigma^h(t_k, \bar{Y}_k)\sqrt{h}\zeta_h, \end{aligned}$$

preserve the comparison property: $\bar{X}_k \leqslant \bar{Y}_k$ *a.s. for all* $k \in \mathbb{N}$*.*

Proof. Inductively suppose $\bar{Y}_k \geqslant \bar{X}_k$ a.s. and take the difference of the two SDEs:

$$\begin{aligned} \bar{Y}_{k+1} - \bar{X}_{k+1} \geqslant&(\bar{Y}_k - \bar{X}_k)(1 - \mu h^{1/2-\alpha/2}A_h) + (\lambda^h(\bar{Y}_k) - \nu^h(\bar{X}_k))h \\ \geqslant&(\bar{Y}_k - \bar{X}_k)(1 - \mu h^{1/2-\alpha/2}A_h) + (\lambda^h(\bar{Y}_k) - \lambda^h(\bar{X}_k))h \\ \geqslant&(\bar{Y}_k - \bar{X}_k)(1 - \mu h^{1-\alpha} - \mu h^{1/2-\alpha/2}A_h). \end{aligned}$$

Require $\alpha < 1$ and $h^{1-\alpha} + h^{1/2-\alpha/2}A_h \leqslant 1/\mu$, and the result follows. □

Condition $\nu^h(t,x) \leqslant \lambda^h(t,x)$ is usually immediately satisfied given $\nu(t,x) \leqslant \lambda(t,x)$, $\forall t,x$. Now let us investigate whether (4.10) is achievable.

If $\lambda(t,x)$ is differentiable in x and $|\partial_x\lambda(t,x)| \vee |\lambda(t,x)| \leqslant K(1+|x|^m)$ for some $K>0, m\geqslant 1$, one multiplies the taming factor $(1+G(x)h^\alpha)^{-1}$ with λ for $G(x)=C|x|^{m-1}$, $C\geqslant 1$, and by the mean value theorem, $|\lambda^h(t,x)-\lambda^h(t,y)| \leqslant |\partial_x\lambda^h(t,\xi)||x-y|$ for some ξ between x and y. Then by the chain rule,

$$\begin{aligned}|\partial_x\lambda^h(t,\xi)| \leqslant & \frac{|\partial_x\lambda^h(t,\xi)|(1+Ch^\alpha|\xi|^{m-1})+C|\lambda(t,\xi)|h^\alpha(m-1)|\xi|^{m-2}}{(1+Ch^\alpha|\xi|^{m-1})^2} \\ \leqslant & Km\frac{(1+|\xi|^{m-1})(1+Ch^\alpha|\xi|^{m-1})+C(1+|\xi|^m)h^\alpha|\xi|^{m-2}}{(1+Ch^\alpha|\xi|^{m-1})^2} \\ = & Km\frac{1+Ch^\alpha|\xi|^{m-2}+(1+Ch^\alpha)|\xi|^{m-1}+2Ch^\alpha|\xi|^{2m-2}}{1+2Ch^\alpha|\xi|^{m-1}+C^2h^{2\alpha}|\xi|^{2m-2}} \\ \leqslant & 2Km\frac{1+2|\xi|^{m-1}+h^\alpha|\xi|^{2m-2}}{Ch^\alpha(1+2|\xi|^{m-1}+h^\alpha|\xi|^{2m-2})} = \frac{2Km}{C}h^{-\alpha},\end{aligned}$$

where the last inequality holds for $Ch^\alpha \leqslant 1$. Thus $|\lambda^h(x)-\lambda^h(y)| \leqslant \mu|x-y|h^{-\alpha}$ where, by choosing a large C, the constant $\mu = 2Km/C$ can be arbitrarily small.

Appendix A. V-integrability applied to strong convergence

Strong L^p-convergence of explicit numerical methods of a SDE

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t,\ t\in[0,T], \tag{A.1}$$

has been well studied in the literature. Although this is not the main topic, we still summarise the framework of it in order to make this article self-contained. For simplicity we may consider L^2 convergence of an explicit numerical scheme $\bar{X}$. A typical proof adopted in [27] is based on splitting the one-step difference into two:

$$\begin{aligned}&X_{t_k,X(t_k)}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) \\ &= X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}) + X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}).\end{aligned}$$

The first difference is the one-step perturbation[6] of the solution X given different initial conditions, which by Lemma 2.2 in [27] can be handled provided that Assumption A.1 below holds. The second difference is the one-step error between $\bar{X}$ and X starting from the same initial condition, and that, as seen from the proof of Lemma 3.2 in [27], can be studied by further decomposing the error as

$$\begin{aligned}&X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) \\ &= X_{t_k,\bar{X}_k}(t_{k+1}) - \tilde{X}_{t_k,\bar{X}_k}(t_{k+1}) + \tilde{X}_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}),\end{aligned} \tag{A.2}$$

where $\tilde{X}$ is the standard Euler scheme

$$\tilde{X}_{t,x}(t+h) = x + b(t,x)h + \sigma(t,x)(W_{t+h}-W_t). \tag{A.3}$$

As is shown in [27], one can achieve optimal rates for the one-step error of (A.3) against the solution X_t without additional assumptions.

