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ARENS–MICHAEL ENVELOPES, HOMOLOGICAL EPIMORPHISMS, AND RELATIVELY QUASI-FREE ALGEBRAS

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Abstract. We describe and investigate Arens–Michael envelopes of associative algebras and their homological properties. We also introduce and study analytic analogs of some classical ring-theoretic constructs: Ore extensions, Laurent extensions, and tensor algebras. For some finitely generated algebras, we explicitly describe their Arens–Michael envelopes as certain algebras of noncommutative power series, and we also show that the embeddings of such algebras in their Arens–Michael envelopes are homological epimorphisms (i.e., localizations in the sense of J. Taylor). For that purpose we introduce and study the concepts of relative homological epimorphism and relatively quasi-free algebra. The above results hold for multiparameter quantum affine spaces and quantum tori, quantum Weyl algebras, algebras of quantum (2×2)-matrices, and universal enveloping algebras of some Lie algebras of small dimensions.

1. Introduction

Noncommutative geometry emerged in the second half of the last century when it was realized that several classical results can be interpreted as some kind of "duality" between geometry and algebra. Informally speaking, those results show that quite often all the information about a geometric "space" (topological space, smooth manifold, affine algebraic variety, etc.) is in fact contained in a properly chosen algebra of functions (continuous, smooth, polynomial...) defined on that space. More precisely, this means that assigning to a space X the corresponding algebra of functions, one obtains an anti-equivalence between a category of "spaces" and a category of commutative algebras. In some cases, the resulting category of commutative algebras admits an abstract description, which *a priori* has nothing to do with functions on any space. A typical example is the first Gelfand–Naimark theorem, asserting that the functor sending a locally compact topological space X to the algebra $C_0(X)$ of continuous functions vanishing at infinity is an anti-equivalence between the category of locally compact spaces and the category of commutative C^*-algebras. Another classical result of this nature is the categorical form of Hilbert's Nullstellensatz asserting that, over an algebraically closed field k, the functor sending an affine algebraic variety X to the algebra of regular (polynomial) functions on X is an anti-equivalence between the category of affine varieties over k and the category of finitely generated commutative k-algebras without nilpotents.

One of the basic principles of noncommutative geometry states that an arbitrary (noncommutative) algebra should be viewed as an "algebra of functions on a noncommutative space". The reason for this is the fact that many important geometric notions and theorems, when translated into the "dual" algebraic language, are also true for noncommutative algebras. These kinds of results gave rise to such branches of mathematics

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as algebraic and operator K-theories, the theory of quantum groups, noncommutative dynamics, etc.

In fact, the term "noncommutative geometry" (as well as the term "geometry") is a name not for one but for several branches of mathematics which share many similar ideas but differ in their objects of study. For example, the natural objects of study in noncommutative measure theory are von Neumann algebras (since commutative von Neumann algebras are exactly the algebras of essentially bounded measurable functions on measure spaces), whereas in noncommutative topology one studies C^*-algebras (in concordance with the mentioned Gelfand–Naimark theorem). Similarly, noncommutative affine algebraic geometry mainly deals with finitely generated algebras (without topology) and noncommutative projective algebraic geometry studies graded algebras. Finally, in noncommutative differential geometry one uses dense subalgebras of C^*-algebras with some special "differential" properties (see [4, 33]).

Notice that the above list of geometric theories misses an important discipline which should be placed "between" differential and algebraic geometries, namely, complex-analytic geometry. To the best of the author's knowledge, there is no theory at the moment that could claim the name of "noncommutative complex-analytic geometry". Perhaps, one of the main reasons for this is that it is not clear what kind of algebras should be viewed as noncommutative generalizations of the algebras of holomorphic functions on complex varieties. Of the few papers related to these issues we mention J. Taylor's [67, 68], which consider various completions of a free algebra, and D. Luminet's [37, 38], dealing with analytic algebras with polynomial identities. Some closely related problems are considered in a series of papers by L. L. Vaksman, S. D. Sinel'shchikov, and D. L. Shklyarov (see, in particular, [71, 63, 70] and [35]), dealing with noncommutative generalizations of function theory in classical domains and using methods from the theory of quantum groups. In particular, in [70], the author introduces a Banach algebra which is a "quantum" analog of the algebra $A(\overline{\mathbb{B}})$ of continuous functions on the closed unit ball $\overline{\mathbb{B}}$ which are holomorphic in its interior, and computes its Shilov boundary. A somewhat different approach to "noncommutative complex analysis", closely related to operator theory in the spirit of Szekefalvi-Nagy and Fojaš, can be traced back to W. Arveson's classical papers [2, 3]. It is being systematically developed G. Popescu, K. Davidson, and others (see, for example, [11, 12, 50, 13, 51]).

The algebras considered in the present paper may, for several reasons, be viewed as noncommutative generalizations of the algebras of holomorphic functions on complex affine algebraic varieties. They are formally defined as the Arens–Michael envelopes of finitely generated associative algebras. By definition, the Arens–Michael envelope of a $\mathbb{C}$-algebra A is its completion $\widehat{A}$ with respect to the system of all submultiplicative prenorms. The Arens–Michael envelopes were introduced by J. Taylor [66, 67] in relation to a problem of multi-operator holomorphic functional calculus. The reason why those algebras may be relevant to noncommutative complex-analytic geometry is that the Arens–Michael envelope of the algebra $\mathcal{O}^{\mathrm{alg}}(V)$ of regular (polynomial) functions on an affine algebraic variety V coincides with the algebra $\mathcal{O}(V)$ of holomorphic functions on V endowed with the standard compact-open topology (see Example 3.6 below). Therefore, if one adopts the point of view of noncommutative algebraic geometry and considers a finitely generated algebra (for example, some algebra of noncommutative polynomials[1]) as an algebra of "regular functions on a noncommutative affine variety", then its Arens–Michael envelope should be considered as the algebra of "holomorphic

[1] Following [20], we use the term "algebra of noncommutative polynomials" not as a synonym of the term "free algebra", but in a different and somewhat informal sense, meaning an algebra obtained from the usual algebra of polynomials in several indeterminates by "deforming" its multiplication.

functions" on the same "variety". This approach is convenient, in particular, because it is functorial: the correspondence $A \mapsto \widehat{A}$ is a functor from the category of algebras to the category of Arens–Michael algebras, which is left adjoint to the forgetful functor acting in the opposite direction. Moreover, this functor extends to the category of A-modules and may therefore be considered as a noncommutative affine analog of the analytization functor [62], widely used in algebraic geometry.

The present paper has two goals. The first is to explicitly compute Arens–Michael envelopes for some standard algebras of noncommutative polynomials, frequently used in noncommutative algebraic geometry. Those include, in particular, Yu. I. Manin's quantum plane, the quantum torus, their higher-dimensional generalizations, quantum Weyl algebras, algebras of quantum matrices, universal enveloping algebras, etc. We will show that for many of those algebras of noncommutative polynomials their Arens–Michael envelope can be described as an algebra of noncommutative power series whose coefficients satisfy certain vanishing conditions at infinity. An interesting point here is that quite often for algebras of noncommutative polynomials which may be isomorphic as vector spaces (but not as algebras), their Arens–Michael envelopes are not isomorphic as topological vector spaces.

The second goal of this paper, speaking informally, stems from the desire to answer the natural question of what are the common properties for an algebra of "noncommutative polynomial functions" and the corresponding algebra of "noncommutative holomorphic functions"? In other words, what properties of a finitely generated algebra A are inherited, in one sense or another, by its Arens–Michael envelope? Of course, there should be more than one approach to this question, depending on which properties of A are singled out and on what should be understood by the word "inherited". The approach used in this paper was suggested by J. Taylor [67] as an essential element of his general view of "noncommutative holomorphic functional calculus". It is based on the notion of *homological epimorphism* (or, in Taylor's own terminology, *localization*) and allows us to compare homological properties of A and of $\widehat{A}$, including in particular their Hochschild (co)homology groups. By definition, a homomorphism of topological algebras $A \to B$ is called a homological epimorphism if the induced functor $\mathsf{D}^b(B) \to \mathsf{D}^b(A)$ between the bounded derived categories of topological modules is fully faithful. Informally this means that all the homological relations between topological B-modules are preserved when they are considered as A-modules under the change of rings $A \to B$.

The notion of homological epimorphism (localization) was introduced by Taylor in [67] and played a key role in the construction of the holomorphic calculus for a set of commuting operators in a Banach space. It looks like, in a purely algebraic setting, homological epimorphisms first appeared almost 20 years later in a paper by W. Geigle and H. Lenzing [17], independently of Taylor's work and in connection with some questions of representation theory of finite-dimensional algebras. In particular, the term "homological epimorphism" was used for the first time in [17].[2] Ten years later homological epimorphisms were independently rediscovered in a preprint [46] by A. Neeman and A. Ranicki, again in a purely algebraic context, and were used to solve some problems of algebraic K-theory. Notice that [46] uses a different terminology: when $A \to B$ is a homological epimorphism in the sense of [17], the authors of [46] say that "*B is stably flat over A*" (see also [48, 49]). Finally, at the end of 2004, R. Meyer [44] introduced, also independently, homological epimorphisms (therein called "isocohomological morphisms") in the context of bornological algebras. The motivation came from the problem of computing Hochschild cohomology groups of various bornological algebras of functions of tempered growth on groups (with convolution multiplication) and more general crossed products.

[2]To the best of the author's knowledge, this term was suggested by W. Crawley-Boevey.

Thus homological epimorphisms were introduced independently in at least four papers in relation with various problems. That, we believe, indicates that this notion is natural and is worth further study.

A basic example of homological epimorphism, which motivated the second half of this paper, is the canonical embedding of the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ (viewed as a topological algebra with respect to the strongest locally convex topology) in the Fréchet algebra of entire functions $\mathcal{O}(\mathbb{C}^n)$. The fact that this embedding is indeed a homological epimorphism (and remains such if $\mathbb{C}^n$ is replaced by an arbitrary polydomain $U \subset \mathbb{C}^n$), was proved by Taylor in [67] and was substantially used in his construction of the holomorphic calculus of commuting operators. Since the algebra of entire functions is precisely the Arens–Michael envelope of the polynomial algebra, the above result brought Taylor to the following natural question: for which algebras A is the canonical homomorphism $A \to \widehat{A}$ a homological epimorphism? In fact, Taylor considered that question as a first step in his general program of constructing analogs of the holomorphic calculus of noncommuting operators. He found that the answer to that question is in the positive when A is a free algebra, and is in the negative when A is the universal enveloping algebra of a semisimple finite-dimensional Lie algebra. The case of universal enveloping algebras was also considered by A. A. Dosiev [16, 15] and the author [48, 49]. In particular, it was shown in [16] that the embedding of the universal enveloping algebra $U(\mathfrak{g})$ in its Arens–Michael envelope is a homological epimorphism, when $\mathfrak{g}$ is metabelian (i.e., $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$). Later, A. A. Dosiev generalized this result to nilpotent Lie algebras satisfying a certain condition of "normal growth" [15], and the author generalized it to positively graded Lie algebras [48]. On the other hand, it was shown in [49] that the embedding of $U(\mathfrak{g})$ in its Arens–Michael envelope can be a homological epimorphism only when $\mathfrak{g}$ is solvable.

In this paper we offer a general method allowing, for many finitely generated algebras A, to prove that the canonical homomorphism $A \to \widehat{A}$ is a homological epimorphism. It is based on the observation that if A is *relatively quasi-free* over a subalgebra R and satisfies certain additional finiteness and projectivity conditions (which can easily be checked in concrete situations), then $A \to \widehat{A}$ is a homological epimorphism whenever $R \to \widehat{R}$ is. This allows us to use induction in problems of that sort. More precisely, if the given algebra A has a chain of subalgebras

$$\mathbb{C} = R_0 \subset R_1 \subset \cdots \subset R_n = A,$$

where each algebra R_i is relatively quasi-free over R_{i-1} and satisfies the above additional finiteness and projectivity conditions, then it follows that $A \to \widehat{A}$ is a homological epimorphism. This method can be applied to many of the above algebras, in particular, the quantum plane, the quantum torus, some of their multi-dimensional generalizations, quantum Weyl algebras, the algebra of quantum (2×2)-matrices, the free algebras, and also, with some modifications, universal enveloping algebras of some solvable Lie algebras of small dimensions.

Notice that the notion of quasi-free algebra, which plays a key role in the described method, was introduced and studied, in the "absolute" case (i.e., when $R = \mathbb{C}$), by J. Cuntz and D. Quillen. A "relative" version of this notion, which technically differs little from the absolute case, is introduced and studied here in Section 7.

Now we describe the contents of the paper. In Section 2 we collect basic facts about topological algebras and modules, and also describe a relative version of homology theory of topological algebras (for the "absolute" version, we refer the reader to [25]; for a purely algebraic version of the "relative" theory, see [28]). We also prove, for later use, some general results about acyclic objects. The Arens–Michael envelopes of topological algebras and modules are defined in Section 3, where we also establish their basic properties and give the first examples. In Section 4 we introduce some constructions of locally convex algebras which are "analytic analogs" of the classical constructions from ring theory: Ore extensions, skew Laurent extensions, and tensor algebras of bimodules. In Section 5 we establish connections between the algebraic and analytic constructions from the preceding section. We prove that, under some additional assumptions, the Arens–Michael envelope of an Ore extension (resp., skew Laurent extension) is an analytic Ore extension (resp., analytic skew Laurent extension), and that the Arens–Michael envelope of the tensor algebra of a bimodule M is an analytic tensor algebra of its Arens–Michael envelope $\widehat{M}$. As a consequence, we give explicit descriptions of the Arens–Michael envelopes for a number of algebras of noncommutative polynomials: the universal enveloping algebras of the two-dimensional solvable Lie algebra and of the three-dimensional Heisenberg Lie algebra, multiparameter quantum affine spaces and quantum tori, quantum Weyl algebras, and the algebra of quantum (2×2)-matrices. In Section 6 we introduce several versions of the notion of homological epimorphism: three "relative" ones (left, right, and two-sided) and an absolute one (which coincides with Taylor's absolute localization from [67]). We give various characterizations of these notions in the spirit of those from [17]. In Section 7 we introduce relatively quasi-free algebras by giving several equivalent definitions (of which the most convenient one is probably the one in the language of noncommutative differential forms, cf. [7]) and establish some of their general properties. In particular, we study the action of the Arens–Michael functor on noncommutative differential forms and on relatively quasi-free algebras, and establish connections with the notion of two-sided relative homological epimorphism. We also compute the bimodules of relative differential forms for the algebras considered in Sections 4–5, and prove that those algebras are relatively quasi-free. Section 8 is of a technical nature and is devoted to various finiteness conditions for topological algebras and modules, which will be used later in the paper. The main results of the paper are contained in Section 9. There we state and prove a general theorem on sufficient conditions for the canonical epimorphism from A to its Arens–Michael envelope $\widehat{A}$ to be a homological homomorphism. We also discuss its various consequences and provide concrete applications (see above). Finally, Section 10 is a detailed appendix containing basic facts about derived categories which are necessary for understating the present paper. We decided to include this appendix because all standard texts on derived categories deal with those of abelian categories, which is not enough for our purposes. For details, see [32].

2. Preliminaries

All vector spaces and algebras in this paper are over the field of complex numbers. They are automatically assumed to be associative and unital (i.e., with identity), and all algebra homomorphisms will preserve identity elements. We do not require that $1 \neq 0$; in other words, the zero algebra is by definition unital. Modules over algebras are also assumed to be unital (i.e., the identity of the algebra acts on modules by identity transformations).

For concepts pertaining to the theory of topological vector spaces see, for example, [60]. Notice that in this paper, unlike [60], locally convex spaces (LCS) are not assumed to be Hausdorff. The completion of an LCS E is defined as the completion of the associated Hausdorff LCS $E/\overline{\{0\}}$ and is denoted either $\widetilde{E}$ or $E\,\tilde{}$. The completed projective tensor product of locally convex spaces E and F is denoted $E \mathbin{\widehat{\otimes}} F$.

By a *topological algebra* we understand a topological space A endowed with a structure of an associative algebra such that the multiplication operator $A \times A \to A$, $(a, b) \mapsto ab$

is separately continuous. The category of topological algebras and continuous homomorphisms will be denoted Topalg. When speaking of continuous homomorphisms between topological algebras we will often omit the word "continuous".

