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FREDHOLM PROPERTY OF GENERAL ELLIPTIC PROBLEMS

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Dedicated to Ya. B. Lopatinskii on the occasion of his 100th birthday anniversary

Abstract. Linear elliptic problems in bounded domains are normally solvable with a finite-dimensional kernel and a finite codimension of the image, that is, satisfy the Fredholm property, if the ellipticity condition, the condition of proper ellipticity and the Lopatinskii condition are satisfied. In the case of unbounded domains these conditions are not sufficient any more. The necessary and sufficient conditions of normal solvability with a finite-dimensional kernel are formulated in terms of limiting problems. Adjoint operators to elliptic operators in unbounded domains are studied and the conditions in order for them to be normally solvable with a finite-dimensional kernel are also formulated by means of limiting problems. The properties of the direct and of the adjoint operators are used to prove the Fredholm property of elliptic problems in unbounded domains. Some special function spaces introduced in this work play an important role in the study of elliptic problems in unbounded domains.

1. Introduction

It is known that elliptic operators in bounded domains satisfy the Fredholm property, that is, the dimension of their kernel is finite, the image is closed, and the codimension of the image is also finite (see [2], [36], [42] and the references therein).

If we consider unbounded domains, then the ellipticity condition, proper ellipticity and the Lopatinskii condition are not sufficient, generally speaking, in order for the operator to satisfy the Fredholm property. Some additional conditions formulated in terms of limiting problems should be imposed. The typical result says that the operator satisfies the Fredholm property if and only if all its limiting operators are invertible. The question is about the classes of operators for which this result is applicable.

Limiting operators and their interrelation with solvability conditions and with the Fredholm property were first studied in [15], [20], [21] (see also [39]) for differential operators on the real axis, and later for some classes of elliptic operators in R^n [8], [25], [26], in cylindrical domains [9], [43], or in some specially constructed domains [6], [7]. Some of these results are obtained for the scalar case, some others for the vector case, under the assumption that the coefficients of the operator stabilize at infinity or without this assumption. This theory is also developed for some classes of pseudodifferential operators [12], [19], [30]–[34], [37], [38] and discrete operators [5], [35]. A survey of this literature is presented in the recent monograph [35].

In spite of the considerable progress in the understanding of properties of elliptic operators in unbounded domains, this question is not yet completely elucidated. The results existing in the literature are formulated for some classes of operators. For example, scalar elliptic problems in unbounded cylinders with constant coefficients at infinity are studied in the works cited above only for some classes of second-order operators. Moreover, in some cases it can be difficult to verify imposed conditions, and even simple problems

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may not belong to considered classes of abstract operators. The theory of md-elliptic operators, for example, appears to not be applicable to the Laplace operator in a strip [12].

In this work we will prove the Fredholm property of general elliptic problems in the Douglis–Nirenberg sense for differential operators in general domains only under the assumption that the coefficients of the operator and the boundary of the domain are sufficiently smooth. Our approach is based on a priori estimates of solutions of direct and adjoint operators and on the introduction of special function spaces.

As before, the results will be formulated in terms of the invertibility of limiting problems. The construction of limiting problems is rather simple. We should consider shifted coefficients and shifted domains and choose locally convergent subsequences (precise definitions will be given below). Limiting problems correspond to the operators with limiting coefficients considered in limiting domains. In the general case, we cannot verify their invertibility. Some examples for which the invertibility can be explicitly verified will be considered at the end of this work.

There are some classes of problems for which the Fredholm property can be studied without the analysis of the detailed structure of limiting problems. We briefly describe this approach. These results will be published in subsequent works. Consider elliptic problems with a parameter at infinity. This means that all limiting problems are elliptic problems with a parameter. Elliptic problems with a parameter are introduced by Agranovich and Vishik [3] for some classes of elliptic problems in bounded domains. Later, some more particular classes of elliptic problems were considered in unbounded domains in relation to sectorial operators (see [4], [11], [18], [23] and the references therein). Our approach consists in the investigation of elliptic problems, which are eliptic problems with a parameter at infinity. In this case, limiting operators are invertible for sufficiently large values of the parameter. Therefore, according to the results of this work, the operator satisfies the Fredholm property. The index of such problems is not necessarily zero. It can be found by the approximation of the original problems by problems in bounded domains for which the value of the index is known.

The formulas in the paper are numbered within subsections. When we refer to a formula within the same subsection, the section number is not indicated.

1.1. **Operators.** Consider the operators

$$A_i u = \sum_{k=1}^{N} \sum_{|\alpha|\le\alpha_{ik}} a_{ik}^{\alpha}(x) D^{\alpha} u_k, \quad i = 1,\dots,N,\ x \in \Omega, \tag{1.1}$$

$$B_j u = \sum_{k=1}^{N} \sum_{|\beta|\le\beta_{jk}} b_{jk}^{\beta}(x) D^{\beta} u_k, \quad i = 1,\dots,m,\ x \in \partial\Omega, \tag{1.2}$$

where $u = (u_1,\dots,u_N)$, and $\Omega \subset R^n$ is an unbounded domain that satisfies certain conditions given below. According to the definition of elliptic operators in the Douglis–Nirenberg sense [13] we suppose that

$$\alpha_{ik} \le s_i + t_k,\ i,k = 1,\dots,N, \quad \beta_{jk} \le \sigma_j + t_k,\ j = 1,\dots,m,\ k = 1,\dots,N$$

for some integers s_i, t_k, σ_j such that $s_i \le 0$, $\max\ s_i = 0$, $t_k \ge 0$.

Denote by E the space of vector-valued functions $u = (u_1,\dots,u_N)$, where u_j belongs to the Sobolev space $W^{l+t_j,p}(\Omega)$, $j = 1,\dots,N$, $1 < p < \infty$, l is an integer, $l \ge \max(0, \sigma_j+1)$,

$E = \prod_{j=1}^{N} W^{l+t_j,p}(\Omega)$. The norm in this space is defined as

$$\|u\|_E = \sum_{j=1}^{N} \|u_j\|_{W^{l+t_j,p}(\Omega)}.$$

The operator A_i acts from E to $W^{l-s_i,p}(\Omega)$ and the operator B_j acts from E to $W^{l-\sigma_j-1/p,p}(\partial\Omega)$. Denote

$$L = (A_1, \dots, A_N, B_1, \dots, B_m), \tag{1.3}$$

$$F = \prod_{i=1}^{N} W^{l-s_i,p}(\Omega) \times \prod_{j=1}^{m} W^{l-\sigma_j-1/p,p}(\partial\Omega).$$

Then $L : E \to F$.

We assume that

$$a_{ik}^{\alpha}(x) \in C^{l-s_i+\theta}(\bar{\Omega}), \quad b_{jk}^{\beta}(x) \in C^{l-\sigma_j+\theta}(\partial\Omega), \tag{1.4}$$

where $0 < \theta < 1$, and that these coefficients can be extended to R^n in such a way that the extended coefficients belong to the same spaces in R^n:

$$a_{ik}^{\alpha}(x) \in C^{l-s_i+\theta}(R^n), \quad b_{jk}^{\beta}(x) \in C^{l-\sigma_j+\theta}(R^n). \tag{1.5}$$

The notation C_0 will be used for functions with bounded support. It is also assumed that the operator is uniformly elliptic.

1.2. **Limiting problems.** To formulate the results of the work we should recall the notions of limiting operators and limiting domains. Limiting operators were first considered in [15], [20], [21] for differential operators on the real axis with quasi-periodic coefficients, and then for elliptic operators in R^n or for domains cylindrical or conical at infinity [8], [25], [26], [39].

In the general case limiting operators and domains are introduced in [44], [45]. Their construction can be briefly described as follows. Let $x_k \in \Omega$ be a sequence, which tends to infinity. Consider the shifted domains Ω_k corresponding to the shifted characteristic functions $\chi(x+x_k)$, where $\chi(x)$ is the characteristic function of the domain Ω. Consider a ball $B_r \subset R^n$ with center at the origin and with radius r. Suppose that for all k there are points of the boundaries $\partial\Omega_k$ inside B_r. If the boundaries are sufficiently smooth, then we can expect that from the sequence $\Omega_k \cap B_r$ we can choose a subsequence that converges to some limiting domain $\hat{\Omega}$. After that we take a larger ball and choose a convergent subsequence of the previous subsequence. The usual diagonal process allows us to extend the limiting domain to the whole space.

To define limiting operators we consider shifted coefficients $a^{\alpha}(x+x_k)$, $b_j^{\alpha}(x+x_k)$ and choose subsequences that converge to some limiting functions $\hat{a}^{\alpha}(x)$, $\hat{b}_j^{\alpha}(x)$ uniformly in every bounded set. The limiting operator is the operator with the limiting coefficients. Limiting operators considered in limiting domains constitute limiting problems. It is clear that the same problem can have a family of limiting problems depending on the choice of the sequence x_k and on the choice of both converging subsequences of domains and coefficients.

1.3. **Function spaces.** An important role in what follows is played by the choice of function spaces. Sobolev spaces $W^{s,p}$ proved to be very convenient in the study of elliptic problems in bounded domains. But more flexible spaces are needed for elliptic problems in unbounded domains. We need some generalization of the space $W^{s,p}$. More exactly, we mean such spaces which coincide with $W^{s,p}$ in bounded domains but have a prescribed behavior at infinity in unbounded domains. It turns out that such spaces can

be constructed for arbitrary Banach spaces of distributions (not only Sobolev spaces) as follows.

Consider first functions defined on R^n. As usual we denote by D the space of infinitely differentiable functions with compact support and by D' its dual. Let $E \subset D'$ be a Banach space; the inclusion is understood both in the algebraic and the topological sense. Denote by E_{loc} the collection of all $u \in D'$ such that $fu \in E$ for all $f \in D$. Let $\omega(x) \in D$, $0 \le \omega(x) \le 1$, $\omega(x) = 1$ for $|x| \le 1/2$, $\omega(x) = 0$ for $|x| \ge 1$.

Definition 1.3.1. E_q $(1 \le q \le \infty)$ is the space of all $u \in E_{loc}$ such that

$$\|u\|_{E_q} := \left(\int_{R^n} \|u(\cdot)\omega(\cdot - y)\|_E^q dy\right)^{1/q} < \infty, \quad 1 \le q < \infty,$$
$$\|u\|_{E_\infty} := \sup_{y \in R^n} \|u(\cdot)\omega(\cdot - y)\|_E < \infty.$$

In what follows we will also use an equivalent definition based on a partition of unity. It is proved that E_q is a Banach space. If Ω is a domain in R^n, then by definition $E_q(\Omega)$ is the space of restrictions of E_q to Ω with the usual norm of restrictions. It is easy to see that if Ω is a bounded domain, then

$$E_q(\Omega) = E(\Omega), \quad 1 \le q \le \infty.$$

In particular, if $E = W^{s,p}$, then we denote $W_q^{s,p} = E_q$ $(1 \le q \le \infty)$. It is proved that

$$W_p^{s,p} = W^{s,p} \quad (s \ge 0,\ 1 < p < \infty).$$

Hence the spaces $W_q^{s,p}$ generalize the Sobolev spaces $(q < \infty)$ and the Stepanov spaces $(q = \infty)$ (see [20], [21]).

1.4. **Normal solvability.** The following condition determines normal solvability of elliptic problems.

Condition NS. Any limiting problem

$$\hat{L}u = 0, \quad x \in \hat{\Omega}, \quad u \in E_\infty(\hat{\Omega})$$

has only the zero solution.

This is a necessary and sufficient condition for general elliptic operators considered in Hölder spaces to be normally solvable with a finite-dimensional kernel [44]. For scalar elliptic problems in Sobolev spaces it was proved in [45]. In [47] we generalize these results for elliptic systems. More precisely, we prove that the elliptic operator L is normally solvable and has a finite-dimensional kernel in the space $W_\infty^{l,p}$ $(1 < p < \infty)$ if and only if Condition NS is satisfied.

In this work we prove normal solvability of adjoint operators. This result is based on a priori estimates. To obtain the estimates of adjoint operators we consider a modified model problem in the half-space where we take the principal terms of operators (2.1), (2.2) and replace the differential operators by some pseudodifferential operators (cf. [14]). In [36] it is proved that these operators are invertible. In the case considered in this work where $l \ge \max(0, \sigma_j + 1)$, that is, the operators act in Sobolev spaces with nonnegative exponents, the proof of the invertibility of the operators can be simplified. We use the approach developed in [42] for differential operators. It allows us to obtain a priori estimates of adjoint operators for general elliptic problems in unbounded domains. In some particular cases (Hilbert spaces, scalar operators) estimates of this type were obtained in [27]. We should note however that these estimates are not sufficient to prove normal solvability and finiteness of the kernel for operators in unbounded domains. In this case we need to introduce an additional condition on limiting problems and obtain some special a priori estimates.

Similarly to Condition NS for the direct operators we introduce Condition NS* for the adjoint operators.

Condition NS*. Any limiting homogeneous problem $\hat{L}^* v = 0$ does not have nonzero solutions in $(F^*(\hat{\Omega}))_\infty$, where $\hat{L}^*$ is the operator adjoint to the limiting operator $\hat{L}$, and $\hat{\Omega}$ is a limiting domain.

We obtain a priori estimates for adjoint operators and prove that if Condition NS* is satisfied, then the operator $L^* : (F^*(\Omega))_\infty \to (E^*(\Omega))_\infty$ is normally solvable with a finite-dimensional kernel.

1.5. **Fredholm property.** We have mentioned above that the same property holds for the operator L if Condition NS is satisfied. However this does not mean that the operator L satisfies the Fredholm property, because we consider the adjoint operator L^* not as acting in the dual spaces from $(F_\infty(\Omega))^*$ to $(E_\infty(\Omega))^*$ but from $(F^*(\Omega))_\infty$ to $(E^*(\Omega))_\infty$. These spaces are different. Detailed analysis of these spaces and of the properties of the operators will allow us to prove that, indeed, Condition NS and Condition NS* imply that the operator L satisfies the Fredholm property in spaces $W^{s,p}$, $1 < p < \infty$, and $W_q^{s,p}$ for some q.

One of the key properties of the function spaces is given by the following relations:

$$(E^*)_\infty = (E_1)^*, \;\; (F^*)_\infty = (F_1)^*.$$

Simplifying the situation we can say that it will allow us to establish a relation between the operators

$$(L^*)_\infty : (E^*)_\infty \to (F^*)_\infty, \quad (L_\infty)^* : (E_\infty)^* \to (F_\infty)^*.$$

We see already here that the same differential expressions considered in different function spaces should be considered as different operators. We come here to the notion of local operators and realization of operators. An operator L is called local if for any $u \in E$ with a bounded support, $\operatorname{supp} Lu \subset \operatorname{supp} u$. Differential operators satisfy obviously this property. If an operator L is local, then the adjoint operator is also local.

For local operators we can define their realization in different spaces,

$$L_q : E_q \to F_q, \;\; 1 \le q \le \infty.$$

We will consider also the operator $L_D : E_D \to F_D$, where E_D and F_D are the spaces obtained as a closure of functions from D in the norms of the spaces E_∞ and F_∞, respectively.

We will first prove that if Conditions NS and NS* are satisfied, then the operator L_D satisfies the Fredholm property. It will allow us to prove next that the operator L_∞ is Fredholm, and then that L_q is Fredholm. The exact formulations of the results are given in Section 5.

2. The space $W_q^{s,p}$

Let $E = W^{s,p}$ be the Sobolev–Slobodetskii space, where $-\infty < s < \infty$, $1 < p < \infty$. In this section we construct and study the space $E_q = W_q^{s,p}$ $(1 \le q \le \infty)$. We do not use any specific properties of the space $W^{s,p}$. Therefore all the results can be generalized to any Banach space of distributions (these generalizations will be published elsewhere). In this paper we confine ourselves to the space $W^{s,p}$ since only these spaces are used here for the elliptic problems under consideration.

2.1. **Systems of functions.**

Definition 2.1.1. A partition of unity is a sequence $\{\phi_i\}$, $i = 1, 2, \dots$ of functions $\phi_i \in D$, $\phi_i(x) \geq 0$ such that

$$\sum_{i=1}^{\infty} \phi_i(x) = 1, \quad x \in R^n.$$

Condition 2.1.2. Let $\{\phi_i\}$, $i = 1, 2, \dots$ be a sequence of functions $\phi_i \in D$. For any i there exist no more than N functions ϕ_j such that $\operatorname{supp} \phi_j \cap \operatorname{supp} \phi_i \neq \emptyset$.

Everywhere below we consider partitions of unity for which Condition 2.1.2 is satisfied.

Definition 2.1.3. Two systems of functions $\{\phi_i\}$, $\{\psi_j\}$, $i = 1, 2, \dots$, $j = 1, 2, \dots$, $\phi_i \in D$, $\psi_j \in D$ are called *equivalent* if there exists a number N such that:
for any i there exist no more than N functions ψ_j such that $\operatorname{supp} \psi_j \cap \operatorname{supp} \phi_i \neq \emptyset$;
for any j there exist no more than N functions ϕ_i such that $\operatorname{supp} \phi_i \cap \operatorname{supp} \psi_j \neq \emptyset$.

Proposition 2.1.4. *The equivalence relation introduced by Definition 2.1.3 is reflexive, symmetric, and transitive.*

The proof is standard. In what follows we consider the equivalence class with the representative which corresponds to a covering of R^n by cubes with centers in some lattice.

We will also use systems of functions satisfying the following condition.

Condition 2.1.5. The system of functions ϕ_i satisfies the following conditions:
1. $\phi_i(x) \geq 0$, $\phi_i \in D$,
2. Condition 2.1.2 is satisfied,
3. $\sup_i \|\phi_i\|_M < \infty$,
4. $\phi(x) = \sum_{i=1}^{\infty} \phi_i(x) \geq m > 0$ for some constant m,
5. the following estimate holds:

$$\sup_x |D^\alpha \phi(x)| \leq M_\alpha,$$

where D^α denotes the operator of differentiation, and the M_α are positive constants.

Here $\|\phi\|_M$ is the norm of a multiplier ϕ:

$$\|\phi u\|_E \leq \|\phi\|_M \, \|u\|_E, \quad \forall u \in E.$$

For $E = W^{s,p}$ it is known that $\|\phi\|_M \leq \|\phi\|_{C^{[|s|]+1}}$, where K is a positive constant.

For the partitions of unity $\{\phi_i\}$ we always suppose that $\sup_i \|\phi_i\|_M < \infty$.

2.2. **The space E_q.**

Definition 2.2.1. E_{loc} is the space of all $u \in D'$ such that $fu \in E$ for all $f \in D$.

Definition 2.2.2. Let $\{\phi_i\}$, $i = 1, 2, \dots$, be a partition of unity. Then E_q is the space of all $u \in E_{loc}$ such that

$$\sum_{i=1}^{\infty} \|\phi_i u\|_E^q < \infty,$$

where $1 \leq q < \infty$, with the norm

$$\|u\|_{E_q} = \left(\sum_{i=1}^{\infty} \|\phi_i u\|_E^q \right)^{1/q}.$$

In what follows we consider two normed spaces to be equal if they are linearly isomorphic and their norms are equivalent.

Proposition 2.2.3. *Let $\{\phi_i^1\}$ and $\{\phi_i^2\}$ be two partitions of unity. Suppose that E_q^1 and E_q^2 are the spaces E_q corresponding to $\{\phi_i^1\}$ and $\{\phi_i^2\}$, respectively. If the partitions of unity are equivalent, then $E_q^1 = E_q^2$.*

Proof. Let $u \in E_q^2$. We have

$$\phi_i^1 u = \phi_i^1 \sum_{j=1}^{\infty} \phi_j^2 u = \sum_{j'} \phi_i^1 \phi_{j'}^2 u,$$

where j' are all the numbers j such that $\operatorname{supp} \phi_i^1 \cap \operatorname{supp} \phi_j^2 \neq \emptyset$. By Definition 2.1.3 the number of such j' is no more than N. We have the estimate

$$\|\phi_i^1 u\|_E^q \leq \left(\sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E \right)^q.$$

Let $a_j \geq 0,\ j = 1, \ldots, m$. Then from convexity of the function s^q we obtain the estimate

$$\left(\sum_{j=1}^{m} a_j \right)^q = m^q \left(\sum_{j=1}^{m} \frac{1}{m} a_j \right)^q \leq m^{q-1} \sum_{j=1}^{m} a_j^q.$$

Therefore

$$\left(\sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E \right)^q \leq m^{q-1} \sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E^q,$$

where m is the number of j'. Since $m \leq N$, then

$$\|\phi_i^1 u\|_E^q \leq N^{q-1} \sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E^q = N^{q-1} \sum_{j=1}^{\infty} \|\phi_i^1 \phi_j^2 u\|_E^q.$$

Let k be a positive integer. We have

$$\begin{aligned} \sum_{i=1}^{k} \|\phi_i^1 u\|_E^q &= N^{q-1} \sum_{i=1}^{k} \sum_{j=1}^{\infty} \|\phi_i^1 \phi_j^2 u\|_E^q = N^{q-1} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \|\phi_i^1 \phi_j^2 u\|_E^q, \\ \sum_{i=1}^{k} \|\phi_i^1 \phi_j^2 u\|_E^q &= \sum_{i'} \|\phi_i^1 \phi_j^2 u\|_E^q \leq \sum_{i'} \|\phi_{i'}^1\|_M^q \|\phi_j^2 u\|_E^q, \end{aligned} \tag{2.1}$$

where the i' are those i for which $\operatorname{supp} \phi_i^1 \cap \operatorname{supp} \phi_j^2 \neq \emptyset$. The number of such i' is less than or equal to N. Let

$$K_j = \sup_i \|\phi_i^j\|_M, \quad j = 1, 2.$$

Then

$$\sum_{i=1}^{k} \|\phi_i^1 \phi_j^2 u\|_E^q \leq N K_1^q \|\phi_j^2 u\|_E^q.$$

It follows from (2.1) that

$$\sum_{i=1}^{k} \|\phi_i^1 u\|_E^q \leq N^q K_1^q \sum_{j=1}^{\infty} \|\phi_j^2 u\|_E^q = N^q K_1^q \|u\|_{E_q^2}^q.$$

From this we obtain

$$\sum_{i=1}^{\infty} \|\phi_i^1 u\|_E^q \leq N^q K_1^q \|u\|_{E_q^2}^q.$$

Hence $u \in E_q^1$ and

$$\|u\|_{E_q^1} \le NK_1\|u\|_{E_q^2}, \quad E_q^2 \subset E_q^1.$$

Similarly we get

$$\|u\|_{E_q^2} \le NK_2\|u\|_{E_q^1}, \quad E_q^1 \subset E_q^2.$$

The proposition is proved. □

Proposition 2.2.4. *The space E_q is complete.*

Proof. Consider a fundamental sequence u_m in the space E_q. Then for any $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$\sum_{i=1}^{\infty} \|(u_k - u_m)\phi_i\|_E^q \le \epsilon \tag{2.2}$$

for any $k, m \ge N(\epsilon)$. Denote $\psi_n = \sum_{i=1}^n \phi_i$. Let Ψ_n be an infinitely differentiable function with a finite support such that $\Psi_n = 1$ in the support of ψ_n. Since E is a Banach space and the sequence $\Psi_n u_m$ is fundamental with respect to m for any fixed n, then $\Psi_n u_m \to v_n$ in E as $m \to \infty$. Obviously $\psi_n u_m \to \psi_n v_n$ in E as $m \to \infty$.

Consider a sequence n_j, $n_j \to \infty$ as $j \to \infty$. We construct a sequence of limiting functions v_{n_j} such that

$$\|\psi_{n_j}(u_m - v_{n_j})\|_E \to 0 \ \text{ as } \ m \to \infty,$$

and for any $j_2 > j_1$,

$$\psi_{n_{j_1}} v_{n_{j_1}} = \psi_{n_{j_1}} v_{n_{j_2}}.$$

Therefore we have constructed a limiting function v defined in R^n. It coincides with v_j in the support of ψ_j. We have

$$\|\psi_{n_j}(u_m - v)\|_E \to 0 \ \text{ as } \ m \to \infty. \tag{2.3}$$

We note that for any $\delta > 0$ there exists $N(\delta)$ and $i_0(\delta)$ such that

$$\sum_{i=i_0(\delta)}^{\infty} \|u_k\phi_i\|_E^q \le \delta \tag{2.4}$$

for any $k \ge N(\delta)$. Indeed, we choose $N(\delta)$ such that

$$\sum_{i=1}^{\infty} \|(u_k - u_m)\phi_i\|_E^q \le C_q\delta \tag{2.5}$$

for any $k, m \ge N(\delta)$. Here $C_q = 2^{-q}$. On the other hand, for a fixed m we can choose $i_0(\delta)$ such that

$$\sum_{i=i_0(\delta)}^{\infty} \|u_m\phi_i\|_E^q \le C_q\delta \tag{2.6}$$

since the corresponding series converges. From (2.5) it follows that for m fixed and any $k \ge N(\delta)$,

$$\sum_{i=i_0(\delta)}^{\infty} \|(u_k - u_m)\phi_i\|_E^q \le C_q\delta. \tag{2.7}$$

From (2.6) and (2.7) we obtain (2.4).

We prove next that

$$\sum_{i=i_0(\delta)}^{\infty} \|v\phi_i\|_E^q \le \delta, \tag{2.8}$$

where $i_0(\delta)$ is the same as in (2.4). Suppose that this estimate is not true. Then there exists $i_1(\delta)$ such that

$$\sum_{i=i_0(\delta)}^{i_1(\delta)} \|v\phi_i\|_E^q > \delta. \tag{2.9}$$

On the other hand from (2.3) we have

$$\sum_{i=i_0(\delta)}^{i_1(\delta)} \|(u_m - v)\phi_i\|_E^q \to 0 \ \text{ as } \ m \to \infty.$$

This convergence and (2.9) contradict (2.4).

From (2.3), (2.4), and (2.8) we conclude that u_m converges to v in E_q. The proposition is proved. □

Proposition 2.2.5. *Let $u_k = \sum_{i=1}^k u\phi_i$. Then $u_k \to u$ in E_q for $1 \le q < \infty$.*

Proof. We have

$$\|u - u_k\|_{E_q}^q = \sum_{i=1}^{\infty} \|\phi_i(u - u_k)\|_E^q = \sum_{i=1}^{\infty} \|\phi_i \sum_{j=k+1}^{\infty} u\phi_j\|_E^q = \sum_{i=k'}^{\infty} \|\phi_i \sum_{j=k+1}^{\infty} u\phi_j\|_E^q = \cdots,$$

where the external sum is taken over all i such that $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j \neq \emptyset$ for all $j \ge k+1$. The value k' depends on k, and $k' \to \infty$ as $k \to \infty$,

$$\cdots = \sum_{i=k'}^{\infty} \|\phi_i \sum_{j'} u\phi_{j'}\|_E^q \le \cdots,$$

where j' denotes all j such that $\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_i \neq \emptyset$ for a given i. Since the number of such j is uniformly bounded, we have the estimate

$$\cdots \le C \sum_{i=k'}^{\infty} \|u\phi_i\|_E^q.$$

The last sum converges to zero as $k \to \infty$. The proposition is proved. □

Corollary 2.2.6. *Infinitely differentiable functions with bounded supports are dense in E_q, $1 \le q < \infty$.*

Proof. It is sufficient to note that D is dense in E, and $u_k \in E$. □

Definition 2.2.7. Let $\{\phi_i\}$, $i = 1, 2, \ldots$ be a system of functions satisfying Condition 2.1.5, and let E_q be the space of all $u \in E_{loc}$ such that

$$\sum_{i=1}^{\infty} \|\phi_i u\|_E^q < \infty,$$

where $1 \le q < \infty$, with the norm

$$\|u\|_{E_q} = \left(\sum_{i=1}^{\infty} \|\phi_i u\|_E^q \right)^{1/q}.$$

The following proposition can be easily proved.

Proposition 2.2.8. *The spaces in Definitions 2.2.2 and 2.2.7 coincide.*

We introduce now one more definition of the norm in the space E_q. Let the norm be given by the equality

$$\|u\|_{E_q} = \left(\int_{R^n} \|u(\cdot)\phi(\cdot - y)\|_E^q dy\right)^{1/q}, \quad \phi \in D. \tag{2.10}$$

We show that this norm is equivalent to the norm defined through a partition of unity. We note first of all that the function

$$s(y) = \|u(\cdot)\phi(\cdot - y)\|_E^q$$

is continuous. Indeed,

$$|s^{1/q}(y) - s^{1/q}(y_0)| \le \|u(\cdot)(\phi(\cdot - y) - \phi(\cdot - y_0))\|_E \to 0 \text{ as } y \to y_0$$

by the properties of multipliers.

We have

$$\|u\|_{E_q}^q = \int_{R^n} s(y)dy = \sum_{i=1}^{\infty} \int_{Q_i} s(y)dy,$$

where the Q_i are unit cubes of the square lattice in R^n,

$$\int_{Q_i} s(y)dy = s(y_i)$$

for some $y_i \in Q_i$ since $s(y)$ is continuous. Hence

$$\|u\|_{E_q}^q = \sum_{i=1}^{\infty} s(y_i). \tag{2.11}$$

This equality is obtained without specific assumptions on the function $\phi(x)$. Suppose now that it equals 1 in the ball of the radius $r = \sqrt{n}$, and 0 outside of the ball with the radius $2r$. Then for any $y_i \in Q_i$,

$$\phi(x - y_i) = 1, \quad x \in Q_i.$$

Therefore the system of functions $\phi_i(x) = \phi(x - y_i)$ satisfies the following conditions:

(1) $m \le \sum_{i=1}^{\infty} \phi_i(x) \le M$ for all $x \in R^n$ and some positive constants m and M,

(2) for each $x \in R^n$ there exists a finite number of functions ϕ_i different from zero at this point. The estimate of this number is independent of x.

Hence the norm (2.11) is equivalent to the norm defined with any other system of functions equivalent to ϕ_i.

We have proved the following proposition.

Proposition 2.2.9. *The norm* (2.10) *is equivalent to the norm in Definition 2.2.2.*

The relation between the space $W_q^{s,p}$ and the corresponding Sobolev–Slobodetskii space is given by the following theorem:

Theorem 2.2.10. *Let s be a real nonnegative number, $1 < p < \infty$. Then $W_p^{s,p} = W^{s,p}$.*

The proof will be published elsewhere.

Consider now the case $q = \infty$.

Definition 2.2.11. Let $\{\phi_i\}$ be a partition of unity. E_∞ is the space of all functions $u \in E_{loc}$ such that

$$\sup_i \|\phi_i u\|_E < \infty,$$

with the norm

$$\|u\|_{E_\infty} = \sup_i \|\phi_i u\|_E.$$

It is proved that Propositions 2.2.3 and 2.2.4 are true also for $q = \infty$.

Other equivalent definitions of the space can be done.

Definition 2.2.12. Let $\eta(x) \in D$ satisfy the following conditions:

1. $0 \le \eta(x) \le 1, \ \ x \in R^n$,
2. $\eta(x) = 1$ in the cube $|x_i| \le a_1$, $i = 1, 2, \dots, n$,
3. $\eta(x) = 0$ outside the cube $|x_i| \le a_2$, $i = 1, 2, \dots, n$, where a_1 and a_2 are given numbers, $a_1 < a_2$.

Denote

$$\eta_y(x) = \eta(x - y), \ \ y \in R^n.$$

The space E_∞ is the set of all $u \in E_{loc}$ such that

$$\sup_{y \in R^n} \|\eta_y u\|_E < \infty.$$

The norm in this space is given by the relation

$$\|u\|_{E_\infty} = \sup_{y \in R^n} \|\eta_y u\|_E.$$

In what follows we use the space $E(G)$, where G is a domain in R^n. The space $E(G)$ is defined as the set of all generalized functions from D'_G which are restrictions to G of generalized functions from E. The norm in this space is

$$\|u\|_{E(G)} = \inf \|v\|_E,$$

where the infimum is taken over all those generalized functions $v \in E$ whose restriction to G coincides with u.

