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SELFADJOINTNESS OF ELLIPTIC DIFFERENTIAL OPERATORS IN $L_2(G)$, AND CORRECTION POTENTIALS

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Abstract. We consider the question of the essential selfadjointness of a symmetric second order elliptic operator L of general form in the space $L_2(G)$ ($D_L = C_0^\infty(G)$), where G is an arbitrary open set in R^n. The main idea is that using the matrix $A(x)$ of the highest order coefficients of L and the domain G, it is possible to construct a function $q_A(x)$ such that the essential selfadjointness of $\bar{L}$ follows from the semiboundedness of the operators L and $L - q_A(x)$. The function $q_A(x)$ is called the correction potential, and we suggest a number of procedures for its construction. We develop a technique which, given a correction potential allows us to establish criteria for the selfadjointness of an elliptic operator in terms of the behaviour of its coefficients. These general results are applied to the Schrödinger operator, which for $G \neq R^n$ leads to new assertions that generalise a number of known theorems.

Contents

1. Introduction

Let G be an arbitrary open set in R^n. In what follows we shall denote the inner product and the norm in an infinite-dimensional Hilbert space by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, and the inner product and the norm in a unitary space E (dim $E<\infty$) by $(\cdot,\cdot)$ and $|\cdot|$. Throughout this paper we consider elliptic differential operators in the space $L_2(G)$ that have one of the following forms:

$$Lu = -\nabla(A(x)\nabla u) + i\big[\nabla\big(\vec{b}(x)u\big) + \big(\nabla u, \vec{b}(x)\big)\big] + q(x)u, \qquad D_L = C_0^\infty(G), \tag{1.1}$$

$$Mu = \big(\nabla - i\vec{b}(x)\big)^*\big(A(x)\big(\nabla - i\vec{b}(x)\big)u\big) + q(x)u, \qquad D_M = C_0^\infty(G). \tag{1.2}$$

Here $A(x)$ is a positive Hermitian matrix function, $\vec{b}(x)$ is an n-component vector function with real components and $q(x)$ is a real-valued function. Local conditions on the coefficients of the operators L and M are described in § 2.

A great many papers are devoted to investigating conditions under which elliptic operators are selfadjoint. However, in the case of $G \neq R^n$, until recently sharp conditions for selfadjointness were only known for the Schrödinger operator in a domain $G = R^n \backslash V$, where V is a set of measure zero of a particular form. In recent years general results for

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operators of the form (1.1), (1.2) were obtained involving the least restrictive sufficient conditions for the Schrödinger operator. This paper is devoted to strengthening these recent results.

In a number of special cases of the operators L and M when $G = R^n$, semiboundedness implies essential selfadjointness. This was first proved by Povzner [1] for the Schrödinger operator with a continuous potential. The well-known theorem due to Berezanskii [2] shows that under quite general local conditions, if M is semibounded, then $\overline{M}$ is essentially selfadjoint provided the local wave perturbation, whose propagation is described by the equation $u''_{tt} + Mu = 0$, cannot reach the boundary of the domain G in finite time (the BFSP condition). When $G = R^n$, this condition had been studied in a number of papers (see, e.g., [3, 4]). As is shown in [4, 5], it is equivalent to the completeness of the Riemannian manifold R^n in the metric defined by the matrix $A^{-1}(x)$. This completeness condition is a typical one in works concerning the selfadjointness of differential operators on manifolds (see, for instance, [6, 7, 8]). If it does not hold, this can lead to the loss of selfadjointness in the case $G = R^n$, $b_j(x) = q(x) \equiv 0$ (see [9, 10]) and even in the case of a semibounded Sturm–Liouville operator in $L_2(-\infty, +\infty)$ (see [11, Remark 1]). We can think of Weyl's theorem [12] for a Sturm-Liouville operator as the first criterion for selfadjointness without a completeness condition. Multidimensional analogs of this theorem together with a number of cognate results are contained in [13, 14, 15].

In § 3 of this paper we consider conditions under which the semiboundedness of the operator L implies its essential selfadjointness without imposing any completeness type conditions. In this context, it is convenient to use the concept of a semimaximal operator introduced in [16].

Definition. A symmetric operator T in a Hilbert space H is called semimaximal if for any $u \in D_{T^*}$ such that $\operatorname{Im}\langle T^*u, u\rangle = 0$, there is a sequence $\{u_k\}_{k=1}^{\infty}$ of elements of D_T, such that $u_k \longrightarrow u$ in H and

$$\lim_{k\to\infty} \langle Tu_k, u_k\rangle = \langle T^*u, u\rangle. \tag{1.3}$$

It is shown in [16] that for semimaximal operators semiboundedness implies essential selfadjointness, while every essentially maximal symmetric operator is semimaximal.

In § 3 we show that under the local conditions formulated in § 2, for an arbitrary open set $G \subseteq R^n$ and a matrix function $A(x)$ of the highest order terms of the operator L, there is a locally bounded function $q_A(x)$ in G, such that if $L - q_A(x)$ is semibounded, then the operator L is semimaximal. (The operator L does not have to be semibounded from below.) The function $q_A(x)$ is called a *correction potential* for $A(x)$ in the domain G. The main result of § 3 (Theorem 3.2) allows us to obtain different methods for constructing correction potentials, which can be semibounded from below or nonsemibounded. The results of § 3 form a foundation for the proofs of the remaining results in this paper and strengthen the statements of [17]. That paper further develops the idea of the first result of this type, Walter's well-known theorem [18], which is a particular case of the statements of § 3.

In § 4 we extend the well-known Kalf–Walter–Schmincke–Simon theorem to the case of an arbitrary open set G and an operator M of the form (1.2) (see [19, theorem X.30]). This theorem deals with a Schrödinger operator in the domain $G = R^n \backslash \{\vec{0}\}$ and gives the least restrictive known selfadjointness conditions in terms of the distance to $\partial G = \{\vec{0}\}$. The first results of this type for $n > 1$ were due to Jörgens [20], while the theorem itself was proved by Simon [21], who generalised theorems of Kalf–Walter [22] and Schmincke [23]. For the case $n = 1$ Friedrichs [24] had already proved a similar theorem (see also [19], theorem X.10). A number of analogs and extensions of the Kalf–Walter–Schmincke–Simon theorem are contained in [25, 26, 27, 28, 29]. All these deal

with the Schrödinger operator in domains of the particular types described before. In § 4 using one of the methods of construction of a correction potential and a generalisation of Hardy's inequality (Theorem 4.1), we study the behaviour of the potential $q(x)$ of a semibounded operator M of the form (1.2) near the boundary which guarantees that $\overline{M}$ is selfadjoint. As we show in §§ 5, 6, Theorem 4.3, which is proved using this, generates the known generalisations of the Kalf–Walter–Schmincke–Simon theorem. Theorem 4.3 strengthens the results of [30] by weakening the smoothness requirements on the coefficients of the operator M.

In § 5 we prove Theorem 5.1, which establishes the essential selfadjointness of an elliptic operator that is not, in general, semibounded, if a certain semibounded operator is selfadjoint. This allows us to extend Theorem 4.3 to the case of nonsemibounded operators (Theorem 5.3). Theorem 5.1 was established in [31] and is a development of an idea in [32].

In § 6 we establish a number of new criteria for the selfadjointness of the Schrödinger operator, which follow from the results of § 5. In particular, we consider the Hamiltonian of a system of strongly interacting particles in an external field with a potential that is unbounded below. The results of § 6 were announced in [33] and they cover a number of theorems in [25, 26, 29] and the survey paper [34]. This last reference contains an extensive bibliography that reflects the history of the topics being addressed here.

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2. Basic local conditions

Let us denote by $\mathrm{Lip}^{(r)}_{\alpha\mathrm{loc}}(G)$ the set of r-component vector functions $\vec{f}(x)$, defined in G and satisfying the condition

$$\left|\vec{f}(x_0+y)-\vec{f}(x_0)\right| = O\left(|y|^{\alpha}\right) \quad \text{as} \quad |y|\to 0, \tag{2.1}$$

for some $\alpha\in(0,1]$ and for any point $x_0\in G$. Here the constant in $O(\cdot)$ in general depends on x_0. In what follows we shall denote the set of functions in $C^1(G)$ with gradients in $\mathrm{Lip}^{(n)}_{\alpha\mathrm{loc}}(G)$ by $C^{(1,\alpha)}(G)$. For $\alpha=1$ the set $\mathrm{Lip}^{(r)}_{\alpha\mathrm{loc}}(G)$ will be denoted by $\mathrm{Lip}^{(r)}_{\mathrm{loc}}(G)$. The set consisting of all the elements that satisfy (2.1) with $\alpha=1$ and such that the constant in $O(\cdot)$ is independent of x_0, will be denoted by $\mathrm{Lip}_r(G)$.

We note that for $\vec{f}(x)\in\mathrm{Lip}^{(r)}_{\mathrm{loc}}(G)$ first partial derivatives exist almost everywhere in G (see [36, p. 295]). Therefore the operators L and M are well defined on $C_0^\infty(G)$ under the conditions

$$a_{ij}(x), b_j(x)\in\mathrm{Lip}^{(1)}_{\mathrm{loc}}(G), \quad q(x)\in L_{2\mathrm{loc}}(G). \tag{2.2}$$

We assume throughout that these conditions are satisfied.

Let $D_{\mathrm{loc}}(L^*)$ denote the set of functions $u\in L_{2\mathrm{loc}}(G)$ for each of which there is a function $g\in L_{2\mathrm{loc}}(G)$ such that for all $\varphi\in C_0^\infty(G)$ we have the equality $\langle u, L\varphi\rangle=\langle g,\varphi\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\mathrm{supp}\,\varphi)$. In particular, $D_{L^*}\subset D_{\mathrm{loc}}(L^*)$.

In addition to (2.2), we assume that the coefficients of the operator L have local properties under which the following condition holds:

Condition A. *For any function $u(x)\in D_{\mathrm{loc}}(L^*)$ and any bounded domain Ω ($\overline{\Omega}\subset G$) there is a sequence $\{u_m\}_{m=1}^\infty$ of functions in $C^2(G)$ for which $u_m\to u$ in $L_2(\Omega)$ and*

$$\lim_{m\to\infty}\mathrm{Re}\langle\psi Lu_m, u_m\rangle_{L_2(\Omega)} = \mathrm{Re}\,\langle\psi g, u\rangle_{L_2(\Omega)}$$

for any real-valued function $\psi\in C_0(\Omega)$.

It is not hard to show that Condition A is satisfied if

$$a_{ij}(x) \in C^2(G), \quad b_j(x) \in C^1(G), \quad q(x) \in L_{\infty\mathrm{loc}}(G).$$

We note that a differential expression M of the form (1.2) can always be written as an expression for L of the form (1.1) with the same matrix $A(x)$, but with different $\vec{b}(x)$ and $q(x)$. The following proposition gives a sufficient criterion for Condition A to hold when $L = M$.

Proposition 2.1. *Let M be an operator of the form (1.2). Suppose that the term $A(x)$ in (1.2) is a positive symmetric (real) matrix function, and*

$$a_{ij}(x) \in C^{(1,\alpha)}(G), \quad b_j(x) \in C^1(G),$$

while the real-valued function $q(x) \in L_{2\mathrm{loc}}(G)$ is locally bounded below. Then Condition A *is satisfied with $L = M$.*

Proof. Using the operator M and a bounded domain $\Omega \Subset G$ we construct a differential operator M_Θ on the space $L_2(R^n)$. Let Ω_0 and Ω_1 be bounded open sets such that $\overline{\Omega} \Subset \Omega_0$, $\overline{\Omega}_0 \Subset \Omega_1$, $\overline{\Omega}_1 \Subset G$. We denote by $A_1(x)$, $\vec{b}_1(x)$ and $q_1(x)$ functions defined in R^n, which coincide with $A(x)$, $\vec{b}(x)$ and $q(x)$ on Ω_1 and vanish outside Ω_1. Let $0 \le \Theta(x) \le 1$ be a $C^\infty(R^n)$ function such that $\Theta(x) \equiv 1$ for $x \in \Omega_0$, and $\Theta(x) \equiv 0$ for $x \notin \Omega_1$. Consider the operator

$$M_\Theta u = \big(\nabla - i\Theta\vec{b}_1(x)\big)^* \big(A_\Theta(x)\big(\nabla - i\Theta\vec{b}_1(x)\big)u\big) + \Theta q_1(x)u, \quad D_{M_\Theta} = C_0^\infty(R^n)$$

in $L_2(R^n)$. Here $A_\Theta(x) = \Theta(x)A_1(x) + (1-\Theta(x))I_n$, where I_n is the unit $n \times n$ matrix. M_Θ is an elliptic operator that satisfies all the conditions of Theorem 1 of [13]. In view of this theorem M_Θ is essentially selfadjoint. Let $u(x) \in D_{\mathrm{loc}}(M^*)$, so that if $x \in \Omega$, $g = Mu = M_\Theta u$. Let $\varphi(x) \in C_0^\infty(R^n)$ be such that $\varphi(x) \equiv 1$ for $x \in \Omega$ and $\varphi(x) \equiv 0$ for $x \notin \Omega_0$. If $u_1 = u$ for $x \in \Omega_0$ and $u_1 = 0$ for $x \notin \Omega_0$, then the function $\varphi u_1 \in L_2(R^n)$. Adapting Kato's method [35], we find that $\varphi u_1 \in D_{M_\Theta^*} = D_{\overline{M}_\Theta}$. Therefore, there is a sequence $\{u_m\}$ of elements in $C_0^\infty(R^n)$ such that $u_m \to \varphi u_1$ and $M_\Theta u_m \to M_\Theta(\varphi u_1)$ in $L_2(R^n)$. Since for $x \in \Omega$, $M_\Theta(\varphi u_1) = Mu = g$, $M_\Theta u_m = Mu_m$, we have that $u_m \to u$, $Mu_m \to Mu$ in $L_2(\Omega)$. Hence it follows that Condition A holds. The proposition is proved. □

Remark 2.1. If Condition A holds for an operator L, it also holds for $T = L + p(x)$, where $p(x) \in L_{\infty\mathrm{loc}}(G)$. In particular, in Proposition 2.1 we can assume that $q(x)$ is essentially locally bounded from below.

Indeed, since $p(x)$ is locally essentially bounded, $D_{\mathrm{loc}}(L^*) = D_{\mathrm{loc}}(T^*)$. If $\{u_m\}$ is any sequence in Condition A for the operator L, then it can also be used for the same $u(x)$ in Condition A for the operator T.

3. Construction of correction potentials

3.1. We can assume that the quadratic form

$$L(\varphi,\varphi) = \int_G \Big[(A(x)\nabla\varphi, \nabla\varphi) - 2\,\mathrm{Im}\,\big(\nabla\varphi, \varphi\vec{b}(x)\big) + q(x)|\varphi|^2\Big]\,dx \tag{3.1}$$

that corresponds to an operator L of the form (1.1) in the space $L_2(G)$ is defined for $\varphi \in C_0(G) \cap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$. It is clear that for $\varphi \in C_0^\infty(G)$ we have the equality $L(\varphi,\varphi) = \langle L\varphi, \varphi\rangle$.

Denote by $\rho(x)$ and $\sigma(x)$ functions in $\mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ such that

$$0 \le \rho(x) \to \infty \quad \text{as} \quad x \to \partial G, \quad 0 \le \sigma(x) \le \text{const}. \tag{3.2}$$

If $\rho(x)$ satisfies (3.2), then for any $N > 0$ there is a compact set $\mathrm{R}_N \subset G$ such that for $x \in G \setminus \mathrm{R}_N$ we have $\rho(x) > N$.

