

**A FUNCTIONAL INTEGRAL  
WITH RESPECT TO A COUNTABLY ADDITIVE MEASURE  
REPRESENTING A SOLUTION OF THE DIRAC EQUATION**

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ABSTRACT. A family  $\{M_p\}_{p \in \mathbb{R}^d}$  of cylindrical measures is constructed on the space of functions (= trajectories)  $f : [0, +\infty) \rightarrow \mathbb{R}^d$  such that for every  $t \geq 0$  the formula

$$(1) \quad \psi(t, p) = \int M_p(df) \psi_0(f(t))$$

represents a solution of the Cauchy problem

$$(2) \quad \frac{\partial}{\partial t} \psi = i\hat{H}\psi \quad (t \geq 0), \quad \psi(0, \cdot) = \psi_0$$

(with respect to the required function  $\psi : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C}^s$ ,  $d, s \in \mathbb{N}$ ), for the general evolution equation from some class containing the classical Dirac equation and the Schrödinger equation in its impulse representation, with “model” potentials that are independent of  $t$ , and are Fourier transforms of countably additive (and in general matrix-valued) measures in the space variables.

The images of the measures  $M_p$  obtained by restricting trajectories to finite intervals  $[0, T]$  have bounded variation and are countably additive.

The integral kernels (“Green’s functions”) of the corresponding solution operators, which can be approximated (using Trotter’s formula) by integrals of finite multiplicity of the expressions explicitly defined by the ingredients of the original equation, are (matrix-valued) transition measures that give cylindrical measures  $M_p$  similarly to the way Markov transition probabilities give the distribution of a Markov process.

A new method using matrix-valued transition measures is applied in this paper, leading to a stronger result for the Dirac equation than those that follow under assumptions analogous to ours from results in previous papers. In [5], uniqueness of solutions is not claimed; in [11] and [12], functional integrals are understood only as generalized integrals; in [13] (and in earlier articles by Ktitarev), the authors proceed under assumptions that are more restrictive when applied to our situation. This method can also be adapted as a generalization, to the case of infinite-dimensional algebras, of values of the coefficients (generally speaking, they depend on the “space” variables) occurring on the right-hand side of the evolution equation; in particular, for the second-order super-differential equations presented in [5], the results of which do not overlap those described below, and in which, in order to construct a functional Poisson distribution analogous to the measure  $M_0$ , the authors use not finite-dimensional distributions (= Trotter approximations) but the Dyson series analogous to that used in [1] to solve the Schrödinger equation. Our method goes back to the similarity indicated in [14] between the properties of complex

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measures of Poisson type constructed in [1] to represent solutions of Schrödinger's equation and the properties of probability distributions on function spaces, generated by homogeneous Markov processes.<sup>1</sup>

### 1. STATEMENT OF THE CAUCHY PROBLEM AND TERMINOLOGY

Evolution equations considered as Cauchy problems (special cases of which are the Schrödinger and Dirac equations with potentials of the type described above) are of the form

$$(3) \quad f'_t = (\mathbf{A}_0 + \mathbf{B}_0)f_t, \quad t \geq 0,$$

where the following conditions are satisfied:

1. For every  $t$ , the value  $f_t \equiv f(t)$  of the required "function"

$$f : [0, \infty) \rightarrow L_2(\mathbb{R}^d, \mathbb{C}^s) \equiv L_2, \quad s = 1, 2, \dots,$$

including the "initial" function  $f_0$ , lie in the complex Hilbert space  $L_2(\mathbb{R}^d, \mathbb{C}^s)$ , denoted from now on by  $L_2$  and consisting of all (classes of) square-summable functions defined on real  $d$ -dimensional ( $d \in \mathbb{N}$ ) space  $\mathbb{R}^d$  (taken to be Euclidean space with scalar product  $q \cdot p = \sum_{j=1}^d q_j p_j$ ,  $q, p \in \mathbb{R}^d$ ) and taking values in complex  $s$ -dimensional space  $\mathbb{C}^s$  ( $s \in \mathbb{N}$ ), to be interpreted in what follows as the space of columns of  $s$  complex numbers with the canonical Hermitian sesquilinear product  $\langle u, v \rangle_{\mathbb{C}^s} = \sum_{j=1}^s u_j \cdot \bar{v}_j$ .

The derivative

$$f'_t \equiv f'(t) \equiv \lim_{\tau \downarrow 0} \tau^{-1} \cdot (f(t + \tau) - f(\tau))$$

of an  $L_2$ -valued function of a real argument is defined relative to the norm in the Hilbert space  $L_2$ .

2.  $\mathbf{B}_0$  is a bounded operator defined everywhere in  $L_2$ ; in fact, it is the operator  $(\mathbf{B}_0\varphi)(x) = V(x)\varphi(x)$  ( $\varphi \in L_2$ ,  $x \in \mathbb{R}^d$ ) of multiplication by a bounded continuous function  $V : \mathbb{R}^d \rightarrow \mathbb{A}_0$ , where  $\mathbb{A}_0$  is a subalgebra of the algebra  $\mathbb{A} = \text{Matr}(s \times s, \mathbb{C})$  of all complex  $s \times s$  matrices,  $\mathbb{A}_0$  contains the identity matrix  $1_{\mathbb{A}}$  and the zero matrix  $0_{\mathbb{A}}$  and is equipped with some Banach norm (for instance the operator norm of the action in the Hermitian space  $\mathbb{C}^s$ )  $\mathbb{A}_0 \ni a \mapsto |a|_0 \in \mathbb{R}$  such that  $|ab|_0 \leq |a|_0 \cdot |b|_0$ ; we assume that  $V$  is the Fourier transform of some countably additive measure  $\nu$  (notation:  $V = \tilde{\nu}$ ) defined on the  $\sigma$ -algebra  $\mathcal{B}_d = \mathcal{B}_{\mathbb{R}^d}$  of all Borel sets in  $\mathbb{R}^d$  and taking values in  $\mathbb{A}_0$ ; that is,

$$(4) \quad V : \mathbb{R}^d \ni x \mapsto \tilde{\nu}(x) \equiv \int_{\mathbb{R}^d} e^{i x \cdot p} \nu(dp) \quad (\in \mathbb{A}_0)$$

is the Fourier transform of  $\nu$ , and the integral of the complex function with respect to the matrix measure is calculated componentwise; thus,  $V$  is continuous and bounded, for instance by the (total) variation [2]

$$\|\nu\| = \sup \left\{ \sum_{j=1}^n |\nu(A_j)|_0 : n \in \mathbb{N}, A_j \in \mathcal{B}_d; j \neq k \Rightarrow A_j \cap A_k = \emptyset \right\}.$$