Alternatively, one can regard the local estimates for one-step perturbation and one-step error as special cases of what is stated in Theorem 1.2 in [10], which holds for two processes at a stopping time.

[6] Or one-step stability, not to be confused with the asymptotic stability of equilibrium.

Assumption A.1. *For SDE* (A.1)*, there exist* $p_0 \geqslant 2$ *and* $\kappa \geqslant 1$*, s.t. for all* $t, s \in [0,T]$, $x, y \in \mathbb{R}^d$*,*

(i) $\langle x-y, b(t,x)-b(t,y)\rangle + \frac{p_0-1}{2}\left\|\sigma(t,x)-\sigma(t,y)\right\|^2 \lesssim |x-y|^2$;
(ii) $|b(t,0)| \vee \|\sigma(t,0)\| \vee \sup_{q>0}\mathbb{E}|X_0|^q < \infty$;
(iii) $|b(t,x)-b(t,y)| \lesssim \left(1+|x|^{\kappa-1}+|y|^{\kappa-1}\right)|x-y|$ *and* $\|\sigma(t,x)-\sigma(t,y)\| \lesssim \left(1+|x|^{(\kappa-1)/2}+|y|^{(\kappa-1)/2}\right)|x-y|$;
(iv) $|b(t,x)-b(s,x)| \lesssim (1+|x|^{\kappa})\,|t-s|$ *and* $\|\sigma(t,x)-\sigma(s,x)\| \lesssim \left(1+|x|^{(\kappa+1)/2}\right)|t-s|$.

Note that (i) and (iii) above provides convenience for the strong and weak estimates of one-step perturbation $X_{t,x}(t+h)-X_{t,y}(t+h)$ for the SDE. If we let $V(\cdot) = |\cdot|^{p_0} \in \bar{\mathcal{V}}^{p_0}_{1/p_0}$, then by (i) and (ii),

$$\mathcal{L}V(x) = |x|^{p_0-2}\left(\langle x, b(t,x)\rangle + \frac{p_0-1}{2}\left\|\sigma(t,x)\right\|^2\right) \lesssim 1+V(x), \tag{A.4}$$

which together with the growth condition implied by (ii) and (iii),

$$|b(t,x)| \lesssim 1+|x|^{\kappa},\ \|\sigma(t,x)\| \lesssim 1+|x|^{(\kappa+1)/2},\ \forall t\in[0,T],\ x\in\mathbb{R}^d, \tag{A.5}$$

can make it possible for the tamed Euler scheme to achieve Theorem 2.5.

Although the argument (A.2) is hidden in the proof of the main result in [27], here we reformulate it as the following.

Theorem A.2. *Let Assumption* A.1 *hold for some even* $p_0 \in \mathbb{N}^+$*. If there is a real number* $p_1 \geqslant 1$ *s.t. a numerical scheme* $\{\bar{X}_k\}$ *with step size* h *is* $|\cdot|^{p_1}$*-integrable and its one-step error against the standard Euler scheme* (A.3) *satisfies for all* $q \geqslant 1$*,*

$$\mathbb{E}\left|\bar{X}_{t,x}(t+h)-\tilde{X}_{t,x}(t+h)\right|^q \lesssim (1+|x|^{\alpha})\,h^{\delta q},$$
$$\left|\mathbb{E}\bar{X}_{t,x}(t+h)-\mathbb{E}\tilde{X}_{t,x}(t+h)\right| \lesssim \left(1+|x|^{\alpha'}\right)h^{\delta+1/2},$$

for some $\alpha, \alpha' > 0$ *and* $\delta > 1/2$*, then for some* $p \in [1, p_1]$*,* $\max_k \left\|\bar{X}_k - X_{t_k}\right\|_{L^p(\Omega)} = O(h^{\delta-1/2})$.