Suppose that

$$\sigma^2(A\nabla\rho, \nabla\rho) + (A\nabla\sigma, \nabla\sigma) \leq \ C\rho^m \cdot \ e^{2\alpha\rho} \tag{3.3}$$

almost everywhere in G, where the constants $C, m > 0$, $\alpha \geq 0$. We introduce the function

$$Q_{\alpha,\varepsilon}(x) = (\alpha + \varepsilon)\, e\, \sigma(A\nabla\rho, \nabla\rho)^{1/2} + (A\nabla\sigma, \nabla\sigma)^{1/2},$$

where the constants $\alpha \geq 0$, $\varepsilon > 0$ are arbitrary, while e is the basis of the natural logarithm. We suppose also that $\varphi \in C_0^\infty(G)$ satisfies the inequality

$$L(\sigma\varphi, \sigma\varphi) + C_1\|\sigma\varphi\|^2 + C_2\|\varphi\|^2 \geq \|Q_{\alpha,\varepsilon}\,\varphi\|^2, \tag{3.4}$$

where the constants $C_1, C_2 \geq 0$, $\varepsilon > 0$, while the constant α is the same as in (3.3). Conditions (3.3), (3.4) can be satisfied for any operator L ($\sigma(x) \equiv 0$, for example). They guarantee the following *a priori* estimates for $u(x) \in D_{L^*}$.

Theorem 3.1. *Assume that Condition* A *holds and also that (3.3) and (3.4) are satisfied. Then for each $u(x) \in D_{L^*}$ the integral*

$$\int_G \sigma^2(A\nabla\rho, \nabla\rho)|u|^2\, dx < \infty \tag{3.5}$$

converges. If in addition in (3.4) the constant $C_2 = 0$, then for any solution $u(x)$ of the equation $L^*u = -\lambda u$, with $\lambda \geq C_1$, we have

$$\sigma(A\nabla\rho, \nabla\rho)|u| = 0 \tag{3.6}$$

almost everywhere in G.

Different *a priori* estimates for $u(x) \in D_{L^*}$ may be found in [37]. As a consequence of Theorem 3.1 we have

Theorem 3.2. *Assume that Condition* A*, and conditions (3.3) and (3.4) hold.*

1°. *If there is a function $\mu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$, $0 \leq \mu(x) \to \infty$ as $x \to \partial G$, for which*

$$\sigma^2(A\nabla\rho, \nabla\rho) \geq (A\nabla\mu, \nabla\mu) \tag{3.7}$$

almost everywhere in G, the operator L is semimaximal, and therefore if L is semibounded, it is essentially selfadjoint.

2°. *Suppose in (3.4) $C_2 = 0$ and there exists $\mu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ such that $0 \leq \mu(x) \to \infty$ as $x \to \partial G$, for which*

$$\mathrm{meas}\big\{x\colon \sigma^2(A\nabla\rho, \nabla\rho) = 0;\ (A\nabla\mu, \nabla\mu) > 0\big\} = 0; \tag{3.8}$$

then, if L is semibounded, it is essentially selfadjoint.

Before proving Theorems 3.1 and 3.2, we present a number of corollaries of Theorem 3.2.

Corollary 3.1. *Let $\eta(x), \rho(x) \geq 0$ be functions in $\mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ such that $\rho(x) \to \infty$ as $x \to \partial G$ and for some constants $C, m > 0$ the inequality*

$$(A\nabla\rho, \nabla\rho) + (A\nabla\eta, \nabla\eta) \leq C\rho^m \cdot e^{2\eta} \tag{3.9}$$

holds almost everywhere in G.

If for some $K \geq 0$ and $\delta > 0$ and all $\varphi \in C_0^\infty(G)$ the operator inequality

$$\langle L\varphi, \varphi\rangle + K\|\varphi\|^2 \geq \int_G \big[\delta(A\nabla\rho, \nabla\rho)^{1/2} + (A\nabla\eta, \nabla\eta)^{1/2}\big]^2\, |\varphi|^2\, dx, \tag{3.10}$$

and Condition A *hold, then the operator L is essentially selfadjoint.*

Proof. The proof consists of verifying that part 2° of Theorem 3.2 holds with $\sigma(x) = e^{-\eta(x)}$ and the same $\rho(x)$. By (3.9), condition (3.3) holds with $\alpha = 0$, and (3.8) is satisfied with $\mu(x) = \rho(x)$. Our operator inequality (3.10) will also hold for $\varphi \in C_0(G) \cap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ if we replace $\langle L\varphi, \varphi\rangle$ by $L(\varphi, \varphi)$. To see this, it is enough to consider the convolution $\varphi_t(x) = (\varphi * \omega_t)(x)$, where $\omega_t(y) \in C_0^\infty(G_t)$ $(G_t = \{y\colon |y| < t\})$ is a convolution kernel with a sufficiently small radius t. In any bounded domain $\Omega \supset \operatorname{supp}\varphi$, $\varphi_t(x) \to \varphi(x)$ as $t \to 0$ uniformly for $x \in \Omega$ and $\varphi_t(x) \to \varphi(x)$ in the Sobolev space $W_2^1(\Omega)$. Therefore $L(\varphi_t, \varphi_t) \to L(\varphi, \varphi)$ as $t \to 0$. From this, using (3.10) we conclude that (3.4) will hold for any function $\varphi(x) = e^{\eta(x)}\psi(x)$, where $\psi(x) \in C_0(G) \cap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$. This set of functions clearly contains $C_0^\infty(G)$. Therefore (3.4) is satisfied and Corollary 3.1 is proved. □

From Corollary 3.1 we derive the next result, which is easier to apply and extends the result of [18].

Corollary 3.2. *Let a function $0 \le \Theta(\tau) \in C([0, \infty))$ be such that $\int_0^\infty \Theta(\tau)\, d\tau = \infty$. If for some $K \ge 0$ and all $\varphi \in C_0^\infty(G)$*

$$\langle L\varphi, \varphi\rangle + K\|\varphi\|^2 \ge \int_G (1 + \Theta(\eta))^2 (A\nabla\eta, \nabla\eta)|\varphi|^2\, dx,$$

where $\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ is such that $0 \le \eta(x) \to \infty$ as $x \to \partial G$ and for some constants $C, m > 0$

$$(1 + \Theta^2(\eta))\,(A\nabla\eta, \nabla\eta) \le C \left(\int_0^{\eta(x)} \Theta(\tau)\, d\tau\right)^m e^{2\eta}, \tag{3.11}$$

then under Condition A *the operator L is essentially selfadjoint.*

Proof. The proof consists of verifying the conditions of Corollary 3.1 for

$$\rho(x) = \int_0^{\eta(x)} \Theta(\tau)\, d\tau$$

and the same $\eta(x)$ as there. □

Lemma 3.1. *If the operator $T = L + p^2(x)$ is essentially selfadjoint, where the function $p(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ and is real-valued and if, for some $N \ge 0$ and for all $\varphi \in C_0^\infty(G)$,*

$$L(p\varphi, p\varphi) - \int_G (A\nabla p, \nabla p)|\varphi|^2\, dx \ge -\frac{1}{2}\|L\varphi\|^2 - N\|\varphi\|^2,$$

then L is also essentially selfadjoint.

Proof. Using integration by parts it is not hard to show that for all $\varphi \in C_0^\infty(G)$

$$L(p\varphi, p\varphi) = \int_G (A\nabla p, \nabla p)|\varphi|^2\, dx + \operatorname{Re}\langle p^2\varphi, L\varphi\rangle.$$

Therefore, under our hypotheses,

$$2\operatorname{Re}\langle p^2\varphi, L\varphi\rangle \ge -\|L\varphi\|^2 - 2N\|\varphi\|^2.$$

Hence

$$\|T\varphi\|^2 + 2N\|\varphi\|^2 = \|L\varphi\|^2 + 2\operatorname{Re}\langle p^2\varphi, L\varphi\rangle + 2N\|\varphi\|^2 + \|p^2\varphi\|^2 \ge \|p^2\varphi\|^2,$$

that is, the operator of multiplication by $-p^2(x)$ is T-bounded with T-bound 1. Hence (see [38, Chapter 5, § 4.2]) $L = T - p^2$ is essentially selfadjoint. □

Corollary 3.3. *Assume that Condition* A *holds. Suppose that for some $K \geq 0$ and all $\varphi \in C_0^\infty(G)$*

$$\langle L\varphi, \varphi\rangle + K\|\varphi\|^2 \geq \int_G (A\nabla\eta, \nabla\eta)|\varphi|^2\, dx,$$

where $\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ is such that $0 \leq \eta(x) \to \infty$ as $x \to \partial G$, and for some $C > 0$

$$(A\nabla\eta, \nabla\eta) \leq Ce^{2\eta}, \tag{3.12}$$

almost everywhere in G. Then L is essentially selfadjoint.

Proof. As in the proof of Corollary 3.1, it is not hard to show that if we substitute $\langle L\varphi, \varphi\rangle$ for $L(\varphi, \varphi)$, the operator inequality holds for $\varphi \in C_0(G) \cap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$. If condition (3.12) holds, (3.11) is satisfied with $\Theta(\tau) = \delta > 0$. Therefore from Corollary 3.2 the operator $T\varphi = L\varphi + K\varphi + \varepsilon e^{2\eta}\varphi$ is essentially selfadjoint for arbitrary $\varepsilon > 0$. Let us apply Lemma 3.1 with $p^2(x) = \varepsilon e^{2\eta}$ to the operator $L\varphi + K\varphi$. Under our conditions

$$\varepsilon L(e^\eta\varphi, e^\eta\varphi) + \varepsilon K\|e^\eta\varphi\|^2 - \varepsilon\int_G (A\nabla\eta, \nabla\eta)|e^\eta\varphi|^2\, dx \geq 0.$$

Therefore the operator $L + K$, and thus also L are essentially selfadjoint. Corollary 3.3 is proved. □

Corollary 3.4. *Let $r(\tau) \geq \delta > 0 \in C^1([0;\infty))$ be such that $r'(\tau) = o(r^2(\tau))$ as $\tau \to \infty$, and suppose that the function $\mu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ is such that $0 \leq \mu(x) \to \infty$ as $x \to \partial G$. Assume that the inequality*

$$(A\nabla\mu, \nabla\mu) \leq \; C\left(\int_0^{\mu(x)} r(\tau)\, d\tau\right)^m \exp\left\{2\int_0^{\mu(x)} r(\tau)\, d\tau\right\} \tag{3.13}$$

holds almost everywhere in G, with some constants $C, m > 0$. If for some $K, k \geq 0$, $\varepsilon > 0$ and all $\varphi \in C_0^\infty(G)$ the operator inequality

$$\langle L\varphi, \varphi\rangle + k\|\varphi\|^2 \geq \int_G r^2(\mu(x))\big[(e^2 + \varepsilon)(A\nabla\mu, \nabla\mu) - K\big]|\varphi|^2\, dx$$

is satisfied, then under Condition A *the operator L is semimaximal.*

Proof. We will check that the hypotheses of part 1° of Theorem 3.2 are satisfied, with

$$\rho(x) = \int_0^{\mu(x)} r(\tau)\, d\tau, \quad \sigma(x) = \big(r(\mu(x))\big)^{-1}.$$

For this choice of $\rho(x)$ and $\sigma(x)$ let us estimate the function $Q_{\alpha,\varepsilon_1}(x)$ with $\alpha = 1$, $\varepsilon_1 > 0$:

$$\begin{aligned} Q^2_{\alpha,\varepsilon_1}(x) &= \left[(1 + \varepsilon_1)e(A\nabla\mu, \nabla\mu)^{1/2} + \frac{|r'(\mu)|}{r^2(\mu)}(A\nabla\mu, \nabla\mu)^{1/2}\right]^2 \\ &\leq (e^2 + \varepsilon)(A\nabla\mu, \nabla\mu) + C_\varepsilon. \end{aligned}$$

This inequality holds for all $\varepsilon > \big(2\varepsilon_1 + \varepsilon_1^2\big)e^2$. From our operator inequality it follows that

$$\begin{aligned} L(\sigma\varphi, \sigma\varphi) + k\|\sigma\varphi\|^2 &\geq \int_G \big[(e^2 + \varepsilon)(A\nabla\mu, \nabla\mu) - K\big]|\varphi|^2\, dx \\ &\geq \|Q_{\alpha,\varepsilon_1}\varphi\|^2 - C_\varepsilon\|\varphi\|^2 - K\|\varphi\|^2. \end{aligned}$$

Hence (3.4) holds with $C_1 = k$ and $C_2 = K + C_\varepsilon$. Inequality (3.3) follows from (3.13). Since here

$$\sigma^2(A\nabla\rho, \nabla\rho) = (A\nabla\mu, \nabla\mu),$$

all the conditions of part 1° of Theorem 3.2 are satisfied. Corollary 3.4 is proved. □

We note that Corollary 3.4 describes correction potentials $q_A(x)$ that may be unbounded below. Thus, for any $\varepsilon > 0$ we could take as a correction potential the expression

$$q_A(x) = (e^2 + \varepsilon)(A\nabla\eta, \nabla\eta) - K\eta^2,$$

where the constant K is ≥ 0, while the function $\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ is such that $0 \leq \eta(x) \to \infty$ as $x \to \partial G$, and satisfies

$$(A\nabla\eta, \nabla\eta) \leq C\eta^\alpha e^{2\eta} \tag{3.14}$$

almost everywhere in G for some constants $C, \alpha \geq 0$. To see that this is the case, it is sufficient to set $\mu(x) = \ln\eta(x)$, $r(\tau) = e^\tau$ in Corollary 3.4.

Remark 3.1. Corollaries 3.1–3.4 allow us to construct a correction potential $q_A(x)$ for any open set $G \subseteq R^n$ and any matrix function $A(x)$ that satisfies one of the conditions (3.10), (3.12)–(3.14). By an appropriate choice of the functions used in the estimates, these conditions can be satisfied for any continuous matrix function $A(x) > 0$.

As an example, we will construct a function $\eta(x)$ satisfying (3.12). Take an arbitrary function $\nu(x) \in C^2(G)$ such that $0 \leq \nu(x) \to \infty$ as $x \to \partial G$. As $\nu(x)$ we can take the function $(\delta(x))^{-1} + \delta_0^2(x)$, where $\delta(x)$ is the regularised distance of a point x from the closed set $R^n \backslash G$ (see [36, p. 203]), while $\delta_0(x)$ is the distance from a fixed point $x_0 \in G$. The function

$$R_1^2(\tau) = \sup\nolimits_{x \in G^\tau}(A\nabla\nu, \nabla\nu),$$

where $G^\tau = \{x\colon \nu(x) < \tau\}$, is nondecreasing, and furthermore we can assume that $R_1(\tau) \to \infty$ as $\tau \to \infty$. Otherwise (3.12) holds with $\eta(x) = \nu(x)$. We can find $R(\tau) \in C^\infty([0,\infty))$ such that $R(\tau) \geq R_1(\tau) + 1$ and $|R'(\tau)| \leq \mathrm{const}\, R^2(\tau)$. We set $\eta(x) = 2\ln R(\nu(x))$. Then

$$(A\nabla\eta, \nabla\eta) \leq R^2(\nu)(A\nabla\nu, \nabla\nu) \leq CR^4(\nu) = Ce^{2\eta},$$

which proves (3.12).