Definition 2.2.13. The space E_∞ is the set of all $u \in E_{loc}$ such that

$$\sup_{y \in R^n} \|u_y\|_{E(G_y)} < \infty, \tag{2.12}$$

where u_y is a restriction of u to G_y, $G \subset R^n$ is a bounded domain containing the origin, and G_y is a shifted domain: the characteristic function of G_y is $\chi(x - y)$, where $\chi(x)$ is the characteristic function of G. The norm in E_∞ is given by

$$\|u\|_{E_\infty} = \sup_{y \in R^n} \|u_y\|_{E(G_y)}.$$

It is proved that the spaces in Definitions 2.2.11–2.2.13 coincide. The same is true if instead of cubes in Definition 2.2.12 we take balls.

2.3. **Bounded sequences in E_∞.**

Definition 2.3.1. A sequence $u_k \in E_{loc}$ is called locally weakly convergent to $u \in E_{loc}$ if for any $\phi \in D$,

$$\phi u_k \to \phi u \ \text{ weakly in } E.$$

Lemma 2.3.2. *If a sequence $u_k \in E_\infty$ is bounded in E_∞ and locally weakly convergent to u, then $u \in E_\infty$.*

Proof. We use Definition 2.2.11 of the space E_∞. Let $\{\phi_i\}$ be a partition of unity. Then $u \in E_\infty$ if

$$\sup_i \|\phi_i u\|_E < \infty.$$

Suppose that $u \notin E_\infty$. Then there is a subsequence i_k of i such that

$$\|\phi_{i_k} u\|_E \to \infty \ \text{ as } \ i_k \to \infty. \tag{3.1}$$

A set in a Banach space is bounded if and only if any functional from the dual space is bounded on it. Hence there exists a functional $F \in E^*$ such that

$$F(\phi_{i_k} u) \to \infty \text{ as } i_k \to \infty.$$

Since u_l is locally weakly convergent to u, then

$$F(\phi_{i_k} u_l) \to F(\phi_{i_k} u) \text{ as } l \to \infty$$

for any i_k. Therefore we can choose l_k such that

$$|F(\phi_{i_k} u_{l_k}) - F(\phi_{i_k} u)| < 1.$$

It follows from (3.1) that

$$F(\phi_{i_k} u_{l_k}) \to \infty \text{ as } i_k \to \infty. \tag{3.2}$$

On the other hand, by assumption u_k is bounded in E_∞. Hence

$$\|u_k\|_{E_\infty} \le M, \quad \|\phi_{i_k} u_k\|_E \le M.$$

This contradicts (3.2). The lemma is proved. □

We present without proof the following theorem.

Theorem 2.3.3. *If $\{u_k\}$, $k = 1, 2, \dots$ is a bounded sequence in E_∞, then there exists a subsequence u_{k_i} of u_k and $u \in E_\infty$ such that*

$$u_{k_i} \to u \text{ locally weakly and in } D'.$$

2.4. **Dual spaces.** For the space E^* dual to E we can define $(E^*)_q$ as is done above for the space E. For example the norm in the space $(E^*)_\infty$ is given by

$$\|v\|_{(E^*)_\infty} = \sup_i \|\phi_i v\|_{E^*}, \tag{4.1}$$

where ϕ_i is a partition of unity.

In the application to elliptic problems here we are interested in the spaces dual to E_∞.

Theorem 2.4.1. *The spaces $(E^*)_\infty$ and $(E_1)^*$ coincide.*

Proof. Let $v \in (E_1)^*$. Then for any $u \in E_1$,

$$\langle v, u\rangle \le \|v\|_{(E_1)^*} \|u\|_{E_1}.$$

Since $v \in E^*_{loc}$ and $u \in E$, then $\langle \phi_i v, u\rangle$ is defined and

$$|\langle \phi_i v, u\rangle| = |\langle v, \phi_i u\rangle| \le \|v\|_{(E_1)^*} \|\phi_i u\|_{E_1} \le M\|v\|_{(E_1)^*} \|u\|_E.$$

Here $\{\phi_i\}$ is a partition of unity, and $\sup_i \|\phi_i\|_M < \infty$.

Therefore

$$\|\phi_i v\|_{E^*} \le M\|v\|_{(E_1)^*}.$$

Consequently,

$$\|v\|_{(E^*)_\infty} \le M\|v\|_{(E_1)^*}.$$

Suppose that $v \in (E^*)_\infty$. Then $v \in E^*_{loc}$. Let $u \in E_1$, $u_k = \sum_{i=1}^k \phi_i u$. Then $u_k \in E$, and

$$|\langle v, u_k\rangle| = |\langle v, \sum_{i=1}^k \phi_i u\rangle| \le \sum_{i=1}^k |\langle v, \phi_i u\rangle| = \sum_{i=1}^k |\langle \phi_i v, \psi_i u\rangle| \le \sum_{i=1}^k \|\phi_i v\|_{E^*} \|\psi_i u\|_E$$
$$\le \|v\|_{(E^*)_\infty} \sum_{i=1}^\infty \|\psi_i u\|_E \le M\|v\|_{(E^*)_\infty} \|u\|_{E_1}.$$

Here $\psi_i \in D$, $\psi_i = 1$ in $\operatorname{supp}\phi_i$. We suppose that the system of functions ψ_i satisfies Condition 2.2.2. We can pass to the limit in the last estimate as $k \to \infty$. Therefore v can be considered as a functional on E_1, and

$$\|v\|_{(E_1)^*} \le M\|v\|_{(E^*)_\infty}.$$

The theorem is proved. □

We note that functionals from both spaces $(E^*)_\infty$ and $(E_1)^*$ are considered in Theorem 2.4.1 on functions from E_1.

Lemma 2.4.2. *Let $\phi \in (E_\infty)^*$, $u_n = \sum_{i=1}^n u\phi_i$, where $u \in E_\infty$ and ϕ_i is a partition of unity. Then the limit $\lim_{n\to\infty}\phi(u_n)$ exists.*

Proof. We have

$$\|u_n\|_{E_\infty} = \sup_j \|u_n\phi_j\|_E = \sup_j \Big\|\Big(\sum_{i=1}^n u\phi_i\Big)\phi_j\Big\|_E$$
$$\le \sup_j \left(\sum_{i:\operatorname{supp}\phi_i\cap\operatorname{supp}\phi_j\ne\emptyset} \|u\phi_i\phi_j\|_E\right) \le MN\sup_j\|u\phi_j\| = MN\|u\|_{E_\infty}.$$

Suppose that the limit $\phi(u_n)$ does not exist. Then there exist two subsequences u_{nk} and u_{nm} such that

$$\phi(u_{nk}) \to C_1,\ \ \phi(u_{nm}) \to C_2,\ \ C_1 \ne C_2.$$

We will construct a bounded sequence in E_∞ such that the functional ϕ will be unbounded on it. This contradiction will prove the existence of the limit.

Without loss of generality we can assume that $C_1 > C_2$. For all k and m sufficiently large,

$$\phi(u_{nk}) \ge C_1 - \epsilon,\ \ \phi(u_{nm}) \le C_2 + \epsilon.$$

For $\epsilon \le (C_1 - C_2)/4$,

$$\phi(u_{nk} - u_{nm}) \ge \frac{C_1 - C_2}{2}\ (= a > 0).$$

We take k and m such that this estimate is satisfied and denote $v_1 = u_{nk} - u_{nm}$. We note that

$$u_{nk} - u_{nm} = \sum_{i=n_m}^{n_k} u\phi_i.$$

Therefore the support of the function v_1 is inside $\bigcup_{i=n_m}^{n_k}\operatorname{supp}\phi_i$.

Similarly, we choose other values of k and m and define the function v_2, $\phi(v_2) \ge a$. Moreover, if the new values k and m are sufficiently large, then $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$. In the same way, we construct other functions v_l such that their supports do not intersect and $\phi(v_l) \ge a$. We put finally

$$w_j = \sum_{l=1}^j v_l.$$

Similar to the sequence u_n, the sequence w_j is uniformly bounded in E_∞. At the same time $\phi(w_j) \to \infty$. This contradicts the assumption that $\phi \in (E_\infty)^*$. The lemma is proved. □

Consider a functional ϕ from $(E_\infty)^*$. We define a new functional $\tilde\phi$ as follows. For any function $u \in E_\infty$ with a bounded support we put

$$\tilde\phi(u) = \phi(u).$$

For any function $u \in E_\infty$, we put

$$\tilde{\phi}(u) = \lim_{n\to\infty} \phi(\sum_{i=1}^{n} u\phi_i).$$

Thus $\tilde{\phi}$ is a weak limit of $\sum_{i=1}^{n} \phi_i\phi$ in $(E_\infty)^*$. From Lemma 2.4.2 it follows that the limit exists. It is easy to verify that $\tilde{\phi}$ is a bounded linear functional on E_∞.

Denote $\phi_0 = \phi - \tilde{\phi}$. Then $\phi_0(u) = 0$ for any function u with a bounded support. Thus we have the following result.

Lemma 2.4.3. *The space $(E_\infty)^*$ can be represented as a direct sum of two subspaces, $(E_\infty)^*_0$ and $(E_\infty)^*_\omega$, where $(E_\infty)^*_0$ consists of functionals equal to 0 on all functions with bounded supports and $(E_\infty)^*_\omega$ consists of the functionals $\tilde{\phi}$ constructed above.*

Proof. It remains to prove that $(E_\infty)^*_\omega$ and $(E_\infty)^*_0$ are closed. Let $v_k \in (E_\infty)^*_\omega$, $v_k \to v$ in $(E_\infty)^*$. We have

$$\langle v_k,\ u\rangle = \lim_{n\to\infty} \langle v_k,\ u_n\rangle,\ \ \forall u \in E_\infty, \tag{4.2}$$

where $u_n = \sum_{i=1}^{n} \phi_i u$. We prove that we can pass to the limit in k in the right-hand side of (4.2). Indeed we have

$$|\langle v_k - v,\ u_n\rangle| \le \|v_k - v\|_{(E_\infty)^*} \|u_n\|_{E_\infty} \le M\|v_k - v\|_{(E_\infty)^*}$$

since $\|u_n\|_{E_\infty}$ is bounded. Hence

$$|\lim_{n\to\infty} \langle v_k - v,\ u_n\rangle| \le M\|v_k - v\|_{(E_\infty)^*} \to 0$$

as $k \to \infty$. Passing to the limit with respect to k in (4.2) we obtain

$$\langle v,\ u\rangle = \lim_{n\to\infty} \langle v,\ u_n\rangle,\ \ \forall u \in E_\infty.$$

Therefore $v \in (E_\infty)^*_\omega$. The completeness of the space $(E_\infty)^*_\omega$ is proved.

It can easily be verified that the second subspace is also closed, and the lemma is proved. □

Lemma 2.4.4. *If $\phi \in (E_\infty)^*$, then $\phi \in (E^*)_1$ and*

$$\|\phi\|_{(E^*)_1} \le M\ \|\phi\|_{(E_\infty)^*}\ , \tag{4.3}$$

where M is a constant independent of ϕ.

Proof. We have $\phi_i\phi \in E^*$ for $\phi_i \in D$ and

$$\|\phi_i\phi\|_{E^*} = \sup_{u\in E,\ \|u\|_E=1} |\phi_i\phi(u)|.$$

Hence there exists $u_i \in E$ such that

$$\|\phi_i\phi\|_{E^*} \le 2\ |\phi_i\phi(u_i)| = 2\ \phi_i\phi(\sigma_i u_i),$$

where $|\sigma_i| = 1$. Therefore

$$\sum_{i=1}^{m} \|\phi_i\phi\|_{E^*} \le 2\ \phi(\sum_{i=1}^{m} \phi_i\sigma_i u_i). \tag{4.4}$$

For any ϕ_k we have

$$\|\sum_{i=1}^{m} \phi_k\phi_i\sigma_i u_i\|_E \le \sum_{i=1}^{m} \|\phi_k\phi_i u_i\|_E \le \sum_{i'} \|\phi_k\phi_{i'} u_{i'}\|_E,$$

where i' are all those numbers i for which supp $\phi_i \cap$ supp $\phi_k \neq \emptyset$. It follows that

$$\|\sum_{i=1}^{m} \phi_k \phi_i \sigma_i u_i\|_E \leq NK^2, \tag{4.5}$$

where N is the number from Condition 2.2.2 and $K = \sup_i \|\phi_i\|_{M(E)}$. Inequality (4.5) implies

$$\|\sum_{i=1}^{m} \phi_i \sigma_i u_i\|_{E_\infty} \leq NK^2.$$

From (4.4) we obtain

$$\sum_{i=1}^{m} \|\phi_i \phi\|_{E^*} \leq 2NK^2 \|\phi\|_{(E_\infty)^*}$$

and (4.3) follows. The lemma is proved. □

Theorem 2.4.5. $(E_\infty)^*_\omega = (E^*)_1$.

Proof. The inclusion $(E_\infty)^*_\omega \subset (E^*)_1$ follows from Lemma 2.4.4. Suppose now that $\phi \in (E^*)_1$. Consider the functionals

$$\Phi_k = \sum_{i=1}^{k} \phi_i \phi,$$

where ϕ_i is a partition of unity. By the definition of the space $(E^*)_1$, the series $\sum_{i=1}^{\infty} \|\phi_i \phi\|_{E^*}$ converges. We show that Φ_k converges to ϕ in $(E^*)_1$. Indeed,

$$\begin{aligned}\|\phi - \Phi_k\|_{(E^*)_1} &= \|\phi - \sum_{i=1}^{k} \phi_i \phi\|_{(E^*)_1} = \sum_{j=1}^{\infty} \|\phi_j(\phi - \sum_{i=1}^{k} \phi_i \phi)\|_{E^*} \\ &= \sum_{j=1}^{\infty} \|\phi_j \phi - \sum_{i=1}^{k} \phi_i(\phi_j \phi)\|_{E^*} = \cdots .\end{aligned}$$

All terms of this sum for which $\sum_{i=1}^{k} \phi_i$ equals 1 in the support of ϕ_j, disappear. The remaining terms begin with some k', where k' depends on k and tends to infinity together with it.

$$\begin{aligned}\cdots &= \sum_{j=k'}^{\infty} \|\phi_j \phi - \sum_{i=1}^{k} \phi_i(\phi_j \phi)\|_{E^*} \leq \sum_{j=k'}^{\infty} \|\phi_j \phi\|_{E^*} + \sum_{j=k'}^{\infty} \sum_{i=1}^{k} \|\phi_i \phi_j \phi\|_{E^*} \\ &= \sum_{j=k'}^{\infty} \|\phi_j \phi\|_{E^*} + \sum_{j=k'}^{\infty} \sum_{i'} \|\phi_{i'} \phi_j \phi\|_{E^*} \\ &\leq \sum_{j=k'}^{\infty} \|\phi_j \phi\|_{E^*} + NM \sum_{j=k'}^{\infty} \|\phi_j \phi\|_{E^*} \to 0 \text{ as } k \to \infty.\end{aligned}$$

Here i' denotes all those i for which the support of ϕ_i intersects the support of ϕ_j for each fixed j. As usual, we use the fact that their number is limited by N.

Thus, the functional ϕ can be represented in the form

$$\phi = \sum_{i=1}^{\infty} \phi_i \phi.$$

Then it is also a continuous functional on E_∞. Indeed, for any $u \in E_\infty$,

$$|\langle \phi, u\rangle| \le \sum_{i=1}^{\infty} |\langle \phi_i\phi, \psi_i u\rangle| \le \sum_{i=1}^{\infty} \|\phi_i\phi\|_{E^*}\|\psi_i u\|_E$$
$$\le C\|u\|_{E_\infty} \sum_{i=1}^{\infty} \|\phi_i\phi\|_{E^*} \le C\|\phi\|_{(E^*)_1}\|u\|_{E_\infty}.$$

Here $\psi_i = 1$ in the support of ϕ_i. Therefore $\phi \in (E_\infty)^*$, and

$$\|\phi\|_{(E_\infty)^*} \le C\|\phi\|_{(E^*)_1}.$$

Let $u \in E_\infty$. Put $u_k = \sum_{i=1}^k \phi_i u$. Then $\phi(u_k) = \Phi_k(u)$. Hence

$$\phi(u) = \lim_{k\to\infty} \Phi_k(u) = \lim_{k\to\infty} \phi(u_k).$$

This means that $\phi \in (E_\infty)^*_\omega$.

It is easy to prove that the spaces $(E_\infty)^*_\omega$ and $(E^*)_1$ are linearly isomorphic. The theorem is proved. □

Consider now the closure E_D of D in the norm E_∞. The following theorem can be proved.

Theorem 2.4.6. $(E_D)^* = (E^*)_1$.

2.5. **The spaces $E_q(\Omega)$ and $E_q(\Gamma)$.** Let Ω be a domain in R^n.

Definition 2.5.1. The space $E_q(\Omega)$ $(1 \le q \le \infty)$ is defined as the set of all those generalized functions from D'_Ω that are restrictions to Ω of generalized functions from E_q. The norm in $E_q(\Omega)$ is defined as

$$\|u\|_{E_q(\Omega)} = \inf \|u^c\|_{E_q},$$

where the minimum is taken over all those $u^c \in E_q$ whose restrictions to Ω coincide with u.

It can be proved that the space $E_q(\Omega)$ in this definition coincides with the space $E_q(\Omega)$ in the following one.

Definition 2.5.2. Let $\{\phi_i\}$ be a partition of unity. The space $E_q(\Omega)$ is defined as the set of generalized functions $u \in D'_\Omega$ such that $\phi_i u \in E(\Omega)$ for all i and

$$\|u\|_{E_q(\Omega)} := (\sum_{i=1}^{\infty} \|\phi_i u\|^q_{E(\Omega)})^{1/q} < \infty \ \ (1 \le q < \infty),$$
$$\|u\|_{E_\infty(\Omega)} := \sup_i \|\phi_i u\|_{E(\Omega)} < \infty.$$

It is obvious that if Ω is a bounded domain, then $E_q(\Omega) = E(\Omega)$ $(1 \le q \le \infty)$. It follows from the definitions that for unbounded domains the space $E_q(\Omega)$ inherits the properties of the space $E_q(R^n)$.

If Γ is an $(n-1)$-dimensional manifold (in particular the boundary of the domain Ω), then the definition of the space $E_q(\Gamma)$ can be given in a standard way using local coordinates and the definition of the space $E_q(R^{n-1})$.

2.6. **Local operators.**

1. Operators in R^n. Let E and F be two Sobolev–Slobodetskii spaces.

Definition 2.6.1. An operator $A : E \to F$ is called local if for every $u \in E$ with a compact support,

$$\operatorname{supp} Au \subset \operatorname{supp} u.$$

Theorem 2.6.2. *If $A : E \to F$ is a bounded local operator, then $A^* : F^* \to E^*$ is also a bounded local operator.*

Proof. The proof follows easily from the definition. □

Let $A : E \to F$ be a local operator. Denote

$$A_{loc}u = \sum_{i,j=1}^{\infty} \phi_j A(\phi_i u), \quad \forall u \in E_{loc},$$

where ϕ_i is a partition of unity. Convergence of the series is understood in the sense of distributions. It can be proved that A_{loc} does not depend on the choice of the partition of unity ϕ_i and it is a linear operator acting from E_{loc} to F_{loc}.

Definition 2.6.3. The operator A_q $(1 \le q \le \infty)$ is a restriction of A_{loc} to E_q.

Theorem 2.6.4. *Let $A : E \to F$ be a bounded local operator. Then A_q is a bounded operator from E_q to F_q.*

Proof. We begin with the case $q = \infty$. Let ϕ_i be a partition of unity, $u \in E_\infty$. We have

$$\phi_i A_{loc} u = \phi_i \sum_{j=1}^{m} A(\phi_j u)$$

for all m sufficiently large. Since

$$\operatorname{supp} A(\phi_j u) \subset \operatorname{supp} \phi_j u \subset \operatorname{supp} \phi_j,$$

then

$$\phi_i A_\infty u = \phi_i A_{loc} u = \phi_i \sum_{j'} A(\phi_{j'} u),$$

where j' are all those j for which supp $\phi_i \cap \operatorname{supp} \phi_j \neq \emptyset$. Therefore

$$\|\phi_i A_\infty u\|_F \le \sum_{j'} \|\phi_i A(\phi_{j'} u)\|_F \le \sum_{j'} \|\phi_i\|_{M(F)} \|A\| \, \|\phi_{j'} u\|_E \le N\|A\| \, \|\phi_i\|_{M(F)} \|u\|_{E_\infty}.$$

Let $\kappa = \sup_i \|\phi_i\|_{M(F)}$. Then

$$\|A_\infty u\|_{F_\infty} \le \kappa N \|A\| \, \|u\|_{E_\infty}.$$

Consider next $1 \le q < \infty$. We have

$$\phi_i A_q u = \phi_i A_{loc} u = \phi_i \sum_{j'} A(\phi_{j'} u),$$

and for any integer m,

$$\begin{aligned}\sum_{i=1}^{m} \|\phi_i A_q u\|_F^q &= \sum_{i=1}^{m} \|\phi_i \sum_{j'} A(\phi_{j'} u)\|_F^q \le \sum_{i=1}^{m} N^{q-1} \sum_{j'} \|\phi_i A(\phi_{j'} u)\|_F^q \\ &= N^{q-1} \sum_{j=1}^{\infty} \sum_{i=1}^{m} \|\phi_i A(\phi_j u)\|_F^q = N^{q-1} \sum_{j=1}^{\infty} \sum_{i'}^{m} \|\phi_{i'} A(\phi_j u)\|_F^q \le \cdots .\end{aligned}$$

Here i' are all those i for which $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j \neq \emptyset$. The number of such i is not greater than N:

$$\cdots \leq N^{q-1}\sum_{j=1}^{\infty}\sum_{i'}^{m}\|\phi_{i'}\|^q_{M(F)}\|A(\phi_j u)\|^q_F \leq N^q\kappa^q\sum_{j=1}^{\infty}\|A(\phi_j u)\|^q_F$$

$$\leq N^q\kappa^q\|A\|^q\sum_{j=1}^{\infty}\|\phi_j u\|^q_E = N^q\kappa^q\|A\|^q\|u\|^q_{E_q}.$$

Passing to the limit as $m \to \infty$, we get

$$\|A_q u\|^q_{F_q} \leq N^q\kappa^q\|A\|^q\|u\|^q_{E_q}.$$

Therefore

$$\|A_q u\|_{F_q} \leq N\kappa\|A\|\ \|u\|_{E_q}.$$

The theorem is proved. □

2. Operators in Ω. Let Ω be a domain in R^n.

Definition 2.6.5. Let $A : E \to F$ be a local bounded operator. Operator $A_q(\Omega)$ $(1 \leq q \leq \infty)$ is the restriction of A_q to $E_q(\Omega)$.

It can be proved that operator $A_q(\Omega)$ is bounded as acting from $E_q(\Omega)$ to $F_q(\Omega)$.

Definition 2.6.6. A linear operator $B : E(\Omega) \to F(\partial\Omega)$ is called local if for any $u \in E(\Omega)$, we have $\operatorname{supp} Bu \subset \operatorname{supp} u$.

Theorem 2.6.7. *Let $B : E(\Omega) \to F(\partial\Omega)$ be a bounded local operator. Then $B^* : (F(\partial\Omega))^* \to (E(\Omega))^*$ is also a bounded local operator.*

The proof follows directly from the definition above.

3. Normal solvability

3.1. **Limiting domains.** We consider an unbounded domain $\Omega \subset R^n$, which satisfies the following condition:

Condition D. For each $x_0 \in \partial\Omega$ there exists a neighborhood $U(x_0)$ such that:

1. $U(x_0)$ contains a sphere with radius δ and center x_0, where δ is independent of x_0.
2. There exists a homeomorphism $\psi(x; x_0)$ of the neighborhood $U(x_0)$ on the unit sphere $B = \{y : |y| < 1\}$ in R^n such that the images of $\Omega \cap U(x_0)$ and $\partial\Omega \cap U(x_0)$ coincide with $B_+ = \{y : y_n > 0,\ |y| < 1\}$ and $B_0 = \{y : y_n = 0,\ |y| < 1\}$ respectively.
3. The function $\psi(x; x_0)$ and its inverse belong to the Hölder space $C^{r+\theta}$, $0 < \theta < 1$. Their $\|\cdot\|_{r+\theta}$ - norms are bounded uniformly in x_0 .

For definiteness we suppose that $\delta < 1$.

To obtain a priori estimates of solutions we suppose that $r \geq \max(l+t_i, l-s_i, l-\sigma_j+1)$.

Let Ω be a domain satisfying Condition D and $\chi(x)$ be its characteristic function. Consider a sequence $x_\nu \in \Omega$, $|x_\nu| \to \infty$ and the shifted domains Ω_ν defined by the shifted characteristic functions $\chi_\nu(x) = \chi(x + x_\nu)$. We suppose that the sequence of domains Ω_ν converges in Ξ_{loc} to some limiting domain $\hat{\Omega}$ (see [45]). We assume that $0 \leq k \leq r$.

Definition 3.1.1. Let $u_\nu \in W^{k,p}_\infty(\Omega_\nu)$, $\nu = 1, 2, \ldots$. We say that u_ν converges to a limiting function $\hat{u} \in W^{k,p}_\infty(\hat{\Omega})$ in $W^{k,p}_{loc}(\Omega_\nu \to \hat{\Omega})$ if there exists an extension $v_\nu(x) \in W^{k,p}_\infty(R^n)$ of $u_\nu(x), \nu = 1, 2, \ldots$ and an extension $\hat{v}(x) \in W^{k,p}_\infty(R^n)$ of $\hat{u}(x)$ such that $v_\nu \to \hat{v}$ in $W^{k,p}_{loc}(R^n)$.

Definition 3.1.2. Let $u_\nu \in W_\infty^{k-1/p,p}(\partial\Omega_\nu)$, $k > 1/p$, $\nu = 1, 2, \ldots$. We say that u_ν converges to a limiting function $\hat{u} \in W_\infty^{k-1/p,p}(\partial\hat{\Omega})$ in $W_{loc}^{k-1/p,p}(\partial\Omega_\nu \to \partial\hat{\Omega})$ if there exists an extension $v_\nu(x) \in W_\infty^{k,p}(R^n)$ of $u_\nu(x), \nu = 1, 2, \ldots$ and an extension $\hat{v}(x) \in W_\infty^{k,p}(R^n)$ of $\hat{u}(x)$ such that $v_\nu \to \hat{v}$ in $W_{loc}^{k,p}(R^n)$.

It is proved in [45] that the limiting function $\hat{u}$ in Definitions 3.1.1 and 3.1.2 does not depend on the choice of the extensions v_ν and $\hat{v}$.

3.2. **Limiting operators.** Suppose that we are given a sequence $\{x_\nu\}, \nu = 1, 2, \ldots, x_\nu \in \Omega$, $|x_\nu| \to \infty$. Consider the shifted domains Ω_ν with the characteristic functions $\chi_\nu(x) = \chi(x + x_\nu)$ where $\chi(x)$ is the characteristic function of Ω, and the shifted coefficients of the operators A_i and B_j:

$$a_{ik,\nu}^\alpha(x) = a_{ik}^\alpha(x + x_\nu), \;\; b_{jk,\nu}^\beta(x) = b_{jk}^\beta(x + x_\nu).$$

We suppose that

$$a_{ik}^\alpha(x) \in C^{l-s_i+\theta}(\bar{\Omega}), \;\; b_{jk}^\beta(x) \in C^{l-\sigma_j+\theta}(\partial\Omega), \tag{2.1}$$

where $0 < \theta < 1$, and that these coefficients can be extended to R^n:

$$a_{ik}^\alpha(x) \in C^{l-s_i+\theta}(R^n), \;\; b_{jk}^\beta(x) \in C^{l-\sigma_j+\theta}(R^n). \tag{2.2}$$

Therefore

$$\|a_{ik,\nu}^\alpha(x)\|_{C^{l-s_i+\theta}(R^n)} \le M, \;\; \|b_{jk,\nu}^\beta(x)\|_{C^{l-\sigma_j+\theta}(R^n)} \le M \tag{2.3}$$

with some constant M independent of ν. It follows from Theorem 3.8 in [45] that there exists a subsequence of the sequence Ω_ν, for which we keep the same notation, such that it converges to a limiting domain $\hat{\Omega}$. From (2.3) it follows that this subsequence can be chosen such that

$$a_{ik,\nu}^\alpha \to \hat{a}_{ik}^\alpha \text{ in } C^{l-s_i}(R^n) \text{ locally}, \;\; b_{jk,\nu}^\beta \to \hat{b}_{jk}^\beta \text{ in } C^{l-\sigma_j}(R^n) \text{ locally}, \tag{2.4}$$

where $\hat{a}_{ik}^\alpha$ and $\hat{b}_{jk}^\beta$ are limiting coefficients,

$$\hat{a}_{ik}^\alpha \in C^{l-s_i+\theta}(R^n), \;\; \hat{b}_{jk}^\beta \in C^{l-\sigma_j+\theta}(R^n).$$

We have constructed limiting operators:

$$\hat{A}_i u = \sum_{k=1}^N \sum_{|\alpha| \le \alpha_{ik}} \hat{a}_{ik}^\alpha(x) D^\alpha u_k, \;\; i = 1, \ldots, N, \; x \in \hat{\Omega}, \tag{2.5}$$

$$\hat{B}_j u = \sum_{k=1}^N \sum_{|\beta| \le \beta_{jk}} \hat{b}_{jk}^\beta(x) D^\beta u_k, \;\; i = 1, \ldots, m, \; x \in \partial\hat{\Omega}, \tag{2.6}$$

$$\hat{L} = (\hat{A}_1, \ldots, \hat{A}_N, \hat{B}_1, \ldots, \hat{B}_m). \tag{2.7}$$

We regard them as acting from $E_\infty(\hat{\Omega})$ to $F_\infty(\hat{\Omega})$. Here and in what follows E and F are the spaces defined in Section 1.1.

3.3. **Condition NS.** We introduce the following condition.

Condition NS. For any limiting domain $\hat{\Omega}$ and any limiting operator $\hat{L}$ the problem

$$\hat{L}u = 0, \quad u \in E_\infty(\hat{\Omega}) \tag{3.1}$$

has only the zero solution.

The following theorems are proved in [47].

Theorem 3.3.1. *Let Condition NS be satisfied. Then there exist numbers M_0 and R_0 such that the following estimate holds:*

$$\|u\|_{E_\infty} \le M_0 \left(\|Lu\|_{F_\infty} + \|u\|_{L^p(\Omega_{R_0})} \right), \quad \forall u \in E_\infty. \tag{3.2}$$

Here $\Omega_{R_0} = \Omega \cap \{|x| \le R_0\}$.

Theorem 3.3.2. *Let Condition NS be satisfied. Then the elliptic operator $L : E_\infty(\Omega) \to F_\infty(\Omega)$ is normally solvable and has a finite-dimensional kernel.*

3.4. **Exponential decay.** Denote

$$\omega_\mu = e^{\mu\sqrt{1+|x|^2}},$$

where μ is a real number.