3.  $\mathbf{A}_0$  is an anti-selfadjoint (skew-Hermitian) differential operator on  $L_2$  with constant coefficients that are elements of  $\mathbb{A}_0$ ; in other words,  $\mathbf{A}_0$  is a polynomial of the form

<sup>1</sup>Another motivation lies in the cadre of ideas of quantum theory, where the measure of the possibility of an event—the amplitude of the probability—is an essentially complex value not reducing to a pair of real numbers (because of the absence of verified experiments measuring the "absolute phase of an amplitude" as the argument of a complex number); the effectiveness of the idea of complex Markov chains [1] required the verification of the applicability of similar constructions to the case of non-commutative values of analogues of complex amplitudes for the probabilities of quantum transitions. In particular, this last consideration has prompted the choice of the term "transition amplitude" for transition measures of Markov type with values that do not commute under multiplication.

$i \cdot P(-i\partial_1, \dots, -i\partial_d)$  in operators that are selfadjoint on  $L_2$  and commute on the dense subspace consisting of the operators  $-i\partial_k$ ,  $k = 1, \dots, d$ ; here, in its turn,  $P$  is a polynomial in the same operators with coefficients that are Hermitian selfadjoint matrices in  $\mathbb{A}_0 \subset \mathbb{A}$ .

In studying questions like this, and in particular the influence of the Fourier inverse image of the potential on the solution, we use the following terminology and associated facts.

**Definition 1.0.1.** Let  $P$  be a semiring of sets,  $(K, +)$  any commutative semigroup, and  $m : P \rightarrow K$  a map. We call such a map a  $(P, K)$ -measure if the finite additivity condition is satisfied: for all natural numbers  $n$  and pairwise disjoint sets  $A_1, \dots, A_n \in P$  such that  $\bigcup_{j=1}^n A_j$  is an element of  $P$ , we have  $\sum_{j=1}^n m(A_j) = m(\bigcup_{j=1}^n A_j)$ .

We say that a  $(P, K)$ -measure is a measure with normed range if  $(K, +)$  is the additive group of a normed field or of a normed vector space over a normed field: the norm of an element  $k \in K$  is denoted by  $|k|_K$ .

The variation  $\text{Var}_m(A)$  of such a measure  $m$  on a set  $A$  (not necessarily contained in  $P$ ) is defined to be the—possibly infinite—least upper bound of all finite or countable sums  $|m(A_1)|_K + |m(A_2)|_K + \dots$ , where the elements of the sequence  $\{A_j : j = 1, 2, \dots\}$  lie in  $P$ , are subsets of  $A$ , and are pairwise disjoint. The total variation of  $m$  is defined to be  $\text{Var}_m(\cup P) (\in [0; +\infty])$ . The function  $P \rightarrow [0; +\infty]$ , taking the value  $\text{Var}_m(A)$  on every set  $A \in P$  is denoted by  $|m|$ , and the total variation of  $m$  by  $\|m\|$ .

**Definition 1.0.2.** If  $m : P \rightarrow (K, +)$  is a measure with normed range, then  $\bar{m}$  is the unique extension of  $m$  from the semiring  $P$  to the smallest ring  $\bar{P}$  containing it, which preserves finite additivity.

Here  $\text{Var}_\mu = \text{Var}_{\bar{\mu}}$ ,  $\|\bar{\mu}\| = \|\mu\|$ , and  $|\mu|$  is a  $(P, [0, \infty])$ -measure. If  $\mu$  is a countably additive  $(P, \mathbb{A})$ -measure, then  $|\mu|$  is countably additive on  $P$ . Suppose that  $P_m = \{A \in P : |m(A)| < \infty\}$ ; then  $P_m$  is a semiring of sets, and is a ring if  $P$  is. The proof proceeds like that for real measures.

**Definition 1.0.3.** Suppose that we are given a norm on  $\mathbb{R}^n$ , where  $n \in \mathbb{N}$ . A  $(\mathcal{B}_d, \mathbb{R}^n)_c$ -measure is a countably additive  $(\mathcal{B}_d, \mathbb{R}^n)$ -measure; for every  $(\mathcal{B}_d, \mathbb{R}^n)_c$ -measure  $\nu$  we define the norm of  $\nu$  to be the total variation  $\|\nu\|$  and note that the vector space of all  $(\mathcal{B}_d, \mathbb{R}^n)_c$ -measures is a Banach space in this norm.

The direct product  $\mu \times \nu$  of  $(\mathcal{B}_d, \mathbb{A}_0)$ -measures  $\mu$  and  $\nu$  is the map

$$A \times B \mapsto \mu(A) \cdot \nu(B),$$

defined on the semi-algebra  $\mathcal{B}_d \oplus \mathcal{B}_d := \{A \times B : A \in \mathcal{B}_{\mathbb{R}^d} \ni B\}$  of “Borel rectangles”.

**Proposition 1.0.1.** The direct product of  $(\mathcal{B}_d, \mathbb{A}_0)$ -measures is a  $(\mathcal{B}_d \oplus \mathcal{B}_d, \mathbb{A}_0)$ -measure; the direct product of  $(\mathcal{B}_d, \mathbb{A}_0)_c$ -measures  $\mu$  and  $\nu$  is a countably additive measure of bounded variation:  $|\mu \times \nu| \leq |\mu| \times |\nu|$ .

The first part is obvious. The proof of countable additivity reduces to a standard fact about non-negative measures; after the measure  $\mu \times \nu$  has been extended to the algebra  $\overline{\mathcal{B}_d \oplus \mathcal{B}_d}$ , the inequality is proved by repeating the proof of the equivalent inequality for alternating numerical measures of bounded variation defined on algebras.

**Definition 1.0.4.** The product of  $(\mathcal{B}_d, \mathbb{A}_0)_c$ -measures  $\mu$  and  $\nu$  is defined to be the (unique)  $(\mathcal{B}_{2d}, \mathbb{A}_0)_c$ -measure  $\mu \otimes \nu$ , extending  $\mu \times \nu$ .

The convolution  $\mu * \nu$  is the image of their product under the Borel map  $+: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  (see [3]).

*Remark 1.0.1.* The convolution of  $(\mathcal{B}_d, \mathbb{A}_0)_c$ -measures  $\mu$  and  $\nu$  is a  $(\mathcal{B}_d, \mathbb{A}_0)_c$ -measure; since  $|\mu \times \nu| \leq |\mu| \times |\nu|$  by Proposition 1.0.1, it follows that  $|\mu * \nu| \leq |\mu| * |\nu|$ .

