

ON COMPLEX WEAKLY COMMUTATIVE HOMOGENEOUS SPACES

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ABSTRACT. Let G be a complex algebraic group and L an algebraic subgroup of G . The quotient space G/L is called weakly commutative if a generic orbit of the action $G : T^*(G/L)$ is a coisotropic submanifold. We classify weakly commutative homogeneous spaces $N \rtimes L/L$ in the case where L is a reductive group and the natural representation $L : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, where \mathfrak{n} is the tangent algebra of the group N , is irreducible.

1. INTRODUCTION

Let G be a complex algebraic group and L an algebraic subgroup of G . The quotient space G/L is called *weakly commutative* if the action of the group G on the symplectic manifold $T^*(G/L)$ is coisotropic, i.e., the orthogonal complement (with respect to the symplectic form) to the tangent space of the generic orbit at some point lies in this tangent space. It is interesting to classify such homogeneous spaces.

In this paper we consider the case where L is a maximal reductive subgroup in G , so that $G = N \rtimes L$, where N is the unipotent radical of the group G .

In papers [1, 2] a similar problem was considered in the case where G is a real Lie group and L is the maximal compact subgroup. In this case the group N is at most two-step nilpotent [1]. Classification of weakly commutative homogeneous spaces of this type for which the natural representation of L in $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ (here \mathfrak{n} is the tangent algebra of the group N) is irreducible, is presented in [2]. The general case can be reduced to this one (although the reduction is quite nontrivial).

In this paper again we only present the classification in the case where the representation $L : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is irreducible (in this case the homogeneous space G/L will be called *irreducible*). Contrary to the case of the compact stabilizer, here the general case cannot be reduced to the irreducible case.

A homogeneous space $X = G/L$ is called *commutative* if the algebra $D(X)^G$ of G -invariant differential operators is commutative. We show that irreducible weakly commutative homogeneous spaces of the form $(N \rtimes L)/L$ are commutative. Let us note that if the group L is not semisimple, commutativity does not imply weak commutativity (see 2.3).

Now we formulate the main classification result. In the case where the group N is commutative, each homogeneous space $(N \rtimes L)/L$ is commutative and weakly commutative. Later on, we will not consider this case.

In the case where N is a two-step nilpotent group, the corresponding homogeneous space G/L is called a space of Heisenberg type. In this case by \mathfrak{z} we denote the L -module $[\mathfrak{n}, \mathfrak{n}]$. Let \mathfrak{v} be an L -submodule in \mathfrak{n} complementing \mathfrak{z} . According to our convention, the L -module \mathfrak{v} is irreducible. The commutator in the algebra \mathfrak{n} is given by a mapping from $\bigwedge^2 \mathfrak{v}$ to \mathfrak{z} .

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In the case where $\dim \mathfrak{z} = 1$, the corresponding homogeneous space G/L is weakly commutative if and only if the representation of the commutator subgroup L' of the group L in \mathfrak{v} is coisotropic (Corollary 2 of Proposition 3). The classification of all irreducible coisotropic representations of reductive groups is obtained in [5].¹ In our case the homogeneous space G/L is commutative (Proposition 6). Therefore, it remains to consider the case where $\dim \mathfrak{z} > 1$.

Now we formulate the main result.

Theorem 1. (1) *The irreducible homogeneous spaces G/L of Heisenberg type with irreducible (resp. reducible) representation $L : \mathfrak{z}$, $\dim \mathfrak{z} > 1$, and the semisimple group L are those listed in Table 1 (resp. Table 2).*

(2) *There exist exactly two weakly commutative homogeneous spaces with the semisimple group L such that the algebra \mathfrak{n} is of nilpotency degree greater than 2. The algebra \mathfrak{n} is 3-step or 4-step nilpotent. The space G/L for which \mathfrak{n} is 4-step nilpotent is obtained as follows:*

$$L = \mathrm{SL}(2), \quad \mathfrak{n} = \bigoplus_{i=1}^4 \mathfrak{n}_i,$$

where $\mathfrak{n}_1, \mathfrak{n}_3$ are two-dimensional $\mathrm{SL}(2)$ -modules and $\mathfrak{n}_2, \mathfrak{n}_4 = \mathfrak{z}(\mathfrak{n})$ are one-dimensional. Moreover, for $x, y \in \mathfrak{n}_1, z \in \mathfrak{n}_2, u \in \mathfrak{n}_3, v \in \mathfrak{n}_4$ the following commutation relations hold:

$$[x, y] = \det(x, y)z, \quad [x, z] = \psi(x),$$

where ψ is an $\mathrm{SL}(2)$ -equivariant isomorphism from \mathfrak{n}_1 to \mathfrak{n}_3 ,

$$[x, u] = \det(x, u)v.$$

The homogeneous space for which \mathfrak{n} is 3-step is $(G/N_4)/L$, where $N_4 \subset N$ is the subgroup corresponding to \mathfrak{n}_4 .

(3) *If the group L is not semisimple and $N \rtimes L/L$ is weakly commutative, the space $N \rtimes L'/L'$ is weakly commutative. The converse statement also holds. (Here L' is the commutator subgroup of the group L .)*

(4) *Each irreducible weakly commutative homogeneous space of Heisenberg type is commutative.*

Here the π_i denote fundamental weights of the first group, and the π'_j , those of the second group. The numbering of fundamental weights is as in the reference section in [3].

We note that in all cases listed in the tables, the Lie algebra structure on \mathfrak{g} is determined by \mathfrak{v} and \mathfrak{z} uniquely up to isomorphism (since in each case the projection from $\bigwedge^2 \mathfrak{v}$ to \mathfrak{z} is essentially unique).

Let us briefly describe the structure and the content of the paper. It is divided into sections, which are divided into subsections, and, sometimes, into subsubsection.

In Section 2 we prove general facts about commutative and weakly commutative homogeneous spaces. The main classification theorem is proved in Section 3.

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Remark. In [5], there are three errors in the classification of reducible coisotropic representations. First, two cases are missing. In the notation of Table 10 in [5], these are linear groups G with one-dimensional center for which $G' = \mathrm{Sp}(2m) \otimes \mathrm{SO}(6) +_{\mathrm{SL}(4)} \mathrm{SL}(4)^{\mathrm{sympl}}$ or $G' = \mathrm{SL}(2) \otimes \mathrm{SO}(6) +_{\mathrm{SL}(4)} (\mathrm{SL}(4) \times \mathrm{SL}(2))^{\mathrm{sympl}}$. In the first case for $m = 1$ the group G' is also coisotropic, and the factor $\mathrm{SL}(2)$ should be distinguished. In addition, in line

¹In [5] several reducible coisotropic representations are missing; see the remark at the end of this section.

TABLE 1. Weakly commutative spaces G/L of Heisenberg type with irreducible representation $L : \mathfrak{z}, \dim \mathfrak{z} > 1$.

N	L	\mathfrak{v}	\mathfrak{z}
1	$SO(n), n > 2, n \neq 4$	$V(\pi_1)$	$V(\pi_2)$
2	G_2	$V(\pi_1)$	$V(\pi_1)$
3	$Spin(7)$	$V(\pi_3)$	$V(\pi_1)$
4	$SL(7)$	$V(\pi_3)$	$V(\pi_6)$
5	$SL(n) \times SO(m), m \neq 2$	$V(\pi_1) \otimes V(\pi'_1)$	$V(\pi_2)$
6	$Spin(13)$	$V(\pi_6)$	$V(\pi_1)$
7	$Spin(14)$	$V(\pi_6)$	$V(\pi_1)$
8	$Sp(2n), n > 1$	$V(\pi_1)$	$V(\pi_2)$
9	$SL(n) \times Spin(7), n$ odd, or $n = 3, 5$	$V(\pi_1) \otimes V(\pi'_3)$	$V(\pi_2)$
10	$SL(n) \times G_2, n = 3, 4$	$V(\pi_1) \otimes V(\pi'_1)$	$V(\pi_1)$
11	$SL(3) \times Spin(9)$	$V(\pi_1) \otimes V(\pi'_4)$	$V(\pi_2)$
12	$Sp(4) \times SO(n), n > 2$	$V(\pi_1) \otimes V(\pi'_1)$	$V(\pi_2)$
13	$Sp(4) \times Spin(7)$	$V(\pi_1) \otimes V(\pi_3)$	$V(\pi_2)$
14	$Sp(2m) \times Sp(2n), m \leq 2$	$V(\pi_1) \otimes V(\pi'_1)$	$V(2\pi_1)$
15	$SL(5) \times SL(2)$	$V(\pi_2) \otimes V(\pi_1)$	$V(\pi_4)$
16	$SL(n) \times Sp(2m), n > 2$	$V(\pi_1) \otimes V(\pi'_1)$	$V(2\pi_1)$
17	$Spin(9) \times SL(2)$	$V(\pi_4) \otimes V(\pi'_1)$	$V(\pi_1)$
18	$Spin(10) \times SL(2)$	$V(\pi_4) \otimes V(\pi'_1)$	$V(\pi_1)$

TABLE 2. Weakly commutative spaces G/L of Heisenberg type with reducible representation $L : \mathfrak{z}$.

no.	L	\mathfrak{v}	\mathfrak{z}
1	$Spin(13)$	$V(\pi_6)$	$V(\pi_1) \oplus V(0)$
2	$Sp(2n), n > 1$	$V(\pi_1)$	$V(\pi_2) \oplus V(0)$
3	$Sp(4) \times SO(n), n > 2$	$V(\pi_1) \otimes V(\pi'_1)$	$V(\pi_2) \oplus V(0)$
4	$Sp(4) \times Spin(7)$	$V(\pi_1) \otimes V(\pi'_3)$	$V(\pi_2) \oplus V(0)$
5	$Spin(9) \times SL(2)$	$V(\pi_4) \otimes V(\pi_1)$	$V(\pi_1) \oplus V(0)$
6	$SL(2) \times SL(2)$	$V(\pi_1) \otimes V(\pi'_1)$	$V(2\pi_1) \oplus V(2\pi'_1)$

23 of Table 10 ($SL(2) \otimes SO(10) +_{Spin(10)} Spin(10)^{sympl}$) the generic stabilizer algebra is incorrectly computed; the correct stabilizer is A_1 . Therefore, in this case the normal subgroup $SL(2)$ does not have to be distinguished. As the present work was being prepared for publication, the preprint [10] appeared, where the correct lists of coisotropic representations are given.

LIST OF NOTATION

G_x , the stabilizer of a point x of a set X under the action $G : X$.

\mathfrak{g}_x , the tangent algebra of the group G_x .

$\Sigma(\lambda_1, \dots, \lambda_k)$, the linear group defined as the image of a simply connected semisimple complex algebraic group with Dynkin diagram Σ under the representation with highest weights $\lambda_1, \dots, \lambda_k$.

$\alpha_1, \dots, \alpha_r$, the simple roots of a simple Lie algebra of rank r . The numbering coincides with that in the reference section in [3].

π_1, \dots, π_r , the fundamental weights of a simple Lie algebra of rank r . The numbering coincides with that in the reference section in [3].

V^μ , the weight μ subspace in the space of a representation of a reductive Lie algebra.

$V(\pi)$, the irreducible module over a semisimple Lie algebra with the highest weight π .

$G \otimes H, G \oplus H, S^k G, \bigwedge^k G$, linear groups, the tensor product and the direct sum of the linear groups G and H , and the symmetric and the exterior powers of the linear group G , respectively.

$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$, the normalizer of a subalgebra \mathfrak{h} in an algebra \mathfrak{g} .

$N_G(\mathfrak{h})$, the normalizer of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in a group G .

$Z_G(H)$, the centralizer of a subgroup H in a group G . For $H = G$ we write $Z(G)$ instead of $Z_G(G)$.

(G, G) the commutator group of a group G .

$\mathfrak{t}(l)$, the one-dimensional commutative diagonalizable Lie algebra.

$\text{Spin}_+(2m), \text{Spin}_-(2m), \text{Spin}(2m)$, the image of a simply connected group of type D_m in the half-spinor and the spinor representations.

G° the connected (= irreducible) component of an algebraic group G .

For a linear group $G \subset \text{GL}(U)$, by G^{symp} and G^{ort} we denote the image of G under the natural imbedding of $\text{GL}(U)$ in $\text{Sp}(U \oplus U^*)$ and $\text{SO}(U \oplus U^*)$ respectively.

$m_G(X)$, the maximal dimension of an orbit of the action $G : X$. If the space X is clear from the context, we write simply m_G .

$\text{def}_G(X)$, the defect of the action of an algebraic group G on a symplectic manifold X , i.e., the dimension of the kernel of the restriction of the form to the tangent space to a generic orbit.

$\text{cork}_G(X)$, the corank of the action of an algebraic group G on a symplectic manifold X , i.e., the rank of the restriction of the form to the orthogonal complement of the tangent space to a generic orbit.

2. PRELIMINARY COMMENTS

2.1. Similarity with the case of a compact stabilizer. In this subsection we outline similarities with the case of a real group G and the case of a compact subgroup L , which was considered in [1, Chapter 2, §4]. The proof of the next proposition is presented there. The subgroup $L \subset G$ is not necessarily a Levi subgroup.

Proposition 1. *The orbit of a point $\alpha \in T_{eL}^*(G/L)$ is coisotropic if and only if the following equivalent conditions are satisfied:*

- (1) $\dim \text{Ad}^*(G)\alpha = 2 \dim \text{Ad}^*(L)\alpha$.
- (2) $\text{ad}^*(\mathfrak{g})\alpha \cap \mathfrak{l}^0 = \text{ad}^*(\mathfrak{l})\alpha$.
- (3) $\alpha([\text{ad}(\mathfrak{l})\alpha]^0, [\text{ad}(\mathfrak{l})\alpha]^0) = 0$.

(Here the superscript 0 denotes the annihilator in the dual space.)

From this point on we will assume that L is the Levi subgroup in the group $G = N \rtimes L$. The proof of the next proposition is a slight modification of the proof of Proposition 10 in [1].

Proposition 2. *The homogeneous space G/L is weakly commutative if and only if for two rational functions $f, g \in \mathbb{C}(\mathfrak{n}^*)^L$ and an arbitrary point $\alpha \in \mathfrak{n}^*$ at which both f and*

g are defined, we have

$$(2.1) \quad \alpha([d_\alpha f, d_\alpha g]) = 0.$$

Corollary 1. *Let $N_1 \subset N$ be a normal subgroup in G . If the homogeneous space G/L is weakly commutative, then the homogeneous space $(G/N_1)/L$ is also weakly commutative.*

Proof. Let $\mathfrak{n}, \mathfrak{n}_1, \mathfrak{l}$ be the tangent algebras of the groups N, N_1, L , and let \mathfrak{m} be the \mathfrak{l} -invariant complement of \mathfrak{n}_1 in \mathfrak{n} . There is a natural imbedding $\mathbb{C}(\mathfrak{m}^*)^L \hookrightarrow \mathbb{C}(\mathfrak{n}^*)^L$ induced by the projection $\mathfrak{n}^* = \mathfrak{n}_1^* \oplus \mathfrak{m}^* \rightarrow \mathfrak{m}^*$. It is clear that for $\alpha \in \mathfrak{m}^*, f \in \mathbb{C}(\mathfrak{m}^*)^L$, the vector $d_\alpha f$ does not depend on whether we view f as an element of $\mathbb{C}(\mathfrak{m}^*)^L$ or of $\mathbb{C}(\mathfrak{n}^*)^L$. \square

Corollary 2. *Let M be an L -invariant subgroup of N . If the homogeneous space $N \rtimes L/L$ is weakly commutative, then the homogeneous space $M \rtimes L/L$ is also weakly commutative.*

Proof. Let \mathfrak{n}_1 be the \mathfrak{l} -invariant complement of \mathfrak{m} in \mathfrak{n} . Then we argue as in the proof of the previous corollary. \square

Corollary 3. *Let L be a reductive group and $L_1 \subset L$ a reductive subgroup. If the homogeneous space $(N \rtimes L_1)/L_1$ is weakly commutative, then the homogeneous space $(N \rtimes L)/L$ is also weakly commutative.*

Proof. The assertion is obvious. \square

It is clear that if the group N is commutative, then the homogeneous space $N \rtimes L/L$ is weakly commutative. Let N be two-step nilpotent, i.e., the commutator subalgebra $[\mathfrak{n}, \mathfrak{n}]$ is in the center of \mathfrak{n} . From this point till the end of Section 2 we will assume this condition to hold (unless explicitly indicated otherwise). Let \mathfrak{v} be an \mathfrak{l} -invariant subspace in \mathfrak{n} complementing $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$.