Regarding moment bounds, Theorem 2.5 plays an essential role in controlling the highest (p_1) moments of $\{\bar{X}_k\}$ needed for L^p convergence. The relation between p_0, p_1 and p depends on what specific taming method one adopts and how one decomposes the global error. This has been studied for various balanced schemes in [9, 25, 27].

Appendix B. Proof of Proposition 2.11 and Corollary 2.12

B.1. **Proof of Proposition 2.11.**

Proof. Since both drift and diffusion are Lipschitz in t, we may assume $b(t,x) = b(x)$, $\sigma(t,x) = \sigma(x)$, $\forall t, x$. Notice that using a more precise growth condition (A.5) rather than Assumption 2.2, we can estimate $|b|h^{1/2}$ and $\|\sigma\|h^{1/4}$ separately in (2.21) and need only choose $r < 1/(2(\kappa-1))$, $q\gamma = 1$.

One only needs to check if $\delta = 1$ in Theorem A.2. Indeed the weak one-step error has estimate, by the Cauchy-Schwarz inequality and Chebyshev's inequality (denote $\Delta W := W_{t+h} - W_t$),

$$\begin{aligned}\left|\mathbb{E}\bar{X}_{t,x} - \mathbb{E}\tilde{X}_{t,x}\right| =& \left|\mathbb{E}\Pi(x + b(x)h + \sigma(x)\Delta W) - \mathbb{E}\left(x + b(x)h + \sigma(x)\Delta W\right)\right| \\ \leqslant& 2\mathbb{E}\left|x + b(x)h + \sigma(x)\Delta W\right| \mathbb{1}_{|x+b(x)h+\sigma(x)\Delta W|>h^{-r}} \\ \leqslant& K\left(\mathbb{E}|x + b(x)h + \sigma(x)\Delta W|^{2+\frac{3}{r}}\right)^{\frac{1}{2}} h^{\frac{3}{2}} \\ \leqslant& K\left(|x|^{1+\frac{3}{2r}} + \left((1+|x|)h^{\frac{1}{2}}\right)^{1+\frac{3}{2r}} + \left((1+|x|)h^{\frac{1}{4}}\right)^{1+\frac{3}{2r}}\right) h^{\frac{3}{2}} \\ \leqslant& K\left(1 + |x|^{1+\frac{3}{2r}}\right) h^{\frac{3}{2}},\end{aligned}$$

where we used (2.21) for $|x| \leqslant h^{-r}$. Similarly,

$$\begin{aligned}\mathbb{E}\left|\bar{X}_{t,x} - \tilde{X}_{t,x}\right|^2 =& \mathbb{E}\left|\Pi(x + b(x)h + \sigma(x)\Delta W) - x - b(x)h - \sigma(x)\Delta W\right|^2 \\ \leqslant& K\mathbb{E}\left|x + b(x)h + \sigma(x)\Delta W\right|^2 \mathbb{1}_{|x+b(x)h+\sigma(x)\Delta W|>h^{-r}} \\ \leqslant& K\left(\mathbb{E}|x + b(x)h + \sigma(x)\Delta W|^{4+\frac{4}{r}}\right)^{\frac{1}{2}} h^2 \\ \leqslant& K\left(|x|^{2+\frac{2}{r}} + \left((1+|x|)h^{\frac{1}{2}}\right)^{2+\frac{2}{r}} + \left((1+|x|)h^{\frac{1}{4}}\right)^{2+\frac{2}{r}}\right) h^2 \\ \leqslant& K\left(1 + |x|^{2+\frac{2}{r}}\right) h^2.\end{aligned}$$

This validates the L^2 convergence of (2.19). □

It is worth mentioning that if set $r = 1/(2\kappa)$, in the end (involving the Cauchy-Schwarz inequality) we have $p_1 = 8\kappa + 4$, which is almost the same p_1 needed for the specific balanced scheme introduced in [27]. However, as shown in Lemma 3.1 therein, $p_0 \geqslant O(p_1\kappa)$, whereas for the projected scheme proposed here we have $p_0 = p_1$. We leave the details of this calculation to the reader.

