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SEMI-INFINITE QUASI-TOEPLITZ MATRICES WITH APPLICATIONS TO QBD STOCHASTIC PROCESSES

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ABSTRACT. Denote by $\mathcal{W}_1$ the set of complex valued functions of the form $a(z) = \sum_{i=-\infty}^{+\infty} a_i z^i$ such that $\sum_{i=-\infty}^{+\infty} |ia_i| < \infty$. We call QT-matrix a quasi-Toeplitz matrix A, associated with a symbol $a(z) \in \mathcal{W}_1$, of the form $A = T(a) + E$, where $T(a) = (t_{i,j})_{i,j\in\mathbb{Z}^+}$ is the semi-infinite Toeplitz matrix such that $t_{i,j} = a_{j-i}$, for $i, j \in \mathbb{Z}^+$, and $E = (e_{i,j})_{i,j\in\mathbb{Z}^+}$ is a semi-infinite matrix such that $\sum_{i,j=1}^{+\infty} |e_{i,j}|$ is finite. We prove that the class of QT-matrices is a Banach algebra with a suitable sub-multiplicative matrix norm. We introduce a finite representation of QT-matrices together with algorithms which implement elementary matrix operations. An application to solving quadratic matrix equations of the kind $AX^2 + BX + C = 0$, encountered in the solution of Quasi-Birth and Death (QBD) stochastic processes with a denumerable set of phases, is presented where A, B, C are QT-matrices.

1. Introduction

Toeplitz matrices, i.e., matrices of the kind $T = (t_{i,j})$ such that $t_{i,j} = a_{j-i}$ for some sequence $\{a_k\}_{k\in\mathbb{Z}}$, are encountered in many applications. In certain stochastic processes, like in the analysis of random walks in the quarter plane [16], [24] or in the analysis of the tandem Jackson queue models [19], [32], one typically encounters semi-infinite Toeplitz matrices, where the indices of the entries range in the set $\mathbb{Z}^+$ of positive integers [32], [27], [20], [35]. In fact, these applications are modeled by a block tridiagonal generator Q of the form

$$Q = \begin{bmatrix} \widehat{A}_0 & \widehat{A}_1 & & & \\ A_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where the blocks A_{-1}, A_0, A_1, $\widehat{A}_0$, $\widehat{A}_1$ are semi-infinite tridiagonal and quasi-Toeplitz matrices. More specifically, they can be written as the sum of a tridiagonal Toeplitz matrix and a correction, that is, $A_i = \mathrm{trid}(\mu_i, \sigma_i, \nu_i) + F_i$, $i = -1, 0, 1$, $\widehat{A}_i = \mathrm{trid}(\widehat{\mu}_i, \widehat{\sigma}_i, \widehat{\nu}_i) + \widehat{F}_i$, $i = 0, 1$. Here, $\mathrm{trid}(\mu, \sigma, \nu)$ denotes a semi-infinite tridiagonal Toeplitz matrix with sub-diagonal, diagonal and super-diagonal entries μ, σ, ν, respectively, and F_i, $\widehat{F}_i$ denote matrices which are possibly nonzero only in the entries of indices $(1,1)$ and $(1,2)$. Generators Q in block tridiagonal form characterize the very wide class of Quasi-Birth-and-Death (QBD) processes [25]. Observe

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that the Toeplitz part $\operatorname{trid}(\mu, \sigma, \nu)$ is uniquely determined by the Laurent polynomial $a(z) = z^{-1}\mu + \sigma + z\nu$ which is called the *symbol* associated with the Toeplitz matrix.

An important problem in the analysis of a QBD process is to compute the minimal nonnegative solutions G and R of the associated matrix equations $A_{-1} + A_0X + A_1X^2 = 0$ and $X^2A_{-1} + XA_0 + A_1 = 0$, respectively; see for instance [25], [7]. If the matrix size is finite, the algorithms of Cyclic Reduction and of Logarithmic Reduction can be effectively used to solve these equations. Other techniques based on fixed point iterations can be used as well. Theoretically, these algorithms can also be used in the case where matrices are semi-infinite. However, in this case, difficult computational issues are encountered because performing arithmetic operations between Toeplitz matrices, generally causes the loss of sparsity and of the Toeplitz structure. This creates the nontrivial problem of storing infinite matrices, with apparently no structure, by means of a finite number of parameters. Another interesting issue is to figure out if the solutions R and G of the associated matrix equations share, in some form, part of the Toeplitz structure.

Let $\mathcal{W}$ be the Wiener class formed by the complex valued functions $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$, defined on the unit circle, such that $\|a\|_{\mathcal{W}} := \sum_{i\in\mathbb{Z}} |a_i|$ is finite. Moreover, define $\mathcal{W}_1 \subset \mathcal{W}$ the subclass of functions $a(z)$ such that $a'(z) = \sum_{i\in\mathbb{Z}} i a_i z^{i-1} \in \mathcal{W}$.

In this work we introduce the class $\mathcal{QT}$ of semi-infinite Quasi-Toeplitz (QT) matrices, that is, matrices of the form $A = T(a) + E$, where $T(a)$ is the Toeplitz matrix associated with the symbol $a(z) \in \mathcal{W}_1$, and the correction $E = (e_{i,j})$ is such that $\|E\|_{\mathcal{F}} := \sum_{i,j\in\mathbb{Z}^+} |e_{i,j}|$ is finite. This class provides a generalization of the structure encountered in QBD problems where the symbol $a(z)$ is a Laurent polynomial and the correction matrix has only a finite number of nonzero entries.

We prove that $\mathcal{QT}$ is a Banach algebra with the norm $\|\cdot\|_{\mathcal{QT}}$ such that $\|T(a) + E\|_{\mathcal{QT}} := \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}} + \|E\|_{\mathcal{F}}$ and $\|AB\|_{\mathcal{QT}} \le \|A\|_{\mathcal{QT}}\|B\|_{\mathcal{QT}}$ for any $A, B \in \mathcal{QT}$.

A nice property of the class $\mathcal{QT}$ is that for any $A \in \mathcal{QT}$ and for any $\epsilon > 0$ there exists $B \in \mathcal{QT}$, determined by a *finite number of parameters*, such that $\|A - B\|_{\mathcal{QT}} \le \epsilon$. This fact allows us to represent any matrix in $\mathcal{QT}$ with a finite number of parameters up to an arbitrarily small error in the QT-norm. We also introduce algorithms that execute the arithmetic operations between QT-matrices, and provide their Matlab implementation. This way, we may extend standard algorithms, valid for finite matrices, to the case of QT-matrices. In particular, we show how the algorithm of Cyclic Reduction [9] can be adapted to solve quadratic matrix equations of the kind $AX^2 + BX + C = 0$, where $A, B, C \in \mathcal{QT}$, which are encountered in QBD processes modeling random walks in the quarter plane [16], [24] and the Jackson Tandem Queue [19], [22]. Some numerical experiments performed with a set of problems presented in [22] and in [24] show the effectiveness of our approach.

The decomposition of a matrix as the sum of a Toeplitz part plus a correction has been used in the literature in different contexts. For instance, in [12, Example 2.28] matrices of the form $T(a) + E$ are considered where E is a compact operator with finite ℓ^2 operator norm and it is shown this set is a Banach algebra in L^2. It is worth pointing out that the boundedness of $\|E\|_2$ does not imply that $\|E\|_{\mathcal{F}} < \infty$ which is required for our computational goals. In the framework of Toeplitz preconditioning and in the analysis of asymptotic spectral properties of finite Toeplitz sequences the decomposition of a Toeplitz matrix in the form $T(a) + E + R$ is considered where $T(a)$ is banded, E and R are corrections of small norm and small rank, respectively; among the many papers on this subject we cite [4], [33], [36] with the many related references.

The analysis and the tools presented in this paper can be used for the effective numerical computation of matrix functions expressed by means of a Taylor expansion, like the exponential function, or expressed by means of an integral representation. These applications are shown in detail in [8], [10]. In particular, in [8] it is shown how this machinery can be extended to the case where matrices are finitely large. The problem of computing the exponential of finite Toeplitz matrices has been recently investigated in [21] relying on the concept of displacement rank.

It is worth pointing out that the definition of matrix function of a QT-matrix A, as well as the algorithms implementing the QT-matrix arithmetic, are somehow related to the decay properties of the coefficients of the matrices $T(f(a))$ and $E_{f(a)}$ such that $f(A) = T(f(a)) + E_{f(a)}$, and also to the numerical rank of the product of two Hankel matrices associated with analytic functions. The analysis of decay properties of matrix functions and of the singular values of some structured matrices, having a displacement rank structure, have recently received much interest and have been investigated in [1], [2], [3], [31].

The paper is organized as follows. In Section 2 we recall some preliminary properties which are needed in our analysis. In Section 3 we prove that $\mathcal{QT}$ is a Banach algebra. In Section 4 we describe the way in which matrix operations can be defined and implemented in $\mathcal{QT}$ and report a few notes on our Matlab implementation of QT-arithmetic. In Section 5 we present an application to solving a matrix equation encountered in QBD stochastic processes together with the results of some numerical experiments which confirm the effectiveness of the class $\mathcal{QT}$. Section 6 draws the conclusions.