To an element $\alpha \in \mathfrak{z}^*$ we associate the skew-symmetric form $\widehat{\alpha}$ on \mathfrak{v} given by the formula

$$(2.2) \quad \widehat{\alpha}(x, y) = \alpha([x, y]).$$

Denote by \mathfrak{w}_α the kernel of the form $\widehat{\alpha}$. The space $\mathfrak{v}_\alpha = \mathfrak{v}/\mathfrak{w}_\alpha$ is endowed with the symplectic form induced by the form $\widehat{\alpha}$. This form on the space \mathfrak{v}_α will be also denoted by $\widehat{\alpha}$.

The next proposition is a version of Proposition 11 in [1, Chapter 2, Section 4].

Proposition 3. *Let α, β be generic points in \mathfrak{z}^* and \mathfrak{w}_α^* respectively (i.e., arbitrary points of appropriate open subsets). Let $L_\alpha = M \rtimes S$ be the Levi decomposition. Denote by $\widehat{\mathfrak{v}}_\alpha$ an S -invariant subspace in \mathfrak{v} complementing \mathfrak{w}_α . Finally, let $\widetilde{\beta}$ be a generic point in $\rho^{-1}(\beta)$, where ρ is the restriction from \mathfrak{v} to \mathfrak{w}_α .*

- (1) *The space G/L is weakly commutative if and only if the subspace $(\mathfrak{l}_\alpha \widetilde{\beta})^0 \subset \mathfrak{v}$ is coisotropic with respect to the form $\widehat{\alpha}$.*
- (2) *If the representation $S_\beta : \mathfrak{v}_\alpha$ is coisotropic, then the homogeneous space G/L is weakly commutative.*
- (3) *If the space G/L is weakly commutative, then*

$$\text{cork}_{S_\beta}(\mathfrak{v}_\alpha) \leq 2 \dim \mathfrak{l}_{\alpha\beta}/\mathfrak{s}_\beta \leq 2 \dim \mathfrak{l}_\alpha/\mathfrak{s}.$$

In particular, $\dim \mathfrak{v}_\alpha \leq 2 \dim \mathfrak{l}_{\alpha\beta}$.

- (4) *Denote by $\widehat{\beta}$ an element of $\widehat{\mathfrak{v}}_\alpha^0 \subset \mathfrak{v}^*$ that projects to β , and let $\widehat{\mathfrak{v}}_{\alpha\beta} = \widehat{\mathfrak{v}}_\alpha \cap (\mathfrak{l}_{\alpha\beta} \widehat{\beta})^0$. Suppose the group M acts trivially on \mathfrak{v}_α and \mathfrak{w}_α . Under these assumptions, the homogeneous space G/L is weakly commutative if and only if for a generic form $\widetilde{\gamma} \in \widehat{\mathfrak{v}}_{\alpha\beta}^*$ the subspace $(\mathfrak{s}_\beta \widetilde{\gamma})^0 \subset \widehat{\mathfrak{v}}_{\alpha\beta}$ is isotropic.*

Proof. Let $\gamma = \tilde{\beta} - \hat{\beta}$ be a generic element in \mathfrak{v}_α^* . We will also view γ as a form on $\widehat{\mathfrak{v}}_\alpha$.

The projection of the space $(\mathfrak{l}(\alpha + \tilde{\beta}))^0$ to \mathfrak{v} coincides with $(\mathfrak{l}_\alpha \tilde{\beta})^0$. Part 3 of Proposition 1 implies that the space G/L is weakly commutative if and only if the space $(\mathfrak{l}_\alpha \tilde{\beta})^0 \subset \mathfrak{v}$ is isotropic with respect to the form $\hat{\alpha}$. This proves part (1) of Proposition 3.

The subspace $(\mathfrak{l}_\alpha \tilde{\beta})^0 \subset \mathfrak{v}$ is isotropic if and only if its projection to \mathfrak{v}_α is isotropic. This projection is $(\mathfrak{l}_{\alpha\beta} \tilde{\beta})^0$ (functions in $\mathfrak{l}_{\alpha\beta} \tilde{\beta}$ vanish on \mathfrak{w}_α , i.e., can be viewed as functions on \mathfrak{v}_α).

The action $S_\beta : \mathfrak{v}_\alpha$ is coisotropic if and only if the subspace $(\mathfrak{s}_\beta \gamma)^0 = (\mathfrak{s}_\beta \tilde{\beta})^0 \subset \widehat{\mathfrak{v}}_\alpha$ is isotropic. This proves part (2) of Proposition 3.

We now prove part (3). Let the space G/L be weakly commutative. Denote by \mathfrak{a} a subspace in $\mathfrak{l}_{\alpha\beta}$ complementing \mathfrak{s}_β . Then $(\mathfrak{l}_{\alpha\beta} \tilde{\beta})^0 = (\mathfrak{a} \tilde{\beta})^0 \cap (\mathfrak{s}_\beta \gamma)^0$. But γ is a generic element in $\widehat{\mathfrak{v}}_\alpha^*$. Note that $(\mathfrak{s}_\beta \gamma)^0$ is nothing but the orthogonal complement of $S_\beta x$, where $x \in \widehat{\mathfrak{v}}_\alpha$ corresponds to $\gamma \in \widehat{\mathfrak{v}}_\alpha^*$ under the isomorphism $\widehat{\mathfrak{v}}_\alpha \cong \widehat{\mathfrak{v}}_\alpha^*$ induced by the form $\hat{\alpha}$. The symplectic form $\hat{\alpha}$ should vanish on the space $(\mathfrak{a} \tilde{\beta})^0 \cap (\mathfrak{s}_\beta \gamma)^0$. Therefore, the rank of the form $\hat{\alpha}$ on the space $(\mathfrak{s}_\beta \gamma)^0$ cannot exceed $2 \dim \mathfrak{a} \tilde{\beta} \leq 2 \dim \mathfrak{a}$. This gives the first inequality.

To prove the remaining inequality, note that

$$2 \dim \mathfrak{l}_{\alpha\beta} = 2 \dim \mathfrak{s}_\beta + 2 \dim \mathfrak{l}_{\alpha\beta} / \mathfrak{s}_\beta \geq 2 \dim \mathfrak{s}_\beta + \text{cork}_{S_\beta} \mathfrak{v}_\alpha \geq \dim \mathfrak{v}_\alpha.$$

Finally, we prove part (4) of Proposition 3. Let $\hat{\gamma}$ be the restriction of γ to the space $\widehat{\mathfrak{v}}_{\alpha\beta}$ and $\mathfrak{l}_{\alpha\beta} \tilde{\beta} = \mathfrak{s}_\beta \gamma + \mathfrak{m} \tilde{\beta} = \mathfrak{s}_\beta \hat{\gamma} + \mathfrak{m} \tilde{\beta}$. We note that $(\mathfrak{l}_{\alpha\beta} \tilde{\beta})^0$ lies in $\widehat{\mathfrak{v}}_{\alpha\beta}$ and coincides with $(\mathfrak{s}_\beta \hat{\gamma})^0$. □

Corollary 4. *If the homogeneous space $(N \rtimes L)/L$ is weakly commutative, then the stabilizer of a generic point for the action $L : \mathfrak{z}^*$ has positive dimension.*

Proof. Otherwise, $\mathfrak{l}_\alpha = 0$ and $(\mathfrak{l}_\alpha \tilde{\beta})^0 = \mathfrak{v}$, contradicting part (1) of Proposition 3. □

Corollary 5. *If $\dim \mathfrak{z} = 1$, then the homogeneous space $(N \rtimes L)/L$ is weakly commutative if and only if the action $L_\alpha : \mathfrak{v}$ is coisotropic (with respect to the symplectic form $\hat{\alpha}$ for a nonzero element $\alpha \in \mathfrak{z}^*$).*

Corollary 6. *For a reductive subgroup $S_1 \subset \text{Sp}(\mathfrak{v}_\alpha)$ containing the group S_β we have*

$$(2.3) \quad \dim S_1 + \text{rk } S_1 + 2 \dim \mathfrak{l}_\alpha / \mathfrak{s} \geq \dim \mathfrak{v}_\alpha.$$

Proof. It follows from part (1) and the relation

$$\text{cork}_{S_\beta}(\mathfrak{v}_\alpha) \geq \text{cork}_{S_1}(\mathfrak{v}_\alpha) = \dim \mathfrak{v}_\alpha - m_{S_1}(\mathfrak{v}_\alpha) - \text{def}_{S_1}(\mathfrak{v}_\alpha) \geq \dim \mathfrak{v}_\alpha - \dim S_1 - \text{rk } S_1.$$

□

Corollary 7. *Suppose the commutator subgroup L' of the group L does not have a dense orbit in \mathfrak{z}^* (this happens, e.g., when the stabilizer of a generic point for the action $L' : \mathfrak{z}^*$ is reductive; see [6]), and the representation $L : \mathfrak{v}$ is reducible. The space $N \rtimes L/L$ is weakly commutative if and only if the space $N \rtimes L'/L'$ is.*

Proof. Since \mathfrak{v} is an irreducible L -module, the center of the group L acts on \mathfrak{z} as multiplication by constants (because \mathfrak{z}^* is a quotient module of the L -module $\bigwedge^2 \mathfrak{v}$). Under the hypotheses of Proposition 3, for a generic vector $\alpha \in \mathfrak{z}^*$ we have $L_\alpha = L'_\alpha$. Indeed, otherwise the generic orbits for the action $L' : \mathfrak{z}^*$ would be conical, and hence $\mathbb{C}[\mathfrak{z}^*]^{L'} = \mathbb{C}$. The latter is equivalent to the existence of a dense orbit for the action $L' : \mathfrak{z}^*$.

Therefore, $\mathfrak{l}_\alpha = \mathfrak{l}'_\alpha$ and the assertion follows from part (1) of Proposition 3. □

Concluding this subsection, we introduce some terminology borrowed from [1, 2]. We say that a homogeneous space G/L is *irreducible* if the natural action of L on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is.

We say that G/L is a space of *Heisenberg type* if \mathfrak{n} is a two-step nilpotent Lie algebra.

In what follows, unless specified otherwise, for an arbitrary space of Heisenberg type, the symbols \mathfrak{v} and \mathfrak{z} mean the same as was defined before.

2.2. Estimates for dimension. Let V be a vector space and W a nonzero subspace of $\bigwedge^2 V^*$. We say that two subspaces $U_1, U_2 \subset V$ are *W-orthogonal* if they are orthogonal with respect to an arbitrary form in W . The largest subspace in V that is *W-orthogonal* to a given subspace $U \subset V$ will be denoted by $U^{\perp W}$. Finally, a subspace $U \subset V$ is called *W-isotropic* if $U \subset U^{\perp W}$.

Proposition 4. *Let $L \subset GL(U)$ be a linear group and W an L -invariant subspace in $\bigwedge^2 U^*$ such that the only W -isotropic L -invariant subspace in U is $\{0\}$ (for example, this happens if the L -module U is irreducible and $W \neq 0$). Then the dimension of an arbitrary W -isotropic subspace in U is at most $(\dim U)/2$.*

Proof. We prove even more. Namely, we establish that if $V_1 \subset V_2$ are *W-orthogonal* subspaces in U , then $\dim V_1 + \dim V_2 \leq \dim U$. Assume the contrary. Let $m > \dim U$ be the largest number such that there are *W-orthogonal* subspaces $V_1 \subset V_2 \subset V$ with $\dim V_1 + \dim V_2 = m$. Choose V_1 and V_2 so that $\dim V_1 + \dim V_2 = m$ and the dimension of V_1 be maximal possible.

Choose an element $g \in L$ and set $V'_1 = V_1 \cap gV_2$, $V'_2 = gV_1 + V_2$. It is clear that $V'_1 \subset V'_2$. The L -invariance of the space W implies that V'_1 and V'_2 are *W-orthogonal*. The dimensions of the spaces V'_1, V'_2 satisfy

$$(2.4) \quad \dim V'_2 + \dim V'_1 = \dim V_1 + \dim V_2 - \dim(V_1 \cap g^{-1}V_2) + \dim(V_1 \cap gV_2).$$

This implies that $\dim V'_1 + \dim V'_2 = m$. Indeed, by the choice of m , $\dim V'_1 + \dim V'_2 \leq m$. If the inequality were strict, replacing, in the definition of V'_1 and V'_2 , the element g with g^{-1} , we would obtain $\dim V'_1 + \dim V'_2 > m$, which is impossible.

Now the choice of V_1 implies that $V'_1 = V_1$, i.e., $V_1 \subset gV_2$ for all $g \in L$. Set $V = \bigcap_{g \in L} gV_2$. The space V is a nonzero (since $V_1 \neq \{0\}$) L -invariant subspace in U . We have $V_1 \subset V^{\perp W}$, and so $V \cap V^{\perp W}$ is a nonzero L -invariant *W-isotropic* subspace in V . We have come to a contradiction, which shows that $m \leq \dim V$. \square

Corollary 8. *In the notation and hypotheses of the previous proposition, there exists a form $\alpha \in W$ such that $\text{rk } \alpha \geq (\dim U)/2$. In particular, if $\alpha \in \mathfrak{z}^*$ is generic and the L -module \mathfrak{v} is irreducible, then $\dim \mathfrak{v} \leq 2 \dim \mathfrak{v}_\alpha$.*

Proof. We prove that for a form $\alpha \in W$ of the maximal rank the subspace $\ker \alpha \subset V$ is *W-isotropic*. Choose a basis in the space V such that the form α is represented by a block matrix $A = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ with a nonsingular X . Let α_1 be an arbitrary form in W , and let $A_1 = \begin{pmatrix} X_1 & Y_1 \\ Z_1 & T_1 \end{pmatrix}$ be its block representation in the chosen basis. We must prove that $T_1 = 0$. The form $t\alpha + \alpha_1$, $t \in \mathbb{C}$, has the matrix $tA + A_1 = \begin{pmatrix} tX + X_1 & Y_1 \\ Z_1 & T_1 \end{pmatrix}$. By the choice of α , we have the inequality $\text{rk}(tX + X_1) \leq \text{rk } X$, which becomes an equality for all but a finite number of values of $t \in \mathbb{C}$. Let $a \neq 0$ be an entry of the matrix T_1 . Choose a submatrix of $tX + X_1$ of order $\text{rk } \alpha + 1$ by taking the first $\text{rk } \alpha$ rows and the row containing a , and using the same rule for the choice of columns. The determinant of this submatrix is a polynomial in t with coefficient $a \det(X)$ at $t^{\text{rk } \alpha}$. Therefore, $a = 0$. \square

As in the previous subsection, let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$.