B.2. **Proof of Corollary 2.12.**

Proof. Suppose we already have a numerical scheme (b^h, σ^h) satisfying the conditions of Theorem A.2. For the composed scheme (2.22) to converge in L^2, one uses the same arguments adopted above to give the one-step estimates

$$\begin{aligned}\left|\mathbb{E}\Pi\left(x + b^h(x)h + \sigma^h(x)\Delta W\right) - \mathbb{E}\left(x + b^h(x)h + \sigma^h(x)\Delta W\right)\right| &= O(h^{\frac{3}{2}}), \\ \mathbb{E}\left|\Pi\left(x + b^h(x)h + \sigma^h(x)\Delta W\right) - x - b^h(x)h - \sigma^h(x)\Delta W\right|^2 &= O(h^2),\end{aligned}$$

and the result follows from the triangle inequality. □

Appendix C. Proof of Lemma 4.1

Proof. Consider $f(x) = x^- = \max(0, -x)$. Take a monotone sequence of smooth functions $\phi_n(x)$ s.t.

$$\phi_n(x) \to f(x),\ \phi_n'(x) \to -\mathbb{1}_{\{x<0\}}(x),\ \phi_n''(x) \to 0,$$

uniformly as $n \to \infty$, and the derivatives satisfy $|\phi_n'(x)| \leqslant 1$, $\phi_n''(x) \lesssim n^{-1}|x|^{-\kappa}$, for all $x \in \mathbb{R}$. Existence of such approximation can be found in, e.g., Section 5.2.C in [13]. By Itō's formula,

$$\phi_n(X_t) = \phi_n(X_0) + \int_0^t \left(\phi_n'(X_s)b(s,X_s) + \frac{1}{2}\phi_n''(X_s)\sigma^2(s,X_s)\right)\mathrm{d}s + \int_0^t \phi_n'(X_s)\sigma(s,X_s)\mathrm{d}W_s. \tag{C.1}$$

From (4.3) one can show that $b(t,x) = b_1(t,x) + b_2(t,x)$, where $b_1(t,x)$ is monotonically decreasing in x, and $b_2(t,x)$ is Lipschitz. One can choose, e.g., $b_2(t,x) = Kx$ and hence

$$\begin{aligned}(x-y)(b_1(t,x) - b_1(t,y)) &= (x-y)(b(t,x) - Kx - b(t,y) + Ky)\\ &= (x-y)(b(t,x) - b(x,y)) - K|x-y|^2 \leqslant 0.\end{aligned}$$

Taking expectation on both sides of (C.1) and letting $n \to \infty$, by the monotone and dominated convergence theorems we find that only one term remains:

$$\begin{aligned}\mathbb{E}X_t^- &\leqslant \mathbb{E}\int_0^t -\mathbb{1}_{\{X_s<0\}} b(s,X_s)\mathrm{d}s = \mathbb{E}\int_0^t -\mathbb{1}_{\{X_s<0\}}\left(b_1(s,X_s) + b_2(s,X_s)\right)\mathrm{d}s\\ &\leqslant \mathbb{E}\int_0^t -\mathbb{1}_{\{X_s<0\}}\left(b_1(s,0) + b_2(s,0) - K|X_s|\right)\mathrm{d}s\\ &= \mathbb{E}\int_0^t \mathbb{1}_{\{X_s<0\}}\left(-b(s,0) + K|X_s|\right)\mathrm{d}s.\end{aligned}$$

Note that $b(s,0) \geqslant 0$, thus

$$\mathbb{E}X_t^- \leqslant \int_0^t K\mathbb{E}X_s^-\mathrm{d}s \;\Rightarrow\; \mathbb{E}X_t^- = 0,\ \forall t \geqslant 0,$$

by Grönwall's inequality, which is validated by checking, for all $t \geqslant 0$,

$$\begin{aligned}\mathbb{E}X_t^- =& \mathbb{E}\mathbb{1}_{X_t<0}\left|X_0 + \int_0^t b(s,X_s)\mathrm{d}s + \sigma(s,X_s)\mathrm{d}W_s\right|\\ \leqslant& \mathbb{E}X_0 + \mathbb{E}\int_0^t |b(s,X_s)|\mathrm{d}s + C\left(\mathbb{E}\int_0^t \sigma^2(s,X_s)\mathrm{d}s\right)^{\frac{1}{2}} < \infty,\end{aligned}$$

for some constant $C > 0$, due to polynomial growth of b and σ^2 and bounded moments of X_t up to the same order. Thus we conclude that $X_t \geqslant 0$ a.s. □

References

[1] J.-F. Chassagneux, A. Jacquier, and I. Mihaylov, *An explicit Euler scheme with strong rate of convergence for non-Lipschitz sdes*, arXiv:1405.3561 (2014).