Theorem 3.4.1. *Let Condition NS be satisfied. Then there exist numbers $M_0 > 0$, $R_0 > 0$ and $\mu_0 > 0$ such that for all μ, $0 < \mu < \mu_0$ the following estimate holds:*

$$\|\omega_\mu u\|_{E_\infty} \le M_0 \left(\|\omega_\mu Lu\|_{F_\infty} + \|\omega_\mu u\|_{L^p(\Omega_{R_0})} \right) \quad \text{if } \omega_\mu u \in E_\infty. \tag{4.1}$$

Proof. According to (3.2) we have

$$\|\omega_\mu u\|_{E_\infty} \le M \left(\|L(\omega_\mu u)\|_{F_\infty} + \|\omega_\mu u\|_{L^p(\Omega_{R_0})} \right). \tag{4.2}$$

The operator L has the form: $L = (A_1, \dots, A_N, B_1, \dots, B_m)$. Consider first the operator

$$A_i(\omega_\mu u) = \sum_{k=1}^{N} \sum_{|\alpha| \le \alpha_{ik}} a_{ik}^\alpha(x) D^\alpha(\omega_\mu u_k), \quad i = 1, \dots, N.$$

We have

$$A_i(\omega_\mu u) = \omega_\mu A_i(u) + \Phi_i, \tag{4.3}$$

where

$$\Phi_i = \sum_{k=1}^{N} \sum_{|\alpha| \le \alpha_{ik}} \sum_{\beta+\gamma=\alpha, |\beta|>0} a_{ik}^\alpha(x) c_{\beta\gamma} D^\beta \omega_\mu D^\gamma u_k,$$

and $c_{\beta\gamma}$ are some constants. Direct calculations give the following estimate:

$$\|\Phi_i\|_{W_\infty^{l-s_i,p}} \le M_1 \mu \|\omega_\mu u\|_{E_\infty(\Omega)}. \tag{4.4}$$

For the boundary operators we have

$$B_j(\omega_\mu u) = \sum_{k=1}^{N} \sum_{|\beta| \le \beta_{jk}} b_{jk}^\beta(x) D^\beta(\omega_\mu u_k).$$

As above we get

$$B_j(\omega_\mu u) = \omega_\mu B_j(u) + \Psi_j, \tag{4.5}$$

$$\|\Psi_j\|_{W_\infty^{l-\sigma_j-1/p,p}} \le M_2 \mu \|\omega_\mu u\|_{E_\infty(\Omega)}. \tag{4.6}$$

From (4.3)–(4.6) we obtain

$$\|L(\omega_\mu u)\|_{F_\infty} \le \|\omega_\mu Lu\|_{F_\infty} + M\mu\|\omega_\mu u\|_{E_\infty}.$$

The assertion of the theorem follows from this estimate and (4.2). The theorem is proved. □

Theorem 3.4.2. *If $0 < \mu < \mu_0$ for some μ_0, $u \in E_\infty$, and $\omega_\mu Lu \in F_\infty$, then $\omega_\mu u \in E_\infty$. In particular, if $u \in E_\infty$ and $Lu = 0$, then $\omega_\mu u \in E_\infty$.*

Proof. Let $\{B_j\}(j = 1, 2, \dots)$ be a covering of R^n by unit balls with centers at the points x_j. Let further θ_j be the corresponding partition of unity, $\operatorname{supp}\theta_j \subset B_j$. We introduce the norms in E_∞ and F_∞ in accordance with this partition of unity. Suppose that functions $\phi_j \in D$ are such that

$$\phi_j(x) = 1 \text{ for } |x - x_j| \le 2, \ \ \operatorname{supp}\phi_j \subset \{|x - x_j| < 3\},$$

and functions $\psi_j \in D$ are such that

$$\psi_j(x) = 1 \text{ for } |x - x_j| \le 3, \ \ \operatorname{supp}\phi_j \subset \{|x - x_j| < 4\}.$$

We introduce next a small parameter $\epsilon > 0$ and denote

$$\phi_j^\epsilon(x) = \phi_j(\epsilon x), \ \ \psi_j^\epsilon(x) = \psi_j(\epsilon x).$$

It follows from Theorem 3.4.1 that

$$\begin{aligned} \|\omega u\phi_j^\epsilon\|_{E_\infty} &\le M_0\left(\|\omega L(u\phi_j^\epsilon)\|_{F_\infty} + \|\omega u\phi_j^\epsilon\|_{L^p(\Omega_R)}\right) \\ &\le M_0\left(\|\omega\phi_j^\epsilon Lu\|_{F_\infty} + \|\omega u\phi_j^\epsilon\|_{L^p(\Omega_R)}\right) + M_0\|\omega(\phi_j^\epsilon Lu - L(u\phi_j^\epsilon))\|_{F_\infty}. \end{aligned} \tag{4.7}$$

(Here and in what follows we write ω instead of ω_μ.) We have

$$\|\omega(\phi_j^\epsilon Lu - L(u\phi_j^\epsilon))\|_{F_\infty} = \|\omega\psi_j^\epsilon(\phi_j^\epsilon Lu - L(u\phi_j^\epsilon))\|_{F_\infty} \le M_1\rho_\epsilon \sup_\alpha \|\omega\psi_j^\epsilon D^\alpha u\|_{F_\infty}, \tag{4.8}$$

where

$$\rho_\epsilon = \sup_{x, 0<|\alpha|\le l+t_k, k=1,\dots,N} |D^\alpha\phi_j^\epsilon(x)|.$$

We estimate the right-hand side in (4.8):

$$\|\omega\psi_j^\epsilon D^\alpha u\|_{F_\infty} \le K\sum_{i'} \|\omega\phi_{i'}^\epsilon u\|_{E_\infty}, \tag{4.9}$$

where K is a constant independent of ϵ, and i' denotes all the i for which

$$\operatorname{supp}\phi_i^\epsilon \cap \operatorname{supp}\psi_j^\epsilon \ne \emptyset.$$

Denote the number of such i by N. It is easy to see that it does not depend on ϵ.

From (4.7)–(4.9) we obtain

$$\|\omega u\phi_j^\epsilon\|_{E_\infty} \le M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^p(\Omega_R)}\right) + M_2\rho_\epsilon\sum_{i'} \|\omega u\phi_{i'}^\epsilon\|_{E_\infty}.$$

In the last term on the right-hand side we take the maximum among the summands:

$$\|\omega u\phi_j^\epsilon\|_{E_\infty} \le M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^p(\Omega_R)}\right) + M_2\rho_\epsilon N\|\omega u\phi_{i(j)}^\epsilon\|_{E_\infty}. \tag{4.10}$$

We rewrite this inequality in the form

$$\|\omega u\phi_{j_1}^\epsilon\|_{E_\infty} \le M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^p(\Omega_R)}\right) + \sigma\|\omega u\phi_{j_2}^\epsilon\|_{E_\infty}, \tag{4.11}$$

where σ is a small constant, and the support of the function ϕ_{j_2} is neighboring to the support of the function ϕ_{j_1}. Since the last estimate is true for any j, then we can write

$$\|\omega u\phi_{j_2}^\epsilon\|_{E_\infty} \le M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^p(\Omega_R)}\right) + \sigma\|\omega u\phi_{j_3}^\epsilon\|_{E_\infty}, \tag{4.12}$$

where the support of the function ϕ_{j_3} is neighboring to the support of the function ϕ_{j_2}. If we continue in the same way, we obtain the inequality

$$\|\omega u\phi^\epsilon_{j_k}\|_{E_\infty} \le M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^P(\Omega_R)}\right) + \sigma\|\omega u\phi^\epsilon_{j_{k+1}}\|_{E_\infty}, \tag{4.13}$$

where the support of the function $\phi_{j_{k+1}}$ is neighboring to the support of the function ϕ_{j_k}. In order to estimate the last summand on the right-hand side of inequality (4.11) we use the inequality (4.12):

$$\|\omega u\phi^\epsilon_{j_1}\|_{E_\infty} \le M_0(1+\sigma)\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^P(\Omega_R)}\right) + \sigma^2\|\omega u\phi^\epsilon_{j_3}\|_{E_\infty}. \tag{4.14}$$

Next we estimate the last summand on the right-hand side of inequality (4.14), and so on. We obtain the estimate:

$$\|\omega u\phi^\epsilon_{j_1}\|_{E_\infty} \le M_0(1+\sigma+\cdots+\sigma^k)\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^P(\Omega_R)}\right) + \sigma^{k+1}\|\omega u\phi^\epsilon_{j_{k+2}}\|_{E_\infty}. \tag{4.15}$$

Let us specify the choice of the functions ϕ_j. Let $\phi(x) \in D$, $0 \le \phi(x) \le 1$, $\phi(x) = 1$ for $|x| \le 2$, $\operatorname{supp}\phi \subset \{|x| < 3\}$. Put $\phi_j(x) = \phi(x - x_j)$. The points x_j are chosen in the nodes of some orthogonal grid. Therefore, the function $\phi^\epsilon_{j_2}$ in (4.11) is shifted with respect to $\phi^\epsilon_{j_1}$ with a value of the shift that does not exceed λ/ϵ, where the constant λ does not depend on x and j. Hence the function $\phi^\epsilon_{j_{k+2}}$ in (4.15) is shifted with respect to $\phi^\epsilon_{j_1}$ with a value of the shift that does not exceed $(k+1)\lambda/\epsilon$. Thus, $\phi^\epsilon_{j_{k+2}}(x) = \phi^\epsilon_{j_1}(x - h_k)$, where

$$|h_k| \le \frac{(k+1)\lambda}{\epsilon}. \tag{4.16}$$

We have, further,

$$\|\omega u\phi^\epsilon_{j_{k+2}}\|_{E_\infty(\Omega)} = \sup_l \|\omega u\theta_l\phi^\epsilon_{j_{k+2}}\|_{E(\Omega)}. \tag{4.17}$$

The following estimate holds:

$$\begin{aligned} S_{k,l} := \|\omega u\theta_l\phi^\epsilon_{j_{k+2}}\|_{E(\Omega)} &= \|\omega u\theta_l\phi^\epsilon_{j_1}(x-h_k)\|_{E(\Omega)} \\ &= \|\omega(x+h_k)u(x+h_k)\theta_l(x+h_k)\phi^\epsilon_{j_1}(x)\|_{E(\Omega_{h_k})} \\ &\le \|\frac{\omega(x+h_k)}{\omega(x)}\|_{M(E)}\|\omega(x)u(x+h_k)\theta_l(x+h_k)\phi^\epsilon_{j_1}(x)\|_{E(\Omega_{h_k})}. \end{aligned}$$

Here Ω_{h_k} is a shifted domain, and $\|\cdot\|_{M(E)}$ is the norm of the multiplier in the space E. It is known that this norm can be estimated by the C-norm of the corresponding derivatives. Therefore

$$\|\frac{\omega(x+h_k)}{\omega(x)}\|_{M(E)} \le ce^{\mu|h_k|},$$

where the constant c is independent of μ and k. Let us return to the estimate of $S_{k,l}$. Since

$$\|\theta_l(x+h_k)\|_{M(E)} \le c_1,$$

then we have

$$S_{k,l} \le c_2 e^{\mu|h_k|}\,\|\omega(x)u(x+h_k)\phi^\epsilon_{j_1}(x)\|_{E(\Omega_{h_k})}. \tag{4.18}$$

Furthermore,

$$\operatorname{supp}\phi^\epsilon_{j_1}(x) \subset \left\{|x - \frac{x_{j_1}}{\epsilon}| < \frac{3}{\epsilon}\right\},$$

such that at the support of the function $\phi^\epsilon_{j_1}$,

$$|x| \le \frac{\rho_{j_1}}{\epsilon} \equiv \frac{3}{\epsilon} + \frac{|x_{j_1}|}{\epsilon}.$$

Let us introduce the function

$$f_\epsilon(x) = \begin{cases} 1, & |x| < \rho_{j_1}/\epsilon, \\ 0, & |x| > 1 + \rho_{j_1}/\epsilon. \end{cases}$$

Then

$$\begin{aligned} \|\omega(x)u(x+h_k)\phi^\epsilon_{j_1}(x)\|_{E(\Omega_{h_k})} &= \|\omega(x)u(x+h_k)\phi^\epsilon_{j_1}(x)f_\epsilon(x)\|_{E(\Omega_{h_k})} \\ &\le \|\omega f_\epsilon\|_{M(E)}\|\phi^\epsilon_{j_1}\|_{M(E)}\|u(x+h_k)\|_{E(\Omega_{h_k})} \le c_3\|\omega f_\epsilon\|_{M(E)}\|u\|_{E(\Omega)} \le c_4 e^{\mu\rho_{j_1}/\epsilon}, \end{aligned}$$

where the constant c_4 does not depend on ϵ for $\epsilon < 1$. From the last inequality, (4.17) and (4.18) we have

$$\|\omega u\phi^\epsilon_{j_{k+2}}\|_{E_\infty(\Omega)} \le c_5 e^{\mu(k+1)\lambda/\epsilon} e^{\mu\rho_{j_1}/\epsilon}. \tag{4.19}$$

Consider inequality (4.15). Taking into account (4.19), we have

$$\|\omega u\phi^\epsilon_{j_1}\|_{E_\infty} \le M_0(1+\sigma+\cdots+\sigma^k)\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^P(\Omega_R)}\right) + c_5\sigma^{k+1}e^{\mu k\lambda/\epsilon}e^{\mu(\rho_{j_1}+\lambda)/\epsilon}. \tag{4.20}$$

Let ϵ be chosen in such a way that $\sigma \le \frac{1}{2}$. Put $\mu_0 < \frac{\epsilon \ln 2}{\lambda}$. Then for $0 < \mu \le \mu_0$ from (4.20) we obtain

$$\|\omega u\phi^\epsilon_{j_1}\|_{E_\infty} \le 2M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^P(\Omega_R)}\right) + \frac{1}{2}c_5\left(\frac{1}{2}e^{\mu_0\lambda/\epsilon}\right)^k e^{\ln 2(\rho_{j_1}+\lambda)/\lambda}.$$

Passing to the limit as $k \to \infty$, we have

$$\|\omega u\phi^\epsilon_{j_1}\|_{E_\infty} \le 2M_0\left(\|\omega Lu\|_{F_\infty} + \|\omega u\|_{L^P(\Omega_R)}\right).$$

The theorem is proved. □

For some classes of elliptic problems satisfying the Fredholm property the exponential decay of solutions is known (see [28] and the references therein). Here it is proved that solutions of general elliptic problems behave exponentially at infinity if the corresponding operator is normally solvable with a finite-dimensional kernel.

4. Adjoint problems

4.1. **Model problems in a half-space.** We use the following notation:

$$D_j = i\partial/\partial x_j, \quad j = 1, \dots, n,$$

$$\hat{D}_j = (F')^{-1}\frac{\xi_j}{|\xi'|}(1+|\xi'|)F', \quad j = 1, \dots, n-1, \quad \hat{D}_n = D_n,$$

where F' is the partial Fourier transform with respect to the variables $x_1, \dots, x_{n-1}$, $\xi' = (\xi_1, \dots, \xi_{n-1})$, $|\xi'| = (\xi_1^2 + \cdots + \xi_{n-1}^2)^{1/2}$.

Denote by $A(D)$ the square $N \times N$ matrix of linear differential operators $A_{ij}(D)$,

$$A_{ij}(D) = \sum_{|\alpha|=\alpha_{ij}} a^\alpha_{ij} D^\alpha$$

with constant coefficients. We suppose that the operator $A(D)$ is elliptic in the Douglis-Nirenberg sense and contains only the principal terms. Then (see [42])

$$A(c\xi) = S(c)A(\xi)T(c) \tag{1.1}$$

for any $\xi = (\xi_1, \dots, \xi_n)$ and any real c. Here S and T are diagonal matrices,

$$S(c) = (\delta_{ij}c^{s_i}), \quad T(c) = \left(\delta_{ij}c^{t_j}\right), \tag{1.2}$$

where δ_{ij} is the Kronecker symbol, $s_1, \dots, s_N, t_1, \dots, t_N$ are given integers, $\alpha_{ij} = s_i + t_j$, $i, j = 1, \dots, N$, $s_i \le 0$.

We consider the system of equations

$$A(\hat{D})u = f \tag{1.3}$$

in the half-space $R^n_+ = \{x \in R^n, x = (x_1, \dots, x_n), x_n > 0\}$, $u(x) = (u_1(x), \dots, u_N(x))$, $f(x) = (f_1(x), \dots, f_N(x))$. We set the boundary conditions

$$B(\hat{D})u = g(x') \tag{1.4}$$

at the boundary Γ of R^n_+, where $g(x') = (g_1(x'), \dots, g_m(x'))$, $B(D)$ is a rectangular $m \times N$ matrix with the elements

$$B_{kj}(D) = \sum_{|\alpha|=\sigma_{kj}} b^\alpha_{kj} D^\alpha,$$

and b^α_{kj} are some constants. The matrix $B(\xi)$ is homogeneous, $B(c\xi) = M(c)B(\xi)T(c)$, where $M(c)$ is a diagonal matrix of order m,

$$M(c) = (\delta_{ij} c^{\sigma_i}), \tag{1.5}$$

$\sigma_i = \max_{1\le j\le N}(\sigma_{ij} - t_j)$, $i = 1, \dots, m$ (see [42]).

We introduce the following spaces:

$$E(\Omega) = \prod_{j=1}^N W^{l+t_j,p}(\Omega), \quad F^d(\Omega) = \prod_{j=1}^N W^{l-s_i,p}(\Omega), \quad F^b(\partial\Omega) = \prod_{j=1}^m W^{l-\sigma_j-1/p,p}(\partial\Omega),$$

where Ω is a domain in R^n, $\partial\Omega$ is its boundary, l is an integer, $l \ge \max_j(\sigma_j + 1)$, $1 < p < \infty$.

The main result of this section is given by the following theorem.

Theorem 4.1.1. *For any $f \in F^d(R^n_+)$ and $g \in F^b(R^{n-1})$ there exists a unique solution $u \in E(R^n_+)$ of problem* (1.3), (1.4).

The proof of this theorem is based on the following result.

Theorem 4.1.2. *For any $u \in E(R^n_+)$ the following estimate holds:*

$$\|u\|_{E(R^n_+)} \le \left(\|A(\hat{D})u\|_{F^d(R^n_+)} + \|B(\hat{D})u\|_{F^b(\Gamma)} \right),$$

where c is a constant independent of u.

The proof of this theorem is given in Section 4.4 and the proof of the previous one in Section 4.5. Theorem 4.1.1 in more general spaces is proved in [36]. We use the approach developed in [42] to give a simpler proof for the case under consideration.

4.2. **A priori estimates for adjoint operators. Model systems.** In this section we consider the operators

$$A^0_i u = \sum_{k=1}^N \sum_{|\alpha|=s_i+t_k} a^\alpha_{ik} D^\alpha u_k, \quad i = 1, \dots, N, \ x \in \Omega,$$

$$B^0_j u = \sum_{k=1}^N \sum_{|\beta|=\sigma_i+t_k} b^\beta_{jk} D^\beta u_k, \quad i = 1, \dots, m, \ x \in \partial\Omega,$$

with constant coefficients a^α_{ik}, b^β_{jk}. We suppose here that the domain Ω is the half-space $R^+_n = \{x_n \ge 0\}$. We will denote by $\hat{A}^0_i$ and $\hat{B}^0_j$ the operators obtained from A^0_i and B^0_j, respectively, if we replace the derivatives D_i, $i = 1, \dots, n-1$ by the operators $\hat{D}_i$. The operator

$$\hat{L}^0 = (\hat{A}^0_1, \dots, \hat{A}^0_N, \hat{B}^0_1, \dots, \hat{B}^0_m)$$

acts from E to $F = F^d \times F^b$.

We consider the operator

$$(L^0)^* : F^* \to E^*$$

adjoint to $L^0 = (A_1^0, \dots, A_N^0, B_1^0, \dots, B_m^0)$. We have

$$E^* = \prod_{j=1}^{N} \dot{W}^{-l-t_j,p'}(\Omega), \quad F^* = \prod_{i=1}^{N} \dot{W}^{-l+s_i,p'}(\Omega) \times \prod_{j=1}^{m} W^{-l+\sigma_j+1/p,p'}(\partial\Omega), \tag{2.1}$$

where $\Omega = R^n_+$, and $\dot{W}^{-s,p'}(\Omega)$ is the closure in $W^{-s,p'}(R^n)$ of infinitely differentiable functions with supports in Ω, $\dot{W}^{-s,p'}(\Omega) = (W^{s,p}(\Omega))^*$, $\frac{1}{p} + \frac{1}{p'} = 1$. Denote

$$F^*_{-1} = \prod_{i=1}^{N} \dot{W}^{-l+s_i-1,p'}(\Omega) \times \prod_{j=1}^{m} W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega). \tag{2.2}$$

Theorem 4.2.1.

$$\|v\|_{F^*} \le C \left(\|(L^0)^* v\|_{E^*} + \|v\|_{F^*_{-1}} \right), \quad \forall v \in F^*, \tag{2.3}$$

where C is a constant independent of v.

Proof. From Theorem 4.1.1 it follows that the operator $\hat{L}^0$ has a bounded inverse,

$$(\hat{L}^0)^{-1} : F \to E.$$

Therefore the operator

$$((\hat{L}^0)^*)^{-1} : E^* \to F^*$$

is also bounded. Hence we have the estimate

$$\|v\|_{F^*} \le C \|(\hat{L}^0)^* v\|_{E^*}, \quad \forall v \in F^*.$$

Therefore

$$\|v\|_{F^*} \le C \left(\|(L^0)^* v\|_{E^*} + \|((\hat{L}^0)^* - (L^0)^*) v\|_{E^*} \right). \tag{2.4}$$

We estimate the second term on the right-hand side of this inequality. Let $\langle , \rangle_E$ be the duality between E and E^*. For $u \in E$ we have

$$|\langle u, ((\hat{L}^0)^* - (L^0)^*) v \rangle_E| = |\langle (\hat{L}^0 - L^0) u, v \rangle_F|. \tag{2.5}$$

Let $v = (v_1, \dots, v_N, w_1, \dots, w_m)$, where

$$v_i \in (W^{l-s_i,p}(\Omega))^*, \quad w_j \in (W^{l-\sigma_j-1/p,p}(\partial\Omega))^*.$$

Then we have

$$\langle (\hat{L}^0 - L^0) u, v \rangle_F = \sum_{i=1}^{N} \langle (\hat{A}_i^0 - A_i^0) u, v_i \rangle + \sum_{j=1}^{m} \langle (\hat{B}_j^0 - B_j^0) u, w_j \rangle. \tag{2.6}$$

Let

$$T_i : W^{l-s_i+1,p}(R^n) \to W^{l-s_i,p}(R^n)$$

be an isomorphism between the two spaces.

Denote by $\tilde{u}_j$ an extension of u_j to $W^{l+t_j,p}(R^n)$ such that

$$\|\tilde{u}_j\|_{W^{l+t_j,p}(R^n)} \le 2 \|u_j\|_{W^{l+t_j,p}(\Omega)}, \tag{2.7}$$

and $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_N)$. Then $(\hat{A}_i^0 - A_i^0)\tilde{u}$ is an extension of $(\hat{A}_i^0 - A_i^0)u$ from $W^{l-s_i+1,p}(\Omega)$ to $W^{l-s_i+1,p}(R^n)$.

We have $(\hat{A}_i^0 - A_i^0)u \in W^{l-s_i+1,p}(\Omega) \subset W^{l-s_i,p}(\Omega)$. Hence v_i can be considered as an element of $(W^{l-s_i+1,p}(\Omega))^*$. It can be extended to an element $\tilde{v}_i \in (W^{l-s_i+1,p}(R^n))^*$ such that

$$\langle (\hat{A}_i^0 - A_i^0) u, v_i \rangle = \langle (\hat{A}_i^0 - A_i^0) \tilde{u}, \tilde{v}_i \rangle$$

and

$$\|\tilde{v}_i\|_{(W^{l-s_i+1,p}(R^n))^*} = \|v_i\|_{(W^{l-s_i+1,p}(\Omega))^*}. \tag{2.8}$$

Then

$$|\langle(\hat{A}_i^0 - A_i^0)u, v_i\rangle| = |\langle T_i^{-1}T_i(\hat{A}_i^0 - A_i^0)\tilde{u}, \tilde{v}_i\rangle| = |\langle T_i(\hat{A}_i^0 - A_i^0)\tilde{u}, (T_i^{-1})^*\tilde{v}_i\rangle|.$$

Since

$$(T_i^{-1})^* : W^{-l+s_i-1,p'}(R^n) \to W^{-l+s_i,p'}(R^n),$$

it follows that

$$\begin{aligned}|\langle(\hat{A}_i^0 - A_i^0)u, v_i\rangle| &\le \|T_i(\hat{A}_i^0 - A_i^0)\tilde{u}\|_{W^{l-s_i,p}(R^n)}\|(T_i^{-1})^*\tilde{v}_i\|_{(W^{l-s_i,p}(R^n))^*}\\ &\le C\|(\hat{A}_i^0 - A_i^0)\tilde{u}\|_{W^{l-s_i+1,p}(R^n)}\|\tilde{v}_i\|_{(W^{l-s_i+1,p}(R^n))^*}\\ &\le C_1\|\tilde{u}\|_{E(R^n)}\|\tilde{v}_i\|_{(W^{l-s_i+1,p}(R^n))^*}.\end{aligned}$$

From (2.7), (2.8),

$$|\langle(\hat{A}_i^0 - A_i^0)u, v_i\rangle| \le C_2\|u\|_{E(\Omega)}\|\|v_i\|_{(W^{l-s_i+1,p}(\Omega))^*} = C_2\|u\|_{E(\Omega)}\|\|v_i\|_{\dot{W}^{-l+s_i-1,p'}(\Omega)}. \tag{2.9}$$

Consider now the boundary operators in (2.6). We have

$$(\hat{B}_j^0 - B_j^0)u \in W^{l-\sigma_j-1/p+1,p}(\partial\Omega).$$

Let

$$S_j : W^{l-\sigma_j-1/p+1,p}(\partial\Omega) \to W^{l-\sigma_j-1/p,p}(\partial\Omega)$$

be an isomorphism between the two spaces. Then

$$\begin{aligned}|\langle(\hat{B}_j^0 - B_j^0)u, w_j\rangle| &= |\langle S_j^{-1}S_j(\hat{B}_j^0 - B_j^0)u, w_j\rangle| = |\langle S_j(\hat{B}_j^0 - B_j^0)u, (S_j^{-1})^*w_j\rangle|\\ &\le \|S_j(\hat{B}_j^0 - B_j^0)u\|_{W^{l-\sigma_j-1/p,p}(\partial\Omega)}\|(S_j^{-1})^*w_j\|_{(W^{l-\sigma_j-1/p,p}(\partial\Omega))^*}\\ &\le C_3\|(\hat{B}_j^0 - B_j^0)u\|_{W^{l-\sigma_j-1/p+1,p}(\partial\Omega)}\|w_j\|_{(W^{l-\sigma_j-1/p+1,p}(\partial\Omega))^*}\\ &\le C_4\|u\|_{E(\Omega)}\|\ \|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega)}.\end{aligned} \tag{2.10}$$

From (2.5), (2.6), (2.9), and (2.10) we obtain

$$\begin{aligned}&|\langle u, ((\hat{L}^0)^* - (L^0)^*)v\rangle_E|\\ &\quad\le C_5\|u\|_{E(\Omega)}\left(\sum_{i=1}^N \|v_i\|_{\dot{W}^{-l+s_i-1,p'}(\Omega)} + \sum_{j=1}^m \|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega)}\right)\\ &\quad= C_5\|u\|_{E(\Omega)}\|v\|_{F^*_{-1}}.\end{aligned}$$

Therefore

$$\|((\hat{L}^0)^* - (L^0)^*)v\|_{E^*} \le C_5\|v\|_{F^*_{-1}}.$$

Estimate (2.3) follows from this estimate and (2.4). The theorem is proved. □

4.3. **General problem in the half-space.** We consider operators (1.1), (1.2) (Section 1) with $\Omega = R^n_+$.

Theorem 4.3.1. *Let $v \in F^*(R^n_+)$ vanish outside the ball $\sigma(\rho) = \{x : |x| < \rho\}$. Then there exists $\rho_0 > 0$ such that for $\rho < \rho_0$ the following estimate holds:*

$$\|v\|_{F^*(R^n_+)} \le C\left(\|L^*v\|_{E^*(R^n_+)} + \|v\|_{F^*_{-1}(R^n_+)}\right). \tag{3.1}$$

Proof. We introduce the notation

$$A_i^0 u = \sum_{k=1}^{N} \sum_{|\alpha|=s_i+t_k} a_{ik}^{\alpha}(0) D^{\alpha} u_k, \quad i = 1, \dots, N, \ x \in R_+^n,$$

$$B_j^0 u = \sum_{k=1}^{N} \sum_{|\beta|=\sigma_j+t_k} b_{jk}^{\beta}(0) D^{\beta} u_k, \quad i = 1, \dots, m, \ x \in R^{n-1},$$

$L^0 = (A_1^0, \dots, A_N^0, B_1^0, \dots, B_m^0) : E \to F$, and $(L^0)^*$ is the adjoint operator, $(L^0)^* : F^* \to E^*$. From Theorem 4.2.1 we have

$$\|v\|_{F^*(R_+^n)} \le C \left(\|(L^0)^* v\|_{E^*(R_+^n)} + \|v\|_{F^*_{-1}(R_+^n)} \right). \tag{3.2}$$

On the other hand,

$$\|(L^0)^* v\|_{E^*} \le \|L^* v\|_{E^*} + \|(L^0)^* v - L^* v\|_{E^*}. \tag{3.3}$$

We estimate the second term in the right-hand side. For any $u \in E$ and $v = (v_1, \dots, v_N, w_1, \dots, w_m)$ we have

$$\begin{aligned} \langle u, ((L^0)^* - L^*) v \rangle_E &= \langle (L^0 - L) u, v \rangle_F \\ &= \sum_{i=1}^{N} \langle (A_i^0 - A_i) u, v_i \rangle + \sum_{j=1}^{m} \langle (B_j^0 - B_j) u, w_j \rangle. \end{aligned} \tag{3.4}$$

Furthermore,

$$|\langle (A_i^0 - A_i) u, v_i \rangle| \le |\langle A_i^1 u, v_i \rangle| + |\langle A_i^2 u, v_i \rangle|, \tag{3.5}$$

where

$$A_i^1 u = \sum_{k=1}^{N} \sum_{|\alpha|=\alpha_{ik}} (a_{ik}^{\alpha}(0) - a_{ik}^{\alpha}(x)) D^{\alpha} u_k, \quad A_i^2 u = \sum_{k=1}^{N} \sum_{|\alpha|<\alpha_{ik}} a_{ik}^{\alpha}(x) D^{\alpha} u_k. \tag{3.6}$$

We estimate first the operator A_i^1. We have

$$\langle A_i^1 u, v_i \rangle = \sum_{k=1}^{N} \sum_{|\alpha|=\alpha_{ik}} \langle (a_{ik}^{\alpha}(0) - a_{ik}^{\alpha}(x)) D^{\alpha} u_k, v_i \rangle,$$

$$\begin{aligned} |\langle (a_{ik}^{\alpha}(0) - a_{ik}^{\alpha}(x)) D^{\alpha} u_k, v_i \rangle| &= |\langle D^{\alpha} u_k, (\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) v_i \rangle| \\ &\le \|D^{\alpha} u_k\|_{W^{l-s_i,p}(R_+^n)} \|(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) v_i\|_{(W^{l-s_i,p}(R_+^n))^*} \\ &\le \|u_k\|_{W^{l+t_k,p}(R_+^n)} \|(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) v_i\|_{(W^{l-s_i,p}(R_+^n))^*}, \end{aligned}$$

and

$$|\langle A_i^1 u, v_i \rangle| \le \sum_{k=1}^{N} \|u_k\|_{W^{l+t_k,p}(R_+^n)} \sum_{|\alpha|=\alpha_{ik}} \|(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) v_i\|_{(W^{l-s_i,p}(R_+^n))^*}. \tag{3.7}$$

We estimate the second sum on the right-hand side. Let $\psi \in D$, $\psi(x) = 1$ in $\sigma(\rho)$, $\psi(x) = 0$ outside $\sigma(2\rho)$. Then we have

$$\begin{aligned} \|(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) v_i\|_{(W^{l-s_i,p}(R_+^n))^*} &= \|(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) \psi v_i\|_{(W^{l-s_i,p}(R_+^n))^*} \\ &= \|(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) \psi v_i\|_{\dot{W}^{-l+s_i,p'}(R_+^n)} \equiv T. \end{aligned}$$

From Lemma 4.3.2 (see below) we obtain

$$T \le C_1 \max_{x \in R^n} |(\overline{a_{ik}^{\alpha}(0)} - \overline{a_{ik}^{\alpha}(x)}) \psi| \ \|v_i\|_{\dot{W}^{-l+s_i,p'}(R_+^n)} + K_{\alpha\rho} \|v_i\|_{\dot{W}^{-l+s_i-1,p'}(R_+^n)}.$$

For any $\epsilon > 0$ we can find $\rho_0 > 0$ such that for $0 < \rho \le \rho_0$ we have

$$T \le \epsilon \|v_i\|_{\dot W^{-l+s_i,p'}(R^n_+)} + K_{\alpha\rho}\|v_i\|_{\dot W^{-l+s_i-1,p'}(R^n_+)}.$$

From (3.7),

$$|\langle A_i^1 u, v_i\rangle| \le \|u\|_{E(R^n_+)}\left(\kappa\epsilon\|v_i\|_{\dot W^{-l+s_i,p'}(R^n_+)} + M\|v_i\|_{\dot W^{-l+s_i-1,p'}(R^n_+)}\right), \tag{3.8}$$

where κ and M are some constants.