2. Notation and preliminaries

Denote by $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ the unit circle in the complex plane, and by $\mathcal{W}$ the Wiener class formed by the functions $a(z) = \sum_{i=-\infty}^{+\infty} a_i z^i : \mathbb{T} \to \mathbb{C}$ such that $\sum_{i=-\infty}^{+\infty} |a_i| < +\infty$. Recall that $\mathcal{W}$ is a Banach algebra, that is, a vector space closed under multiplication, endowed with the norm $\|a\|_{\mathcal{W}} := \sum_{i\in\mathbb{Z}} |a_i|$ which makes the space complete and such that $\|ab\|_{\mathcal{W}} \le \|a\|_{\mathcal{W}} \|b\|_{\mathcal{W}}$ for any $a(z), b(z) \in \mathcal{W}$. We refer the reader to the first chapter of the book [12] for more details.

In the following, we denote by $a^+(z)$ and by $a^-(z)$ the power series defined by the coefficients of $a(z)$ with positive and with negative powers, respectively, that is, $a^+(z) = \sum_{i\in\mathbb{Z}^+} a_i z^i$ and $a^-(z) = \sum_{i\in\mathbb{Z}^+} a_{-i} z^i$, so that $a(z) = a_0 + a^+(z) + a^-(z^{-1})$. We associate with the Laurent series $a(z)$, and with the power series $b(z) = \sum_{i=0}^{\infty} b_i z^i$ the following semi-infinite matrices,

$$\begin{aligned} T(a) &= (t_{i,j})_{i,j}, \quad t_{i,j} = a_{j-i}, \\ H(b) &= (h_{i,j})_{i,j}, \quad h_{i,j} = b_{i+j-1}, \qquad i, j \in \mathbb{Z}^+, \end{aligned}$$

respectively. Observe that $T(a)$ is a Toeplitz matrix while $H(b)$ is Hankel.

Finally, denote by $\mathcal{F}$ the class of semi-infinite matrices $F = (f_{i,j})_{i,j\in\mathbb{Z}^+}$ such that $\|F\|_{\mathcal{F}} := \sum_{i,j\in\mathbb{Z}^+} |f_{i,j}|$ is finite. The norm that we use in this case is just the 1-norm if we look at the matrix F as an infinite vector.

Observe that $\mathcal{F}$ is a vector space, closed under rows-by-columns multiplication, and $\|F\|_{\mathcal{F}}$ is a norm over $\mathcal{F}$ which is endowed of the sub-multiplicative property. In the following, we write $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ to denote the linear space $\mathcal{F}$ endowed with the norm $\|\cdot\|_{\mathcal{F}}$. We have the following.

Lemma 2.1. $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ *equipped with matrix sum and multiplication is a Banach algebra over* $\mathbb{C}$.

Proof. We need to show that given $E, F \in \mathcal{F}$ and $\alpha \in \mathbb{C}$ it holds that

(i) $\alpha E \in \mathcal{F}$,
(ii) $E + F \in \mathcal{F}$,
(iii) $EF \in \mathcal{F}$ and $\|EF\|_{\mathcal{F}} \le \|E\|_{\mathcal{F}} \|F\|_{\mathcal{F}}$,
(iv) $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is a complete metric space.

Clearly, $\sum_{i,j\in\mathbb{Z}^+} |\alpha e_{i,j}| = |\alpha| \sum_{i,j\in\mathbb{Z}^+} |e_{i,j}| < +\infty$ which proves (i). By the triangular inequality one obtains that $\sum_{i,j\in\mathbb{Z}^+} |e_{i,j} + f_{i,j}| \le \sum_{i,j\in\mathbb{Z}^+} |e_{i,j}| + \sum_{i,j\in\mathbb{Z}^+} |f_{i,j}| < +\infty$ which implies (ii). If $H = EF = (h_{i,j})$, then $h_{i,j} = \sum_{r\in\mathbb{Z}^+} e_{i,r} f_{r,j}$ so that, defining $\alpha_r = \sum_{i\in\mathbb{Z}^+} |e_{i,r}|$, and $\beta_r = \sum_{j\in\mathbb{Z}^+} |f_{r,j}|$, for the quantity $\|EF\|_{\mathcal{F}} = \sum_{i,j\in\mathbb{Z}^+} |h_{i,j}|$ we have

$$\|EF\|_{\mathcal{F}} \le \sum_{i,j,r\in\mathbb{Z}^+} |e_{i,r}| \cdot |f_{r,j}| = \sum_{r\in\mathbb{Z}^+} \alpha_r \beta_r \le \left(\sum_{r\in\mathbb{Z}^+} \alpha_r \right) \left(\sum_{r\in\mathbb{Z}^+} \beta_r \right) = \|E\|_{\mathcal{F}} \cdot \|F\|_{\mathcal{F}},$$

which shows (iii). Finally, we observe that any matrix $E \in \mathcal{F}$ can be viewed as a vector $v = (v_k)_{k\in\mathbb{Z}^+}$ obtained by suitably ordering the entries $e_{i,j}$. Moreover, the norm $\|\cdot\|_{\mathcal{F}}$ corresponds to the ℓ^1-norm in the space of infinite vectors having finite sum of their moduli. This way, the space $\mathcal{F}$ actually coincides with ℓ^1, which is a Banach space. Thus, we get (iv). $\square$

Observe that the condition $\|F\|_{\mathcal{F}} < +\infty$ implies that for any $\epsilon > 0$ there exists an integer $k > 0$ such that $\sum_{\max(i,j)\ge k} |f_{i,j}| < \epsilon$, that is, the entries of the matrix F decay to zero as either $i \to \infty$ or $j \to \infty$ so that F can be approximated with an arbitrarily small error by a matrix with finite support. This property is of fundamental importance in order to represent the matrix F with a finite number of parameters up to an error which is smaller than a given bound, say smaller than the machine precision.

Any semi-infinite matrix $S = (s_{i,j})_{i,j\in\mathbb{Z}^+}$ can be viewed as a linear operator, acting on semi-infinite vectors $v = (v_i)_{i\in\mathbb{Z}^+}$, which maps the vector v onto the vector u such that $u_i = \sum_{j\in\mathbb{Z}^+} s_{i,j} v_j$, provided that the results of the summations are finite.

Indeed, the matrices $F \in \mathcal{F}$ define linear operators on the space ℓ^1 of semi-infinite vectors $v = (v_i)$ such that $\|v\|_1 = \sum_{i\in\mathbb{Z}^+} |v_i|$ is finite, since

$$\sum_{i\in\mathbb{Z}^+} |\sum_{j\in\mathbb{Z}^+} f_{i,j} v_j| \le \sum_{i,j\in\mathbb{Z}^+} |f_{i,j} v_j| \le \sum_{i,j\in\mathbb{Z}^+} |f_{i,j}| \cdot \sup_k |v_k|,$$

which is finite as the product of two finite terms.

For any integer $p \ge 1$, we may wonder if also the matrices $T(a)$, $H(a^+)$, and $H(a^-)$ define linear operators acting on the Banach space ℓ^p formed by vectors v such that the ℓ^p-norm $\|v\|_p = (\sum_{i\in\mathbb{Z}^+} |v_i|^p)^{1/p}$ is finite. In this case we may evaluate the p-norm of the operator S (operator norm) as $\|S\|_p := \sup_{\|v\|_p=1} \|Sv\|_p$. The answer to this question is given by the following result of [12] which relates the matrix $T(a)T(b)$ with $T(ab)$, $H(a^-)$ and $H(a^+)$.

Theorem 2.2. *For $a(z), b(z) \in \mathcal{W}$ let $c(z) = a(z)b(z)$. Then we have*

$$T(a)T(b) = T(c) - H(a^-)H(b^+).$$

Moreover, for any $a(z) \in \mathcal{W}$ and for any $p \geq 1$, including $p = \infty$, we have

$$\|T(a)\|_p \leq \|a\|_{\mathcal{W}}, \quad \|H(a^-)\|_p \leq \|a^-\|_{\mathcal{W}}, \quad \|H(a^+)\|_p \leq \|a^+\|_{\mathcal{W}}.$$

A direct consequence of the above result is that the product of two Toeplitz matrices can be written as a Toeplitz matrix plus a correction whose ℓ^p-norm is bounded by $\|a\|_{\mathcal{W}}\|b\|_{\mathcal{W}}$.

A similar property holds for matrix inversion in the case where the function $a(z)$ is nonzero for $|z| = 1$ and its winding number is zero. In fact, in this case we may apply another classical result (we refer to the book [11] for more details) which relates the invertibility of the operator $T(a)$ to the winding number κ of $a(z)$, that is, the (integer) number of times that the complex number $a(\cos\theta + \mathbf{i}\sin\theta)$, where $\mathbf{i}^2 = -1$, winds around the origin as θ moves from 0 to 2π.

Theorem 2.3 ([17], [13])**.** *Let $a(z)$ be a continuous function from $\mathbb{T}$ in $\mathbb{C}$. Then the linear operator $T(a)$ is invertible if and only if the winding number of $a(z)$ is zero and $a(z)$ does not vanish on $\mathbb{T}$.*

Thus, under the assumptions of the above theorem, it follows that $T(a)$ is invertible and we have $T(a)^{-1} = T(a^{-1}) + E$, where $\|E\|_p$ is bounded from above by a constant [12, Proposition 1.18].