3.2. To prove Theorem 3.1 we need two lemmas.

Lemma 3.2 ([17]). *Let $f(\tau)$ be a function defined for $\tau \in [0,\infty)$, which satisfies*

$$0 \leq f(\tau) \leq C\tau^l(\ln\tau)^m$$

for some constants $C, l, m \geq 0$. Then for any numbers $r > 1$, $\delta > 0$ there is a sequence of points $\{\tau_k\}_{k=1}^\infty$, with $\tau_k \to \infty$ as $k \to \infty$, for which

$$f(\tau_k) \leq r^l(1+\delta)f\left(\frac{\tau_k}{r}\right).$$

Proof. Let us assume the opposite, that is, that for some $r_0 > 1$, $\delta_0 > 0$ and for all $\tau \geq \tau_0 > 0$

$$f(\tau) > r_0^l(1+\delta_0)f\left(\frac{\tau}{r_0}\right).$$

Then the sequence $\tau_k' = \tau_0 r_0^k$ satisfies

$$f\big(\tau_0 r_0^k\big) > r_0^{lk}(1+\delta_0)^k f(\tau_0).$$

From the hypothesis of the lemma we obtain

$$C\tau_0^l r_0^{lk}(\ln\tau_0 + k\,\ln r_0)^m > r_0^{lk}(1+\delta_0)^k f(\tau_0),$$

which is impossible for sufficiently large k if $f(\tau_0) \neq 0$. If there are arbitrarily large τ_0 such that $f(\tau_0) \neq 0$, we obtain a contradiction, so that the statement of the lemma holds. In the opposite case the statement is obvious. Lemma 3.2 is proved. □

We now introduce the function

$$\psi_\gamma(x,\tau) = \left[1 - \left(\frac{e^{\rho(x)}}{\tau}\right)^\gamma\right]_+. \tag{3.15}$$

The support of this function depends on the parameter $\tau > 0$ and γ is an arbitrary positive constant.

Consider the integral

$$J_\gamma(\tau) = \int_{\Omega^\tau} \left[\gamma\sigma(A\nabla\rho,\nabla\rho)^{1/2} + \psi_\gamma(A\nabla\sigma,\nabla\sigma)^{1/2}\right]^2 |u|^2\,dx, \tag{3.16}$$

where $\Omega^\tau = \{x\colon e^{\rho(x)} < \tau\}$, $u(x) \in L_2(G)$.

Lemma 3.3 ([17])**.** *If (3.3) holds, then for any $u(x) \in L_2(G)$ and $\varepsilon > 0$ there exists a sequence $\{\tau_k\}_{k=1}^\infty$, $0 < \tau_k \to \infty$, and a number $\gamma > 0$ such that*

$$J_\gamma(\tau_k) \le \left\|Q_{\alpha,\varepsilon}(x)\,\psi_\gamma(x,\tau_k)\,u(x)\right\|^2.$$

Proof. The integral $J_\gamma(\tau)$ can be written in the form

$$\begin{aligned} J_\gamma(\tau) &= \int_{\Omega^\tau} \psi_\gamma^2(A\nabla\sigma,\nabla\sigma)|u|^2\,dx \\ &\quad + \int_{\Omega^\tau} \left[2\gamma\sigma\psi_\gamma(A\nabla\sigma,\nabla\sigma)^{1/2}(A\nabla\rho,\nabla\rho)^{1/2} + \gamma^2\sigma^2(A\nabla\rho,\nabla\rho)\right]|u|^2\,dx \\ &= J_1(\tau) + J_2(\tau). \end{aligned}$$

By (3.3) we have

$$J_2(\tau) \le C\int_{\Omega^\tau} \left[\sigma^2(A\nabla\rho,\nabla\rho) + (A\nabla\sigma,\nabla\sigma)\right]|u|^2\,dx \le C_u\tau^{2\alpha}(\ln\tau)^m.$$

According to Lemma 3.2, for arbitrary $r > 1$, $\delta > 0$, $\gamma > 0$ we can find a sequence $\{\tau_k\}$ that satisfies

$$\begin{aligned} J_2(\tau_k) &\le r^{2\alpha}(1+\delta)J_2\left(\frac{\tau_k}{r}\right) \\ &= r^{2\alpha}(1+\delta) \\ &\quad \times \int_{\Omega^{\tau_k/r}} \left[2\gamma\sigma\psi_\gamma\left(x,\frac{\tau_k}{r}\right)(A\nabla\sigma,\nabla\sigma)^{1/2}(A\nabla\rho,\nabla\rho)^{1/2} + \gamma^2\sigma^2(A\nabla\rho,\nabla\rho)\right]|u|^2\,dx. \end{aligned}$$

For $x \in \Omega^{\tau_k/r}$ we have

$$\psi_\gamma\left(x,\frac{\tau_k}{r}\right) = \left(1 - \frac{r^\gamma e^{\gamma\rho}}{\tau_k^\gamma}\right)_+ \le \left(1 - \frac{e^{\gamma\rho}}{\tau_k^\gamma}\right)_+ = \psi_\gamma(x,\tau_k).$$

Moreover, since $e^{\gamma\rho} \le \left(\frac{\tau_k}{r}\right)^\gamma$ for $x \in \Omega^{\tau_k/r}$, we obtain

$$\psi_\gamma(x,\tau_k) = \left(1 - \frac{e^{\gamma\rho}}{\tau_k^\gamma}\right)_+ \ge \frac{r^\gamma - 1}{r^\gamma}.$$

From the above estimates, we obtain the inequality

$$\begin{aligned} J_2(\tau_k) &\le r^{2\alpha}(1+\delta)\int_{\Omega^{\tau_k/r}} \psi_\gamma^2(x,\tau_k) \\ &\quad \times \left[2\gamma\sigma\frac{r^\gamma}{r^\gamma - 1}(A\nabla\sigma,\nabla\sigma)^{1/2}(A\nabla\rho,\nabla\rho)^{1/2} + \gamma^2\sigma^2\frac{r^{2\gamma}}{(r^\gamma-1)^2}(A\nabla\rho,\nabla\rho)\right]|u|^2\,dx, \end{aligned}$$

and so, since $\Omega^{\tau k/r} \subset \Omega^{\tau k}$, we have

$$\begin{aligned} J_\gamma(\tau_k) &= J_1(\tau_k) + J_2(\tau_k) \\ &\le \int_{\Omega^{\tau_k}} \psi_\gamma^2(x,\tau_k) \left[\frac{\gamma r^{\gamma+\alpha}(1+\delta)}{r^\gamma - 1}\, \sigma\, (A\nabla\rho, \nabla\rho)^{1/2} + (A\nabla\sigma, \nabla\sigma)^{1/2} \right]^2 |u|^2\, dx. \end{aligned}$$

Taking $\gamma > 0$ fixed, let us find the minimum of the function $\beta(r) = \frac{\gamma r^{\gamma+\alpha}}{(r^\gamma - 1)}$. Obviously $\beta_{\min} = \alpha\left(1 + \frac{\gamma}{\alpha}\right)^{1+\alpha/\gamma}$ for $\alpha > 0$ and $\beta_{\min} = \gamma$ for $\alpha = 0$. Therefore for any $\varepsilon > 0$ we can choose $r > 1$, $\delta > 0$, $\gamma > 0$ such that

$$\gamma r^{\gamma+\alpha} \frac{1+\delta}{r^\gamma - 1} \le (\alpha + \varepsilon) e.$$

For these fixed r, δ, γ there is a sequence $\{\tau_k\}_{k=1}^\infty$ for which

$$J_\gamma(\tau_k) \le \int_{\Omega^{\tau_k}} Q_{\alpha,\varepsilon}^2(x) \psi_\gamma^2(x,\tau_k) |u(x)|^2\, dx,$$

as required. This completes the proof of Lemma 3.3. □

Proof of Theorem 3.1. Using integration by parts, it is not hard to show that any real-valued function $\psi(x) \in C_0(G) \cap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ and any function $\nu(x) \in C^2(G)$ satisfy the equality

$$L(\psi\nu, \psi\nu) = \int_G (A\nabla\psi, \nabla\psi)|\nu|^2\, dx + \mathrm{Re}\langle \psi^2 L\nu, \nu\rangle. \tag{3.17}$$

In particular, in this equality we can substitute $\sigma\psi$ for $\psi(x)$. Applying (3.4) we obtain

$$\begin{aligned} &\int_G \big(A\nabla(\sigma\psi), \nabla(\sigma\psi)\big)|\nu|^2\, dx + \mathrm{Re}\int_G \sigma^2\psi^2(L\nu)\bar\nu\, dx + C_1\|\sigma\psi\nu\|^2 + C_2\|\psi\nu\|^2 \\ &\qquad \ge \int_G Q_{\alpha,\varepsilon}^2(x)|\psi\nu|^2\, dx. \end{aligned} \tag{3.18}$$

Let us set $\psi = \psi_\gamma(x,\tau)$ in (3.18), where $\psi_\gamma(x,\tau)$ was defined in (3.15). For $x \in \Omega^\tau = \{x\colon e^{\rho(x)} < \tau\}$ we have

$$\begin{aligned} \big(A\nabla(\sigma\psi_\gamma), \nabla(\sigma\psi_\gamma)\big) &= (A\nabla\sigma, \nabla\sigma)\psi_\gamma^2 + \gamma^2\sigma^2\tau^{-2\gamma}e^{2\gamma\rho}(A\nabla\rho, \nabla\rho) \\ &\quad - 2\psi_\gamma\gamma\sigma\tau^{-\gamma}e^{\gamma\rho}\,\mathrm{Re}(A\nabla\sigma, \nabla\rho). \end{aligned}$$

For $x \in \Omega^\tau$, $e^{\gamma\rho}\tau^{-\gamma} \le 1$. Furthermore,

$$\big|\mathrm{Re}(A\nabla\sigma, \nabla\rho)\big| \le (A\nabla\sigma, \nabla\sigma)^{1/2}(A\nabla\rho, \nabla\rho)^{1/2}.$$

Therefore

$$\big(A\nabla(\sigma\psi_\gamma), \nabla(\sigma\psi_\gamma)\big) \le \big[\psi_\gamma(A\nabla\sigma, \nabla\sigma)^{1/2} + \gamma\sigma(A\nabla\rho, \nabla\rho)^{1/2}\big]^2.$$

Hence, using (3.18) we obtain

$$J_\gamma(\tau) + \mathrm{Re}\left\langle \sigma^2\psi_\gamma^2 L\nu, \nu\right\rangle + C_1\|\sigma\psi_\gamma\nu\|^2 + C_2\|\psi_\gamma\nu\|^2 \ge \int_{\Omega^\tau} \psi_\gamma^2 Q_{\alpha,\varepsilon}^2(x)|\nu|^2\, dx \tag{3.19}$$

($J_\gamma(\tau)$ was defined in (3.16) for $u(x) = \nu(x)$). By Condition A for every $\tau > 0$ there is a sequence $\{u_m\}$ of functions in $C^2(G)$, with $u_m \to u(x)$ in $L_2(\Omega^\tau)$ and such that $\lim_{m\to\infty} \mathrm{Re}\left\langle \sigma^2\psi_\gamma^2 Lu_m, u_m\right\rangle = \mathrm{Re}\left\langle \sigma\psi_\gamma^2 L^* u, u\right\rangle$, where $u(x)$ is an arbitrary function in D_{L^*}. Substituting $\nu(x) = u_m(x)$ in (3.19) and passing to the limit as $m \to \infty$, we get

$$J_\gamma(\tau) + \mathrm{Re}\left\langle \sigma^2\psi_\gamma^2 L^* u, u\right\rangle + C_1\|\sigma\psi_\gamma u\|^2 + C_2\|\psi_\gamma u\|^2 \ge \int_{\Omega^\tau} \psi_\gamma^2 Q_{\alpha,\varepsilon}^2(x)|u|^2\, dx.$$

By Lemma 3.2, for any $\varepsilon_1 > 0$ we can find a sequence $\{\tau_k\}$ and a number $\gamma > 0$ for which

$$\begin{aligned}\operatorname{Re}\left\langle \sigma\psi_\gamma^2 L^* u, u\right\rangle + C_1\|\sigma\psi_\gamma u\|^2 + C_2\|\psi_\gamma u\|^2 \\ + \int_{\Omega^{\tau_k}} \psi_\gamma^2 Q^2_{\alpha,\varepsilon_1}(x)|u|^2\,dx \geq \int_{\Omega^{\tau_k}} \psi_\gamma^2 Q^2_{\alpha,\varepsilon}(x)|u|^2\,dx.\end{aligned} \tag{3.20}$$

Since $\psi_\gamma \leq 1$, $\sigma \leq \text{const}$, for some $C > 0$ we have

$$C + \int_{\Omega^{\tau_k}} \psi_\gamma^2 Q^2_{\alpha,\varepsilon_1}(x)|u|^2\,dx \geq \int_{\Omega^{\tau_k}} \psi_\gamma^2 Q^2_{\alpha,\varepsilon}(x)|u|^2\,dx.$$

For $\varepsilon_1 < \varepsilon$ it follows from this that for every $u \in D_{L^*}$ the integral (3.5) converges. If $C_2 = 0$, from (3.20) it follows that for $\lambda \geq C_1$ we have the inequality

$$\operatorname{Re}\int_{\Omega^{\tau_k}} \sigma^2\psi_\gamma^2(L^*u + \lambda u)\bar{u}\,dx + \int_{\Omega^{\tau_k}} \psi_\gamma^2 Q^2_{\alpha,\varepsilon_1}|u|^2\,dx \geq \int_{\Omega^{\tau_k}} \psi_\gamma^2 Q^2_{\alpha,\varepsilon}|u|^2\,dx.$$

If $u(x)$ is a solution of the equation $L^*u = -\lambda u$, then setting $\varepsilon_1 < \varepsilon$ in this inequality, we derive (3.6). Theorem 3.1 is proved. □

Proof of Theorem 3.2. To prove that the operator L is semimaximal, it suffices to show that for any $u \in D_{L^*}$ there is a sequence $\{u_k\}$, $u_k \in D_{\bar{L}}$, such that $u_k \to u$ in $L_2(G)$ and

$$\langle Lu_k, Lu_k\rangle \to \operatorname{Re}\langle L^*u, u\rangle. \tag{3.21}$$

We will use the local Condition A taking

$$\psi(x) = \varphi_k^2(x) = \left(1 - \frac{\mu(x)}{k}\right)_+^8.$$

The sequence $u_{km} \in C^2(G)$ corresponds to the set $\Omega = \operatorname{supp}\varphi_k$. It is clear that $\varphi_k(x)u_{km} \in D_{\bar{L}}$. Putting $\psi = \varphi_k$, $\nu = u_{km}$ in (3.17), we find that

$$\begin{aligned}&\left\langle L(\varphi_k u_{km}), \varphi_k u_{km}\right\rangle \\ &\quad = \frac{16}{k^2}\int_{G^k}\left(1 - \frac{\mu(x)}{k}\right)^6 (A\nabla\mu, \nabla\mu)|u_{km}|^2\,dx + \operatorname{Re}\int_{G^k}\left(1 - \frac{\mu(x)}{k}\right)^8 Lu_{km}\bar{u}_{km}\,dx,\end{aligned}$$

where $G^k = \{x\colon \mu(x) < k\}$. For each k we can choose m_k so that

$$\begin{aligned}&\left\langle L(\varphi_k u_{km_k}), \varphi_k u_{km_k}\right\rangle \\ &\quad = \frac{16}{k^2}\int_{G^k}\left(1 - \frac{\mu(x)}{k}\right)^6 (A\nabla\mu, \nabla\mu)|u|^2\,dx + \operatorname{Re}\langle\varphi_k^2 L^*u, u\rangle + O\left(\frac{1}{k}\right),\end{aligned} \tag{3.22}$$

as $k \to \infty$.