Proposition 5. *Suppose the L -module \mathfrak{v} is irreducible. Then*

$$(2.5) \quad \dim \mathfrak{v} \geq 2(\dim \mathfrak{v} - m_L(\mathfrak{v})).$$

Proof. Consider the space $U = \{d_\beta f\}$ for a fixed generic point $\beta \in \mathfrak{v}^*$ and all rational functions $f \in \mathbb{C}(\mathfrak{v}^*)^L$ defined at β . The dimension $\dim U$ equals $\text{tr. deg } \mathbb{C}(\mathfrak{v})^L$. Proposition 2 implies that the subspace $U \subset \mathfrak{v}$ is \mathfrak{z}^* -isotropic. It remains to note that $\text{tr. deg } \mathbb{C}(\mathfrak{v})^L$ equals the codimension of a generic orbit for the action $L : \mathfrak{v}$. \square

Corollary 9. $\dim \mathfrak{v} \leq 2m_L(\mathfrak{v}) \leq 2 \dim L$.

2.3. Commutative homogeneous spaces. Let G be an arbitrary algebraic group and L an algebraic subgroup of G .

The homogeneous space $X = G/L$ is called *commutative* if the algebra $D(X)^G$ of G -invariant differential operators is commutative. In the case where $G = N \rtimes L$ is the Levi decomposition, the commutativity of the homogeneous space X is equivalent to the commutativity of the algebra $U(\mathfrak{n})^L$; see, e.g., [1] (here $U(\mathfrak{n})$ denotes the universal enveloping algebra of the Lie algebra \mathfrak{n}).

The algebra $U(\mathfrak{n})$ is a filtered algebra, and the associated graded algebra is the algebra $S(\mathfrak{n}) \cong \mathbb{C}[\mathfrak{n}^*]$ endowed with the Poisson bracket $\{f, g\}(\alpha) = \alpha(d_\alpha f, d_\alpha g)$. This implies that if the algebra $U(\mathfrak{n})^L$ is commutative, the algebra $\text{gr } U(\mathfrak{n})^L = \mathbb{C}[\mathfrak{n}^*]^L$ is also commutative. Therefore, if generic orbits of the group L in \mathfrak{n} are separated by polynomial invariants (or, equivalently, a generic fiber of the factorization morphism $\pi_L : \mathfrak{n} \rightarrow \mathfrak{n}/L$ contains a dense orbit), the commutativity of the homogeneous space implies weak commutativity. The converse is not true. Indeed, let $\mathfrak{n} = \bigoplus_{i \in \mathbb{N}} \mathfrak{n}_i$ be a graded nilpotent Lie algebra. We have the action of $L = \mathbb{C}^*$ on \mathfrak{n} by the formula $t \cdot x = t^i x$, $x \in \mathfrak{n}_i$. It is clear that $\mathbb{C}[\mathfrak{n}^*]^L = U(\mathfrak{n})^L = \mathbb{C}$. Part 1 of Proposition 1 implies that the homogeneous space $(N \rtimes \mathbb{C}^*)/\mathbb{C}^*$ is not, in general, weakly commutative.

Now let \mathfrak{n} be a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, as before.

Proposition 6. *Suppose the representation $L : \mathfrak{v}$ is symplectic and $\dim \mathfrak{z} = 1$. The following statements are equivalent:*

- (1) *The representation $L : \mathfrak{v}$ is coisotropic.*
- (2) *The Poisson algebra $\mathbb{C}[\mathfrak{v}]^L$ is commutative.*
- (3) *The homogeneous space G/L is weakly commutative.*
- (4) *The homogeneous space G/L is commutative.*

Proof. First of all, we prove that for an arbitrary symplectic representation $L : V$, generic orbits are separated by polynomial invariants. If the L -module V does not contain a direct summand of the form $U \oplus U^*$, then the image of L in $\text{GL}(V)$ is a semisimple group, and in this case the result is known (see, e.g., [4]). Now let $V = V_1 \oplus V_2$ and $V_1 = U \oplus U^*$ as L -modules. The action $L : V_1$ is stable, i.e., a generic orbit is closed (see, e.g., [5]). Denote by H a generic stabilizer for the action $L : V_1$. By the étale slice theorem (see, e.g., [4]), the existence of a dense orbit in a generic fiber of the factorization morphism for the action $H : V_2$ implies the existence of a dense orbit in a generic fiber of the factorization morphism for the action $L : V$. It remains to use induction.

This implies that (1) and (2) are equivalent. Equivalence of (1) and (3) follows from Proposition 3. The implication (4) \Rightarrow (3) was discussed earlier.

Similarly to [1] one can prove that the commutativity of the space G/L is equivalent to the commutativity of the algebra $W(\mathfrak{v})^L$, where $W(\mathfrak{v})$ is the Weyl algebra. Now the implication (1) \Rightarrow (4) follows from a result of Rybnikov ([7]; see also [10]). \square

Proposition 7. *Using the notation of Proposition 3 assume that the group S_β is reductive. Under the assumptions of part (3) of Proposition 3, the homogeneous space G/L is commutative. The same holds under the assumptions of part (4) of Proposition 3 provided that the restriction of the form $\hat{\alpha}$ to the space $\hat{\mathfrak{v}}_{\alpha\beta}$ is nondegenerate.*

Proof. We need to prove that the algebra $U(\mathfrak{n})^L$ is commutative. We use the same method as in the proof of Proposition 6 in [1]. The algebra $U(\mathfrak{n})$ can be viewed as the set of global sections of the bundle over \mathfrak{z}^* with the fiber $W_\alpha(\mathfrak{v}) = U(\mathfrak{n})/(x - \alpha(x), x \in \mathfrak{z})$ over $\alpha \in \mathfrak{z}^*$. Similarly, the algebra $W_\alpha(\mathfrak{v})$ can be viewed as the set of global sections of the bundle over \mathfrak{w}_α^* with the fiber $W_{\alpha\beta}(\mathfrak{v}_\alpha) = W_\alpha(\mathfrak{v})/(x - \beta(x), x \in \mathfrak{w}_\alpha)$. We have the action of the group L_α on the algebra $W_\alpha(\mathfrak{v})$ induced by the action of the group L on $U(\mathfrak{n})$, or, which is the same, by the action of L_α on \mathfrak{v} . Next we have the action of the group $L_{\alpha\beta}$ by automorphisms of the algebra $W_{\alpha\beta}(\mathfrak{v}_\alpha)$, which now does not coincide with the action induced by an action on $L_{\alpha\beta} : \mathfrak{v}_\alpha$. It is easy to see that if for generic α, β the algebra $W_{\alpha\beta}(\mathfrak{v}_\alpha)^{L_{\alpha\beta}}$ is commutative, then the algebra $W_\alpha(\mathfrak{v})^{L_\alpha}$, hence also the algebra $U(\mathfrak{n})^L$, is commutative.

If the hypotheses of part (2) of Proposition 3 hold, then the algebra $W_{\alpha\beta}(\mathfrak{v}_\alpha)^{S_\beta}$ is commutative (by Proposition 6), so that the subalgebra $W_{\alpha\beta}(\mathfrak{v}_\alpha)^{L_{\alpha\beta}}$ of this algebra is also commutative.

Now assume that the action of the group M on the spaces \mathfrak{v}_α and \mathfrak{w}_α is trivial and the restriction of the form $\widehat{\alpha}$ to the subspace $\widehat{\mathfrak{v}}_{\alpha\beta} \subset \mathfrak{v}_\alpha$ is nondegenerate. Denote by $\mathfrak{w}_{\alpha\beta}$ the orthogonal complement of $\widehat{\mathfrak{v}}_{\alpha\beta}$ in $\widehat{\mathfrak{v}}_\alpha$. We obtain an S_β -equivariant morphism of algebras $W_{\alpha\beta}(\mathfrak{v}_\alpha) \cong W(\widehat{\mathfrak{v}}_{\alpha\beta}) \otimes W(\mathfrak{w}_{\alpha\beta})$.

We have a homomorphism of the algebra $\mathfrak{l}_{\alpha\beta}$ to the algebra $\text{Der}(W_{\alpha\beta}(\mathfrak{v}_\alpha))$ of derivations of the algebra $W_{\alpha\beta}(\mathfrak{v}_\alpha)$. Denote by D_x the derivation corresponding to an element $x \in \mathfrak{m}$. By the definition of the action of the group $L_{\alpha\beta}$, $D_x(v) = \beta(xv)$ for $v \in \mathfrak{v}_\alpha$ (here the spaces \mathfrak{v}_α and $\widehat{\mathfrak{v}}_\alpha$ are identified). In particular, for $x \in \widehat{\mathfrak{v}}_{\alpha\beta}$ we have $D_x(v) = 0$, hence $D_x(W(\widehat{\mathfrak{v}}_{\alpha\beta})) = 0$. On the other hand, the subalgebra of \mathfrak{m} -invariants in the algebra $W(\mathfrak{w}_{\alpha\beta})$ coincides with the center of the latter algebra, which, as one can easily see, is trivial. Therefore, $W_{\alpha\beta}(\mathfrak{v}_\alpha)^{L_{\alpha\beta}} = W(\widehat{\mathfrak{v}}_{\alpha\beta})^{S_\beta}$ and we obtain the desired result. \square

3. CLASSIFICATION OF WEAKLY COMMUTATIVE SPACES

In this section we prove Theorem 1. In Subsection 3.1 we obtain restrictions on possible representations of L in \mathfrak{z} . In 3.3 and 3.4 we classify all weakly commutative spaces of Heisenberg type with irreducible and reducible L -module \mathfrak{z} respectively. Finally, in 3.4 we consider the spaces of non-Heisenberg type.

3.1. The form of the representation $L' : \mathfrak{z}$. Here we consider the case where the representation $L : \mathfrak{v}$ is irreducible. Let λ be the highest weight of the L -module \mathfrak{v} , and λ^* the highest weight of the L -module \mathfrak{v}^* . The L' -module \mathfrak{z} can be imbedded in $\mathfrak{v} \otimes \mathfrak{v}$. Formula (F11) in the reference section of [3] shows that the multiplicity of an irreducible submodule $\widetilde{\mathfrak{z}} \subset \mathfrak{z}$ in the L' -module $\mathfrak{v} \otimes \mathfrak{v}$ equals

$$(3.1) \quad \dim \{v \in \widetilde{\mathfrak{z}}^{\lambda - \lambda^*} \mid e_i^{\lambda_i^* + 1} v = 0\}.$$

Here λ_i^* is the value of the highest weight λ^* on the i th simple root α_i , and e_i is a nonzero element in \mathfrak{l}_{α_i} .

Formula (3.1) implies, in particular, that each irreducible submodule of the L' -module \mathfrak{z} contains the weight $\lambda - \lambda^*$.

Moreover, if μ is the highest weight of \mathfrak{z} , then $\mu - 2\lambda \in Q$, where Q is the root lattice of \mathfrak{l} . Therefore, possible restrictions of μ to simple ideals in \mathfrak{l} satisfy the condition that the numbers in the second line of Table 3 are even.

Denote by ρ and φ the representations of the group L' in \mathfrak{z} and \mathfrak{v} respectively. We assume that both representations φ and ρ are irreducible. Let \mathfrak{l}^2 be the tangent algebra of the kernel of the representation ρ , and \mathfrak{l}^1 the ideal complementing \mathfrak{l}^2 in \mathfrak{l}' . Next, let ρ^1 be the restriction of the representation ρ to \mathfrak{l}^1 , and let $\varphi = \varphi^1 \otimes \varphi^2$, where φ^i is an

TABLE 3. Condition on simple components of the representation $L' : \mathfrak{z}$.

A_{2l-1}	B_l	C_l	D_{2l-1}	D_{2l}	E_7
$\sum i\mu_i$	μ_l	$\sum \mu_{2i-1}$	$\mu_{2l-1} - \mu_{2l}$	$\mu_1 + \mu_3 + \cdots + \mu_{2l-1}, \mu_{2l-1} + \mu_{2l}$	$\mu_1 + \mu_3 + \mu_7$

irreducible representation of the algebra \mathfrak{l}^i , $i = 1, 2$. Two cases are possible:

(Alt) The representation φ^2 is orthogonal (possibly trivial) and ρ^1 is a subrepresentation in $\bigwedge^2 \varphi^1$. In this case we will say that the homogeneous space G/L is a *space of skew-symmetric Heisenberg type*.

(Sym) The representation φ^2 is symplectic and ρ^1 is a subrepresentation of $S^2\varphi^1$. In this case we will say that G/L is a *space of symmetric Heisenberg type*.

Throughout the rest of the paper, we denote by $\mathfrak{v}^1, \mathfrak{v}^2$ the space of the representation φ^1, φ^2 respectively, so that $\mathfrak{v} = \mathfrak{v}^1 \otimes \mathfrak{v}^2$. Let α be a generic element of \mathfrak{z}^* , and $\hat{\alpha}$ the form given by formula (2.2). We have $\hat{\alpha} = \hat{\alpha}^1 \otimes \hat{\alpha}^2$, where $\hat{\alpha}^i$ is a symmetric or skew-symmetric form on \mathfrak{v}^i , $i = 1, 2$. Since the form $\hat{\alpha}^2$ is invariant, it is nondegenerate. By \mathfrak{w}_α^1 we denote the kernel of the form $\hat{\alpha}^1$, so that $\mathfrak{w}_\alpha = \mathfrak{w}_\alpha^1 \otimes \mathfrak{v}^2$. Finally, we set $\mathfrak{v}_\alpha^1 = \mathfrak{v}^1/\mathfrak{w}_\alpha^1$.

Proposition 8. *Suppose L -modules \mathfrak{v} and \mathfrak{z} are irreducible, $\dim \mathfrak{z} > 1$, and \mathfrak{l}_α^1 is a commutative diagonalizable subalgebra (possibly zero). Then the homogeneous space G/L is commutative if and only if*

- 1) $L' = \mathrm{SO}(n)$ and $L' : \mathfrak{v}$ is the tautological representation of $\mathrm{SO}(n)$, whereas $\mathfrak{z} = \bigwedge^2 \mathfrak{v}$.
- 2) $L' = \mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$, $m = 1, 2$, $\mathfrak{v} = V(\pi_1) \otimes V(\pi'_1)$, $\mathfrak{z} = V(2\pi_1)$.

All the spaces listed in this proposition are commutative.

Proof. Using part (2) of Proposition 3 and results of [5, 10] one can verify that all the above spaces G/L with $L = L'$ are indeed weakly commutative. By Proposition 7 they are commutative. We now prove that there are no other weakly commutative spaces of the above-mentioned type.

First consider the case where the representation $L' : \mathfrak{z}$ is locally exact. The subalgebra $\mathfrak{l}_\alpha \subset \mathfrak{l}$ is reductive. Denote by β a generic element of \mathfrak{w}_α . Part (3) of Proposition 3 implies that the action $L_{\alpha\beta} : \mathfrak{v}_\alpha$ is coisotropic. The subalgebra $\mathfrak{l}_{\alpha\beta} \subset \mathfrak{l}_\alpha$ is the kernel of the representation \mathfrak{l}_α in \mathfrak{w}_α . As an $\mathfrak{l}_{\alpha\beta}$ -module, \mathfrak{v}_α is isomorphic to $U \oplus U^*$, where the action $L_{\alpha\beta} : U$ is spherical, i.e., the weights of the representation $\mathfrak{l}_{\alpha\beta} : U$ are linearly independent. This implies that the linear algebra $\mathfrak{l} \subset \mathfrak{gl}(\mathfrak{v})$ contains a semisimple operator x with only two nonzero eigenvalues $+1$ and -1 , each of multiplicity one. Now we describe all \mathfrak{l} -modules \mathfrak{v} with this property such that in $\bigwedge^2 \mathfrak{v}$ there is a submodule \mathfrak{z} for which the algebra \mathfrak{l}_α is diagonalizable.