In the analysis that we are going to perform in the next section, the above properties concerning the ℓ^p-norms are very useful, but are not enough to arrive at an algorithmic implementation concerning Toeplitz and quasi-Toeplitz matrices. In fact, our request is to write the product and the inverse of Toeplitz matrices as a Toeplitz matrix plus a correction whose entries have a decay along the diagonals. In fact, in this case, the correction can be approximated with any precision by using a finite number of parameters. Observe that, for the matrix product, this property is satisfied if $E = H(a^-)H(b^+) \in \mathcal{F}$ in view of Theorem 2.2.

Finally, we recall a result concerning the Wiener-Hopf factorization of $a(z)$ which will be useful next [13, Theorem 1.14].

Theorem 2.4. *Let $a(z) \in \mathcal{W}$ be a function which does not vanish for $z \in \mathbb{T}$ and such that its winding number is κ. Then $a(z)$ admits the Wiener-Hopf factorization*

$$a(z) = u(z)z^\kappa \ell(z),$$

where $u(z) = \sum_{i=0}^{\infty} u_i z^i$, $\ell(z) = \sum_{i=0}^{\infty} \ell_i z^{-i}$ are in $\mathcal{W}$ and $u(z)$, $\ell(z^{-1})$ do not vanish in the closed unit disk. If $\kappa = 0$, the factorization is said canonical.

3. Quasi-Toeplitz matrices

In this section we introduce the classes of quasi-Toeplitz matrices and analyze their properties.

Definition 3.1. We denote $\mathcal{W}_1 = \{a(z) \in \mathcal{W} : \ a(z) \text{ continuous, and } a'(z) \in \mathcal{W}\}$, and define the norm

$$\|a\|_{\mathcal{W}_1} = \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}}.$$

We recall that $\mathcal{W}_1$ is a Banach algebra with the norm $\|a\|_{\mathcal{W}_1}$; see [13].

Definition 3.2. We say that the semi-infinite matrix A is a *quasi-Toeplitz matrix (QT-matrix)* if it can be written in the form

$$A = T(a) + E,$$

where $a(z) = \sum_{i=-\infty}^{+\infty} a_i z^i \in \mathcal{W}_1$ and $E = (e_{i,j}) \in \mathcal{F}$. We refer to $T(a)$ as the Toeplitz part of A, and to E as the correction. We denote by $\mathcal{QT}$ the class of QT-matrices. Moreover, we define the following norm on $\mathcal{QT}$

$$\|T(a) + E\|_{\mathcal{QT}} := \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}} + \|E\|_{\mathcal{F}}.$$

Observe that given $A \in \mathcal{QT}$ there is a unique way to decompose it in the sense of Definition 3.2. In fact, suppose by contradiction that there exist $a_1(z), a_2(z) \in \mathcal{W}_1$ and $E_1, E_2 \in \mathcal{F}$ with $a_1 \neq a_2$ and $E_1 \neq E_2$ such that

$$A = T(a_1) + E_1 = T(a_2) + E_2.$$

Then we should have $E_1 - E_2 = T(a_2) - T(a_1) = T(a_2 - a_1)$, hence $\|E_1 - E_2\|_{\mathcal{F}} = \|T(a_2 - a_1)\|_{\mathcal{F}}$. On the other hand, since $T(a_2 - a_1) \neq 0$ we have $\|T(a_2 - a_1)\|_{\mathcal{F}} = \infty$, which contradicts the fact that $E_1 - E_2 \in \mathcal{F}$.

Lemma 3.3. *The set $\mathcal{QT}$ endowed with the norm $\|\cdot\|_{\mathcal{QT}}$ is a Banach space.*

Proof. The set of quasi-Toeplitz matrices is clearly isomorphic to the direct sum $\mathcal{QT} \simeq \mathcal{W}_1 \oplus \mathcal{F}$. Since both $\mathcal{W}_1$ and $\mathcal{F}$ are Banach spaces the composition of the 1-norm of $\mathbb{R}^2$ with the vector valued function $T(a) + E \to (\|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}}, \|E\|_{\mathcal{F}})$ makes $\mathcal{W}_1 \oplus \mathcal{F}$ a complete metric space. $\square$

The class $\mathcal{QT}$ includes all the matrices encountered in QBD processes, formed by a banded Toeplitz part, and by a correction E such that $e_{i,j} = 0$ for $i, j > k$ for some integer k.

The goal of this section is to prove that the class of QT-matrices is a normed matrix algebra, i.e., a vector space closed under matrix multiplication. We provide a few results which are useful to prove this property. The following lemma shows that the product of two semi-infinite Toeplitz matrices associated with symbols in $\mathcal{W}_1$ belongs to $\mathcal{QT}$.

Lemma 3.4. *Let $a(z), b(z) \in \mathcal{W}_1$, and set $c(z) = a(z)b(z)$. Then $T(a)T(b) = T(c) + E_c$, where $E_c \in \mathcal{F}$; moreover,*

$$\|E_c\|_{\mathcal{F}} \leq \|H(a^-)\|_{\mathcal{F}} \cdot \|H(b^+)\|_{\mathcal{F}} = \sum_{i \in \mathbb{Z}^+} i|a_{-i}| \sum_{i \in \mathbb{Z}^+} i|b_i|.$$

Proof. From Theorem 2.2 we deduce that $T(a)T(b) = T(c) + E_c$ where we set $E_c = -H(a^-)H(b^+)$. Let us prove that $H(a^-), H(b^+) \in \mathcal{F}$. We have $\|H(b^+)\|_{\mathcal{F}} = \sum_{i,j \in \mathbb{Z}^+} |b_{i+j-1}|$. Setting $k = i+j-1$ we may write $\|H(b^+)\|_{\mathcal{F}} = \sum_{k \in \mathbb{Z}^+} k|b_k|$ which is finite since $b(z) \in \mathcal{W}_1$. The same argument applies to $H(a^-)$. In view of Lemma 2.1, $\mathcal{F}$ is a normed matrix algebra therefore $\|E_c\|_{\mathcal{F}} \leq \|H(a^-)\|_{\mathcal{F}} \cdot \|H(b^+)\|_{\mathcal{F}} < +\infty$. $\square$

Remark 3.5. Observe that the quantities $\sum_{i \in \mathbb{Z}^+} i|a_{-i}|$ and $\sum_{i \in \mathbb{Z}^+} i|b_i|$ coincide with the $\mathcal{W}$-norms of the first derivatives of the functions $a^-(z)$ and $b^+(z)$, respectively. This way we may rewrite the bound given in Lemma 3.4 as

$$\|E_c\|_{\mathcal{F}} \leq \|(a^-)'\|_{\mathcal{W}} \|(b^+)'\|_{\mathcal{W}} \leq \|a'\|_{\mathcal{W}} \|b'\|_{\mathcal{W}}. \tag{3.1}$$

The condition $a(z), b(z) \in \mathcal{W}_1$ is needed to prove Lemma 3.4, as it is demonstrated by the following example. Consider the case where $a(z) = \sum_{i=0}^{+\infty} a_{-i} z^{-i}$, $b(z) = \sum_{i=0}^{+\infty} b_i z^i$, $a_{-i} = b_i = i^{-3/2}$. Clearly $a(z), b(z) \in \mathcal{W}$ but $a(z)'$ and $b(z)'$ are not in $\mathcal{W}$ since $\sum_{i\in\mathbb{Z}^+} i a_{-i}$ and $\sum_{i\in\mathbb{Z}^+} i b_i$ are not convergent. Moreover,

$$\|H(a^-)H(b^+)\|_{\mathcal{F}} = \sum_{i,j\in\mathbb{Z}^+} \sum_{r=0}^{+\infty} \frac{1}{(i+r)^{3/2}} \frac{1}{(r+j)^{3/2}} = \sum_{r=0}^{+\infty} \left(\sum_{k=r+1}^{+\infty} \frac{1}{k^{3/2}} \right)^2.$$

This is the sum of the squares of the remainders of the series $\sum_{i=1}^{+\infty} \frac{1}{i^{3/2}}$. This sum diverges since these remainders behave like $\int_r^{+\infty} \frac{1}{x^{3/2}} dx = \frac{2}{\sqrt{r}}$.

Now we can prove the main result of this section which states that $\mathcal{QT}$ is closed under multiplication.