Consider now the operator A_i^2 in (3.6). We have $A_i^2 : E(R_n^+) \to W^{l-s_i,p}(R^n_+)$. We can extend its coefficients in such a way that the extended operator

$$\tilde A_i^2 : E(R^n) \to W^{l-s_i,p}(R^n)$$

is bounded. Furthermore, let T_i be a bounded linear operator with a bounded inverse acting from $W^{l-s_i+1,p}(R^n)$ to $W^{l-s_i,p}(R^n)$. Then

$$(T_i^{-1})^* : (W^{l-s_i+1,p}(R^n))^* \to (W^{l-s_i,p}(R^n))^*$$

is also bounded.

Let $u \in E(R_n^+)$, and let $\tilde u$ be its extension to $E(R^n)$ such that $\|\tilde u\|_{E(R^n)} \le 2\|u\|_{E(R^n_+)}$. Suppose that

$$v_i \in (W^{l-s_i,p}(R^n_+))^* = \dot W^{-l+s_i,p'}(R^n_+).$$

We consider the extension $\tilde v_i \in W^{-l+s_i-1,p'}(R^n)$. Then we have

$$\begin{aligned}
|\langle A_i^2 u, v_i\rangle| &= |\langle \tilde A_i^2 \tilde u, \tilde v_i\rangle| = |\langle T_i^{-1} T_i \tilde A_i^2 \tilde u, \tilde v_i\rangle| = |\langle T_i \tilde A_i^2 \tilde u, (T_i^{-1})^* \tilde v_i\rangle| \\
&\le \|T_i \tilde A_i^2 \tilde u\|_{W^{l-s_i,p}(R^n)} \|(T_i^{-1})^* \tilde v_i\|_{W^{-l+s_i,p'}(R^n)} \\
&\le C_1 \|\tilde A_i^2 \tilde u\|_{W^{l-s_i+1,p}(R^n)} \|\tilde v_i\|_{W^{-l+s_i-1,p'}(R^n)} \\
&\le C_2 \|u\|_{E(R^n_+)} \|v_i\|_{\dot W^{-l+s_i-1,p'}(R^n_+)}.
\end{aligned} \tag{3.9}$$

From this estimate, (3.5), and (3.8) it follows that

$$|\langle (A_i^0 - A_i)u, v_i\rangle| \le \|u\|_{E(R_n^+)}\left(\kappa\epsilon\|v_i\|_{\dot W^{-l+s_i,p'}(R^n_+)} + M_1\|v_i\|_{\dot W^{-l+s_i-1,p'}(R^n_+)}\right). \tag{3.10}$$

Consider now the second term on the right-hand side of (3.4). We have

$$(B_j^0 - B_j)u = B_j^1 u + B_j^2 u, \tag{3.11}$$

where

$$B_j^1 u = \sum_{k=1}^N \sum_{|\beta|=\beta_{jk}} (b_{jk}^\beta(0) - b_{jk}^\beta(x)) D^\beta u_k, \quad B_j^2 u = -\sum_{k=1}^N \sum_{|\beta|<\beta_{jk}} b_{jk}^\beta(x) D^\beta u_k. \tag{3.12}$$

Consider first the operator B_j^1:

$$\langle B_j^1 u, w_j\rangle = \sum_{k=1}^N \sum_{|\beta|=\beta_{jk}} \langle (b_{jk}^\beta(0) - b_{jk}^\beta(x)) D^\beta u_k, w_j\rangle,$$

$$\begin{aligned}
|\langle (b_{jk}^\beta(0) - b_{jk}^\beta(x)) D^\beta u_k, w_j\rangle| &= |\langle D^\beta u_k, (\overline{b_{jk}^\beta(0)} - \overline{b_{jk}^\beta(x)}) w_j\rangle| \\
&\le \|D^\beta u_k\|_{W^{l-\sigma_j-1/p,p}(R^{n-1})} \|(\overline{b_{jk}^\beta(0)} - \overline{b_{jk}^\beta(x)}) w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})} \\
&\le \|u_k\|_{W^{l+t_k,p}(R^n_+)} \|(\overline{b_{jk}^\beta(0)} - \overline{b_{jk}^\beta(x)}) w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})}.
\end{aligned}$$

Hence
(3.13)
$$|\langle B_j^1 u, w_j\rangle| \le \sum_{k=1}^{N} \|u_k\|_{W^{l+t_k,p}(R_+^n)} \sum_{|\beta|=\beta_{jk}} \|(\overline{b_{jk}^{\beta}(0)} - \overline{b_{jk}^{\beta}(x)})w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})}.$$

We estimate the second sum on the right-hand side. We have
$$\begin{aligned}&\|(\overline{b_{jk}^{\beta}(0)} - \overline{b_{jk}^{\beta}(x)})w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})}\\ &\quad = \|(\overline{b_{jk}^{\beta}(0)} - \overline{b_{jk}^{\beta}(x)})\psi w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})} \equiv T_1.\end{aligned}$$

Then by Lemma 4.3.2,
$$\begin{aligned}T_1 \le C_1 &\max_{x\in R^{n-1}} |(b_{jk}^{\beta}(0) - b_{jk}^{\beta}(x))\psi|\ \|w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})}\\ &+ K_{\beta\rho}\|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}.\end{aligned}$$

For any $\epsilon > 0$ we can find $\rho_0 > 0$ such that for $0 < \rho \le \rho_0$ the following inequality holds:
$$T_1 \le \epsilon\ \|w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})} + K_{\beta\rho}\|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}.$$

From (3.13),
(3.14)
$$|\langle B_j^1 u, w_j\rangle| \le \|u\|_{E(R_+^n)} \left(\kappa\epsilon\|w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})} + M\|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}\right).$$

Consider now the operator B_j^2 in (3.11),
$$B_j^2 : E(R_+^n) \to W^{l-\sigma_j-1/p+1,p}(R^{n-1}).$$

Let
$$S_j : W^{l-\sigma_j-1/p+1,p}(R^{n-1}) \to W^{l-\sigma_j-1/p,p}(R^{n-1})$$
be an isomorphism. We have
$$\begin{aligned}\langle B_j^2 u, w_j\rangle &= \langle S_j^{-1} S_j B_j^2 u, w_j\rangle = \langle S_j B_j^2 u, (S_j^{-1})^* w_j\rangle,\\ |\langle B_j^2 u, w_j\rangle| &\le \|S_j B_j^2 u\|_{W^{l-\sigma_j-1/p,p}(R^{n-1})}\|(S_j^{-1})^* w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})}\\ &\le C\|B_j^2 u\|_{W^{l-\sigma_j-1/p+1,p}(R^{n-1})}\|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}\\ &\le C_1\|u\|_{E(R_+^n)}\|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}.\end{aligned} \tag{3.15}$$

From this estimate, (3.11), and (3.14) we obtain
$$\begin{aligned}&|\langle (B_j^0 - B_j)u, w_j\rangle|\\ &\quad \le \|u\|_{E(R_+^n)} \left(\kappa\epsilon\|w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})} + M_1\|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}\right).\end{aligned} \tag{3.16}$$

From (3.4), (3.10), and (3.16),
$$\begin{aligned}&|\langle (u, ((L^0)^* - L^*)v\rangle_E|\\ &\quad \le \|u\|_{E(R_n^+)} \left(\kappa\epsilon \sum_{i=1}^{N} \|v_i\|_{\dot W^{-l+s_i,p'}(R_+^n)} + \sum_{i=1}^{N} M_1\|v_i\|_{\dot W^{-l+s_i-1,p'}(R_+^n)}\right)\\ &\qquad + \|u\|_{E(R_n^+)} \left(\kappa\epsilon \sum_{j=1}^{m} \|w_j\|_{W^{-l+\sigma_j+1/p,p'}(R^{n-1})} + M_1 \sum_{j=1}^{m} \|w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(R^{n-1})}\right).\end{aligned}$$

Using the notation in (2.1) and (2.2) we can write this estimate as
$$|\langle (u, ((L^0)^* - L^*)v\rangle_E| \le \|u\|_{E(R_n^+)} \left(\kappa\epsilon\|v\|_{F^*(R_+^n)} + M_1\|v\|_{F_{-1}^*(R_+^n)}\right).$$

Hence

$$\|((L^0)^* - L^*)v\|_{E^*(R_n^+)} \le \kappa\epsilon\|v\|_{F^*(R_+^n)} + M_1\|v\|_{F^*_{-1}(R_+^n)}.$$

Estimate (3.1) follows from the last estimate, (3.2), and (3.3). The theorem is proved. □

In the proof of this theorem we used the following lemma.

Lemma 4.3.2. *Suppose v has bounded support, $a \in C_0^m(R^n)$, $v \in H^{s,p}(R^n)$, and $1-m \le s \le 0$. Then*

$$\|av\|_{H^{s,p}(R^n)} \le c_1 \max_{x\in R^n} |a(x)|\|v\|_{H^{s,p}(R^n)} + c_2(a)\|v\|_{H^{s-1,p}(R^n)}, \tag{3.17}$$

where the constant c_1 does not depend on v and a, and $c_2 = 0$ if $s = 0$.

A similar estimate holds for a function $v \in B^{s,p}(R^{n-1})$:

$$\|av\|_{B^{s,p}(R^{n-1})} \le c_1 \max_{x\in R^{n-1}} |a(x)|\|v\|_{B^{s,p}(R^{n-1})} + c_2(a)\|v\|_{B^{s-1,p}(R^{n-1})}. \tag{3.18}$$

Proof. The proof in [36] (Section 1.12) is given for the spaces $H^{s,p}(R^n)$.

Let us prove estimate (3.18) for the case of noninteger s which we use below. All necessary elements of the proof are given in [36]. We recall that the Besov spaces coincide in this case with the Sobolev–Slobodetskii spaces.

We note first of all that for a positive noninteger s the estimate

$$\|av\|_{W^{s,p}(R^n)} \le c_1 \max_{x\in R^n} |a(x)|\|v\|_{W^{s,p}(R^n)} + c_2(a)\|v\|_{W^{s-\sigma,p}(R^n)}, \tag{3.19}$$

where $\sigma = s-k$, $k = [s]$ can be verified directly from the definition of the space $W^{s,p}(R^n)$. Indeed,

$$\|av\|^p_{W^{s,p}(R^n)} = \|av\|^p_{W^{k,p}(R^n)} + \sum_{|\alpha|=k} \int_{R^n}\int_{R^n} \frac{|D^\alpha(a(x)v(x)) - D^\alpha(a(y)v(y))|^p}{|x-y|^{n+p\sigma}} dxdy,$$

$$\begin{aligned}
\|av\|_{W^{k,p}(R^n)} &\le C \sum_{|\alpha|+|\beta|\le k} \|D^\alpha a D^\beta v\|_{L^p(R^n)} \\
&= C \sum_{|\beta|\le k} \|aD^\beta v\|_{L^p(R^n)} + C \sum_{|\alpha|+|\beta|\le k, |\alpha|>0} \|D^\alpha a D^\beta v\|_{L^p(R^n)} \\
&\le c_1 \sup_x |a(x)|\|v\|_{W^{k,p}(R^n)} + c_2(a)\|v\|_{W^{k-1,p}(R^n)},
\end{aligned}$$

$$\int_{R^n}\int_{R^n} \frac{|D^\alpha a(x) D^\beta v(x) - D^\alpha a(y) D^\beta v(y)|^p}{|x-y|^{n+p\sigma}} dxdy \le M_1(I_1 + I_2),$$

where

$$I_1 = \int_{R^n}\int_{R^n} \frac{|D^\alpha a(x)|^p |D^\beta v(x) - D^\beta v(y)|^p}{|x-y|^{n+p\sigma}} dxdy,$$

$$I_2 = \int_{R^n}\int_{R^n} \frac{|D^\beta v(y)|^p |D^\alpha a(x) - D^\alpha a(y)|^p}{|x-y|^{n+p\sigma}} dxdy.$$

If $|\alpha| = 0$, then

$$I_1 \le \|a\|^p_{C^0(R^n)} \|v\|^p_{W^{s,p}(R^n)}.$$

If $|\alpha| > 0$, then

$$I_1 \le \|a\|^p_{C^k(R^n)} \|v\|^p_{W^{s-1,p}(R^n)}.$$

We now estimate I_2:

$$I_2 = \int_{R^n} |D^\beta v(y)|^p \left(\int_{R^n} \frac{|D^\alpha a(x) - D^\alpha a(y)|^p}{|x-y|^{n+p\sigma}} dx \right) dy.$$

Let us prove that

$$J = \int_{R^n} \frac{|D^\alpha a(x) - D^\alpha a(y)|^p}{|x-y|^{n+p\sigma}} dx \le M_2,$$

where M_2 is a constant. We have

$$J = \int_{R^n} \frac{|D^\alpha a(y+z) - D^\alpha a(y)|^p}{|z|^{n+p\sigma}} dz = J_1 + J_2,$$

where

$$J_1 = \int_{|z|\le 1} \frac{|D^\alpha a(y+z) - D^\alpha a(y)|^p}{|z|^{n+p\sigma}} dz, \quad J_2 = \int_{|z|>1} \frac{|D^\alpha a(y+z) - D^\alpha a(y)|^p}{|z|^{n+p\sigma}} dz.$$

The integral J_1 is bounded since

$$|D^\alpha a(y+z) - D^\alpha a(y)| \le K|z|, \quad |z| \le 1, \ y \in R^n,$$

and J_2 is bounded since $|D^\alpha a(x)| \le M$. Here K and M are some positive constants.

Thus, $I_2 \le M_2 \|v\|^p_{W^{k,p}(R^n)}$. This completes the proof of estimate (3.19) for positive s. We recall that it is proved for $\sigma = s - [s]$. It can now be obtained for any positive σ with the help of the estimate

$$\|v\|_{W^{s,p}(R^n)} \le \epsilon \|v\|_{W^{s_1,p}(R^n)} + C_\epsilon \|v\|_{W^{s_2,p}(R^n)},$$

which holds for any $s_2 < s < s_1$ and any $\epsilon > 0$.

We now prove a similar estimate for the dual spaces. It is shown in [36] that there exists an operator χ_N which satisfies the following properties:

(i) it is a continuous operator from $B^{s,p}(R^{n-1})$ to $B^{s+t,p}(R^{n-1})$ for any real s and $t > 0$;

(ii) the following estimate holds:

$$\|(I - \chi_N)u\|_{B^{s,p}(R^{n-1})} \le M \|u\|_{B^{s,p}(R^{n-1})}$$

with a constant M independent of u and N;

(iii) for any $\epsilon > 0$ and $\sigma_0 > 0$ there exists $N(\epsilon, \sigma_0) > 0$ such that

$$\|(I - \chi_N)u\|_{B^{s-\sigma,p}(R^{n-1})} \le \epsilon \|u\|_{B^{s,p}(R^{n-1})}$$

for any $N \ge N(\epsilon, \sigma_0)$ and $\sigma \ge \sigma_0$.

Substituting in (3.19) p' instead of p and $(I - \chi_N)u$ instead of v and using the properties of the operator χ_N, we obtain the estimate

$$\|a(I - \chi_N)u\|_{B^{s,p'}(R^{n-1})} \le M \max_{x \in R^{n-1}} |a(x)| \|u\|_{B^{s,p'}(R^{n-1})} \quad (s > 0). \tag{3.20}$$

Let $u \in B^{-s,p}(R^{n-1})$, $w \in B^{s,p}(R^{n-1})$. Then for a positive s,

$$|\langle (I - \chi_N)au, w\rangle| = |\langle u, \bar{a}(I - \chi_N)w\rangle| \le \|u\|_{B^{-s,p}(R^{n-1})} \|\bar{a}(I - \chi_N)w\|_{B^{s,p'}(R^{n-1})}$$

from (3.20)

$$\le M \max_{x \in R^{n-1}} |a(x)| \|u\|_{B^{-s,p}(R^{n-1})} \|w\|_{B^{s,p'}(R^{n-1})}.$$

Therefore

$$\|(I - \chi_N)au\|_{B^{-s,p}(R^{n-1})} \le M \max_{x \in R^{n-1}} |a(x)| \|u\|_{B^{-s,p}(R^{n-1})} \quad (s > 0). \tag{3.21}$$

Finally for $s > 0$, $\sigma > 0$,

$$\begin{aligned}\|au\|_{B^{-s,p}(R^{n-1})} &\le \|(I-\chi_N)au\|_{B^{-s,p}(R^{n-1})} + \|\chi_N au\|_{B^{-s,p}(R^{n-1})}\\ &\le M_1\left(\max_{x\in R^{n-1}}|a(x)|\|u\|_{B^{-s,p}(R^{n-1})} + \|au\|_{B^{-s-\sigma,p}(R^{n-1})}\right)\\ &\le M_1\left(\max_{x\in R^{n-1}}|a(x)|\|u\|_{B^{-s,p}(R^{n-1})} + c_2(a)\|u\|_{B^{-s-\sigma,p}(R^{n-1})}\right).\end{aligned}$$

Here we use (3.21), the fact that the operator χ_N is continuous from $B^{-s-\sigma,p}(R^{n-1})$ to $B^{-s,p}(R^{n-1})$, and the estimate

$$\|au\|_{B^{-s-\sigma,p}(R^{n-1})} \le c_2(a)\|u\|_{B^{-s-\sigma,p}(R^{n-1})} \quad (\sigma < 1).$$

To obtain (3.18) we use the inequality

$$\|u\|_{B^{-s-\sigma,p}(R^{n-1})} \le \epsilon\|u\|_{B^{-s,p}(R^{n-1})} + c(\epsilon)\|u\|_{B^{-s-1,p}(R^{n-1})},$$

where $\epsilon = (\max|a(x)|)/c_2(a) > 0$ (in the case $a(x) = 0$, (3.18) is obvious). The lemma is proved. □

4.4. **General problem in unbounded domains.** Consider the operators A_i, B_j, and L defined by (1.1)–(1.3) (Section 1). We will use the spaces E and F introduced in Sections 1.1 and 4.2, and the corresponding ∞-spaces:

$$E_\infty(\Omega) = \prod_{j=1}^{N} W_\infty^{l+t_j,p}(\Omega),$$

$$F_\infty(\Omega) = \prod_{i=1}^{N} W_\infty^{l-s_i,p}(\Omega) \times \prod_{j=1}^{m} W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega),$$

$$(E^*(\Omega))_\infty = \prod_{j=1}^{N} (\dot{W}^{-l-t_j,p'}(\Omega))_\infty,$$

$$(F^*(\Omega))_\infty = \prod_{i=1}^{N} (\dot{W}^{-l+s_i,p'}(\Omega))_\infty \times \prod_{j=1}^{m} (W^{-l+\sigma_j+1/p,p'}(\partial\Omega))_\infty,$$

$$(F^*_{-1}(\Omega))_\infty = \prod_{i=1}^{N} (\dot{W}^{-l+s_i-1,p'}(\Omega))_\infty \times \prod_{j=1}^{m} (W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega))_\infty.$$

We assume that the domain Ω satisfies the following condition.

Condition D. For each $x_0 \in \partial\Omega$ there exists a neighborhood $U(x_0)$ such that:

1. $U(x_0)$ contains a sphere with radius δ and center x_0, where δ is independent of x_0.
2. There exists a homeomorphism $\psi(x; x_0)$ of the neighborhood $U(x_0)$ on the unit sphere $B = \{y : |y| < 1\}$ in R^n such that the images of $\Omega \cap U(x_0)$ and $\partial\Omega \cap U(x_0)$ coincide with $B_+ = \{y : y_n > 0,\ |y| < 1\}$ and $B_0 = \{y : y_n = 0,\ |y| < 1\}$ respectively.
3. The function $\psi(x; x_0)$ and its inverse belong to the Hölder space $C^{r+\theta}$, $0 < \theta < 1$. Their $\|\cdot\|_{r+\theta}$-norms are bounded uniformly in x_0.

Here $r \ge \max(l + t_i, l - s_i, l - \sigma_j + 1)$, where the first expression under the maximum is required for a priori estimates of solutions [1], the second and the third ones for the proof of convergence in Lemma 4.5.4. For definiteness we suppose that $\delta < 1$.

Theorem 4.4.1. *For any $v \in (F^*(\Omega))_\infty$ the following estimate holds:*

$$\|v\|_{(F^*(\Omega))_\infty} \le M\left(\|L^*v\|_{(E^*(\Omega))_\infty} + \|v\|_{(F^*_{-1}(\Omega))_\infty}\right) \tag{4.1}$$

with a constant M independent of v.

The proof of the theorem will be given after some preliminary considerations. Let δ and ψ be the same as in Condition D, $B_\delta(x_0) = \{x : |x - x_0| < \delta\}$, $G_{x_0} = \psi(B_\delta(x_0))$. We introduce the operator of change of variables,

$$T : W^{s,p}(\Omega \cap B_\delta(x_0)) \to W^{s,p}(G_{x_0} \cap \{y_n > 0\}), \quad s \geq 0.$$

We will use the same notation also for the operator of change of variables in the space $W^{s,p}(\Gamma)$ $(s \geq 0, \Gamma = \partial\Omega)$ defined on functions with support in $B_\delta(x_0)$,

$$T : W^{s,p}(\Gamma) \to W^{s,p}(R^{n-1}_{y'}).$$

We have for functions with supports in $B_\delta(x_0)$:

$$T : F(\Omega) \to F(R^n_+), \quad T^{-1} : F(R^n_+) \to F(\Omega),$$
$$T : E(\Omega) \to E(R^n_+), \quad T^{-1} : E(R^n_+) \to E(\Omega),$$
$$L : E(\Omega) \to F(\Omega), \quad \tilde{L} = TLT^{-1} : E(R^n_+) \to F(R^n_+).$$

Consider the adjoint operators. We have

$$(\tilde{L})^* = (T^{-1})^* L^* T^* : F^*(R^n_+) \to E^*(R^n_+).$$

Here

$$T^* : F^*(R^n_+) \to F^*(\Omega), \quad (T^*)^{-1} : F^*(\Omega) \to F^*(R^n_+),$$
$$T^* : E^*(R^n_+) \to E^*(\Omega), \quad (T^*)^{-1} : E^*(\Omega) \to E^*(R^n_+).$$

Let $\tilde{v} \in F^*(R^n_+)$ satisfy the conditions of Theorem 4.3.1, and $v = T^*\tilde{v} \in F^*(\Omega)$. From Theorem 4.3.1 we have

$$\|\tilde{v}\|_{F^*(R^n_+)} \leq C \left(\|(\tilde{L})^*\tilde{v}\|_{E^*(R^n_+)} + \|\tilde{v}\|_{F^*_{-1}(R^n_+)} \right). \tag{4.2}$$

Since

$$(\tilde{L})^*\tilde{v} = (T^{-1})^* L^* v,$$

then from (4.2),

$$\|v\|_{F^*(\Omega)} \leq \|T^*\| \, \|\tilde{v}\|_{F^*(R^n_+)} \leq C\|T^*\| \left(\|(T^{-1})^* L^* v\|_{E^*(R^n_+)} + \|(T^*)^{-1} v\|_{F^*_{-1}(R^n_+)} \right).$$

Therefore

$$\|v\|_{F^*(\Omega)} \leq C_1 \left(\|L^* v\|_{E^*(\Omega)} + \|v\|_{F^*_{-1}(\Omega)} \right). \tag{4.3}$$

Suppose $\phi(x) \in C^\infty(\Gamma_\delta)$, $\operatorname{supp}\phi \subset B_\epsilon(x_0)$, Γ_δ is the δ-neighborhood of Γ, and $\epsilon > 0$ is taken such that $\epsilon \leq \delta/2$ and $\psi(B_\epsilon(x_0)) \subset \sigma_\rho$ with the same ρ as in Theorem 4.3.1. Then the previous estimate gives

$$\|\phi v\|_{F^*(\Omega)} \leq C_1 \left(\|L^*(\phi v)\|_{E^*(\Omega)} + \|\phi v\|_{F^*_{-1}(\Omega)} \right). \tag{4.4}$$

Let us estimate the difference $L^*(\phi v) - \phi L^* v$. For any $u \in E(\Omega)$,

$$\begin{aligned} \langle u, L^*(\phi v) - \phi L^* v \rangle &= \langle \phi L u - L(\phi u), v \rangle \\ &= \sum_{i=1}^N \langle \phi A_i u - A_i(\phi u), v_i \rangle + \sum_{j=1}^m \langle \phi B_j u - B_j(\phi u), w_j \rangle, \end{aligned} \tag{4.5}$$

where $v = (v_1, \ldots, v_N, w_1, \ldots, w_m)$. We begin with the first term on the right-hand side of (4.5). The operator A_i acts from $E(\Omega)$ to $W^{l-s_i+1,p}(\Omega)$. Let

$$T_i : W^{l-s_i+1,p}(R^n) \to W^{l-s_i,p}(R^n)$$

be a linear isomorphism between the two spaces. Then

$$(T_i^{-1})^* : (W^{l-s_i+1,p}(R^n))^* \to (W^{l-s_i,p}(R^n))^*.$$

Consider a function $\psi \in D$ such that $\operatorname{supp}\psi \in B_\delta(x_0)$, $\psi(x) \geq 0$, $\psi(x) = 1$ for $x \in B_{\delta/2}(x_0)$. Denote by $\tilde{u}$ an extension of u to $E(R^n)$ such that

$$\|\tilde{u}\|_{E(R^n)} \leq 2\|u\|_{E(\Omega)}. \tag{4.6}$$

We have $\psi v_i \in (W^{l-s_i,p}(\Omega))^* \subset (W^{l-s_i+1,p}(\Omega))^*$. Hence there exists an extension $\widehat{\psi v_i} \in (W^{l-s_i+1,p}(R^n))^*$ such that

$$\langle \phi A_i u - A_i(\phi u), \psi v_i\rangle = \langle \phi A_i \tilde{u} - A_i(\phi\tilde{u}), \widehat{\psi v_i}\rangle.$$

Here we assume that the coefficients of the operator A_i are extended to R^n. Hence

$$\begin{aligned}
|\langle \phi A_i u - A_i(\phi u), v_i\rangle| &= |\langle \phi A_i u - A_i(\phi u), \psi v_i\rangle| \\
&= |\langle T_i^{-1} T_i(\phi A_i \tilde{u} - A_i(\phi\tilde{u})), \widehat{\psi v_i}\rangle| \\
&= |\langle T_i(\phi A_i \tilde{u} - A_i(\phi\tilde{u})), (T_i^{-1})^* \widehat{\psi v_i}\rangle| \\
&\leq \|T_i(\phi A_i \tilde{u} - A_i(\phi\tilde{u}))\|_{W^{l-s_i,p}(R^n)} \|(T_i^{-1})^* \widehat{\psi v_i}\|_{(W^{l-s_i,p}(R^n))^*} \\
&\leq C_1 \|\tilde{u}\|_{E(R^n)} \|\widehat{\psi v_i}\|_{(W^{l-s_i+1,p}(R^n))^*} \\
&\leq C_2 \|u\|_{E(\Omega)} \|\psi v_i\|_{(W^{l-s_i+1,p}(\Omega))^*}
\end{aligned} \tag{4.7}$$

according to (4.6).

We obtain similar estimates for the operators B_j in (4.5):

$$\begin{aligned}
|\langle \phi B_j u - B_j(\phi u), w_j\rangle| &= |\langle S_j^{-1} S_j(\phi B_j u - B_j(\phi u)), \psi w_j\rangle| \\
&= |\langle S_j(\phi B_j u - B_j(\phi u)), (S_j^{-1})^*(\psi w_j)\rangle| \\
&\leq \|S_j(\phi B_j u - B_j(\phi u))\|_{W^{l-\sigma_j-1/p,p}(\Gamma)} \|(S_j^{-1})^*(\psi w_j)\|_{(W^{l-\sigma_j-1/p,p}(\Gamma))^*} \\
&\leq C_3 \|u\|_{E(\Omega)} \|\psi w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(\Gamma)}.
\end{aligned} \tag{4.8}$$

From (4.5), (4.7), (4.8) we obtain

$$\begin{aligned}
&|\langle u, L^*(\phi v) - \phi L^* v\rangle| \\
&\quad \leq C_4 \|u\|_{E(\Omega)} \| \left(\sum_{i=1}^N \|\psi v_i\|_{(W^{l-s_i+1,p}(\Omega))^*} + \sum_{j=1}^m \|\psi w_j\|_{W^{-l+\sigma_j+1/p-1,p'}(\Gamma)} \right) \\
&\quad = C_4 \|u\|_{E(\Omega)} \|\psi v\|_{F^*_{-1}(\Omega)}.
\end{aligned}$$

Therefore

$$\|L^*(\phi v) - \phi L^* v\|_{E^*(\Omega)} \leq C_4 \|\psi v\|_{F^*_{-1}(\Omega)}. \tag{4.9}$$

From this estimate and (4.4) it follows that

$$\|\phi v\|_{F^*(\Omega)} \leq C_5 \left(\|\phi L^* v\|_{E^*(\Omega)} + \|\phi v\|_{F^*_{-1}(\Omega)} + \|\psi v\|_{F^*_{-1}(\Omega)} \right). \tag{4.10}$$

Proof of Theorem 4.4.1. Let δ be the same as in Condition D. We cover the boundary Γ of the domain Ω by a countable number of balls B_ϵ of radius ϵ, where $\epsilon \leq \delta/2$ is the number that appears in the proof of estimate (4.10), and extend this covering to a covering of Ω. Let $V_j, j = 1, 2, \ldots,$ be all the balls of the covering, and let $\tilde{V}_j$ be the balls with the same centers as V_j but with radius δ. We suppose that there exists a number N such that each of the balls $\tilde{V}_j$ has a nonempty intersection with at most N other balls.

Furthermore, let $\phi_j(x)$ and $\tilde{\phi}_j(x)$ be systems of nonnegative functions such that

$$\phi_j(x) \in C^\infty(R^n),\ \operatorname{supp}\phi_j \subset V_j, \qquad \tilde{\phi}_j(x) \in C^\infty(R^n),\ \operatorname{supp}\tilde{\phi}_j \subset \tilde{V}_j,$$

and $\tilde{\phi}_j(x) = 1$ for $x \in V_j$. For the balls V_j with centers at Γ by virtue of (4.10) we have the estimate

$$\|\phi_j v\|_{F^*(\Omega)} \le M_0 \left(\|\phi_j L^* v\|_{E^*(\Omega)} + \|\phi_j v\|_{F^*_{-1}(\Omega)} + \|\tilde{\phi}_j v\|_{F^*_{-1}(\Omega)} \right) \tag{4.11}$$

with a constant M_0 independent of j and v.

The covering of the domain Ω can be constructed in such a way that all other balls, with centers outside of Γ, do not contain points of the boundary. We can obtain a similar estimate for them. It is even simpler because we do not have to take into account the boundary operators.