We note that $x \in \mathfrak{l} \cap \mathfrak{sl}(\mathfrak{v}) = \mathfrak{l}'$, so that we can assume that the algebra \mathfrak{l} is semisimple. Suppose \mathfrak{l} is not simple and let $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ be a decomposition into a sum of ideals. Next, let $\mathfrak{v} = \mathfrak{v}_1 \otimes \mathfrak{v}_2$, where \mathfrak{v}_i is an irreducible \mathfrak{l}_i -module of dimension n_i , $x = x_1 + x_2$, for $x_i \in \mathfrak{l}_i$ and let λ_j^i , $i = 1, 2$, $j = 1, \dots, n_i$, be the eigenvalues of the element x_i in \mathfrak{v}_i . The eigenvalues of x in \mathfrak{v} are $\lambda_i^1 + \lambda_j^2$, where the indices i and j take all possible values. Among these eigenvalues there is one eigenvalue -1 , one eigenvalue $+1$, and all other eigenvalues are 0, whence $n_1 = n_2 = 2$. Therefore, $\mathfrak{l} = \mathfrak{so}(4)$ and \mathfrak{v} is the tautological $\mathfrak{so}(4)$ -module. The diagonalizability of \mathfrak{l}_α implies that $\mathfrak{z} = \bigwedge^2 \mathfrak{v}$.

It remains to consider the case where the algebra \mathfrak{l} is simple. It is clear that the tautological representation of $\mathfrak{so}(n)$ for $n > 2$ satisfies all the necessary conditions, and,

moreover, $\mathfrak{z} = \bigwedge^2 \mathfrak{v}$. Consider the remaining cases. Let $\mathfrak{v} = V(\lambda)$. Without loss of generality, one can assume that x lies in the positive Weyl chamber of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}$. In this case $\lambda(x) = 1$, $-\lambda^*(x) = -1$ (where λ^* is the highest weight of the module \mathfrak{v}^*), and all other weights of the \mathfrak{l} -module \mathfrak{v} vanish on x . The case where the \mathfrak{l} -module \mathfrak{v} is orthogonal is considered in [2, Lemma 2 of Proposition 2]. The case of a symplectic module can be considered similarly. In this case we have $\mathfrak{l} = \mathfrak{sp}(\mathfrak{v})$, but the stabilizer of a generic element of the \mathfrak{l} -module $\bigwedge^2 \mathfrak{v}$ is not commutative. If the \mathfrak{l} -module \mathfrak{v} is not self-conjugate, then the Dynkin diagram of the algebra \mathfrak{l} is simply laced. Now we prove that λ is a fundamental weight (i.e., the Dynkin diagram of the representation of \mathfrak{l} in \mathfrak{v} has one nonzero mark, which equals 1). Indeed, one can easily see that none of the marks is greater than 1 (otherwise, there exists an index i such that all weights λ , $\lambda - \alpha_i$, $\lambda - 2\alpha_i$, $\lambda - 2\alpha_i - \alpha_j$, where α_j is a simple root not orthogonal to α_i , are weights of \mathfrak{v} ; computing the values of these weights on x we get a contradiction). Therefore, $\lambda = \sum_{i \in I} \pi_i$ for some set I , $|I| > 1$. Let the indices i_0, i_1, \dots, i_m are such that $i_j \in I$ if and only if $j = 0, m$, and $(\alpha_{i_k}, \alpha_{i_l}) < 0$ if and only if $l = k \pm 1$. One can easily see that $\lambda - \sum_{k=0}^j \alpha_{i_k}$, $j = 0, \dots, m$, $\lambda - \sum_{k=0}^m \alpha_{i_k} - \alpha_{i_m}$ are weights of \mathfrak{v} contradicting the properties of x . Therefore, $\lambda = \pi_i$. We must consider the cases where i is arbitrary for A_l , $i = l - 1$ for D_l , and $i = 1, 2$ for E_6 . This can be done directly using information from the reference section of [3]. The only case where there exists a required element x is the tautological representation of the algebra $\mathfrak{sl}(n)$. But in this case the stabilizer of a generic element of the \mathfrak{l} -module $\bigwedge^2 \mathfrak{v}$ is not diagonalizable.

Now assume that the representation $L' : \mathfrak{z}$ is not exact. If $\dim \mathfrak{v}^1 = 2$, then $\mathfrak{l}^1 = \mathfrak{sl}_2$, $\dim \mathfrak{z} = 3$. The weak commutativity of the homogeneous space G/L is equivalent to the fact that the action of the group L_α on \mathfrak{v} is not coisotropic. This is equivalent to the fact that the linear group $\mathbb{C}^* \otimes L^2$ is spherical. Therefore, $L^2 = \text{Sp}(2m)$ (see [5, Table 8]).

In the sequel we will assume that $\dim \mathfrak{v}^1 > 2$. According to a result in [5], in the case (Alt) we have $\dim \mathfrak{v}_\alpha^1 \leq 2$ and $\mathfrak{l}_\alpha^1 \neq 0$ (otherwise the action $L_\alpha : \mathfrak{v}_\alpha$ is not coisotropic). In the case (Sym), we have $\dim \mathfrak{v}_\alpha^1 \leq 4$, and if $\dim \mathfrak{l}_\alpha^1 \leq 1$, then $\dim \mathfrak{v}_\alpha^1 \leq 2 + \dim \mathfrak{l}_\alpha^1$.

Now we prove that $\dim \mathfrak{w}_\alpha^1 < k$, where $k = 1$ in the case (Alt) and $k = 2$ in the case (Sym). If this inequality does not hold, we take a k -dimensional subspace $U \subset \mathfrak{v}^2$ such that the restriction of the \mathfrak{l}^2 -invariant form on the space \mathfrak{v}^2 (symplectic or orthogonal) to U is nondegenerate. Take a generic element $\beta \in \mathfrak{w}_\alpha^1 \otimes U$. The connected component of the group $\varphi(L_{\alpha\beta})$ lies in $L_\alpha^1 \otimes (\mathbb{C}^{*\text{sympl}} \oplus \text{Sp}(2m - 2))$ (in the case (Sym)) and in $L_\alpha^1 \otimes \text{SO}(n - 1)$ (in the case (Alt)). According to the results in [5], these two linear groups are not coisotropic. Together with part (3) of Proposition 12, this implies that $\dim \mathfrak{w}_\alpha^1 < k$ (indeed, the inequality $\text{cork}_{S_\beta}(\mathfrak{v}_\alpha) > 0$ for some $\beta \in \mathfrak{w}_\alpha$ implies a similar inequality for a generic β). In the case (Alt) this implies that $\dim \mathfrak{v}^1 = 2$ completing the proof. In the case (Sym) we obtain $\dim \mathfrak{v}^1 \leq 5$.

Therefore, it remains to analyze the following representations φ^1 :

- (1) The tautological representation of $\mathfrak{sl}(k)$, $k = 3, 4, 5$.
- (2) The tautological representation of $\mathfrak{so}(k)$, $k = 3, 4, 5$.
- (3) The tautological representation of $\mathfrak{sp}(4)$.

In case (1) the algebra \mathfrak{l}_α^1 is not diagonalizable. In case (2), we have $\mathfrak{v}_\alpha^1 = \mathfrak{v}^1$ and $\mathfrak{l}_\alpha^1 = 0$ (for $k = 4$ we must use that the representation ρ^1 is exact). □

The notation $\mathfrak{h} + V$ in the second column means that the algebra \mathfrak{l}_α^1 is the semidirect product of the reductive algebra \mathfrak{h} and the \mathfrak{h} -module V . In line 8, \mathbb{C}^l is the \mathfrak{f} -module obtained by restricting the tautological representation of $\mathfrak{sl}(l)$ to \mathfrak{f} .

TABLE 4. Possible irreducible representations $L' : \mathfrak{g}$.

no.	$\rho(L')$	\mathfrak{l}_α^1
1	$\mathrm{SL}(2n+1), n > 1$	$\mathfrak{sl}(2n) + V(\pi_{2n})$
2	$\bigwedge^2 \mathrm{SL}(n)$	n even: $\mathfrak{sp}(n)$ n odd: $\mathfrak{sp}(n-1) + V(\pi_1)$
3	$S^2 \mathrm{SL}(n)$	$\mathfrak{so}(n)$
4	$C_l(\pi_2), l > 2$	$\mathfrak{sl}(2)$
5	$\mathrm{SO}(n), n$ is not divisible by 4, $n \neq 6, n > 3$	$\mathfrak{so}(n-1)$
6	$G_2(\pi_1)$	A_2
7	$F_4(\pi_1)$	D_4
8	$\mathrm{SL}(2n+1) \otimes F,$ $F \subset \mathrm{SL}(l), 2 < l \leq 2n+1$	$(\mathfrak{sl}(2n+1-l) \times \mathfrak{f}) + V(\pi_{2n-l}) \otimes \mathbb{C}^l$
9	$\mathrm{SO}(n) \otimes F,$ n is not divisible by 4, $F \subset \mathrm{SL}(l), l < n$	$F = \mathrm{SL}(l) : \mathfrak{so}(n-l) \times \mathfrak{so}(l)$ $F \subsetneq \mathrm{SL}(l) : \mathfrak{so}(n-l)$

Proposition 9. *Let the space G/L be weakly commutative and let the representation ρ^1 be irreducible. If \mathfrak{l}_α^1 is a nonzero nondiagonalizable algebra, then $\rho(L')$ is one of the groups in Table 4.*

Proof. First we consider the case where the group $\rho(L')$ is simple. The results in [8] and restrictions listed in Table 3 show that $\rho(L')$ is one of the groups listed in Table 4 or $\bigwedge^3 \mathrm{SL}(7), E_6(\pi_1)$.

Let $\rho_1(L') = \bigwedge^3 \mathrm{SL}(7)$ or E_6 . Then the connected component H° of the stabilizer of a generic element in the linear group $\rho(L')$ is G_2 or F_4 respectively. Part (3) of Proposition 3 shows that there exists a linear representation ψ of the group H° such that the linear group $\psi(H^\circ) \otimes \varphi^2(L')$ is coisotropic. According to [5], we have $\psi(H^\circ) = G_2(\pi_1)$, $\varphi^2(L') = \mathrm{Sp}(2m)$, $m < 3$. Corollary to Proposition 4 shows that $\dim \mathfrak{v} \leq 28m$. Therefore, $\dim \varphi^1 \leq 14$, so that φ^1 is a tautological representation of the algebra \mathfrak{sl}_7 . However, this is impossible since $\rho^1 \notin S^2 \varphi^1$.

It remains to consider the case where the group $\rho(L')$ is not simple. We use the results of [9, §3]. Cases 1, 5 in [9, §3] do not satisfy the assumptions of the proposition, and Case 4 does not satisfy the conditions of Table 3. Let $\rho(L')$ be one of the linear groups that satisfy Case 6 and the parity conditions in Table 3. In this case the group H is reductive. If the homogeneous space G/L is weakly commutative, then one of the linear groups $\varphi^1(H) \otimes \mathrm{Sp}(2l)$ or $\varphi^1(H) \otimes \mathrm{SO}(l)$ is coisotropic. Direct verification using the table in [9, §3] and the classification results of [5] show that this is impossible (except in the cases indicated in the proposition). \square

Weakly commutative homogeneous spaces for which both representations ρ and φ are orthogonal, are complexifications of weakly commutative Riemannian homogeneous spaces, and were essentially classified in [2]. All these spaces are commutative. This gives spaces 1–3 in Table 1. Therefore, in what follows we assume that at least one of the representations φ, ρ is not orthogonal.

Finally, we note that if the linear group $\rho^1(L')$ is different from $SL(2n + 1) \otimes F$, $F \subset SL(l)$, $1 \leq l < 2n + 1$, $\bigwedge^2 SL(2n + 1)$, then we can assume that the group L is semisimple (see Corollary 4 to Proposition 3).

3.2. Lowering the dimension.

Proposition 10. *Let $L = L^1 \times Sp(2m)$, let φ^2 be the tautological representation of the group $Sp(2m)$, and let the normal subgroup $Sp(2m) \subset L$ act trivially on \mathfrak{z} . The space $N \times L/L$ is weakly commutative if and only if the similar homogeneous space $L = L^1 \times Sp(2m_1)$, where $m_1 = \min\{[(\dim \varphi^1 + 1)/2], m\}$, is weakly commutative. A similar result holds for $L = L^1 \times SO(m)$.*

Proof. The homogeneous space $N \times L/L$ is weakly commutative if and only if, for generic elements $\alpha \in \mathfrak{z}^*$, $\beta \in \mathfrak{v}^*$, the subspace $(\mathfrak{l}_\alpha^1 \times \mathfrak{sp}(2m) \cdot \beta)^0 \subset \mathfrak{v}$ is isotropic with respect to the form $\widehat{\alpha}$. Denote by ω the symplectic form on the space \mathfrak{v}^2 entering the definition of the group $Sp(2m)$. Let \mathfrak{v}_1^2 be a $2m_1$ -dimensional subspace in \mathfrak{v}^2 such that the restriction of ω to \mathfrak{v}_1^2 is nondegenerate, and $\mathfrak{v}_2^2 = (\mathfrak{v}_1^2)^\perp$. Similarly to Proposition 10 in [5] one can prove that a generic element in $(\mathfrak{v}^1 \otimes \mathfrak{v}^2)^*$ is conjugate to an element in $(\mathfrak{v}_1 \otimes \mathfrak{v}_1^2)^*$ under the action of the group $Sp(2m)$, and that for a generic $\beta_1 \in (\mathfrak{v}_1 \otimes \mathfrak{v}_1^2)^*$ we have

$$(\mathfrak{l}_\alpha^1 \times \mathfrak{sp}(\mathfrak{v}^2))\beta_1 = (\mathfrak{l}_\alpha^1 \times \mathfrak{sp}(\mathfrak{v}_1^2))\beta_1 \oplus (\mathfrak{v}_1 \otimes \mathfrak{v}_2^2)^*,$$

i.e., $(\mathfrak{l}_\alpha^1 \times \mathfrak{sp}(\mathfrak{v}^2))\beta_1^0$ coincides with the annihilator of the space $(\mathfrak{l}_\alpha^1 \times \mathfrak{sp}(\mathfrak{v}_1^2))\beta_1$ in the space $\mathfrak{v}_1 \otimes \mathfrak{v}_2^2$, as required.

In the case $L = L^1 \times SO(n)$ the proof is similar. □

The next proposition generalizes Proposition 10 in [5]; the proof is similar.

Proposition 11. *Let $G = Sp(2m) \otimes G^{(2)}$, $G^{(2)} \subset SO(n)$, $2m \geq n$. The corank of the action of the linear group G does not depend on $m \geq n/2$. Similarly, the corank of the action of the linear group $G^{(1)} \otimes SO(n)$, $G^{(1)} \subset Sp(2m)$, does not depend on $n \geq 2m$.*

3.3. The case of irreducible \mathfrak{z} . We assume that the homogeneous space $N \times L/L$ is weakly commutative.

Suppose the case (Alt) holds, i.e., φ^2 is an orthogonal representation of dimension m , where $m = 1$ or $m \geq 3$.

According to Proposition 10, we can assume that $m \leq \dim \varphi^1$. By the corollary to Proposition 5 we have

$$(3.2) \quad m \dim \varphi^1 \leq m^2 - m + 2(\dim \mathfrak{l}^1 + \dim \mathfrak{z}(\mathfrak{l}))$$

$$(3.3) \quad \Rightarrow \dim \varphi^1 \leq 2(\dim \mathfrak{l}^1 + \mathfrak{z}(\mathfrak{l})).$$

Next, set $m_1 = \min\{\dim \mathfrak{v}_\alpha^1, m\}$. By Proposition 11 and Corollary 3 to Proposition 3 we have

$$(3.4) \quad m_1 \dim \mathfrak{v}_\alpha^1 \leq \dim \mathfrak{s}^1 + \text{rk } \mathfrak{s}^1 + [m_1^2/2] + 2 \dim \mathfrak{l}_\alpha^1/\mathfrak{s}^1 + 2 \dim \mathfrak{z}(\mathfrak{l})$$

$$(3.5) \quad \Rightarrow \dim \mathfrak{v}_\alpha^1 \leq \dim \mathfrak{s}^1 + \text{rk } \mathfrak{s}^1 + 2 \dim \mathfrak{l}_\alpha^1/\mathfrak{s} + 2 \dim \mathfrak{z}(\mathfrak{l}),$$

where $\mathfrak{l}_\alpha^1 \subset \mathfrak{l}^1$ is the stabilizer of the form $\alpha \in \mathfrak{z}^*$ and \mathfrak{s}^1 is the maximal reductive subalgebra in \mathfrak{l}_α^1 .