Theorem 3.6. *Let $A, B \in \mathcal{QT}$, where $A = T(a) + E_a$, $B = T(b) + E_b$. Then we have $C = AB = T(c) + E_c \in \mathcal{QT}$ with $c(z) = a(z)b(z)$. Moreover,*

$$\|E_c\|_{\mathcal{F}} \le \|H(a^-)\|_{\mathcal{F}} \cdot \|H(b^+)\|_{\mathcal{F}} + \|a\|_{\mathcal{W}} \|E_b\|_{\mathcal{F}} + \|b\|_{\mathcal{W}} \|E_a\|_{\mathcal{F}} + \|E_a\|_{\mathcal{F}} \cdot \|E_b\|_{\mathcal{F}}.$$

Proof. We have $C = AB = (T(a) + E_a)(T(b) + E_b)$. Applying Theorem 2.2 yields

$$C = T(c) - H(a^-)H(b^+) + T(a)E_b + E_a T(b) + E_a E_b =: T(c) + E_c,$$

where

$$E_c = -H(a^-)H(b^+) + T(a)E_b + E_a T(b) + E_a E_b. \tag{3.2}$$

Therefore, it is sufficient to prove that $\|E_c\|_{\mathcal{F}}$ is finite. From Lemmas 3.4 and 2.1 it follows that both $\|H(a^-)H(b^+)\|_{\mathcal{F}}$ and $\|E_a E_b\|_{\mathcal{F}}$ are finite. It remains to show that $\|E_a T(b)\|_{\mathcal{F}}$ and $\|T(a)E_b\|_{\mathcal{F}}$ are finite. We prove this property only for $\|T(a)E_b\|_{\mathcal{F}}$ since the boundedness of the other matrix norm follows by transposition. In fact, for any $F \in \mathcal{F}$ one has $\|F\|_{\mathcal{F}} = \|F^T\|_{\mathcal{F}}$ and $T(a)^T = T(\widehat{a})$, where $\widehat{a}(z) = a(z^{-1})$ and $\|a\|_{\mathcal{W}} = \|\widehat{a}\|_{\mathcal{W}}$. Denote $H = T(a)E_b = (h_{i,j})$ and $E_b = (e_{i,j})$. We have $h_{i,j} = \sum_{r=1}^{+\infty} a_{r-i} e_{r,j}$ so that

$$\|H\|_{\mathcal{F}} = \sum_{i,j\in\mathbb{Z}^+} |h_{i,j}| \le \sum_{i,j\in\mathbb{Z}^+} \sum_{r=1}^{+\infty} |a_{r-i} e_{r,j}|.$$

Substituting $k = r - i$ yields

$$\|H\|_{\mathcal{F}} \le \sum_{k\in\mathbb{Z}} |a_k| \sum_{j=1}^{+\infty} \sum_{i=-k+1}^{+\infty} |e_{k+i,j}|.$$

Since $\sum_{j=1}^{+\infty} \sum_{i=-k+1}^{+\infty} |e_{k+i,j}| = \sum_{j=1}^{+\infty} \sum_{i=1}^{+\infty} |e_{i,j}| = \|E_b\|_{\mathcal{F}}$ for any k, we have

$$\|H\|_{\mathcal{F}} \le \sum_{k\in\mathbb{Z}} |a_k| \|E_b\|_{\mathcal{F}} = \|a\|_{\mathcal{W}} \|E_b\|_{\mathcal{F}} < +\infty.$$

Thus, taking norms in (3.2) yields

$$\|E_c\|_{\mathcal{F}} \le \|H(a^-)\|_{\mathcal{F}} \cdot \|H(b^+)\|_{\mathcal{F}} + \|a\|_{\mathcal{W}} \|E_b\|_{\mathcal{F}} + \|E_a\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{W}} + \|E_a\|_{\mathcal{F}} \cdot \|E_b\|_{\mathcal{F}},$$

which completes the proof. $\square$

Observe that in view of Remark 3.5 we may write

$$\|E_c\|_{\mathcal{F}} \le \|a'\|_{\mathcal{W}} \|b'\|_{\mathcal{W}} + \|a\|_{\mathcal{W}} \|E_b\|_{\mathcal{F}} + \|E_a\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{W}} + \|E_a\|_{\mathcal{F}} \cdot \|E_b\|_{\mathcal{F}}. \tag{3.3}$$

Now, our next goal is to prove that the class $\mathcal{QT}$ is a Banach algebra.

Theorem 3.7. *The class $\mathcal{QT}$ equipped with the norm $\|\cdot\|_{\mathcal{QT}}$ is a Banach algebra over $\mathbb{C}$. Moreover, $\|AB\|_{\mathcal{QT}} \le \|A\|_{\mathcal{QT}}\|B\|_{\mathcal{QT}}$ for any matrices $A, B \in \mathcal{QT}$.*

Proof. Theorem 3.6 ensures the closure of $\mathcal{QT}$ under matrix multiplication. To prove the sub-multiplicative property of the norm, i.e.,

$$\|AB\|_{\mathcal{QT}} \le \|A\|_{\mathcal{QT}} \cdot \|B\|_{\mathcal{QT}}$$

for any $A, B \in \mathcal{QT}$, $A = T(a) + E_a$, $B = T(b) + E_b$, observe that

$$\begin{aligned} \|ab\|_{\mathcal{W}_1} &= \|ab\|_{\mathcal{W}} + \|(ab)'\|_{\mathcal{W}} = \|ab\|_{\mathcal{W}} + \|a'b + ab'\|_{\mathcal{W}} \\ &\le \|a\|_{\mathcal{W}}\|b\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}}\|b\|_{\mathcal{W}} + \|a\|_{\mathcal{W}}\|b'\|_{\mathcal{W}}. \end{aligned} \tag{3.4}$$

Since $\|AB\|_{\mathcal{QT}} = \|ab\|_{\mathcal{W}_1} + \|E_c\|_{\mathcal{F}}$, for $c(z) = a(z)b(z)$ and E_c defined as in Theorem 3.6, by applying (3.3) and (3.4) we obtain

$$\begin{aligned} \|AB\|_{\mathcal{QT}} &\le \|ab\|_{\mathcal{W}_1} + \|a'\|_{\mathcal{W}}\|b'\|_{\mathcal{W}} + \|a\|_{\mathcal{W}}\|E_b\|_{\mathcal{F}} + \|b\|_{\mathcal{W}}\|E_a\|_{\mathcal{F}} + \|E_a\|_{\mathcal{F}}\|E_b\|_{\mathcal{F}} \\ &\le \|a\|_{\mathcal{W}}\|b\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}}\|b\|_{\mathcal{W}} + \|a\|_{\mathcal{W}}\|b'\|_{\mathcal{W}} \\ &\quad + \|a'\|_{\mathcal{W}}\|b'\|_{\mathcal{W}} + \|a\|_{\mathcal{W}}\|E_b\|_{\mathcal{F}} + \|b\|_{\mathcal{W}}\|E_a\|_{\mathcal{F}} + \|E_a\|_{\mathcal{F}}\|E_b\|_{\mathcal{F}} \\ &= (\|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}})(\|b\|_{\mathcal{W}} + \|b'\|_{\mathcal{W}}) \\ &\quad + \|a\|_{\mathcal{W}}\|E_b\|_{\mathcal{F}} + \|b\|_{\mathcal{W}}\|E_a\|_{\mathcal{F}} + \|E_a\|_{\mathcal{F}}\|E_b\|_{\mathcal{F}} \\ &\le (\|a\|_{\mathcal{W}_1} + \|E_a\|_{\mathcal{F}})(\|b\|_{\mathcal{W}_1} + \|E_b\|_{\mathcal{F}}) \\ &= \|A\|_{\mathcal{QT}}\|B\|_{\mathcal{QT}}. \end{aligned}$$

Concerning the completeness, we have observed that the set of QT-matrices is isomorphic to the direct sum $\mathcal{QT} \simeq \mathcal{W}_1 \oplus \mathcal{F}$. Since both $\mathcal{W}_1$ and $\mathcal{F}$ are Banach spaces, the composition of the 1-norm of $\mathbb{R}^2$ with the vector valued function $T(a) + E \to (\|a\|_{\mathcal{W}_1}, \|E\|_{\mathcal{F}})$ makes $\mathcal{W}_1 \oplus \mathcal{F}$ a complete metric space. ☐

In the next section we represent the inverse matrix of an infinite Toeplitz matrix $T(a)$ in terms of the Wiener-Hopf factorization of $a(z)$.

3.1. **Matrix inversion.** Assume that $a(z) \in \mathcal{W}_1$ does not vanish on the unit circle and its winding number is zero, so that in view of Theorem 2.4 there exists the canonical Wiener-Hopf factorization $a(z) = u(z)\ell(z)$. It can be shown [26] that $u(z), \ell(z) \in \mathcal{W}_1$. From this factorization we deduce the following matrix factorization

$$T(a) = T(u)T(\ell),$$

where $T(\ell)$ is lower triangular and $T(u)$ is upper triangular. Since $u(z)$ and $\ell(z^{-1})$ do not vanish in the unit disk, the functions $u(z)$ and $\ell(z)$ have inverses in $\mathcal{W}_1$ [26]. By Theorem 2.2 one has $T(u)T(u^{-1}) = T(u^{-1})T(u) = I$ and $T(\ell)T(\ell^{-1}) = T(\ell^{-1})T(\ell) = I$, so that

$$T(a)^{-1} = T(\ell)^{-1}T(u)^{-1} = T(\ell^{-1})T(u^{-1}).$$

In view of Lemma 3.4, we have

$$T(a)^{-1} = T(a^{-1}) - H((\ell^{-1})^-)H((u^{-1})^+) = T(a^{-1}) - H(\ell^{-1})H(u^{-1}) \in \mathcal{QT}. \tag{3.5}$$

That is, a semi-infinite Toeplitz matrix associated with a symbol $a(z) \in \mathcal{W}_1$, with null winding number, which does not annihilate in $\mathbb{T}$, is invertible and its inverse is a QT-matrix.

This fact, together with the available algorithms to compute the Wiener-Hopf factorization of $a(z)$, enables us to implement the inversion of QT-matrices in a very efficient manner. We will see this in the next section.

4. QT-matrix arithmetic

The properties that we have described in the previous sections imply that any finite computation which takes as input a set of QT-matrices and that performs matrix additions, multiplications, inversions, and multiplications by a scalar, generates results that belong to $\mathcal{QT}$. If the computation can be carried out with no breakdown, say, caused by singularity, then the output still belongs to $\mathcal{QT}$.