By the definition of the spaces $(E^*(\Omega))_\infty$ and $(F^*_{-1}(\Omega))_\infty$ we have for any j:

$$\|\phi_j L^* v\|_{E^*(\Omega)} \le M_1 \|L^* v\|_{(E^*(\Omega))_\infty}, \quad \|\phi_j v\|_{F^*_{-1}(\Omega)} \le M_2 \|v\|_{(F^*_{-1}(\Omega))_\infty},$$
$$\|\tilde{\phi}_j v\|_{F^*_{-1}(\Omega)} \le M_3 \|v\|_{(F^*_{-1}(\Omega))_\infty},$$

where the constants M_1, M_2, M_3 do not depend on v and j. Therefore (4.11) gives

$$\|\phi_j v\|_{F^*(\Omega)} \le M \left(\|L^* v\|_{(E^*(\Omega))_\infty} + \|v\|_{(F^*_{-1}(\Omega))_\infty} \right).$$

Estimate (4.1) follows from this. The theorem is proved. □

4.5. **Estimates with Condition NS*.** Normal solvability of elliptic problems in unbounded domains is determined not only by the ellipticity condition (including proper ellipticity and the Lopatinskii condition) but also by Condition NS introduced in Section 3.3. We introduce a similar condition for adjoint problems.

Condition NS*. Any limiting homogeneous problem $\hat{L}^* v = 0$ does not have nonzero solutions in $(F^*(\hat{\Omega}))_\infty$, where $\hat{L}^*$ is the operator adjoint to the limiting operator $\hat{L}$, and $\hat{\Omega}$ is a limiting domain.

In this section we prove the following theorem.

Theorem 4.5.1. *Let L be an elliptic operator, and let Condition NS* be satisfied. Then there exist positive numbers M and ρ such that for any $v \in (F^*(\Omega))_\infty$ the following estimate holds:*

$$\|v\|_{(F^*(\Omega))_\infty} \le M \left(\|L^* v\|_{(E^*(\Omega))_\infty} + \|v\|_{F^*_{-1}(\Omega_\rho)} \right). \tag{5.1}$$

Here

$$F^*_{-1}(\Omega_\rho) = \prod_{i=1}^{N} \dot{W}^{-l+s_i-1,p'}(\Omega_\rho) \times \prod_{j=1}^{m} \dot{W}^{-l+\sigma_j+1/p-1,p'}(\Gamma_\rho),$$

and Ω_ρ and Γ_ρ are the intersections of Ω and Γ with the ball $|x| < \rho$.

Proof. Suppose that the assertion of the theorem is not right. Let $M_k \to \infty$, $\rho_k \to \infty$ be given sequences. Then there exist $v_k \in (F^*(\Omega))_\infty$ such that

$$\|v_k\|_{((F(\Omega))^*)_\infty} \ge M_k \left(\|L^* v_k\|_{((E(\Omega))^*)_\infty} + \|v_k\|_{F^*_{-1}(\Omega_{\rho_k})} \right). \tag{5.2}$$

We can assume that

$$\|v_k\|_{((F(\Omega))^*)_\infty} = 1. \tag{5.3}$$

Then from (5.2),

$$\|L^* v_k\|_{((E(\Omega))^*)_\infty} + \|v_k\|_{F^*_{-1}(\Omega_{\rho_k})} < \frac{1}{M_k} \to 0 \text{ as } k \to \infty. \tag{5.4}$$

From (4.1),

$$\|L^* v_k\|_{((E(\Omega))^*)_\infty} + \|v_k\|_{(F^*_{-1}(\Omega))_\infty} \ge \frac{1}{M}. \tag{5.5}$$

Estimate (5.4) implies that

$$\|L^* v_k\|_{((E(\Omega))^*)_\infty} \to 0, \quad \|v_k\|_{F^*_{-1}(\Omega_{\rho_k})} \to 0 \text{ as } k \to \infty. \tag{5.6}$$

Hence

$$\|v_k\|_{(F^*_{-1}(\Omega))_\infty} > \frac{1}{2M} \tag{5.7}$$

for k sufficiently large. The norm

$$\|v_k\|_{(F^*_{-1}(\Omega))_\infty} = \sup_{y\in\Omega} \|v_k\|_{F^*_{-1}(\Omega\cap B_y)}$$

is equivalent to that given in the introduction. It follows from (5.7) that there exists $y_k \in \Omega$ such that

$$\|v_k\|_{F^*_{-1}(\Omega\cap B_{y_k})} > \frac{1}{2M}\,. \tag{5.8}$$

From this and (5.6) we conclude that $|y_k| \to \infty$. Denote

$$L^* v_k = z_k. \tag{5.9}$$

From (5.6) it follows that

$$\|z_k\|_{(E^*(\Omega))_\infty} \to 0 \text{ as } k \to \infty.$$

Let T_h be the operator of translation in $(E^*(\Omega))_\infty$, $h \in R^n$. We apply T_{y_k} to (5.9):

$$T_{y_k} L^* v_k = T_{y_k} z_k. \tag{5.10}$$

The shifted functions are defined in shifted domains Ω_k. We will pass to the limit in this equality as $k \to \infty$. We have

$$T_{y_k} L^* T_{-y_k} T_{y_k} v_k = T_{y_k} z_k, \tag{5.11}$$

where the operators T_{y_k} act in the corresponding spaces. Denote

$$w_k = T_{y_k} v_k, \quad L^*_k = T_{y_k} L^* T_{-y_k}, \quad T_{y_k} z_k = \zeta_k. \tag{5.12}$$

From (5.11) we obtain

$$L^*_k w_k = \zeta_k. \tag{5.13}$$

Let $v_k = (v_{1k}, \dots, v_{Nk}, v^b_{1k}, \dots, v^b_{mk})$. From (5.3) we have

$$\begin{aligned} &\|v_{ik}\|_{(\dot W^{-l+s_i,p'}(\Omega))_\infty} \le 1, \qquad i = 1, \dots, N, \\ &\|v^b_{jk}\|_{(W^{-l+\sigma_j+1/p,p'}(\Gamma))_\infty} \le 1, \qquad j = 1, \dots, m. \end{aligned} \tag{5.14}$$

Denoting $w_k = (w_{1k}, \dots, w_{Nk}, w^b_{1k}, \dots, w^b_{mk})$, we have

$$w_{ik} = T_{y_k} v_{ik}, \quad i = 1, \dots, N, \qquad w^b_{jk} = T_{y_k} v^b_{jk}, \quad j = 1, \dots, m.$$

Then from (5.14),

$$\|w_{ik}\|_{(\dot W^{-l+s_i,p'}(\Omega_k))_\infty} \le 1, \quad i = 1, 2, \dots, N, \quad \|w^b_{jk}\|_{(W^{-l+\sigma_j+1/p,p'}(\partial\Omega_k))_\infty} \le 1, \tag{5.15}$$

where Ω_k is the shifted domain, $j = 1, 2, \dots, m$. Consider first the functions w_{ik}. Since

$$(\dot W^{-l+s_i,p'}(\Omega_k))_\infty \subset W^{-l+s_i,p'}_\infty(R^n),$$

then $w_{ik} \in W^{-l+s_i,p'}_\infty(R^n)$, and

$$\|w_{ik}\|_{W^{-l+s_i,p'}_\infty(R^n)} \le 1. \tag{5.16}$$

It follows from Theorem 2.3.3 that there exists a subsequence w_{ik_j} and a function $\bar w_i \in W^{-l+s_i,p'}_\infty(R^n)$ such that

$$\phi w_{ik_j} \to \phi \bar w_i \text{ weakly in } W^{-l+s_i,p'}(R^n) \text{ as } k_j \to \infty \tag{5.17}$$

for any $\phi \in D$. Moreover the sequence w_{ik_j} can be chosen such that for an $\epsilon > 0$,

$$\phi w_{ik_j} \to \phi \bar{w}_i \text{ strongly in } W^{-l+s_i-\epsilon,p'}(R^n) \text{ as } k_j \to \infty \tag{5.18}$$

for any $\phi \in D$.

For what follows we need a *special covering* of the boundary $\partial\Omega_*$ of the limiting domain Ω_*. Let $x_0 \in \partial\Omega_*$. Then there exists a sequence $\hat{x}_k$ such that $\hat{x}_k \to x_0$, $\hat{x}_k \in \partial\Omega_k$. For each point $\hat{x}_k$ and domain Ω_k there exists a neighborhood $U(\hat{x}_k)$ and a function $\psi_k(x) = \psi(x; \hat{x}_k)$ defined in Condition D. It maps $U(\hat{x}_k)$ on the unit ball $B \subset R^n$ with center at 0. This mapping is a bijection. Denote $\phi_k = \psi_k^{-1}$. By Condition D, the functions ϕ_k are uniformly bounded in $C^{r+\theta}(B)$. Hence this sequence has a convergent in $C^r(B)$ subsequence: $\phi_{k_i} \to \phi_0$ in $C^r(B)$. Denote $U(x_0) = \phi_0(B)$. The mapping $\phi_0 : B \to U(x_0)$ is also a bijection. Denote $\psi_0 = \phi_0^{-1} : U(x_0) \to B$. Then $\psi_{k_i} \to \psi_0 \in C^{r+\theta}(U(x_0))$, and $U(x_0)$ is an open set that contains a sphere $S(x_0)$ of radius δ.

Consider now a sequence $x_j \in \partial\Omega_*$, and denote by $S(x_j)$ the spheres of radius $\delta/2$ and with centers at x_j. We can take the points x_j such that the union W of the spheres $S(x_j)$ covers the $\delta/4$-neighborhood of the boundary $\partial\Omega_*$. We repeat the construction above for the point x_1 of this sequence. We choose a subsequence of the previous sequence (denoted also k_i) such that

$$\psi_{k_i} \to \psi^1, \ \ \phi_{k_i} \to \phi^1, \ \ U(x_1) = \phi^1(B), \ \ S(x_1) \subset U(x_1).$$

We then repeat the same construction for the point x_2 and so on, and take the diagonal subsequence. Therefore we construct neighborhoods $U(x_j)$ of all points x_j. Moreover this construction can be done in such a way that for some number N, any N different sets $U(x_j)$ have an empty intersection, and for any compact $K \subset R^n$ the number of the sets for which $K \cap U(x_j) \neq \emptyset$ is finite.

The covering $V = \bigcup_{j=1}^{\infty} U(x_j)$ is called a special covering of $\partial\Omega_*$. Hence we have a sequence Ω_k, points $x_k^j \in \partial\Omega_k$, neighborhoods $U(x_k^j)$ of the points x_k^j, and mappings $\psi_k^j : U(x_k^j) \to B$, $\phi_k^j = (\psi_k^j)^{-1}$, such that

$$\psi_k^j \to \psi^j, \ \ \phi_k^j \to \phi^j, \ \ U(x_j) = \phi^j(B).$$

Consider a sequence g_k defined on $\partial\Omega_k$ such that

$$\|g_k\|_{(E(\partial\Omega_k))_\infty} \leq K, \tag{5.19}$$

where K is a constant independent of k, $E = W^{-s,p}$, $s > 0$. The norm in (5.19) is defined as follows. Let $\eta(x) \in C^\infty(R^n)$ be such that

$$\eta(x) \geq 0, \ \eta(x) = 1, \ |x| < \frac{\delta}{2}, \quad \eta(x) = 0, \ |x| > \delta.$$

Denote $\eta_z = \eta(x - z)$, $z \in R^n$. Then

$$\|g_k\|_{(E(\partial\Omega_k))_\infty} = \sup_{z \in \partial\Omega_k} \|(\eta_z g_k) \circ \psi_z^{-1}\|_{E(R^{n-1}_{y'})}, \tag{5.20}$$

where ψ_z maps the neighborhood $U(z)$ to the ball B.

Consider the neighborhood $U(x_k^1)$ of the point $x_k^1 \in \partial\Omega_k$. The function ψ_k^1 maps $U(x_k^1) \cap \partial\Omega_k$ onto $B_0 = B \cap \{y^n = 0\}$. We can define a generalized function $\tilde{g}_k^1$ on D_{B_0} by the equality

$$\tilde{g}_k^1 = (\eta_{x_k^1} g_k) \circ (\psi_k^1)^{-1}.$$

We extend it to $D_{R^{n-1}}$ by zero outside B_0. It follows from (5.19), (5.20) that

$$\|\tilde{g}_k^1\|_{E(R^{n-1}_{y'})} \leq K, \tag{5.21}$$

where $E(R^{n-1}_{y'}) = W^{-s,p}(R^{n-1}_{y'})$. Since this space is reflexive, we can find a subsequence $\tilde{g}^1_{k_j}$ and a function $\tilde{h}^1 \in E(R^{n-1}_{y'})$ such that $\tilde{g}^1_{k_j} \to \tilde{h}^1$ weakly in $E(R^{n-1}_{y'})$ and $\tilde{g}^1_{k_j} \to \tilde{h}^1$ strongly in $E_{-1}(R^{n-1}_{y'})$ $(= W^{-s-1,p}(R^{n-1}_{y'}))$. We use here the compact embedding of E into E_{-1} in bounded domains. The generalized function $\tilde{h}^1$ is defined on D_{B_0}. Denote by h^1 the corresponding generalized function defined on $U(x_1) \cap \partial\Omega_*$: $h^1 = \tilde{h}^1 \circ \psi^1$. We extend h^1 by zero outside $U(x_1) \cap \partial\Omega_*$ on $\partial\Omega_*$.

We construct next a generalized function h^2 on $D_{U(x_2)\cap\partial\Omega_*}$. The construction is the same, but we consider a subsequence of the previous subsequence. We continue this construction for all x_j and take a diagonal subsequence. Denote this subsequence k_l.

Thus we have the following result. There exists a subsequence k_l of k such that for any j there exists a generalized function h^j on $D_{U(x_j)\cap\partial\Omega_*}$ defined by the equality $h^j = \tilde{h}^j \circ \psi^j$,

$$\|\tilde{g}^j_{k_l} - \tilde{h}^j\|_{E_{-1}(R^{n-1}_{y'})} \to 0 \text{ as } k_l \to \infty,$$

$$\tilde{g}^j_{k_l} \to \tilde{h}^j \text{ weakly in } E(R^{n-1}_{y'}) \text{ as } k_l \to \infty.$$

Moreover,

$$\|\tilde{g}^j_{k_l}\|_{E(R^{n-1}_{y'})} \le K. \tag{5.22}$$

The points $x_j \in \partial\Omega_*$ and the functions $\eta_{x^j_k}$ can be chosen such that $g_k = \sum_j \eta_{x^j_k} g_k$, $x \in \partial\Omega_k$.

Denote

$$h = \sum_{j=1}^{\infty} h^j.$$

This is the limiting function for the sequence g_k. We note that for any $\phi \in D_{\partial\Omega_*}$ we have

$$\langle h, \phi\rangle = \sum_{j'} \langle h^{j'}, \phi\rangle,$$

where the j' are those j for which $\operatorname{supp}\phi \cap U(x_j) \neq \emptyset$. By the construction of $U(x_j)$, the number of such j' is finite. □

Lemma 4.5.2. *The limiting generalized function h belongs to $(W^{-s,p}(\partial\Omega_*))_\infty$, that is,*

$$\|h\|_{(E(\partial\Omega_*))_\infty} = \sup_{z\in\partial\Omega_*} \|(\eta_z h)\circ\psi_z^{-1}\|_{E(R^{n-1}_{y'})} < \infty. \tag{5.23}$$

Proof. Suppose that it is not so. Then there is a sequence $z_i \in \partial\Omega_*$ such that

$$\|(\eta_{z_i} h)\circ\psi_{z_i}^{-1}\|_{E(R^{n-1}_{y'})} \to \infty \text{ as } i \to \infty. \tag{5.24}$$

Since $\eta_{z_i} h^j = 0$ for all j except for a finite number of them less than or equal to N, then there is a sequence h^{j_i} such that

$$\|(\eta_{z_i} h^{j_i})\circ\psi_{z_i}^{-1}\|_{E(R^{n-1}_{y'})} \to \infty \text{ as } i \to \infty.$$

Therefore

$$\|\eta_{z_i}(\phi_{z_i}(y))\tilde{h}^{j_i}\|_{E(R^{n-1}_{y'})} \to \infty \text{ as } i \to \infty. \tag{5.25}$$

From this we can conclude that there exists a functional $F \in E^*(R^{n-1}_{y'})$ such that

$$F(\eta_{z_i}(\phi_{z_i}(y))\tilde{h}^{j_i}) \to \infty \text{ as } i \to \infty.$$

From the weak convergence

$$\tilde{g}^{j_i}_k \to \tilde{h}^{j_i} \text{ as } k \to \infty$$

it follows that for some sequence k_i,

$$F(\eta_{z_i}(\phi_{z_i}(y))\tilde{g}^{j_i}_{k_i}) \to \infty \text{ as } i \to \infty.$$

This contradicts estimate (5.22). The lemma is proved. □

Thus from (5.15) and Lemma 4.5.2 we can conclude that there exist limiting functions

$$\bar{w}^b_j \in (W^{-l+\sigma_j+1/p,p'}(\partial\Omega_*))_\infty, \ \ j = 1, \dots, m.$$

Existence of the limits $\bar{w}_i \in W^{-l+s_i,p'}_\infty(R^n)$, $j = 1, \dots, N$ was proved above. Denote $\bar{w} = (\bar{w}_1, \dots, \bar{w}_N, \bar{w}^b_1, \dots, \bar{w}^b_m)$.

Lemma 4.5.3. *The limiting function $\bar{w}$ is a solution of the problem adjoint to a limiting problem.*

Proof. Consider equation (5.13). It is supposed that we have done a special covering of $\partial\Omega_*$. Let ψ^i_k and ϕ^i_k be the functions from the special covering,

$$\psi^j_k \to \psi^j, \ \ \phi^j_k \to \phi^j, \ \ U(x^j_k) = \phi^j_k(B), \ \ U(x_j) = \phi^j(B).$$

Let $\theta(x) = (\theta^1(x), \dots, \theta^N(x))$, where $\theta^i(x) \in D(R^n)$, $\operatorname{supp}\theta^i \subset U(x_j)$, and let $\theta_k(x)$ be the corresponding function with support in $U(x^j_k)$: $\theta_k = \theta(\phi^j(\psi^j_k))$. From (5.13) we have

$$\langle L^*_k w_k, \theta_k\rangle = \langle \zeta_k, \theta_k\rangle$$

or

$$\langle w_k, T_{y_k} L T_{-y_k}\theta_k\rangle = \langle \zeta_k, \theta_k\rangle.$$

We can rewrite this equality in the form

$$\langle w_k, L_k\theta_k\rangle = \langle \zeta_k, \theta_k\rangle, \tag{5.26}$$

where

$$L_k = T_{y_k} L T_{-y_k}, \qquad L_k = (A_{1k}, \dots, A_{Nk}, B_{1k}, \dots, B_{mk}),$$
$$A_{ik} = T_{y_k} A_i T_{-y_k}, \qquad B_{jk} = T_{y_k} B_j T_{-y_k},$$
$$T_{y_k} A_i u = \sum_{l=1}^N \sum_{|\alpha|\le\alpha_{il}} T_{y_k}(a^\alpha_{il}(x)) T_{y_k} D^\alpha u_l = \sum_{l=1}^N \sum_{|\alpha|\le\alpha_{il}} a^\alpha_{il}(x + y_k) T_{y_k} D^\alpha u_l.$$

Hence

$$A_{ik}u = \sum_{l=1}^N \sum_{|\alpha|\le\alpha_{il}} a^\alpha_{ilk}(x) D^\alpha u_l, \ \ B_{jk}u = \sum_{l=1}^N \sum_{|\beta|\le\beta_{jl}} b^\beta_{jlk}(x) D^\beta u_l,$$

where

$$a^\alpha_{ilk}(x) = a^\alpha_{il}(x + y_k), \ \ b^\beta_{jlk}(x) = b^\beta_{jl}(x + y_k).$$

We can rewrite (5.26) in the form

$$\sum_{i=1}^N \langle w_{ik}, A_{ik}\theta_k\rangle + \sum_{j=1}^m \langle w^b_{jk}, B_{jk}\theta_k\rangle = \langle \zeta_k, \theta_k\rangle. \tag{5.27}$$

We will pass to the limit in this equality. We begin with the first term on the left-hand side. From (5.17) we have the weak convergence

$$\phi w_{ik} \to \phi\bar{w}_i \text{ in } W^{-l+s_i,p'}(R^n) \text{ as } k \to \infty \tag{5.28}$$

for any $\phi \in D$ (we write k instead of k_j). By the definition of the limiting problem,

$$a^\alpha_{ipk}(x) \to \hat{a}^\alpha_{ip}(x) \text{ in } C^{l-s_i}_{loc}(R^n), \ \ p = 1, \dots, N.$$

Here $\hat{a}^{\alpha}_{ip}(x)$ are the coefficients of the limiting operator. From the definition of $\theta_k(x)$ we have

$$\lim_{k\to\infty} \theta_k(x) = \theta(\phi^j(\psi^j(x))) = \theta(x),$$

where this limit is supposed to be in C^r. We suppose that $\psi^j_k(x)$ and $\psi^j(x)$ are extended on a ball which contains $U(x_j)$ and $U(x^j_k)$ with k sufficiently large. Then

$$A_{ik}\theta_k \to \sum_{p=1}^{N} \sum_{|\alpha|\le\alpha_{ip}} \hat{a}^{\alpha}_{ip}(x) D^{\alpha}\theta^p = \hat{A}_i\theta, \quad k\to\infty$$

in $C^{l-s_i}(R^n)$. Here $\hat{A}_i$ is the limiting operator. From (5.28) it follows that

$$\sum_{i=1}^{N} \langle w_{ik}, A_{ik}\theta_k\rangle \to \sum_{i=1}^{N} \langle \bar{w}_i, \hat{A}_i\theta\rangle, \quad k\to\infty. \tag{5.29}$$

Now consider $\langle w^b_{ik}, B_{ik}\theta_k\rangle$. Let η_z be the function which is used in the definition of the limiting function h above. Instead of the functions g_k considered above we take functions w^b_{ik}, and instead of k_l we write k. We obtain a sequence of functions

$$\tilde{w}^{bj}_{ik} = (\eta_{x^1_k} w^b_{ik}) \circ \phi^j_k,$$

where $\phi^j_k = (\psi^1_k)^{-1}$.

As above $\tilde{w}^{bj}_{ik} \to \tilde{w}^{bj}_i$ weakly in $W^{-l+\sigma_i+1/p,\, p'}(R^{n-1}_{y'})$ as $k\to\infty$. Denote $w^{bj}_i = \tilde{w}^{bj}_i \circ \psi^j$ and

$$\bar{w}^b_i = \sum_{j=1}^{\infty} w^{bj}_i.$$

This is the limiting function for the sequence w^b_{ik}.

Suppose that $x \in \partial\Omega_k \cap U(x^j_k)$. Denote $f_{ik} = B_{ik}\theta_k$. Since $\operatorname{supp}\theta_k \subset U(x^j_k)$, then $\operatorname{supp} f_{ik} \subset U(x^j_k)$. We have

$$\langle \eta_{x^j_k} w^b_{ik}, f_{ik}\rangle = \langle \tilde{w}^{bj}_{ik}, (f_{ik}\rho_k)\circ\phi^j_k\rangle,$$

where ρ_k is the density for the manifold Ω_k. The density is the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega_k$ written in local coordinates. Furthermore,

$$f_{ik}(\phi^j_k(y)) = \sum_{p=1}^{N} \sum_{|\beta|\le\beta_{ip}} b^{\beta}_{ipk}(\phi^j_k(y)) D^{\beta}\theta^p_k(\phi^j_k(y)).$$

By the definition of limiting problems, the functions $b^{\beta}_{ip}(x)$ are extended to R^n and

$$b^{\beta}_{ipk}(x) \to \hat{b}^{\beta}_{ip}(x) \ \text{ in } C^{l-\sigma_i}_{loc}(R^n) \ \text{ as } k\to\infty,$$

where $\hat{b}^{\beta}_{ip}(x)$ are the limiting coefficients. Since

$$\lim_{k\to\infty} \theta_k(\phi^j_k(y)) = \lim_{k\to\infty} \theta(\phi^j(\psi^j_k(\phi^j_k(y)))) = \theta(\phi^j(\psi^j(\phi^j(y)))) = \theta(\phi^j(y)),$$

then

$$f_{ik}(\phi^j_k(y)) \to \sum_{p=1}^{N} \sum_{|\beta|\le\beta_{ip}} \hat{b}^{\beta}_{ip}(\phi^j(y)) D^{\beta}\theta^p(\phi^j(y)) \equiv f_i(\phi^j(y)).$$

This convergence is in $C^{l-\sigma_i}$.

Therefore

$$\langle \tilde{w}^{bj}_{ik}, (f_{ik}\rho_k)\circ\phi^j_k\rangle \to \langle \tilde{w}^{bj}_i, (f_i\rho_*)\circ\phi^j\rangle \ \text{ as } \ k\to\infty,$$

where ρ_* is the density for the manifold Ω_*. It follows that

$$\langle \eta_{x_k^j} w_{ik}^b, f_{ik}\rangle \to \langle \tilde{w}_i^{bj}, (f_i\rho_*)\circ\phi^j\rangle = \langle w_i^{bj}, f_i\rangle,$$

which is the duality on $\partial\Omega_*$.

Taking the sum in j we get

$$\langle w_{ik}^b, f_{ik}\rangle \to \langle \bar{w}_i^b, \sum_{p=1}^N \sum_{|\beta|\le\beta_{ip}} \hat{b}_{ip}^\beta D^\beta\theta^p\rangle.$$

From this and (5.29) it follows that

$$\sum_{i=1}^N \langle w_{ik}, A_{ik}\theta_k\rangle + \sum_{i=1}^m \langle w_{ik}^b, B_{ik}\theta_k\rangle \to \sum_{i=1}^N \langle \bar{w}_i, \hat{A}_i\theta\rangle + \sum_{i=1}^m \langle \bar{w}_i^b, \hat{B}_i\theta\rangle,$$

where

$$\hat{B}_i\theta = \sum_{p=1}^N \sum_{|\beta|\le\beta_{ip}} \hat{b}_{ip}^\beta D^\beta\theta^p.$$

We will prove that

$$\sum_{i=1}^N \langle \bar{w}_i, \hat{A}_i\theta\rangle + \sum_{i=1}^m \langle \bar{w}_i^b, \hat{B}_i\theta\rangle = 0. \tag{5.30}$$

It is sufficient to show that

$$\langle \zeta_k, \theta_k\rangle \to 0, \quad k\to\infty \tag{5.31}$$

(see (5.27)). We recall that

$$\|\zeta_k\|_{(E^*(\Omega_k))_\infty} \to 0, \quad k\to\infty.$$

Since the diameters of $\operatorname{supp}\theta_k$ are uniformly bounded, convergence (5.31) follows from the last convergence and from the boundedness of the norm $\|\theta_k\|_{E(R^n)}$ independently of k. The lemma is proved. □

To finish the proof of the theorem it remains to prove the following lemma.

Lemma 4.5.4. *The solution $\bar{w}$ of the limiting problem* (5.30) *is different from* 0.

Proof. If $(\bar{w}_1,\dots,\bar{w}_N)\ne 0$, then the lemma is proved. Consider the case $(\bar{w}_1,\dots,\bar{w}_N) = 0$. From (5.8), (5.12) we get

$$\|w_k\|_{F_{-1}^*(\Omega_k\cap B_0)} > \frac{1}{2M}. \tag{5.32}$$

Therefore

$$\sum_{j=1}^m \|w_{jk}^b\|_{W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega_k\cap B_0)} > \frac{1}{2M}$$

for $k>k_0$ if k_0 is sufficiently large. For any $k>k_0$ there exists $j=j_k$ such that

$$\|w_{jk}^b\|_{W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega_k\cap B_0)} > \frac{1}{2Mm}.$$

Passing to a subsequence if necessary we can assume that j is the same for all k. Hence

$$\|\bar{w}_j^b\|_{W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega_*\cap B_0)} \ge \frac{1}{2Mm}, \tag{5.33}$$

and $\bar{w}_j$ is different from 0 as an element of the space $W^{-l+\sigma_j+1/p-1,p'}(\partial\Omega_* \cap B_0)$. Consequently, it is also different from 0 as an element of $W^{-l+\sigma_j+1/p,p'}(\partial\Omega_* \cap B_0)$. Indeed, if it is not so, then

$$\langle \bar{w}_j^b, \phi\rangle = 0 \ \ \forall \phi \in W^{l-\sigma_j-1/p,p}(\partial\Omega_* \cap B_0).$$

Then the same equality is true for all $\phi \in W^{l-\sigma_j-1/p+1,p}(\partial\Omega_* \cap B_0)$. But this contradicts (5.33). The lemma is proved. □

Thus, assuming that (5.1) does not hold, we have obtained a nonzero solution of a limiting problem, which contradicts Condition NS*. The theorem is proved.

Corollary 4.5.5. *If Condition NS* is satisfied, then the operator $L^* : (E^*(\Omega))_\infty \to (F^*(\Omega))_\infty$ is normally solvable with a finite-dimensional kernel.*

5. Fredholm theorems

5.1. **Abstract operators.** Let $E = E(\Omega)$ and $F^d(\Omega)$ be Banach spaces of functions defined in a domain Ω, $F^b(\partial\Omega)$ be a space of functions defined at the boundary $\partial\Omega$, $F = F^d(\Omega) \times F^b(\partial\Omega)$. We assume that these spaces satisfy the conditions of Section 2.

Furthermore, let $L : E \to F$ be a local operator in the sense of Section 2.6. Then we can define its realization in various spaces:

$$L_\infty : E_\infty \to F_\infty, \ \ L_D : E_D \to F_D, \ \ L_q : E_q \to F_q, \ 1 \le q < \infty.$$

Here E_D and F_D are the closures of D in E_∞ and F_∞, respectively.

We will consider also the adjoint operators

$$(L_\infty)^* : (F_\infty)^* \to (E_\infty)^*, \ \ (L_D)^* : (F_D)^* \to (E_D)^*, \ \ (L_q)^* : (F_q)^* \to (E_q)^*, \ 1 \le q < \infty$$

and the operator

$$(L^*)_\infty : (F^*)_\infty \to (E^*)_\infty.$$

We recall that

$$(E^*)_\infty = (E_1)^*, \quad (F^*)_\infty = (F_1)^*.$$

Therefore by the definition of local operators $(L^*)_\infty = (L_1)^*$. This equality is understood as

$$\langle (L^*)_\infty w, \theta\rangle = \langle w, L_1\theta\rangle$$

for any $w \in (F_1)^*$, and any $\theta \in E_1$. It is sufficient to consider it for $\forall\theta \in D$.

We suppose that there exist Banach spaces of distributions $\mathcal{E}$ and $\mathcal{F}$ such that the spaces E and F are imbedded in them locally compactly. This means that for any ball B_ρ with radius ρ the restriction $E(\Omega_\rho)$ of the spaces $E(\Omega)$ to $\Omega_\rho = \Omega \cap B_\rho$ is compactly imbedded into the space $\mathcal{E}(\Omega_\rho)$. A similar property holds for the spaces F and $\mathcal{F}$.

We note that $E_q(\Omega_\rho) = E(\Omega_\rho)$. Therefore $E_q(\Omega_\rho)$ is also compactly imbedded in $\mathcal{E}(\Omega_\rho)$.

Lemma 5.1.1. *Suppose that the following estimate*

$$\|u\|_{E_q} \le M\left(\|L_q u\|_{F_q} + \|u\|_{\mathcal{E}(\Omega_\rho)}\right) \tag{1.1}$$

holds for some positive constants M and ρ, and any $u \in E_q$. Then the operator L_q is proper; that is, the inverse image of a compact set is compact in any bounded closed ball. Here $1 \le q \le \infty$.

Proof. Let $L_q u_k = f_k$, $f_k \to f_0$ in F_q, and $\|u_k\|_{E_q} \le C$ for some constant C and all k. Let us take ρ for which (1.1) is satisfied. Then there exists a subsequence u_{kn} fundamental in $\mathcal{E}(\Omega_\rho)$:

$$\|u_{kn} - u_{km}\|_{\mathcal{E}(\Omega_\rho)} \to 0 \text{ as } m, n \to \infty.$$

From (1.1) it follows that the same subsequence is fundamental in E_q. The lemma is proved. □

Corollary 5.1.2. *The operator L_q is normally solvable with a finite-dimensional kernel.*

We repeat the same construction for the adjoint operators. We suppose that there exist spaces $\mathcal{E}^*$ and $\mathcal{F}^*$ such that the spaces E^* and F^* are imbedded in them locally compactly.