[2] D. J. Higham, X. Mao, and L. Szpruch, *Convergence, non-negativity and stability of a new Milstein scheme with applications to finance*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 8, 2083–2100, DOI 10.3934/dcdsb.2013.18.2083. MR3082312

[3] D. J. Higham, *A-stability and stochastic mean-square stability*, BIT **40** (2000), no. 2, 404–409, DOI 10.1023/A:1022355410570. MR1765744

[4] D. J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numer. Anal. **38** (2000), no. 3, 753–769, DOI 10.1137/S003614299834736X. MR1781202
[5] D. J. Higham, X. Mao, and A. M. Stuart, *Exponential mean-square stability of numerical solutions to stochastic differential equations*, LMS J. Comput. Math. **6** (2003), 297–313, DOI 10.1112/S1461157000000462. MR2051587
[6] D. J. Higham, X. Mao, and A. M. Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations*, SIAM J. Numer. Anal. **40** (2002), no. 3, 1041–1063, DOI 10.1137/S0036142901389530. MR1949404
[7] D. J. Higham, X. Mao, and C. Yuan, *Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations*, SIAM J. Numer. Anal. **45** (2007), no. 2, 592–609, DOI 10.1137/060658138. MR2300289
[8] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, *Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2011), no. 2130, 1563–1576, DOI 10.1098/rspa.2010.0348. MR2795791
[9] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, *Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients*, Ann. Appl. Probab. **22** (2012), no. 4, 1611–1641, DOI 10.1214/11-AAP803. MR2985171
[10] M. Hutzenthaler and A. Jentzen, *On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients*, arXiv:1401.0295 (2014).
[11] M. Hutzenthaler and A. Jentzen, *Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients*, Mem. Amer. Math. Soc. **236** (2015), no. 1112, v+99, DOI 10.1090/memo/1112. MR3364862
[12] M. Hutzenthaler, A. Jentzen, and X. Wang, *Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations*, arXiv:1309.7657 (2014).
[13] I. Karatzas and S. E. Shreve, *Browian Motion and Stochastic Calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940
[14] R. Z. Khasminski, *Stochastic Stability of Differential Equations*, Kluwer Academic Publishers, 1980.
[15] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Applications of Mathematics (New York), vol. 23, Springer-Verlag, Berlin, 1992. MR1214374
[16] R. Sh. Liptser and A. N. Shiryayev, *Theory of Martingales*, Mathematics and its Applications (Soviet Series), vol. 49, Kluwer Academic Publishers Group, Dordrecht, 1989. MR1022664
[17] X. Mao, *Stability of stochastic differential equations with respect to semimartingales*, Pitman Research Notes in Mathematics Series, vol. 251, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1991. MR1110584
[18] X. Mao, *Stochastic versions of the LaSalle theorem*, J. Differential Equations **153** (1999), no. 1, 175–195, DOI 10.1006/jdeq.1998.3552. MR1682267
[19] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Pub Ltd, 2007.
[20] X. Mao and L. Szpruch, *Strong convergence rates for backward Euler-Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients*, Stochastics **85** (2013), no. 1, 144–171, DOI 10.1080/17442508.2011.651213. MR3011916
[21] X. Mao and L. Szpruch, *Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients*, J. Comput. Appl. Math. **238** (2013), 14–28, DOI 10.1016/j.cam.2012.08.015. MR2972586
[22] J. C. Mattingly, A. M. Stuart, and D. J. Higham, *Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise*, Stochastic Process. Appl. **101** (2002), no. 2, 185–232, DOI 10.1016/S0304-4149(02)00150-3. MR1931266
[23] G. N. Milstein and M. V. Tretyakov, *Stochastic numerics for mathematical physics*, Scientific Computation, Springer-Verlag, Berlin, 2004. MR2069903
[24] S. Sabanis, *A note on tamed Euler approximations*, Electron. Commun. Probab. **18** (2013), no. 47, 10, DOI 10.1214/ECP.v18-2824. MR3070913
[25] S. Sabanis, *Euler approximations with varying coefficients: The case of superlinearly growing diffusion coefficients*, arXiv:1308.1796 (2014).

[26] Y. Shen, Q. Luo, and X. Mao, *The improved LaSalle-type theorems for stochastic functional differential equations*, J. Math. Anal. Appl. **318** (2006), no. 1, 134–154, DOI 10.1016/j.jmaa.2005.05.026. MR2210878

[27] M. V. Tretyakov and Z. Zhang, *A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications*, SIAM J. Numer. Anal. **51** (2013), no. 6, 3135–3162, DOI 10.1137/120902318. MR3129758

[28] F. Wu, X. Mao, and L. Szpruch, *Almost sure exponential stability of numerical solutions for stochastic delay differential equations*, Numer. Math. **115** (2010), no. 4, 681–697, DOI 10.1007/s00211-010-0294-7. MR2658159

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