If A is locally convex and complete and the multiplication on A is jointly continuous, then it is called a $\widehat{\otimes}$*-algebra*. In this case the multiplication operator $A \times A \to A$ gives rise to a continuous linear operator $m_A \colon A \mathbin{\widehat{\otimes}} A \to A$, which is uniquely determined by the condition $a \otimes b \mapsto ab$.

A locally convex algebra is called a *Fréchet algebra* if it is metrizable and complete, i.e., if its underlying LCS is a Fréchet space. Since any separately continuous bilinear map of Fréchet spaces is automatically jointly continuous (see [60, III.5.1]), any Fréchet algebra is a $\widehat{\otimes}$-algebra.

A prenorm $\|\cdot\|$ on an algebra A is said to be *submultiplicative*, if $\|ab\| \le \|a\|\|b\|$ for all $a, b \in A$. This is equivalent to saying that the "unit ball" $U = \{a \in A : \|a\| \le 1\}$ with respect to this prenorm is an *idempotent set*, i.e., $U^2 \subset U$. Therefore submultiplicative prenorms are precisely Minkowski functionals of convex absorbing circled idempotent sets. In particular, it now easily follows that for any two-sided ideal $I \subset A$ and any submultiplicative prenorm on A, its quotient prenorm on A/I is also submultiplicative. A locally convex algebra A is said to be *locally multiplicatively-convex* (or *locally* m*-convex*, or *multinormed*) if its topology can be defined by a system of submultiplicative prenorms (or, equivalently, if in A there is a basis of convex circled idempotent neighborhoods of zero). If A is also complete, then it is called an *Arens–Michael algebra*. Any Banach algebra is an Arens–Michael algebra, and any Arens–Michael algebra is a $\widehat{\otimes}$-algebra. It follows from the above that a quotient algebra of a locally m-convex algebra (resp., the completion of a quotient algebra of an Arens–Michael algebra) is locally m-convex (resp., is an Arens–Michael algebra).

Notice that any nonzero submultiplicative prenorm on A is equivalent to some submultiplicative prenorm $\|\cdot\|$ such that $\|1\| = 1$ (this is proved the same way as for norms; see, for example, [24, 1.2.5]). Therefore when speaking of nonzero submultiplicative prenorms we will always assume that they satisfy the above identity.

To check whether an algebra is locally m-convex, it is often convenient to use the following lemma from [45].

Lemma 2.1. *Let A be a locally convex topological algebra whose topology is defined by a system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$. Suppose that for any $\nu \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ such that $\|a_1 \cdots a_n\|_\nu \le C^n \|a_1\|_\mu \cdots \|a_n\|_\mu$ for any $a_1, \ldots, a_n \in A$. Then A is locally m-convex.*

Corollary 2.2. *Let A be a locally convex topological algebra whose topology is defined by a system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$. Suppose that for any $\nu \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ such that $\|ab\|_\nu \le C\|a\|_\mu \|b\|_\nu$ for any $a, b \in A$. Then A is locally m-convex.*

Let A be a topological algebra, and $M_n(A)$ the algebra of $(n \times n)$-matrices with coefficients in A. Identifying $M_n(A)$ with A^{n^2}, we may endow $M_n(A)$ with the direct product topology, with respect to which, as can easily be checked, $M_n(A)$ is a topological algebra. If A is locally convex, (resp., locally m-convex, is a Fréchet algebra, is an Arens–Michael algebra), then so is $M_n(A)$. Indeed, if $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ is a system of prenorms defining the topology on A, then the topology on $M_n(A)$ is determined by the system of prenorms

$$\|a\|_\nu\widehat{\ } = \max_{1 \le j \le n} \sum_{i=1}^n \|a_{ij}\|_\nu \quad \bigl(a = (a_{ij}) \in M_n(A),\ \nu \in \Lambda\bigr).$$

A trivial calculation (which we omit) shows that if the prenorms $\|\cdot\|_\nu$ are submultiplicative, then so are $\|\cdot\|_\nu\widehat{\ }$.

A *left topological module* over a topological algebra A is a topological vector space X endowed with a structure of a left A-module such that the operator $A \times X \to X$, $(a, x) \mapsto a \cdot x$ is separately continuous. If A is a $\widehat{\otimes}$-algebra, the underlying space of the module X is locally convex and complete, and the above operator is jointly continuous, then X is called a *left* A-$\widehat{\otimes}$-*module*. In this case we have a well-defined continuous linear operator $A \widehat{\otimes} X \to X$, $a \otimes x \mapsto a \cdot x$.

If X and Y are left topological A-modules, then a *morphism* from X to Y is a continuous linear map $\varphi : X \to Y$, such that $\varphi(a \cdot x) = a \cdot \varphi(x)$ for all $a \in A$ and $x \in X$. The morphisms from X to Y form a vector space, denoted ${}_A\mathbf{h}(X, Y)$. When $Y = X$ that space will be denoted simply ${}_A\mathbf{h}(X)$.

Similarly, one defines right topological modules, topological bimodules, and their morphisms. The space of morphisms from a right topological A-module X to a right topological A-module Y (resp., from a topological A-bimodule X to a topological A-bimodule Y) is denoted $\mathbf{h}_A(X, Y)$ (resp., ${}_A\mathbf{h}_A(X, Y)$).

Suppose X and Y are left topological A-modules. We endow the space Y^X of all maps from X to Y with the Tikhonov topology and consider ${}_A\mathbf{h}(X, Y)$ as a subspace of Y^X. The topology on ${}_A\mathbf{h}(X, Y)$ induced from Y^X is called the *topology of simple convergence*. The subsets of the form

$$M(S, U) = \bigl\{\varphi \in {}_A\mathbf{h}(X, Y) : \varphi(S) \subset U\bigr\}, \tag{2.1}$$

where $U \subset Y$ is a neighborhood of zero and $S \subset X$ is a finite set, form a basis of neighborhoods of zero in that space. It is easy to check that the canonical isomorphism of vector spaces ${}_A\mathbf{h}(A, X) \cong X$, $\varphi \mapsto \varphi(1)$ is a topological isomorphism. Similarly, using X^n as notation for the direct sum of n copies of a module X (with the standard direct sum topology), we have a topological isomorphism ${}_A\mathbf{h}(A^n, X) \cong X^n$. When $X = A^n$ this isomorphism can be viewed as an isomorphism of topological algebras ${}_A\mathbf{h}(A^n) \cong M_n(A)$. In particular, if A is an Arens–Michael algebra, then so is ${}_A\mathbf{h}(A^n)$.

Later on we will give a slight generalization of the last observation (see Lemma 8.2).

Suppose A is a $\widehat{\otimes}$-algebra, X a right A-$\widehat{\otimes}$-module, and Y a left A-$\widehat{\otimes}$-module. A *projective A-module tensor product* $X \widehat{\otimes}_A Y$ is defined as the completion of the quotient space of $X \widehat{\otimes} Y$ by the closure of the linear hull of all elements of the form $x \cdot a \otimes y - x \otimes a \cdot y$ ($x \in X$, $y \in Y$, $a \in A$). For any two continuous prenorms on X and, respectively, on Y, their projective tensor product is a prenorm on $X \widehat{\otimes} Y$ and, after factorization and extension by continuity, yields a prenorm on $X \widehat{\otimes}_A Y$, which we shall also call a projective tensor product of the given prenorms. Notice that, as in a purely algebraic situation, the projective A-module tensor product admits an abstract categorical description (see [25, II.4.1]).

Recall that the *strongest locally convex topology* on a vector space E is defined by the system of all prenorms on E. We will denote the space E endowed with this topology by E_s. Clearly, if E is an LCS, then its topology is strongest locally convex if and only if any linear map from E to any LCS is continuous. Later in this paper we shall often use some basic properties of the strongest locally convex topology. For ease of reference, we collect those properties in a separate proposition.

Proposition 2.3. (i) *Any strongest LCS is complete.*
(ii) *If E is a strongest LCS and $F \subset E$ is a vector subspace, then the quotient topology on E/F is the strongest one.*
(iii) *If E is a strongest LCS and $F \subset E$ is a vector subspace, then F is closed and complemented in E and the topology on F induced from E is the strongest one.*
(iv) *If E, F are strongest LCS of at most countable dimension, then any bilinear map from $E \times F$ to an arbitrary LCS is jointly continuous.*

(v) *Any algebra A of at most countable dimension is a $\widehat{\otimes}$-algebra with respect to the strongest locally convex topology and any A-module of at most countable dimension is an A-$\widehat{\otimes}$-module with respect to the strongest locally convex topology.*

(vi) *Suppose A is a $\widehat{\otimes}$-algebra, X is a right A-$\widehat{\otimes}$-module, Y is a left A-$\widehat{\otimes}$-module. Suppose also that X and Y are of at most countable dimension and the topology on each one of them is strongest locally convex. Then the algebraic tensor product $X \otimes_A Y$ endowed with the strongest locally convex topology coincides with $X \widehat{\otimes}_A Y$.*

Proof. (i) Just notice that E is isomorphic to the topological direct sum of one-dimensional spaces and use the fact that completeness is preserved under direct sums (see [60, II.6.2]).

(ii) Just notice that the canonical map from E to $(E/F)_s$ is continuous and therefore factors through E/F.

(iii) Just consider an arbitrary (and automatically continuous) projector from E to F and use (ii).

(iv) See [5], Corollary A.2.8.

(v), (vi) These follow from (iv) and (i). ☐

Remark 2.1. Notice that (v) applies, in particular, to any finitely generated algebra A and any finitely generated A-module.

Let A be an algebra, X an A-bimodule, and α an endomorphism of A. We define a new structure of an A-bimodule on X by setting $a \circ x = a \cdot x$ and $x \circ a = x \cdot \alpha(a)$ for $a \in A$, $x \in X$. This bimodule is denoted X_α. In a similar way we define a bimodule ${}_\alpha X$. It is easy to see that if A is a topological algebra (resp., $\widehat{\otimes}$-algebra), α is continuous, and X is a topological A-bimodule (resp., A-$\widehat{\otimes}$-bimodule), then X_α and ${}_\alpha X$ are also topological A-bimodules (resp., A-$\widehat{\otimes}$-bimodules).

Suppose R is an algebra. Recall that an *R-algebra* is a pair (A, η_A), where A is an algebra and $\eta_A : R \to A$ is a homomorphism. If, in addition, R and A are topological algebras and η_A is continuous, then (A, η_A) is called a *topological R-algebra*. Finally, if both A and R are $\widehat{\otimes}$-algebras, then we shall say that (A, η_A) is an *R-$\widehat{\otimes}$-algebra*. When speaking of R-algebras we shall often abuse the language by saying that A is an R-algebra (without explicitly mentioning η_A); it should not lead to confusion. A similar convention applies to topological R-algebras and R-$\widehat{\otimes}$-algebras.

A homomorphism $\varphi : A \to B$ between two R-algebras is called an *R-homomorphism* if $\varphi\eta_A = \eta_B$. If A is an R-algebra and B is an S-algebra, then an *R-S-homomorphism* from A to B is defined as a pair of homomorphisms $f : A \to B$ and $g \colon R \to S$ making the diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \\
R \arrow[u, "\eta_A"] \arrow[r, "g"] & S \arrow[u, "\eta_B"']
\end{tikzcd}
\tag{2.2}
$$

commute. If g is fixed, then we shall sometimes say that $f \colon A \to B$ is an R-S-homomorphism. The category of topological R-algebras and continuous R-homomorphisms will be denoted R-Topalg.

Every topological R-algebra (resp., R-$\widehat{\otimes}$-algebra) A will be considered as a topological R-bimodule (resp., R-$\widehat{\otimes}$-bimodule) with respect to the homomorphism $\eta_A \colon R \to A$. We remark that R-homomorphisms are precisely algebra homomorphisms which are R-bimodule homomorphisms (or in this case, equivalently, homomorphisms of left or right R-modules).

If A is a topological algebra, and X is a topological A-bimodule, then the direct product $A \times X$ becomes a topological algebra with respect to multiplication

$$(a, x)(b, y) = (ab, a \cdot y + x \cdot b).$$

Algebra thus obtained is called the *semidirect product* of A and X or the *trivial extension* of A by X. It is easy to see that if A is a $\widehat{\otimes}$-algebra, and X is an A-$\widehat{\otimes}$-bimodule, then $A \times X$ is also a $\widehat{\otimes}$-algebra. If A is a topological R-algebra, then $A \times X$ also becomes a topological R-algebra with respect to the homomorphism

$$\eta_{A\times X} \colon R \to A \times X, \qquad \eta_{A\times X}(r) = (\eta_A(r), 0).$$

Notice that the projection $p_1 \colon A \times X \to A$ to the first factor is an R-homomorphism.

Suppose A is an algebra and X is an A-bimodule. Recall that a linear map $D \colon A \to X$ is called a *derivation* if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for any $a, b \in A$. Henceforth all derivations of topological algebras with values in topological bimodules will be assumed to be continuous. For a topological algebra A and a topological A-bimodule X the vector space of all derivations from A to X is denoted $\mathrm{Der}\,(A, X)$. If A is a topological R-algebra, then a derivation $D \colon A \to X$ is called an R-*derivation* if $D\eta_A = 0$. It is not difficult to see that this means precisely that D is a morphism of R-bimodules (or in this case, equivalently, of left or right R-modules). The subspace of $\mathrm{Der}\,(A, X)$ consisting of R-derivations will be denoted $\mathrm{Der}_R(A, X)$.

There is a close relationship between R-derivations from A to X and R-homomorphisms from A to $A \times X$. More precisely, as direct calculation shows, a continuous linear map $\varphi \colon A \to A \times X$ of the form $\varphi(a) = (a, D(a))$ is an R-homomorphism if and only if D is an R-derivation. In other words, there is an isomorphism of vector spaces

$$\mathrm{Der}_R(A, X) \cong \bigl\{\varphi \in \mathrm{Hom}_{R\text{-}\mathsf{Topalg}}(A, A \times X) : p_1\varphi = \mathbf{1}_A\bigr\}. \tag{2.3}$$

Moreover, the isomorphism (2.3) is in fact an isomorphism between functors of X acting from the category of topological A-bimodules to the category of vector spaces.

If A is a $\widehat{\otimes}$-algebra, then the category of left A-$\widehat{\otimes}$-modules (resp., right A-$\widehat{\otimes}$-modules, A-$\widehat{\otimes}$-bimodules) and (continuous) morphisms between them will be denoted A-mod (resp., mod-A, A-mod-A). These categories are additive and have both kernels and cokernels (see [25, 0.4.1]). More precisely, for any morphism $\varphi \colon X \to Y$ its kernel is a pair $(\mathrm{Ker}\,\varphi, \ker\varphi)$, where $\mathrm{Ker}\,\varphi = \varphi^{-1}(0)$ (with the topology induced from X) and $\ker\varphi \colon \mathrm{Ker}\,\varphi \to X$ is the canonical embedding. Similarly, the cokernel of φ is a pair $(\mathrm{Coker}\,\varphi, \mathrm{coker}\,\varphi)$, where $\mathrm{Coker}\,\varphi$ is the completion of the quotient module $Y/\overline{\varphi(X)}$, and $\mathrm{coker}\,\varphi \colon Y \to \mathrm{Coker}\,\varphi$ is the quotient map.

Suppose $f \colon A \to B$ is a homomorphism of $\widehat{\otimes}$-algebras. Recall that any left B-$\widehat{\otimes}$-module X is a left A-$\widehat{\otimes}$-module under the action $a \cdot x = f(a) \cdot x$ ($a \in A$, $x \in X$). Clearly, any morphism of B-$\widehat{\otimes}$-modules $\varphi \colon X \to Y$ is a morphism of A-$\widehat{\otimes}$-modules. Thus we have a functor from B-mod to A-mod, which we will denote $f^\bullet$ and call the *restriction of scalars* along f. A similar functor from mod-B to mod-A will also be denoted $f^\bullet$.