Now set $u_k(x) = \varphi_k u_{km_k}$. By Theorem 3.1 and (3.7), if $u \in D_{L^*}$, the following integral is finite:

$$\int_G (A\nabla\mu, \nabla\mu)|u|^2\,dx < \infty. \tag{3.23}$$

Hence from (3.22) it follows that (3.21) holds as $k \to \infty$. Thus, (1.3) is satisfied and the operator L is semimaximal. Part 1° is proved.

We now assume that under the conditions of part 2°, L is semibounded. By Theorem 3.1 and (3.8)

$$(A\nabla\mu, \nabla\mu)|u|^2 = 0 \tag{3.24}$$

almost everywhere in G for each solution $u \in L_2(G)$ of the equation $L^*u + \lambda u = 0$ with $\lambda \geq C_1$. We choose $\lambda \geq C_1$ sufficiently large that the inequality

$$\langle L\varphi + \lambda\varphi, \varphi\rangle \geq \|\varphi\|^2 \tag{3.25}$$

is satisfied for all $\varphi \in D_{\bar{L}}$. Suppose that the operator L is not essentially selfadjoint. Then there is a function $u \in D_{L^*}$, not identically zero, such that $L^*u + \lambda u = 0$. By Condition A, as before we can construct a sequence u_{km_k} for which (3.22) holds with L replaced by $L + \lambda$. By (3.24) and (3.25) we have

$$\|\varphi_k u_{km_k}\|^2 = O\left(\frac{1}{k}\right) \quad (k \to \infty).$$

Since $\varphi_k u_{km_k} \to u$ in $L_2(G)$, we have that $u(x) \equiv 0$, which contradicts the assumption that L is not selfadjoint. This completes the proof of Theorem 3.2. □

Remark 3.2. If Condition A is satisfied and there is a function $\mu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ such that $0 \le \mu(x) \to \infty$ as $x \to \partial G$, for which $(A\nabla\mu, \nabla\mu) \le \mathrm{const}$ almost everywhere in G, then the operator L is semimaximal without any additional conditions.

Indeed, semimaximality follows from (3.23) which is satisfied here automatically.

4. Behaviour near the boundary of the potential of a semibounded elliptic operator which guarantees its essential selfadjointness

In this section we establish conditions on the real-valued function $q(x)$ under which an operator M of the form (1.2) is essentially selfadjoint. As we mentioned before, the operator M can be recast as an operator L of the form (1.1) with different $\vec{b}(x)$ and $q(x)$ satisfying (2.2). The results given above are thus applicable to the operator M.

We shall denote by $\vec{f}(x)$ a vector field with real components that is defined in G. In what follows, it will be assumed throughout that $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$. Let us note that such a vector field has divergence $\nabla\vec{f}$ defined almost everywhere in G.

Theorem 4.1 ([30]). *Let $A(x)$ be a positive Hermitian $(n \times n)$-matrix function and $\vec{b}(x)$ a vector function with n real components, both defined on an open set $G \subseteq R^n$. Suppose that the elements of $A(x)$ and the components of $\vec{b}(x)$ are continuous in G. Then for any vector field $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$ and any function $\varphi(x) \in C^1_0(G)$,*

$$\int_G \big(A(\nabla\varphi - i\vec{b}\varphi), \nabla\varphi - i\vec{b}\varphi\big)\,dx \ge \int_G \big(\nabla\vec{f} - (A^{-1}\vec{f}, \vec{f})\big)|\varphi|^2\,dx. \tag{4.1}$$

Proof. For an arbitrary $\varphi(x) \in C^1_0(G)$ consider the integral

$$\begin{aligned} J &= \int_G \big|A^{1/2}(\nabla\varphi - i\vec{b}\varphi) + A^{-1/2}(\varphi\vec{f})\big|^2\,dx \\ &= \int_G \Big(\big|A^{1/2}(\nabla\varphi - i\vec{b}\varphi)\big|^2 + \big|A^{-1/2}\varphi\vec{f}\big|^2 + 2\,\mathrm{Re}\,\big((\nabla\varphi, \varphi\vec{f}) - i(\vec{b}, \vec{f})|\varphi|^2\big)\Big)\,dx \\ &= \int_G \big(A(\nabla\varphi - i\vec{b}\varphi), (\nabla\varphi - i\vec{b}\varphi)\big)\,dx + \int_G (A^{-1}\vec{f}, \vec{f})|\varphi|^2\,dx + \int_G 2\,\mathrm{Re}(\nabla\varphi, \varphi\vec{f})\,dx. \end{aligned}$$

Subtract and add the quantity $\nabla\vec{f}|\varphi|^2$ in the third integral:

$$\begin{aligned} &\int_G \big(2\,\mathrm{Re}(\nabla\varphi, \varphi\vec{f}) + \nabla\vec{f}|\varphi|^2 - \nabla\vec{f}|\varphi|^2\big)\,dx \\ &\qquad = \int_G \big(\nabla(|\varphi|^2\vec{f}) - \nabla\vec{f}|\varphi|^2\big)\,dx = -\int_G \nabla\vec{f}|\varphi|^2\,dx. \end{aligned}$$

Since $J \ge 0$, we have that

$$\int_G \Big|\big(A(\nabla\varphi - i\vec{b}\varphi), (\nabla\varphi - i\vec{b}\varphi)\big) + (A^{-1}\vec{f}, \vec{f})|\varphi|^2 - \nabla\vec{f}|\varphi|^2\Big|\,dx \ge 0,$$

which is equivalent to (4.1). This completes the proof. □

As a consequence of Theorem 4.1 we have the next theorem.

Theorem 4.2 ([30])**.** *Suppose the potential of an operator M of the form (1.2) satisfies the inequality*

$$q(x) \geq q_A(x) + \left(A^{-1}\vec{f}, \vec{f}\right) - \nabla\vec{f} \tag{4.2}$$

almost everywhere in G, where $q_A(x)$ is a correction potential for the matrix $A(x)$ in the domain G, while $\vec{f}(x)$ is a vector field in $\mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$. Then if M is semibounded, it is essentially selfadjoint.

Proof. It follows from (4.2) and Theorem 4.1 that the operator $M - q_A(x)$ is nonnegative, and therefore if M is semibounded, it is essentially selfadjoint. Let us also note that from Theorem 4.1 it follows that in the case $q_A(x) \geq \mathrm{const}$, (4.2) implies that the operator M is semibounded, and therefore it is essentially selfadjoint. □

From Corollary 3.3 we conclude that we can take

$$q_A(x) = (A\nabla\eta, \nabla\eta) - K$$

as a correction potential. Here the constant K is ≥ 0, while $\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ is such that $0 \leq \eta(x) \to \infty$ as $x \to \partial G$ and (3.12) is satisfied. Note that such a function $\eta(x)$ can be found for any matrix function $A(x)$ that is continuous in G.

The next result is an immediate consequence of Theorem 4.2.

Theorem 4.3. *Assume that Condition* A *holds for $L = M$. If the potential of the operator M of the form (1.2) satisfies the inequality*

$$q(x) \geq (A\nabla\eta, \nabla\eta) + \left(A^{-1}\vec{f}, \vec{f}\right) - \nabla\vec{f} - K \tag{4.3}$$

for some $K \geq 0$ almost everywhere in G, for some function $\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G), 0 \leq \eta(x) \to \infty$ as $x \to \partial G$, such that (3.12) holds, and for some vector field $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$, then the operator M is essentially selfadjoint.

Note that (4.3) automatically guarantees that M is semibounded from below.

As an example of an application of Theorem 4.3 consider the following particular case. Let $G = \left\{x \in R^n,\, x_j > 0,\, j = 1, \ldots, n\right\}$. Consider an operator $M \in L_2(G)$ of the form (1.2) with matrix $A(x) = \mathrm{diag}\left\{x_1^{2k_1}, x_2^{2k_2}, \ldots, x_n^{2k_n}\right\}$, and with $b_j(x) \in C^1(G)$. We denote this operator by $S^{(k)}$.

Corollary 4.1. *Suppose that the exponents $k_j\,(j = 1, \ldots, n)$ in the operator $S^{(k)}$ are any real numbers, and that its potential $q(x) \in L_{2\mathrm{loc}}(G)$ satisfies the inequality*

$$q(x) \geq \sum_{k_j \neq 1} \left(\frac{3}{4} - k_j\right) x_j^{-2+2k_j} - K \tag{4.4}$$

almost everywhere in G, for some $K \geq 0$. Then the operator $S^{(k)}$ is essentially selfadjoint.

Proof. Since by (4.4) $q(x)$ is locally bounded below, appealing to Proposition 2.1 we conclude that Condition A holds here automatically. Let us use Theorem 4.3 choosing

$$\eta(x) = \sum_{j=1}^{n} l_j |\ln x_j| + 1; \quad f_i(x) = \frac{1}{2}(2k_i - 1)x_i^{2k_i - 1}.$$

Here $l_j = |k_j - 1|$ if $k_j \neq 1$ and $l_j = 1$ otherwise. We have

$$\begin{aligned}
&(A\nabla\eta, \nabla\eta) + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} \\
&\quad = \sum_{j=1}^{n} \left[l_j^2 x_j^{2k_j-2} + \frac{1}{4}(2k_j-1)^2 x_j^{2k_j-2} - \frac{1}{2}(2k_j-1)^2 x_j^{2k_j-2} \right] \\
&\quad = \sum_{k_j \neq 1} \left[(k_j-1)^2 + \frac{1}{4}(2k_j-1)^2 - \frac{1}{2}(2k_j-1)^2 \right] x_j^{2k_j-2} + C_0 \\
&\quad = \sum_{k_j \neq 1} \left(\frac{3}{4} - k_j \right) x_j^{-2+2k_j} + C_0,
\end{aligned}$$

where $C_0 \geq 0$. Furthermore $e^{2\eta} = e^2 \prod_{j=1}^{n} x_j^{2l_j \operatorname{sgn}(x_j-1)}$ and in this expression all the factors are ≥ 1. Therefore

$$e^{2\eta} \geq \frac{e^2}{n} \sum_{j=1}^{n} x_j^{2l_j \operatorname{sgn}(x_j-1)} \geq \frac{e^2}{n \max_j l_j^2} \sum_{j=1}^{n} l_j^2 x_j^{2k_j-2}.$$

Since $(A\nabla\eta, \nabla\eta) = \sum_{j=1}^{n} l_j^2 x_j^{2k_j-2}$, we have $(A\nabla\eta, \nabla\eta) \leq e^{-2} n \big(\max_j l_j^2\big) e^{2\eta}$.

Thus we have shown that the function $\eta(x)$ satisfies (3.12). Hence all the hypotheses of Theorem 4.3 are satisfied and the operator $S^{(k)}$ is essentially selfadjoint. Corollary 4.1 is proved. □

It is not hard to verify the following remark for specific examples.

Remark 4.1. For $\vec{b}(x) = \vec{0}$ none of the constants $\frac{3}{4} - k_j$ in (4.4) can be taken smaller for any $k_j \neq 1$. Nonzero deficiency indices of the operator $S^{(k)}$ can appear if condition (4.4) is violated both close to a finite part of the boundary and at infinity.

In the case $q(x) \geq \text{const}$, Corollary 4.1 guarantees that $\bar{S}^{(k)}$ is selfadjoint for $k_j \geq \frac{3}{4}$ $(j = 1, \dots, n)$. It follows from the results of [39] that if $\frac{1}{4} < k_j < \frac{3}{4}$ even for one j, the selfadjointness of an operator $\bar{S}^{(k)}$ with a potential that is bounded in a neighbourhood of ∂G can be compromised. Properties of deficiency indices of elliptic operators that degenerate at interior points of the domain $G = R^n$ were studied in [40].

5. Essential selfadjointness for elliptic operators in $L_2(G)$ that are not semibounded

In the current literature there are many different notations for writing down a symmetric elliptic operator. To make our exposition consistent, in this section we shall consider the operator

$$Lu = L_2 u + L_1 u + q(x)u, \quad D_L = C_0^\infty(G), \tag{5.1}$$

where

$$\begin{aligned}
L_2 u &= \big(\nabla - i\vec{b}(x)\big)^* \big(A(x)\big(\nabla - i\vec{b}(x)\big)u\big), \\
L_1 u &= i\big[\nabla\big(\vec{a}(x)u\big) + \big(\nabla u, \vec{a}(x)\big)\big].
\end{aligned} \tag{5.2}$$

Here $A(x)$ is a positive Hermitian matrix function, and $\vec{a}(x)$, $\vec{b}(x)$ are n-component vector functions with real-valued components. We assume that

$$a_{ij}(x) \in \operatorname{Lip}^{(1)}_{\text{loc}}(G); \quad \vec{a}(x), \vec{b}(x) \in \operatorname{Lip}^{(n)}_{\text{loc}}(G); \quad q(x) \in L_{2\text{loc}}(G).$$

Unless explicitly stated otherwise, in this section we do not assume that Condition A holds.

We have already considered the question of selfadjointness for semibounded operators. In Theorem 5.1 we show that if we replace the condition that L be semibounded by a certain operator inequality, the selfadjointness of $\overline{L}$ follows from the essential selfadjointness of an operator that is necessarily semibounded.

The first paper to use an operator inequality as a condition for a nonsemibounded second order elliptic operator to be selfadjoint was [41]. For higher order operators it was first used in [11]. Theorem 5.1 is an analog of results obtained in [41]–[49]. In these papers, for $G = R^n$ and under some additional conditions, it is established that elliptic operators are selfadjoint if some kind of operator inequality holds, while for the corresponding semibounded operators it can be guaranteed automatically.

Conditions for the selfadjointness of general nonsemibounded elliptic operators in $L_2(G)$ were studied in [20, 50, 51]. The requirements on the behaviour of the coefficients near the finite part of the boundary are more stringent in these papers than they are in a number of well-known theorems for semibounded operators. Theorem 5.1 allows us to fill this gap.

5.1. Using an operator L of the form (5.1)–(5.2), a nonnegative function $\nu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ and a vector field $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$, we construct the function

$$W_{\nu,f}(x) = \nu^2(A\nabla\nu, \nabla\nu) + \nu\,|(\vec{a}, \nabla\nu)| + \left(A^{-1}\vec{f}, \vec{f}\right) - \nabla\vec{f},$$

as well as the operator

$$L_{\nu,f}u = L_2u + W_{\nu,f}(x)u,$$

which is positive on $C_0^\infty(G)$. Its positivity follows from Theorem 4.1.

Theorem 5.1 ([31]). *Let $0 \le \nu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$, and $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$. Assume also that the inequalities*

$$\gamma \le (A\nabla\nu, \nabla\nu) \le \mu\nu^4 + C \tag{5.3}$$

hold almost everywhere in G, with some constants $\gamma, \mu, C \ge 0$. Suppose that for all $\varphi \in C_0^\infty(G)$ the operator inequality

$$\left\langle (L + t\nu^2)\varphi, \varphi \right\rangle \ge \varepsilon\langle L_{\nu,f}\varphi, \varphi\rangle - K\|\varphi\|^2 \tag{5.4}$$

holds, with some constants t, $K \ge 0$, $\varepsilon > 0$. Then if the operator $L + \lambda\nu^2$ in $C_0^\infty(G)$ is essentially selfadjoint for $\lambda \ge \max\{2(t + \mu - \varepsilon\gamma), t\}$, the operator L is essentially selfadjoint.