Next assume that the case (Sym) holds, i.e., the representation φ^2 is symplectic of dimension $2m$.

Denote $m_1 = \min\{[(\dim \mathfrak{v}_\alpha^1 + 1)/2], m\}$. As in the case (Alt), we have

$$(3.6) \quad 2m_1 \dim \mathfrak{v}_\alpha^1 \leq \dim \mathfrak{s}^1 + \text{rk } \mathfrak{s}^1 + 2m_1^2 + 2m_1 + 2 \dim \mathfrak{l}_\alpha^1/\mathfrak{s}^1 + 2 \dim \mathfrak{z}(\mathfrak{l}).$$

Inequality (3.6) implies that for $\dim \mathfrak{v}_\alpha^1 \geq 4$ (according to the corollary to Proposition 4, the latter always holds if $\dim \mathfrak{v}^1 \geq 7$) we have

$$(3.7) \quad 2 \dim \mathfrak{v}_1^\alpha \leq \dim \mathfrak{s}^1 + \text{rk } \mathfrak{s}^1 + 4 + 2 \dim \mathfrak{l}_\alpha^1 / \mathfrak{s}^1 + 2 \dim \mathfrak{z}(\mathfrak{l}).$$

In this subsection, λ always denotes the highest weight of the representation φ^1 , and μ , the highest weight of the representation ρ^1 .

3.3.1. *The case $\mathfrak{l}^1 = \mathfrak{sl}(2n + 1)$, $\mu = \pi_{2n}$, $n > 1$.* In this case $\lambda - \lambda^*$ is a weight of the representation $L' : \mathfrak{z}$. Therefore, one can easily verify that $\lambda = \pi_n + \nu$, where $\nu = \nu^*$ is a dominant weight. Thus,

$$\dim \mathfrak{s}^1 = 4n^2 - 1, \quad \text{rk } \mathfrak{s}^1 = 2n - 1, \quad \dim \mathfrak{l}_\alpha^1 / \mathfrak{s}^1 = 2n.$$

In the case (Alt) inequality (3.3) implies that $\dim \mathfrak{v}^1 \leq 2(2n + 1)^2$. In the case (Sym) the corollary to Proposition 4 and inequality (3.7) implies that

$$(3.8) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq 4n^2 + 6n + 4.$$

Direct verification (using, e.g., the Weyl formula for the dimension of an irreducible representation) shows that the only \mathfrak{l}^1 -module of required dimension is the module $V(\pi_n) = \bigwedge^n V(\pi_1)$ for $n \leq 4$.

Formula (3.1) implies that the multiplicity of the \mathfrak{l}^1 -module \mathfrak{z} in the \mathfrak{l}^1 -module $\mathfrak{v} \otimes \mathfrak{v}$ does not exceed 1.

For n even the \mathfrak{l}^1 -module $V(\pi_{2n})$ is a submodule of the \mathfrak{l}^1 -module $S^2V(\pi_n)$, and for n odd, a submodule of $\bigwedge^2 V(\pi_n)$.

First, let us consider the case (Alt). Here $n = 3$.

Let $m = 1$. We show that in this case the homogeneous space $N \backslash L/L$ is weakly commutative whenever the group L is semisimple. The restriction of the \mathfrak{l} -module \mathfrak{v} to the reductive part of the algebra $\mathfrak{s} = \mathfrak{sl}(2n) \subset \mathfrak{l}_\alpha$ is the direct sum $V(\pi_3) \oplus V(\pi_2)$ with $\mathfrak{w}_\alpha = V(\pi_2)$. The action $S_\beta : \mathfrak{v}_\alpha$ is coisotropic (the corresponding linear group is $\bigwedge^3 \text{Sp}(6)$) and it remains to use part (2) of Proposition 3. Therefore, the homogeneous space G/L is also commutative (see Proposition 7).

Now let $m \geq 3$. The generic stabilizer for the action of the group S on the space \mathfrak{w}_α is one-dimensional for $m = 3$ and equals $\mathfrak{so}(m - 15)$ for $m > 16$. In the remaining cases the stabilizer is trivial. Part (3) of Proposition 3 implies that the homogeneous space G/L is not weakly commutative.

Now consider the case (Sym). Inequality (3.7) holds only for $n = 2$. Similarly to the case $n = 3$, we have $\dim \mathfrak{v}_\alpha^1 = 6$, $\dim \mathfrak{w}_\alpha^1 = 4$. The restriction of the L^1 -module \mathfrak{v}^1 to the maximal reductive subgroup $S^1 = \text{SL}(4) \subset L_\alpha^1$ is isomorphic to $V(\pi_1) \oplus V(\pi_2)$. We identify the space \mathfrak{w}_α^1 with $V(\pi_1)$ and \mathfrak{v}_α^1 with $V(\pi_2)$.

First let $m \geq 2$ (but, according to our convention, $m \leq 5$). For a generic vector $\beta \in \mathfrak{w}$ the algebra \mathfrak{s}_β is isomorphic to $\mathfrak{sp}(4) \times \mathfrak{sp}(2m - 4)$ (see [9]). According to part (3) of Proposition 3, we have

$$(3.9) \quad 12m = \dim \mathfrak{v}_\alpha \leq \dim \mathfrak{s}_\beta + \text{rk } \mathfrak{s}_\beta + 2 \dim \mathfrak{l}_\alpha / \mathfrak{s} = 12 + (m - 2)^2 + 2(m - 2) + 8.$$

For $2 \leq m \leq 5$ this inequality does not hold.

It remains to consider the case $m = 1$. We show that in this case for $L = \text{SL}(5) \times \text{SL}(2)$ we have $\mathbb{C}(\mathfrak{n})^L = \mathbb{C}$, thus proving the commutativity of the homogeneous space G/L for a semisimple group L . According to a result in [8], the generic stabilizer for the action $\text{SL}(5) : \mathfrak{n}$ is one-dimensional and consists of nilpotent elements. Moreover, the generic stabilizer for the action $L : \mathfrak{v}$ is a semidirect product of the subalgebra $\mathfrak{sl}(2)$ (such that the projection of this subalgebra to both factors of the algebra \mathfrak{l} is nontrivial) and a four-dimensional nilpotent radical. Therefore, the dimension of the generic stabilizer for the action $L : \mathfrak{n}$ is at most 2. Hence, $m_L(\mathfrak{n}) \geq 24 + 3 - 2 = \dim \mathfrak{n}$, as required.

3.3.2. $\mathfrak{l}^1 = \mathfrak{sl}(n)$, $\mu = \pi_2$, $n > 2$. In this case $\lambda - \lambda^* = \varepsilon_i + \varepsilon_{n+1-i}$, $i < \frac{n+1}{2}$ (we use the notation of the reference section in [3]). Therefore, $\lambda = \pi_i + \pi_{n+1-i} + \nu$, where $\nu = \nu^*$ is a dominant weight (we use the convention $\pi_n = 0$).

In the case where n is even, the algebra \mathfrak{l}_α^1 is reductive and is imbedded in $\mathfrak{sl}(n)$ as the subalgebra $\mathfrak{sp}(n)$. According to (3.5), in the case (Alt) we have

$$(3.10) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq n^2 + 2n.$$

In the case (Sym) we use (3.7) to see that

$$(3.11) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq \frac{n^2}{2} + n + 4.$$

Now, consider the case of an odd n . The maximal reductive subalgebra of \mathfrak{l}_α^1 is isomorphic to $\mathfrak{sp}(n-1)$, and the unipotent radical of \mathfrak{l}_α^1 is commutative and has dimension $n - 1$. In the case (Alt) we use (3.3) and (3.5) to see that

$$(3.12) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq \min\{2n^2, n^2 + 4n - 1\}.$$

In the case (Sym) we have

$$(3.13) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq \frac{n^2 - 1}{2} + 2n + 4.$$

Using tables presented in [3] we see that the only possible alternative is (Alt), and the following weights can occur:

- 1) $\lambda = \pi_1$;
- 2) $\lambda = 2\pi_1 + \pi_2$, $n = 3$, $\dim \mathfrak{v}^1 = 15$;
- 3) $\lambda = \pi_2 + \pi_3$, $n = 4$, $\dim \mathfrak{v}^1 = 20$.

First, consider the case $\lambda = \pi_1$ with the tautological representation φ^2 of the algebra $\mathfrak{so}(m)$. In this case for n even the group S_β acts in the space \mathfrak{v}_α as $\mathrm{Sp}(n) \otimes \mathrm{SO}(m)$, whereas for n odd it acts as $\mathrm{Sp}(n-1) \otimes (\mathrm{SO}(m-1) \oplus \mathrm{SO}(1))$. In this case part (3) of Proposition 3 and the results of [5] show that the homogeneous space G/L is weakly commutative, and hence, by Proposition 7, commutative.

Now we consider the remaining cases. From inequality (3.4) it follows that we can take $m = 1$.

Let $n = 3$, $\mathfrak{v} = R(2\pi_1 + \pi_2)$. We can assume that the action of the group L in the space \mathfrak{v}^1 coincides with the restriction of the natural action of the group $\mathrm{GL}(3)$ to the invariant 15-dimensional subspace in $S^2\mathbb{C}^3 \otimes \mathbb{C}^{3*}$. Then the action of the group L in the space \mathfrak{z}^* is $g \mapsto \det(g)^{-1}g$, and for L_α we can take the subgroup consisting of the matrices of the form $\begin{pmatrix} x & y & z \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$, $x \neq 0$, $ad - bc = 1$. The S -module \mathfrak{v}^1 is isomorphic to $(3\pi_1)^0 \oplus R(2\pi_1)^1 \oplus R(2\pi_1)^{-1} \oplus R(\pi_1)^0 \oplus R(\pi_1)^2 \oplus 1^1$ (the upper index denotes the character corresponding to the action of the center). Since the last two summands cannot occur in the symplectic module \mathfrak{v}_α , we have $\dim \mathfrak{w}_\alpha > 2$, $\dim \mathfrak{l}_{\alpha\beta} \leq 4$. Part (3) of Proposition 3 implies that $\dim \mathfrak{v}_\alpha \leq 8$, whereas the corollary to Proposition 4 implies that $\dim \mathfrak{v}_\alpha \geq 8$; i.e., $\dim \mathfrak{w}_\alpha = 7$, hence $\dim \mathfrak{l}_{\alpha\beta} < 4$. This contradicts part (3) of Proposition 3.

In case 3) we have $S = \mathrm{Sp}(4)$. The maximal dimension of a coisotropic representation of the group S equals 8. This fact, together with the corollary to Proposition 4, shows that the space G/L is not weakly commutative.

Now let $\lambda = \pi_1$ again. Let \mathfrak{l}^2 be an irreducible linear orthogonal algebra, $\mathfrak{l}^2 \subsetneq \mathfrak{so}(m)$, and $L^2 \subset \mathrm{SO}(m)$ a real algebraic group with the tangent algebra \mathfrak{l}^2 . In the case where n is even, weak commutativity (and commutativity for a semisimple group L) of the space G/L is equivalent to the fact that the linear group $\mathrm{Sp}(n) \otimes L^2$ is coisotropic. All groups with this property are listed in Theorem 4 in [5].

TABLE 5.

no.	\mathfrak{l}^2	$\mathfrak{l}_\beta^2 \hookrightarrow \mathfrak{l}^2$
1	$\mathfrak{spin}(7)$	$G_2(\pi_1)$
2	$\mathfrak{spin}(9)$	$B_3(\pi_3, 0)$
3	$C_l(\pi_2), l > 2$	$\mathfrak{sl}(2)$
4	$G_2(\pi_1)$	$A_2(\pi_1, \pi_2, 0)$
5	$F_4(\pi_1)$	$D_4(\pi_1, \pi_3, \pi_4, 0, 0)$

Now consider the case of even n . We show that in this case the homogeneous space G/L is weakly commutative if and only if the linear group $\mathrm{Sp}(n-1) \otimes H$ is coisotropic. Here $H = L_x^2$ is the stabilizer of a generic element $x \in \mathbb{C}^m$ in the linear group $L^2 \subset \mathrm{SO}(m)$; we do not view H as a linear group acting in the space x^\perp .

Let us note that all the hypotheses of part (4) of Proposition 3 hold. The desired result is a direct consequence of this part (4) (the relevant action of $\mathrm{Sp}(n-1) \otimes H$ is just the action of the group S_β in the space $\widehat{\mathfrak{v}}_{\alpha\beta}$). Moreover, the additional hypotheses of Proposition 7 also hold, i.e., the homogeneous spaces G/L are, in addition, commutative.

Now we find all algebras \mathfrak{l}^2 such that the linear group $\mathrm{Sp}(n-1) \otimes H$ is coisotropic. Suppose the algebra \mathfrak{l}^2 is not simple. We show that in this case the homogeneous space G/L is not weakly commutative. It suffices to consider two cases: $\mathfrak{l}^2 = \mathfrak{so}(m_1) \otimes \mathfrak{so}(m_2)$, $3 \leq m_1 \leq m_2$, $m_1 m_2 = m$, and $\mathfrak{l}^2 = \mathfrak{sp}(2m_1) \otimes \mathfrak{sp}(2m_2)$, $m = 4m_1 m_2$, $m_2 > 1$. In the first case, $\mathfrak{l}_\beta^2 = \mathfrak{so}(m_2 - m_1) \subset \mathfrak{so}(m)$ (see [9, Theorem 5]) and the representation of the algebra \mathfrak{l}_β^2 in $\langle \beta \rangle^\perp \subset \mathfrak{v}^2$ has at least three irreducible components. The linear group $\mathrm{Sp}(n-1) \otimes L_\beta^2$ is not coisotropic due to the remark at the end of 2.4.2 in [5].

Now let $\mathfrak{l}^2 = \mathfrak{sp}(2m_1) \otimes \mathfrak{sp}(2m_2)$. According to Theorems 6 and 7 in [9], $\mathfrak{l}_\beta^2 = (m_1 \mathfrak{sl}(2) \times \mathfrak{sp}(2(m_2 - m_1)))$, where the ideal $m_1 \mathfrak{sl}(2) \subset \mathfrak{l}_\beta^2$ is imbedded in $\mathfrak{l}^2 = \mathfrak{sp}(2m_1) \times \mathfrak{sp}(2m_2)$ diagonally, and the ideal $\mathfrak{sp}(2(m_2 - m_1)) \subset \mathfrak{l}_\beta^2$ is imbedded in $\mathfrak{sp}(2m_2) \subset \mathfrak{l}^2$. Similarly to the case $\mathfrak{l}^2 = \mathfrak{so}(n_1) \otimes \mathfrak{so}(n_2)$, we can prove that $m_1 = 1$. In this case the linear group $\mathrm{Sp}(n-1) \otimes H$ is $\mathrm{Sp}(n-1) \otimes (\mathrm{SO}(3) \oplus \mathrm{SL}(2) \otimes \mathrm{Sp}(2(m_2 - 1)))$, which is not coisotropic for $m_2 > 1$.

Therefore, it remains to consider the case where the algebra \mathfrak{l}^2 is simple.