### 1.1. Examples.

1.1.1. *The Dirac equation for a relativistic electron.* This is the equation

$$(5) \quad \left[ p_0 + \frac{e}{c}A_0 + \sum_{j=1}^3 a_j \left( p_j + \frac{e}{c}A_j \right) + a_4 mc \right] \psi = 0,$$

where the unknown function  $\psi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$  is a function of the variables  $t \in \mathbb{R}$  and  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ ,

$$a_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the  $A_j : \mathbb{R} \times \mathbb{R}^3$  ( $j = 0, \dots, 3$ ) are the real components of the electromagnetic 4-potential, which depends, in general, on the time variable  $t \in \mathbb{R}$  and on the space variables  $x_1, x_2, x_3$ ;

$$p_0 = -i \frac{\partial}{\partial t} \equiv -i\partial_0, \quad p_j = -i \frac{\partial}{\partial x_j} = -i\partial_j, \quad j = 1, 2, 3.$$

The electron charge  $e$ , its mass  $m$  and the speed of light  $c$  are constants. Without loss of generality we may assume that  $e = c = 1$  (or that  $e/c$  is included in the corresponding components of the potential), and in what follows the last coefficient  $mc$  will be denoted by  $m$ . The equation now takes the form

$$(6) \quad \left[ p_0 + A_0 + \sum_{j=1}^3 a_j(p_j + A_j) + ma_4 \right] \psi = 0,$$

or, after taking the derivative with respect to time and separating the differential polynomial from the product by the matrix-valued function,

$$(7) \quad \partial_t \psi = -i \left[ ma_4 + \sum_{j=1}^3 a_j p_j \right] \psi + (-i) \left[ A_0 + \sum_{j=1}^3 a_j A_j \right] \psi.$$

The first summand gives a selfadjoint operator (this follows from the permutability of the pairs of selfadjoint factors occurring in the four summands in the bracket), while the second gives the operation of multiplication by the matrix-valued function

$$V = -i \left( A_0 + \sum_{j=1}^3 a_j A_j \right),$$

and, with a suitable choice of the potentials  $A_j$  ( $j = 0, 1, 2, 3$ ) (namely, when they are independent of time and have the property of being Fourier transforms of numerical or  $\mathbb{A}$ -valued countably additive Borel measures on  $\mathbb{R}^d$ ), we arrive at equation (3).

1.1.2. *Schrödinger's equation with matrix potential independent of time.* This is the equation

$$(8) \quad \partial_t \psi = -\frac{i}{2} \Delta_x \psi + V \cdot \psi$$

with respect to the function  $\psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}^s$  of variables  $t \in \mathbb{R}$  and  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ . Here,  $\Delta_x = \sum_{j=1}^d \left(\frac{\partial}{\partial x_j}\right)^2$  is a selfadjoint operator on  $L_2(\mathbb{R}^d, \mathbb{C}^s)$ , namely the Laplacian. If, as before, the factor  $V = \tilde{\nu} : \mathbb{R}^d \rightarrow \mathbb{A}$  is given as above by the Fourier transform of the countably additive measure  $\nu : B_{\mathbb{R}^d} \rightarrow \mathbb{A}$ , we again arrive at equation (3).

1.2. **Impulse representation of equation (3).** To find a solution of the evolution equation (3) with initial condition  $f_0 \in L_2$  we use the Fourier transform in  $L_2$ , which is the inverse in  $L_2$  of a bounded operator that is a multiple of an isometry and is given by the formula  $\tilde{\varphi} = \widetilde{\varphi \cdot \text{Leb}}$  for every  $\varphi$  from the subspace  $L_2^c \subset L_2$  consisting of elements of  $L_2$  with bounded support; here  $\text{Leb}$  is canonical Lebesgue measure on  $\mathbb{R}^d$ . The standard properties of such a Fourier transform lead to the following Cauchy problem, equivalent to the Cauchy problem for equation (3):

$$(9) \quad g'_t = (\mathbf{A} + \mathbf{B})g_t \equiv i \cdot P \cdot g_t + \nu * g_t \quad (t \geq 0); \quad g_0 = \tilde{f}_0,$$

where  $g : [0, \infty) \rightarrow L_2$  is the desired function, and  $P$  is a polynomial on  $\mathbb{R}^d$  with Hermitian selfadjoint coefficients (and thus also values) in  $\mathbb{A}_0$  (we shall call such polynomials selfadjoint). Furthermore, to every finite continuous function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}^s$  the value of the convolution  $\nu * \varphi : \mathbb{R}^d \rightarrow \mathbb{C}^s$  at a point  $p \in \mathbb{R}^d$  is given by

$$(\nu * \varphi)(p) = \int_{\mathbb{R}^d} \nu(dx) \varphi(p - x).$$

We note that the operator  $\mathbf{A}$  of multiplication by  $iP$  is anti-selfadjoint on its natural domain ( $(i\mathbf{A})^* = i\mathbf{A}$ ), while the operator  $\mathbf{B}$  of convolution with  $\varphi$  is bounded in  $L_2$ .

Moreover, there exist representations for bounded semigroups with generators  $\mathbf{A}$  and  $\mathbf{B}$ : the unitary operators  $e^{t\mathbf{A}}$  are simply operators of multiplication by a matrix-valued function,  $(e^{t\mathbf{A}}\varphi)(p) = e^{iP(p)}\varphi(p)$ . For the bounded operator  $\mathbf{B}$ , the Maclaurin series

$$e^{t\mathbf{B}} = \mathbf{1} + \sum_{j=1}^{\infty} \frac{t^j}{j!} \mathbf{B}^j$$

converges in the operator norm, and because of the properties of convolution, for every  $\varphi \in L_2^c$ , it has the representation  $e^{t\mathbf{B}}\varphi = \mu^t * \varphi$ , where, as is easy to see, for every  $t \geq 0$  the measure  $\mu^t$  is represented by the series

$$\mu^t = \delta_0 + \sum_{j=1}^{\infty} \frac{t^j}{j!} \cdot \nu^{*j},$$

which converges in the variation norm. Here the convolution powers are defined inductively:

$$\nu^{*0}(Y) = \delta_0(Y) := \begin{cases} 1_{\mathbb{A}}, & 0 \in Y, \\ 0_{\mathbb{A}}, & 0 \notin Y \end{cases} \quad (Y \in B_{\mathbb{R}^d}) \text{ and } \nu^{*(n+1)} = \nu * (\nu^{*n}) \quad (n = 0, 1, \dots).$$

To get a uniform representation of these operators in terms of generalized integral kernels used in later estimates, we need the following objects and facts.