Suppose now that (A, η_A) is an R-$\widehat{\otimes}$-algebra. Set $\mathcal{A} = A$-mod, $\mathcal{R} = R$-mod and endow $\mathcal{R}$ with the split exact structure (see Appendix, Example 10.1). Let $\Box \colon \mathcal{A} \to \mathcal{R}$ denote the restriction-of-scalars functor along η_A; clearly, it is additive and preserves kernels and cokernels. Hence A-mod can be made an exact category as in Example 10.3 (see Appendix). The exact category $\mathcal{A}_{\mathcal{R}}$ thus obtained will be denoted (A, R)-mod (cf. [28]). Thus the admissible sequences in (A, R)-mod are those that split in R-mod. In particular, when $R = \mathbb{C}$, we recover the standard definition of an admissible (or $\mathbb{C}$-split) sequence of A-$\widehat{\otimes}$-modules used in the homological theory of topological algebras (see [25, 66]).

For a $\widehat{\otimes}$-algebra A let A^{op} denote the opposite algebra and A^e the enveloping algebra $A \mathbin{\widehat{\otimes}} A^{\mathrm{op}}$. If A is an R-$\widehat{\otimes}$-algebra and B is an S-$\widehat{\otimes}$-algebra, then we set

$$\mathsf{mod}\text{-}(A,R) = (A^{\mathrm{op}}, R^{\mathrm{op}})\text{-}\mathsf{mod},$$
$$(A,R)\text{-}\mathsf{mod}\text{-}(B,S) = (A \mathbin{\widehat{\otimes}} B^{\mathrm{op}}, R \mathbin{\widehat{\otimes}} S^{\mathrm{op}}).$$

The exact category $(A,\mathbb{C})$-mod will be denoted simply A-mod; this should not lead to confusion. We adopt a similar convention for the categories mod-$A = \mathsf{mod}$-$(A,\mathbb{C})$ and A-mod-$A = (A,\mathbb{C})$-mod-$(A,\mathbb{C})$.

Suppose as before that (A,η_A) is an R-$\widehat{\otimes}$-algebra and let $\square\colon A\text{-}\mathsf{mod} \to R\text{-}\mathsf{mod}$ be the restriction-of-scalars functor along η_A. It has a left adjoint functor, namely, $A \mathbin{\widehat{\otimes}}_R (\,\cdot\,)$. Therefore, taking account of Example 10.8 (see Appendix), for any $E \in R$-mod the A-$\widehat{\otimes}$-module $A \mathbin{\widehat{\otimes}}_R E$ is projective in (A,R)-mod, (A,R)-mod has enough projectives, and a module X is projective in (A,R)-mod if and only if the multiplication morphism $A \mathbin{\widehat{\otimes}}_R X \to X$ is a retraction of left A-$\widehat{\otimes}$-modules.

Modules of the form $A \mathbin{\widehat{\otimes}}_R E$, where $E \in R$-mod, will be called *free* in (A,R)-mod. For future reference we remark that, under the standard identification of right A-$\widehat{\otimes}$-modules with left A^{op} -$\widehat{\otimes}$-modules and of A-$\widehat{\otimes}$-bimodules with left A^e-$\widehat{\otimes}$-modules, free objects in the exact categories of right A-$\widehat{\otimes}$-modules and A-$\widehat{\otimes}$-bimodules are of the form:

$$\begin{array}{rll}
E \mathbin{\widehat{\otimes}}_R A & (E \in \mathsf{mod}\text{-}R) & \text{— in } \mathsf{mod}\text{-}(A,R),\\
A \mathbin{\widehat{\otimes}}_R E \mathbin{\widehat{\otimes}}_R A & (E \in R\text{-}\mathsf{mod}\text{-}R) & \text{— in } (A,R)\text{-}\mathsf{mod}\text{-}(A,R),\\
M \mathbin{\widehat{\otimes}}_R A & (M \in A\text{-}\mathsf{mod}\text{-}R) & \text{— in } (A,A)\text{-}\mathsf{mod}\text{-}(A,R),\\
A \mathbin{\widehat{\otimes}}_R M & (M \in R\text{-}\mathsf{mod}\text{-}A) & \text{— in } (A,R)\text{-}\mathsf{mod}\text{-}(A,A).
\end{array}$$

Remark 2.2. Of course, all this has an obvious purely algebraic analog (see [28]). In particular, if A is an R-algebra, then the exact category of left A-modules (A,R)-algmod is the category of all left A-modules whose admissible sequences are precisely those that split as sequences of left R-modules. Notice that if A and the left A-module X are of at most countable dimension, then X is projective in (A,R)-algmod if and only if the strongest locally convex A_s-$\widehat{\otimes}$-module X_s is projective in (A_s,R_s)-mod. This follows immediately from the description of projective objects in terms of free ones (see above) and from the isomorphism $(A \otimes_R X)_s \cong A_s \mathbin{\widehat{\otimes}}_{R_s} X_s$ (see Prop. 2.3). Clearly, similar assertions are also true for right modules and for bimodules.

Since the above categories have enough projective objects, it follows that any additive covariant (resp., contravariant) functor on those categories with values in an exact category has a left (resp., right) derived functor (see Appendix).

In particular, let Vect be the abelian category of vector spaces and LCS the quasiabelian category of locally convex spaces (see Appendix, Example 10.2). Then, for a fixed $Y \in (A,R)$-mod, the morphism functor

$${}_A\mathbf{h}(\,\cdot\,,Y)\colon (A,R)\text{-}\mathsf{mod} \to \mathsf{Vect}$$

has a right derived functor

$$\mathrm{R}_A\mathbf{h}(\,\cdot\,,Y)\colon \mathsf{D}^-((A,R)\text{-}\mathsf{mod}) \to \mathsf{D}^+(\mathsf{Vect}).$$

The corresponding n-th classical derived functor is denoted $\operatorname{Ext}^n_{A,R}(\,\cdot\,,Y)$. Similarly, for a fixed $X \in \mathsf{mod}$-(A,R), the tensor product functor

$$X \mathbin{\widehat{\otimes}}_A (\,\cdot\,)\colon (A,R)\text{-}\mathsf{mod} \to \mathsf{LCS}$$

has a left derived functor

$$X \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A (\,\cdot\,)\colon \mathsf{D}^-((A,R)\text{-}\mathsf{mod}) \to \mathsf{D}^-(\mathsf{LCS}).$$

The corresponding n-th classical derived functor is denoted $\operatorname{Tor}_n^{A,R}(X,\,\cdot\,)$. When $R=\mathbb{C}$ the constructed functors are denoted $\operatorname{Ext}_A^n(\,\cdot\,,Y)$ and $\operatorname{Tor}_n^A(X,\,\cdot\,)$, respectively; see [25, III.4].

Remark 2.3. Since the forgetful functor from Vect to the category $\mathcal{A}b$ of abelian groups preserves kernels and cokernels, it is easy to see that the underlying abelian group of $\operatorname{Ext}_{A,R}^n(X,Y)$ is naturally isomorphic to the group $\operatorname{Ext}_{(A,R)\text{-}\mathsf{mod}}^n(X,Y)$ (see Appendix, Example 10.9).

For $X\in\mathsf{mod}\text{-}(A,R)$ and $Y\in(A,R)\text{-}\mathsf{mod}$ the canonical morphism $X\mathbin{\widehat{\otimes}}^{\mathrm{L}}_A Y\to X\mathbin{\widehat{\otimes}}_A Y$ (see Appendix) gives rise, when passing to cohomology, to a morphism $\operatorname{Tor}_0^{A,R}(X,Y)\to X\mathbin{\widehat{\otimes}}_A Y$ of LCS. It is an open map onto its image, which is dense in $X\mathbin{\widehat{\otimes}}_A Y$, and its kernel is the closure of zero in $\operatorname{Tor}_0^{A,R}(X,Y)$. We omit the proof because it is identical to that in the case $R=\mathbb{C}$; see [25, III.4.26]. Therefore, $X\mathbin{\widehat{\otimes}}_A Y$ can be identified with the completion of $\operatorname{Tor}_0^{A,R}(X,Y)$.

If B is an S-$\widehat{\otimes}$-algebra and $X\in(B,S)\text{-}\mathsf{mod}\text{-}(A,R)$, then $X\mathbin{\widehat{\otimes}}_A(\,\cdot\,)$ can be viewed as a functor from $(A,R)\text{-}\mathsf{mod}$ to $(B,S)\text{-}\mathsf{mod}$. The corresponding left derived functor with values in $\mathsf{D}^-((B,S)\text{-}\mathsf{mod})$ will be denoted by the same symbol $X\mathbin{\widehat{\otimes}}^{\mathrm{L}}_A(\,\cdot\,)$ as before. It will not lead to confusion because we will always indicate the domain and codomain of the given functor.

Remark 2.4. As in a purely algebraic situation (see, for example, [23, I.6] or [30, 1.10]), one can show that $\mathrm{R}_A\mathbf{h}$ and $\widehat{\otimes}^{\mathrm{L}}_A$ extend to bifunctors

$$\mathrm{R}_A\mathbf{h}\colon \mathsf{D}^-((A,R)\text{-}\mathsf{mod})\times\mathsf{D}^+((A,R)\text{-}\mathsf{mod})\to\mathsf{D}^+(\mathsf{Vect}),$$
$$\widehat{\otimes}^{\mathrm{L}}_A\colon \mathsf{D}^-(\mathsf{mod}\text{-}(A,R))\times\mathsf{D}^-((A,R)\text{-}\mathsf{mod})\to\mathsf{D}^-(\mathsf{LCS}).$$

Remark 2.5. On the morphism spaces ${}_A\mathbf{h}(X,Y)$ one can consider various topologies of uniform convergence on families of bounded subsets (for example, all bounded subsets, or precompact subsets, or finite subsets ...); see [60, III.3]. This allows us to view $\mathrm{R}_A\mathbf{h}$ as a functor with values in $\mathsf{D}^+(\mathsf{LCS})$ and $\operatorname{Ext}_{A,R}^n(\,\cdot\,,\,\cdot\,)$ as a functor with values in LCS. We will not go into detail since for our purpose, we do not need such a level of generality.

The *relative homological dimension* of an A-$\widehat{\otimes}$-module X is by definition

$$\operatorname{dh}_{A,R}X=\min\Bigl\{n\in\mathbb{Z}_+ :\ \operatorname{Ext}_{A,R}^{n+1}(X,Y)=0\ \forall Y\in(A,R)\text{-}\mathsf{mod}\Bigr\}.$$

If such an n does not exist, then one sets $\operatorname{dh}_{A,R}X=+\infty$. Notice that X is projective in $(A,R)\text{-}\mathsf{mod}$ if and only if $\operatorname{dh}_{A,R}X=0$. The relative homological dimension may also be defined as the length of a shortest projective resolution (see Appendix) of X in $(A,R)\text{-}\mathsf{mod}$. Finally, for any $p>\operatorname{dh}_{A,R}X$ and any $Y\in(A,R)\text{-}\mathsf{mod}$ we have $\operatorname{Ext}_{A,R}^p(X,Y)=0$. We omit the proofs of these assertions because they are identical to those in the case $R=\mathbb{C}$ (see [25, III.5]).

The *relative left global dimension* of an R-$\widehat{\otimes}$-algebra A is defined as

$$\operatorname{dg}_R A=\sup\bigl\{\operatorname{dh}_{A,R}X :\ X\in(A,R)\text{-}\mathsf{mod}\bigr\}.$$

When $R=\mathbb{C}$ the dimensions of modules and algebras just defined are denoted $\operatorname{dh}_A X$ and $\operatorname{dg}A$, respectively; see [25, III.5].

We end this section with several results about acyclic objects (see Appendix) which will be needed later.

Proposition 2.4. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be exact categories and $F\colon\mathcal{A}\to\mathcal{B}$ and $G\colon\mathcal{B}\to\mathcal{C}$ additive functors having left derived functors. Suppose that the conditions of the theorem* *on the composition of derived functors (see Appendix) hold for F and G. Let $X \in \mathrm{Ob}\,(\mathcal{A})$ be an F-acyclic object. Then X is GF-acyclic if and only if $F(X)$ is G-acyclic.*

Proof. By the assumption, the canonical morphism $\mathrm{L}F(X) \to F(X)$ is an isomorphism in $\mathsf{D}^-(\mathcal{B})$. Therefore, $(\mathrm{L}G \circ \mathrm{L}F)(X) \to \mathrm{L}G(F(X))$ is an isomorphism in $\mathsf{D}^-(\mathcal{C})$. In the commutative diagram

$$\begin{tikzcd} (\mathrm{L}G \circ \mathrm{L}F)(X) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathrm{L}G(F(X)) \arrow[d] \\ \mathrm{L}(G \circ F)(X) \arrow[r] & G(F(X)) \end{tikzcd} \tag{2.4}$$

the top horizontal and the left vertical arrows are isomorphisms. Therefore, the lower horizontal arrow is an isomorphism if and only if the right vertical arrow is an isomorphism. The rest is clear. ☐

Corollary 2.5. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F_1"] & \mathcal{B} \\ \mathcal{R} \arrow[u, "G_1"] \arrow[r, "F_2"'] & \mathcal{S} \arrow[u, "G_2"'] \end{tikzcd}$$

be a commutative diagram of exact categories and additive functors having left derived functors. Suppose that the theorem on the composition of derived functors is true for (F_2, G_2) and (G_1, F_1). Suppose $X \in \mathrm{Ob}\,(\mathcal{R})$ is acyclic with respect to F_2 and G_1, and $F_2(X)$ is acyclic with respect to G_2. Then $G_1(X)$ is acyclic with respect to F_1.

Proposition 2.6. *Any R-$\widehat{\otimes}$-algebra A is acyclic with respect to the tautological functors*

$$(A, R)\text{-}\mathsf{mod}\text{-}(A, R) \to (A, A)\text{-}\mathsf{mod}\text{-}(A, R),$$
$$(A, R)\text{-}\mathsf{mod}\text{-}(A, R) \to (A, R)\text{-}\mathsf{mod}\text{-}(A, A).$$

Moreover, any projective resolution of A in (A, R)-mod-(A, R) is automatically a projective resolution in (A, A)-mod-(A, R) and (A, R)-mod-(A, A).

Proof. Clearly, it suffices to consider the first of the two functors. Fix a projective resolution

$$0 \leftarrow A \leftarrow P_\bullet \tag{2.5}$$

in (A, R)-mod-(A, R). The augmented complex (2.5) in the exact category (A, R)-mod-(R, R) is admissible and all of its objects are projective Therefore, it splits in A-mod-R; i.e., it is admissible in (A, A)-mod-(A, R). This proves the acyclicity. The second assertion follows at once from the fact that the above functors send free modules to free modules and therefore projectives to projectives. ☐

Proposition 2.7. *Let A be an R-$\widehat{\otimes}$-algebra, $\mathcal{B}$ an exact category, and $F\colon A$-mod-$A \to \mathcal{B}$ an additive covariant functor. Suppose functors*

$$F_1\colon (A, A)\text{-}\mathsf{mod}\text{-}(A, R) \to \mathcal{B},$$
$$F_2\colon (A, R)\text{-}\mathsf{mod}\text{-}(A, R) \to \mathcal{B},$$
$$F_3\colon (A, R)\text{-}\mathsf{mod}\text{-}(A, A) \to \mathcal{B}$$

act the same way as F. Then $\mathrm{L}F_1(A) \cong \mathrm{L}F_2(A) \cong \mathrm{L}F_3(A)$ in $\mathsf{D}^-(\mathcal{B})$.

Proof. Let $I\colon (A,R)\text{-}\mathsf{mod}\text{-}(A,R) \to (A,A)\text{-}\mathsf{mod}\text{-}(A,R)$ denote the tautological functor. Clearly, $F_2 = F_1 \circ I$. In $\mathsf{D}^-(\mathcal{B})$ we have a chain of morphisms

$$\mathrm{L}F_2(A) = \mathrm{L}(F_1 \circ I)(A) \leftarrow \mathrm{L}F_1(LI(A)) \to \mathrm{L}F_1(I(A)) = \mathrm{L}F_1(A).$$

Since I sends projectives to projectives (see the proof of Proposition 2.6), the first arrow in this chain is an isomorphism. By the same proposition, the second arrow is also an isomorphism. Therefore, $\mathrm{L}F_1(A) \cong \mathrm{L}F_2(A)$, as required. The isomorphism $\mathrm{L}F_3(A) \cong \mathrm{L}F_2(A)$ is established similarly. □

Corollary 2.8. *Let A be an R-$\widehat{\otimes}$-algebra and X an A-$\widehat{\otimes}$-bimodule. Then for any $n \in \mathbb{Z}$, we have isomorphisms of vector spaces*

$$\mathrm{Tor}_n^{A^e,\, R^e}(X,A) \cong \mathrm{Tor}_n^{A^e,\, A \widehat{\otimes} R^{\mathrm{op}}}(X,A) \cong \mathrm{Tor}_n^{A^e,\, R \widehat{\otimes} A^{\mathrm{op}}}(X,A),$$
$$\mathrm{Ext}^n_{A^e,\, R^e}(A,X) \cong \mathrm{Ext}^n_{A^e,\, A \widehat{\otimes} R^{\mathrm{op}}}(A,X) \cong \mathrm{Ext}^n_{A^e,\, R \widehat{\otimes} A^{\mathrm{op}}}(A,X).$$

Remark 2.6. If we endow the Ext-spaces from Corollary 2.8 with the locally convex topology as in Remark 2.5, then the above isomorphisms will also be topological isomorphisms.