This observation makes it possible to compute functions of semi-infinite QT-matrices in an efficient way or to solve quadratic matrix equations where the coefficients are QT-matrices. In order to do that, we have to provide a simple and effective way of representing, up to an arbitrarily small error, QT-matrices by means of a finite number of parameters. This is done in this section.

Given a QT-matrix $A = T(a) + E_a$, since the symbol $a(z)$ belongs to the Wiener class, and since the correction matrix E_a has entries with finite sum of their moduli, we may write A through its *truncated form* $\widetilde{A} = \mathrm{trunc}(A)$. That is, for any $\epsilon > 0$ there exist integers n_-, n_+, k_-, k_+ such that

$$\begin{aligned} &A = \widetilde{A} + \mathcal{E}_a, \quad \|\mathcal{E}_a\|_{\mathcal{QT}} \le \epsilon, \\ &\widetilde{A} = T(\widetilde{a}) + \widetilde{E}_a, \\ &\widetilde{a}(z) = \sum_{i=-n_-}^{n_+} a_i z^i, \end{aligned} \tag{4.1}$$

where $\widetilde{E}_a = (\widetilde{e}_{i,j})$, is such that $\widetilde{e}_{i,j} = e_{i,j}$ for $i = 1, \ldots, k_-$, $j = 1, \ldots, k_+$, while $\widetilde{e}_{i,j} = 0$ elsewhere.

In this way, we can approximate any given QT-matrix A, to any desired precision, with a QT-matrix $\widetilde{A}$ where the Toeplitz part is banded and the correction $\widetilde{E}_a$ has a finite dimensional nonzero part. The QT-matrix $\widetilde{A}$ can be easily stored with a finite number of memory locations. The "finite approximation" $\widetilde{A}$ of a QT-matrix A is the computational counterpart with which we are going to work in practice.

Observe that, if $A \in \mathcal{QT}$ and the symbol $a(z)$ is analytic, for the exponential decay of the coefficients $|a_i|$, the values of $n_\pm$ are $O(\log \epsilon^{-1})$. Concerning the values of $k_\pm$, unless we make additional assumptions on the decay of the entries $|e_{i,j}|$ as i, j tend to infinity, the values that $k_\pm$ can assume are as large as $1/\epsilon$. Think for instance the case where $e_{i,j} = 1/(i+j)^p$ for $p > 2$ and $k_\pm$ are of the order of $\epsilon^{-1/p}$. The same qualitative bounds hold for the coefficients a_i if we simply assume that $a(z) \in \mathcal{W}_1$.

Here and in the sequel, we do not care much to give *a priori* bounds to the values of $n_\pm$ and $k_\pm$ since these values can be determined automatically at run time during the computation.

Another observation concerns the truncated correction $\widetilde{E}_a$. In fact, from the computational point of view, it is convenient to express the matrix $\widetilde{E}_a$ by means of a factorization of the kind $\widetilde{E}_a = F_a G_a^T$, where matrices F_a and G_a have a number of columns given by the rank of $\widetilde{E}_a$ and infinitely many rows. In this way, in presence of low-rank corrections, the storage is reduced together with the computational cost for performing matrix arithmetic. This representation in product form can be obtained by means of SVD up to some error which can be controlled at run time and which can be included in $\mathcal{E}_a$. Observe also that the truncation operates both on the function $a(z)$ and in the correction E_a by means of compression.

In the following, we represent a QT-matrix $A = T(a) + E_a$ in the form (4.1) with $\widetilde{E}_a = F_a G_a^T$ where F_a has f_a nonzero rows and k_a columns, G_a has g_a nonzero rows and k_a columns, and the error $\mathcal{E}_a$ has a sufficiently small norm. This way, $\widetilde{E}_a$ has f_a nonzero rows, g_a nonzero columns and rank at most k_a.

With this notation we may easily implement the operations of addition, subtraction, multiplication and inversion of two QT-matrices $\widetilde{A}$, $\widetilde{B}$ which are the truncated representations of two QT-matrices A and B, i.e.,

$$\begin{aligned} A &= \widetilde{A} + \mathcal{E}_a, \quad \widetilde{A} = \operatorname{trunc}(A) = T(\widetilde{a}) + \widetilde{E}_a, \\ B &= \widetilde{B} + \mathcal{E}_b, \quad \widetilde{B} = \operatorname{trunc}(B) = T(\widetilde{b}) + \widetilde{E}_b, \end{aligned}$$

denote by $\star$ any arithmetic operation, define $C = A \star B$, $\widehat{C} = \widetilde{A} \star \widetilde{B}$ and $\widetilde{C} = \operatorname{trunc}(\widehat{C})$.

We define *total error* in the operation $\star$ as $\mathcal{E}_c^{tot} = C - \widetilde{C}$, the *local error* as $\mathcal{E}_c^{loc} = \widehat{C} - \widetilde{C}$ and the *inherent error* as $\mathcal{E}_c^{in} = C - \widehat{C}$, so that $\mathcal{E}_c^{tot} = \mathcal{E}_c^{in} + \mathcal{E}_c^{loc}$. Observe that the inherent error is the result of $\mathcal{E}_a$ and $\mathcal{E}_b$ through the performed matrix operation, the local error is generated by the truncation of the matrix arithmetic operation $\widetilde{A} \star \widetilde{B}$, while the total error is the sum of the two errors. Formally, these errors behave like the inherent error and the round-off error in the standard floating point arithmetic.

In our study we do not analyze the growth of the inherent error in each arithmetic operation, but rather we limit ourselves to operate the truncation and compression in such a way that the norm of the local error is bounded by a given value ϵ, say the machine precision. Moreover, we do not consider the errors generated by the floating point arithmetic.

4.1. **Addition.** Let $A = \widetilde{A} + \mathcal{E}_a$ and $B = \widetilde{B} + \mathcal{E}_b$ be QT-matrices where $\widetilde{A} = T(\widetilde{a}) + \widetilde{E}_a$, $\widetilde{B} = T(\widetilde{b}) + \widetilde{E}_b$ with $\widetilde{a}(z)$, $\widetilde{b}(z)$ Laurent polynomials of degrees $n_a^{\pm}$ and $n_b^{\pm}$ respectively, and $\widetilde{E}_a = F_a G_a^T$, $\widetilde{E}_b = F_b G_b^T$.

If A and B have the above representation, then, for the matrix $C = A + B$ we have the representation

$$C = \widetilde{A} + \widetilde{B} + \mathcal{E}_a + \mathcal{E}_b,$$

from which we deduce that the inherent error is $\mathcal{E}_c^{in} = \mathcal{E}_a + \mathcal{E}_b$. On the other hand, concerning $\widehat{C} = \widetilde{A} + \widetilde{B}$ we have

$$\widehat{C} = T(\widetilde{a} + \widetilde{b}) + \widetilde{E}_a + \widetilde{E}_b,$$

where $\widetilde{a}(z) + \widetilde{b}(z)$ is a Laurent polynomial of degrees $n_c^- = \max(n_a^-, n_b^-)$, $n_c^+ = \max(n_a^+, n_b^+)$, while

$$\begin{aligned} E_c &:= \widetilde{E}_a + \widetilde{E}_b = F_c G_c^T, \\ F_c &= [F_a, F_b], \quad G_c = [G_a, G_b], \end{aligned}$$

where $f_c = \max(f_a, f_b)$ and $g_c = \max(g_a, g_b)$ are the number of nonzero rows of F_c and G_c, respectively, and $k_c = k_a + k_b$ is the number of columns of F_c and G_c.

The Laurent polynomial $\widetilde{a}(z)+\widetilde{b}(z)$ can be truncated and replaced by a Laurent polynomial $\widetilde{c}(z)$ of possibly less degree. Also the value of k_c, can be reduced and the matrices F_c, G_c can be compressed, by using a compression technique which guarantees a local error with norm bounded by a given ϵ. This technique, based on computing SVD and QR factorization is explained in the next section. Denoting by $\widetilde{F}_c$, $\widetilde{G}_c$ the matrices obtained after compressing F_c and G_c, respectively, we have

$$\widetilde{C} = \operatorname{trunc}(\widehat{C}) = T(\widetilde{c}) + \widetilde{E}_c + \mathcal{E}_c^{loc}, \quad \widetilde{E}_c = \widetilde{F}_c \widetilde{G}_c^T,$$

where $\mathcal{E}_c^{loc}$ denotes the local error due to truncation and compression, i.e., $\mathcal{E}_c^{loc} = \widetilde{A} + \widetilde{B} - \operatorname{trunc}(\widetilde{A}+\widetilde{B})$. This way we have

$$A + B = T(\widetilde{c}) + \widetilde{E}_c + \mathcal{E}_c^{loc} + \mathcal{E}_c^{in}.$$