**Definition 1.2.1.** An  $\mathbb{R}^n$ -valued transition amplitude on  $\mathbb{R}^d$  (an  $(\mathbb{R}^d, \mathbb{R}^n)$ -TA for short) is any function  $f : \mathbb{R}^d \times \mathcal{B}_d \rightarrow \mathbb{R}^n$  satisfying the following three conditions:

- 1°  $f(\mathbb{R}^d \times \mathcal{B}_d)$  is a bounded subset of  $\mathbb{R}^n$ ;

- 2° for every  $x \in \mathbb{R}^d$ , the map  $\mathcal{B}_d \ni Y \mapsto f(x, Y) \in \mathbb{R}^n$  is countably additive (that is, it is a  $(\mathcal{B}_d, \mathbb{R}^n)_c$ -measure);
- 3° for every  $Y \in \mathcal{B}_d$  the map  $\mathbb{R}^d \ni x \mapsto f(x, Y) \in \mathbb{R}^n$  is a Borel map (is measurable with respect to Borel  $\sigma$ -algebras in  $\mathbb{R}^n$  and  $\mathbb{R}^d$ ).

The convolution of  $\mathbb{A}_0$ -valued transition amplitudes  $p$  and  $q$  is the map

$$\mathbb{R}^d \times \mathcal{B}_d \ni (x, Y) \mapsto (p * q)(x, Y) = \int_{z \in S} p(x, dz) q(z, Y)$$

where the integral is calculated componentwise.

We denote the vector space of all  $(\mathbb{R}^d, \mathbb{R}^n)$ -TA by  $\mathfrak{A}(\mathbb{R}^d, \mathbb{R}^n)$ .

**Example 1.** The function given on  $\mathbb{R}^d \times \mathcal{B}_d$  by the formula

$$\delta(x, Y) = 1_Y(x) = \delta_x(Y) = \begin{cases} 1_{\mathbb{A}}, & x \in Y, \\ 0_{\mathbb{A}}, & x \notin Y, \end{cases}$$

is a  $(\mathcal{B}_d, \mathbb{A}_0)$ -TA, referred to below as the “delta-TA”.

**Example 2.** Let  $\mu$  be a  $(\mathcal{B}_d, \mathbb{A}_0)$ -measure. Then the function  $f_\mu : \mathbb{R}^d \times \mathcal{B}_d \rightarrow \mathbb{A}_0$  given by the formula  $f_\mu(x, Y) = \mu(Y - x)$  is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA.

The proof requires only measurability with respect to  $x$ , which can be proved like this. Since  $+^{-1}(Y) \in \mathcal{B}_{2d}$  and

$$Y - x = \{y \in \mathbb{R}^d : (x, y) \in (+^{-1}(Y)) \subset \mathbb{R}^{2d}\},$$

the problem reduces to establishing Borel measurability of the function

$$\mathbb{R}^d \ni x \mapsto \mu\{y : (x, y) \in B\} \in \mathbb{A}_0 \quad (B \in \mathcal{B}_{2d});$$

this follows immediately since a Borel set can be approximated by polyhedra in the non-negative measure  $|\mu|$ .

In such a situation, we shall be interested in  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA of the form  $f_{\mu^t}$ , where  $t \geq 0$  and  $\mu$  is a  $(\mathcal{B}_d, \mathbb{A}_0)_c$ -measure; it is clear that  $f_{\mu^0} = f_{\delta_0} = \delta$ .

*Remark 1.2.1.* Suppose we are given natural numbers  $k, m, n$ , a bilinear operator  $b : \mathbb{R}^k \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  between finite-dimensional spaces, and a bounded Borel function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^k$ ; then for every  $p \in \mathfrak{A}(\mathbb{R}^d, \mathbb{R}^m)$  the map

$$\mathbb{R}^d \times \mathcal{B}_d \ni (x, Y) \mapsto \int_Y b(\varphi(y), p(x, dy))$$

is an  $(\mathbb{R}^d, \mathbb{R}^n)$ -TA.

Measurability in  $x$  follows since  $\varphi$  can be uniformly approximated by simple measurable functions, while the remaining TA-properties are obvious.

*Remark 1.2.2.* We note that a  $(\mathcal{B}_d, \mathbb{A}_0)$ -measure is simply a matrix of complex-valued measures; that an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA is a matrix of complex-valued transition amplitudes; and that the convolution of amplitudes is well defined.

**Proposition 1.2.1.** *If  $p, q \in \mathfrak{A}(\mathbb{R}^d, \mathbb{A}_0)$ , then  $p * q \in \mathfrak{A}(\mathbb{R}^d, \mathbb{A}_0)$ .*

*Proof.* Clearly, it is sufficient to consider the case where  $p$  and  $q$  take real values; property 1° for  $p * q$  is obvious. Borel measurability of the map  $x \mapsto (p * q)(x, Y)$  (for fixed  $Y \in \mathcal{B}_d$ ) follows as in Remark 1.2.1, since the Lebesgue integral can be approximated by integral sums corresponding to the uniform approximation of the function  $z \mapsto q(z, Y)$ , which has totally bounded range, by simple measurable functions. Finally, countable additivity of the map  $Y \mapsto (p * q)(x, Y)$  (for fixed  $x \in \mathbb{R}^d$ ) follows in particular from the following lemma, which is of independent interest. □

**Definition 1.2.2.** Suppose that  $p \in \mathfrak{A}(\mathbb{R}^d, \mathbb{R}^n)$ . For every  $x \in \mathbb{R}^d$  the symbol  $p_x$  denotes the  $(\mathcal{B}_d, \mathbb{R}^n)_c$ -measure  $Y \mapsto p(x, Y)$ . Set  $\|p\| = \sup_{x \in \mathbb{R}^d} \|p_x\|$ . The map  $\mathbb{R} \times \mathcal{B}_d \ni (x, Y) \mapsto |p_x|(Y) \geq 0$  is denoted by  $|p_\bullet|$ , so that  $|p_\bullet|(x, Y) = |p_x|(Y)$ .