Definition 2.1. Suppose A is an R-$\widehat{\otimes}$-algebra and X is an A-$\widehat{\otimes}$-bimodule. The vector spaces

$$\mathcal{H}_n(A,R;X) = \mathrm{Tor}_n^{A^e,\, R^e}(X,A), \quad \mathcal{H}^n(A,R;X) = \mathrm{Ext}^n_{A^e,\, R^e}(A,X)$$

are called, respectively, the *n-th relative Hochschild homology (cohomology) group* of A with coefficients in X.

Definition 2.2. The number $\mathrm{db}_R\, A = \mathrm{dh}_{A^e,\, R^e}\, A$ is called the *relative bidimension* of the R-$\widehat{\otimes}$-algebra A is. When $R = \mathbb{C}$, it is denoted $\mathrm{db}\, A$ (see [25, III.5]).

Remark 2.7. In view of Corollary 2.8, the above definitions of relative homology and cohomology and relative bidimension are equivalent to Hochschild's original definitions in [28]. See also [19].

3. Arens–Michael envelopes

Arens–Michael envelopes (under a different name) were introduced in [66]. Here we adopt the terminology from [24].

Definition 3.1. Suppose A is a topological algebra. A pair $(\widehat{A}, \iota_A)$ consisting of an Arens–Michael algebra $\widehat{A}$ and a continuous homomorphism $\iota_A\colon A \to \widehat{A}$ is called the *Arens–Michael envelope* of A if for any Arens–Michael algebra B and any continuous homomorphism $\varphi\colon A \to B$ there exists a unique continuous homomorphism $\widehat{\varphi}\colon \widehat{A} \to B$ making the following diagram commute:

$$\begin{tikzcd} \widehat{A} \arrow[r, dashed, "\widehat{\varphi}"] & B \\ A \arrow[u, "\iota_A"] \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{3.1}$$

We then say that $\widehat{\varphi}$ *extends* φ (even though ι_A is in general not injective; see Example 3.4 below).

In other words, if AM denotes the full subcategory of Topalg consisting of Arens–Michael algebras and Set denotes the category of sets, then the Arens–Michael envelope of a topological algebra A is a representing object for the functor $\mathrm{Hom}_{\mathsf{Topalg}}(A, \cdot\,)\colon \mathsf{AM} \to \mathsf{Set}$. The Arens–Michael envelope is clearly unique in the sense that if $(\widehat{A}, \iota_A)$ and $(\overline{A}, j_A)$

are Arens–Michael envelopes of the same topological algebra A, then there exists a unique isomorphism of topological algebras $\varphi\colon \widehat{A} \to \overline{A}$ making the following diagram commute:

$$\begin{tikzcd} \widehat{A} \arrow[rr, dashed, "\varphi"] & & \overline{A} \\ & A \arrow[ul, "\iota_A"] \arrow[ur, "j_A"'] & \end{tikzcd}$$

Remark 3.1. Notice that it suffices to check the universal property of the Arens–Michael envelope expressed by diagram (3.1) only for homomorphisms with values in Banach algebras. This follows easily from the fact that each Arens–Michael algebra is the inverse limit of a system of Banach algebras (see, for example, [24, Ch. V]). Indeed, suppose A is a topological algebra, $\widehat{A}$ is an Arens–Michael algebra, and $\iota_A\colon A \to \widehat{A}$ is a homomorphism such that for any Banach algebra B and any homomorphism $\varphi\colon A \to B$ there is a unique homomorphism $\widehat{\varphi}\colon \widehat{A} \to B$ making diagram (3.1) commutative. Take an arbitrary Arens–Michael algebra C and represent it as the inverse limit $\varprojlim C_\nu$ of a system of Banach algebras. Then

$$\begin{aligned} \operatorname{Hom}_{\mathsf{Topalg}}(A, C) &= \operatorname{Hom}_{\mathsf{Topalg}}(A, \varprojlim C_\nu) = \varprojlim \operatorname{Hom}_{\mathsf{Topalg}}(A, C_\nu) \\ &= \varprojlim \operatorname{Hom}_{\mathsf{AM}}(\widehat{A}, C_\nu) = \operatorname{Hom}_{\mathsf{AM}}(\widehat{A}, \varprojlim C_\nu) = \operatorname{Hom}_{\mathsf{AM}}(\widehat{A}, C). \end{aligned}$$

Therefore, $(\widehat{A}, \iota_A)$ is the Arens–Michael envelope of A.

Every topological algebra A has an Arens–Michael envelope (see [66] or [24, Ch. V]). More precisely, $(\widehat{A}, \iota_A)$ can be obtained as the completion of A with respect to a new topology defined by the family of all continuous submultiplicative prenorms on A. In particular, this implies that the image of the canonical homomorphism $\iota_A\colon A \to \widehat{A}$ is dense. It is easy to see that the correspondence $A \mapsto \widehat{A}$ is a functor from Topalg to AM and that it is the left adjoint of the embedding functor from AM to Topalg. Henceforth we shall call it the *Arens–Michael functor*.

If A is an algebra without topology, then we define its Arens–Michael envelope as the one with respect to the strongest locally convex algebra A_s (see Section 2).

We now give several basic examples of Arens–Michael envelopes. Other examples will be given in Section 5.

Example 3.1. Suppose the polynomial algebra $A = \mathbb{C}[z_1, \dots, z_n]$, is endowed with the strongest locally convex topology. Taylor [66] noticed that its Arens–Michael envelope coincides with the algebra of entire functions $\mathcal{O}(\mathbb{C}^n)$ endowed with the compact-open topology and that the homomorphism ι_A is the tautological embedding. Since this example serves as the motivation for the rest of the paper, we sketch the basic idea behind the proof of the last assertion. For simplicity, we consider the case $n = 1$.

The Cauchy inequality for the coefficients of a power series implies that the compact-open topology on $\mathcal{O}(\mathbb{C})$ can be defined by the family of norms

$$\|f\|_\rho = \sum_i |c_i|\rho^i \quad \Big(f = \sum_i c_i z^i \in \mathcal{O}(\mathbb{C});\ 0 < \rho < \infty\Big).$$

For any Banach algebra B and homomorphism $\varphi\colon \mathbb{C}[z] \to B$, set $b = \varphi(z)$. Then for any entire function $f = \sum_i c_i z^i \in \mathcal{O}(\mathbb{C})$ the series $\sum_i c_i b^i$ converges absolutely in B to some $f(b) \in B$; moreover, $\|f(b)\| \le \|f\|_\rho$ for any $\rho \ge \|b\|$. The map $f \mapsto f(b)$ is the desired homomorphism from $\mathcal{O}(\mathbb{C})$ to B extending φ.

The case of an arbitrary n is argued similarly.

Example 3.2. Let $A = F_n$ be the free $\mathbb{C}$-algebra with n generators $\zeta_1, \dots, \zeta_n$. For any set of integers $\alpha = (\alpha_1, \dots, \alpha_k)$ from $[1, n]$, let ζ_α be the element $\zeta_{\alpha_1} \cdots \zeta_{\alpha_k}$ of F_n. It is convenient to assume that the identity of the algebra corresponds to the set of zero length ($k = 0$). We also set $|\alpha| = k$. The algebra $\mathcal{F}_n$ of noncommutative power series consists of all formal expressions $a = \sum_\alpha \lambda_\alpha \zeta_\alpha$ (where $\lambda_\alpha \in \mathbb{C}$) such that

$$\|a\|_\rho = \sum_\alpha |\lambda_\alpha| \rho^{|\alpha|} < \infty \quad \text{for all} \quad 0 < \rho < \infty.$$

The system of prenorms $\{\| \cdot \|_\rho : 0 < \rho < \infty\}$ makes $\mathcal{F}_n$ into a Fréchet–Arens–Michael algebra. Clearly, $\mathcal{F}_n$ contains F_n as a subalgebra. Taylor showed [67] (cf. also the previous example), that $\mathcal{F}_n$ is the Arens–Michael envelope of F_n. Notice that when $n = 1$, we have $F_1 = \mathbb{C}[z]$ and $\mathcal{F}_1 \cong \mathcal{O}(\mathbb{C})$.

Example 3.3. Let $\mathfrak{g}$ be a semisimple Lie algebra and $A = U(\mathfrak{g})$ its universal enveloping algebra. Taylor showed [67] that the Arens–Michael envelope of $U(\mathfrak{g})$ is topologically isomorphic to the direct product of a countable family of full matrix algebras. More precisely, each finite-dimensional irreducible representation π_λ of $\mathfrak{g}$ yields a homomorphism $\varphi_\lambda \colon U(\mathfrak{g}) \to M_{d_\lambda}(\mathbb{C})$ (here d_λ is the dimension of π_λ). Let $\widehat{\mathfrak{g}}$ be the set of equivalence classes of irreducible finite-dimensional representations of $\mathfrak{g}$; we then have a homomorphism

$$\prod_{\lambda \in \widehat{\mathfrak{g}}} \varphi_\lambda \colon U(\mathfrak{g}) \to \prod_{\lambda \in \widehat{\mathfrak{g}}} M_{d_\lambda}(\mathbb{C}).$$

The algebra $\prod_\lambda M_{d_\lambda}(\mathbb{C})$ endowed with the direct product topology together with the above homomorphism is the Arens–Michael envelope of $U(\mathfrak{g})$.

Notice that for the algebras in Examples 3.1–3.3 the canonical homomorphism ι_A to the Arens–Michael envelope is injective. The reason for this is that those algebras have sufficiently many Banach representations. On the other hand, it may happen that a given algebra A has no nonzero Banach representations, and in that case its Arens–Michael envelope is trivial. For illustration, consider the following example.

Example 3.4. Suppose A is a Weyl algebra (endowed, as the algebras from the above examples, with the strongest locally convex topology). Recall that it is generated (as a unital $\mathbb{C}$-algebra) by generators ∂ and x, subject to the relation $[\partial, x] = 1$. It is well known (see, for example, [59, 13.6]) that a nonzero normed algebra cannot contain elements ∂ and x with such a property. Hence there are no nonzero submultiplicative prenorms on A and therefore $\widehat{A} = 0$.

Other examples of algebras with trivial Arens–Michael envelopes can be found in [24, 5.2.26].

Example 3.5. Choose any Banach algebra A such that all homomorphisms from it to any other Banach algebra are continuous. For example, this could be the algebra $\mathcal{B}(H)$ of bounded operators in a Hilbert space or, more generally, in a Banach space E such that $E \cong E \oplus E$; see [10, 5.4.12]. Endowing A with the strongest locally convex topology we have the topological algebra A_s. Then, clearly, we have an isomorphism $\widehat{A_s} \cong A$.

When describing Arens–Michael envelopes it is often convenient to use the fact that the Arens–Michael functor commutes with the quotient operation on algebras. The following proposition is proved in [48].

Proposition 3.1. *Suppose A is a topological algebra and I is a two-sided ideal in A. Let J denote the closure of the set $\iota_A(I)$ in $\widehat{A}$. Then J is a two-sided ideal in $\widehat{A}$, and* *the homomorphism $A/I \to \widehat{A}/J$ induced by the homomorphism $\iota_A \colon A \to \widehat{A}$ extends to an isomorphism of topological algebras*

$$\widehat{A/I} \cong (\widehat{A}/J)^{\sim}.$$

Since the quotient space of any Fréchet space by a closed subspace is complete, we have the following.

Corollary 3.2. *Under the assumptions of Proposition* 3.1*, assume in addition that $\widehat{A}$ is a Fréchet algebra. Then $\widehat{A/I} \cong \widehat{A}/J$.*

Corollary 3.3. *If A is a finitely generated algebra, then $\widehat{A_s}$ is a trace class Fréchet algebra.*

Proof. Since A is finitely generated, it is isomorphic to a quotient algebra of the free algebra F_n with finitely many generators. Now apply Example 3.2 and Corollary 3.2, as well as the fact that the Fréchet algebra $\mathcal{F}_n$ is trace class (see [37]). $\square$

Example 3.6. Suppose V is an affine complex algebraic variety embedded in $\mathbb{C}^N$. Let $\mathcal{O}^{\mathrm{alg}}(V)$ denote the algebra of regular (polynomial) functions on V. Generalizing Example 3.1, we shall show that its Arens–Michael envelope is the Fréchet algebra of holomorphic functions $\mathcal{O}(V)$. To this end, notice that $\mathcal{O}^{\mathrm{alg}}(V) = \mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N)/I$, where I is the ideal in $\mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N)$ of all functions vanishing on V. Since the algebra $\mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N) = \mathbb{C}[z_1, \ldots, z_N]$ is Noetherian, there is an exact sequence

$$\big(\mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N)\big)^r \to \mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N) \to \mathcal{O}^{\mathrm{alg}}(V) \to 0 \tag{3.2}$$

of $\mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N)$-modules. It gives rise to an exact sequence

$$(\mathcal{O}^{\mathrm{alg}}_{\mathbb{C}^N})^r \to \mathcal{O}^{\mathrm{alg}}_{\mathbb{C}^N} \to i_* \mathcal{O}^{\mathrm{alg}}_V \to 0 \tag{3.3}$$

of sheaves of $\mathcal{O}^{\mathrm{alg}}_{\mathbb{C}^N}$-modules, where i denotes the embedding of V in $\mathbb{C}^N$. Applying the analytization functor to (3.3) (see, for example, [69, 13.4] or [62]), we have an exact sequence

$$(\mathcal{O}_{\mathbb{C}^N})^r \to \mathcal{O}_{\mathbb{C}^N} \to i_* \mathcal{O}_V \to 0$$

of $\mathcal{O}_{\mathbb{C}^N}$-modules. Finally, applying the global section functor we have, by Cartan's Theorem B, an exact sequence

$$\big(\mathcal{O}(\mathbb{C}^N)\big)^r \to \mathcal{O}(\mathbb{C}^N) \to \mathcal{O}(V) \to 0 \tag{3.4}$$

of $\mathcal{O}(\mathbb{C}^N)$-modules. Let $J \subset \mathcal{O}(\mathbb{C}^N)$ denote the ideal of all functions vanishing on V. Comparing (3.2) and (3.4), we see that if $I \subset \mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N)$ is generated by r functions, then the same functions generate J as an ideal in $\mathcal{O}(\mathbb{C}^N)$. Therefore, J is the smallest ideal in $\mathcal{O}(\mathbb{C}^N)$ containing I. In particular, $J \subset \overline{I}$. Since the inverse inclusion is obvious by the closedness of J, we have $J = \overline{I}$. Now it remains to apply Corollary 3.2, according to which

$$\big(\mathcal{O}^{\mathrm{alg}}(V)\big)\widehat{\ } = \big(\mathcal{O}^{\mathrm{alg}}(\mathbb{C}^N)/I\big)\widehat{\ } = \mathcal{O}(\mathbb{C}^N)/J = \mathcal{O}(V).$$

Next, we shall establish several auxiliary results about modules over Arens–Michael algebras. Suppose A is an algebra and X is a left A-module. We shall say that a prenorm $\|\cdot\|_X$ on X is *m-compatible* if there is a submultiplicative prenorm $\|\cdot\|_A$ on A such that $\|a \cdot x\|_X \le \|a\|_A \|x\|_X$ for all $a \in A$ and $x \in X$. A similar definition makes sense when X is a right module or a bimodule. In the latter case we shall simultaneously require that $\|a \cdot x\|_X \le \|a\|_A \|x\|_X$ and $\|x \cdot a\|_X \le \|x\|_X \|a\|_A$ for all $a \in A$ and $x \in X$. If A is a topological algebra and X is a topological A-module, then we shall also require that both prenorms $\|\cdot\|_X$ and $\|\cdot\|_A$ be continuous. Notice that if Y is a submodule of X, then the quotient prenorm on X/Y induced by an m-compatible prenorm on X is also m-compatible.