4.2. **Multiplication.** A similar expression holds for multiplication. For the product $C = AB$ we have the equation

$$AB = \widetilde{A}\widetilde{B} + \widetilde{A}\mathcal{E}_b + \mathcal{E}_a\widetilde{B} + \mathcal{E}_a\mathcal{E}_b$$

from which we deduce that the inherent error is $\mathcal{E}_c^{in} = \widetilde{A}\mathcal{E}_b + \mathcal{E}_a\widetilde{B} + \mathcal{E}_a\mathcal{E}_b$. Moreover, we have

$$\begin{aligned} \widehat{C} = \widetilde{A}\widetilde{B} &= T(\widetilde{a})T(\widetilde{b}) + T(\widetilde{a})\widetilde{E}_b + \widetilde{E}_a T(\widetilde{b}) + \widetilde{E}_a\widetilde{E}_b \\ &= T(\widetilde{a}\widetilde{b}) - H(\widetilde{a}^-)H(\widetilde{b}^+) + T(\widetilde{a})\widetilde{E}_b + \widetilde{E}_a T(\widetilde{b}) + \widetilde{E}_a\widetilde{E}_b \\ &=: T(\widetilde{a}\widetilde{b}) + E_c. \end{aligned}$$

Observe that, since $\widetilde{a}^-(z)$ and $\widetilde{b}^+(z)$ are polynomials, the matrices $H(\widetilde{a}^-)$ and $H(\widetilde{b}^+)$ have a finite number of nonzero entries. Therefore, we may factorize the product $H(\widetilde{a}^-)H(\widetilde{b}^+)$ in the form FG^T. Thus, we find that the matrix E_c can be written as $E_c = F_c G_c^T$, where

$$F_c = [F, T(\widetilde{a})F_b, F_a], \quad G_c = [G, G_b, T(\widetilde{b})^T G_a + G_b(F_b^T G_a)].$$

This provides the finite representation of the product $\widehat{C} = \widetilde{A}\widetilde{B}$ where $n_c^- = n_a^- + n_b^-$, $n_c^+ = n_a^+ + n_b^+$, while the number of nonzero rows of F_c and G_c is given by $f_c = \max(f_b + n_a^-, f_a)$ and $g_c = \max(n_b^+, g_b, g_a + n_b^-)$, respectively; moreover, $k_c = k_a + k_b + n_b^+$.

Also in this case we may apply a compression technique, based on SVD for reducing the memory storage of the correction and for reducing the degree of the Laurent polynomial $\widetilde{a}(z)\widetilde{b}(z)$. Operating in this way, we introduce a local error $\mathcal{E}_c^{loc} = \widetilde{A}\widetilde{B} - \operatorname{trunc}(\widetilde{A}\widetilde{B})$. Denoting by $\widetilde{c}(z)$ the truncation of the Laurent polynomial $\widetilde{a}(z)\widetilde{b}(z)$ and with $\widetilde{F}_c\widetilde{G}_c^T$ the compression of $F_c G_c^T$, we have

$$\widehat{C} = \widetilde{A}\widetilde{B} = T(\widetilde{c}) + \widetilde{F}_c\widetilde{G}_c^T + \mathcal{E}_c^{loc}.$$

This way we have

$$C = AB = T(\widetilde{c}) + \widetilde{F}_c\widetilde{G}_c^T + \mathcal{E}_c^{loc} + \mathcal{E}_c^{in},$$

which expresses the result C of the multiplication in terms of the approximated value $\widetilde{C} = T(\widetilde{c}) + \widetilde{E}_c$, the local error $\mathcal{E}_c^{loc}$ and the inherent error $\mathcal{E}_c^{in}$. The overall error is given by $\mathcal{E}_c = \mathcal{E}_c^{loc} + \mathcal{E}_c^{in}$.

4.3. **Matrix inversion.** It is worth paying particular attention to the operation of matrix inversion since it is less immediate than multiplication and addition.

First, we consider the problem of inverting the matrix $A = T(a)$, i.e., we assume that $E_a = 0$. The general case will be treated afterwords.

Recall that, if $a(z) \in \mathcal{W}_1$ does not vanish in the unit circle and if it has a zero winding number, then Theorem 2.3 implies that the matrix $T(a)$ is invertible and, in view of Theorem 2.4, there exists the canonical Wiener-Hopf factorization $a(z) = u(z)\ell(z)$ so that (3.5) holds. Thus, a finite representation of A^{-1} is obtained by truncating the Laurent series of $1/a(z)$ to a Laurent polynomial and by approximating the Hankel matrices $H((\ell^{-1})^-)$ and $H((u^{-1})^+)$ by means of matrices having a finite number of nonzero entries, an infinite number of rows and the same finite number of columns. The latter operation can be achieved by truncating the power series $\ell^{-1}(z)$ and $u^{-1}(z)$ to polynomials and by numerically compressing the product of the Hankel matrices obtained this way. This operation can be effectively performed by reducing the Hankel matrices to tridiagonal form by means of Lanczos method with orthogonalization. This procedure takes advantage of the Hankel structure since the matrix-vector product can be computed by means of FFT in $O(n \log n)$ operations where n is the size of the Hankel matrix. The advantage of this compression is that the cost grows as $O(r^2 n \log n)$ where r is the numerical rank of the matrix.

If $a(z)$ is analytic in the annulus $\mathbb{A}(r_a, R_a) = \{z \in \mathbb{C} : \quad r_a < |z| < R_a\} \supset \mathbb{T}$, then its coefficients have an exponential decay so that $|a_i^+| \le \gamma\lambda_+^i$, $|a_i^-| \le \gamma\lambda_-^i$, $|u_i| \le \gamma\lambda_+^i$, $|\ell_i^-| \le \gamma\lambda_-^i$, for some positive γ and for $1/R_a < \lambda_+ < 1$, $r_a < \lambda_- < 1$. Thus, we find that for the truncated approximation of the matrix A the values of n^+, n^-, f, g are bounded by $\log(\gamma^{-1}\epsilon^{-1})/\log(\lambda_\pm^{-1})$.

Performing numerical experiments it turns out that the singular values of the principal submatrices of the Hankel matrices $H(\ell^-)$ and $H(u^+)$ associated with power series having coefficients with an exponential decay, have an exponential decay themselves. So that also the truncation on the value of the numerical rank k of $H(\ell^-)H(u^+)$ can be performed efficiently.

The analysis of the inherent error due to inversion is related to the analysis of the condition number of semi-infinite Toeplitz matrices. We do not carry out this analysis, we refer the reader to the books [12], [13] on this regard.

Now consider the more general case of the matrix $A = T(a) + F_a G_a^T$ which we assume already in its truncated form. Assume $T(a)$ invertible and write $A = T(a)(I + T(a)^{-1} F_a G_a^T)$. Denoting for simplicity $U = T(u)$, $L = T(\ell)$ we have

$$(T(a) + F_a G_a^T)^{-1} = T(a)^{-1} - L^{-1}(U^{-1}F_a)Y^{-1}(G_a^T L^{-1})U^{-1},$$
$$Y = I + G_a^T L^{-1} U^{-1} F_a,$$

where Y is a finite matrix which is invertible if and only if A is invertible. This way, the algorithm for computing A^{-1} in its finite QT-matrix representation is given by the following steps:

(1) compute the spectral factorization $a(z) = u(z)\ell(z)$;
(2) compute the coefficients of the power series $\widetilde{u}(z) = 1/u(z)$ and $\widetilde{\ell}(z) = 1/\ell(z)$, so that $L^{-1} = T(\widetilde{\ell})$, $U^{-1} = T(\widetilde{u})$;
(3) represent the matrix $H = L^{-1}U^{-1}$ as $T(c) + F_h G_h^T$, where $c(z) = \widetilde{\ell}(z)\widetilde{u}(z)$ by means of Theorem 2.2;
(4) compute the products: $G_1 = T(\widetilde{\ell})G_a$, $F_1 = T(\widetilde{u})F_a$;

(5) compute $Y = I + G_1^T F_1$, $F_2 = F_1 Y^{-1}$, $F_3 = T(\widetilde{\ell})F_2$, $G_2 = T(\widetilde{u})G_1$;
(6) output the coefficients of $c(z)$ and the matrices $F_c = [F_h, F_3]$, $G_c = [G_h, G_2]$.

For computing the spectral factorization of $a(z)$ we rely on the algorithm of [5] which employs evaluation/interpolation techniques at the Fourier points.

4.4. **Compression.** Given the matrix E in the form $E = FG^T$ where F and G are matrices of size $m \times k$ and $n \times k$, respectively, we aim to reduce the size k and to approximate E in the form $\widetilde{F}\widetilde{G}^T$, where $\widetilde{F}$ and $\widetilde{G}$ are matrices of size $m \times \widetilde{k}$ and $n \times \widetilde{k}$, respectively, with $\widetilde{k} < k$.

We use the following procedure. Compute the pivoted (rank-revealing) QR factorizations $F = Q_f R_f P_f$, $G = Q_g R_g P_g$, where P_f and P_g are permutation matrices, Q_f and Q_g are orthogonal and R_f, R_g are upper triangular; remove the last negligible rows from the matrices R_f and R_g, remove the corresponding columns of Q_f and Q_g. In this way we obtain matrices $\hat{R}_f$, $\hat{R}_g$, $\hat{Q}_f$, $\hat{Q}_g$ such that, up to within a small error, satisfy the equations $F = \hat{Q}_f \hat{R}_f P_f$, $G = \hat{Q}_g \hat{R}_g P_g$. Then, in the factorization $FG^T = \hat{Q}_f(\hat{R}_f P_f P_g^T \hat{R}_g^T)\hat{Q}_g^T$, compute the SVD of the matrix in the middle $\hat{R}_f P_f P_g^T \hat{R}_g^T = U\Sigma V^T$ and replace U, Σ, and V with matrices $\hat{U}$, $\hat{\Sigma}$, $\hat{V}$, obtained by removing the singular values σ_i and the corresponding singular vectors if $\sigma_i < \epsilon\sigma_1$, where ϵ is a given tolerance. In output, the matrices $\widetilde{F} = \hat{Q}_f \hat{U}\hat{\Sigma}^{1/2}$, $\widetilde{G} = \hat{Q}_g \hat{V}\hat{\Sigma}^{1/2}$ are delivered.