**Lemma 1.2.1.**  $\mathfrak{A}(\mathbb{R}^d, \mathbb{A}_0) \ni p \mapsto \|p\| \geq 0$  is a norm on  $\mathfrak{A}(\mathbb{R}^d, \mathbb{A}_0)$ . If  $p, q \in \mathfrak{A}(\mathbb{R}^d, \mathbb{A}_0)$ , then  $|p_\bullet| \in \mathfrak{A}(\mathbb{R}^d, \mathbb{R})$ ,  $|(p * q)_\bullet| \leq |p_\bullet| * |q_\bullet|$ , and thus  $\|p * q\| \leq \|p\| \cdot \|q\|$ .

*Proof.* The norm properties are clear, for example, from the fact that the total variation of a measure is a norm on the space of  $(\mathcal{B}_d, \mathbb{A}_0)_c$ -measures. Further, we use the well-known fact that the total variation on the space of such measures  $\nu$  is the adjoint norm to the uniform norm on the (separable with respect to the latter) space  $C_0(\mathbb{R}^d; (\mathbb{A}_0)^*)$  of finite continuous functions on  $\mathbb{R}^d$  taking values in the real-adjoint space  $(\mathbb{A}_0)^*$  to the normed space  $\mathbb{A}_0$ . In other words,

$$\|\nu\| = \sup_n \int_{\mathbb{R}^d} b_0(f_n(y), \nu(dy)),$$

where  $b_0$  is the form of the real duality between  $(\mathbb{A}_0)^*$  and  $\mathbb{A}_0$ , and  $\{f_j : j = 1, 2, \dots\}$  is a countable set of functions dense in the central unit ball in  $C_0(\mathbb{R}^d; (\mathbb{A}_0)^*)$ .

To prove that  $|p_\bullet| \in \mathfrak{A}(\mathbb{R}^d, \mathbb{R})$ , all we need to do is to establish that the map  $\mathbb{R}^d \ni x \mapsto |p_x|(A)$  is Borel measurable for every fixed  $A \in \mathcal{B}_d$ . We do this first in the case where  $A = \mathbb{R}^d$ , that is, we show that the map  $\mathbb{R}^d \ni x \mapsto \|p_x\|$  is Borel measurable. Since

$$\|p_x\| = \sup_n \int_{\mathbb{R}^d} b_0(f_j(y), p_x(dy)),$$

where the sequence  $\{f_j : j = 1, 2, \dots\}$  introduced above does not depend on  $x$ , Remark 1.2.1 shows that measurability follows from the boundedness of the  $f_j$ .

Suppose now that  $A \in \mathcal{B}_d$  is an arbitrary fixed set. The map

$$p^A : \mathbb{R}^d \times \mathcal{B}_d \ni (x, Y) \mapsto p(x, Y \cap A) = \int_Y 1_A(y) \cdot p(x, dy)$$

is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA again by Remark 1.2.1, and moreover  $|p_x|(A) = \|(p^A)_x\|$ , so that the desired result is immediate. This proves the lemma.  $\square$

**Example 3.** If  $f$  is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA and  $g : \mathbb{R}^d \rightarrow \mathbb{A}_0$  is a bounded measurable function, then the function  $(gf)(x, Y) := g(x) \cdot f(x, Y)$  is also an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA.

**Definition 1.2.3.** Let  $f : \mathbb{R}^d \times B_{\mathbb{R}^d} \rightarrow \mathbb{A}$  be an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA, and let  $g : \mathbb{R}^d \rightarrow \mathbb{A}_0$  and  $h : \mathbb{R}^d \rightarrow \mathbb{C}^s$  be finitary continuous functions. The convolutions  $f * g$ ,  $g * f$ , and  $f * h$  are defined to be the measurable bounded functions

$$\begin{aligned} \mathbb{R}^d \ni x &\mapsto \int f(x, dy)g(y) \equiv (f * g)(x) \in \mathbb{A}, \\ \mathbb{R}^d \ni x &\mapsto \int g(y)f(x, dy) \equiv (g * f)(x) \in \mathbb{A}, \\ \mathbb{R}^d \ni x &\mapsto \int f(x, dy)h(y) \equiv (f * h)(x) \in \mathbb{C}^s, \end{aligned}$$

where the integrals are understood componentwise.

For the types  $g$  and  $h$  described, the maps  $f^* : g \mapsto f * g$ ,  $*f : g \mapsto g * f$ , and  $f^* : h \mapsto f * h$  as well as the continuous extensions of these maps to wider function spaces, are called operators with integral kernel  $f$ .

The operator  $F^*$  (of convolution) with integral kernel  $F$  is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA and will be denoted by  $O_F$ .

As an example we note that the delta-TA taken as an integral kernel generates the identity operators in  $L_p(\mathbb{R}^d, \mathbb{A})$  and  $L_p(\mathbb{R}^d, \mathbb{C}^s)$ , even though the explicit formula  $\int \delta_x(dy)g(y)$  is meaningless for discontinuous elements  $g$  of these spaces.

*Remark 1.2.3.* Let  $b : \mathbb{R}^d \rightarrow \mathbb{A}_0$  be a bounded Borel function. For  $g_0 \in C(\mathbb{R}^d, \mathbb{A})$  or  $h_0 \in C(\mathbb{R}^d, \mathbb{C}^s)$  we have the equalities

$$\begin{aligned}(b \cdot g_0)(x) &= b(x)g_0(x) = ((b \cdot \delta) * g_0)(x), \\ (g_0 \cdot b)(x) &= g_0(x)b(x) = (g_0 * (b \cdot \delta))(x), \\ (b \cdot h_0)(x) &= b(x)h_0(x) = ((b \cdot \delta) * h_0)(x)\end{aligned}$$

representing the operator of multiplication by  $b$  with integral kernel  $b \cdot \delta$ , which is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA.

Moreover, for every  $(\mathcal{B}_d, \mathbb{A}_0)$ -measure and the same  $g_0$  and  $h_0$ , the equalities  $\mu * g_0 = f_\mu * g_0$ ,  $\mu * h_0 = f_\mu * h_0$ , and so on, show that the operator of convolution with an  $\mathbb{A}_0$ -distribution is also realized via an integral kernel that is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA.

In particular, this is the case for  $e^{t\mathbf{A}}$  and  $e^{t\mathbf{B}}$ .