Proposition 3.4. *Suppose A is an Arens–Michael algebra and X is a left A-$\widehat{\otimes}$-module. Then the topology on X can be defined by some family of m-compatible prenorms.*

Proof. Since every left A-$\widehat{\otimes}$-module is a quotient of a free module, it suffices to prove the assertion for free modules. Thus let $X = A \widehat{\otimes} E$ be a free A-module. For each continuous submultiplicative prenorm on A and each continuous prenorm on E their projective tensor product is clearly an m-compatible prenorm on X. On the other hand, the prenorms of this form do define the topology on X. □

Remark 3.2. Suppose $\{\|\cdot\|_\lambda : \lambda \in \Lambda_1\}$ is a directed system of submultiplicative prenorms defining the topology on A and $\{\|\cdot\|_\mu : \mu \in \Lambda_2\}$ is a directed system of prenorms defining the topology on X. For each $\lambda \in \Lambda_1$ and each $\mu \in \Lambda_2$ we define a new prenorm on X by setting

$$\|x\|_{\lambda,\mu} = \inf\left\{ \sum_{i=1}^n \|a_i\|_\lambda \|x_i\|_\mu : \ x = \sum_{i=1}^n a_i x_i, \ a_i \in A, \ x_i \in X, \ n \in \mathbb{N} \right\}.$$

It is easy to check that this prenorm is m-compatible and that the system of all prenorms on X of this form is equivalent to the original one.

Remark 3.3. Notice that the completeness of the algebras and modules in Proposition 3.4 is not essential.

Proposition 3.5. *Suppose that A is an Arens–Michael algebra. Then any left A-$\widehat{\otimes}$-module X is isomorphic to the inverse limit of some system of Banach A-$\widehat{\otimes}$-modules.*

Proof. For each m-compatible prenorm $\|\cdot\|_X$ on X the set of $x \in X$ such that $\|x\|_X = 0$ is a submodule of X. With the aid of Proposition 3.4, the rest of the proof is argued similarly to the proof in the case $A = \mathbb{C}$ (see, for example, [60, II.5.4]). □

Remark 3.4. Assertions similar to Propositions 3.4 and 3.5 for right modules and for bimodules also hold.

Suppose E is a locally convex space whose topology is defined by a system of prenorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$. We can make Λ a partially ordered set by defining $\lambda \prec \mu$ if $\|x\|_\lambda \le \|x\|_\mu$ for all $x \in E$. Henceforth we shall assume that the indexing set for prenorms on a locally convex space is ordered this way.

Proposition 3.6. *Suppose A is an Arens–Michael algebra and X is an A-$\widehat{\otimes}$-bimodule. Then the trivial extension $A \times X$ is also an Arens–Michael algebra.*

Proof. Let $\{\|\cdot\|_\lambda : \lambda \in \Lambda_1\}$ be the system of all continuous submultiplicative prenorms on A and $\{\|\cdot\|_\mu : \mu \in \Lambda_2\}$ the system of all continuous m-compatible prenorms on X. In view of Proposition 3.4, these systems define the topologies on A and X, respectively. For each $\lambda \in \Lambda_1$ and each $\mu \in \Lambda_2$ we define a prenorm on $A \times X$ by setting $\|(a,x)\|_{\lambda,\mu} = \|a\|_\lambda + \|x\|_\mu$. Clearly, the system of all such prenorms defines the direct product topology on $A \times X$. Using the definition of an m-compatible prenorm, for any $\mu \in \Lambda_2$ we choose an element $\varphi(\mu) \in \Lambda_1$ such that $\|a \cdot x\|_\mu \le \|a\|_{\varphi(\mu)} \|x\|_\mu$ and $\|x \cdot a\|_\mu \le \|x\|_\mu \|a\|_{\varphi(\mu)}$ for any $a \in A$ and any $x \in X$. Set $\Lambda = \{(\lambda,\mu) \in \Lambda_1 \times \Lambda_2 : \varphi(\mu) \prec \lambda\}$. Clearly, Λ is cofinal in $\Lambda_1 \times \Lambda_2$ and therefore the topology on $A \times X$ is defined by the system of prenorms $\{\|\cdot\|_{\lambda,\mu} : (\lambda,\mu) \in \Lambda\}$. For any elements (a,x) and (b,y) in $A \times X$ and any $(\lambda,\mu) \in \Lambda$ we have

$$\begin{aligned}\|(a,x)(b,y)\|_{\lambda,\mu} &= \|(ab, a\cdot y + x \cdot b)\|_{\lambda.\mu} \\ &= \|ab\|_\lambda + \|a\cdot y + x\cdot b\|_\mu \le \|a\|_\lambda \|b\|_\lambda + \|a\|_\lambda \|y\|_\mu + \|x\|_\mu \|b\|_\lambda \\ &\le (\|a\|_\lambda + \|x\|_\mu)(\|b\|_\lambda + \|y\|_\mu) = \|(a,x)\|_{\lambda,\mu} \|(b,y)\|_{\lambda,\mu}.\end{aligned}$$

Therefore, each prenorm $\|\cdot\|_{\lambda,\mu}$ for $(\lambda,\mu)\in\Lambda$ is submultiplicative and $A\times X$ is an Arens–Michael algebra. $\square$

Using the preceding result it is not difficult to show that the Arens–Michael envelope has the extension property not just for homomorphisms but also for derivations defined on the original algebra. Before giving a rigorous statement and a proof of this assertion we want to make some observations. Suppose A is a topological R-algebra, B is a topological S-algebra, and (f,g) is an R-S-homomorphism from A to B (see (2.2)). It is easy to see that for any topological B-bimodule X and any S-derivation $D\colon B\to X$ the composition $D\circ f\colon A\to X$ is an R-derivation. Thus we have a map

$$\operatorname{Der}_S(B,X)\to\operatorname{Der}_R(A,X),\quad D\mapsto D\circ f, \tag{3.5}$$

obviously giving rise to a morphism of functors of X acting from the category of topological B-bimodules to the category of vector spaces.

Recall that a morphism $f\colon X\to Y$ in an arbitrary category $\mathcal{A}$ is said to be an *epimorphism* if for any $Z\in\operatorname{Ob}(\mathcal{A})$ and any $g,h\colon Y\to Z$ we have $g=h$ whenever $gf=hf$. If this condition is only satisfied for those Z which are objects of a certain full subcategory $\mathcal{C}\subset\mathcal{A}$, then we shall say that f is an *epimorphism relative to* $\mathcal{C}$. For ease of reference we state the following lemma, the proof of which is obvious.

Lemma 3.7. *Suppose that in some category $\mathcal{A}$ we are given a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "h"] & C \\ R \arrow[u, "\alpha"] \arrow[r, "g"] & S \arrow[u, "\beta"] \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{3.6}$$

with a commutative left square. Suppose further that the diagram obtained by removing β is also commutative. If g is an epimorphism relative to some full subcategory $\mathcal{C}\subset\mathcal{A}$ containing C, then the whole diagram (3.6) *is commutative.*

Proposition 3.8. *Suppose A,R,S are topological algebras and $g\colon R\to S$ is an epimorphism in* Topalg *relative to a subcategory of the category of Arens–Michael algebras* AM. *Suppose that A is a topological R-algebra and its Arens–Michael envelope $\widehat{A}$ is a topological S-algebra such that the pair (ι_A,g) is an R-S-homomorphism from A to $\widehat{A}$. Then for any $\widehat{A}$-$\widehat{\otimes}$-bimodule X the canonical map*

$$\operatorname{Der}_S(\widehat{A},X)\to\operatorname{Der}_R(A,X),\quad D\mapsto D\circ\iota_A \tag{3.7}$$

is a bijection. In other words, any R-derivation from A to X uniquely extends to an S-derivation from $\widehat{A}$ to X.

Proof. The injectivity of (3.7) is obvious since $\operatorname{Im}\iota_A$ is dense in $\widehat{A}$. To establish the surjectivity, take an arbitrary R-derivation $D\colon A\to X$ and construct an R-homomorphism $\varphi\colon A\to A\times X$, $\varphi(a)=(a,D(a))$ (see (2.3)). Composing it with the R-S-homomorphism $\iota_A\times\mathbf{1}_X\colon A\times X\to\widehat{A}\times X$ we have an R-S-homomorphism $\psi\colon A\to\widehat{A}\times X$. Since $\widehat{A}\times X$ is an Arens–Michael algebra (Proposition 3.6), there exists a unique continuous homomorphism $\widehat{\psi}\colon\widehat{A}\to\widehat{A}\times X$ extending ψ. Thus for any $a\in A$ we have

$$\widehat{\psi}(\iota_A(a))=\psi(a)=(\iota_A(a),D(a)). \tag{3.8}$$

Since $\operatorname{Im}\iota_A$ is dense in $\widehat{A}$ it follows that $\widehat{\psi}(b)=(b,\widehat{D}(b))$ for some uniquely determined map $\widehat{D}\colon\widehat{A}\to X$. This map is a derivation by virtue of (2.3) and we also have $\widehat{D}\circ\iota_A=D$

because of (3.8). It remains to show that $\widehat{D}$ is an S-derivation, or, equivalently, that $\widehat{\psi}$ is an S-homomorphism. To this end, consider the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_A"] & \widehat{A} \arrow[r, "\widehat{\psi}"] & \widehat{A} \times X \\ R \arrow[u, "\eta_A"] \arrow[r, "g"] & S \arrow[u, "\eta_{\widehat{A}}"] \arrow[ur, "\eta_{\widehat{A}\times X}"'] & \end{tikzcd} \tag{3.9}$$

Its left square is commutative by assumption. The diagram obtained from it by removing $\eta_{\widehat{A}}$ is also commutative, because $\widehat{\psi}\iota_A = \psi$ is an R-S-homomorphism. Thus we are under the assumptions of Lemma 3.7 and therefore the right triangle in (3.9) is commutative, i.e., $\widehat{\psi}$ is an S-homomorphism. □

Remark 3.5. Clearly, any homomorphism between topological algebras with a dense image is an epimorphism relative to AM. Further results about epimorphisms between topological algebras will be mentioned in Section 6.

Corollary 3.9. *For any topological R-algebra A and any $\widehat{A}$-$\widehat{\otimes}$-bimodule X the canonical map* $\operatorname{Der}_{\widehat{R}}(\widehat{A}, X) \to \operatorname{Der}_R(A, X)$, $D \mapsto D\iota_A$ *is a bijection.*

Later on we will need a notion of the Arens–Michael envelope for modules. For definiteness, we only consider left modules; the cases of right modules and bimodules are dealt with similarly.

Definition 3.2. Let A be a topological algebra, $\widehat{A}$ its Arens–Michael envelope, and X a left topological A-module. A pair $(\widehat{X}, \iota_X)$ consisting of a left $\widehat{A}$-$\widehat{\otimes}$-module $\widehat{X}$ and a continuous morphism $\iota_X \colon X \to \widehat{X}$ of A-modules is called the *Arens–Michael envelope* of X if for any left $\widehat{A}$-$\widehat{\otimes}$-module Y and any continuous morphism $\alpha \colon X \to Y$ of A-modules, there is a unique continuous morphism $\widehat{\alpha} \colon \widehat{X} \to Y$ of $\widehat{A}$-$\widehat{\otimes}$-modules making the following diagram commute:

$$\begin{tikzcd} \widehat{X} \arrow[r, dashed, "\widehat{\alpha}"] & Y \\ X \arrow[u, "\iota_X"] \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{3.10}$$

We shall also say that $\widehat{\alpha}$ *extends* α.

To construct the Arens–Michael envelope of an A-module X, we define a new topology on X using the system of all continuous m-compatible prenorms. The LCS thus obtained will be denoted X_m and its completion, $\widehat{X}$. We obviously have a continuous linear map $\iota_X \colon X \to \widehat{X}$. Furthermore, let A_m denote the algebra A with topology defined by the system of all continuous submultiplicative prenorms. Recall that its completion is the Arens–Michael envelope $\widehat{A}$ of A. It easily follows from the definition of an m-compatible prenorm that X_m is a left topological A_m-module and that the action of A_m on X_m is jointly continuous. We conclude that there is a unique structure of left $\widehat{A}$-$\widehat{\otimes}$-module on $\widehat{X}$ such that the canonical map $X_m \to \widehat{X}$ is a morphism of A_m-modules. Therefore, $\iota_X \colon X \to \widehat{X}$ is a morphism of A-modules.

Proposition 3.10. *The pair $(\widehat{X}, \iota_X)$ constructed above is the Arens–Michael envelope of X.*

Proof. Suppose Y is a left $\widehat{A}$-$\widehat{\otimes}$-module and $\alpha \colon X \to Y$ is a morphism of A-modules. Then for any m-compatible prenorm $\|\cdot\|$ on Y the prenorm $x \mapsto \|\alpha(x)\|$ on X is also, as is easy to check, m-compatible. Since the topology on Y is defined by m-compatible prenorms (Proposition 3.4), we see that α is continuous as a map from X_m to Y. Therefore, there is a unique continuous linear map $\widehat{\alpha}\colon \widehat{X} \to Y$ extending α. The fact that $\widehat{\alpha}$ is a morphism of $\widehat{A}$-modules is obvious since the images of ι_A in $\widehat{A}$ and of ι_X in $\widehat{X}$ are dense. $\square$

Remark 3.6. The Arens–Michael envelope of a left topological A-module can also be obtained using the construction mentioned in the proof of Proposition 3.5.

As in the case of algebras, it is easy to see that the correspondence $X \mapsto \widehat{X}$ is a functor from the category of left topological A-modules to the category of left $\widehat{A}$-$\widehat{\otimes}$-modules, which is left adjoint to the restriction-of-scalars functor along the homomorphism ι_A. In other words, for any left topological A-module X and any left $\widehat{A}$-$\widehat{\otimes}$-module Y, we have a natural (in X and Y) isomorphism

$$ {}_A\mathbf{h}(X,Y) \cong {}_{\widehat{A}}\mathbf{h}(\widehat{X},Y). \tag{3.11} $$

Notice that the functor $X \mapsto \widehat{X}$ is additive.

From (3.11) we immediately have the following.

Proposition 3.11. *If A is a $\widehat{\otimes}$-algebra and X a left A-$\widehat{\otimes}$-module, then there is a natural isomorphism of left $\widehat{A}$-$\widehat{\otimes}$-modules*

$$ \widehat{X} \cong \widehat{A} \mathbin{\widehat{\otimes}}_A X. \tag{3.12} $$

Remark 3.7. It is trivial to check that if A is a topological algebra, then the Arens–Michael envelope $(\widehat{A}, \iota_A)$ is also the Arens–Michael envelope of A as a left A-module. Therefore, for any $n \in \mathbb{N}$, the Arens–Michael envelope of the free A-module A^n is the pair $(\widehat{A}^n, \iota_A^n)$.

The next proposition is a "module-theoretic analog" of Proposition 3.1.

Proposition 3.12. *Suppose A is a topological algebra, X is a left topological A-module, and $Y \subset X$ is a submodule. Let Z denote the closure of $\iota_X(Y)$ in $\widehat{X}$. Then Z is an $\widehat{A}$-submodule of $\widehat{X}$ and the morphism $X/Y \to \widehat{X}/Z$ induced by the morphism $\iota_X\colon X \to \widehat{X}$ extends to an isomorphism of $\widehat{A}$-$\widehat{\otimes}$-modules*

$$ \widehat{X/Y} \cong (\widehat{X}/Z)^{\sim}. $$

In particular, if $\widehat{X}$ is metrizable, then $\widehat{X/Y} \cong \widehat{X}/Z$.