Before proving Theorem 5.1, we present two of its corollaries.

Theorem 5.2 ([31]). *Let $\nu(x) \ge 0$ lie in $\mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ and let $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$. Suppose that for an operator M of the form (1.2) the inequalities*

$$\gamma \le (A\nabla\nu, \nabla\nu) \le C, \tag{5.5}$$

$$q(x) \ge \left(A^{-1}\vec{f}, \vec{f}\right) - \nabla\vec{f} - k\nu^2 - K \tag{5.6}$$

hold almost everywhere in G, for some constants $C, k > 0$, and $\gamma, K \ge 0$. Then if the operator $M + \lambda\nu^2$ is essentially selfadjoint for some $\lambda > 2k$ (or if $\gamma = C \ge 1$, for some $\lambda \ge 2k$), M is essentially selfadjoint.

Proof. An operator M of the form (1.2) is a particular case of an operator L given by (5.1), (5.2) with $\vec{a} \equiv \vec{0}$. We shall show that under the hypotheses of Theorem 5.2, the conditions of Theorem 5.1 are satisfied. By (5.5), condition (5.3) holds with $\mu = 0$. We

will show that if (5.6) holds, then (5.4) is true for any $t > k$ and for some $\varepsilon > 0$ depending on t. Moreover, it suffices to prove this for $t - k < 1$. Let $t - k = \delta > 0$. We have

$$\begin{aligned} M + t\nu^2 &= L_2 + q(x) + t\nu^2 \\ &= L_2 + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} + q(x) - \big(A^{-1}\vec{f}, \vec{f}\big) + \nabla\vec{f} + t\nu^2 \\ &= \delta L_2 + \delta\big[\big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f}\big] + \delta\nu^2 + (1-\delta)\big[L_2 + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f}\big] \\ &\quad + q(x) - \big(A^{-1}\vec{f}, \vec{f}\big) + \nabla\vec{f} + k\nu^2. \end{aligned}$$

By Theorem 4.1, $L_2 + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} \geq 0$. Therefore (5.6) implies that

$$M + t\nu^2 \geq \delta L_2 + \delta\big[\big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} + \nu^2\big] - K \geq \varepsilon L_{\nu,f} - K,$$

where $0 < \varepsilon \leq \frac{\delta}{\max\{C,1\}}$.

Thus the conditions of Theorem 5.1 are satisfied for any $t > k$ and Theorem 5.2 is proved for $\gamma \leq C$. In the case $\gamma = C \geq 1$, by Theorem 5.1, if the operator $\overline{M + \lambda\nu^2}$ is selfadjoint, then $\overline{M}$ is selfadjoint for

$$\lambda \geq \max\{2(t - \varepsilon\gamma), t\} = \max\{2(k + \delta - \varepsilon C), k + \delta\}.$$

For $\varepsilon = \frac{\delta}{C}$ and $\delta < k$ this maximum equals $2k$. Theorem 5.2 is proved. □

The last theorem allows us to extend Theorem 4.3 to the case of operators that are not semibounded. Let the functions $\eta(x), \nu(x)$ and the vector field $\vec{f}(x)\colon G \to R^n$ be such that

1) $\eta(x), \nu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$; $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$;
2) $0 < \eta(x) \to \infty$ as $x \to \partial G$, $\nu(x) \geq 0$;
3) the inequalities

$$(A\nabla\eta, \nabla\eta) \leq Ce^{2\eta}, \quad (A\nabla\nu, \nabla\nu) \leq C$$

hold almost everywhere in G for some $C > 0$.

Theorem 5.3. *Assume that Condition* A *holds for $L = M$. If the potential of the operator M of the form (1.2) satisfies the inequality*

$$q(x) \geq (A\nabla\eta, \nabla\eta) + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} - k\nu^2 - K \tag{5.7}$$

almost everywhere in G, for some $k, K \geq 0$, where the functions $\eta(x)$ and $\nu(x)$ and the vector field $\vec{f}(x)$ satisfy conditions 1)–3), then the operator M is essentially selfadjoint.

Proof. Condition (5.7) implies that (5.6) holds, while 3) implies that (5.5) holds with $\gamma = 0$. By Theorem 5.2, $\overline{M}$ will be selfadjoint if $\overline{M + \lambda\nu^2}$ is selfadjoint for $\lambda > 2k$. But by Theorem 4.3 the operator $M + \lambda\nu^2$ is essentially selfadjoint for $\lambda \geq k$, since for this λ, by (5.7),

$$q(x) + \lambda\nu^2 \geq (A\nabla\eta, \nabla\eta) + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} - K.$$

However, the operator $M + \lambda\nu^2$ has to satisfy Condition A. As $\nu^2(x)$ is locally bounded and M satisfies Condition A, it follows from Remark 2.1 that this condition also holds for $M + \lambda\nu^2$. By Theorem 5.2 the operator $\overline{M}$ is selfadjoint. The theorem is proved. □

5.2. To prove Theorem 5.1 we need three lemmas. The first is one of the forms of the commutator theorem (see [19, Theorem X.37]).

Lemma 5.1 ([31]). *Let B and T be densely defined symmetric operators in a Hilbert space H with domain $D \subset H$. Assume also that the operator $B+T$ is essentially selfadjoint on D. If for all $\varphi \in D$ the inequalities*

$$\alpha\|B\varphi\|^2 + 2\,\mathrm{Re}\langle B\varphi, T\varphi\rangle + \|T\varphi\|^2 \geq -k\|\varphi\|^2, \tag{5.8}$$

$$\big\langle (B+T)\varphi, \varphi\big\rangle \geq \delta\big|\mathrm{Im}\langle B\varphi, T\varphi\rangle\big| - k\|\varphi\|^2 \tag{5.9}$$

hold for some constants $\alpha < 1$, $\delta > 0$, $k \geq 0$, then the operator B is essentially selfadjoint on D.

Proof. According to Theorem X.37 of [19], if an essentially selfadjoint operator $N \geq I$ and a symmetric operator A, defined on D, satisfy the inequalities ($\varphi \in D$)

$$\|A\varphi\| \leq C\|N\varphi\|, \tag{5.10}$$

$$\big|\mathrm{Im}\langle A\varphi, N\varphi\rangle\big| \leq K\langle N\varphi, \varphi\rangle \tag{5.11}$$

with constants C and K, then A is essentially selfadjoint on D. We shall show that under conditions (5.8), (5.9), conditions (5.10), (5.11) hold with $A = B$ and $N = B + T + mI$, where $m > 0$ is sufficiently large. By (5.9) the operator $B+T$ is semibounded from below. Therefore, if $m > 0$ is sufficiently large, then $N \geq I$. We can rewrite (5.8) in the form

$$\big\|(B+T)\varphi\big\|^2 + k\|\varphi\|^2 \geq (1-\alpha)\|B\varphi\|^2.$$

On the other hand, as $B+T$ is bounded below, we can choose $m > 0$ so large that

$$\big\|(B+T)\varphi + m\varphi\big\|^2 \geq \big\|(B+T)\varphi\big\|^2 + k\|\varphi\|^2.$$

Therefore (5.10) holds with $C = (1-\alpha)^{-1/2}$. If we choose $m > k$, then (5.11) follows from (5.9) with $K = \frac{1}{\delta}$, since

$$\mathrm{Im}\langle B\varphi, N\varphi\rangle = \mathrm{Im}\langle B\varphi, T\varphi\rangle.$$

Thus the operator B is essentially selfadjoint if $B+T$ is. Lemma 5.1 is proved. □

Lemma 5.2 ([31]). *Suppose that conditions (5.3), (5.4) of Theorem 5.1 hold, with $0 \leq \nu(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$, $\vec{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$ and $K = 0$. Then for all $\varphi \in C_0^\infty(G)$ the inequality*

$$\mathrm{Re}\langle L\varphi, \nu^2\varphi\rangle \geq (\varepsilon\gamma - \mu - t)\|\nu^2\varphi\|^2 - C\|\varphi\|^2 \tag{5.12}$$

holds for the constants $\varepsilon > 0$, $\mu, t, \gamma, C \geq 0$, which appear in (5.3), (5.4).

Proof. For any function $p(x) \in C^\infty(G)$ and all $\varphi \in C_0^\infty(G)$,

$$\mathrm{Re}\langle L\varphi, p^2\varphi\rangle = \big\langle L(p\varphi), p\varphi\big\rangle - \int_G (A\nabla p, \nabla p)|\varphi|^2\,dx.$$

By (5.4)

$$\big\langle L(p\varphi), p\varphi\big\rangle \geq \varepsilon\big\langle L_{\nu,f}(p\varphi), p\varphi\big\rangle - t\|\nu p\varphi\|^2.$$

Since $L_2 + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} \geq 0$, by (5.3)

$$\big\langle L_{\nu,f}(p\varphi), p\varphi\big\rangle \geq \gamma\|\nu p\varphi\|^2.$$

Therefore

$$\mathrm{Re}\langle L\varphi, p^2\varphi\rangle \geq (\varepsilon\gamma - t)\|\nu p\varphi\|^2 - \int_G (A\nabla p, \nabla p)|\varphi|^2\,dx.$$

In this inequality we assume that for $x \in \mathrm{supp}\,\varphi$, $p(x) = p_\delta(x) = \nu * \theta_\delta$, where $\theta_\delta(|x-y|)$ is a convolution kernel with the radius of convolution $\delta > 0$ sufficiently small. Then $p_\delta(x) \in C^\infty(\mathrm{supp}\ \varphi)$ and as $\delta \to 0$, $p_\delta^2(x) \to \nu^2(x)$ uniformly in $x \in \mathrm{supp}\ \varphi$.

Furthermore $p_\delta(x) \to \nu(x)$ in the metric of the space $W_2^1(\operatorname{supp}\varphi)$. Passing to the limit, we obtain

$$\operatorname{Re}\langle L\varphi, \nu^2\varphi\rangle \geq (\varepsilon\gamma - t)\|\nu^2\varphi\|^2 - \int_G (A\nabla\nu, \nabla\nu)|\varphi|^2\,dx.$$

Using this and the right-hand inequality in (5.3) we obtain (5.12). The lemma is proved. □

Lemma 5.3 ([31]). *An operator L of the form (5.1)–(5.2) satisfies the inequality*

$$\big|\operatorname{Im}\langle L\varphi, \nu^2\varphi\rangle\big| \leq 2\langle L_{\nu,f}\varphi, \varphi\rangle, \tag{5.13}$$

for any $\varphi \in C_0^\infty(G)$, $0 \leq \nu(x) \in \operatorname{Lip}^{(1)}_{\mathrm{loc}}(G)$, $\vec{f}(x) \in \operatorname{Lip}^{(n)}_{\mathrm{loc}}(G)$.

Proof. We note that

$$\big|\operatorname{Im}\langle L\varphi, \nu^2\varphi\rangle\big| \leq \big|\operatorname{Im}\big\langle L_2\varphi, \nu^2\varphi\big\rangle\big| + \big|\operatorname{Im}\big\langle L_1\varphi, \nu^2\varphi\big\rangle\big|.$$

We estimate every term separately:

$$\begin{aligned}\big|\operatorname{Im}\big\langle L_1\varphi, \nu^2\varphi\big\rangle\big| &= \frac{1}{2}\big|\big\langle L_1\varphi, \nu^2\varphi\big\rangle - \big\langle \nu^2\varphi, L_1\varphi\big\rangle\big| \\ &= \frac{1}{2}\Big|\big\langle \nabla(\vec{a}\varphi), \nu^2\varphi\big\rangle + \big\langle (\nabla\varphi, \vec{a}), \nu^2\varphi\big\rangle + \big\langle \nu^2\varphi, \nabla(\vec{a}\varphi)\big\rangle + \big\langle \nu^2\varphi, (\nabla\varphi, \vec{a})\big\rangle\Big| \\ &= 2\big|\big\langle \nu(\vec{a}, \nabla\nu)\varphi, \varphi\big\rangle\big|.\end{aligned}$$

Therefore

$$\big|\operatorname{Im}\big\langle L_1\varphi, \nu^2\varphi\big\rangle\big| \leq 2\int_G \nu|(\vec{a}, \nabla\nu)| \cdot |\varphi|^2\,dx. \tag{5.14}$$

To estimate $\big|\operatorname{Im}\big\langle L_2\varphi, \nu^2\varphi\big\rangle\big|$, consider the integral

$$I_\pm = \int_G \Big(A\big(\nabla\varphi - i\big(\vec{b} \pm \nabla\nu^2\big)\varphi\big), \nabla\varphi - i\big(\vec{b} \pm \nabla\nu^2\big)\varphi\Big)\,dx.$$

It is not hard to see that

$$I_\pm = \langle L_2\varphi, \varphi\rangle \pm 2\operatorname{Im}\big\langle L_2\varphi, \nu^2\varphi\big\rangle + 4\big\langle \nu^2(A\nabla\nu, \nabla\nu)\varphi, \varphi\big\rangle.$$

By Theorem 4.1,

$$I_\pm \geq \int_G \big(\nabla\vec{f} - \big(A^{-1}\vec{f}, \vec{f}\big)\big)|\varphi|^2\,dx.$$

Therefore

$$\big|\operatorname{Im}\big\langle L_2\varphi, \nu^2\varphi\big\rangle\big| \leq \frac{1}{2}\langle L_2\varphi, \varphi\rangle + \frac{1}{2}\int_G \big(\big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f}\big)|\varphi|^2\,dx + 2\big\langle \nu^2(A\nabla\nu, \nabla\nu)\varphi, \varphi\big\rangle.$$

This inequality, together with (5.14), shows that (5.13) holds. Lemma 5.3 is proved. □

Proof of Theorem 5.1. Since the operators L and $L+KI$ have the same deficiency indices, in (5.4) we can take $K = 0$. We will apply Lemma 5.1 with $B = L$, $T = \lambda\nu^2 I$. We will show that (5.8) holds. By Lemma 5.2

$$\begin{aligned}2\operatorname{Re}\langle B\varphi, T\varphi\rangle + \|T\varphi\|^2 &= 2\operatorname{Re}\langle L\varphi, \lambda\nu^2\varphi\rangle + \lambda^2\|\nu^2\varphi\|^2 \\ &\leq \big[2\lambda(\varepsilon\gamma - \mu - t) + \lambda^2\big]\|\nu^2\varphi\|^2 - 2C\lambda\|\varphi\|^2 \geq -2C\lambda\|\varphi\|^2.\end{aligned}$$

Therefore (5.8) holds with $\alpha = 0$, $k = 2C\lambda$. Let us verify (5.9). By Lemma 5.3,

$$\varepsilon\langle L_{\nu,f}\varphi, \varphi\rangle \geq \frac{\varepsilon}{2}\big|\operatorname{Im}\langle L\varphi, \nu^2\varphi\rangle\big| \geq \frac{\varepsilon}{2(\lambda+1)}\big|\operatorname{Im}\langle L\varphi, \lambda\nu^2\varphi\rangle\big|.$$

Hence, by (5.4), taking into account that $\lambda \geq t$, we see that (5.9) holds with $\delta = \frac{\varepsilon}{2(\lambda+1)}$, $k = 0$. Therefore by Lemma 5.1 if $\overline{L + \lambda\nu^2}$ is selfadjoint, so is $\overline{L}$. Theorem 5.1 is proved. □

6. The case of the Schrödinger operator

All the statements in this section follow from Theorem 5.3 for a particular choice of the functions $\eta(x)$, $\nu(x)$ and the vector field $\vec{f}(x)$.