We state some lemmas here that follow immediately from the definitions and facts already established:

**Lemma 1.2.2.** *If  $p$  and  $q$  are  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA, then  $O_{p*q} = O_p \cdot O_q$ .*

**Lemma 1.2.3.** *If  $t, s, s_1, \dots, s_n$  are non-negative numbers,  $\mu^s = e^{*s\nu}$  for some  $(\mathcal{B}_d, \mathbb{A})$ -measure  $\nu$ , and  $v_1, \dots, v_n$  are continuous functions defined on  $\mathbb{R}^d$  and taking selfadjoint values in  $\mathbb{A}$ , then  $\mu_x^t * \mu_y^s = \mu_{x+y}^{t+s}$ ,  $f_{\mu^s} * f_{\mu^t} = f_{\mu^{s+t}}$ ,  $|\mu^s| \leq e^{*s|\nu|}$  (on every set) and*

$$\left| (f_{\mu^{s_1}} * (e^{iv_1} \cdot \delta) * f_{\mu^{s_2}} * (e^{iv_2} \cdot \delta) * \dots * f_{\mu^{s_n}} * (e^{iv_n} \cdot \delta))_x \right| \leq (f_{e^{*(s_1+\dots+s_n)|\nu|}})_x.$$

The continuity condition can be weakened to measurability, but we shall not need this.

**1.3. Trotter's formula.** The fact that  $\mathbf{A}$  is anti-selfadjoint and  $\mathbf{B}$  is bounded guarantees [10] the existence of a semigroup of bounded operators  $e^{t(\mathbf{A}+\mathbf{B})}$  that are solutions of (9), as well as the fact that the Trotter–Lie formula holds: for every  $\varphi \in L_2$

$$(10) \quad e^{t(\mathbf{A}+\mathbf{B})}\varphi = \lim_{n \rightarrow \infty} (e^{(t/n)\mathbf{A}} e^{(t/n)\mathbf{B}})^n \varphi$$

(where the limit is taken in  $L_2$ ).

Next we give a representation of the solution  $g_t = e^{t(\mathbf{A}+\mathbf{B})}g_0$  using integrals with respect to some countably additive measures.

## 2. INTEGRAL REPRESENTATIONS

**2.1. General remarks on bounded operators in  $L_2$ .** Suppose that

$$\langle f, g \rangle_{L_2} = \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle_{\mathbb{C}^s} dx \quad (f, g \in L_2)$$

and let  $B_{\mathbb{R}^d, b}$  be the ring of bounded Borel subsets of  $\mathbb{R}^d$ ; let  $1_B : \mathbb{R}^d \rightarrow \{0, 1\} \subset \mathbb{R}$  be the indicator function of the subset  $B \subset \mathbb{R}^d$ . Let  $e_1, \dots, e_s$  be the canonical basis in  $\mathbb{C}^s$ .

In what follows the operator

$$J_a : f \mapsto \int a(dy)f(y)$$

(where the integral is that of a function  $f \in L_2$  with respect to the  $L_2(\mathbb{R}^d, \mathbb{A})$ -valued measure  $a : B_{\mathbb{R}^d, b} \rightarrow L_2(\mathbb{R}^d, \mathbb{A})$ ) is understood like this. If  $f = \mathbf{v} \cdot 1_B$  is a vector-function

with values in  $\mathbb{C}^s$ , a multiple with coefficient  $\mathbf{v} \in \mathbb{C}^s$  of the indicator function of a bounded Borel set  $B \subset \mathbb{R}^d$ , we set

$$\int a(dy)(\mathbf{v}1_B)(y) = a(B) \cdot \mathbf{v} = (A(\mathbf{v}1_B)).$$

The integral is extended by linearity to the everywhere dense subspace  $L_{2, \text{step}}$  of  $L_2$ , generated by functions of the form  $\mathbf{v} \cdot 1_B$ , and if the map thus obtained is continuous on  $L_{2, \text{step}}$  in the  $L_2$ -norm, then it extends by continuity to the whole of  $L_2$  and the resulting extension is also called an integral operator.

**Lemma 2.1.1.** *Suppose a continuous linear operator  $Q : L_2 \rightarrow L_2$  is given; then the  $L_2(\mathbb{R}^d, \text{Matr}(s \times s, \mathbb{C}))$ -valued measure with components*

$$a_{j,k} : B_{\mathbb{R}^d, b} \ni B \mapsto \langle Q(1_B e_k), e_j \rangle \in L_2(\mathbb{R}^d, \mathbb{C}^1)$$

*is countably additive, and  $Qf = \int a(dy)f(y)$  for every  $f \in L_2$ .*

*Proof.* This follows from the computation

$$\begin{aligned} \left\langle \int a(dy)(1_B e_k)(y), e_j \right\rangle_{\mathbb{C}^s} &= \sum_{\ell=1}^s \int a_{j,\ell}(dy) \langle 1_B(y) e_k, e_\ell \rangle_{\mathbb{C}^s} \\ &= \int a_{j,k}(dy) 1_B(y) = a_{j,k}(B) = \langle Q(1_B e_k), e_j \rangle_{\mathbb{C}^s}, \end{aligned}$$

the linearity of the integral, and the continuity of the operator. □

*Remark 2.1.1.* In this lemma, for a fixed  $Y \in B_{\mathbb{R}^d, b}$ , the value  $a(Y) \in L_2(\mathbb{R}^d, \mathbb{A})$  of the measure gives a function that is defined only almost everywhere; for continuum many sets  $Y$  there may not exist a common point in the domains of representatives of such elements  $a(Y)$  (which are classes of almost everywhere equal  $\mathbb{A}$ -valued functions on  $\mathbb{R}^d$ ), while for any choice of everywhere defined representatives  $a(Y, \cdot) \in a(Y)$  the map  $Y \mapsto a(Y, x)$  may not be additive for any  $x \in \mathbb{R}^d$ . In what follows we shall need to choose modifications  $a(Y, \cdot) : \mathbb{R}^d \rightarrow \mathbb{A}$  of these functions such that, for (almost) every  $x \in \mathbb{R}^d$ , the map  $K \ni Y \rightarrow a(Y, x)$  is actually additive on a subring  $K \subset B_{\mathbb{R}^d, b}$  sufficient for our purposes. The next Lemma 2.1.2 is used in this context.

Further,  $L_{2, \mathbb{A}} = L_2(\mathbb{R}^d, \mathbb{A})$  (on  $\mathbb{A}$  we can take, for example, the Hilbert–Schmidt Hermitian product  $\langle u, v \rangle_{\mathbb{A}} = \text{Tr}(u \cdot v^*)$ ), and  $P$  is a countable subsemiring of  $B_{\mathbb{R}^d, b}$  consisting of certain bounded rectangular parallelepipeds with edges parallel to the coordinate axes and generating the whole Borel sigma-algebra; here it suffices that the set of vertices of these parallelepipeds be dense in  $\mathbb{R}^d$ ; for example, the vertices may run over all points with rational coordinates.