Lemma 5.1.1′. *Suppose that the following estimate*

$$\|u\|_{(E^*)_q} \le M \left(\|(L^*)_q u\|_{(F^*)_q} + \|u\|_{\mathcal{E}^*(\Omega_\rho)} \right) \tag{1.2}$$

holds for some positive constants M and ρ, and any $u \in (E^*)_q$. Then the operator $(L^*)_q$ is proper. Here $1 \le q \le \infty$ or $q = D$.

Corollary 5.1.2′. *The operator $(L^*)_q$ is normally solvable with a finite-dimensional kernel.*

Lemma 5.1.3. *Let the operator $L_\infty : E_\infty \to F_\infty$ be proper. Then the operator $L_D : E_D \to F_D$ is also proper.*

Proof. Let $L_\infty u_k = f_k$, $f_k \to f_0$ in F_∞, $u_k \in E_D$, $f_k, f_0 \in F_D$, $\|u_k\|_{E_\infty} \le C$ for some constant C and all k. Since L_∞ is proper, then there exists a subsequence u_{kn} and $u_0 \in E_\infty$ such that $u_{kn} \to u_0$ in E_∞. Since $u_{kn} \in E_D$, then u_0 also belongs to E_D. The lemma is proved. □

Theorem 5.1.4. *Suppose that estimates* (1.1) *and* (1.2) *are satisfied for the operators L_∞ and $(L^*)_\infty$, respectively, in the corresponding spaces. Then L_D is a Fredholm operator.*

Proof. It follows from Lemma 5.1.3 that L_D is normally solvable with a finite-dimensional kernel. It remains to show that the adjoint operator $(L_D)^*$ also has a finite-dimensional kernel.

We note that $\mathrm{Ker}(L^*)_\infty$ is finite-dimensional by virtue of Corollary 4.1.2 for the operator $(L^*)_\infty$. We will show that $\mathrm{Ker}(L_D)^* \subset \mathrm{Ker}(L^*)_\infty$. Indeed, let $(L_D)^* v = 0$ for some $v \in (F_D)^*$. This means that

$$\langle v, L_D u \rangle = 0, \quad \forall u \in E_D.$$

Then for any $u \in E_1 \subset E_D$,

$$\langle v, L_1 u \rangle = \langle v, L_D u \rangle = 0.$$

The functional on the left-hand side is well defined because $v \in (F_D)^* \subset (F_1)^*$. Thus $(L_1)^* v = 0$. By definition,

$$\langle (L^*)_\infty v, u \rangle = \langle (L_1)^* v, u \rangle, \quad \forall u \in E_1. \tag{1.3}$$

Since $(L_1)^* v = 0$, then $(L^*)_\infty v$ also equals zero as an element of $(E^*)_\infty$. Indeed, if it is different from zero, then there exists $\phi \in D$ such that $\phi (L^*)_\infty v \ne 0$. On the other hand $\phi (L^*)_\infty v \in E^*$. Hence for some $w \in E$,

$$\langle (L^*)_\infty v, \phi w \rangle = \langle \phi (L^*)_\infty v, w \rangle \ne 0.$$

This contradicts (1.3) since $\phi w \in E_1$. The theorem is proved. □

Corollary 5.1.5. *The equation*

$$L_D u = f, \quad f \in F_D(\Omega) \tag{1.4}$$

is solvable in $E_D(\Omega)$ *if and only if* $\phi_i(f) = 0$ *for a finite number of linearly independent functionals* $\phi_i \in (F_D(\Omega))^*$ *that are solutions of the homogeneous adjoint problem* $(L_D)^* v = 0$.

In the remaining part of this section we study the operator L_∞. If it satisfies estimate (1.1), then it is normally solvable with a finite-dimensional kernel. A priori we do not know whether the codimension of its image is finite. We will use the normal solvability of this operator and the Fredholm property of the operator L_D to show that it is finite.

From the normal solvability of the operator L_∞ we conclude that the equation

$$L_\infty u = f, \quad f \in F_\infty \tag{1.5}$$

is solvable in E_∞ if and only if $\phi(f) = 0$ for all $\phi \in \Phi$, where Φ is a set in $(F_\infty)^*$.

Consider the functionals $\phi_i,\ i = 1, \dots, N$ that provide the solvability conditions for equation (1.4). By the Hahn–Banach theorem they can be extended from $F_D(\Omega)$ to $(F(\Omega))_\infty$. Denote these new functionals by $\hat{\phi}_i$. Since $\hat{\phi}_i \in ((F(\Omega))_\infty)^*$, then by virtue of Lemma 2.4.2 we can define functionals $\tilde{\phi}_i \in ((F(\Omega))_\infty)^*$ as follows: $\tilde{\phi}_i(f) = \hat{\phi}_i(f)$ for functions $f \in (F(\Omega))_\infty$ with bounded support,

$$\tilde{\phi}_i(f) = \lim_{k\to\infty} \hat{\phi}_i\Big(\sum_{j=1}^{k} \theta_j f\Big), \quad \forall f \in (F(\Omega))_\infty. \tag{1.6}$$

Here θ_j is a partition of unity.

We note that the functionals $\hat{\phi}_i$ are not uniquely defined. However the functionals $\tilde{\phi}_i$ are uniquely defined. Indeed, if there are two different functionals $\tilde{\phi}_i^1$ and $\tilde{\phi}_i^2$ that correspond to the same $\hat{\phi}_i$, then the difference $\tilde{\phi}_i^1 - \tilde{\phi}_i^2$ vanishes on all functions with bounded support. Therefore the limit in (1.6) is also zero.

By the definition of $\tilde{\phi}_i$,

$$\tilde{\phi}_i(f) = \phi_i(f), \quad \forall f \in F_D(\Omega). \tag{1.7}$$

Lemma 5.1.6. *The restriction* ϕ_D *of a functional* $\phi \in \Phi$ *from the solvability condition for equation* (1.5) *to* $F_D(\Omega)$ *is a linear combination of functionals* ϕ_i *from the solvability condition for equation* (1.4).

Proof. For any $f \in F_D(\Omega)$, the equation

$$Lu = f - \sum_{i=1}^{N} \langle \phi_i, f\rangle e_i, \tag{1.8}$$

where $e_i, i = 1, \dots, N$ are such that $\langle \phi_i, e_j\rangle = \delta_{ij},\ e_j \in F_D(\Omega)$, is solvable in $E_D(\Omega)$. Therefore it is also solvable in $E_\infty(\Omega)$. Hence for any $\phi \in \Phi$,

$$\phi\left(f - \sum_{i=1}^{N} \langle \phi_i, f\rangle e_i\right) = 0, \quad \forall f \in F_D(\Omega).$$

Denote $c_i = \phi(e_i)$. Then from the previous equality,

$$\phi(f) = \sum_{i=1}^{N} c_i \phi_i(f), \quad \forall f \in F_D(\Omega). \tag{1.9}$$

Here $\phi_i(f) = \langle \phi_i, f\rangle$. The lemma is proved. □

Corollary 5.1.7. *For any $\phi \in \Phi$,*

$$\phi = \sum_{i=1}^{N} c_i \tilde{\phi}_i + \psi, \quad c_i = \phi(e_i), \tag{1.10}$$

where $\psi \in ((F(\Omega))_\infty)^*$, $\psi(f) = 0$ for any $f \in F_D(\Omega)$.

Proof. We construct the functionals $\tilde{\phi}_i \in ((F(\Omega))_\infty)^*$ on the basis of the functionals $\phi_i \in (F_D(\Omega))^*$. Set $\psi = \phi - \sum_{i=1}^{N} c_i \tilde{\phi}_i$. From (1.7) and (1.9) we conclude that $\psi(f) = 0$ for any $f \in F_D(\Omega)$. The corollary is proved. □

Condition C. Let $Lu_n = f_n$, $(f_n - f_0)\theta \to 0$ in $(F(\Omega))_\infty$ for any infinitely differentiable function θ with a bounded support, $f_n, f_0 \in (F(\Omega))_\infty$, and $\|u_n\|_{(E(\Omega))_\infty} \le M$. Then there exists $u_0 \in (E(\Omega))_\infty$ such that $Lu_0 = f_0$.

Lemma 5.1.8. *Let Condition C be satisfied. Then the functional ψ in* (1.10) *equals zero.*

Proof. Let $f \in (F(\Omega))_\infty$, $f_k = \sum_{i=1}^{k} \theta_i f$. The equation

$$Lu = f_k - \sum_{i=1}^{N} \langle \phi_i, f_k \rangle e_i \tag{1.11}$$

is solvable in $E_D(\Omega)$. The operator $L_D : E_D(\Omega) \to F_D(\Omega)$ has a bounded inverse defined on the image $R(L_D) \subset F_D(\Omega)$ and acting to the subspace of $E_D(\Omega)$ supplementary to the kernel. Therefore

$$\|u_k\|_{E_D(\Omega)} \le \|(L_D)^{-1}\| \, \|f_k - \sum_{i=1}^{N} \langle \phi_i, f_k \rangle e_i\|_{F_D(\Omega)},$$

where u_k is a solution of (1.11) in the subspace supplementary to the kernel.

We note that the norm in $F_D(\Omega)$ is the same as in $(F(\Omega))_\infty$. Hence

$$\|f_k - \sum_{i=1}^{N} \langle \phi_i, f_k \rangle e_i\|_{F_D(\Omega)} \le C_1 \|f_k\|_{F_D(\Omega)} \le C_2 \|f\|_{(F(\Omega))_\infty}.$$

Thus $\|u_k\|_{(E(\Omega))_\infty} \le M$ for some constant M.

We can use now Condition C. Passing to the limit in (1.11), we obtain that the equation

$$Lu = f - \sum_{i=1}^{N} \langle \tilde{\phi}_i, f \rangle e_i \tag{1.12}$$

is solvable in $(E(\Omega))_\infty$ for any $f \in (F(\Omega))_\infty$. Then for any $\phi \in \Phi$,

$$\phi\left(f - \sum_{i=1}^{N} \langle \tilde{\phi}_i, f \rangle e_i \right) = 0.$$

Hence

$$\phi(f) = \sum_{i=1}^{N} c_i \tilde{\phi}_i(f), \quad \forall f \in (F(\Omega))_\infty.$$

From (1.10) we conclude that $\psi = 0$. The lemma is proved. □

Thus we have proved the following theorem.

Theorem 5.1.9. *Suppose that the operators L_∞ and $(L^*)_\infty$ satisfy estimates* (1.1) *and* (1.2)*, respectively, in the corresponding spaces, and Condition C is satisfied. Then the operator L_∞ is Fredholm. Equation* (1.5) *is solvable in $(E(\Omega))_\infty$ if and only if $\phi(f) = 0$ for a finite number of functionals $\phi \in ((F(\Omega))_\infty)^*$. They satisfy the homogeneous adjoint equation $(L_\infty)^*\phi = 0$. The restriction ϕ_D of these functionals to $F_D(\Omega)$ coincides with the functionals ϕ_i in the solvability conditions for equation* (1.4)*.*

Remark 5.1.10. The space $((F(\Omega))_\infty)^*$ contains "bad" functionals that vanish at all functions from $F_D(\Omega)$ and do not belong to D'. Theorem 5.1.9 shows that these functionals do not enter the solvability conditions.

5.2. **Elliptic problems in spaces $W^{s,p}_\infty(\Omega)$.** Consider the operators

$$A_i u = \sum_{k=1}^{N} \sum_{|\alpha|\le\alpha_{ik}} a^\alpha_{ik}(x) D^\alpha u_k, \quad i = 1, \dots, N, \ x \in \Omega, \tag{2.1}$$

$$B_j u = \sum_{k=1}^{N} \sum_{|\beta|\le\beta_{jk}} b^\beta_{jk}(x) D^\beta u_k, \quad j = 1, \dots, m, \ x \in \partial\Omega, \tag{2.2}$$

where $u = (u_1, \dots, u_N)$, $\Omega \subset R^n$ is an unbounded domain that satisfies Condition D. According to the definition of elliptic operators in the Douglis–Nirenberg sense [13] we suppose that

$$\alpha_{ik} \le s_i + t_k, \ i, k = 1, \dots, N, \ \ \beta_{jk} \le \sigma_j + t_k, \ j = 1, \dots, m, \ k = 1, \dots, N$$

for some integers s_i, t_k, σ_j such that $s_i \le 0$, $\max\ s_i = 0$, $t_k \ge 0$.

Denote by E the space of vector-valued functions $u = (u_1, \dots, u_N)$, where u_j belongs to the Sobolev space $W^{l+t_j,p}(\Omega)$, $j = 1, \dots, N$, $1 < p < \infty$, l is an integer, $l \ge \max(0, \sigma_j+1)$, $E = \prod_{j=1}^N W^{l+t_j,p}(\Omega)$. The norm in this space is defined as

$$\|u\|_E = \sum_{j=1}^{N} \|u_j\|_{W^{l+t_j,p}(\Omega)}.$$

The operator A_i acts from E to $W^{l-s_i,p}(\Omega)$, and the operator B_j acts from E to $W^{l-\sigma_j-1/p,p}(\partial\Omega)$. Denote

$$\begin{gathered} L = (A_1, \dots, A_N, B_1, \dots, B_m), \\ F = \prod_{i=1}^{N} W^{l-s_i,p}(\Omega) \times \prod_{j=1}^{m} W^{l-\sigma_j-1/p,p}(\partial\Omega). \end{gathered} \tag{2.3}$$

Then $L : E \to F$.

Lemma 5.2.1. *The operator L_q, $1 \le q \le \infty$ is a bounded operator from E_q to F_q.*

The proof is standard.

We will apply the results of the previous section. The properness of the operator L_∞ is proved in [47] (see Section 3). The estimates of the operator $(L^*)_\infty$ are obtained in Section 4.

It remains to check Condition C.

Let $Lu_\nu = f_\nu$ $(\nu = 1, 2, \dots)$, $(f_\nu - f_0)\theta \to 0$ in $(F(\Omega))_\infty$ for any infinitely differentiable function with a bounded support as $\nu \to \infty$, f_ν, $f_0 \in (F(\Omega))_\infty$, and

$$\|u_\nu\|_{(E(\Omega))_\infty} \le M_0, \quad \forall\nu. \tag{2.4}$$

Let $u_\nu = (u_{1\nu}, \dots, u_{N\nu})$. It follows from Theorem 2.3.3, which is true also for domains in R^n, and from (2.4) that there exists a subsequence of $u_{i\nu}$ and $u_{i0} \in W^{l+t_i,p}_\infty(\Omega)$ such that for $\epsilon > 0$,

$$u_{i\nu} \to u_{i0} \text{ in } W^{l+t_i-\epsilon,p}(\Omega) \text{ locally,} \tag{2.5}$$

$$u_{i\nu} \to u_{i0} \text{ in } W^{l+t_i,p}(\Omega) \text{ locally weakly} \tag{2.6}$$

as $\nu \to \infty$, $i = 1, \dots, N$. We retain the same notation for the subsequence. Denote $u_0 = (u_{10}, \dots, u_{N0})$. We prove that

$$Lu_0 = f_0. \tag{2.7}$$

Indeed, we have

$$A_i u_\nu = f^d_{i\nu}, \quad i = 1, \dots, N, \tag{2.8}$$

$$B_j u_\nu = f^b_{i\nu}, \quad i = 1, \dots, m, \tag{2.9}$$

where $f_\nu = (f^d_{1\nu}, \dots, f^d_{N\nu}, f^b_{1\nu}, \dots, f^b_{m\nu})$. Denote $f_0 = (f^d_{10}, \dots, f^d_{N0}, f^b_{10}, \dots, f^b_{m0})$. By (2.6) for any $\theta \in C^\infty_0(\Omega)$ we have

$$\theta A_i u_\nu \to \theta A_i u_0 \text{ as } \nu \to \infty \text{ weakly in } W^{l-s_i,p}(\Omega).$$

Hence

$$\theta A_i u_0 = \theta f^d_{i0} \ \ (i = 1, \dots, N).$$

Therefore

$$A_i u_0 = f^d_{i0} \ \ (i = 1, \dots, N) \text{ in } W^{l-s_i,p}(\Omega). \tag{2.10}$$

Now consider (2.9). We can assume that the coefficients b^β_{jk} of the operator B_j are extended to Ω such that $b^\beta_{jk} \in C^{l-\sigma_j+\delta}(\Omega)$. From (2.5) it follows that for $\theta \in D$ we have

$$\theta B_j u_\nu \to \theta B_j u_0 \text{ in } W^{l-\sigma_j-\epsilon,p}(\Omega) \text{ as } \nu \to \infty.$$

Hence

$$\theta B_j u_\nu \to \theta B_j u_0 \text{ in } W^{l-\sigma_j-\epsilon-1/p,p}(\partial\Omega) \text{ as } \nu \to \infty.$$

By assumption of Condition C, $\theta f^b_{j\nu} \to \theta f^b_{j0}$ in $W^{l-\sigma_j-1/p,p}(\partial\Omega)$. Therefore

$$\theta B_j u_0 = \theta f^b_{j0} \text{ in } W^{l-\sigma_j-\epsilon-1/p,p}(\partial\Omega). \tag{2.11}$$

Since $u_0 \in (E(\Omega))_\infty$ and $f_0 \in (F(\Omega))_\infty$, it follows from (2.11) that

$$B_j u_0 = f^b_{j0} \text{ in } W^{l-\sigma_j-1/p,p}(\partial\Omega), \quad j = 1, \dots, m.$$

This and (2.10) imply (2.7).

Thus we have proved that the operator L defined by (2.3) satisfies Condition C. Hence Theorem 5.1.9 is applicable for the elliptic operators. We obtain the following result.

Theorem 5.2.2. *Let Conditions NS and NS* be satisfied. Then the realizations L_D and L_∞ of the operator L are Fredholm operators.*

The equation $L_D u = f$, $f \in F_D(\Omega)$ is solvable in $E_D(\Omega)$ if and only if $\phi(f) = 0$ for any solution $\phi \in (F_D(\Omega))^*$ of the problem $(L_\infty)^*\phi = 0$.

The equation $L_\infty u = f$, $f \in (F(\Omega))_\infty$ is solvable in $(E(\Omega))_\infty$ if and only if $\phi(f) = 0$ for any solution $\phi \in ((F(\Omega))^*)_1$ of the problem $(L_\infty)^*\phi = 0$.

Let v be a vector-valued function, $v \in F$, $v = (v^d_1, \dots, v^d_N, v^b_1, \dots, v^b_m)$. We use Definitions 3.1.1 and 3.1.2 for v^d_i in $W^{l-s_i,p}(\Omega)$ and for v^b_j in $W^{l-\sigma_j-1/p,p}(\partial\Omega)$.

Denote by T_y the translation operator $T_y u(x) = u(x+y)$. Then we can define the operator with shifted coefficients, $L_y v = T_y L T_y^{-1} v$. It acts on functions defined in the shifted domain Ω_y.

We use the following condition.

Condition CL. Let $L_{y_k}u_k = f_k$, $u_k \in (E(\Omega_{y_k}))_\infty$, $f_k \in (F(\Omega_{y_k}))_\infty$, $(f_k - f_0)\theta \to 0$ in $F(\Omega_{y_k} \to \hat{\Omega})$ for any infinitely differentiable function θ with a bounded support, $\|u_k\|_{(E(\Omega_{y_k}))_\infty} \le M$, $L_{y_k} \to \hat{L}$. Then there exists a function $u_0 \in (E(\hat{\Omega}))_\infty$ such that $\hat{L}u_0 = f_0$.

Condition CL is satisfied for the elliptic operators (see the proof of Theorem 4.1 in [47]).

Theorem 5.2.3. *If the operator L_∞ satisfies the Fredholm property, Condition CL and Condition NS, then any limiting operator $\hat{L}_\infty$ is invertible.*

Proof. For any function $f_0 \in (F(\hat{\Omega}))_\infty$ there exists a sequence of functions $f_k \in (F(\Omega))_\infty$, $\|f_k\|_{(F(\Omega))_\infty} \le M$ and of points $y_k \in \Omega$, $|y_k| \to \infty$ such that

$$(f_k(x + y_k) - f_0(x))\theta \to 0 \ \text{ in } \ F(\Omega_{y_k} \to \hat{\Omega})$$

for any infinitely differentiable function θ with a finite support.

Indeed, let $f_0 \in (F(\hat{\Omega}))_\infty$. Then $f_0 = (f^d_{10}, \dots, f^d_{N0}, f^b_{10}, \dots, f^b_{m0})$, where

$$f^d_{i0} \in W^{l-s_i,p}_\infty(\hat{\Omega}),\ i = 1, \dots, N,\ \ f^b_{j0} \in W^{l-\sigma_j-1/p,p}_\infty(\partial\hat{\Omega}),\ i = 1, \dots, m.$$

We can extend these functions to R^n in such a way that for the extended functions $\tilde{f}^d_{i0}$ and $\tilde{f}^b_{j0}$ we have

$$\tilde{f}^d_{i0} \in W^{l-s_i,p}_\infty(R^n),\ \ \tilde{f}^b_{j0} \in W^{l-\sigma_j,p}_\infty(R^n).$$

Let $y_k, k = 1, 2, \dots$ be a sequence such that $y_k \in \Omega$, $|y_k| \to \infty$, $\Omega_{y_k} \to \hat{\Omega}$, where Ω_{y_k} are the shifted domains.

Denote

$$\tilde{f}^d_{ik}(x) = \tilde{f}^d_{i0}(x - y_k),\ i = 1, \dots, N,\ \ \tilde{f}^b_{jk} = \tilde{f}^b_{j0}(x - y_k),\ j = 1, \dots, m.$$

Let $f^d_{ik}(x)$ be the restriction of $\tilde{f}^d_{ik}(x)$ to Ω, f^b_{jk} be the trace of $\tilde{f}^b_{jk}$ on $\partial\Omega$. Then it is easy to verify that the sequence

$$f_k(x) = (f^d_{1k}(x), \dots, f^d_{Nk}(x), f^b_{1k}(x), \dots, f^b_{mk}(x))$$

satisfies the conditions above.

Since the operator $L_\infty : E_\infty(\Omega) \to F_\infty(\Omega)$ satisfies the Fredholm property, then the equation

$$L_\infty u = f_m - \sum_{i=1}^N \langle v_i, f_m\rangle\ e_i \tag{2.12}$$

is solvable in $(E(\Omega))_\infty$. Here v_i, $i = 1, \dots, N$ are all linearly independent solutions of the homogeneous adjoint equation, $(L_\infty)^* v_i = 0$, and $e_i \in F_\infty$, $i = 1, \dots, N$ are functions biorthogonal to the functionals v_j, $j = 1, \dots, N$. We can assume that e_i, $i = 1, \dots, N$ have bounded supports (see Lemma 5.4.1 below).

Denote by u_m the solution of the equation (2.12). The numbers $a_{im} = \langle v_i, f_m\rangle$ are uniformly bounded because the sequence f_m is bounded in $(F(\Omega))_\infty$. The equation

$$L_{y_m} v = f_m(x + y_m) - \sum_{i=1}^N a_{im}\ e_i(x + y_m) \tag{2.13}$$

has a solution $v_m(x) = u_m(x+y_m) \in (E(\Omega_{y_m}))_\infty$. Since $e_i(x+y_m) \to 0$ in $F(\Omega_{y_m} \to \hat{\Omega})$, then by virtue of Condition CL there exists a solution $v_0 \in (E(\hat{\Omega}))_\infty$ of the equation $\hat{L}_\infty v_0 = f_0$. It remains to note that the homogeneous equation has only the zero solution since Condition NS is necessary for normal solvability. The theorem is proved. $\square$

Remark 5.2.4. In the proof of the theorem we use the existence of functions e_i biorthogonal to functionals v_j and such that they have bounded supports. We will prove this in Lemma 4.4.1 below using Condition NS. Therefore we have to assume in the formulation of Theorem 5.2.3 that it is satisfied. Otherwise we can assume that Conditions NS and NS* are satisfied and not assume that the operator is Fredholm (see Theorem 5.2.2).

If instead of the operator L_∞ we consider the operator L_D, then the functions e_i belong by assumption to F_D. Though they do not necessarily have bounded supports, the convergence

$$e_i(x+y_m) \to 0 \text{ in } F(\Omega_{y_m} \to \hat{\Omega})$$

remains valid. This allows us to prove the following theorem.

Theorem 5.2.5. *If the operator L_D satisfies the Fredholm property and Condition CL, then any limiting operator $\hat{L}_D$ is invertible.*

The proof is the same as the proof of the previous theorem.

Theorem 5.2.6. *If all limiting operators $\hat{L}_D$ are invertible, then Conditions NS and NS* for the operator L_∞ are satisfied, and consequently the operators L_D and L_∞ are Fredholm.*

Proof. We prove first that for all limiting operators $\hat{L}$ the equation

$$\hat{L}_1 u = f, \quad u \in E_1(\hat{\Omega}), \quad f \in F_1(\hat{\Omega}) \tag{2.14}$$

is solvable. Indeed, consider the equation

$$\hat{L} u = \theta_j f, \tag{2.15}$$

where θ_j, $j = 1, 2, \dots$ is a partition of unity, $f \in F_1(\hat{\Omega})$. Since $F_1(\hat{\Omega}) \subset F_D(\hat{\Omega})$, then there exists a solution $u = u_j \in E_D(\hat{\Omega})$ of equation (2.15).

Let $\omega_\delta(x) = e^{\delta\sqrt{1+|x|^2}}$. Then according to Lemma 4.4.4 below for $\delta > 0$ sufficiently small the following estimate holds:

$$\|u_j(\cdot)\omega_\delta(\cdot - y_j)\|_{E_1(\hat{\Omega})} \le C\|f\theta_j\|_{F(\hat{\Omega})},$$

where B_j is a unit ball with center at y_j, $\operatorname{supp}\theta_j \subset B_j$ and the constant C is independent of j. Since

$$\|u_j\|_{E_1(\hat{\Omega})} \le C_1\|u_j(\cdot)\omega_\delta(\cdot - y_j)\|_{E_1(\hat{\Omega})},$$

we get

$$\|u_j\|_{E_1(\hat{\Omega})} \le C_2\|f\theta_j\|_{F(\hat{\Omega})}.$$

It follows that the series $u = \sum_{j=1}^\infty u_j$ is convergent in $E_1(\hat{\Omega})$, and

$$\|u\|_{E_1(\hat{\Omega})} \le \sum_{j=1}^\infty \|u_j\|_{E_1(\hat{\Omega})} \le C_2 \sum_{j=1}^\infty \|f\theta_j\|_{F(\hat{\Omega})} = C_2\|f\|_{F_1(\hat{\Omega})}.$$

From (2.15) we conclude that

$$\hat{L}_1 u = \sum_{j=1}^\infty \hat{L}_1 u_j = \sum_{j=1}^\infty \theta_j f = f.$$

Therefore we have proved that equation (2.14) has a solution for any $f \in F_1(\hat{\Omega})$. Hence the equation

$$(\hat{L}_1)^* v = 0, \quad v \in (F_1(\hat{\Omega}))^*$$

has only the zero solution. Since $(\hat{L}_1)^* = (\hat{L}^*)_\infty$, then the equation

$$(\hat{L}^*)_\infty v = 0, \quad v \in (F^*(\hat{\Omega}))_\infty$$

also has only the zero solution. Thus we have proved that Condition NS* is satisfied.

We now prove that Condition NS is satisfied. Let u be a solution of the equation

$$\hat{L}_\infty u = 0, \quad u \in E_\infty(\hat{\Omega})$$

for a limiting operator $\hat{L}$. Then $\tilde{u} = S_{-\delta} u$ is a solution of the equation

$$\hat{L}_\delta \tilde{u} = 0, \tag{2.16}$$

where $\hat{L}_\delta = S_{-\delta}\hat{L}S_\delta$, and S_δ is an operator of multiplication by $\omega_\delta(x)$. Equation (2.16) can be written in the form

$$(\hat{L} + \delta K)\tilde{u} = 0,$$

where $K : E_D(\hat{\Omega}) \to F_D(\hat{\Omega})$ is a bounded operator. Since the operator $\hat{L}$ is invertible by the assumption of the theorem, then for δ sufficiently small, $\hat{L} + \delta K$ is also invertible. Hence $\tilde{u} = 0$, and consequently, $u = 0$. The theorem is proved. □

5.3. **Exponential decay. Continuation.** Exponential decay of solutions is proved in Section 3.4 in the case of normally solvable operators with a finite-dimensional kernel. In this section we consider Fredholm operators and prove exponential decay of solutions also for the homogeneous adjoint equation. Moreover, the exponent is not supposed to be small as in Section 3.4. It belongs to some domain of the complex plane described below.

Consider the spaces $E_q, 1 \le q \le \infty$. Denote by S_μ the operator of multiplication by

$$\omega_\mu(x) = \exp(\mu\sqrt{1 + |x|^2}), \quad \mu \in \mathbb{C}.$$

If $\mathrm{Re}\mu \le 0$, then S_μ is a bounded operator in the spaces under consideration.

Consider operator (2.3). Define the operator $T_\mu = S_\mu L S_{-\mu}$ on functions with compact support. It can be directly verified that $T_\mu = L + \mu K(\mu)$, where $K(\mu) : E_q \to F_q$ is a bounded operator that depends on μ polynomially. For functions $u \in E_q$ with compact supports we have

$$S_\mu T_\mu u = T_{\mu+\nu} S_\mu u, \quad \mu, \nu \in \mathbb{C}. \tag{3.1}$$

If $\mathrm{Re}\,\mu \le 0$, then all operators in (3.1) are bounded in the corresponding spaces. Since these are local operators, then (3.1) is true for all $u \in E_q$. The operator T_μ is a holomorphic operator function with respect to the complex variable μ. If L is a Fredholm operator, then T_μ is also Fredholm in some domain G of the μ-plane, $0 \in G$. Its index $\kappa(L_\mu)$ is constant in G, $\alpha(L_\mu)$ and $\beta(L_\mu)$ are also constant with a possible exception of some isolated points where they have greater values [16].

Lemma 5.3.1. *Equation*

$$T_\mu u = 0 \tag{3.2}$$

has the same number of linearly independent solutions for all $\mu \in G$.

Proof. Suppose that for some $\mu_0 \in G$ the number of linearly independent solutions of equation (3.2) in E_q is greater than for other μ in a small neighborhood of μ_0. Denote these solutions by $u_1, \dots, u_m$. Then

$$\tilde{u}_i \equiv S_\delta u_i \in E_\infty, \quad i = 1, \dots, m$$

for real negative δ. On the other hand, $\tilde{u}_i$ are linearly independent solutions of the equation

$$T_{\mu_0+\delta} u = 0.$$

Indeed, by virtue of (3.1),

$$T_{\mu_0+\delta}\tilde{u}_i = T_{\mu_0+\delta} S_\delta u_i = S_\delta T_{\mu_0} u_i = 0.$$

We obtain a contradiction with the assertion that the number of solutions is the same for all μ except for isolated values. This contradiction proves the lemma. □

Theorem 5.3.2. *For any $\mu \in G$, $\operatorname{Re}\mu \le 0$ all solutions of the equation $Lu = 0$, $u \in E_q$ can be represented in the form*

$$u = v \exp(\mu\sqrt{1+|x|^2}),$$

where $v \in E_q$.

Proof. Let $m = \dim \operatorname{Ker} L$. Consider the equation $T_\mu u = 0$, $u \in E_q$. Denote by $v_1, \dots, v_m$ its linearly independent solutions and put $w_i = S_\mu v_i$. We have

$$Lw_i = LS_\mu v_i = S_\mu T_\mu v_i = 0, \quad i = 1, \dots, m.$$

Since the w_i are linearly independent, each solution u of the equation $Lu = 0$ is their linear combination. The theorem is proved. □

Consider now the adjoint operator $L^* : (F_q)^* \to (E_q)^*$. Denote by T_μ^* the operator adjoint to T_μ. Since the index $\kappa(T_\mu)$ is independent of μ for all $\mu \in G$, and also the dimension of the kernel $\alpha(T_\mu)$, then the codimension of the image $\beta(L_\mu)$ is also independent of μ. On the other hand, the dimension of the kernel of the adjoint operator $\alpha(T_\mu^*)$ equals $\beta(T_\mu)$. Therefore we have proved the following lemma.