Proof. That Z is an $\widehat{A}$-submodule of $\widehat{X}$ follows from the fact that $\operatorname{Im}\iota_X$ is dense in $\widehat{X}$. For any left $\widehat{A}$-$\widehat{\otimes}$-module W we have natural isomorphisms

$$ \begin{aligned} {}_A\mathbf{h}(X/Y,W) &\cong \bigl\{\varphi \in {}_A\mathbf{h}(X,W) : \varphi\big|_Y = 0\bigr\} \\ &\cong \bigl\{\psi \in {}_{\widehat{A}}\mathbf{h}(\widehat{X},W) : \psi\big|_{\iota_X(Y)} = 0\bigr\} = \bigl\{\psi \in {}_{\widehat{A}}\mathbf{h}(\widehat{X},W) : \psi\big|_Z = 0\bigr\} \\ &\cong {}_{\widehat{A}}\mathbf{h}(\widehat{X}/Z,W) \cong {}_{\widehat{A}}\mathbf{h}\bigl((\widehat{X}/Z)^{\sim},W\bigr). \end{aligned} $$

Moreover, for $W = (\widehat{X}/Z)^{\sim}$, the morphism $X/Y \to W$ mentioned in the statement of the proposition corresponds to the identity morphism $\mathbf{1}_W$. The rest is clear. $\square$

Remark 3.8. The definition and construction of the Arens–Michael envelope of left modules carries over, with obvious modifications, to the case of right modules or bimodules. In particular, in the case of bimodules, the natural isomorphism (3.12) becomes

$$ \widehat{X} \cong \widehat{A} \mathbin{\widehat{\otimes}}_A X \mathbin{\widehat{\otimes}}_A \widehat{A}. \tag{3.13} $$

4. Analytic analogs of classical ring-theoretic constructions

4.1. **Analytic Ore extensions.** We recall some standard facts about Ore extensions (see, for example, [31, 1.7]). Suppose R is an arbitrary associative algebra (without topology) and $\alpha\colon R \to R$ is an endomorphism. A linear map $\delta\colon R \to R$ is called an *α-derivation* if $\delta(ab) = \delta(a)b+\alpha(a)\delta(b)$ for any $a, b \in R$. In other words, δ is an α-derivation if and only if it is a derivation with values in the R-bimodule ${}_\alpha R$ (see Section 2). The *Ore extension* (or the *algebra of skew polynomials*) $R[t;\alpha,\delta]$ as a left R-module consists of all polynomials $\sum_{i=0}^n r_i t^i$ in t with coefficients in R. Multiplication on $R[t;\alpha,\delta]$ is uniquely determined by the relation $tr = \alpha(r)t + \delta(r)$ for $r \in R$ and the requirements that the natural embeddings $R \hookrightarrow R[t;\alpha,\delta]$ and $\mathbb{C}[t] \hookrightarrow R[t;\alpha,\delta]$ be algebra homomorphisms. Notice that the former embedding makes $R[t;\alpha,\delta]$ an R-algebra. The algebra $R[t;\alpha,\delta]$ and its element t have the following universal property, which completely characterizes them: for any R-algebra A and any $x \in A$ such that $x \cdot r = \alpha(r)\cdot x + \eta_A(\delta(r))$ $(r \in R)$, there exists a unique R-homomorphism $f\colon R[t;\alpha,\delta] \to A$ such that $f(t) = x$. When $\alpha = \mathbf{1}_R$, the algebra $R[t;\alpha,\delta]$ is denoted $R[t;\delta]$, and when $\delta = 0$ it is denoted $R[t;\alpha]$.

Next, we recall a useful formula for the multiplication on $R[t;\alpha,\delta]$. For each $k, n \in \mathbb{Z}_+$ and each $k \le n$, let $S_{n,k}\colon R \to R$ denote the operator which is the sum of all compositions of k copies of δ and $n-k$ copies of α (with a total of $\binom{n}{k}$ summands). Then for any $r \in R$ we have

$$t^n r = \sum_{k=0}^n S_{n,k}(r) t^{n-k}. \tag{4.1}$$

In particular, when $\delta = 0$ the last formula turns into $t^n r = \alpha^n(r)t^n$, and when $\alpha = \mathbf{1}_R$ it becomes

$$t^n r = \sum_{k=0}^n \binom{n}{k} \delta^k(r) t^{n-k}.$$

Later on we shall construct a locally convex analog of $R[t;\alpha,\delta]$, called an *analytic Ore extension* $\mathcal{O}(\mathbb{C}, R;\alpha,\delta)$. In Section 5 we shall show that, under some additional assumptions, the Arens–Michael envelope of an Ore extension is an analytic Ore extension.

Suppose E is a vector space and $\mathcal{T}$ is a family of linear operators in E.

Definition 4.1. A prenorm $\|\cdot\|$ on E will be called *$\mathcal{T}$-stable* if for any $T \in \mathcal{T}$ there is a $C > 0$ such that $\|Tx\| \le C\|x\|$ for all $x \in X$.

Clearly, a scalar multiple of a $\mathcal{T}$-stable prenorm and the sum of two $\mathcal{T}$-stable prenorms is $\mathcal{T}$-stable. It is also easy to see that if a subfamily $\mathcal{S} \subset \mathcal{T}$ is such that $\mathcal{T}$ is contained in the subalgebra generated by $\mathcal{S}$, then any $\mathcal{S}$-stable prenorm is also $\mathcal{T}$-stable.

Suppose now that E is a locally convex space.

Definition 4.2. A family $\mathcal{T} \subset \mathcal{L}(E)$ is said to be *localizable* if the topology on E can be defined by a family of $\mathcal{T}$-stable prenorms. An operator $T \in \mathcal{L}(E)$ is said to be *localizable*[3] if $\{T\}$ is.

If E is complete, then it is not difficult to see that $\mathcal{T}$ is localizable if and only if E is the inverse limit $\varprojlim E_\nu$ of Banach spaces E_ν $(\nu \in \Lambda)$ such that each operator $T \in \mathcal{T}$ is of the form $T = \varprojlim T_\nu$ for some compatible family of operators $\{T_\nu \in \mathcal{L}(E_\nu) : \nu \in \Lambda\}$. This was the motivation for the term "localizable family".

Notice that the notion of localizable operator is closely related to the notion of *tame* operator (see [22]). More precisely, it is easy to see that an operator T in a Fréchet

[3]Operators with this property were independently introduced by W. Zelazko [73] under the name of *m-topologizable* operators.

space E is localizable if and only if E admits a grading (in the sense of [22], i.e., an increasing defining sequence of prenorms) such that T is tame of order 0.

Proposition 4.1. *Suppose E is a locally convex space and $\mathcal{T} \subset \mathcal{L}(E)$ is a finite set of operators. Then the following conditions are equivalent:*[4]

(i) *$\mathcal{T}$ is localizable;*
(ii) *each operator $T \in \mathcal{T}$ is localizable;*
(iii) *if $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ is a directed system of prenorms defining the topology on E, then for any $T \in \mathcal{T}$ and any $\nu \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ such that $\|T^n x\|_\nu \le C^n \|x\|_\mu$ for any $x \in E$ and $n \in \mathbb{Z}_+$;*
(iv) *if $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ is a directed system of prenorms defining the topology on E, then for any $\nu \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ such that $\|T^n x\|_\nu \le C^n \|x\|_\mu$ for any $T \in \mathcal{T}$, $x \in E$, and $n \in \mathbb{Z}_+$.*

Proof. (i)$\Longrightarrow$(ii). This is obvious.

(ii)$\Longrightarrow$(iii). Fix an arbitrary $\nu \in \Lambda$. As T is localizable, it follows that there are T-stable prenorms $\|\cdot\|$ on E, an element $\mu \in \Lambda$, and a constant $C_1 \ge 1$ such that $\|\cdot\|_\nu \le \|\cdot\| \le C_1 \|\cdot\|_\mu$. Choose a $C_2 > 0$ such that $\|Tx\| \le C_2\|x\|$ for all $x \in E$. Then for any $x \in E$ and any $n \in \mathbb{Z}_+$ we have $\|T^n x\|_\nu \le \|T^n x\| \le C_2^n \|x\| \le C_1 C_2^n \|x\|_\mu$. It remains to set $C = C_1 C_2$.

(iii)$\Longrightarrow$(iv). This easily follows from the finiteness of $\mathcal{T}$.

(iv)$\Longrightarrow$(i). Let $\mathcal{T} = \{T_1, \ldots, T_k\}$. For each finite set $\alpha = (\alpha_1, \ldots, \alpha_s)$ of an arbitrary length, where $\alpha_i \in \{1, \ldots, k\}$ for each i, set $|\alpha| = s$ and $T_\alpha = T_{\alpha_1} \ldots T_{\alpha_s}$. We also assume that the identity operator corresponds to the set of zero length ($s = 0$) (cf. the notation of Example 3.2). Fix an arbitrary $\nu \in \Lambda$ and choose $C > 0$ and $\mu \in \Lambda$ satisfying (iii). Then $\|T_\alpha x\|_\nu \le C^{|\alpha|}\|x\|_\mu$ for any set α of the above form and any $x \in E$. Therefore, for the prenorm $\|\cdot\|'_\nu$ on E defined by the formula

$$\|x\|'_\nu = \sup_\alpha \frac{\|T_\alpha x\|_\nu}{C^{|\alpha|}}, \tag{4.2}$$

we have $\|\cdot\|_\nu \le \|\cdot\|'_\nu \le \|\cdot\|_\mu$. It now follows that the topology on E is defined by the system of prenorms $\{\|\cdot\|'_\nu : \nu \in \Lambda\}$. On the other hand, it easily follows from (4.2) that $\|T_i x\|'_\nu \le C\|x\|'_\nu$ for any $x \in E$ and any $i = 1, \ldots, k$. Therefore, $\mathcal{T}$ is localizable. $\square$

Remark 4.1. Recall (see, for example, [72] or [57]) that a linear operator T in an LCS E is called *regular* if for some $\lambda > 0$ the set of operators $\{(\lambda T)^n : n \in \mathbb{Z}_+\}$ is equicontinuous. If the topology of E is defined by a system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$, then T is regular if and only if there is a $C > 0$ such that for any $\nu \in \Lambda$ there is a $\mu \in \Lambda$ such that $\|T^n x\|_\nu \le C^n \|x\|_\mu$ for any $x \in E$ and any $n \in \mathbb{Z}_+$. Comparing this with condition (iii) of the preceding proposition, we see that any regular operator is localizable. The converse is however not true; for example, it is not difficult to check that the operator of multiplication by an independent variable in the space $\mathcal{O}(\mathbb{C})$ of entire functions is localizable (because of the local m-convexity of $\mathcal{O}(\mathbb{C})$) but not regular (since there is no $\lambda > 0$ such that the family of functions $\{z \mapsto (\lambda z)^n : n \in \mathbb{Z}_+\}$ is bounded in $\mathcal{O}(\mathbb{C})$).

Suppose now that R is a $\widehat{\otimes}$-algebra, $\alpha\colon R \to R$ is a (continuous) endomorphism, and $\delta\colon R \to R$ is a (continuous) derivation. Assume in addition that α and δ are localizable. Our immediate goal is to show that on the space $\mathcal{O}(\mathbb{C}, R)$ of R-valued entire functions there is a jointly continuous multiplication which coincides with multiplication on $R[z; \alpha, \delta]$ when it is restricted to polynomials. To this end, we first recall that $\mathcal{O}(\mathbb{C}, R)$ is isomorphic (as an LCS and as a left R-$\widehat{\otimes}$-module) to the projective tensor product

[4] In the case of one operator this proposition was independently proved by W. Zelazko [73].

$R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$ and that for any system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on R the topology on $\mathcal{O}(\mathbb{C}, R)$ is defined by all prenorms of the form

$$\|f\|_{\nu,\rho} = \sum_{n=0}^{\infty} \|c_n\|_\nu \rho^n \quad (\nu \in \Lambda,\ \rho > 0), \tag{4.3}$$

where $f(z) = \sum_n c_n z^n$ is the Taylor expansion of $f \in \mathcal{O}(\mathbb{C}, R)$.

Lemma 4.2. *Let R be a $\widehat{\otimes}$-algebra, $\alpha\colon R \to R$ a localizable endomorphism, and $\delta\colon R \to R$ a localizable derivation. Then there is a unique continuous linear map*

$$\tau\colon \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \to R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}),$$

such that

$$\tau(z^n \otimes r) = \sum_{k=0}^{n} S_{n,k}(r) \otimes z^{n-k} \tag{4.4}$$

for all $r \in R$ and $n \in \mathbb{Z}_+$.

Proof. The uniqueness is obvious. To prove the existence, we first define the map on the elements $z^n \otimes r$ by (4.4) and extend it by linearity to polynomials. Since polynomials form a dense subspace of $\mathcal{O}(\mathbb{C}, R)$, it remains to check the continuity of the constructed map.

By Proposition 4.1 (implication (ii)$\Longrightarrow$(i)), there is a system of $\{\alpha, \delta\}$-stable prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on R. Fix an arbitrary $\nu \in \Lambda$ and choose a constant $C > 0$ such that $\|\alpha(r)\|_\nu \le C\|r\|_\nu$ and $\|\delta(r)\|_\nu \le C\|r\|_\nu$ for any $r \in R$. Then for any $k, n \in \mathbb{Z}_+$, $k \le n$, we have

$$\|S_{n,k}(r)\|_\nu \le \binom{n}{k} C^n \|r\|_\nu. \tag{4.5}$$

For any $\rho > 0$, let $\|\cdot\|_{\rho,\nu}$ be the prenorm on $\mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R$ corresponding to the prenorm $\|\cdot\|_{\nu,\rho}$ on $R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$ under the permutation isomorphism $\mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \cong R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$. Then (4.4), (4.3), and (4.5) imply that

$$\begin{aligned}\|\tau(z^n \otimes r)\|_{\nu,\rho} &= \left\| \sum_{k=0}^{n} S_{n,k}(r) \otimes z^{n-k} \right\|_{\nu,\rho} = \sum_{k=0}^{n} \|S_{n,k}(r)\|_\nu \rho^{n-k} \\ &\le C^n \left(\sum_{k=0}^{n} \binom{n}{k} \rho^{n-k} \right) \|r\|_\mu = C^n (\rho+1)^n \|r\|_\mu = \|z^n \otimes r\|_{C(\rho+1),\mu}.\end{aligned}$$

This, together with (4.3), imply that $\|\tau(f)\|_{\nu,\rho} \le \|f\|_{C(\rho+1),\mu}$ for any polynomial $f = \sum_n z^n \otimes r_n$, and therefore τ is continuous on the subspace of polynomials, which is dense in $R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$. Extending τ by continuity to $\mathcal{O}(\mathbb{C}, R)$, we have the desired assertion. $\square$

Remark 4.2. If $\alpha = \mathbf{1}_R$, then the map τ from the preceding lemma can also be defined as follows. As δ is localizable, the formula

$$f \cdot r = \sum_{n=0}^{\infty} c_n \delta^n(r) \quad \Big(f \in \mathcal{O}(\mathbb{C}),\ f(z) = \sum_{n=0}^{\infty} c_n z^n,\ r \in R\Big) \tag{4.6}$$

defines a structure of a left $\mathcal{O}(\mathbb{C})$-$\widehat{\otimes}$-module on R. Consider the maps

$$m_R\colon \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \to R, \quad f \otimes r \mapsto f \cdot r,$$
$$\Delta\colon \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \cong \mathcal{O}(\mathbb{C} \times \mathbb{C}), \quad (\Delta f)(z, w) = f(z + w).$$

It is not difficult to check that the composition

$$\begin{aligned}(4.7)\qquad &\mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \xrightarrow{\Delta\otimes\mathbf{1}_R} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R\\ &\xrightarrow{c_{23}} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \xrightarrow{m_R\otimes\mathbf{1}_{\mathcal{O}(\mathbb{C})}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}),\end{aligned}$$

where c_{23} is the transposition of the 2-nd and 3-rd tensor factors, coincides with the map τ from the previous lemma.

Remark 4.3. Suppose now that the endomorphism α is arbitrary and $\delta = 0$. Replacing δ by α in (4.6), we have another structure of a left $\mathcal{O}(\mathbb{C})$-$\widehat{\otimes}$-module on R. Consider the map

$$\Delta' \colon \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \cong \mathcal{O}(\mathbb{C}\times\mathbb{C}), \quad (\Delta' f)(z,w) = f(zw).$$

Now to get the map τ from the preceding lemma it suffices to replace Δ by Δ' in (4.7).