6.1. Consider the operator

$$Su = -\big(\nabla - i\vec{b}(x)\big)^2 u + q(x)u, \quad D_S = C_0^\infty(G)$$

in the case when $G \neq R^n$ and $b_j(x) \in C^1(G)$. Then S is the special case of the operator M with $A(x) \equiv I_n$. Let $\delta(x)$ denote the regularised distance from a point x to the set $R^n \setminus G$. By the regularised distance we mean a function $\delta(x) \in C^2(G)$ which satisfies the inequalities

$$C_1 \mathrm{dist}\,(x, R^n \setminus G) \leq \delta(x) \leq C_2 \mathrm{dist}\,(x, R^n \setminus G),$$

for some C_1, $C_2 > 0$, as well as the conditions

$$|\nabla\delta| \leq \mathrm{const}; \quad |\Delta\delta| \leq \frac{\mathrm{const}(1+\delta)}{\delta}. \tag{6.1}$$

Such a function always exists and can be constructed using the procedure described in [36, p. 303], for instance. When $d(x) = \mathrm{dist}\,(x, R^n \setminus G) \in C^2(G)$, we will take $\delta(x) = d(x)$. Let $\alpha(x)$, $\nu(x)$ be functions that are defined in G and satisfy a global Lipschitz condition,

$$\alpha(x), \nu(x) \in \mathrm{Lip}_1(G). \tag{6.2}$$

We assume that

$$\nu(x) \geq 0, \quad |\alpha(x)| \leq \mathrm{const}, \tag{6.3}$$

and also, that for some $\varepsilon > 0$

$$\inf_{x \in \Omega_\varepsilon} |\nabla\delta| > 0, \tag{6.4}$$

where $\Omega_\varepsilon = \{x\colon x \in G,\ \mathrm{dist}\,(x, R^n \setminus G) < \varepsilon\}$.

Theorem 6.1. *If $q(x)$, the potential of the operator S, lies in $L_{2\mathrm{loc}}(G)$ and satisfies*

$$q(x) \geq \frac{1-\alpha+\alpha^2}{\delta^2}|\nabla\delta|^2 + \alpha\frac{\Delta\delta}{\delta} - C\left(\nu^2 + \frac{1}{\delta}\right) \tag{6.5}$$

almost everywhere in G, where $C \geq 0$ is constant and the functions $\alpha(x)$, $\nu(x)$ and $\delta(x)$ satisfy (6.2)–(6.4), then the operator S is essentially selfadjoint.

Proof. Condition A is fulfilled here automatically in view of the fact that $q(x)$ is locally semibounded and by Proposition 2.1. It remains to show that under condition (6.5) we can choose $\eta(x)$ and $\vec{f}(x)$ so that for the function $\nu(x)$ that appears in (6.5), conditions 1)–3) and (5.7) of Theorem 5.3 will be satisfied. We assume that for $x \in G$

$$\eta(x) = -\ln\delta(x) + R\delta(x) + \rho(x), \qquad \vec{f}(x) = \alpha(x)\nabla(-\ln\delta(x)) = -\frac{\alpha(x)}{\delta(x)} \cdot \nabla\delta(x),$$

where $R = \mathrm{const} > 0$, $\rho(x) \equiv 0$ if G is a bounded domain, $0 \leq \rho(x) \in C^2(R^n)$ otherwise, and $\rho(x) \to \infty$ as $|x| \to \infty$ and $|\nabla\rho| \leq \mathrm{const}$. For sufficiently large R, $0 < \eta(x) \to \infty$ (as $x \to \partial G$). Furthermore,

$$\begin{aligned}(A\nabla\eta, \nabla\eta) &= |\nabla\eta|^2 \\ &= \frac{|\nabla\delta|^2}{\delta^2} - \frac{2R|\nabla\delta|^2 + 2(\nabla\delta, \nabla\rho)}{\delta} + R^2|\nabla\delta|^2 + 2R(\nabla\delta, \nabla\rho) + |\nabla\rho|^2 \\ &\leq \frac{1}{\delta^2}\big(|\nabla\delta|^2 + N_1\delta + N_2\delta^2\big),\end{aligned}$$

where the constants $N_1, N_2 \geq 0$. Since for some $N > 0$

$$|\nabla\delta|^2 + N_1\delta + N_2\delta^2 \leq Ne^{2(R\delta+\rho)},$$

we have that

$$(A\nabla\eta, \nabla\eta) \leq Ne^{2\eta} = \frac{N}{\delta^2}e^{2(R\delta+\rho)}.$$

Therefore the functions $\eta(x)$ and $\nu(x)$ we have chosen and the vector field $\vec{f}(x)$ satisfy conditions 1)–3) of Theorem 5.3. We will now find an upper bound for the quantity

$$\begin{aligned} T(x) &= (A\nabla\eta, \nabla\eta) + \big(A^{-1}\vec{f}, \vec{f}\big) - \nabla\vec{f} \\ &= (1-\alpha+\alpha^2)\frac{|\nabla\delta|^2}{\delta^2} + \alpha\frac{\Delta\delta}{\delta} \\ &\quad + \frac{1}{\delta}\big((\nabla\alpha, \nabla\delta) - 2(\nabla\delta, \nabla\rho)\big) - \frac{2R|\nabla\delta|^2}{\delta} + R^2|\nabla\delta|^2 + 2R(\nabla\delta, \nabla\rho) + |\nabla\rho|^2. \end{aligned}$$

In view of the properties of the functions $\alpha(x)$, $\delta(x)$ and $\rho(x)$, we have

$$\begin{aligned} T_1(x) &= \frac{(\nabla\alpha, \nabla\rho) - 2(\nabla\delta, \nabla\rho)}{\delta} - \frac{2R|\nabla\delta|^2}{\delta} \leq \frac{M_0 - 2R|\nabla\delta|^2}{\delta}, \\ T_2(x) &= R^2|\nabla\delta|^2 + 2R(\nabla\delta, \nabla\rho) + |\nabla\rho|^2 \leq M_R, \end{aligned}$$

where $M_0, M_R \geq 0$ are constants, the first of which does not depend on R, while the second does. Due to (6.4), the constant R can be chosen so large that

$$T(x) \leq \frac{(1-\alpha+\alpha^2)|\nabla\delta|^2}{\delta^2} + \frac{\alpha\Delta\delta}{\delta} - \frac{C}{\delta} + M_R.$$

Thus, if (6.5) holds, all the conditions of Theorem 5.3 are satisfied and the operator S is essentially selfadjoint. Theorem 6.1 is proved. □

6.2. To each point x_0 of a k-dimensional differentiable manifold $\Gamma \subset R^n$ without boundary we associate a Cartesian system of coordinates in which the first k coordinate axes are in the tangent space to Γ at the point x_0. We denote by $\aleph_{r,C}$ the class of differentiable manifolds $\Gamma \subset R^n$ without boundary such that for each point $x_0 \in \Gamma$, a sphere of radius r with centre at x_0 cuts out of Γ a piece Γ_{x_0} which in this system of coordinates with origin at x_0 is given by the equations

$$x_j = f_j(x_1, x_2, \ldots, x_k), \quad j = k+1, \ldots, n,$$

where the functions $f_j(x_1, x_2, \ldots, x_k)$ are in $C^2(R^k)$ and for some $C > 0$

$$\big|D^\alpha f_j(x_1, x_2, \ldots, x_k)\big| \leq C, \quad j = k+1, \ldots, n,$$

for all multiindices α, $|\alpha| \leq 2$. Here r and C are independent of x_0.

A linear manifold lies in $\aleph_{r,C}$ for any r and $C > 0$. Any compact $C^{(2)}$ manifold without boundary is in $\aleph_{r,C}$ for some r and C.

Let G be a domain such that

$$\partial G = \bigcup_{i\in N} \Gamma_i, \tag{6.6}$$

where N is an at most a countable set, while the Γ_i $(i \in N)$ are manifolds of dimension $0 \leq k_i \leq n-1$ that belong to $\aleph_{r,C}$ for some r and $C > 0$ independent of i. Set

$$\gamma = \inf_{\substack{i,j\in N \\ i\neq j}} \operatorname{dist}(\Gamma_i, \Gamma_j) > 0. \tag{6.7}$$

The next statement is an immediate generalisation of a result in [26].

Theorem 6.2. *Assume that in a domain G of the form (6.6), (6.7) the potential of the operator S has the form*

$$q(x) = q_1(x) + q_2(x),$$

where $q_1(x) \in L_{2\mathrm{loc}}(G)$, $q_2(x) \in L_\infty(G)$. Suppose that for some constants ε $(0 < \varepsilon < \frac{\gamma}{2})$ and $K \geq 0$ and a function $1 \leq \nu(x) \in \mathrm{Lip}_1(G)$, the following inequalities are satisfied:

$$q_1(x) \geq -\frac{(n-k_i)(n-k_i-4)}{4}\left[\mathrm{dist}(x,\Gamma_i)\right]^{-2} - K\left[(\mathrm{dist}(x,\Gamma_i))^{-1} + \nu^2(x)\right], \tag{6.8}$$

for any $i \in N$, $0 < \mathrm{dist}(x,\Gamma_i) < \varepsilon$; and

$$q_1(x) \geq -K\nu^2(x), \tag{6.9}$$

for $x \in \bigcap_{i\in N}\{x \in G, \mathrm{dist}(x,\Gamma_i) \geq \varepsilon\}$. Then the operator S is essentially selfadjoint.

To prove this theorem, we shall need two lemmas that are obvious for linear manifolds. In their proof, the manifold Γ is taken to be nonlinear, so that $\dim\Gamma \geq 1$.

Lemma 6.1. *For any manifold $\Gamma \in \aleph_{r,C}$ there is a constant $\varepsilon > 0$, which depends only on r, C, such that in the set*

$$\Omega_\varepsilon = \{x \in R^n,\ \mathrm{dist}\,(x,\Gamma) < \varepsilon\}$$

a single-valued projection mapping onto Γ can be defined, which associates with every point $x \in \Omega_\varepsilon$ the point y which is the nearest point of Γ. This mapping $y(x)$ is differentiable and its Jacobian $\frac{\partial y}{\partial x}$ satisfies the inequality

$$\left|\mathrm{Tr}\,\frac{\partial y}{\partial x} - k\right| \leq K\,\mathrm{dist}\,(x,\Gamma), \tag{6.10}$$

where k is the dimension of the manifold Γ, while K is a constant that depends only on r and C.

Proof. It suffices to show that for any point $x_0 \in \Gamma$ there is a neighbourhood $U_\varepsilon(x_0)$, with radius $\varepsilon > 0$ independent of x_0, such that for each point $x \in U_\varepsilon(x_0)$ there is a unique nearest point y in Γ, the mapping $y(x)$ is continuously differentiable and for $x \in U_\varepsilon(x_0)$ the inequality (6.10) is satisfied. In the choice of the system of coordinates corresponding to the definition of the class $\aleph_{r,C}$, the functions f_j are in C^2 and $f_j(\vec{0}) = 0$;

$$\frac{\partial f_j}{\partial x_i}(\vec{0}) = 0 \quad (j = k+1,\dots,n;\ i = 1,\dots,k).$$

Let $U_{r/2}$ be a neighbourhood of the origin of radius $\frac{r}{2}$. For any point $x \in U_{r/2}$ the nearest point of Γ, $y \in \Gamma_{x_0}$, and its coordinates $\{y_1, y_2, \dots, y_k\} = \hat{y}$ satisfy

$$y_i = x_i + \sum_{j=k+1}^{n}\big(x_j - f_j(\hat{y})\big)\frac{\partial f_j}{\partial y_i}(\hat{y}), \quad i = 1,\dots,k. \tag{6.11}$$

These are necessary conditions for the function $|x-y|^2$ to have a constrained minimum for a fixed $x \in U_{r/2}$. We will show that there exists $\varepsilon > 0$ independent of x_0 such that for $|x| < \varepsilon$ the system of equations (6.11) has a unique solution $\hat{y}(x)$. From this it follows that there exists a single-valued projection mapping $y(x)$ on $U_\varepsilon(x_0)$.

Note that we can only assume that the functions $f_j(\hat{y})$ are defined under the condition

$$|\hat{y}|^2 + \sum_{j=k+1}^{n} f_j^2(\hat{y}) \leq r^2.$$

Since for any differentiable function $g(\hat{y})$

$$\big|g(\hat{y}_2) - g(\hat{y}_1)\big| = \left|\int_{\hat{y}_1}^{\hat{y}_2} dg\right| \leq (\max|\nabla g|)\cdot|\hat{y}_2 - \hat{y}_1|\,, \tag{6.12}$$

setting $\hat{y}_1 = \vec{0}$, $\hat{y}_2 = \hat{y}$, we obtain

$$\sum_{j=k+1}^{n} f_j^2(\hat{y}) \le (n-k)k^2C^2|\hat{y}|^2.$$

Therefore, for a fixed point $x \in U_{r/2}$ the functions

$$F_i(x,\hat{y}) = x_i + \sum_{j=k+1}^{n} \big(x_j - f_j(\hat{y})\big)\frac{\partial f_j}{\partial y_i}, \quad i = 1,\dots,k,$$

can be taken to be defined in the ball

$$|\hat{y}| \le \frac{r}{\sqrt{1+(n-k)k^2C^2}} = \rho.$$

We observe that for $|x| < \frac{\rho}{2}$ the nearest point of Γ also satisfies the inequality $|\hat{y}| \le \rho$. We will show that for some $0 < \varepsilon < \frac{\rho}{2}$, for each x such that $|x| < \varepsilon$ the mapping

$$\vec{F}(x,\hat{y}) = \big\{F_1(x,\hat{y}), F_2(x,\hat{y}), \dots, F_k(x,\hat{y})\big\}$$

is a contraction on the ball $|\hat{y}| < \rho$ which maps this ball into itself. Hence there exists a unique fixed point in the ball $|\hat{y}| \le \rho$, that is, a unique solution of the system (6.11).

Assuming that $|x| < \varepsilon$, using (6.12) and the fact that $\frac{\partial f_j}{\partial y_i}(\vec{0}) = 0$, we find that

$$\begin{aligned}\left|\frac{\partial F_i}{\partial y_l}\right| &= \left|\sum_{j=k+1}^{n} \big(x_j - f_j(\hat{y})\big)\frac{\partial^2 f_j}{\partial y_i \partial y_l} - \frac{\partial f_j}{\partial y_i}\cdot\frac{\partial f_j}{\partial y_l}\right| \\ &\le (\varepsilon + r)(n-k)^{1/2}C + (n-k)k^2C^2|\hat{y}|^2.\end{aligned}$$

Using the fact that $|\hat{y}| \le \rho$, $n-k \ge 1$, and also the relation between ρ and r, we have

$$\left|\frac{\partial F_i}{\partial y_l}\right| \le (\varepsilon + r)(n-k)C + r^2. \tag{6.13}$$

Since by (6.12) for $i = 1, 2, \dots, k$

$$\big|F_i(x,\hat{y}_2) - F_i(x,\hat{y}_1)\big| \le \max_{|\hat{y}|\le\rho} |\nabla F_i(x,\hat{y})| \cdot |\hat{y}_2 - \hat{y}_1|,$$

we obtain

$$\big|\vec{F}(x,\hat{y}_2) - \vec{F}(x,\hat{y}_1)\big| \le k^2\big[(\varepsilon + r)(n-k)C + r^2\big] \cdot |\hat{y}_2 - \hat{y}_1|.$$

By taking r smaller we just make the class $\aleph_{r,C}$ larger; therefore, we can assume that $rk^2(n-k)C + k^2r^2 < 1$. Let us choose ε such that

$$0 < \varepsilon < \frac{1 - Ck^2(n-k)r - k^2r^2}{Ck^2(n-k)}.$$

Here $q = k^2\big[(\varepsilon + r)(n-k)C + r^2\big] < 1$ and our mapping is a strict contraction. For the ball $|\hat{y}| \le \rho$ to be invariant under the mapping $\vec{F}(x,\hat{y})$, it is sufficient that the following inequality holds [52, p. 10]:

$$\left|\vec{F}(x,\vec{0})\right| \le (1-q)\rho.$$

Since $F_i(x,\vec{0}) = x_i$, it is enough to choose $\varepsilon > 0$ such that $\varepsilon \le (1-q)\rho$.