Lemma 5.3.3. *The dimension $\alpha(L_\mu^*)$ of the kernel of the operator T_μ^* is independent of μ for all $\mu \in G$.*

Theorem 5.3.4. *For any $\mu \in G$, $\operatorname{Re}\mu < 0$ all solutions of the equation $L^*\phi = 0$ can be represented in the form*

$$\phi = \psi \exp(\mu\sqrt{1+|x|^2}),$$

where $\psi \in (F_q)^*$.

The proof is the same as the proof of Theorem 5.3.2. We note that we do not use the reflexivity of the spaces E and F.

5.4. **The space E_q.** Suppose that the operator $L_\infty : E_\infty \to F_\infty$ satisfies the Fredholm property. This means that the equation

$$L_\infty u = f \tag{4.1}$$

is solvable if and only if

$$\langle f, v_i \rangle = 0, \quad i = 1, \dots, N, \tag{4.2}$$

and the homogeneous equation ($f = 0$) has a finite number of linearly independent solutions. Here v_i, $i = 1, \dots, N$ are all linearly independent solutions of the adjoint homogeneous equation,

$$(L_\infty)^* v_i = 0,$$

where $(L_\infty)^* : (F_\infty)^* \to (E_\infty)^*$ is the adjoint operator.

We study in this section the operator L acting from E_q to F_q. To show its dependence on the spaces we denote it by L_q. We begin with some auxiliary results. Let $e_i \in F_\infty$, $i = 1, \dots, N$ be functions biorthogonal to the functionals $v_j, j = 1, \dots, N$,

$$\langle e_i, v_j \rangle = \delta_{ij}, \tag{4.3}$$

where δ_{ij} is the Kronecker symbol.

Lemma 5.4.1. *There exist functions $e_i, i = 1, \dots, N$ with bounded supports satisfying* (4.3).

Proof. Let $e_i \in F_\infty,\ i = 1, \dots, N$ satisfy (4.3). We will construct new functions $\tilde{e}_i \in E_\infty,\ i = 1, \dots, N$ with bounded supports such that

$$\langle \tilde{e}_i, v_j \rangle = \delta_{ij}. \tag{4.4}$$

Denote

$$\hat{e}_i = \sum_{k=1}^{m} e_i \theta_k, \quad i = 1, \dots, N,$$

where θ_i is a partition of unity. We put

$$\tilde{e}_i = c_{i1}\hat{e}_1 + \dots + c_{iN}\hat{e}_N.$$

Then (4.4) is a system of equations with respect to $c_{i1}, \dots, c_{iN}$. Its matrix has the elements

$$\langle \hat{e}_i, v_j \rangle = \langle \sum_{k=1}^{m} e_i \theta_k, v_j \rangle,\ j = 1, \dots, N.$$

Since $v_j \in (F_\infty)^*_\omega$, then

$$\langle \sum_{k=1}^{m} e_i \theta_k, v_j \rangle \to \delta_{ij}$$

as $m \to \infty$. Therefore for m sufficiently large the determinant of this matrix is different from 0, and the system has a solution. The lemma is proved. □

Lemma 5.4.2. *If $w \in E_\infty$, then $u = S^{-1}w \in E_1$ for any $\mu > 0$, and*

$$\|u\|_{E_1} \le C(\mu)\|w\|_{E_\infty}.$$

The proof of the lemma is based on the definition of the spaces and on the properties of multipliers.

Lemma 5.4.3. *Let an operator L acting from a Banach space E to another space F have a bounded inverse defined on its image $R(L) \subset F$. Suppose that the equation*

$$L_\mu u = f$$

has a solution, where

$$L_\mu = L + \mu K,$$

$K : E \to F$ is a bounded operator, $\|K\| \le M$. Then for μ sufficiently small,

$$\|u\|_E \le C\|f\|_F,$$

where the constant C depends on μ and M but does not depend on the operator K.

Proof. Since the equation

$$Lu + \mu Ku = f$$

is solvable, then $f - \mu Ku \in R(L)$. Therefore

$$u = L^{-1}(f - \mu Ku).$$

The assertion of the lemma follows from the estimate

$$\|u\|_E \le \|L^{-1}\|\|f - \mu Ku\|_F \le \|L^{-1}\|(\|f\|_F + |\mu|\|K\|\|u\|_E).$$

The lemma is proved. □

We generalize here the approach developed in [26] for the operators acting in $H^s(R^n)$. As above we use the function

$$\omega_\delta(x) = e^{\delta\sqrt{1+|x|^2}}.$$

Lemma 5.4.4. *Let θ_j be a partition of unity, v_i and e_i be the same as in Lemma 5.4.1, and e_i have a bounded support. Then for any $f \in F_\infty$ there exists a solution u_j of the equation*

$$Lu = \theta_j f - \sum_{i=1}^{N} \langle \theta_j f, v_i \rangle e_i, \tag{4.5}$$

and for δ sufficiently small the following estimate holds:

$$\|u_j(\cdot)\omega_\delta(\cdot - y_j)\|_{E_q} \le C\|f\theta_j\|_F, \tag{4.6}$$

where B_j is a unit ball with center at y_j, $\operatorname{supp}\theta_j \subset B_j$, and the constant C is independent of j.

Proof. Since the operator $L : E_\infty \to F_\infty$ satisfies the Fredholm property, then the equation

$$Lu = g - \sum_{i=1}^{N} \langle g, v_i \rangle e_i \tag{4.7}$$

is solvable for any $g \in F_\infty$. Let $\operatorname{supp} g \in B_j$. Consider the function

$$\tilde{g}(x) = \left(g(x) - \sum_{i=1}^{N} \langle g, v_i \rangle e_i(x) \right) \omega_\delta(x - y_j).$$

We show that its norm in F_∞ is independent of j. We note first of all that $\omega_\delta(x - y_j)$ is bounded in B_j together with all derivatives independently of j. Therefore

$$\|g(x)\omega_\delta(x - y_j)\|_{F_\infty} \le C\|g(x)\|_{F_\infty}$$

with a positive constant C independent of j. We use here that the norms of multipliers in Ω and $\partial\Omega$ can be estimated by the norms in C^k. We have next

$$|\langle g, v_i \rangle| = |\langle g, \psi_j v_i \rangle|$$

where $\psi_j = \psi(x - y_j)$ is a function with a finite support equal 1 in B_j,

$$= |\langle g, \omega_{-\mu}(x)\psi_j w_i \rangle|$$

where $w_i \in (F_\infty)^*$ (see Theorem 5.3.4),

$$\le \|g\|_{F_\infty} \|\omega_{-\mu}(x)\psi_j w_i\|_{(F_\infty)^*} \le \|g\|_{F_\infty} \, \|\omega_{-\mu}(x)\psi_j\|_M \, \|w_i\|_{(F_\infty)^*},$$

where $\|\cdot\|_M$ is the norm in the space of multipliers. By virtue of the properties of this norm,

$$\|\omega_{-\mu}(x)\psi_j\|_M \le K\|\omega_{-\mu}(x + y_j)\psi(x)\|_M \le C\omega_{-\mu}(y_j)$$

with constants K and C independent of j, $\mu > 0$. For $\delta \le \mu$ the product $\omega_{-\mu}(y_j)\omega_\delta(x - y_j)$ is bounded independently of $y_j \in R^n$ and of $x \in \operatorname{supp} e_i(x)$. Hence

$$\|\tilde{g}\|_{F_\infty} \le C\|g\|_{F_\infty},$$

where the constant C depends on the diameter of the supports of e_i but is independent of j.

Since u is a solution of equation (4.7), then $\tilde{u} = S_\delta u$ is a solution of the equation

$$L_\delta \tilde{u} = \tilde{g}, \tag{4.8}$$

where $L_\delta = S_\delta L S_{-\delta}$, and S_δ is the operator of multiplication by $\omega_\delta(x - y_j)$. On the other hand, $L_\delta = L + \delta K$, where K is a bounded operator, $\|K\| \le C$, where C does not depend on j and on δ for δ sufficiently small. By virtue of Lemma 5.4.3 the solution

of (4.8), which belongs to the subspace supplementary to the kernel of the operator L, admits the estimate

$$\|\tilde{u}\|_{E_\infty} \le C_1\|\tilde{g}\|_{F_\infty} \le C_2\|g\|_{F_\infty}$$

independent of j. Let $\delta = \delta_1 + \delta_2$, where δ_1 and δ_2 are positive. Then

$$\|u(x)\omega_{\delta_1}(x-y_j)\|_{E_q} = \|\tilde{u}(x)\omega_{-\delta_2}(x-y_j)\|_{E_q} \le C_3\|\tilde{u}(x)\|_{E_\infty} \le C_4\|g\|_{F_\infty}.$$

Applying this estimate to equation (4.5), we obtain

$$\|u(x)\omega_{\delta_1}(x-y_j)\|_{E_q} \le C_5\|f\theta_j\|_{F_\infty} \le C_6\|f\theta_j\|_F.$$

The lemma is proved. □

Assumption 5.4.5. Let $u_j \in E_q$, $j = 1, 2, \dots$, and

$$\sum_{j=1}^{\infty} \|u_j\ \omega_\delta(x-y_j)\|_{E_q}^q < \infty.$$

Then the series $u = \sum_{j=1}^{\infty} u_j$ is convergent, and the following estimate holds:

$$\|u\|_{E_q}^q \le C\sum_{j=1}^{\infty} \|u_j\ \omega_\delta(x-y_j)\|_{E_q}^q. \tag{4.9}$$

If this assumption is satisfied, then from the estimate in Lemma 4.4.4 we obtain

$$\|u\|_{E_q} \le C\|f\|_{F_q}.$$

Therefore for any $f \in F_q$ ($\subset F_\infty$) there exists a solution $u \in E_q$ of the equation

$$Lu = f - \sum_{i=1}^{N} \langle f, v_i\rangle e_i. \tag{4.10}$$

From this it follows that the operator L_q is normally solvable and the codimension of its image is finite. Its kernel is also finite-dimensional, since it is so for the operator L_∞. Hence L_q is a Fredholm operator.

We note that estimate (4.9) characterizes the function spaces and is not related to the operators under consideration. In the remaining part of this section we show that it is satisfied for Sobolev spaces.

Lemma 5.4.6 (Elementary inequality)**.** *Let $u_i \ge 0$. Then*

$$(u_1^s + u_2^s + \cdots)^{1/s} \le u_1 + u_2 + \cdots \quad (s \ge 1),$$
$$(u_1^s + u_2^s + \cdots)^{1/s} \ge u_1 + u_2 + \cdots \quad (s \le 1).$$

(See for example [17]*.)*

Lemma 5.4.7. *Let $u = \sum_{i=1}^{\infty} u_i$, y_i be centers of an orthogonal lattice in R^n, $1 \le p < \infty$. Then the following estimate holds:*

$$\|u\|_{L^p(R^n)}^p \le C\sum_{i=1}^{\infty} \|u_i\ \omega_\delta(x-y_i)\|_{L^p(R^n)}^p. \tag{4.11}$$

Proof. Let $k = [p] + 1$, $p = ks$. Here k is an integer, $s < 1$. If p is an integer, we do not need to introduce k. All estimates below can be done directly for p. We have

$$\begin{aligned}\|u\|_{L^p(R^n)}^p &= \int_{R^n} |u|^{ks}dx = \int_{R^n} |\sum_{i=1}^{\infty} u_i|^{ks}dx \le \int_{R^n} \left(\sum_{i=1}^{\infty} |u_i|^s\right)^k dx \\ &= \int_{R^n} \sum_{i_1,i_2,\dots,i_k=1}^{\infty} |u_{i_1}|^s|u_{i_2}|^s\cdots|u_{i_k}|^s dx.\end{aligned} \tag{4.12}$$

By virtue of the inequality between the geometrical and arithmetic mean values,

$$|u_{i_1}|^s|u_{i_2}|^s\cdots|u_{i_k}|^s \le \frac{1}{k}\left(|u_{i_1}|^{ks}+\cdots+|u_{i_k}|^{ks}\right).$$

The same inequality with any positive a_{i_j} gives

$$|u_{i_1}|^s|u_{i_2}|^s\cdots|u_{i_k}|^s \le \frac{1}{k}\left(|u_{i_1}|^{ks}a_{i_1}^{k-1}a_{i_2}^{-1}\cdots a_{i_k}^{-1}+\cdots+|u_{i_k}|^{ks}a_{i_1}^{-1}a_{i_2}^{-1}\cdots a_{i_k}^{k-1}\right). \tag{4.13}$$

Put $a_{i_k}(x)=\omega_{\delta_1}(x-y_{i_k})$. Then

$$\sum_{i_k=1}^{\infty} a_{i_k}^{-1}(x)\le C \ \ \forall x,$$

and substituting (4.13) into the right-hand side of (4.12) and taking into account that there are k similar summands, we obtain

$$\int_{R^n}\sum_{i_1,i_2,\dots,i_k}|u_{i_1}|^s|u_{i_2}|^s\dots|u_{i_k}|^s dx \le \sum_{i_1,i_2,\dots,i_k}\int_{R^n}|u_{i_1}|^{ks}a_{i_1}^{k-1}a_{i_2}^{-1}\dots a_{i_k}^{-1}dx$$
$$\le C^{k-1}\int_{R^n}\sum_{i_1}|u_{i_1}|^p\omega_{\delta_1}(x-y_{i_1})^{k-1}dx.$$

Replacing C^{k-1} by C, we obtain (4.11) for $\delta=\delta_1(k-1)/p$. The lemma is proved. □

Similarly we prove the lemma for the spaces $W^{l,p}(R^n)$ and $W^{l,p}(\Omega)$ with an integer $l\ge 0$.

Lemma 5.4.8. *The following estimate*

$$\|u\|^p_{W^{k,p}(\Omega)} \le C\sum_{i=1}^{\infty}\|u_i\ \omega_\delta(x-y_i)\|^p_{W^{k,p}(\Omega)} \tag{4.14}$$

holds with $1\le p<\infty$ *and integer* $k\ge 0$.

This lemma proves that Assumption 5.4.5 holds for the Sobolev spaces. This allows us to prove the Fredholm theorems for elliptic operators in Sobolev spaces. Thus we have proved the following theorem.

Theorem 5.4.9. *Suppose that Conditions NS and NS* are satisfied. Then for* $q=p$ *the equation* $L_q u=0$ *has a finite number of linearly independent solutions in* $(E(\Omega))_q$, *and the equation*

$$Lu=f,\ \ f\in(F(\Omega))_q$$

has a solution $u\in(E(\Omega))_q$ *if and only if*

$$\langle f,v_i\rangle=0,\ \ i=1,\dots,N,$$

where $v_i\in(F_q)^*$ *are linearly independent solutions of the equation*

$$(L_q)^*v=0. \tag{4.15}$$

Proof. Equation (4.15) should be considered in $(F_\infty)^*$. However, if $v_i\in(F_\infty)^*$, then $v_i\in(F_q)^*$. Moreover, all solutions of this equation from $(F_q)^*$ belong also to $(F_\infty)^*$. Indeed, suppose that there exists $w\in(F_q)^*$ such that $L^*w=0$ and w is not a linear combination of $v_i,\ i=1,\dots,N$. Then we can find $g\in F_q$ such that

$$v_i(g)=0,\ i=1,\dots,N,\ \ w(g)\ne 0.$$

By virtue of the solvability conditions, the equation $L_q u=g$ has a solution in E_q. Applying the functional w to both sides, we obtain a contradiction.

We recall finally that if $E = W^{k,q}$, then $E_q = E = W^{k,q}$ $(1 < q < \infty)$. Thus we obtain the Fredholm property for elliptic operators in Sobolev spaces. The theorem is proved. □

Remark 5.4.10. Solvability conditions in the spaces $W^{k,q}$ do not depend on q.

5.5. **The space E_q. Continuation.** In this section we prove the main theorem about the Fredholm property in spaces E_q. We begin with the following lemma.

Lemma 5.5.1. *Let E be a Banach space such that D is dense in E, and let $\phi_i \in E^*$ be linearly independent functionals, $i = 1, \dots, N$, and $\phi_i(f) = 0$ for some $f \in E$. Then for any $\epsilon > 0$ there exists $f_0 \in D$ such that $\|f - f_0\|_E \le \epsilon$ and $\phi_i(f_0) = 0$, $i = 1, \dots, N$.*

Proof. We show first that there exists a system of functions θ_j, $j = 1, \dots, N$ biorthogonal to ϕ_i and such that $\theta_j \in D$. To do this we note that there exist functions $\theta_j \in D$ such that the matrix $\Phi_N = (\phi_i(\theta_j))$ is invertible. We prove it by induction in the number of functionals. For a single functional it is obvious. Suppose that for the functionals $\phi_1, \dots, \phi_{N-1}$ there exist functions θ_j, $j = 1, \dots, N-1$ such that the corresponding matrix Φ_{N-1} is invertible. We show that for a functional ϕ_N linearly independent with the functionals $\phi_1, \dots, \phi_{N-1}$ we can choose θ_N such that the matrix Φ_N is invertible. Indeed, otherwise, from the equality of its determinant to zero we obtain

$$c_N \phi_N(\theta_N) = c_1 \phi_1(\theta_N) + \dots + c_{N-1} \phi_{N-1}(\theta_N), \ \forall \theta_N \in D,$$

where the coefficients c_j are determined by $\phi_i(\theta_j)$ with $j = 1, \dots, N-1$. We note that $c_N \neq 0$ since $\det \Phi_{N-1} \neq 0$. Hence ϕ_N is linearly dependent on $\phi_1, \dots, \phi_{N-1}$ since D is dense in E. This contradiction proves the existence of functions $\theta_j \in D$ such that the matrix Φ_N is invertible.

The construction of the biorthogonal system of functions is now obvious. We put

$$\tilde{\theta}_j = k_1 \theta_1 + \dots + k_N \theta_N$$

and choose k_i such that $\phi_i(\tilde{\theta}_j) = \delta_{ij}$. We omit the tilde in what follows.

Let $f \in E$ be such that $\phi_i(f) = 0$, $i = 1, \dots, N$. Consider a sequence $f_n \in D$ converging to f. Put

$$\tilde{f}_n = f_n - \sum_{j=1}^{N} \phi_j(f_n) \theta_j.$$

Then $\tilde{f}_n \in D$ and $\phi_i(\tilde{f}_n) = 0$. Moreover, $\tilde{f}_n$ converges to f. As a function f_0 from the formulation of the lemma we take $\tilde{f}_n$ for n sufficiently large. The lemma is proved. □

Since D is dense in E_q, then the lemma is applicable, and we can choose a system of functions $e_j \in D$, $j = 1, \dots, N$ biorthogonal to functionals $v_i \in (F_q)^*$, $i = 1, \dots, N$. Moreover, Theorem 5.3.5 can be proved for the spaces E_q. Lemma 5.4.4 is applicable for these spaces. If we assume that the operator $L_q : E_q \to F_q$ satisfies the Fredholm property, then the equation

$$Lu = \theta_j f_j - \sum_{i=1}^{N} \langle \theta_j f_j, v_i \rangle e_i \tag{5.1}$$

is solvable in E_q for any $f_j \in F_q$, and its solution u_j satisfies the estimate

$$\|u_j(\cdot) \omega_\delta(\cdot - y_j)\|_{E_q} \le C \|f_j \theta_j\|_F, \tag{5.2}$$

where C depends on the diameters of the supports of e_i but is independent of j, and the support of θ_j belongs to a unit ball B_j with center at y_j. Since $E_q \subset E_\infty$, we have also the estimate

$$\|u_j(\cdot) \omega_\delta(\cdot - y_j)\|_{E_\infty} \le C \|f_j \theta_j\|_F. \tag{5.3}$$

Let $\theta_0(x) \in C_0^\infty(R^n)$, $\operatorname{supp}\theta_0 \subset B_0$, where B_0 is the unit ball with center at the origin, $f_0 \in (F(\hat{\Omega}))_q$. We use the construction similar to that in the proof of Theorem 4.2.3. We extend the function f_0 to $F_q(R^n)$. Let it be $\tilde{f}_0$. Denote

$$\tilde{f}_j(x) = \tilde{f}_0(x - y_j), \;\; \theta_j(x) = \theta_0(x - y_j).$$

Then $\operatorname{supp}\theta_j \subset B_j$. As functions $f_j(x)$ we take restrictions of $\tilde{f}_j(x)$ to Ω. It can be proved that

$$(\theta_j(x + y_j)f_j(x + y_j) - \theta_0(x)f_0(x))\theta \to 0 \;\text{ in }\; F(\Omega_{y_j} \to \hat{\Omega})$$

for any $\theta \subset C_0^\infty(R^n)$, where $\hat{\Omega}$ is a limiting domain.

Now, consider sequences θ_j and f_j such that

$$\|\theta_j f_j\|_F \le M$$

with some constant M, and where $y_j \to \infty$. This means that the support of θ_j moves to infinity. Instead of this, we can shift the domain Ω in such a way that B_j does not change. Let it be the unit ball with the center at y_0. As in the proof of Theorem 4.2.3 we can pass to the limit in equation (5.1):

$$\hat{L}u = \theta_0 \hat{f}. \tag{5.4}$$

The second term on the right-hand side disappears since the functions e_i have bounded supports. Since the sequence u_j is uniformly bounded in E_∞, and the operator L_∞ satisfies Condition CL, then equation (5.4) has a solution $u_0 \in E_\infty(\hat{\Omega})$. The sequence u_j converges to u_0 locally weakly in E_∞.

On the other hand, the sequence $v_j(x) = u_j(x)\omega_\delta(x - y_0)$ is uniformly bounded in E_∞. Therefore, there exists its subsequence that converges locally weakly to some $v_0 \in E_\infty(\hat{\Omega})$. Hence $u_0(x)\omega_\delta(x - y_0) \in E_\infty(\hat{\Omega})$. As in the proof of Lemma 5.4.4 we conclude that

$$\|u_0(x)\omega_{\delta_1}(x - y_0)\|_{E_q} \le C_1\|\theta_0 \hat{f}\|_F$$

for some positive constant C_1 and $0 < \delta_1 \le \delta$.

Let θ^j be a partition of unity. As above, the equation

$$\hat{L}u = \theta^j \hat{f} \tag{5.5}$$

has a solution. Denote it by u^j. Then $u = \sum_{j=1}^\infty u^j$ is a solution of the equation $Lu = f$. Lemmas 5.4.7 and 5.4.8 allow us to conclude that $u \in E_q$ for $q = p$. Thus we have proved the following lemma.

Lemma 5.5.2. *Let the operator L_q be Fredholm, $q = p$. Then any limiting problem*

$$\hat{L}_q u = f, \;\; x \in \hat{\Omega}$$

is solvable in E_q for any $f \in F_q$.

We recall that for $q = p$ the spaces E_q and F_q coincide, respectively, with the spaces E and F defined in Section 1.

Theorem 5.5.3. *Let q be a given number, $1 < q < \infty$, $q = p$, and let L be an elliptic operator. Then the following assertions are equivalent:*

(i) *The operator L_q is Fredholm.*

(ii) *All limiting operators $\hat{L}_q$ are invertible.*

(iii) *Conditions NS and NS* are satisfied.*

Proof. 1. (i) $\to$ (ii). Consider the equation

$$\hat{L}_q u = f_0, \quad u \in E_q(\hat{\Omega}), \quad f_0 \in F_q(\hat{\Omega}).$$

The solvability of this equation for any $f_0 \in F_q(\hat{\Omega})$ follows from Lemma 5.5.2.

It remains to prove that the equation

$$\hat{L}_q u = 0, \quad u \in E_q(\hat{\Omega}) \tag{5.6}$$

has only the zero solution. Suppose that it is not so. To obtain a contradiction it is sufficient to prove that the operator $L_q : E_q(\Omega) \to F_q(\Omega)$ is not proper. Consider a nonzero solution $u = u_0$ of equation (5.6). We can assume that u_0 is extended to $E_q(R^n)$. Let $v_n(x) = \phi_n(x)u_0(x+x_n)$, where $\phi_n(x)$ are functions with compact supports, $x_n \in \Omega$, and $|x_n| \to \infty$ is the sequence for which the shifted domains converge to the limiting domain $\hat{\Omega}$. Moreover, we assume that $\operatorname{supp}\phi_n$ are balls with radius $r_n \to \infty$, and all derivatives of $\phi_n(x)$ tend to zero as $n \to \infty$. We have

$$L_q v_n = \phi_n L_q u_0(\cdot + x_n) + \cdots . \tag{5.7}$$

The terms on the right-hand side of (5.7) that are not written tend to zero because of the assumption on ϕ_n that their derivatives tend to zero. The supports of the functions ϕ_n can be chosen in such a way that the first term on the right-hand side of (5.7) tends to zero as $n \to \infty$ (see [45] for more details). Hence $L_q v_n \to 0$. It can be easily proved that the sequence v_n is not compact in $E_q(\Omega)$. Therefore the operator L_q is not proper. This contradiction show that equation (5.6) has only the zero solution. Thus the invertibility of the operator $\hat{L}_q$ is proved.

2. (ii) $\to$ (iii). The proof is the same as the proof of Theorem 5.2.6.

3. (iii) $\to$ (i). This follows from Theorem 5.4.9.

The theorem is proved. □

We will show now that if the Fredholm property is satisfied for some value of p, then it is also satisfied for other p assuming that the domain and the coefficients are sufficiently smooth. Suppose that the operator L_p is Fredholm for some $p = p^0$. Then from (i) of Theorem 5.5.3 we have (ii) and (iii) for the same p_0.

We can prove that Conditions NS and NS* are satisfied in other spaces. Let us begin with Condition NS.

Suppose that it is not satisfied for some l^1, p^1; that is, there exists a nonzero solution of the equation

$$\hat{L}u = 0, \quad u \in \prod_{j=1}^{N} W_\infty^{l^1+t_j,p^1}(\hat{\Omega}).$$

Then obviously $u \in \prod_{j=1}^{N} W_\infty^{l+t_j,p^1}(\hat{\Omega})$ for $\max(0, \sigma_j + 1) \le l < l^1$. But also for $l > l^1$. This follows from a priori estimates of solutions in ∞-spaces. From the embedding theorems it follows that u belongs to $\prod_{j=1}^{N} W_\infty^{l+t_j,p}(\hat{\Omega})$ with other p also.

Hence if Condition NS is not satisfied in some space, then it is not satisfied in other spaces either.

Consider now Conditions NS*. We note first of all that from Theorem 5.5.3 it follows that limiting operators are invertible. Therefore the equation

$$\hat{L}u = f$$

is solvable for any $f \in D$. Its solution belongs to $\prod_{j=1}^{N}(W^{l+t_j,p}(\hat{\Omega}))_1$ for any l, p. Locally it follows from a priori estimates, and we have the required behavior at infinity since f has a bounded support.

Suppose that Conditions NS* are not satisfied for some l^1, p^1; that is, there exists $v \neq 0$ such that

$$\hat{L}^* v = 0, \quad v \in (F^*)_\infty$$

or

$$\hat{L}^* v = 0, \quad v \in (F_1)^*.$$

Since v belongs to the space dual to F_1, then there exists $f \in D$ such that

$$\langle v, f \rangle \neq 0.$$

Hence the equation $\hat{L}u = f$ is not solvable in E_1. Indeed, otherwise we apply the functional v to both sides of this equality and obtain a contradiction.

However, it was shown above that this equation is solvable. This contradiction shows that Conditions NS* is satisfied for all l, p.

This result shows in particular that if the Fredholm property is verified for elliptic problems in L^p, then it can be used also in L^2, which is sometimes more convenient. On the other hand, if the Fredholm property is verified in L^2, then it can be done also in L^p.

Let us show now that the Fredholm property holds not only for $q = p$ but also for $q \leq p$. We will verify Assumption 5.4.5. Let

$$E = L^p, \quad E_q = (L^p)_q.$$

Then

$$\|u\|^q_{E_q} = \sum_j \|\phi_j u\|^q_{L^p}.$$

If $u = \sum_i u_i$, then from Lemma 5.4.7,

$$\|\phi_j u\|^p_{L^p} \leq C \sum_i \|\phi_j u_i \omega_\delta(x - y_i)\|^p_{L^p}.$$

From this estimate and the previous equality we have

$$\|u\|^q_{E_q} \leq C^{q/p} \sum_j \left(\sum_i \|\phi_j u_i \omega_\delta(x - y_i)\|^p_{L^p} \right)^{q/p}.$$

On the other hand,

$$\sum_i \|u_i \omega_\delta(x - y_i)\|^q_{E_q} = \sum_i \sum_j \|\phi_j u_i \omega_\delta(x - y_i)\|^q_{L^p}.$$

Therefore, to verify Assumption 5.4.5 it is sufficient to satisfy the estimate

$$\left(\sum_i \|\phi_j u_i \omega_\delta(x - y_i)\|^p_{L^p} \right)^{q/p} \leq \sum_i \|\phi_j u_i \omega_\delta(x - y_i)\|^q_{L^p}.$$

It is satisfied if $q \leq p$ (see Lemma 5.4.6). We can now apply Theorem 5.4.9 for any $q \leq p$.

5.6. **Weighted spaces.** Let $\mu(x)$ be a positive infinitely differentiable function defined for all $x \in R^n$ and satisfying the following condition:

$$|\frac{1}{\mu(x)} D^\beta \mu(x)| \to 0 \text{ as } |x| \to \infty$$

for any multi-index $\beta, |\beta| > 0$. We can take for example $\mu(x) = (1 + |x|^2)^s$, where $s \in R$.

For any Banach space E we introduce the space E_μ with the norm

$$\|u\|_{E_\mu} = \|\mu u\|_E. \tag{6.1}$$

This means that $u \in E_\mu$ if and only if $\mu u \in E$ (see [44]). Consider weighted Sobolev spaces. Let

$$E = \prod_{j=1}^{N} W^{l+t_j,p}(\Omega), \tag{6.2}$$

$$F = \prod_{j=1}^{N} W^{l-s_i,p}(\Omega) \times \prod_{j=1}^{m} W^{l-\sigma_j-1/p,p}(\partial\Omega). \tag{6.3}$$

Then the spaces E_μ and F_μ are defined.

Denote by S the operator of multiplication by μ. We have

$$S : E_\mu \to E, \quad S^{-1} : E \to E_\mu,$$
$$S : F_\mu \to F, \quad S^{-1} : F \to F_\mu.$$

If $v \in E_\mu$, then $\|Sv\|_E = \|\mu v\|_E = \|v\|_{E_\mu}$.

Consider the elliptic operators (1.1)–(1.3) (Section 1), $L : E \to F$, where E and F are the spaces (6.2), (6.3).

Proposition 5.6.1. *The operator L is a bounded operator from E_μ to F_μ.*

Proof. Let $u \in W^{r,p}_\mu(\Omega)$, where r is a positive integer. Then $v = u\mu \in W^{r,p}(\Omega)$, and

$$\mu \frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i} - \mu^{-1}\frac{\partial \mu}{\partial x_i} \in W^{r-1,p}(\Omega).$$

Therefore $u \in W^{r-1,p}_\mu(\Omega)$; that is, the operator of differentiation is bounded from $W^{r,p}_\mu(\Omega)$ to $W^{r-1,p}_\mu(\Omega)$. Hence D^α is a bounded operator from $W^{r,p}_\mu(\Omega)$ to $W^{r-|\alpha|,p}_\mu(\Omega)$, and A_i is a bounded operator from E_μ to $W^{l-s_i,p}_\mu(\Omega)$.