We now return to the general case and set $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$. Define a multiplication operator $m_A \colon A \mathbin{\widehat{\otimes}} A \to A$ as the composition

$$\begin{aligned}(4.8)\qquad &A \mathbin{\widehat{\otimes}} A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})\\ &\xrightarrow{\mathbf{1}_R\otimes\tau\otimes\mathbf{1}_{\mathcal{O}(\mathbb{C})}} R \mathbin{\widehat{\otimes}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \xrightarrow{m_R\otimes m_{\mathcal{O}(\mathbb{C})}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) = A.\end{aligned}$$

Proposition 4.3. *The operator $m_A \colon A \mathbin{\widehat{\otimes}} A \to A$ introduced above makes A a $\widehat{\otimes}$-algebra. Moreover the tautological embedding $i \colon R[z;\alpha,\delta] \hookrightarrow A$ is a homomorphism.*

Proof. Combining (4.1), (4.4), and (4.8), we see that the diagram

$$(4.9)\qquad \begin{tikzcd} R[z;\alpha,\delta] \otimes R[z;\alpha,\delta] \arrow[r, "m"] \arrow[d, "i\otimes i"'] & R[z;\alpha,\delta] \arrow[d, "i"] \\ A \mathbin{\widehat{\otimes}} A \arrow[r, "m"'] & A \end{tikzcd}$$

is commutative. This, together with the fact that the image of i is dense in A, implies that the multiplication on A is associative and that i is a homomorphism. $\square$

Definition 4.3. The algebra $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$ with the above multiplication will be denoted $\mathcal{O}(\mathbb{C}, R;\alpha,\delta)$ and called the *analytic Ore extension* of R. When $\alpha = \mathbf{1}_R$ this algebra will be denoted $\mathcal{O}(\mathbb{C}, R;\delta)$ and when $\delta = 0$ it will be denoted $\mathcal{O}(\mathbb{C}, R;\alpha)$.

Notice that $\mathcal{O}(\mathbb{C}, R;\alpha,\delta)$ contains R as a closed subalgebra and is therefore an R-$\widehat{\otimes}$-algebra.

Remark 4.4. It follows from Remarks 4.2 and 4.3 that the algebras $\mathcal{O}(\mathbb{C}, R;\delta)$ and $\mathcal{O}(\mathbb{C}, R;\alpha)$ are particular cases of the analytic smash product of R and the $\widehat{\otimes}$-bialgebra H acting on it (see [49]), the same way as $R[t;\delta]$ and $R[t;\alpha]$ are particular cases of the algebraic smash product (see [65]). In the case of $\mathcal{O}(\mathbb{C}, R;\delta)$ (resp., $R[t;\delta]$) for H one takes the bialgebra $(\mathcal{O}(\mathbb{C}),\Delta)$ (resp., $(\mathbb{C}[t],\Delta)$), and in the case of $\mathcal{O}(\mathbb{C}, R;\alpha)$ (resp., $R[t;\alpha]$) one takes the bialgebra $(\mathcal{O}(\mathbb{C}),\Delta')$ (resp., the bialgebra $(\mathbb{C}[t],\Delta')$ isomorphic to the semigroup bialgebra $\mathbb{C}\mathbb{Z}_+$).

Remark 4.5. Notice that α and δ being localizable is not only a sufficient but also a necessary condition for the multiplication on $R[z;\alpha,\delta]$ to extend to $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$. Indeed, if $m \colon A \mathbin{\widehat{\otimes}} A \to A$ is a continuous linear operator making diagram (4.9) commute, then the map

$$\tau \colon \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \to R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}), \quad \tau(a\otimes r) = m(1\otimes a\otimes r\otimes 1)$$

must satisfy (4.4). Suppose $\{\|\cdot\|_\nu : \nu\in\Lambda\}$ is a system of prenorms defining the topology on R. Fix an arbitrary $\nu\in\Lambda$. It follows from the continuity of τ that there are $\mu\in\Lambda$

and constants $C, \rho > 0$ such that $\|\tau(u)\|_{\nu,1} \le C\|u\|_{\rho,\mu}$ for all $u \in \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R$. Therefore for any $r \in R$ and any $n \in \mathbb{Z}_+$ we have an estimate

$$\|\delta^n(r)\|_\nu \le \sum_{k=0}^n \|S_{n,k}(r)\|_\nu = \left\| \sum_{k=0}^n S_{n,k}(r) \otimes z^{n-k} \right\|_{\nu,1}$$
$$= \|\tau(z^n \otimes r)\|_{\nu,1} \le C\|z^n \otimes r\|_{\rho,\mu} = C\rho^n \|r\|_\mu.$$

Taking account of Proposition 4.1 (implication (iii)$\Longrightarrow$(ii)) we see that δ is localizable. A similar argument shows that α is localizable.

Proposition 4.4. *Suppose R is a $\widehat{\otimes}$-algebra, $\alpha\colon R \to R$ is a localizable endomorphism, and $\delta\colon R \to R$ is a localizable α-derivation. Suppose that B is an R-$\widehat{\otimes}$-algebra which is also Arens–Michael and that $x \in B$ is such that $x \cdot r = \alpha(r) \cdot x + \eta_B(\delta(r))$ for any $r \in R$. Then there exists a unique continuous R-homomorphism $f\colon \mathcal{O}(\mathbb{C}, R; \alpha, \delta) \to B$ such that $f(z) = x$.*

Proof. For an arbitrary element $a = \sum_n r_n z^n \in \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ we set $f(a) = \sum_n r_n \cdot x^n$. Since B is an Arens–Michael algebra, it is easy to see that the latter series converges absolutely in B and that $f\colon \mathcal{O}(\mathbb{C}, R; \alpha, \delta) \to B$ is a continuous linear map. The fact that f is an R-homomorphism (and that f is a unique continuous R-homomorphism with the required properties) easily follows from the universal property of the algebra $R[z; \alpha, \delta]$ (see above) and the density of $R[z; \alpha, \delta]$ in $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$. ☐

Next, we want to determine under what conditions the algebra $\mathcal{O}(\mathbb{C}, R; \delta)$ is Arens–Michael.

Definition 4.4. Suppose R is an Arens–Michael algebra. We shall say that a family $\mathcal{T} \subset \mathcal{L}(R)$ is *m-localizable* if the topology on R can be defined by a family of $\mathcal{T}$-stable submultiplicative prenorms. We shall say that an operator $T \in \mathcal{L}(R)$ is *m-localizable* if the family $\{T\}$ is m-localizable.

Example 4.1. It is easy to see that for any r in the Arens–Michael algebra R the multiplication operators $L_r\colon s \mapsto rs$ and $R_r\colon s \mapsto sr$ are m-localizable. Of course, the same is true for their compositions and linear combinations and, in particular, for inner derivations and inner automorphisms of R.

Proposition 4.5. *Let R be an Arens–Michael algebra, $\alpha\colon R \to R$ an endomorphism, and $\delta\colon R \to R$ an α-derivation. Suppose that the family $\{\alpha, \delta\}$ is m-localizable. Then $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ is an Arens–Michael algebra.*

Proof. Suppose $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ is a system of submultiplicative $\{\alpha, \delta\}$-stable prenorms defining the topology on R. Fix an arbitrary $\nu \in \Lambda$ and choose a constant $C > 0$ such that $\|\alpha(r)\|_\nu \le C\|r\|_\nu$ and $\|\delta(r)\|_\nu \le C\|r\|_\nu$ for any $r \in R$. Then for any $k, n \in \mathbb{Z}_+$ and $k \le n$, inequality (4.5) holds.

We shall show that for any $a, b \in A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ and any $\rho > 0$ we have

$$\|ab\|_{\nu,\rho} \le \|a\|_{\nu, C(\rho+1)} \|b\|_{\nu,\rho}. \tag{4.10}$$

If $a = s \otimes z^n$ and $b = r \otimes z^m$, then

$$\|s \otimes z^n \cdot r \otimes z^m\|_{\nu,\rho} = \left\| \sum_{k=0}^n s S_{n,k}(r) \otimes z^{n+m-k} \right\|_{\nu,\rho} = \sum_{k=0}^n \|s S_{n,k}(r)\|_\nu \rho^{n+m-k}$$
$$\le \|s\|_\nu C^n \left(\sum_{k=0}^n \binom{n}{k} \rho^{n-k} \right) \|r\|_\nu \rho^m = \|s\|_\nu C^n (\rho+1)^n \|r\|_\nu \rho^m$$
$$= \|s \otimes z^n\|_{\nu, C(\rho+1)} \|r \otimes z^m\|_{\nu,\rho}.$$

Thus, (4.10) holds for monomials. If a and b are polynomials, then they can be written as finite sums $a = \sum_i a_i$ and $b = \sum_j b_j$, where a_i and b_j are monomials; moreover, $\|a\|_{\nu,\rho} = \sum_i \|a_i\|_{\nu,\rho}$ and $\|b\|_{\nu,\rho} = \sum_j \|b_j\|_{\nu,\rho}$ for any $\nu \in \Lambda$ and any $\rho > 0$. This implies that

$$\begin{aligned} \|ab\|_{\nu,\rho} &= \Big\|\sum_{i,j} a_i b_j\Big\|_{\nu,\rho} \le \sum_{i,j} \|a_i\|_{\nu, C(\rho+1)} \|b_j\|_{\nu,\rho} \\ &= \Big(\sum_i \|a_i\|_{\nu, C(\rho+1)}\Big)\Big(\sum_j \|b_j\|_{\nu,\rho}\Big) = \|a\|_{\nu, C(\rho+1)} \|b\|_{\nu,\rho}. \end{aligned}$$

Therefore, (4.10) holds for all polynomials and therefore for arbitrary $a, b \in A$. Using Corollary 2.2, we conclude that A is Arens–Michael. $\square$

Remark 4.6. Suppose R, α, δ are the same as in the preceding proposition. Then the property of $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ established in Proposition 4.4 is obviously universal and completely characterizes the pair $(A,\ z \in A)$. In other words, if R-AM denotes the category of R-$\widehat{\otimes}$-algebras which are Arens–Michael, then we have a bijection

$$\mathrm{Hom}_{R\text{-}\mathsf{AM}}(\mathcal{O}(\mathbb{C}, R; \alpha, \delta), B) \cong \big\{x \in B : \ x \cdot r = \alpha(r) \cdot x + \eta_B(\delta(r)) \ \forall r \in R\big\},$$

giving rise to an isomorphism of functors of B.

Remark 4.7. The author does not know if an analog of Proposition 4.1 holds for m-localizable families and if there are localizable families of the form $\{\alpha, \delta\}$ which are not m-localizable.

We remark that the prenorms (4.3) defining the topology on $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ are not necessarily submultiplicative, even in the case when $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ is Arens–Michael (cf. (4.10)). Thus it is convenient to single out a special case when the norms are submultiplicative.

Definition 4.5. Suppose R is an Arens–Michael algebra. An operator $T \in \mathcal{L}(R)$ is said to be *m-contracting* (resp., *m-isometric*) if the topology on R can be defined by a family $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ of submultiplicative prenorms such that $\|Tr\|_\nu \le \|r\|_\nu$ (resp., $\|Tr\|_\nu = \|r\|_\nu$) for all $r \in R$.

Proposition 4.6. *Suppose R is an Arens–Michael algebra and $\alpha\colon R \to R$ is an m-contracting endomorphism. Fix a system of submultiplicative prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on R and such that $\|\alpha(r)\|_\nu \le \|r\|_\nu$ for all $r \in R$ and $\nu \in \Lambda$. Then each of the prenorms* (4.3) *on $\mathcal{O}(\mathbb{C}, R; \alpha)$ is submultiplicative.*

Proof. For any $s, r \in R$ we have

$$\begin{aligned} \|s \otimes z^n \cdot r \otimes z^m\|_{\nu,\rho} &= \|s\alpha^n(r) \otimes z^{n+m}\|_{\nu,\rho} \\ &\le \|s\|_\nu \|r\|_\nu \rho^{n+m} = \|s \otimes z^n\|_{\nu,\rho} \|r \otimes z^m\|_{\nu,\rho}. \end{aligned}$$

This implies, as in the proof of Proposition 4.5, that $\|ab\|_{\nu,\rho} \le \|a\|_{\nu,\rho}\|b\|_{\nu,\rho}$ for any $a, b \in \mathcal{O}(\mathbb{C}, R; \alpha)$. $\square$

Remark 4.8. If $\alpha = \mathbf{1}_R$, then it is not difficult to show that the conditions of Proposition 4.5 are not only sufficient but also necessary for $\mathcal{O}(\mathbb{C}, R; \delta)$ to be Arens–Michael. Indeed, if $\mathcal{O}(\mathbb{C}, R; \delta)$ is Arens–Michael, then so is its closed subalgebra R. Furthermore, the derivation δ coincides with the restriction to R of the inner derivation ad_z and therefore is m-localizable (see Example 4.1).

Similarly, if $\delta = 0$, α is an automorphism, and $\mathcal{O}(\mathbb{C}, R; \alpha)$ is Arens–Michael, then α, being the restriction to R of the inner automorphism Ad_z, is m-localizable.

Finally, if $\delta = 0$, $\|1\|_\nu = 1$ for all $\nu \in \Lambda$, and each of the prenorms (4.3) on $\mathcal{O}(\mathbb{C}, R; \alpha)$ is submultiplicative, then α is necessarily m-contracting. This follows at once from the chain of inequalities

$$\|\alpha(r)\|_\nu = \|\alpha(r)z\|_{\nu,1} = \|zr\|_{\nu,1} \le \|z\|_{\nu,1}\|r\|_{\nu,1} = \|r\|_\nu.$$

4.2. **Analytic tensor algebras.** Suppose R is an arbitrary associative algebra (without topology) and M is an R-bimodule. For any $n \in \mathbb{N}$ set

$$M^{\otimes n} = \underbrace{M \otimes_R \cdots \otimes_R M}_{n}. \tag{4.11}$$

We also set $M^{\otimes 0} = R$. Recall that the *tensor algebra* of M is defined as the R-bimodule

$$T_R(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n}$$

together with the multiplication uniquely determined by the rule

$$ab = a \otimes b \quad \text{for} \quad a \in M^{\otimes m},\ b \in M^{\otimes n}$$

(here we used the canonical isomorphisms $M^{\otimes m} \otimes_R M^{\otimes n} \cong M^{\otimes (m+n)}$ that hold for any $m, n \in \mathbb{Z}_+$).

For any $n \in \mathbb{Z}_+$ let $i_n \colon M^{\otimes n} \hookrightarrow T_R(M)$ denote the canonical embedding. Clearly, i_n is a morphism of R-bimodules and also i_0 is an algebra homomorphism making $T_R(M)$ an R-algebra.

The tensor algebra can also be defined as a pair $(T_R(M), i_1)$ consisting of an R-algebra $T_R(M)$ and an R-bimodule morphism $i_1 \colon M \to T_R(M)$ such that for any R-algebra A and any R-bimodule morphism $\alpha \colon M \to A$ there exists a unique R-algebra homomorphism $\psi \colon T_R(M) \to A$ making the following diagram commute:

$$\begin{tikzcd} T_R(M) \arrow[r, dashed, "\psi"] & A \\ M \arrow[u, "i_1"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Example 4.2. Suppose R is an algebra and $\alpha \colon R \to R$ is an endomorphism. Then the Ore extension $R[t; \alpha]$ is canonically isomorphic to the tensor algebra $T_R(M)$, where $M = R_\alpha$ (see Section 2); under this isomorphism $1 \in M$ corresponds to $t \in R[t; \alpha]$.

Later we shall describe an "analytic analog" of the tensor algebra, the *analytic tensor algebra* $\widehat{T}_R(M)$, which is a generalization of Taylor's algebra of power series in several free variables (see Example 3.2) and also of Cuntz's "smooth tensor algebra" [8, 9]. We shall then show that the Arens–Michael envelope of the tensor algebra is an analytic tensor algebra.