4.5. **The Matlab code.** The arithmetic operations on QT-matrices have been implemented in Matlab. The package can be obtained upon request from the authors. It includes the functions `qt_add`, `qt_mul`, `qt_inv`, `qt_compress` for performing matrix arithmetic and compression. A QT-matrix A is stored by means of the variables `am, ap, aF, aG`, where `am` and `ap` are the vectors containing the coefficients of the Laurent polynomial $a(z) = \sum_{i=-h}^{k} a_i z^i$ so that `am =` $(a_0, a_{-1}, \dots, a_{-h})$, `ap =` $(a_0, a_1, \dots, a_k)$, the variables `aF` and `aG` contain the values of the nonnegligible entries in the correction matrices F and G, respectively.

In each function, after performing an arithmetic operation, the compression of the matrices F and G is applied.

5. Solving certain semi-infinite quadratic matrix equations by means of Cyclic Reduction

In the analysis of certain QBD queueing processes like the tandem Jackson queue [19] or bi-dimensional random walks in the quarter plane [27], [20], [24] one has to find the invariant probability vector of a stochastic process with a discrete two-dimensional state space. The two coordinates of the latter—usually called level and phase—are both countably infinite. Typically, the allowed transitions from a state are limited to a subset of the adjacent states. Moreover, the probability of a certain transition is homogeneous in time and—except for some boundary conditions—depends only on the distance between the departure and the arrival. This makes the model representable with a generator of the kind

$$Q = \begin{bmatrix} \widehat{A}_0 & \widehat{A}_1 & & & \\ A_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \quad A_i, \widehat{A}_i \in \mathcal{QT}.$$

The matrix analytic methods, designed in this framework for finding the invariant distribution, require to find the minimal non negative solutions G and R of the semi-infinite matrix equations

$$A_1X^2 + A_0X + A_{-1} = 0, \quad X^2A_{-1} + XA_0 + A_1 = 0, \tag{5.1}$$

respectively. It can be proved that, under very mild assumptions, the minimal nonnegative solutions of the above equations exist and are unique. We refer to the books [7], [25], [29], for more details.

When the blocks A_is are finite, one of the most reliable and fast algorithms for performing this computation is the Cyclic Reduction (CR) [9, 15, 18]. This is an iterative method based on generating the following matrix sequences:

$$\begin{aligned}
&A_0^{(h+1)} = A_0^{(h)} - A_1^{(h)}S^{(h)}A_{-1}^{(h)} - A_{-1}^{(h)}S^{(h)}A_1^{(h)}, \quad S^{(h)} = (A_0^{(h)})^{-1},\\
&A_1^{(h+1)} = -A_1^{(h)}S^{(h)}A_1^{(h)}, \quad A_{-1}^{(h+1)} = -A_{-1}^{(h)}S^{(h)}A_{-1}^{(h)},\\
&\widetilde{A}^{(h+1)} = \widetilde{A}^{(h)} - A_{-1}^{(h)}S^{(h)}A_1^{(h)}, \quad \widehat{A}^{(h+1)} = \widehat{A}^{(h)} - A_1^{(h)}S^{(h)}A_{-1}^{(h)}
\end{aligned} \tag{5.2}$$

for $h = 0, 1, 2\ldots$, with $A_0^{(0)} = \widetilde{A}^{(0)} = \widehat{A}^{(0)} = A_0$, $A_1^{(0)} = A_1$, $A_{-1}^{(0)} = A_{-1}$. The sequences

$$G^{(h)} := -(\widetilde{A}^{(h)})^{-1}A_{-1}, \quad R^{(h)} := -A_1(\widehat{A}^{(h)})^{-1} \tag{5.3}$$

converge to the minimal nonnegative solutions G and R of the matrix equations (5.1).

These convergence properties are valid also in the case where the blocks A_{-1}, A_0, A_1 are semi-infinite where convergence holds componentwise. We refer the reader to [23] for more details.

The arithmetic developed in Section 4 paves the way to the use of CR when $A_i \in \mathcal{QT}$, $i = -1, 0, 1$. Observe that, since $\mathcal{QT}$ is an algebra, all the matrices generated by CR belong to $\mathcal{QT}$. Moreover, the Toeplitz part of these matrices have associated symbols $a_{-1}^{(h)}(z)$, $a_0^{(h)}(z)$, $a_1^{(h)}(z)$, $\widetilde{a}^{(h)}(z)$, $\widehat{a}^{(h)}(z)$, which satisfy the same recurrence equations as (5.2). More precisely we have the *scalar* functional relations

$$\begin{aligned}
&a_0^{(h+1)}(z) = a_0^{(h)}(z) - 2a_1^{(h)}(z)a_{-1}^{(h)}(z)/a_0^{(h)}(z),\\
&a_1^{(h+1)}(z) = -a_1^{(h)}(z)^2/a_0^{(h)}(z), \quad a_{-1}^{(h+1)}(z) = -a_{-1}^{(h)}(z)^2/a_0^{(h)}(z),\\
&\widetilde{a}^{(h+1)}(z) = \widetilde{a}^{(h)}(z) - a_1^{(h)}(z)a_{-1}^{(h)}(z)/a_0^{(h)}(z),
\end{aligned}$$

with $h = 0, 1, \ldots$, where $a_i^{(0)}(z) = a_i(z)$, $i = -1, 0, 1$ and $\widetilde{a}^{(0)}(z) = a_0(z)$. Observe that since all the quantities in the above recurrence are scalar functions, they commute so that $\widehat{a}^{(h)}(z)$ coincides with $\widetilde{a}^{(h)}(z)$.

As pointed out in [6], [9], in the scalar case CR reduces to the celebrated Graeffe iteration whose properties have been investigated in [30]. Thus, in order to analyze the convergence of the sequences defined above, we rely on the convergence properties of the Graeffe iteration applied to quadratic polynomials. In particular, we know that if, for a given $z \in \mathbb{T}$ the polynomial $p_z(x) := a_1(z)x^2 + a_0(z)x + a_{-1}(z)$ associated with the triple $(a_{-1}(z), a_0(z), a_1(z))$, has one root inside the unit disk and one root outside, then the sequence $-(a_{-1}(z)/\widetilde{a}^{(h)}(z))$ has a limit $g(z)$ which coincides with the root of the polynomial $p_z(x)$ inside the unit disk.

The following theorem provides mild conditions which ensure the above properties, and are generally satisfied in the applications.

Theorem 5.1. *Let $a_i(z) = a_{i,-1}z^{-1} + a_{i,0} + a_{i,1}z$, for $i = -1, 0, 1$, be such that $\sum_{i,j=-1}^{1} a_{i,j} = 0$, $a_{0,0} < 0$, $a_{i,j} \geq 0$, otherwise. If*

(i) *$a_{-1,0} > 0$ or $a_{1,0} > 0$,*
(ii) *$a_{ij} \neq 0$ for at least a pair (i, j), with $j \neq 0$,*

then for any $z \in \mathbb{T}$, $z \neq 1$, the quadratic polynomial $p_z(x) = a_1(z)x^2 + a_0(z)x + a_{-1}(z)$, has a root of modulus less than 1 and a root of modulus greater than 1.

Proof. Without loss of generality we may assume that the entries $a_{i,j}$ belong to the interval $[-1, 1]$. If not, we may scale equation (5.1) by a suitable constant and reduce it to this case. As a first step we show that there are no roots of modulus 1. Assume by contradiction that x is a root of modulus 1. Obviously, we have $p_z(x) = 0$ if and only $p_z(x) + x = x$. Observe that, if $z \in \mathbb{T}$, the left-hand side of the previous equation is a convex combination of the points in the discrete set $\mathcal{C}_{x,z} := \{x^i z^j,\ i = 0, 1, 2,\ j = -1, 0, 1\} \subset \mathbb{T}$. If $z \neq 1$, condition (i) and the fact that $-1 \leq a_{0,0} < 0$ ensure that the convex combination involves at least two different points of the unit circle, either x and 1 or x and x^2. Therefore, this convex combination $p_z(x) + x$ is equal to a point which belongs to the interior of the unit disc. This contradicts the fact that $|p_z(x) + x| = |x| = 1$. This argument excludes roots on $\mathbb{T}$ for $z \in \mathbb{T} \setminus \{1\}$. We conclude by showing that there is exactly one root of modulus less than 1. In order to prove this, we first show that $|a_0(z)| > |a_{-1}(z) + a_1(z)|$ holds for any $z \in \mathbb{T} \backslash \{1\}$. Therefore, by applying the Rouché Theorem one finds that the functions $f(x) = a_0(z)x$ and $p_z(x)$ have the same number of zeros in the open unit disc. To prove the inequality $|a_0(z)| > |a_{-1}(z) + a_1(z)|$ we observe that

$$\begin{aligned} |a_{0,-1}z^{-1} + a_{0,0} + a_{0,1}z| &\geq |a_{0,0}| - |a_{0,-1}z^{-1}| - |a_{0,1}z| = -a_{0,0} - a_{0,-1} - a_{0,1} \\ &= a_{-1,-1} + a_{-1,0} + a_{-1,1} + a_{1,-1} + a_{1,0} + a_{1,1} \\ &\geq |a_{-1,-1}z^{-1} + a_{-1,0} + a_{-1,1}z + a_{1,-1}z^{-1} + a_{1,0} + a_{1,1}z|, \end{aligned}$$

where at least one of the two above inequalities is strict because of condition (ii). $\square$