Thus we have shown that given $r, C > 0$ we can choose a number $\varepsilon > 0$ such that in a neighbourhood $U_\varepsilon(x_0)$ of any point of the manifold Γ we can define a single-valued projection operator onto Γ. It is given by the functions

$$y_i = \hat{y}_i(x_1, x_2, \dots, x_n) \quad (i = 1,\dots,k); \qquad y_j = f_j(\hat{y}(x)) \quad (j = k+1,\dots,n)$$

and is continuous in $U_\varepsilon(x_0)$ (see [52, p. 22]). We will show that if we take ε even smaller, this mapping is differentiable on $U_\varepsilon(x_0)$. This is proved in the same way as the implicit

function theorem (see [53]). We show that if the matrix $B = \left(I_k - \frac{\partial \vec{F}}{\partial \hat{y}}\right)$ is invertible in $U_\varepsilon(x_0)$, our mapping is differentiable. Here I_k is the identity $(k \times k)$-matrix and

$$\frac{\partial \vec{F}}{\partial \hat{y}} = \left(\frac{\partial F_i}{\partial y_j}\right)_{\substack{i=1,\dots,k\\ j=1,\dots,k}}$$

Since the elements of this matrix satisfy (6.13), r and ε can be taken to be so small that B is invertible in $U_\varepsilon(x_0)$ for any point $x_0 \in \Gamma$. Since $\vec{F}(x, \hat{y})$ is a differentiable function of its arguments, the vector of increments $\Delta\hat{y} = \{\Delta y_1, \dots, \Delta y_k\}$ of the projection operator satisfies

$$\Delta\hat{y} = \frac{\partial \vec{F}}{\partial \hat{y}} \cdot \Delta\hat{y} + \frac{\partial \vec{F}}{\partial x}\, d\vec{x} + o\left(|d\vec{x}| + |\Delta\hat{y}|\right),$$

where $\frac{\partial \vec{F}}{\partial x}$ is the rectangular $(k \times n)$-matrix $\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,k\\ j=1,\dots,k}}$. From this, as in [53], it is not hard to obtain

$$\Delta\hat{y} = \left(I_k - \frac{\partial \vec{F}}{\partial \hat{y}}\right)^{-1} \frac{\partial \vec{F}}{\partial x}\, d\vec{x} + o(|d\vec{x}|).$$

Similarly, if we introduce the vector of increments $\Delta\bar{y} = \{\Delta y_{k+1}, \dots, \Delta y_n\}$ and an $((n-k) \times k)$-matrix $\frac{\partial f}{\partial \hat{y}} = \left(\frac{\partial f_l}{\partial y_j}\right)_{\substack{i=k+1,\dots,n\\ j=1,\dots,k}}$, we obtain

$$\Delta\bar{y} = \frac{\partial f}{\partial \hat{y}}\Delta\hat{y} + o(|\Delta\hat{y}|) = \frac{\partial f}{\partial \hat{y}}\left(I_k - \frac{\partial \vec{F}}{\partial \hat{y}}\right)^{-1} \frac{\partial \vec{F}}{\partial x}\, d\vec{x} + o(|d\vec{x}|).$$

Therefore, we can take the $(n \times n)$-matrix function

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \left(I_k - \dfrac{\partial \vec{F}}{\partial \hat{y}}\right)^{-1} \cdot \dfrac{\partial \vec{F}}{\partial x} \\ \dfrac{\partial f}{\partial \hat{y}} \cdot \left(I_k - \dfrac{\partial \vec{F}}{\partial \hat{y}}\right)^{-1} \cdot \dfrac{\partial \vec{F}}{\partial x} \end{pmatrix} = \Phi(x, \hat{y}(x))$$

to be the Jacobian matrix of the projection operator. Note that if $\varepsilon > 0$ is sufficiently small, the elements of this matrix are bounded in $U_\varepsilon(x_0)$ by a constant independent of x_0.

To complete the proof of Lemma 6.1, that is, to prove (6.10), we observe that the matrix $\frac{\partial y}{\partial x}$ is the matrix of a linear operator, namely the differential of the projection operator. Hence its trace,

$$\operatorname{Tr} \frac{\partial y}{\partial x} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_i}$$

is independent of the system of coordinates. To find the trace at a point $x \in U_\varepsilon(x_0)$ we choose a system of coordinates with origin at the point $y \in \Gamma$, which is the closest point to x, while the first k coordinate axes are chosen to lie in the tangent space to Γ at the point y. The manifold Γ is given by the other functions $\tilde{f}_j \in C^2$, where

$$\tilde{f}_j(\vec{0}) = 0, \quad \frac{\partial \tilde{f}_j}{\partial \tilde{x}_i}(\vec{0}) = 0 \quad (j = k+1, \dots, n).$$

The system of equations (6.11) must be satisfied in any system of coordinates, and it must be satisfied identically by the projection mapping; therefore, taking into account

that $\tilde{y}(x) = \vec{0}$ at the point x, we have

$$\frac{\partial \tilde{y}_i}{\partial \tilde{x}_i} = 1 + \sum_{j=k+1}^{n} \tilde{x}_j \sum_{l=1}^{k} \frac{\partial^2 \tilde{f}_j}{\partial \tilde{y}_i \partial \tilde{y}_l} \cdot \frac{\partial \tilde{y}_l}{\partial \tilde{x}_i} \qquad \text{for} \quad i = 1, \dots, k;$$

$$\frac{\partial \tilde{y}_i}{\partial \tilde{x}_i} = \sum_{l=1}^{k} \frac{\partial \tilde{f}_i}{\partial \tilde{y}_l} \cdot \frac{\partial \tilde{y}_l}{\partial \tilde{x}_i} = 0 \qquad \text{for} \quad i = k+1, \dots, n.$$

Thus

$$\operatorname{Tr} \frac{\partial y}{\partial x} = k + \sum_{j=k+1}^{n} \tilde{x}_j \left(\sum_{i=1}^{k} \sum_{l=1}^{k} \frac{\partial^2 \tilde{f}_j}{\partial \tilde{y}_i \partial \tilde{y}_l} \cdot \frac{\partial \tilde{y}_l}{\partial \tilde{x}_i} \right).$$

Hence it follows that

$$\left| \operatorname{Tr} \frac{\partial y}{\partial x} - k \right| \le K \operatorname{dist}(x, \Gamma).$$

Lemma 6.1 is proved. □

Lemma 6.2. *If the manifold $\Gamma \in \aleph_{r,C}$, then $d(x) = \operatorname{dist}(x, \Gamma) \in C^2(\Omega_\varepsilon \setminus \Gamma)$, where Ω_ε is a neighbourhood in which there is a single-valued differential projection mapping $y(x)$ (see Lemma 6.1). Moreover at each point $x \in \Omega_\varepsilon \setminus \Gamma$*

$$|\nabla d| = 1, \quad \Delta d = \frac{n-1}{d} - \frac{\operatorname{Tr} \frac{\partial y}{\partial x}}{d}. \tag{6.14}$$

Proof. From Lemma 6.1 it follows that the function

$$d(x) = \sqrt{\textstyle\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $y_i = y_i(x_1, \dots, x_n) \in C^1(U_\varepsilon(x_0))$ $(i = 1, \dots, n)$, lies in $C^1(U_\varepsilon(x_0) \setminus \Gamma)$. Obviously

$$\frac{\partial d}{\partial x_i} = \frac{1}{d(x)} \left[x_i - y_i - \sum_{j=1}^{n} (x_j - y_j) \frac{\partial y_j}{\partial x_i} \right].$$

Taking into account the way the mapping $y(x)$ was constructed in Lemma 6.1, we obtain

$$\begin{aligned}
\sum_{j=1}^{n} (x_j - y_j) \frac{\partial y_j}{\partial x_i} &= \sum_{j=1}^{k} (x_j - y_j) \frac{\partial y_j}{\partial x_i} + \sum_{j=k+1}^{n} (x_j - y_j) \frac{\partial y_j}{\partial x_i} \\
&= \sum_{j=1}^{k} (x_j - y_j) \frac{\partial y_j}{\partial x_i} + \sum_{j=k+1}^{n} (x_j - y_j) \sum_{l=1}^{k} \frac{\partial f_j}{\partial y_l} \frac{\partial y_l}{\partial x_i} \\
&= \sum_{l=1}^{k} (x_l - y_l) \frac{\partial y_l}{\partial x_i} + \sum_{l=1}^{k} \frac{\partial y_l}{\partial x_i} \sum_{j=k+1}^{n} (x_j - y_j) \frac{\partial f_j}{\partial y_l} \\
&= \sum_{l=1}^{k} \frac{\partial y_l}{\partial x_i} \left(x_l - y_l + \sum_{j=k+1}^{n} \big(x_j - f_j(\hat{y})\big) \frac{\partial f_j}{\partial y_l} \right) = 0.
\end{aligned}$$

Therefore

$$\frac{\partial d}{\partial x_i} = \frac{x_i - y_i}{d(x)}.$$

Hence it follows that $|\nabla d| = 1$, and also that $\frac{\partial d}{\partial x_i} \in C^1(U_\varepsilon(x_0) \setminus \Gamma)$ and

$$\frac{\partial^2 d}{\partial x_i^2} = \frac{1}{d^2(x)} \left[\left(1 - \frac{\partial y_i}{\partial x_i} \right) d(x) - (x_i - y_i) \frac{x_i - y_i}{d(x)} \right].$$

Therefore

$$\Delta d = \frac{n-1}{d} - \frac{\operatorname{Tr}\frac{\partial y}{\partial x}}{d}.$$

Lemma 6.2 is proved. □

Proof of Theorem 6.2. We will now show that under our hypotheses, the conditions of Theorem 6.1 are satisfied. To do this, we construct a particular regularised distance $\delta(x)$, which coincides with $d(x) = \operatorname{dist}(x, \partial G)$ in a neighbourhood of ∂G, and a function $\alpha(x)$ that satisfies (6.2) and (6.3). Let $\Omega_\varepsilon = \{x\colon x \in G,\, d(x) < \varepsilon\}$, and suppose that $\psi_1(x)$ and $\psi_2(x)$ are functions in $C^\infty(G)$ satisfying the conditions

1) $\psi_1(x) = 1$ for $x \in \Omega_{\varepsilon/2}$, $\psi_1(x) = 0$ for $x \in G \setminus \Omega_\varepsilon$;
2) $\psi_2(x) = 1$ for $x \in G \setminus \Omega_\varepsilon$, $\psi_2(x) = 0$ for $x \in \Omega_{\varepsilon/2}$;
3) $\psi_1(x)$, $\psi_2(x) \geq 0$, $\psi_1(x) + \psi_2(x) = 1$.

Set

$$\delta(x) = \psi_1(x)d(x) + \psi_2(x)\Delta_t(x),$$

where $\Delta_t(x) = (d * \omega_t)(x)$ for $x \notin \Omega_{\varepsilon/2}$, while $\omega_t(y)$ is a convolution kernel of radius $t < \frac{\varepsilon}{4}$. It is not hard to verify that $\delta(x)$ satisfies (6.1). To construct the function $\alpha(x)$, for each $i \in N$ we take the function

$$\beta_i(x) = \begin{cases} -\frac{n-k_i-2}{2} & \text{for } x \in \Omega^{(i)}_{\varepsilon/2} = \left\{x\colon \operatorname{dist}(x, \Gamma_i) < \frac{\varepsilon}{2}\right\}, \\ 0 & \text{for } x \in G \setminus \Omega^{(i)}_{\varepsilon/2}. \end{cases}$$

We set $\alpha_i(x) = (\beta_i * \omega_t)(x)$ for some $t < \frac{\varepsilon}{4}$. Consider $\alpha(x) = \sum_{i\in N} \alpha_i(x)$, where the sum on the right has only one term for each x, and for $x \in \Omega^{(i)}_{\varepsilon/4}$, $\alpha(x) = -\frac{n-k_i-2}{2}$. The function $\alpha(x) \in C^\infty(G)$ is bounded and has a bounded gradient. Let us check that the function $\delta(x)$ constructed in this way satisfies condition (6.4) in Ω_{ε_1} for sufficiently small ε_1 $(0 < \varepsilon_1 < \frac{\varepsilon}{4})$. First we note that $\Omega_{\varepsilon_1} = \bigcup_{i\in N}\left(\Omega^{(i)}_{\varepsilon_1} \cap G\right)$ and for $x \in \Omega^{(i)}_{\varepsilon_1}$, $\delta(x) = d(x) = \operatorname{dist}(x, \Gamma_i)$, $\alpha(x) = -\frac{n-k_i-2}{2}$. By Lemma 6.2, $\varepsilon_1 > 0$ can be chosen so small that for $x \in \Omega^{(i)}_{\varepsilon_1}$, $|\nabla\delta| = |\nabla d| = 1$, that is, (6.4) is satisfied. It remains to show that the functions $\alpha(x)$, $\delta(x)$ and $\nu(x) \in \operatorname{Lip}_1(G)$ constructed above satisfy (6.5). We will find an upper bound for

$$T(x) = \frac{1+\alpha+\alpha^2}{\delta^2}|\nabla\delta|^2 + \alpha\frac{\Delta\delta}{\delta}.$$

For $x \in G \setminus \Omega_{\varepsilon_1}$, $T(x) \leq \text{const}$. For $x \in \Omega^{(i)}_{\varepsilon_1} \cap G$ by Lemmas 6.1 and 6.2 we have

$$\begin{aligned} T(x) &= \frac{1-\alpha+\alpha^2}{d^2} + \alpha\left(\frac{n-1-\operatorname{Tr}\frac{\partial y}{\partial x}}{d^2}\right) \\ &\leq \frac{1-\alpha+\alpha^2}{d^2} + \alpha\frac{n-k_i-1}{d^2} + \frac{K_1}{d} = -\frac{(n-k_i)(n-k_i-4)}{4d^2} + \frac{K_1}{d}. \end{aligned}$$

Thus, if (6.8) holds, $q_1(x) + K\nu^2 + \frac{K_2}{d} \geq T(x)$ for $x \in \Omega_{\varepsilon_1}$. Therefore, for some $C > 0$, $q(x) \geq T(x) - C\left(\nu^2 + \frac{1}{d}\right)$ for almost all $x \in \Omega_{\varepsilon_1}$. For $x \in G \setminus \Omega_{\varepsilon_1}$ this last inequality also holds for sufficiently large $C > 0$ due to (6.9) and the fact that $T(x) \leq \text{const}$. Therefore all the hypotheses of Theorem 6.1 are satisfied and so S is essentially selfadjoint. Theorem 6.2 is proved. □

We let L^k denote a linear manifold of dimension $k < n$ in R^n.