Let φ^2 be the adjoint representation of the algebra \mathfrak{l}^2 . The remark at the end of 2.4.2 in [5] implies that if the space G/L is weakly commutative, then the restriction of the adjoint representation of the algebra \mathfrak{l}^2 to the Cartan subalgebra $\mathfrak{l}_\beta^2 \subset \mathfrak{l}^2$ is not decomposed into the direct sum of three orthogonal representations, which means that

$$(3.14) \quad \frac{\dim \mathfrak{l}^2 - \mathrm{rk} \mathfrak{l}^2}{2} + \mathrm{rk} \mathfrak{l}^2 - 1 \leq 2.$$

This implies that $\dim \mathfrak{l}^2 + \mathrm{rk} \mathfrak{l}^2 \leq 6$, which holds only if $\mathfrak{l}^2 = \mathfrak{sl}(2)$, and this case was analyzed earlier.

Since $\mathfrak{l}_\beta^2 \neq 0$, it remains to consider the cases (see [8]) listed in Table 5. In this table, the column marked \mathfrak{l}^2 contains the image of the algebra \mathfrak{l}^2 under the representation φ^2 , and the column marked $\mathfrak{l}_\beta^2 \hookrightarrow \mathfrak{l}^2$ contains the linear algebra that is the image of the algebra \mathfrak{l}_β^2 under the simplest representation \mathfrak{l}^2 .

The classification presented in [5] shows that the space G/L is weakly commutative only in the cases listed in line 1 of Table 5 for $n = 3, 5$ and in lines 2, 4 for $n = 3$.

3.3.3. *The case $\mathfrak{l}^1 = \mathfrak{sl}(n)$, $\mu = 2\pi_1$, $n > 2$.* Here $\mathfrak{l}_\alpha^1 = \mathfrak{so}(n)$. As in 3.3.2, $\lambda = \pi_i + \pi_{n+1-i} + \nu$, $i \leq \frac{n+1}{2}$, and $\nu = \nu^*$ is a dominant weight.

First we consider the case (Alt). The classification results in [5] show that a symplectic \mathfrak{l}_α^1 -module V such that the $\mathfrak{l}_\alpha^1 \times \mathfrak{so}(m)$ -module $V \otimes \mathbb{C}^m$ (for some $m \neq 2$) is coisotropic, exists if and only if $n = 3, 4, 5, 6, 10, 11, 12, 13$. The dimension of the space \mathfrak{v}_α^1 is at most 4, 8, 8, 8, 32, 32, 64, 64 respectively (according to [5]) and $\dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1$ in view of the corollary to Proposition 4.

Using tables in [3] and the Weyl formula for the dimension of an irreducible representation, one can directly prove that the restrictions on the dimension listed in the previous paragraph are satisfied only for the representation φ^1 with the highest weight π_1 . However, $\rho^1 \notin \bigwedge^2 \varphi^1$.

It remains to consider the case (Sym). If $\lambda = \pi_1$, then Theorem 4 in [5] implies that φ^2 is the tautological representation of the algebra $\mathfrak{sp}(2m)$. The corresponding homogeneous space G/L is weakly commutative (according to part (1) of Proposition 3 and the results of [5]). Moreover, Proposition 7 shows that the space G/L is commutative.

Inequality (3.7) can be rewritten as follows:

$$(3.15) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq \left\lfloor \frac{n^2}{2} \right\rfloor + 4.$$

One can easily see that the only (except the tautological) representation with the highest weight of the required form and the dimension satisfying (3.15) is the representation of the algebra \mathfrak{sl}_3 with the highest weight $2\pi_2$. However, the restriction of the representation φ^1 to $\mathfrak{l}_\alpha^1 = \mathfrak{so}(3) \subset \mathfrak{sl}(3)$ is isomorphic to the direct sum of the 5-dimensional and the one-dimensional representations of the algebra \mathfrak{sl}_2 , which implies that $\dim \mathfrak{v}_\alpha^1 \geq 5$, contradicting (3.15).

3.3.4. *The case $\mathfrak{l}^1 = \mathrm{Sp}(2n)$, $n > 2$, $\mu = \pi_2$.* Here $\mathfrak{l}_\alpha^1 = n\mathfrak{sl}(2)$.

First we consider the case (Alt). Let $\lambda = \pi_1$. In this case $\mathfrak{w}_\alpha^1 = 0$. Indeed, otherwise we can find (see the proof of the corollary to Proposition 4) vectors $v_1, v_2 \in \mathfrak{v}$ such that $\hat{\alpha}(v_1, v_2) = 0$ for each $\alpha \in \mathfrak{z}^*$. If ω is an L^1 -invariant form on \mathfrak{v}^1 , then $\omega(v_1, v_2) \neq 0$ (otherwise v_1, v_2 would be orthogonal with respect to an arbitrary skew-symmetric form on \mathfrak{v}). Therefore, Witt's theorem shows that for arbitrary $v'_1, v'_2 \in \mathfrak{v}$ such that $\omega(v'_1, v'_2) \neq 0$ we have $\hat{\alpha}(v'_1, v'_2) = 0$ for all $\alpha \in \mathfrak{z}^*$, which is impossible.

According to [5] (remark at the end of 2.4.2), the representation $L_\alpha : \mathfrak{v} = \mathfrak{v}_\alpha$ is coisotropic if and only if $m = 1$. By Proposition 7, in this case the homogeneous space G/L is commutative.

Now let $\lambda \neq \pi_1$. From (3.5) we see that $\dim \mathfrak{v} \leq 8n$. Since we do not consider the case $\mathfrak{v} = V(\pi_2)$ (in this case \mathfrak{v} and \mathfrak{z} are the spaces of orthogonal representations of \mathfrak{l}), it remains to consider the case $n = 3$, $\lambda = \pi_3$. Tables in [3] show that in this case ρ^1 is not a subrepresentation of $\varphi^1 \otimes \varphi^1$.

Now let us consider the case (Sym). In this case $\lambda \neq \pi_1$. Inequality (3.7) can be rewritten in the form

$$(3.16) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq 4l + 4.$$

Therefore, it remains to consider the case $n = 3, \lambda = \pi_2$. Applying inequality (3.6), we see that $m = 1$. Inequality (3.16) implies that $\dim \mathfrak{v}_\alpha^1 \leq 8$. Denote by ι_i the tautological representation of the i th factor $\mathfrak{sl}(2)$ in \mathfrak{l}_α^1 . The restriction of the representation φ^1 to \mathfrak{l}_α^1 is isomorphic to the representation $\iota_1 \otimes \iota_2 + \iota_1 \otimes \iota_3 + \iota_2 \otimes \iota_3 + 2$. Therefore, without loss of generality we can assume that the representation of the algebra \mathfrak{l}_α^1 in \mathfrak{w}_α^1 is $\iota_2 \otimes \iota_3 + 2$. Thus, for a generic $\beta \in \mathfrak{w}_\alpha^*$ the algebra $\mathfrak{l}_{\alpha\beta}$ is isomorphic to $\mathfrak{sl}(2)$. Using part (3) of Proposition 3, we see that in this case the space G/L is not weakly commutative.

3.3.5. *The case* $\mathfrak{l}^1 = \mathfrak{so}(n)$, $\mu = \pi_1$, $n > 3$, n not divisible by 4, $n \neq 6$. In this case $\mathfrak{l}_\alpha^1 = \mathfrak{so}(n-1)$.

Denote $l = \text{rk } \mathfrak{so}(n) = [n/2]$. According to (3.1), for odd n the coefficient λ_l of the weight λ is different from 0. For n even, one of the coefficients λ_l, λ_{l-1} is different from 0. In the case where φ^1 is either the spinor or the half-spinor representation, the inclusion $\rho^1 \subset \bigwedge^2 \varphi^1$ (respectively $\rho^1 \subset S^2 \varphi^1$) happens if and only if n is congruent to 5, 6, 7 (respectively 1, 2, 3) modulo 8.

First of all, let us prove that if φ^1 is either the spinor or the half-spinor representation, then \mathfrak{z} as a subspace in $\bigwedge^2 \mathfrak{v}^1$ (or $S^2 \mathfrak{v}^1$) contains a nondegenerate form. For n even this follows from the fact that the restriction of the representation φ^1 to \mathfrak{l}_α^1 is irreducible. For n odd we note that the restriction of the representation φ^1 to the normalizer of $\langle \alpha \rangle$ (which contains an involutive outer automorphism of \mathfrak{l}_α^1) is also irreducible.

Consider the case (Alt). By (3.5), $\dim \mathfrak{v}_\alpha^1 \leq [(n-1)^2/2]$, hence $\dim \mathfrak{v}^1 \leq (n-1)^2$. Using tables in [3] one can show that the only possible cases are:

- 1) $n = 5, 13, 14$, $\lambda = \pi_l$ (we do not consider the case $n = 7$ since all representations of the algebra $\mathfrak{l} = \mathfrak{so}(7)$ are orthogonal).
- 2) $n = 5$, $\lambda = \pi_1 + \pi_2$.

In case 1), for $m = 1$ the corresponding homogeneous spaces G/L are weakly commutative (by part (2) of Proposition 3 and the classification of coisotropic representations). For $n \neq 5$, $m \geq 3$ the corresponding homogeneous spaces G/L are not weakly commutative (by classification in [5] and part (3) of Proposition 3). Finally, for $n = 5$ the image of the group L_α under the representation in \mathfrak{v} is the linear group $(\text{SL}(2) \oplus \text{SL}(2)) \otimes L^2$. Therefore, the classification in [5] shows that for $L^2 = \text{SO}(m)$ the representation $L_\alpha : \mathfrak{v}$ is coisotropic. For the irreducible subgroup $L^2 \subset \text{SO}(m)$ the above representation of the group L_α is coisotropic only for $m = 8$ and $L^2 = \text{Spin}(7)$.

According to Proposition 7, all these weakly commutative homogeneous spaces are in fact commutative.

Consider the case $n = 5$, $\lambda = \pi_1 + \pi_2$. Here we have $\dim \mathfrak{v}^1 = 16$, $\dim \mathfrak{v}_\alpha^1 = 8$. Inequality (3.5) implies that $m = 1$. We have the isomorphism of L -modules $R(\lambda) \oplus R(\pi_2) = R(\pi_1) \otimes R(\pi_2)$. Therefore the restriction of the representation φ to the subalgebra $\mathfrak{l}_\alpha = \mathfrak{so}(4) \subset \mathfrak{l}$ is isomorphic to $(\iota_1 \oplus \iota_2) \otimes \iota_1 \otimes \iota_2$, where ι_1, ι_2 are tautological representations of the first and the second summands $\mathfrak{sl}(2)$ respectively. From this one can easily deduce that the representation of the algebra $\mathfrak{l}_{\alpha\beta}$ in the space \mathfrak{v}_α is not coisotropic.

Now we consider the case where the homogeneous space G/L is of type (Sym). Inequality (3.7) can be rewritten as

$$(3.17) \quad \dim \mathfrak{v}^1 \leq \left[\frac{(n-1)^2}{2} \right] + 4.$$

Representations φ^1 satisfying this condition and the condition $\rho^1 \subset S^2 \varphi^1$ are the spinor representation for $n = 9, 11$ and the half-spinor representation for $n = 10$.

The group L_α acts on the space \mathfrak{v} as $\text{Spin}(n-1) \otimes \text{Sp}(2m)$. Part (1) of Proposition 3 and the results of [5] imply that the homogeneous space G/L is weakly commutative if and only if $n = 9, 10$ and $m = 1$. Proposition 7 implies that both these spaces are commutative.

3.3.6. *The cases* $\mathfrak{l}^1 = G_2, F_4$, $\mu = \pi_1$. In these cases all representations of the algebra \mathfrak{l}^1 are orthogonal. Therefore, we consider homogeneous spaces of type (Sym) only.

In the case $\mathfrak{l}^1 = G_2$ inequality (3.7) can be rewritten as follows:

$$(3.18) \quad \dim \mathfrak{v}^1 \leq 14.$$

However, the algebra G_2 does not have representations φ^1 that are of required dimension and satisfy $\rho^1 \subset S^2\varphi^1$.

In the case $\mathfrak{l}^1 = F_4$ inequality (3.7) can be rewritten in the form

$$(3.19) \quad \dim \mathfrak{v}^1 \leq 2 \dim \mathfrak{v}_\alpha^1 \leq 36.$$

The only representation satisfying this restriction is the simplest representation φ^1 of dimension 26. We have $\rho^1 \subset S^2\varphi^1$. The algebra \mathfrak{l}_α^1 is isomorphic to \mathfrak{so}_8 . After restricting to \mathfrak{l}_α^1 , the space φ is isomorphic to the sum of the tautological representation, the spinor representation, and the two-dimensional trivial representation.

The representation of the normalizer $N_{L^1}(\mathfrak{l}_\alpha^1)$ in the space \mathfrak{v}^1 has an irreducible subrepresentation of dimension 24. Inequality (3.19) implies that the form \hat{a} vanishes on the corresponding subspace. The corollary to Proposition 4 implies that the homogeneous space G/L is not weakly commutative.

3.3.7. *The case $\rho^1(L^1) = \mathrm{SL}(2n + 1) \otimes F$, where $F \subset \mathrm{SL}(l)$, $2 \leq l \leq 2n + 1$.* Let $\mathfrak{l}^1 = \mathfrak{sl}(2n + 1) \times \mathfrak{f}$, where \mathfrak{f} is the tangent algebra of the group F and $\varphi^1 = \varphi^{11} \otimes \varphi^{12}$, $\rho^1 = \rho^{11} \otimes \rho^{12}$, where φ^{11} , ρ^{11} are irreducible representations of the algebra $\mathfrak{sl}(2n + 1)$, and φ^{12} , ρ^{12} are irreducible representations of the algebra \mathfrak{f} .

First we consider the case $n = 1$. The restrictions in Table 3 show that $\dim \rho^{12} = 3$. In this case the algebras \mathfrak{l}_α^1 and \mathfrak{f} are isomorphic. In the case (Alt) we have $F = \mathrm{SO}(3)$ (otherwise $\rho^1 \not\subset \bigwedge^2 \varphi^1$). Inequality (3.5) is rewritten in the form $\dim \mathfrak{v}_\alpha^1 \leq 4$. This implies $3 \dim \varphi^{12} = \dim \varphi \leq 8$; hence $\dim \varphi^{12} = 2$. The group $L_\alpha^1 = \mathrm{SL}(2)$ acts on \mathfrak{v}^1 as on $V(\pi_1) \oplus V(3\pi_1)$. Therefore, $\mathfrak{m}_\alpha^1 = V(\pi_1)$, $\mathfrak{v}_\alpha^1 = V(3\pi_1)$. Part (3) of Proposition 3 implies that the homogeneous space G/L is not weakly commutative for any m .

Now we consider the case (Sym). In the case where $F = \mathrm{SO}(3)$ as a linear group, inequality (3.7) shows that $\dim \varphi^1 \leq 8$; hence $\dim \varphi^{12} = 2$. This contradicts the condition $\rho^1 \subset S^2\varphi^1$. It remains to consider the case where $F = \mathrm{SL}(3)$. In this case inequality (3.7) gives $\dim \varphi^{11} \dim \varphi^{12} \leq 14$. This implies that φ^1 is the product of two tautological representations of the algebra \mathfrak{sl}_3 . The restriction of the representation φ^1 to the subalgebra $\mathfrak{l}_\alpha^1 \subset \mathfrak{sl}(3) \times \mathfrak{sl}(3)$ is the direct sum of the adjoint representation and the one-dimensional trivial representation. Therefore, $\dim \mathfrak{v}_\alpha^1 = 8$. The action $L_{\alpha\beta} : \mathfrak{v}_\alpha$ cannot be coisotropic. Part (3) of Proposition 3 implies that the homogeneous space G/L is not weakly commutative.