Corollary 5.2. *Under the conditions of Theorem 5.1, if $a_1(z) \neq 0$ for any $z \in \mathbb{T}$ and $a_{-1}(1) \neq a_1(1)$, then $g(z) = \lim_h -a_1(z)/\widetilde{a}^{(h)}(z)$ is an analytic function.*

Proof. We recall that the roots of a monic polynomial are analytic functions of the coefficients, on the set where the polynomial does not have multiple roots [14]. Thus, in order to prove the analyticity of $g(z)$, it is sufficient to show that $p_z(x)$ has no multiple root $\forall z \in \mathbb{T}$. This follows from Theorem 5.1 if $z \in \mathbb{T} \setminus \{1\}$. Moreover, observe that for $z = 1$, $p_1(x)$ has roots 1 and $\frac{a_{-1}(1)}{a_1(1)}$ where the latter is real, nonnegative and different from 1 by assumption. $\square$

With the information that we have collected so far, we cannot yet say if the matrix G belongs to $\mathcal{QT}$. In fact, in principle, writing $G = T(g) + E_g$, it is not ensured that $\|E_g\|_{\mathcal{F}} < \infty$. The boundedness of $\|E_g\|_{\mathcal{F}}$ can be proved if E_g has all entries with the same sign. This analysis is part of the subject of our future research. On this regard, it is worth citing the paper [34] where, relying on probabilistic arguments, it is proved that the matrices G and R asymptotically share the Toeplitz structure.

TABLE 1. Parameters values of the test examples for the two node Jackson tandem network.

Case	λ_1	λ_2	μ_1	μ_2	p	q
1	1	0	1.5	2	1	0
2	1	0	2	1.5	1	0
3	0	1	1.5	2	0	1
4	0	1	2	1.5	0	1
5	1	1	2	2	0.1	0.8
6	1	1	2	2	0.8	0.1
7	1	1	2	2	0.4	0.4
8	1	1	10	10	0.5	0.5
9	1	5	10	15	0.4	0.9
10	5	1	15	10	0.9	0.4

5.1. **Numerical validation.** In order to validate our analysis, we consider ten instances of the two-node Jackson network, analyzed in [28]. In detail, we assume

$$A_{-1} = \begin{bmatrix} (1-q)\mu_2 & q\mu_2 & & \\ & (1-q)\mu_2 & q\mu_2 & \\ & & \ddots & \ddots \end{bmatrix},$$

$$A_0 = \begin{bmatrix} -(\lambda_1+\lambda_2+\mu_2) & \lambda_1 & & \\ (1-p)\mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & \\ & \ddots & \ddots & \ddots \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \lambda_2 & & \\ p\mu_1 & \lambda_2 & \\ & \ddots & \ddots \end{bmatrix},$$

where the parameters $p, q, \lambda_1, \lambda_2, \mu_1, \mu_2$ are chosen according to Table 1. These examples are also studied in [32] where the bad effect of truncation in approximating the stationary distribution is shown. Different decay properties of the invariant probability distribution correspond to the different values of the parameters.

We have applied CR in all the 10 cases and computed the minimal nonnegative solution G represented in the QT form as $T(g) + U_g V_g^T$. In the results of the tests that we have performed, we report, besides the CPU time in seconds, also the norm of the residual error $E = A_1 G^2 + A_0 G + A_{-1}$ where we used both the infinity norm $\|E\|_\infty$ and the QT norm $\|E\|_{\mathcal{QT}}$.

In order to analyze the intrinsic complexity of the problem, we also report the bandwidth of the matrix $T(g)$, that is the number of nonnegligible coefficients of the Laurent series $\sum_{i\in\mathbb{Z}} g_i z^i$, the number of the nonzero rows of the matrices U_g and V_g and the number of their columns that is their rank.

All this information is reported in Table 2. We may observe that a high CPU time, like for instance in the case of problem 7, corresponds to large values of the bandwidth in the matrix $T(g)$ or to large sizes of the correction. The large values of these two components of the QT representation of G imply that the entries $g_{i,j}$ have a low decay speed as $i, j \to \infty$.

TABLE 2. Features of the computed solutions by means of CR.

Case	CPU time	Res_∞	$Res_{\mathcal{QT}}$	Band	Rows	Columns	Rank
1	2.61	$8.6 \cdot 10^{-16}$	$6.1 \cdot 10^{-13}$	561	541	138	8
2	2.91	$1.5 \cdot 10^{-15}$	$7.9 \cdot 10^{-13}$	561	555	145	8
3	0.29	$1.1 \cdot 10^{-16}$	$2.7 \cdot 10^{-14}$	143	89	66	8
4	2.32	$6.8 \cdot 10^{-16}$	$6.1 \cdot 10^{-13}$	463	481	99	9
5	0.48	$1.2 \cdot 10^{-15}$	$1.1 \cdot 10^{-13}$	233	108	148	9
6	7.96	$1.9 \cdot 10^{-14}$	$6.7 \cdot 10^{-13}$	455	462	153	10
7	29	$4.3 \cdot 10^{-15}$	$6.9 \cdot 10^{-12}$	1,423	1,543	247	13
8	1.01	$1.1 \cdot 10^{-15}$	$4.3 \cdot 10^{-13}$	366	348	40	6
9	0.3	$5.4 \cdot 10^{-16}$	$2.5 \cdot 10^{-14}$	157	81	86	8
10	1.25	$1.1 \cdot 10^{-15}$	$3.4 \cdot 10^{-14}$	268	241	107	8

It must be said that in the Tandem Jackson queue that we tested, the invariant measure π can be expressed in product form [24], [28]. This representation is useful since it allows to compute directly π without approximating the semi-infinite matrix G. It is interesting to test our approach in the case where π does not admit a product-form representation. This happens, for instance for random walks in the quarter plane where the transition probabilities in the boundary of the domain do not satisfy specific restrictions like in the Tandem Jackson queue. This is the case treated in the next experiment where we have considered Example 4.1 of [24] which models a 2-d reflecting random walk with a solution nonrepresentable in product form, together with 5 other cases obtained by randomly generating nonzero values for the transition probabilities. More precisely, denoting (i, j) the coordinates of the generic particle in the quarter plane, $i, j = 0, 1, \ldots$, we have chosen random nonzero values for the rates of the transitions $(i, j) \to (i', j')$ with

- $i, j > 0$, $i' \in \{i-1, i, i+1\}$, $j' \in \{j-1, 1, j+1\}$,
- $i = 0, j > 0$, $i' \in \{0, 1\}$, $j' \in \{j-1, j, j+1\}$,
- $i > 0, j = 0$, $i' \in \{i-1, i, i+1\}$, $j' \in \{0, 1\}$,
- $i = j = 0$, $i', j' \in \{0, 1\}$.

This way, the matrices A_{-1}, A_0, and A_1 are tridiagonal. The distribution in the random numbers generation has been chosen uniform, with the only exception that we have multiplied by 2 the values of the rates of transition to lower levels in order to increase the chance to have a positive recurrent process.

Table 3 reports timings, residual errors, bandwidth, size and rank of the correction for the tested Example 4.1 of [24], followed by 5 random instances generated as described above.

All the experiments have been performed over a laptop with an i5 processor under the Linux system.

Table 3. Random walk in the quarter plane where the solution cannot be expressed in product form. For cases 1–5, the transition probabilities have been chosen randomly.

Case	CPU time	Res_∞	$Res_{\mathcal{QT}}$	Band	Rows	Columns	Rank
[24, Ex. 4.1]	1.23	$5.1 \cdot 10^{-15}$	$5.6 \cdot 10^{-13}$	324	321	58	12
1	0.38	$7.8 \cdot 10^{-15}$	$8.2 \cdot 10^{-14}$	112	55	73	13
2	0.22	$1.5 \cdot 10^{-15}$	$2.8 \cdot 10^{-14}$	85	54	44	12
3	0.13	$3.7 \cdot 10^{-15}$	$2.7 \cdot 10^{-14}$	65	40	33	8
4	0.41	$3.4 \cdot 10^{-15}$	$1.6 \cdot 10^{-13}$	181	45	153	10
5	0.44	$3.2 \cdot 10^{-15}$	$9.3 \cdot 10^{-14}$	172	144	73	18

6. Conclusion

We have introduced the class of semi-infinite quasi-Toeplitz matrices and proved that it is a Banach algebra with a suitable norm. These properties have been used to define a matrix arithmetic on the algebra of semi-infinite QT-matrices. These tools have been used to design methods for solving quadratic matrix equations with semi-infinite matrix coefficients encountered in the analysis of a class of QBD stochastic processes.

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