Similarly, B_j is a bounded operator from E_μ to $W^{l-\sigma_j,p}_\mu(\Omega)$, and hence to $W^{l-\sigma_j-1/p,p}_\mu(\partial\Omega)$. The proposition is proved. □

Theorem 5.6.2. *If operator $L : E \to F$ is Fredholm, then the operator $L : E_\mu \to F_\mu$ is Fredholm.*

Proof. Consider operator L as acting from E_μ to F_μ. Then the operator $M = SLS^{-1}$ acts from E to F. We have for $u \in E$, $\omega = 1/\mu$:

$$SA_iS^{-1}u = SA_i(\omega u) = A_i u + \sum_{k=1}^{n} \sum_{|\alpha|\le\alpha_{ik}} a^\alpha_{ik}(x) \sum_{\beta+\gamma\le\alpha,\beta\ne 0} c_{\beta\gamma}\mu D^\beta \omega D^\gamma u_k.$$

Since for $|\beta| > 0$,

$$\mu(x)D^\beta\omega(x) \to 0 \text{ as } |x| \to \infty,$$

we conclude that the limiting operators for SA_iS^{-1} coincide with the limiting operators for A_i. The same is true for the boundary operators. Hence the operators $M : E \to F$ and $L : E \to F$ have the same limiting operators.

If the operator $L : E \to F$ is Fredholm, then Conditions NS and NS* are satisfied for it. Hence they are satisfied also for the operator $M : E \to F$. Therefore the operator M is Fredholm.

It remains to prove that if $M : E \to F$ is Fredholm, then the operator $L : E_\mu \to F_\mu$ also satisfies the Fredholm property.

Indeed, let $u_i \in E, i = 1, \dots, k$ be all the linearly independent solutions of the equation $Mu = 0$. Then $v_i = S^{-1}u_i \in E_\mu, i = 1, \dots, k$ are solutions of the equation

$$Lv = 0. \tag{6.4}$$

Conversely, if $v \in E_\mu$ is a solution of the equation (6.4), then $u = Sv$ is a solution of the equation $Mu = 0$. Hence $u = \sum_{i=1}^k c_i u_i$, and it follows that

$$v = S^{-1}u = \sum_{i=1}^k c_i S^{-1} u_i = \sum_{i=1}^k c_i v_i.$$

Therefore $v_i, i = 1, \dots, k$ are all the linearly independent solutions of (6.4).

Now consider the adjoint operators:

$$L^* : F_\mu^* \to E_\mu^*, \quad M^* : F^* \to E^*.$$

We have

$$S^* : F^* \to F_\mu^*, \quad (S^{-1})^* : F_\mu^* \to F^*.$$

Let $\phi_j \in F^*, j = 1, \dots, l$ be linearly independent solutions of the equation

$$M^*\phi = 0. \tag{6.5}$$

Then $\psi_j = S^*\phi_j \in F_\mu^*, j = 1, \dots, l$ are solutions of the equation $L^*\psi_j = 0$ since $M^* = (S^{-1})^* L^* S^* : F^* \to E^*$. If $\psi \in F_\mu^*$ is an arbitrary solution of the equation

$$L^*\psi = 0, \tag{6.6}$$

then $\phi = (S^{-1})^*\psi \in F^*$ is a solution of the equation $M^*\phi = 0$. Hence $\phi = \sum_{j=1}^l c_j \phi_j$. Therefore

$$\psi = S^*\phi = \sum_{j=1}^l c_j S^* \phi_j = \sum_{j=1}^l c_j \psi_j.$$

We have proved that $\psi_j, j = 1, \dots, l$ is a complete system of linearly independent solutions of equation (6.6).

Consider the equation

$$Lv = g, \ v \in E_\mu, \ g \in F_\mu. \tag{6.7}$$

Suppose that

$$\langle g, \psi_j \rangle = 0, \quad j = 1, \dots, l, \tag{6.8}$$

where $\psi_j \in F_\mu^*$ are all the linearly independent solutions of the equation (6.6). Then

$$\phi_j = (S^*)^{-1}\psi_j \in F^*, \ j = 1, \dots, l$$

are all the linearly independent solutions of the equation $M^*\phi = 0$. It follows from (6.8) that

$$\langle g, S^*\phi_j \rangle = 0, \quad j = 1, \dots, l.$$

Consequently,

$$\langle Sg, \phi_j \rangle = 0. \tag{6.9}$$

Denote $f = Sg \in F$. Since the operator M is Fredholm, then from (6.9) it follows that the equation $Mu = f$ has a solution $u \in E$. We have $SLS^{-1}u = f$. Therefore $LS^{-1}u = S^{-1}f = g$. Hence $v = S^{-1}u \in E_\mu$ is a solution of equation (6.7). We have proved that from (6.8) it follows that equation (6.7) has a solution. Therefore the operator $L : E_\mu \to F_\mu$ is Fredholm. The theorem is proved. □

Remark 5.6.3. In bounded domains Ω the weighted space E_μ coincides with the space E. Choosing a proper μ we can obtain a prescribed behavior at infinity in the case of unbounded domains.

6. Examples

We show in this work that elliptic operators in unbounded domains satisfy the Fredholm property if and only if all their limiting operators are invertible. In the general case, the invertibility of limiting operators cannot be verified explicitly. In some particular cases it can be done. We consider below some examples.

6.1. **Limiting domains.** We begin with some examples in which limiting domains can be constructed explicitly.

1. If $\Omega = R^n$, then the only limiting domain is the whole R^n. If Ω is an exterior domain for some bounded domain, then, as before, the only limiting domain is R^n.

2. In the case of the half-space, $\Omega = R^n_+$, $n > 1$, there are two limiting domains, the same half-space and the whole space. For the half-line, the limiting domain is R^1.

3. If Ω is an unbounded cylinder with a bounded cross section, then the only limiting domain is the same cylinder (up to a shift). It is also a limiting domain for a half-cylinder.

Consider a domain in R^2 given by

$$\Omega = \{(x, y), y > f(x)\},$$

where $f(x)$, $x \in R^1$ is a given function continuous with its first derivative. Suppose that $f(x)$ and $f'(x)$ have limits (finite or infinite) as $x \to \pm\infty$. Then the tangent to the boundary $\partial\Omega$ has limits. The half-planes limited by the limiting tangents are limiting domains. These half-planes and the whole plane form all limiting domains.

If, for example, $f(x) = x^2$, then there are three types of limiting domains: the whole plane, the right half-plane, the left half-plane. For the exponential, $f(x) = e^x$, the limiting domains are the whole plane, the left half-plane, the upper half-plane.

For periodic and quasi-periodic functions $f(x)$, for which limits for the function and for its derivative at infinity do not exist, the limiting domains are either the domain Ω itself (up to a sift) or the whole plane.

We will introduce a special class of domains for which the limiting domains can be either the whole space or a half-space. We denote by $\nu(x)$ the inward normal unit vector to the boundary at $x \in \partial\Omega$.

Condition R. For any sequence $x_k \in \partial\Omega$, $k = 1, 2, \dots$, $|x_k| \to \infty$ and for any given number $r > 0$ there exists a subsequence x_{k_i} such that the limit $\lim_{k_i\to\infty} \nu(x_{k_i} + h_{k_i})$ exists for all h_{k_i}: $h_{k_i} \in R^n$, $|h_{k_i}| < r$, $x_{k_i} + h_{k_i} \in \partial\Omega$, and it does not depend on h_{k_i}.

We will study the structure of limiting domains Ω_* for domains Ω that satisfy Condition R. Let Ω_* be a limiting domain and $x_k \in \Omega$, $k = 1, 2, \dots$, $|x_k| \to \infty$, be the sequence that determines this limiting domain. Denote by d_k the distance from x_k to the boundary $\partial\Omega$. Consider two cases:

1. If the sequence d_k is unbounded, then there exists a subsequence $d_{k_i} \to \infty$. The sequence x_{k_i} determines the same limiting domain Ω_*. Then $\Omega_* = R^n$.

2. If the sequence d_k is bounded, then we can assume, choosing a subsequence if necessary, that $d_k \to d < \infty$.

It is convenient to reformulate Condition R in the following form.

Condition R. If the boundary $\partial\Omega$ is unbounded, then for any sequence $x_k \in \partial\Omega$, $k = 1, 2, \dots$, $|x_k| \to \infty$ there exists a subsequence x_{k_i} such that for any given number $r > 0$ the limit $\lim_{k_i\to\infty} \nu(x_{k_i}+h_{k_i})$ exists for all h_{k_i}: $h_{k_i} \in R^n$, $|h_{k_i}| < r$, $x_{k_i}+h_{k_i} \in \partial\Omega$, and it does not depend on h_{k_i}.

In other words, the subsequence can be chosen independently of r. To prove that the second definition follows from the first one, it is sufficient to take a sequence $r_j \to \infty$. For each value of j we can take a subsequence according to the first definition in such a way that it is a subsequence of the previous one. Then we use a diagonal process.

Denote by y_k the point of the boundary $\partial\Omega$ such that the distance from y_k to x_k equals d_k. Obviously, $|y_k| \to \infty$.

Let y_{k_i} be a subsequence chosen according to Condition R. Instead of the sequence x_k we consider the subsequence x_{k_i}. The limiting domain Ω_* remains the same. We can use Theorem 3.3 from [45]. Let $f(x)$ be a function that satisfies the conditions of the theorem. Then, taking a subsequence if necessary, according to Theorem 3.8 from [45] we can find a function $f_*(x)$ such that $f_{k_i}(x) \equiv f(x + x_{k_i}) \to f_*(x)$ in $C^1_{loc}(\mathbb{R}^n)$, and domain $\Omega_* = \{x : f_*(x) > 0\}$ is the limiting domain under consideration.

For convenience we write k instead of k_i:

$$f_k(x) \equiv f(x + x_k) \to f_*(x) \tag{1.1}$$

in $C^1_{loc}(\mathbb{R}^n)$. The limit

$$\lim_{k\to\infty} \nu(x_k + h_k) = \mu \tag{1.2}$$

exists for all h_k: $h_k \in \mathbb{R}^n$, $|h_k| < r$, $x_k + h_k \in \partial\Omega$ (where $r > 0$ is a given number, μ is some constant), and it is independent of h_k.

It will be shown below that for any $z_0 \in \partial\Omega_*$ the inward unit normal vector equals μ. The following lemma allows us to conclude that Ω_* is a half-space.

Lemma 6.1.1. *If the domain $\Omega \subset \mathbb{R}^n$ satisfies Condition D and all inward normal unit vectors to the boundary $\partial\Omega$ coincide up to a shift, then Ω is a half-space.*

Proof. Denote $\Gamma = \partial\Omega$. Let us show that any point $z \in \Gamma$ has a neighborhood U such that $U \cap \Gamma$ coincides with $U \cap T(z)$, where $T(z)$ is a tangent plane to Γ at the point z. Consider the local coordinate $y = (y_1, \dots, y_n)$ in the vicinity of the point z such that the axis y_n goes along the inward normal vector to Γ at the point z, and all other coordinate axes are in the tangent plane. We can assume that this system of coordinates is obtained from the original one by a translation of the origin and by rotation. In this case, if the neighborhood U is sufficiently small, then the surface Γ in U can be given by the equation

$$y_n = f(y'), \quad y' = (y_1, \dots, y_{n-1}). \tag{1.3}$$

Since all normal vectors to Γ_* are parallel to each other, then all normal vectors to the surface (1.3) are parallel to the y_n-axis. This means that $\partial f/\partial y_i \equiv 0$, $i = 1, \dots, n-1$; that is, $f(y')$ is a constant. Since $f(0) = 0$, then $f(y') \equiv 0$. Thus, it is proved that $U \cap \Gamma = U \cap T(z)$.

Take an arbitrary point $z_0 \in \Gamma$. Denote by $\Gamma(z_0)$ the part of the manifold Γ such that its points can be connected by a continuous curve on Γ. Let $z \in \Gamma(z_0)$ and $\gamma \subset \Gamma$ be a continuous curve connecting z_0 and z. For each point $\zeta \in \gamma$ choose a neighborhood as indicated above. We can also choose a finite covering of the curve γ with such neighborhoods. If we consider consecutive neighborhoods from this covering and take into account that the vectors $\nu(\zeta)$ are equal, we obtain $T(z_0) = T(z)$. Therefore $\gamma \subset T(z_0)$. Since z is an arbitrary point in $\Gamma(z_0)$, then

$$\Gamma(z_0) \subset T(z_0). \tag{1.4}$$

Let us show that

$$\Gamma(z_0) = T(z_0). \tag{1.5}$$

Suppose that this equality is not true. Let $z_1 \in T(z_0), z_1 \notin \Gamma(z_0)$. Let us connect the points z_0 and z_1 by an interval l. Then there is a point $z_* \in l$ such that in each of its neighborhoods there are points from $\Gamma(z_0)$ and points that do not belong to $\Gamma(z_0)$. Since

$[z_0, z_*) \in \Gamma$ and Γ is a closed set, then $z_* \in \Gamma$. As shown above, z_* has a neighborhood U where

$$U \cap \Gamma = U \cap T(z_*) \tag{1.6}$$

Therefore $[z_0, z_*] \subset \Gamma$. This means that $z_* \in \Gamma(z_0)$. From (1.4) it follows that $z_* \in T(z_0)$. As above, we obtain $T(z_*) = T(z_0)$. By virtue of (1.6), $U \cap \Gamma = U \cap T(z_0)$. Hence, we can conclude that all points of the interval l, sufficiently close to z_* belong to Γ and consequently to $\Gamma(z_0)$. This contradiction proves (1.5).

Denote by $\Pi_+(z_0)$ ($\Pi_-(z_0)$) the half-space bounded by the plane $T(z_0)$ and located in the direction of the inward (outward) normal vector to Γ at the point z_0. Let us show that

$$\Pi_+(z_0) \subset \Omega. \tag{1.7}$$

If this is not the case, then there exists a point $z \in \Pi_+(z_0), z \notin \Omega$. Let $\tilde{z}$ be the projection of the point z on the plane $T(z_0)$. We have $\tilde{z} \in \Gamma$. Since Condition D is satisfied, then some interval $\nu(\tilde{z})$ belongs to Ω. Therefore, there is a point ξ in the interval $[\tilde{z}, z]$ such that $(\tilde{z}, \xi) \in \Omega$ and $\xi \notin \Omega$. Hence, $\xi \in \Gamma$. Consider the inward normal vector $\nu(\xi)$ at the point ξ. According to the condition of the lemma, $\nu(\xi)$ and $\nu(\tilde{z})$ coincide up to a shift. Moreover, by virtue of Condition D, none of the points of the outward normal vector sufficiently close to ξ belongs to Ω. This contradiction proves (1.7).

We show that

$$\Pi_+(z_0) = \Omega. \tag{1.8}$$

Indeed, otherwise there exists a point $z \in \Omega, z \notin \Pi_+(z_0)$. This means that $z \in \Pi_-(z_0)$. Let $\tilde{z}$ be the projection of the point z on the plane $T(z_0)$. There is a point ξ in the interval $[z, \tilde{z}]$ such that $[z, \xi)$ belongs to Ω, $\xi \notin \Omega$. Therefore, $\xi \in \Gamma$ and the inward normal vector $\nu(\xi)$ has a direction opposite to the interval $[z, \xi]$. Since it is not possible, we obtain a contradiction which proves (1.8). The lemma is proved. □

We will show now that for any point $z_0 \in \partial\Omega_*$ the inward normal unit vector equals μ. Denote $B(x_0, r) = \{x \in \mathbb{R}^n, |x - x_0| < r\}$. Let Γ_k be the intersection of $\partial\Omega$ with $B(x_k, r)$, where $r > d$. Then Γ_k is not empty. As above, we take the point $y_k \in \partial\Omega$, such that its distance to x_k equals d_k. All points of the set Γ_k have the form $y_k + h$, where $|h| < 2r$. Therefore, we have (1.2).

Let us shift the point x_k to the origin and denote the shifted domain by Ω_k. Denote further $\tilde{\Gamma}_k = \partial\Omega_k \cap B(0, r)$. For any point $z \in \partial\Omega_* \cap B(0, r)$ we can indicate a sequence of points $z_k \to z$, $z_k \in \tilde{\Gamma}_k$. Moreover, $\nu_k(z_k) \to \nu_*(z)$, where $\nu_k(z_k)$ ($\nu(z_*)$) is the inward normal unit vector to $\tilde{\Gamma}_k$ ($\partial\Omega_*$) at the point z_k (z). From (1.2) it follows that $\nu_*(z) = \mu$. The assertion is proved.

We have proved the following theorem.

Theorem 6.1.2. *If Conditions D and R are satisfied for a domain $\Omega \subset \mathbb{R}^n$, then each of its limiting domains is either the whole space $\mathbb{R}^n$ or some half-space.*

The inverse theorem also holds.

Theorem 6.1.3. *Suppose that a domain $\Omega \subset \mathbb{R}^n$ satisfies Condition D and each of its limiting domain is either the space $\mathbb{R}^n$ or some half-space. Then Condition R is satisfied.*

Consider domains Ω in the space $\mathbb{R}^{n+1}$ with coordinates (x, y), where $x = (x_1, \ldots, x_n)$, determined by the inequality $y > f(x)$, where $f(x)$ is a function defined for all $x \in \mathbb{R}^n$ and continuous with its first derivatives. As usual, put $\nabla f(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n)$ and consider the spherical coordinates $x = r\theta$ where $r = |x|$.

We present the following theorem without proof.

Theorem 6.1.4. *For any $\theta_0, |\theta_0| = 1$, let one of the following conditions be satisfied:*

1. *The limit $\lim_{r\to\infty,\theta\to\theta_0} \nabla f(r\theta)$ exists and $\lim_{r\to\infty,\theta\to\theta_0} |\nabla f(r\theta)| < \infty$.*
2. $\lim_{r\to\infty,\theta\to\theta_0} |\nabla f(r\theta)| = \infty$ *and the limit*

$$\lim_{r\to\infty,\theta\to\theta_0} \frac{\nabla f(r\theta)}{|\nabla f(r\theta)|}.$$

exists.

Then the domain Ω satisfies Condition R.

Consider the following examples (see also [34]).

1. Let $f(x) = g(|x|)$, where $g(t)$ is a continuously differentiable function defined for $t \geq 0$. Suppose that the limit $\lim_{t\to\infty} g'(t) = g'(\infty)$ exists. If $|g'(\infty)| < \infty$, then we have the first case of Theorem 6.1.4; if $|g'(\infty)| = \infty$, then we have the second case.

2. Let $f(x) = f_0(x) + f_1(x)$ (for $|x| \geq \sigma > 0$), where $f_0(x)$ is a homogeneous function of a positive order $\alpha \geq 1$,

$$f_0(\rho x) = \rho^\alpha f_0(x),\ \rho > 0,$$

and $f_1(x)$ satisfies the condition

$$\frac{\nabla f_1(x)|}{|x|^{\alpha-1}} \to 0, \quad |x| \to \infty$$

(for example, $f(x)$ can be a polynomial). These functions are supposed to be continuously differentiable and $|\nabla f(\theta)| \neq 0$ for all $|\theta| = 1$. Then it can be verified that the conditions of the theorem are satisfied.

6.2. **Invertibility.** To verify the invertibility of limiting operators we need to show that the corresponding homogeneous equation has only the zero solution and that the nonhomogeneous equation is solvable for any right-hand side. In some cases, in particular for scalar operators, the solvability conditions can be formulated with the help of formally adjoint operators [46].

If the coefficients of the operator have limits at infinity and the limiting domains are invariant with respect to translation, then we can use a full or partial Fourier transform. We will limit ourselves to the following examples.

1. One-dimensional case with constant limits at infinity.

Consider the operator

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

assuming that the coefficients are sufficiently smooth and that the limits

$$a_\pm, b_\pm, c_\pm = \lim_{x\to\pm\infty} a(x), b(x), c(x)$$

exist. The limiting equations

$$L_\pm u \equiv a_\pm u'' + b_\pm u' + c_\pm u = 0$$

have nonzero solutions if and only if one of the functions

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm,\ \xi \in \mathbb{R}$$

becomes zero for some ξ.

For the formally adjoint operator

$$L^* u = (a(x)u)'' - (b(x)u)' + c(x)u$$

(under the assumption that the derivatives of the coefficients tend to zero at infinity) the functions $\lambda_\pm(\xi)$ do not change. Thus, the limiting operators are invertible if and only if

$$\lambda_\pm(\xi) \neq 0 \quad \forall \xi \in \mathbb{R}.$$

The curves $\lambda_\pm(\xi)$ on the complex plane determine the essential spectrum of the operator L, that is, the set of all λ for which the operator $L-\lambda$ does not satisfy the Fredholm property.

2. Problems in cylinders.

Consider the operator

$$Lu = a(x)\Delta u + b(x)\frac{\partial u}{\partial x_1} + c(x)u$$

in an unbounded cylinder $\Omega \subset \mathbb{R}^n$ with axis x_1 and the orthogonal variables $x' \in G$, where $G \subset \mathbb{R}^{n-1}$ is a bounded domain with a sufficiently smooth boundary. Consider for example the homogeneous Dirichlet boundary conditions though it is not essential for what follows. Assuming that the limits

$$a_\pm(x'), b_\pm(x'), c_\pm(x') = \lim_{x_1\to\pm\infty} a(x), b(x), c(x)$$

exist, we can apply the partial Fourier transform with respect to the variable x_1 to the homogeneous limiting equations. The essential spectrum of the operator L is given by the eigenvalues $\lambda_\pm(\xi)$ of the problem

$$a_\pm(x')\Delta' u + (-a_\pm(x')\xi^2 + b_\pm(x')i\xi + c_\pm(x'))u = \lambda_\pm(\xi)u, \quad u|_{x'\in G} = 0.$$

Here Δ' is the Laplace operator with respect to the variables x'.

If the coefficients of the limiting operators and the limiting domains are not invariant with respect to translation and the Fourier transform cannot be done, then the uniqueness of the solution of the homogeneous equation can be shown, in some cases, by some other methods. Consider the same operator as in the previous example and suppose first that solutions of the equation

$$Lu = \lambda u$$

decay exponentially as $x_1 \to \pm\infty$. Multiplying this equation by u and integrating over Ω, we obtain that for real positive and sufficiently large λ it has only the zero solution. If its solutions are bounded in the corresponding spaces but we do not assume them to decay exponentially at infinity, then we can introduce a weighted space with a small exponential weight and reduce this case to the previous one (see [45]).

This example is related to elliptic problems with a parameter. If we assume that all limiting problems for the operator L are elliptic problems with a parameter, then we can obtain their invertibility and, consequently, the Fredholm property of the operator L for sufficiently large values of the parameter. Here we use the ellipticity with a parameter in unbounded domains. This question will be studied in subsequent works.

6.3. **Non-Fredholm operators.** If one of the limiting operators is not invertible, then the original operator is not Fredholm. It can still satisfy the Fredholm property in some weighted spaces.

1. Problems in cylinders.

Consider the equation

$$\Delta u + cu = f \tag{3.1}$$

in an unbounded cylinder $\Omega \subset \mathbb{R}^n$ with the axis x_1 and with the orthogonal variables $x' = (x_2, \dots, x_n)$. Here $c \geq 0$ is some constant. Consider the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0, \tag{3.2}$$

where n is an outer normal vector. The corresponding operator L, acting from the space

$$E = \{W^{2,2}(\Omega),\ \ \frac{\partial u}{\partial n} = 0\}$$

to the space $F = L^2(\Omega)$, does not satisfy the Fredholm property since the limiting problem

$$\Delta u + cu = 0, \quad \frac{\partial u}{\partial n} = 0 \tag{3.3}$$

has a nonzero solution. As is shown in [46], it is sufficient to verify Conditions NS and NS* in the classes of smooth bounded functions.

To solve this problem put $v(x) = p(x_1)\phi(x')$, where $\phi(x')$ is an eigenfunction of the Laplace operator in the section of the cylinder with the homogeneous Neumann boundary condition corresponding to the eigenvalue σ. Then

$$p'' + (c + \sigma)p = 0. \tag{3.4}$$

This equation has a bounded for all $x_1 \in \mathbb{R}$ solution if $c + \sigma \geq 0$. Denote the eigenvalues of the Laplace operator in the section of the cylinder by $\sigma_i, i = 0, 1, \dots$ and suppose that

$$c + \sigma_i > 0,\ i = 1, \dots, k, \quad c + \sigma_i < 0,\ i = k + 1, \dots .$$

Then the bounded solutions of problem (3.3) are

$$u_i^1(x) = \cos(a_i x_1)\phi_i(x'), \quad u_i^2(x) = \sin(a_i x_1)\phi_i(x'), \quad i = 1, \dots, k, \ \ a_i = \sqrt{c + \sigma_i}.$$

Let us introduce the weighted spaces

$$E_\mu = \{u(x) : u(x)\mu(x_1) \in E\}, \quad F_\mu = \{u(x) : u(x)\mu(x_1) \in F\},$$

where

$$\mu(x_1) = e^{\mu\sqrt{1+x_1^2}}.$$

Denote

$$v(x) = u(x)e^{\mu\sqrt{1+x_1^2}}, \quad g(x) = f(x)e^{\mu\sqrt{1+x_1^2}}.$$

Then v satisfies the equation

$$\Delta v - 2\mu\frac{x_1}{\sqrt{1+x_1^2}}\,\frac{\partial v}{\partial x_1} + \left(\mu^2\frac{x_1^2}{1+x_1^2} - \mu\,\frac{1}{(1+x_1^2)^{3/2}} + c\right)v = g. \tag{3.5}$$

Consider the operator

$$L_\mu v = \Delta v - 2\mu\frac{x_1}{\sqrt{1+x_1^2}}\,\frac{\partial v}{\partial x_1} + \left(\mu^2\frac{x_1^2}{1+x_1^2} - \mu\,\frac{1}{(1+x_1^2)^{3/2}} + c\right)v,$$

acting from E to F, and the corresponding limiting problems

$$L_\mu^\pm v = \Delta v \mp 2\mu\frac{\partial v}{\partial x_1} + (\mu^2 + c)v, \quad \frac{\partial v}{\partial n} = 0.$$

Denote

$$p(x_1) = \int_G v(x)\phi(x')dx',$$

where G is the cross section of the cylinder. Multiplying the limiting equation $L_\mu^\pm v = 0$ by $\phi(x')$ and integrating by parts over G, we obtain the equation

$$p'' \mp 2\mu p' + (\sigma + c + \mu^2)p = 0.$$

Here σ is the eigenvalue corresponding to the eigenfunction $\phi(x')$. Since the eigenvalues of the Laplace operator in the section of the cylinder form a discrete set, then we can choose a value $\mu \neq 0$ such that $\sigma + c + \mu^2 \neq 0$ for any eigenvalue σ. Hence bounded solutions $p(x_1)$ of the last equation do not exist and the limiting problems do not have nonzero bounded solutions.

The same remains valid for the formally adjoint operator. Consequently, we can use the results of [46] in order to conclude that the operator $L_\mu : E \to F$ satisfies the Fredholm property. Equation (3.5) is solvable if and only if the solvability conditions

$$\int_\Omega g(y)v(y)dy = 0 \tag{3.6}$$

are verified for any solution v of the formally adjoint equation

$$\Delta v + 2\mu \frac{\partial}{\partial x_1}\left(\frac{x_1}{\sqrt{1+x_1^2}}\, v\right) + \left(\mu^2 \frac{x_1^2}{1+x_1^2} - \mu\, \frac{1}{(1+x_1^2)^{3/2}} + c\right) v = 0.$$

If we write it in the form

$$\Delta v + 2\mu \frac{x_1}{\sqrt{1+x_1^2}} \frac{\partial v}{\partial x_1} + \left(\mu^2 \frac{x_1^2}{1+x_1^2} + \mu\, \frac{1}{(1+x_1^2)^{3/2}} + c\right) v = 0, \tag{3.7}$$

we can easily see that it can be obtained from (3.5) with the change of μ by $-\mu$.

Any solution $u \in E_\mu$ of equation (3.1) has the form

$$u = \exp(-\mu\sqrt{1+x_1^2})\, v,$$

where $v \in E$ is a solution of equation (3.5). Any solution $u \in E_{-\mu}$ of the equation formally adjoint to (3.1) (it coincides with (3.1)) has the form

$$u = \exp(\mu\sqrt{1+x_1^2})\, v,$$

where $v \in E$ is a solution of equation (3.7).

We have proved the following theorem.

Theorem 6.3.1. *Problem* (3.1), (3.2), *where* $u \in E_\mu$, $f \in F_\mu$ *satisfies the Fredholm property for any real* $\mu \neq 0$ *such that* $\mu^2 + c + \sigma \neq 0$ *for all eigenvalues* σ.

This problem is solvable in E_μ *for* $f \in F_\mu$ *if and only if*

$$\int_\Omega \overline{u_0(x)}\, f(x)\, dx \;=\; 0$$

for any solution $u_0 \in E_{-\mu}$ *of the homogeneous problem* (3.3).

It follows from the theorem that the number of solvability conditions can grow when μ increases. This can be explained as follows. The space E_μ narrows when μ increases. Therefore, we need more conditions on f in order for the solution to belong to this space.

These results can easily be generalized for operators with variable coefficients having limits at infinity.

2. Problems in $\mathbb{R}^2$.

Consider the operator

$$Lu = \Delta u + c\frac{\partial u}{\partial x_2} + b(x_2)u$$

in $\mathbb{R}^2$. Here c is some constant, the function b depends only on x_2, $b(\pm\infty) < 0$. Suppose that the zero eigenvalue of the operator

$$L_0 u = u'' + cu' + b(x_2)u$$

is simple (with an eigenfunction $u_0(x_2)$) and all other points of its spectrum are in the left half-plane. Here $'$ denotes the differentiation with respect to x_2.

There are three limiting problems:

$$\Delta u + c\frac{\partial u}{\partial x_2} + b(\pm\infty)u = 0$$

and

$$\Delta u + c\frac{\partial u}{\partial x_2} + b(x_2)u = 0.$$

The last one has a nonzero bounded solution $u(x) = u_0(x_2)$. Therefore, the operator L is not Fredholm. Consider the equation

$$Lu = f.$$

We will use the same notation v and g as in the previous example. Then

$$\Delta v + c\frac{\partial v}{\partial x_2} - 2\mu\frac{x_1}{\sqrt{1+x_1^2}}\,\frac{\partial v}{\partial x_1} + \left(\mu^2\frac{x_1^2}{1+x_1^2} - \mu\,\frac{1}{(1+x_1^2)^{3/2}} + b(x_2)\right)v = g. \tag{3.8}$$

The limiting problems

$$\Delta v + c\frac{\partial v}{\partial x_2} \mp 2\mu\,\frac{\partial v}{\partial x_1} + \left(\mu^2 + b(\pm\infty)\right)v = 0$$

have constant coefficients. For sufficiently small μ they have only the zero solutions. Consider next the limiting problems

$$\Delta v + c\frac{\partial v}{\partial x_2} \mp 2\mu\,\frac{\partial v}{\partial x_1} + \left(\mu^2 + b(x_2)\right)v = 0.$$

We use the partial Fourier transform with respect to x_1:

$$\tilde{v}'' + c\tilde{v}' + \left(\mu^2 - \xi^2 \mp 2\mu i\xi + b(x_2)\right)\tilde{v} = 0.$$

This equation can be written in the form

$$L_0\tilde{v} = -\left(\mu^2 - \xi^2 \mp 2\mu i\xi\right)\tilde{v}.$$

Since zero is a simple eigenvalue of the operator L_0 and the rest of its spectrum is in the left half-plane, this equation does not have nonzero solutions for small positive μ.

For a fixed x_1 and x_2 going to infinity we obtain another type of limiting problems (x_1 can be replaced here by $x_1 + h$):

$$\Delta v + c\frac{\partial v}{\partial x_2} - 2\mu\frac{x_1}{\sqrt{1+x_1^2}}\,\frac{\partial v}{\partial x_1} + \left(\mu^2\frac{x_1^2}{1+x_1^2} - \mu\,\frac{1}{(1+x_1^2)^{3/2}} + b(\pm\infty)\right)v = 0.$$

Since $b(\pm\infty) < 0$, then for sufficiently small μ this equation has only the zero solution in the class of bounded functions.

The formally adjoint operator can be studied in the same way. The operator L considered in the weighted space satisfies the Fredholm property.

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