In the remaining part of 3.3.7 we assume that $n > 1$.

Now we consider the case of a homogeneous space of type (Sym). Inequality (3.7) can be rewritten in the form

$$(3.20) \quad 2 \dim \mathfrak{v}_\alpha^1 \leq (2n + 1 - l)^2 + (2n - 1 - l) + \dim \mathfrak{f} + \mathrm{rk} \mathfrak{f} + 2(2n + 1 - l)l + 4.$$

From this inequality one can deduce that

$$(3.21) \quad \dim \mathfrak{v}^1 \leq (2n + 1)^2 + 2n + 2.$$

The representation ρ^{11} is a subrepresentation of either $S^2\varphi^{11}$ or $\bigwedge^2 \varphi^{11}$. Similarly to 3.3.1, we obtain that the highest weight of φ^{11} is π_n . We have

$$(3.22) \quad \dim \varphi^{12} C_{2n+1}^n \leq (2n + 1)^2 + 2n + 2.$$

Therefore, $n \leq 3$. In the case $n = 3$ inequality (3.22) implies that $\dim \varphi^{12} = 2$. Since $\rho^{11} \subset \bigwedge^2 \varphi^{11}$, we have $\rho^{12} \subset \bigwedge^2 \varphi^{12}$. But the dimension of the latter representation equals 1, contradicting our assumptions. Therefore, $n = 2$.

In this case, $\rho^{12} \subset S^2\varphi^{12}$. The above inequality implies that $\dim \varphi^{12} \leq 3$. Since $\rho^{12} \subset S^2\varphi^{12}$, the algebra \mathfrak{f} is isomorphic to $\mathfrak{sl}(2)$. In the case where $\dim \varphi^{12} = 3$, the representation ρ^{12} is isomorphic to the five-dimensional irreducible representation of the algebra $\mathfrak{sl}(2)$, and in this case the inequality (3.7) does not hold (since the algebra

\mathfrak{l}_α^1 is isomorphic to $\mathfrak{sl}(2)$). Therefore, φ^{12}, ρ^{12} are the two-dimensional and the three-dimensional representations of $\mathfrak{sl}(2)$ respectively. Inequality (3.6) shows that $m = 1$.

The largest reductive subalgebra $\mathfrak{s}^1 \subset \mathfrak{l}_\alpha^1$ is isomorphic to $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$. The restriction of the representation φ^1 to the subalgebra \mathfrak{s} is isomorphic to $2\iota_2 + \iota_1 \otimes S^3\iota_2 + \iota_1 \otimes \iota_2 + S^3\iota_2$, where ι_1, ι_2 denote the tautological representation of the first and the second factor $\mathfrak{sl}(2)$ of \mathfrak{s}^1 . The representation $S^3\iota_2$ is not orthogonal. Therefore, $S^3\iota_2 \otimes \iota$ (where ι denotes the tautological representation of the algebra $\mathfrak{l}^2 \cong \mathfrak{sl}_2$) is a subrepresentation of \mathfrak{v}_α . The stabilizer of a generic vector of the representation $S^3\iota_2 \otimes \iota$ is trivial. Therefore, $\dim \mathfrak{s}_\beta \leq 4$ and $\dim \mathfrak{v}_\alpha \leq 6 + 2 \dim \mathfrak{l}_\alpha/\mathfrak{s} = 18$. To prove that the homogeneous space G/L is not weakly commutative, it remains to use the corollary to Proposition 4. Therefore, the case (Sym) is completely analyzed.

Now we consider the case where the homogeneous space G/L is of type (Alt). First, we prove that $\dim \varphi^{11} \leq (2n + 1)^2 + 3$. Indeed, inequality (3.3) shows that

$$(3.23) \quad \dim \varphi^{11} \dim \varphi^{12} \leq 2(\dim \mathfrak{l}^1 + 1) \leq 2(2n + 1)^2 + 2(\dim \varphi^{12})^2 - 2,$$

and we have the desired result.

We now show that $m = 1$. Assume that $m \geq 3$. Inequality (3.4) shows that

$$\dim \mathfrak{v}_\alpha^1 m_1 \leq \left[\frac{m_1^2}{2} \right] + (2n + 1 - l)^2 + l^2 - 1 + 2n + 2(2n + 1 - l)l = \left[\frac{m_1^2}{2} \right] + (2n + 1)^2 + (2n - 1),$$

where $m_1 = \min\{m, \dim \mathfrak{v}_\alpha^1\}$; hence

$$(3.24) \quad m_1 \left(\dim \mathfrak{v}_\alpha^1 - \frac{m_1}{2} \right) \leq 4n^2 + 6n.$$

The left-hand side of this inequality takes the minimum value at $m_1 = 3$ ($\dim \mathfrak{v}_\alpha^1 > 2$ since otherwise $\dim \mathfrak{v}^1 \leq 4$, which is impossible). The results of 3.3.1 show that $\dim \mathfrak{v}^{11} \geq \binom{2n+1}{n}$. We get

$$(3.25) \quad 3 \binom{2n+1}{n} \dim \mathfrak{v}^{12} \leq 8n^2 + 12n + 9.$$

One can easily see that in this case $n \leq 2$. If $n = 2$, then $\dim \mathfrak{v}^{12} = 2$ and ρ^1 is not a subrepresentation of $\wedge^2 \varphi^1$. We conclude that if the space G/L is weakly commutative, then $m = 1$.

Now we show that $n \leq 3$. The dimension of the representation φ^{11} is at most $\binom{2n+1}{n}$. As we indicated earlier, $\dim \varphi^{11} \leq (2n + 1)^2 + 3$ and the desired statement is proved.

Case $n = 2$. In this case $l = 3$ or 5 (according to the restrictions on ρ imposed by Table 3). First let $l = 5$. In this case the stabilizer of a generic element for the action $L : \mathfrak{z}^*$ is isomorphic to F , so that $\dim \mathfrak{v} \leq 2 \dim \mathfrak{v}_\alpha \leq 2(\dim F + \text{rk } F)$. On the other hand, $\dim \mathfrak{v} \geq 10 \dim \varphi^{12}$. The only possibility here is $\mathfrak{f} \cong \mathfrak{sl}_5$, $\dim \varphi^{12} = 5$. However, this contradicts the requirement $\rho^{12} \subset \varphi^{12} \otimes \varphi^{12}$. Therefore, $l = 3$.

It suffices to consider the case $F = \text{SL}(3)$, $\dim \varphi^{12} = 3$ (according to part (3) of Proposition 2). We can assume that as a subgroup in $\text{GL}(\mathfrak{v})$, L is $\wedge^2 \text{SL}(5) \otimes \text{GL}(3)$ (otherwise the condition $\dim \varphi^{11} \leq 5^2 + 3 = 28$ does not hold). Direct verification shows that the maximal reductive subalgebra $\mathfrak{s} \subset \mathfrak{l}_\alpha$ is isomorphic to $\mathfrak{t}(1) \times \mathfrak{sl}(2) \times \mathfrak{sl}(3)$ with $\mathfrak{t}(1)$ imbedded in $\mathfrak{sl}(5) \oplus \mathfrak{gl}(3)$ as the subspace $\{\text{diag}(-2x, -2x, -2x, 3x, 3x), \text{diag}(x, x, x)\}$. The restriction of the representation φ to \mathfrak{s} is isomorphic to the representation $(\iota_1 \otimes \iota_2 \otimes \iota_2^*)^0 + \iota_2^5 + \iota_2^{*-5} + (S^2\iota_2^*)^{-5}$, where ι_1, ι_2 are tautological representations of the algebras $\mathfrak{sl}(2), \mathfrak{sl}(3)$ respectively, and the upper index denotes the character determined by the action of the center. The representation $(S^2\iota_2^*)^{-5}$ is a subrepresentation of \mathfrak{v}_α . Therefore, the subalgebra \mathfrak{s}_β is a subalgebra of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Corollary 3 to Proposition 3 implies that $\dim \mathfrak{v}_\alpha \leq 20$. From the decomposition of the representation \mathfrak{v}_α into irreducible

components we see that $\iota_2^5 + \iota_2^{*-5}$ is a subrepresentation of \mathfrak{w}_α . Thus, \mathfrak{s}_β is a subalgebra of \mathfrak{sl}_2 , $\dim \mathfrak{v}_\alpha = 16$, and the representation of \mathfrak{s} in \mathfrak{v}_α is $(\iota_1 \otimes \iota_2 \otimes \iota_2^*)^0$. We see that \mathfrak{s}_β is a one-dimensional subalgebra and $\dim \mathfrak{v}_\alpha \leq 14$. We arrive at a contradiction.

Case $n = 3$. We recall that according to Proposition 4,

$$(3.26) \quad \dim \varphi^{11} \dim \varphi^{12} \leq 98 + 2 \dim F \leq 98 + 2 \min\{l, \dim \mathfrak{v}^{12}\}^2.$$

Since $\dim \varphi^{11} \geq \binom{7}{3} = 35$, we have $\dim \varphi^{12} \leq 3$. Therefore, $\dim \varphi^{11} \leq 45$, so that the highest weight of the representation φ^{11} is π_3 .

Let us show that $\mathfrak{f} \cong \mathfrak{sl}(2)$, $\dim \varphi^{12} = 2$, $\dim \rho^{12} = 3$. Indeed, if this is not the case, then $\dim \varphi^{12} = 3$ and $\dim \mathfrak{f} + \text{rk } \mathfrak{f} \leq 10$. Inequality (3.5) shows that

$$(3.27) \quad \dim \varphi^{11} \dim \varphi^{12} / 2 \leq \dim \mathfrak{v}_\alpha^1 \leq 10 + (7-l)^2 + (7-l) + 2l(7-l),$$

whence $l \leq 4$. However, it is easy to see that this contradicts the condition $\rho^{12} \subset S^2 \varphi^{12}$. Therefore, φ^{12} and ρ^{12} are irreducible representations of the algebra $\mathfrak{sl}(2)$ of dimension 2 and 3 respectively.

We can assume that the group L acts on the space \mathfrak{v} as the linear group $\bigwedge^3 \text{SL}(7) \otimes \text{GL}(2)$. The maximal reductive subalgebra $\mathfrak{s} \subset \mathfrak{l}_\alpha$ is isomorphic to $\mathfrak{sl}(4) \times \mathfrak{sl}(2) \times \mathfrak{t}(1)$ and $\dim \mathfrak{l}_\alpha / \mathfrak{s} = 12$. Denote by ι_1 and ι_2 the tautological representation of the algebra $\mathfrak{sl}(2)$ and $\mathfrak{sl}(4)$ respectively. The restriction of the representation φ to the subalgebra $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{l}$ is

$$\iota_1 \otimes \bigwedge^3 (S^2 \iota_1 + \iota_2) = \iota_1 + S^3 \iota_1 \otimes \iota_2 + \iota_1 \otimes \iota_2 + S^3 \iota_1 \otimes \bigwedge^2 \iota_2 + \iota_1 \otimes \bigwedge^2 \iota_2 + \iota_1 \otimes \iota_2^*.$$

The representation $S^3 \iota_1 \otimes \iota_2$ is not self-adjoint. The stabilizer of a generic element for this representation is isomorphic to $\mathfrak{sl}(2)$. Therefore, Corollary 3 of Proposition 3 implies the inequality

$$(3.28) \quad 4 + 2 + 2 \cdot 12 = 30 \geq \dim \mathfrak{v}_\alpha,$$

which is impossible according to the corollary of Proposition 4.

3.3.8. *The case $\rho^1(L^1) = \text{SO}(n) \otimes F$, where $F \subset \text{SL}(l)$, $2 \leq l < n$, and n is not divisible by 4.* The stabilizer L_α^1 is reductive and is isomorphic to $\text{SO}(n-l) \times F_1$, where F_1 is a generic stabilizer for the action $F : S^2(\mathbb{C}^l)$ (see [9]). In the cases we are interested in, the group F_1 has positive dimension only in the case $F = \text{SL}(2k+1)$, and in this case $F_1 = \text{SO}(2k+1)$.

Let $\varphi^1 = \varphi^{11} \otimes \varphi^{12}$, where φ^{11} is an irreducible representation of the algebra $\mathfrak{so}(n)$ and φ^{12} is an irreducible representation of the tangent algebra \mathfrak{f} of the group F .

First we consider the case where $F \neq \text{SL}(2k+1)$ as a linear group. According to Proposition 8, the algebra \mathfrak{l}_α^1 is not commutative. Therefore, $n \geq 6$. As shown in 3.3.5, we have the inequality $\dim \varphi^{11} \geq 2^{\lfloor \frac{n-1}{2} \rfloor}$. Moreover, $\dim \varphi^{12} \geq 2$. On the other hand, $\dim \mathfrak{l}_\alpha^1 + \text{rk } \mathfrak{l}_\alpha^1 \leq \lfloor (n-3)^2 / 2 \rfloor$.

In the case (Sym), inequality (3.7) shows that

$$(3.29) \quad 2^{\lfloor \frac{n+1}{2} \rfloor} \leq \left\lfloor \frac{(n-3)^2}{2} \right\rfloor + 4,$$

which is possible only if $n = 6$, where φ^{11} is the tautological representation of the group $\text{SL}(4)$ and φ^{12} is the tautological representation of the group $\text{SL}(2)$. In this case, ρ^1 is not a representation in $S^2 \varphi^1$.

In the case (Alt), we have the inequality

$$(3.30) \quad 2^{\lfloor \frac{n-1}{2} \rfloor} \leq \left\lfloor \frac{(n-3)^2}{2} \right\rfloor.$$

This inequality holds only in the cases $n = 6, 7$. Moreover, in these cases the inequality becomes an equality, which implies that $2 \dim \mathfrak{v}_\alpha^1 = \dim \mathfrak{v}^1$. However, as before, φ^{12} is the tautological representation of the algebra $\mathfrak{sl}(2)$, and φ^{11} is the tautological representation of the algebra $\mathfrak{sl}(4)$ or the spinor representation of the algebra $\mathfrak{so}(7)$. Let $\mathfrak{z} = \mathfrak{z}^1 \otimes \mathfrak{z}^2$, where $\mathfrak{z}^1, \mathfrak{z}^2$ are irreducible representations of l^{11}, l^{12} -modules. Choose generic forms α^1, α^2 in $\mathfrak{z}^{1*}, \mathfrak{z}^{2*}$ and consider the form $\alpha_0 = \alpha^1 \otimes \alpha^2 \in \mathfrak{z}^*$. According to 3.3.5, the form $\widehat{\alpha}_0$ is the product of two nondegenerate forms (a skew-symmetric and a symmetric one), whence itself is nondegenerate. Therefore, $\mathfrak{w}_\alpha^1 = 0$. This contradicts the weak commutativity of the space G/L .

It remains to consider the case $F = \mathrm{SL}(2k+1)$. In this case

$$\dim \varphi^{12} \geq \binom{2k+1}{k}, \quad \dim \mathfrak{t}_\alpha^1 + \mathrm{rk} \mathfrak{t}_\alpha^1 = \left\lceil \frac{(n-2k-1)^2}{2} \right\rceil + 2k^2 + 2k.$$

In the case (Sym), inequality (3.7) implies

$$(3.31) \quad \left\lceil \frac{(n-2k-1)^2}{2} \right\rceil + 2k^2 + 2k + 4 \geq 2^{\lfloor \frac{n-1}{2} \rfloor} \binom{2k+1}{k}.$$

This inequality holds only if $n = 6, k = 1$, in which case it becomes an equality. Therefore, $\dim \mathfrak{v}^1 = 2 \dim \mathfrak{v}_\alpha^1$. Similarly to the proof given earlier in this subsection, we can show that $\dim \mathfrak{w}_\alpha^1 \leq 4$, a contradiction.