Remark 6.1. The operator

$$S_\gamma u = -\Delta u + \frac{\gamma}{(\operatorname{dist}(x, \mathsf{L}^k))^2}u,$$

acting on the space $L_2(R^n) = L_2(R^n \setminus \mathsf{L}^k)$ with domain $C_0^\infty(R^n \setminus \mathsf{L}^k)$ and constant γ such that

$$-\frac{(n-k-1)(n-k-3)}{4} \le \gamma < -\frac{(n-k)(n-k-4)}{4},$$

is nonnegative and has nonzero deficiency indices. Therefore, in Theorem 6.2, when $\partial G = \mathsf{L}^k$, it is impossible to have a smaller constant than $-\frac{(n-k)(n-k-4)}{4}$.

Indeed, denoting $\operatorname{dist}(x, \mathsf{L}^k)$ by $d(x)$ and applying (4.1) with the vector field $\vec{f} = \left(\frac{n-k-2}{2d}\right)\nabla d$, it is not hard to show that

$$\langle S_\gamma \varphi, \varphi\rangle \ge \left[\frac{(n-k-2)^2 + 4\gamma}{4}\right] \left\|\frac{\varphi}{d}\right\|^2.$$

Therefore the operator S_γ is nonnegative. As Δ and $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ are invariant with respect to rotations and translations of the system of coordinates, we can assume that the first k coordinate axes lie in L^k and that $d(x) = \sqrt{x_{k+1}^2 + \dots + x_n^2}$. The operator S_γ can be written in the form $-\Delta_k - \Delta_{n-k} + \gamma\left(x_{k+1}^2 + \dots + x_n^2\right)^{-1}$, where Δ_k and Δ_{n-k} are Laplacians in the variables $x_1, \dots, x_k$ and $x_{k+1}, \dots, x_n$, respectively. By [19, Theorem X.11], the operator $-\Delta_{n-k} + \gamma\left(x_{k+1}^2 + \dots + x_n^2\right)^{-1}$ with domain $C_0^\infty\left(R^{n-k} \setminus \{\vec{0}\}\right)$ has nonzero deficiency indices in the space $L_2(R^{n-k})$. Therefore there is a nonzero function $\nu(x) \in C^2\left(R^{n-k} \setminus \{\vec{0}\}\right) \cap L_2$ such that

$$-\Delta_{n-k}\nu + \gamma\left(x_{k+1}^2 + \dots + x_n^2\right)^{-1}\nu = i\nu.$$

Consider the function

$$u = \varphi_0(x_1, \dots, x_k)\nu(x_{k+1}, \dots, x_n) \in L_2(R^n),$$

where the nonzero function $\varphi_0(x_1, \dots, x_k) \in C_0^2(R^k)$. Clearly, $u \in D_{S_\gamma^*}$ and

$$\left\langle S_\gamma^* u, u\right\rangle = \left(\|\nabla\varphi_0\|_k^2 + i\|\varphi_0\|_k^2\right) \cdot \|u\|_{n-k}^2,$$

where $\|\cdot\|_k$ and $\|\cdot\|_{n-k}$ are the norms in $L_2(R^k)$ and $L_2(R^{n-k})$ respectively. Since $\operatorname{Im}\left\langle S_\gamma^* u, u\right\rangle \ne 0$, the operator S_γ has nonzero deficiency indices. Thus Remark 6.1 is true.

Note that by Theorem 6.2, the operator $-\Delta$ on $C_0^\infty(G)$ is essentially selfadjoint when G has the form (6.6), (6.7), provided that

$$n - k_i \ge 4, \quad i \in N. \tag{6.15}$$

A result in [54] shows that at least in the case when ∂G is composed of a single C^∞ manifold, (6.15) is a necessary and sufficient condition for the operator $-\Delta$ on $C_0^\infty(G)$ to be essentially selfadjoint. Here ∂G does not have to belong to $\aleph_{r,C}$.

6.3. Consider a quantum system consisting of $N > 1$ interacting particles, any two of which cannot occupy the same position because of the repulsive forces that act when they are close. Let the position of the kth particle be described by a vector $\vec{x}_k \in R^m$. This only has physical significance when $m = 3$, but we can take m to be any natural number. The state of the system is described by the mN-dimensional vector $\vec{x} = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N\} \in R^{mN}$. We denote by $R_{k,j}$ the subspace $R_{k,j} = \{\vec{x}\colon \vec{x} \in R^{mN}, \vec{x}_k = \vec{x}_j\}$. Let the domain G be

$$G = R^{mN} \setminus \bigcup_{k,j} R_{k,j}. \tag{6.16}$$

In the space $L_2(G)$ we will investigate the operator

$$H = -\sum_{j=1}^{N} a_j \big(\nabla_j - i\vec{b}_j(\vec{x})\big)^2 + q(\vec{x}) \tag{6.17}$$

with domain $D_H = C_0^\infty(G)$. Here $a_j > 0$ are constants, ∇_j is the gradient in the variables $\vec{x}_j = \{x_1^{(j)}, x_2^{(j)}, \dots, x_m^{(j)}\}$, and $\vec{b}_j(\vec{x})$ are real-valued m-component vector functions in $C^1(G)$.

Theorem 6.3. *If H is an operator of the form (6.16), (6.17), and its potential $q(\vec{x}) \in L_{2\mathrm{loc}}(G)$ satisfies the inequality*

$$q(\vec{x}) \geq \sum_{j<k} \left[\frac{m(4-m)(a_j + a_k)}{4|\vec{x}_j - \vec{x}_k|^2} - \frac{\gamma(a_j + a_k)}{|\vec{x}_j - \vec{x}_k|} \right] - K\nu^2(\vec{x}) \tag{6.18}$$

almost everywhere in G for some constants $\gamma, K \geq 0$ and a function $\nu(x) \in \mathrm{Lip}_1(G)$, then the operator H is essentially selfadjoint.

Proof. The operator H is a particular case of the operator M in (1.2) where A is a constant diagonal matrix:

$$A = \mathrm{diag}\,\{a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_N, \dots, a_N\}.$$

We will show that under our conditions, the hypotheses of Theorem 5.3 hold. Condition A is satisfied because $q(\vec{x})$ is locally semibounded, in view of Proposition 2.1. For $\eta(\vec{x})$, we take the expression

$$\eta(\vec{x}) = \sum_{j<k} \left(-\ln|\vec{x}_j - \vec{x}_k| + R|\vec{x}_j - \vec{x}_k| + \rho(\vec{x}_j) + \rho(\vec{x}_k) \right),$$

where $R > 0$, and the function $\rho(\vec{x}_j)$ is defined in R^m so that $0 \leq \rho(\vec{x}_j) \to \infty$ as $|\vec{x}_j| \to \infty$, $\rho \in C^1(R^m)$ and $|\nabla\rho| \leq \mathrm{const}$ in R^m. As the vector field $\vec{f}(\vec{x})$ we choose

$$\vec{f}(\vec{x}) = \frac{m-2}{2} A \cdot \nabla \left(\sum_{j<k} \ln|\vec{x}_j - \vec{x}_k| \right).$$

We will show that for sufficiently large R, the $\eta(\vec{x})$ and $\vec{f}(\vec{x})$ we have chosen, as well as $\nu(\vec{x}) \in \mathrm{Lip}_1(G)$, satisfy conditions 1)–3) of Theorem 5.3. To that end we note that in our case the operator ∇ can be represented as

$$\nabla = \nabla_1 \oplus \nabla_2 \oplus \cdots \oplus \nabla_N.$$

In what follows we shall assume that $\vec{x}_j$ and $\vec{x}_k \in R^m$, but when we apply the operator ∇ to a function that only depends on $\vec{x}_j$, we will obtain a vector in R^{mN} such that its m nonzero coordinates occupy the jth position. To distinguish between vectors in R^m and vectors in R^{mN}, which have similar notation, for a vector in R^{mN} we shall write $\overline{\vec{x}}$. For example, $\overline{\nabla_j\rho}(\vec{x}_j)$ and $\overline{\nabla_k\rho}(\vec{x}_k)$ are in R^{mN} and their nonzero coordinates are in the jth and kth places, respectively. The position of the coordinates for the corresponding vector in R^{mN} is determined by which index is encountered first. For example, $\vec{x}_j - \vec{x}_k$, $\vec{x}_k - \vec{x}_j \in R^m$, but $\overline{\vec{x}_j - \vec{x}_k}$ and $\overline{\vec{x}_k - \vec{x}_j} \in R^{mN}$, and their nonzero coordinates are in the jth and kth places, respectively. On this basis, we can write

$$\nabla\eta = \sum_{j<k} \left[-\frac{\overline{(\vec{x}_j - \vec{x}_k)} + \overline{(\vec{x}_k - \vec{x}_j)}}{|\vec{x}_j - \vec{x}_k|^2} + R\frac{\overline{(\vec{x}_j - \vec{x}_k)} + \overline{(\vec{x}_k - \vec{x}_j)}}{|\vec{x}_j - \vec{x}_k|} + \overline{\nabla_j\rho}(\vec{x}_j) + \overline{\nabla_k\rho}(\vec{x}_k) \right].$$

Taking into account the form of the matrix A, we have

$$(A\nabla\eta,\nabla\eta)=\sum_{j<k}\Bigg[\frac{a_j+a_k}{|\vec{x}_j-\vec{x}_k|^2}+R(a_j+a_k)+a_j|\nabla_j\rho(\vec{x})|^2+a_k|\nabla_k\rho(\vec{x})|^2-\frac{2R(a_j+a_k)}{|\vec{x}_j-\vec{x}_k|}$$
$$-\frac{2\big[a_k\big((\vec{x}_j-\vec{x}_k),\nabla_k\rho(\vec{x}_k)\big)+a_j\big((\vec{x}_k-\vec{x}_j),\nabla_j\rho(\vec{x}_j)\big)\big]}{|\vec{x}_j-\vec{x}_k|^2}\big(1-R|\vec{x}_j-\vec{x}_k|\big)\Bigg].$$

Hence, making use of the properties of the function ρ, we conclude that

$$(A\nabla\eta,\nabla\eta)\le D\sum_{j<k}\frac{1}{|\vec{x}_j-\vec{x}_k|^2}\big(1+|\vec{x}_j-\vec{x}_k|+|\vec{x}_j-\vec{x}_k|^2\big)$$

for some constant $D>0$. On the other hand,

$$e^{2\eta}=\prod_{j<k}\Bigg[\frac{1}{|\vec{x}_j-\vec{x}_k|^2}\exp\big\{2\big(R|\vec{x}_j-\vec{x}_k|+\rho(\vec{x}_j)+\rho(\vec{x}_k)\big)\big\}\Bigg]$$
$$\ge\frac{2}{N(N-1)}\sum_{j<k}\frac{1}{|\vec{x}_j-\vec{x}_k|^2}\exp\big\{2\big(R|\vec{x}_j-\vec{x}_k|+\rho(\vec{x}_j)+\rho(\vec{x}_k)\big)\big\}.$$

This inequality is true for R so large that all the factors in the product for $e^{2\eta}$ are no less that one. Now it is clear that the inequality $(A\nabla\eta,\nabla\eta)\le Ce^{2\eta}$ holds for some $C>0$. Since for a sufficiently large R, $0<\eta(\vec{x})\to\infty$ as $\vec{x}\to\partial G$, the functions $\eta(\vec{x})$, $\nu(\vec{x})$ and the vector field $\vec{f}(\vec{x})$ that we have chosen satisfy conditions 1)–3) of Theorem 5.3. Now we will show that if (6.18) holds for sufficiently large R, condition (5.7) in Theorem 5.3 is satisfied. To find an upper bound for the quantity

$$T(x)=(A\nabla\eta,\nabla\eta)+\big(A^{-1}\vec{f},\vec{f}\big)-\nabla\vec{f}$$

consider each of the terms:

$$(A\nabla\eta,\nabla\eta)\le\sum_{j<k}\Bigg[\frac{a_j+a_k}{|\vec{x}_j-\vec{x}_k|^2}+\frac{M_0-2R(a_j+a_k)}{|\vec{x}_j-\vec{x}_k|}\Bigg]+M_R,$$

where $M_0,M_R>0$ are two constants, the first of which does not depend on R, while the second one does;

$$\big(A^{-1}\vec{f},\vec{f}\big)=\left(\frac{m-2}{2}\right)^2\sum_{j<k}\frac{a_j+a_k}{|\vec{x}_j-\vec{x}_k|^2},$$

$$-\nabla f=-\frac{m-2}{2}\nabla\left[\sum_{j<k}\frac{a_j\overline{(\vec{x}_j-\vec{x}_k)}+a_k\overline{(\vec{x}_k-\vec{x}_j)}}{|\vec{x}_j-\vec{x}_k|^2}\right]=-\frac{(m-2)^2}{2}\sum_{j<k}\frac{a_j+a_k}{|\vec{x}_j-\vec{x}_k|^2}.$$

Thus we have the inequality

$$T(x)\le\sum_{j<k}\left\{\left[1-\frac{(m-2)^2}{2}+\left(\frac{m-2}{2}\right)^2\right]\frac{a_j+a_k}{|\vec{x}_j-\vec{x}_k|}+\frac{M_0-2R(a_j+a_k)}{|\vec{x}_j-\vec{x}_k|}\right\}+M_R.$$

If we choose $R\ge\frac{M_0+\gamma}{2(a_j+a_k)}$, we have the inequality

$$T(x)\le\sum_{j<k}\left[\frac{m(4-m)(a_j+a_k)}{4|\vec{x}_j-\vec{x}_k|^2}-\frac{\gamma(a_j+a_k)}{|\vec{x}_j-\vec{x}_k|}\right]+M_R.$$

If (6.18) holds, we have

$$q(\vec{x})+K\nu^2(\vec{x})\ge T(\vec{x})-M_R,$$

that is, all the conditions of Theorem 5.3 are satisfied and the operator H is essentially selfadjoint. Theorem 6.3 is proved. □

Remark 6.2. It is impossible to take a smaller constant than $\frac{m(4-m)}{4}$ in Theorem 6.3, at least for a system of two identical particles. ($N = 2$, $a_1 = a_2 = a$) in the case when $\vec{b}_1(\vec{x}_1) = \vec{b}_2(\vec{x}_2) = \vec{0}$.

For, suppose we set

$$q(\vec{x}) = \frac{2\beta a}{|\vec{x}_2 - \vec{x}_1|^2} \quad \text{and} \quad \beta = \text{const},$$

in the operator H. Since $\partial G = R_{1,2} = \{\vec{x} \in R^{2m},\ \vec{x}_1 = \vec{x}_2\}$ is an m-dimensional linear manifold in R^{2m} the distance to which is $d(\vec{x}) = \frac{|\vec{x}_2 - \vec{x}_1|}{\sqrt{2}}$, we have $H = a\big(-\Delta + \frac{\beta}{d^2}\big)$. By Remark 6.1 for $-\frac{(m-1)(m-3)}{4} \le \beta < \frac{m(4-m)}{4}$ the operator H has nonzero deficiency indices.

The question of essential selfadjointness for the internal Hamiltonian of a system of identical particles with a strong pairwise interaction has been studied in [25]. The results there cannot easily be compared with Theorem 6.3. Unlike the case treated in that paper, Theorem 6.3 covers the case of systems of particles with differing masses subjected to Coulomb attraction forces and an external field, such as, for example, a magnetic field.

In conclusion, we note the relatively recent paper [55], where an analog of the Kalf–Walter–Schmincke–Simon theorem was obtained for a nonsemibounded Schrödinger operator on Riemannian manifolds with a one-point boundary.

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