**Lemma 2.1.2.** *Under the assumptions of the preceding lemma, there exist representative functions  $a(Y, \cdot) \in a(Y) \in L_{2, \mathbb{A}}$  ( $Y \in K$ ) such that the measure  $a(\cdot, x)$  is additive on the semiring  $P$  for every  $x \in \mathbb{R}^d$ .*

*Proof.* Let  $e_n \in L_2$  ( $n \in \mathbb{N}$ ) be continuous or smooth functions forming an orthonormal basis of the (separable) space  $L_{2, \mathbb{A}}$ . Then the series

$$(11) \quad Af = \sum_{n \in \mathbb{N}} \langle Af, e_n \rangle \cdot e_n$$

converges in  $L_2$  for every  $f \in L_2$ ; in particular for any indexed sequence of indicators  $j \mapsto P_j$  of parallelepipeds in  $P$  we can use the diagonal process to choose a subsequence of partial sums that converges (to a numerical function) almost everywhere, that is, at points of  $\mathbb{R} \setminus N$  for some Borel set  $N \subset \mathbb{R}$  of measure zero. Similarly, we may assume that

$N$  contains all points outside which the limit measure is additive on  $P$ . Choosing modified basis functions  $e_j$  in (11) that vanish on  $N$ , and viewing  $f$  as  $1_{P_j}$ , we get an everywhere convergent sequence of functions with values in the (finitely additive) measures on  $P$ ; the limit function is therefore also finitely additive for fixed  $x$ . Of course, additivity is preserved under the canonical extension to the ring  $\bar{P}$  whose elements are finite unions of sets from  $P$ . □

**Lemma 2.1.3.** *In the notation of the preceding lemmas, let  $a_0$  be an  $(\mathbb{R}^d, \mathbb{R})$ -TA, and let  $a_n$  ( $n = 1, 2, \dots$ ) be a sequence in  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA such that*

- 1)  $|(a_n)_\bullet| \leq (a_0)_\bullet$ ;
- 2) the operators  $O_{a_n}$  are bounded and everywhere defined;
- 3) the sequence of operators  $O_{a_n}$  converges in the strong operator topology to some operator  $Q$ .

Then  $Q = O_a$ , where the kernel  $a$  is an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA such that  $|a_\bullet| \leq (a_0)_\bullet$ .

*Proof.* Firstly, by the Uniform Boundedness Theorem,  $Q$  is bounded on  $L_2$ . Further, arguments similar to those used in the proof of the preceding lemma show that there exists a map  $q : \mathbb{R}^d \times P \rightarrow \mathbb{A}$ , with the following properties:

- 1) for every  $x \in \mathbb{R}^d$  the measure  $q(x, \cdot)$  is additive on  $P$  and on  $\bar{P}$ ;
- 2) for every  $Y \in P$ , the function  $q(\cdot, Y)$  is square-summable on  $\mathbb{R}^d$ ; we shall denote the equivalence class of this function by  $q_o(Y)$  ( $\in L_2$ );
- 3) the map  $q : P \ni Y \mapsto q(\cdot, Y) \in L_2$  is countably additive and extends to an  $L_2$ -valued measure  $q_o : B_{\mathbb{R}^d, b} \rightarrow L_2$  such that  $q_o(Y) = Q(1_Y)$  for all  $Y \in B_{\mathbb{R}^d, b}$ ;
- 4) there exists a sequence  $a_{n_j}$  ( $j = 1, 2, \dots$ ) and a set  $N \subset \mathbb{R}^d$  of Lebesgue measure zero such that for every  $Y \in P$  and for all  $x \in \mathbb{R}^d \setminus N$  we have  $\lim a_{n_j}(x, Y) = q(x, Y)$ ; hence
- 5)  $|q(x, Y)| \leq a_0(x, Y)$  for all  $x \in \mathbb{R}^d$  and  $Y \in \bar{P}$ , which means that
- 6)  $q$  extends to an  $\mathbb{A}_0$ -TA  $a$  such that  $|q_\bullet| \leq a_\bullet$  and  $Q = O_a$ , thus proving the required result.

This completes the proof of the lemma. □

**2.2. Bounds for the product of  $e^{tA}$  and  $e^{tB}$ .** We recall that

$$(e^{tA}\varphi)(p) = e^{iP(p)}\varphi(p) \text{ for } \varphi \in L_2$$

and

$$(e^{tB}\varphi)(p) = (\mu^t * \varphi)(p) = \int \mu^t(dz)\varphi(p - z) \text{ for } \varphi \in L_2^c.$$

It follows from the estimates proved earlier that, for every  $t \geq 0$ , for a sequence  $Q_n^t = (e^{\frac{t}{n}B}e^{\frac{t}{n}A})^n$  ( $n = 1, 2, \dots$ ) of operators, there exists a sequence of defining them kernels  $q_n^t$  that are  $\mathbb{A}_0$ -TA and satisfy  $|(q_n^t)_x| \leq e_x^{*t|\nu|}$  ( $x \in \mathbb{R}^d$ ). This fact, the lemma just proved and Trotter's formula lead immediately to the following theorem.

**Theorem 2.2.1.** *The operator  $Q^t = e^{t(A+B)}$  is also given by some  $\mathbb{A}_0$ -TA  $G^t$  (to be called Green's amplitude of equation (9) in what follows), that is,  $Q^t = O_{G^t}$ , where*

$$(12) \quad |(G^t)_x| \leq e_x^{*t|\nu|} \quad (x \in \mathbb{R}^d).$$

**2.3. Functional distributions.**

**Lemma 2.3.1.**  $G^s * G^t = G^{s+t}$  for all  $s, t \in \mathbb{R}$ .

*Proof.* This is obvious. □

**Lemma 2.3.2.** *Every one-parameter semigroup  $\{P^t\}_{t \geq 0}$  consisting of  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA (in particular, the semigroup  $\{G^t\}_{t \geq 0}$ ) generates a family  $M_p$  ( $p \in \mathbb{R}^d$ ) of finitely additive cylindrical measures (with respect to the functionals of the calculation) on  $(\mathbb{R}^d)^{[0, \infty)}$  such that for every natural number  $n$ , Borel sets  $B_1, \dots, B_n \subset \mathbb{R}^d$ , and  $0 \leq t_1 < \dots < t_n < \infty$ ,*

$$(13) \quad M_p \{f : [0, t] \rightarrow \mathbb{R}^d; \forall j = 1, \dots, n, f(t_j) \in B_j\} \\ = \int_{B_1} P^{t_1}(p, dx_1) \int_{B_2} P^{t_2-t_1}(x_1, dx_2) \dots \int_{B_n} P^{t_n-t_{n-1}}(x_{n-1}, dx_n).$$

*Proof.* The assertion follows from the semigroup property of the family  $\{P^t\}_{t \geq 0}$ . □

We denote by  $M_{p,T}$  the cylindrical measure (with respect to the operators of the calculation) on  $(\mathbb{R}^d)^{[0,T]}$  that is the cylindrical image of the restriction  $M_p \upharpoonright_{C_T}$  of the measure  $M_p$  to the algebra of Borel cylinders generated by the calculation operators up to time  $T$  inclusive, under the isomorphism of this algebra with the algebra of subsets of  $(\mathbb{R}^d)^{[0,T]}$  generated by all calculation operators.