In the case (Alt) inequality (3.5) shows that

$$(3.32) \quad (n-2k-1)^2 + (2k+1)^2 \geq 2^{\lfloor \frac{n-1}{2} \rfloor} \binom{2k+1}{k}.$$

Now one can easily see that $k = 1, n \leq 8$. The representation ρ_1 is a subrepresentation of $S^2\varphi_1$, and the tables in the reference section in [3] show that $n = 1, 2, 3$. Hence we conclude that the space G/L is not weakly commutative.

3.4. The case where \mathfrak{z} is reducible. All homogeneous spaces in Table 2 are weakly commutative (for lines 1–5 this follows from the fact that, as we have verified, the representation $L_\alpha : \mathfrak{v}$ is coisotropic; for the remaining line this follows from part (2) of Proposition 3). Now let $\mathfrak{z} = \mathfrak{z}_1 \oplus \mathfrak{z}_2$ be a decomposition into a direct sum of L -invariant subspaces. We note that no representation φ except for the tautological representations of the algebras $\mathfrak{sp}(2m), \mathfrak{sp}(2n), n, m = 1, 2$, enters Table 1 twice, and that for all homogeneous spaces in Table 1 the multiplicity of the subrepresentation ρ in $\bigwedge^2 \varphi$ equals 1 (this follows from (3.1)). Therefore, it remains to show that the homogeneous spaces with φ equal to the tensor product of the tautological representations of the algebras $\mathfrak{sp}(2m), \mathfrak{sp}(4), m = 1, 2$, and with ρ equal to the direct sum of the adjoint representations of these algebras, are not weakly commutative. This follows from part (3) of Proposition 3.

3.5. The case where the algebra \mathfrak{n} is k -step nilpotent, $k > 2$. We will need the following lemma.

Lemma 1. *Let G be an algebraic group acting linearly in the spaces V_1 and V_2 . Let β and α be generic elements in V_1^* and V_2^* respectively. The following two spaces coincide:*

- (1) *The space U_1 generated by the projections to V_2 of the vectors $d_{\alpha+\beta}\varphi$ for the functions $\varphi \in \mathbb{C}(V_1^* \oplus V_2^*)^G$ defined at the point $\alpha + \beta$.*
- (2) *The space U_2 generated by the vectors $d_\alpha\psi$ for the functions $\psi \in \mathbb{C}(V_2^*)^{G_\beta}$ defined at the point α .*

Proof. We note that for a rational function $\varphi \in \mathbb{C}(V_1^* \oplus V_2^*)$ defined at the point $\alpha + \beta$ the projection $d_{\alpha+\beta}\varphi$ to V_2^* equals $d_\alpha\psi$, where ψ is the rational function on V_2^* obtained

by restricting φ to the affine space $\{\beta\} \times V_2^*$ (this follows from the Leibniz formula for the derivative of a product). Therefore, $U_1 \subset U_2$.

To prove the opposite inclusion it suffices to prove that the spaces U_1 and U_2 have the same dimension. Indeed,

$$\dim U_2 = \text{tr. deg } \mathbb{C}(V_2^*)^{G_\beta} = \dim V_1^* - m_{G_\beta}(V_1^*),$$

$$\dim U_1 = \text{tr. deg } \mathbb{C}(V_1^* \oplus V_2^*)^G - \text{tr. deg } \mathbb{C}(V_1^*)^G = \dim V_1^* - m_G(V_1^* \oplus V_2^*) + m_G(V_1^*).$$

Therefore, it suffices to prove that

$$m_G(V_1^* \oplus V_2^*) = m_G(V_1^*) + m_{G_\beta}(V_2^*),$$

which is obvious. □

Let us introduce certain notation. Define the sequence of ideals $\mathcal{B}^i(\mathfrak{n})$ in the algebra \mathfrak{n} by the following inductive procedure: take $\mathcal{B}^0(\mathfrak{n}) = \mathfrak{n}$, and for $i > 0$ set $\mathcal{B}^i(\mathfrak{n}) = [\mathfrak{n}, \mathcal{B}^{i-1}(\mathfrak{n})]$. Next, by \mathfrak{n}_i , $i \geq 1$, we denote the L -submodule in $\mathcal{B}^{i-1}(\mathfrak{n})$ complementing $\mathcal{B}^i(\mathfrak{n})$. The fact that \mathfrak{n} is k -step nilpotent means that $\mathfrak{n}_i \neq 0$ if and only if $i \leq k$.

Now let $N \rtimes L/L$ be a weakly commutative homogeneous space such that the representation $L : \mathfrak{n}_1$ is irreducible and $\mathfrak{n}_3 \neq 0$. First we consider the case where the algebra \mathfrak{n} is 3-step nilpotent. We show (by contradiction) that in this case $\dim \mathfrak{n}_1 = 2$.

We can assume (by Corollary 1 of Proposition 2) that the L -modules $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$ are irreducible. Denote by $\varphi_1, \varphi_2, \varphi_3$ the corresponding representations of the algebra \mathfrak{l} .

Proposition 12. *The action of the commutator subgroup L' of the group L on one of the spaces $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$ has a dense orbit.*

Proof. We argue by contradiction. For a generic $\gamma \in \mathfrak{n}_3^*$, denote $L'_\gamma = L_\gamma$. Lemma 1 and Proposition 2 imply that the weak commutativity properties of the homogeneous spaces $N \rtimes L/L$ and $N \rtimes L'/L'$ are equivalent. Let $\varphi \in \mathbb{C}(\mathfrak{n}_1^*)^{L'}$, $\psi \in \mathbb{C}(\mathfrak{n}_2^*)^{L'}$ be nonconstant rational functions and $\alpha \in \mathfrak{n}_1^*$, $\beta \in \mathfrak{n}_2^*$ generic elements in the corresponding spaces. According to Proposition 2, for an arbitrary function $\gamma \in \mathfrak{n}_3^*$ we have $\gamma([d_\alpha \varphi, d_\beta \psi]) = 0$, which is impossible. □

We note that for a simple group L' the hypotheses of Proposition 12 are satisfied only for the tautological representations of the algebras $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$, the exterior square of the tautological representation of the algebra $\mathfrak{sl}(2n + 1)$, and the half-spinor representations of the algebra $\mathfrak{so}(10)$.

Proposition 13. *The stabilizer of a generic element for the action $L' : \mathfrak{n}_3^*$, where L' is as above, has a positive dimension.*

Proof. We argue by contradiction. According to Lemma 1 and Proposition 2, for generic $\alpha \in \mathfrak{n}_1^*$, $\beta \in \mathfrak{n}_2^*$, $\gamma \in \mathfrak{n}_3^*$ and arbitrary rational functions $\varphi \in \mathbb{C}(\mathfrak{n}_1^*)^{L_\gamma}$, $\psi \in \mathbb{C}(\mathfrak{n}_2^*)^{L_\gamma}$, we have $\gamma([d_\alpha \varphi, d_\beta \psi]) = 0$. By assumption, the vectors $d_\alpha \varphi$ generate the entire space \mathfrak{n}_1 , and the same is true for $d_\beta \psi$. We arrive at a contradiction. □

Now let \mathfrak{l}^2 be the kernel of the representation φ_2 , and \mathfrak{l}_1 an ideal in \mathfrak{l} complementing \mathfrak{l}^2 . Denote by φ_i^1, φ_i^2 , $i = 1, 3$, the representations of $\mathfrak{l}^1, \mathfrak{l}^2$ such that $\varphi_i = \varphi_i^1 \otimes \varphi_i^2$. Let $\mathfrak{n}_i^1, \mathfrak{n}_i^2$, $i = 1, 3$, be the spaces of the representations φ_i^1, φ_i^2 . One can easily see that $\varphi_1^2 = \varphi_3^2$. Moreover, the space \mathfrak{n}_1^2 admits an \mathfrak{l}^2 -invariant symmetric or skew-symmetric form.

Proposition 14. *We have $\dim \varphi_1^2 \leq 2$. In the case where this inequality becomes an equality, the representation φ_1^1 is a subrepresentation of $S^3 \varphi_1^2$. In the general case, denoting by τ the projection from $\mathfrak{n}_1 \otimes \mathfrak{n}_1 \otimes \mathfrak{n}_1$ to \mathfrak{n}_3 , $\tau(v_1 \otimes v_2 \otimes v_3) = [v_1, [v_2, v_3]]$, we have*

$\pi \circ \tau = 0$, where

$$\pi(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2.$$

In particular, $\pi(\wedge^3 \mathfrak{n}_1) = 0$.

Proof. The equality $\pi \circ \tau = 0$ is a reformulation of the Jacobi identity.

Let $\dim \varphi_1^2 \geq 3$, and let x, y, z be linearly independent vectors in the space \mathfrak{n}_1^2 , and a, b, c arbitrary vectors in the space \mathfrak{n}_1^1 . The space \mathfrak{n}_3 is identified with $\mathfrak{n}_3^1 \otimes \mathfrak{n}_1^2$. It is clear that $[a \otimes x, [b \otimes y, c \otimes z]] = \omega(y, z)\tau_1(a \otimes b \otimes c) \otimes x$, where τ_1 is a linear mapping from $\mathfrak{n}_1^1 \otimes \mathfrak{n}_1^1 \otimes \mathfrak{n}_1^1$ to \mathfrak{n}_3^1 , and ω is an ℓ^2 -invariant form on the space \mathfrak{n}_2^2 . Now the Jacobi identity implies $\tau_1 = 0$, which is impossible.

Finally, let $\dim \mathfrak{n}_1^2 = 2$, so that φ_1^2 is the tautological representation of the algebra $\mathfrak{sl}(2)$. The desired result follows from the fact that \mathfrak{n} is a Lie algebra. \square

Denote by \tilde{G} the quotient group of G by the subgroup $N_3 = (N, (N, N))$. The homogeneous space \tilde{G}/L is weakly commutative. We assume that $\dim \mathfrak{n}_2 = 1$, i.e., on the space \mathfrak{n}_1 there exists a nondegenerate skew-symmetric form ω such that $[x, y] = \omega(x, y)$ for all $x, y \in \mathfrak{n}_1$. The Jacobi identity implies that $\dim \mathfrak{n}_1 = 2$, which contradicts our assumptions.

Therefore, \tilde{G}/L is one of the spaces in Table 1. First, consider the case (Alt). In this case the representation $\dim \varphi^{12}$ is trivial (by Proposition 14). Table 1 shows that the group L' is simple. One of the spaces $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$ contains a dense L' -orbit (Proposition 13). Therefore, it remains to consider the triples in lines 1, 4, 5 (for $m = 1$).

For line 1, the representation $L' : \mathfrak{n}_3$ is different from the spinor (or half-spinor) representation and cannot have a dense orbit and a nontrivial stabilizer. Indeed, the highest weight of the representation φ_3 is congruent modulo the root lattice to the sum of highest weights of the representations φ_1 and φ_2 .

For line 5, L' -module \mathfrak{n}_3 is $R(\pi_1 + \pi_2)$ (this follows from Proposition 14) and the stabilizer of a generic element for the action $L' : \mathfrak{n}_3$ is of dimension zero in all cases except for the case $n = 3$. In this case, the L' -module \mathfrak{n}_3 is the coadjoint module. Choose an element $\gamma \in \mathfrak{n}_3^*$ with the stabilizer in \mathfrak{l} isomorphic to the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}'$. If the space G/L is weakly commutative, then $\gamma([d_\alpha f, d_\beta g]) = 0$ for generic $\alpha \in \mathfrak{n}_1^*$, $\beta \in \mathfrak{n}_2^*$ and $f \in \mathbb{C}(\mathfrak{n}_1^*)^t$, $g \in \mathbb{C}(\mathfrak{n}_2^*)^t$. Since the maximal torus in L' does not have a dense orbit in $\mathfrak{n}_1^*, \mathfrak{n}_2^*$, the vectors $d_\alpha f, d_\beta g$ span the corresponding spaces. Applying Lemma 1 and Proposition 2, we come to the contradiction with the fact that the projection $\mathfrak{n}_1 \otimes \mathfrak{n}_2 \rightarrow \mathfrak{n}_3$ is surjective.

Finally, consider line 4, i.e., the case where $L' = \text{SL}(7)$ and the weights of the representations φ_1 and φ_2 are π_3 and π_6 respectively. Proposition 13 shows that the highest weights of the representation of the group L' in \mathfrak{n}_3 is π_2 . Moreover, the mapping $x \otimes y \otimes z \mapsto [x, [y, z]]$ is nothing but

$$\begin{aligned} (x_1 \wedge x_2 \wedge x_3) \otimes (y_1 \wedge y_2 \wedge y_3) \otimes (z_1 \wedge z_2 \wedge z_3) &\mapsto \det(x_1, y_1, y_2, y_3, z_1, z_2, z_3)x_2 \wedge x_3 \\ &+ \det(x_2, y_1, y_2, y_3, z_1, z_2, z_3)x_3 \wedge x_1 + \det(x_3, y_1, y_2, y_3, z_1, z_2, z_3)x_1 \wedge x_2. \end{aligned}$$

However, this mapping does not satisfy the Jacobi identity.

Now we consider the case where the homogeneous space \tilde{G}/L is of type (Sym). Proposition 14 implies that $\dim \varphi_1^2 = 2$, $\varphi_3^1 \subset S^3 \varphi_1^1$. By Proposition 13, in the cases where \tilde{G}/L is in lines 14 and 16, the homogeneous space G/L is not weakly commutative. Proposition 12 and the results of [9] show that it remains to consider the homogeneous space \tilde{G}/L in line 15. The proof that in this case the homogeneous space G/L is not weakly commutative is similar to the case $L' = \text{SL}(7)$ considered before.

Now we prove that homogeneous space G/L with $\varphi^1(L') = \mathrm{SL}(2)$ and the 5-step nilpotent Lie algebra \mathfrak{n} is not weakly commutative.

For $\alpha \in \mathcal{B}^3(\mathfrak{n})^*$ we have $L_\alpha = \{e\}$. Lemma 1 shows that the set of projections to \mathfrak{n}_1 of the vectors $d_{\alpha_1+\alpha}(\varphi)$ with $\alpha_1 \in \mathfrak{n}_2^*$, $\varphi \in \mathbb{C}(n_1 \oplus n_3 \oplus n_4 \oplus n_5)^L$ is the entire space \mathfrak{n}_1 . This contradicts Proposition 2.

It remains to prove that the homogeneous space G/L with $L = \mathrm{SL}(2)$ and the 4-step nilpotent algebra \mathfrak{n} is weakly commutative. Let x, y be a basis of the space \mathfrak{n}_1 , $z = [x, y]$, $u = [z, x]$, $v = [z, y]$, $w = [x, v] = [u, y]$. We can assume that $\mathfrak{n}_2 = \langle z \rangle$, $\mathfrak{n}_3 = \langle u, v \rangle$, $\mathfrak{n}_4 = \langle w \rangle$. The decomposition $\mathfrak{n} = \sum_{i=1}^4 \mathfrak{n}_i$ defines a \mathbb{Z} -grading of the algebra \mathfrak{n} and the commutation relations remain the same as in the formulation of the theorem.

Now we prove that the algebra $U(\mathfrak{n})^L$ is commutative. This algebra is generated by the following three elements: $f_1 = z, f_2 = xv - yu, f_3 = w$ (since the graded algebra associated with $U(\mathfrak{n})^L$ is $\mathbb{C}(\mathfrak{n}^*)^L$, and the images of f_1, f_2, f_3 under the canonical mapping $U(\mathfrak{n})^L \rightarrow \mathbb{C}(\mathfrak{n}^*)^L$ generate the algebra $\mathbb{C}(\mathfrak{n}^*)^L$). Now the commutativity of $U(\mathfrak{n})^L$ is verified directly.

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