**Lemma 2.3.3.** *The cylindrical measure  $M_{p,T}$  and (therefore) all its cylindrical images have finite variations:  $\|M_{p,T}\| \leq e^{T\|v\|}$ .*

*Proof.* The lemma follows from (12). □

**Lemma 2.3.4.** *The cylindrical measure  $M_{p,T}$  and therefore all its cylindrical images are countably additive measures.*

*Proof.* This follows from the Kolmogorov-type theorem in [3]. □

**Lemma 2.3.5.** *The countably additive numerical measures  $N_{p,T}$ , obtained similarly to the  $M_{p,T}$  on replacing  $P_x^t$  by  $|\mu_x|^t$ , satisfy the inequality  $|M_{p,T}| \leq N_{p,T}$ .*

*Proof.* Countable additivity of the non-negative measures  $N_{p,t}$  follows from the preceding lemmas, and the required inequality on every cylindrical set again follows from (12). □

**Lemma 2.3.6.** *For  $p \in \mathbb{R}^d$  and  $T > 0$  there exists a countably additive non-negative cylindrical (with respect to the same calculation functionals) measure  $n_{p,T}$  on the space  $C_T$  of  $\mathbb{R}^d$ -valued right continuous functions on  $[0, T]$ , with only finitely many points of discontinuity with left limits, whose cylindrical image with respect to embedding in  $(\mathbb{R}^d)^{[0,T]}$  is the measure  $M_{p,T}$ .*

*Proof.* The lemma is a special case of Theorem 3.2 in [14]. □

**Lemma 2.3.7.** *For  $p \in \mathbb{R}^d$  and  $T > 0$ , on  $C_T$  there exists an  $\mathbb{A}_0$ -valued countably additive cylindrical (with respect to (the restrictions of) the same calculation functionals) measure  $m_{p,T}$  whose cylindrical image with respect to embeddings in  $(\mathbb{R}^d)^{[0,T]}$  is the measure  $M_{p,T}$ .*

*Proof.* The existence of a finitely additive measure  $m_{p,T}$  on  $C_T$  whose image is  $M_{p,T}$  (and the estimate  $|m_{p,T}| \leq n_{p,T}$ ) is proved in the same way as for  $M_{p,T}$  and  $N_{p,T}$ . The countable additivity of  $m_{p,T}$  follows from this estimate and the fact that  $n_{p,T}$  is countably additive. □

Similarly, there exists a cylindrical measure  $m_p$  on the space

$$C_\infty = \{f : [0, \infty) \rightarrow \mathbb{A}_0; \forall T > 0, f \upharpoonright_{[0,T]} \in C_T\}$$

such that  $m_{p,T}$  is its image under restriction of trajectories to  $[0; T]$ .

**Theorem 2.3.1.** For  $0 < t < T$ , the following equality in  $L_2$  holds for the Cauchy problem given by equation (9) and continuous finitary initial condition  $g_0$ :

$$\begin{aligned} g_t(p) &= \int_{(\mathbb{R}^d)^{[0,T]}} M_{p,T}(df) g_0(f(t)) \\ &= \int_{C_T} m_{p,T}(df) g_0(f(t)) = \int M_p(df) g_0(f(t)) = \int m_p(df) g_0(f(t)). \end{aligned}$$

*Proof.* Taking into account properties of the measures with respect to which the initial condition is integrated, the equalities to be proved are equivalent to  $g_t = G^t * g_0$ , which follows from Theorem 2.2.1.  $\square$

Let  $h$  be an  $(\mathbb{R}^d, \mathbb{A}_0)$ -TA generating the operator  $H = h*$  on  $L_2$ , and let  $F$  be the unitary operator of Fourier transform in  $L_2$ , and  $F^{-1}$  its inverse. The pseudodifferential operator (PDO)  $\hat{h}$  in  $L_2$  with symbol  $h$  (depending only on the impulse variables) is a product  $F^{-1}HF$  of operators (in a sense analogous to the standard one; see [5]). We shall refer to such a PDO as a  $p$ -PDO.

**Example 4.** If  $h = p \cdot \delta$ , where  $p$  is a polynomial, so that  $H$  is the operator of multiplication by  $p$ , then  $\hat{h}$  is a differential operator.

Our final theorem follows immediately from the theorem just proved.

**Theorem 2.3.2.** Suppose that the TA  $h$ , which is the symbol of a  $p$ -PDO  $\hat{h}$ , is of the form  $h = ip \cdot \delta + f_\nu$ , where  $p$  is a selfadjoint polynomial and  $\nu$  is an  $\mathbb{A}_0$ -distribution. Then  $\hat{h}$  is a generator of a semigroup of bounded operators each of which is a  $p$ -PDO; that is,  $e^{t\hat{h}} = \widehat{G^t}$ .

### 3. CONCLUDING REMARKS.

1. A generalized (measurable) version of matrix transition measures is presented in [14]; there, the Fourier transform of measures analogous to  $M_{0,T}$  is calculated in the complex case.
2. To prove countable additivity (bounded variation) of the measures  $M_{p,T}$ , it would be possible to use the well-known (Feynman–)Dyson series instead of Trotter’s formula (in particular, this happens in similar situations in the cited papers by Maslov and Ktitarev). However, this would hide the Markov character of the functional distribution thus obtained, which is expressed by the independence of the increments of the “non-commutative stochastic processes” corresponding to these distributions (see [15]).
3. The method used to construct the measure  $M_p$  allows the phase space  $\mathbb{R}^d$  to be replaced not only by an infinite-dimensional space, but even by a commutative Lie group. For commuting values of the coefficients of equations of Schrödinger type, the method recently proposed by O. G. Smolyanov using Hamiltonian Feynman surface integrals gives another approach to representing solutions.

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