Suppose R is a $\widehat{\otimes}$-algebra and M is an R-$\widehat{\otimes}$-bimodule. Replacing the algebraic tensor product $\otimes_R$ in (4.11) by the projective tensor product $\widehat{\otimes}_R$ (see Section 1) we have a family of R-$\widehat{\otimes}$-bimodules $M^{\widehat{\otimes} n}$ ($n \in \mathbb{Z}_+$). Suppose $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ is a directed system of prenorms defining the topology on M. For any $\nu \in \Lambda$ and any $n \in \mathbb{N}$, let $\|\cdot\|_\nu^{\otimes n}$ denote the prenorm on $M^{\widehat{\otimes} n}$ which is the projective tensor product of n copies of $\|\cdot\|_\nu$. Consider the space $\widehat{T}_R(M)^+$ consisting of all $x = (x_n) \in \prod_{n=1}^\infty M^{\widehat{\otimes} n}$ such that

$$\|x\|_{\nu,\rho} = \sum_{n=1}^{\infty} \|x_n\|_\nu^{\otimes n} \rho^n < \infty \ \forall \rho > 0.$$

The system of prenorms $\{\|\cdot\|_{\nu,\rho} : \nu \in \Lambda,\ \rho > 0\}$ makes $\widehat{T}_R(M)^+$ an LCS which is, as is easily seen, complete. It is also easy to show that the topology on $\widehat{T}_R(M)^+$ does not depend on the defining system $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ of prenorms on M.

We now set

$$\widehat{T}_R(M) = R \oplus \widehat{T}_R(M)^+.$$

The elements of $\widehat{T}_R(M)$ which belong to $M^{\widehat{\otimes} n}$ for some $n \in \mathbb{Z}_+$ will be called *homogeneous*. We shall use a similar convention for $T_R(M)$.

Consider the canonical map $i\colon T_R(M) \to \widehat{T}_R(M)$ sending each elementary tensor $x_1 \otimes \cdots \otimes x_n \in M^{\otimes n}$ to $x_1 \otimes \cdots \otimes x_n \in M^{\widehat{\otimes} n}$. It is easy to see that its image is dense in $\widehat{T}_R(M)$.

Proposition 4.7. *There is a unique multiplication on $\widehat{T}_R(M)$ making it a $\widehat{\otimes}$-algebra and such that i is a homomorphism. On the homogeneous elements this multiplication coincides with the canonical map*

$$M^{\widehat{\otimes} m} \times M^{\widehat{\otimes} n} \to M^{\widehat{\otimes}(m+n)}, \quad (a, b) \mapsto a \otimes b. \tag{4.12}$$

Moreover $\widehat{T}_R(M)^+$ is an ideal of $\widehat{T}_R(M)$ and a (nonunital) Arens–Michael algebra.

If in addition R is an Arens–Michael algebra, then so is $\widehat{T}_R(M)$.

Proof. Let $\widehat{T}_R(M)_{\mathrm{fin}}$ denote the subspace of $\widehat{T}_R(M)$ consisting of compactly supported sequences. Clearly, there exists a unique bilinear operator

$$\widehat{T}_R(M)_{\mathrm{fin}} \times \widehat{T}_R(M)_{\mathrm{fin}} \to \widehat{T}_R(M)_{\mathrm{fin}}, \quad (x, y) \mapsto xy\,, \tag{4.13}$$

coinciding on homogeneous elements with canonical map (4.12). Let $\bar{x} = (x_n)_{n\geq 1}$ and $\bar{y} = (y_n)_{n \geq 1}$ be elements of $\widehat{T}_R(M)_{\mathrm{fin}} \cap \widehat{T}_R(M)^+$. Then for any $\nu \in \Lambda$ and any $\rho > 0$ we have:

$$\begin{aligned} \|\bar{x}\bar{y}\|_{\nu,\rho} &= \sum_n \Big\| \sum_{i+j=n} x_i y_j \Big\|_\nu^{\otimes n} \rho^n \\ &\leq \sum_n \sum_{i+j=n} \|x_i y_j\|_\nu^{\otimes n} \rho^n \leq \sum_n \sum_{i+j=n} \|x_i\|_\nu^{\otimes i} \|y_j\|_\nu^{\otimes j} \rho^n \\ &= \Big(\sum_i \|x_i\|_\nu^{\otimes i} \rho^i\Big)\Big(\sum_j \|y_j\|_\nu^{\otimes j} \rho^j\Big) = \|\bar{x}\|_{\nu,\rho} \|\bar{y}\|_{\nu,\rho}. \end{aligned} \tag{4.14}$$

Furthermore, as the action of R on M is jointly continuous, we have that for any $\nu \in \Lambda$ there are $\mu \in \Lambda$ and a continuous prenorm $\|\cdot\|$ on R such that

$$\|a \cdot x\|_\nu \leq \|a\| \|x\|_\mu \quad \text{and} \quad \|x \cdot a\|_\nu \leq \|x\|_\mu \|a\| \quad (a \in R,\ x \in M).$$

We may assume that $\|\cdot\|_\mu$ majorizes $\|\cdot\|_\nu$. This, together with the properties of the projective tensor product, imply that

$$\|a \cdot x\|_\nu^{\otimes n} \leq \|a\| \|x\|_\mu^{\otimes n} \quad \text{and} \quad \|x \cdot a\|_\nu^{\otimes n} \leq \|x\|_\mu^{\otimes n} \|a\|$$
$$(a \in R,\ x \in M^{\widehat{\otimes} n},\ n \in \mathbb{N}),$$

whence

$$\|a \cdot x\|_{\nu,\rho} \leq \|a\| \|x\|_{\mu,\rho} \text{ and } \|x \cdot a\|_{\nu,\rho} \leq \|x\|_{\mu,\rho} \|a\| \quad (a \in R,\ x \in \widehat{T}_R(M)^+). \tag{4.15}$$

As the multiplication on R is jointly continuous, (4.14) and (4.15) show that the bilinear operator (4.13) is also jointly continuous. Therefore, it extends to a jointly continuous bilinear operator $m_{\widehat{T}}\colon \widehat{T}_R(M)\times\widehat{T}_R(M)\to\widehat{T}_R(M)$. Let m_T denote the multiplication on $T_R(M)$; then it follows easily from the construction that the diagram

$$\begin{tikzcd} T_R(M)\times T_R(M) \arrow[r, "m_T"] \arrow[d, "i\times i"'] & T_R(M) \arrow[d, "i"] \\ \widehat{T}_R(M)\times\widehat{T}_R(M) \arrow[r, "m_{\widehat{T}}"] & \widehat{T}_R(M) \end{tikzcd}$$

is commutative. Since the image of i is dense in $\widehat{T}_R(M)$, we see that $m_{\widehat{T}}$ is associative (hence $\widehat{T}_R(M)$ is a $\widehat{\otimes}$-algebra), and that i is a homomorphism. Thus we have a multiplication on $\widehat{T}_R(M)$ with the required properties. Its uniqueness is obvious because the image of i is dense. That $\widehat{T}_R(M)^+$ is a closed ideal of $\widehat{T}_R(M)$ is clear from the construction. Finally, it follows from (4.14) that $\widehat{T}_R(M)^+$ is Arens–Michael. This proves the first part of the proposition.

Assume now that R is an Arens–Michael algebra, and choose a directed system of submultiplicative prenorms $\{\|\cdot\|_\lambda : \lambda\in\Lambda_1\}$ defining the topology on R. By Proposition 3.4 (see also Remark 3.4), without loss of generality we may assume that for any $\nu\in\Lambda$ the prenorm $\|\cdot\|_\nu$ on M is m-compatible. As in the proof of Proposition 3.6, with each $\nu\in\Lambda$ we associate $\varphi(\nu)\in\Lambda_1$ such that

$$\|a\cdot x\|_\nu \le \|a\|_{\varphi(\nu)}\|x\|_\nu \quad\text{and}\quad \|x\cdot a\|_\nu\le\|x\|_\nu\|a\|_{\varphi(\nu)} \quad (a\in R,\ x\in M),$$

and set $\Lambda'=\{(\lambda,\nu)\in\Lambda_1\times\Lambda : \varphi(\nu)\prec\lambda\}$. As in (4.15), it follows from the last formula that

$$\|a\cdot x\|_{\nu,\rho}\le\|a\|_\lambda\|x\|_{\nu,\rho} \text{ and } \|x\cdot a\|_{\nu,\rho}\le\|x\|_{\nu,\rho}\|a\|_\lambda \quad (a\in R,\ x\in\widehat{T}_R(M)^+) \tag{4.16}$$

for any $(\lambda,\nu)\in\Lambda'$ and any $\rho>0$. Now we write an arbitrary element $x=(x_n)\in\widehat{T}_R(M)$ in the form $x=(x_0,\bar{x})$, where $\bar{x}=(x_n)_{n\ge1}\in\widehat{T}_R(M)^+$, and for any $(\lambda,\nu)\in\Lambda'$ and any $\rho>0$ define a prenorm $\|\cdot\|_{\lambda,\nu,\rho}$ on $\widehat{T}_R(M)$ by the formula

$$\|x\|_{\lambda,\nu,\rho}=\|x_0\|_\lambda+\|\bar{x}\|_{\nu,\rho}. \tag{4.17}$$

As in the proof of Proposition 3.6, we have that the topology on $\widehat{T}_R(M)$ is defined by the system of prenorms of the form (4.17). On the other hand, it follows from (4.14) and (4.16) that

$$\begin{aligned}\|xy\|_{\lambda,\nu,\rho}&=\|x_0y_0\|_\lambda+\|x_0\bar{y}+\bar{x}y_0+\bar{x}\bar{y}\|_{\nu,\rho}\\ &\le\|x_0\|_\lambda\|y_0\|_\lambda+\|x_0\|_\lambda\|\bar{y}\|_{\nu,\rho}+\|\bar{x}\|_{\nu,\rho}\|y_0\|_\lambda+\|\bar{x}\|_{\nu,\rho}\|\bar{y}\|_{\nu,\rho}\\ &=(\|x_0\|_\lambda+\|\bar{x}\|_{\nu,\rho})(\|y_0\|_\lambda+\|\bar{y}\|_{\nu,\rho})=\|x\|_{\lambda,\nu,\rho}\|y\|_{\lambda,\nu,\rho}\end{aligned}$$

for any $x,y\in\widehat{T}_R(M)$, any $(\lambda,\nu)\in\Lambda'$, and any $\rho>0$. Therefore, all prenorms $\{\|\cdot\|_{\lambda,\nu,\rho} : (\lambda,\nu)\in\Lambda',\ \rho>0\}$ are submultiplicative and $\widehat{T}_R(M)$ is Arens–Michael. □

Definition 4.6. The $\widehat{\otimes}$-algebra $\widehat{T}_R(M)$ thus constructed will be called the *analytic tensor algebra* of M.

Remark 4.9. When $A=\mathbb{C}$ the algebra $\widehat{T}_{\mathbb{C}}(M)$ coincides with Cuntz's "smooth tensor algebra" [8]. If in addition $n=\dim M<\infty$, we recover Taylor's algebra $\mathcal{F}_n$ of power series in n free variables (see Example 3.2).

Now suppose, as before, that R is a $\widehat{\otimes}$-algebra and M is an R-$\widehat{\otimes}$-bimodule. As in the case of the usual tensor algebra $T_R(M)$, we have the canonical morphisms $i_n\colon M^{\widehat{\otimes}n}\hookrightarrow\widehat{T}_R(M)$ $(n\in\mathbb{Z}_+)$ of R-$\widehat{\otimes}$-bimodules; moreover $i_0\colon R\to\widehat{T}_R(M)$ is an algebra homomorphism making $\widehat{T}_R(M)$ an R-$\widehat{\otimes}$-algebra.

Proposition 4.8. *Suppose R is a $\widehat{\otimes}$-algebra and M is an R-$\widehat{\otimes}$-bimodule. Then for any R-$\widehat{\otimes}$-algebra A which is Arens–Michael, and any morphism of R-$\widehat{\otimes}$-bimodules $\alpha\colon M \to A$ there exists a unique continuous R-homomorphism $\psi\colon \widehat{T}_R(M) \to A$ making the following diagram commute:*

$$
\begin{tikzcd}
\widehat{T}_R(M) \arrow[r, dashed, "\psi"] & A \\
M \arrow[u, "i_1"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
\tag{4.18}
$$

Proof. We fix a directed system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on M and also choose a system of submultiplicative prenorms $\{\|\cdot\|_\mu : \mu \in \Lambda_2\}$ defining the topology on A. For any $n \in \mathbb{N}$ consider the morphism

$$m_A^n\colon A^{\widehat{\otimes} n} \to A, \quad a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n.$$

As the prenorms $\|\cdot\|_\mu$ are submultiplicative, we have

$$\|m_A^n(u)\|_\mu \le \|u\|_\mu^{\otimes n} \quad \text{for all} \quad u \in A^{\widehat{\otimes} n},\ n \in \mathbb{N},\ \mu \in \Lambda_2. \tag{4.19}$$

Define a map $\psi\colon \widehat{T}_R(M) \to A$ by the formula

$$\psi(x) = \eta_A(x_0) + \sum_{n=1}^{\infty} (m_A^n \circ \alpha^{\otimes n})(x_n).$$

To show that ψ is well-defined and continuous, we fix an arbitrary $\mu \in \Lambda_2$ and, using the continuity of α, choose $\nu \in \Lambda$ and $C > 0$ such that $\|\alpha(x)\|_\mu \le C\|x\|_\nu$ for all $x \in M$. Then, obviously,

$$\|\alpha^{\otimes n}(x_n)\|_\mu^{\otimes n} \le C^n \|x_n\|_\nu^{\otimes n} \quad \text{for all} \quad x_n \in M^{\widehat{\otimes} n},\ n \in \mathbb{N}. \tag{4.20}$$

Using (4.19) and (4.20), for each $x \in \widehat{T}_R(M)$ we have

$$\begin{aligned}
\|\psi(x)\|_\mu &\le \|\eta_A(x_0)\|_\mu + \sum_{n=1}^{\infty} \|(m_A^n \circ \alpha^{\otimes n})(x_n)\|_\mu \\
&\le \|\eta_A(x_0)\|_\mu + \sum_{n=1}^{\infty} C^n \|x_n\|_\nu^{\otimes n} = \|\eta_A(x_0)\|_\mu + \|\bar{x}\|_{\nu, C}.
\end{aligned}$$

Since η_A is continuous, the latter estimate implies that ψ is well-defined and continuous. The commutativity of diagram (4.18) follows directly from the construction. A simple calculation on elementary tensors (which we omit) shows that ψ is a homomorphism of R-algebras. Finally, the uniqueness of ψ with the required properties follows easily from the fact that the images of i_0 and i_1 generate a dense subalgebra of $\widehat{T}_R(M)$. ☐

Suppose R is a $\widehat{\otimes}$-algebra and α is a (continuous) endomorphism of it. We shall show that the analytic tensor algebra $\widehat{T}_R(R_\alpha)$ can be realized as some algebra of "skew power series". For this we fix a directed system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on R, and for each $\nu \in \Lambda$ and each $n \in \mathbb{N}$ define a prenorm $\|\cdot\|_\nu^{(n)}$ on R by the formula

$$\|r\|_\nu^{(n)} = \inf \sum_{i=1}^{k} \|r_{0,i}\|_\nu \cdots \|r_{n-1,i}\|_\nu, \tag{4.21}$$

where inf is taken over all representations of r in the form

$$r=\sum_{i=1}^{k} r_{0,i}\,\alpha(r_{1,i})\alpha^2(r_{2,i})\cdots\alpha^{n-1}(r_{n-1,i}) \quad (r_{j,i}\in R,\ k\in\mathbb{N}). \tag{4.22}$$

We also set $\|\cdot\|_\nu^{(0)}=\|\cdot\|_\nu$. It is not difficult to see that for a fixed n the system of prenorms $\{\|\cdot\|_\nu^{(n)} : \nu\in\Lambda\}$ on R is equivalent to the original one. Consider the space $R\{t;\alpha\}$ consisting of all formal expressions $a=\sum_{n=0}^\infty r_n t^n$ such that

$$\|a\|_{\nu,\rho}=\sum_{n=0}^{\infty}\|r_n\|_\nu^{(n)}\rho^n<\infty \quad \forall\rho>0.$$

The system of prenorms $\{\|\cdot\|_{\nu,\rho} : \nu\in\Lambda,\ \rho>0\}$ makes $R\{t;\alpha\}$ an LCS, which, as is easy to see, is complete.

Proposition 4.9. *There is a unique structure of a $\widehat{\otimes}$-algebra on $R\{t;\alpha\}$ such that the embedding $R[t;\alpha]\hookrightarrow R\{t;\alpha\}$ is a homomorphism. Moreover, there exists a unique R-$\widehat{\otimes}$-algebra isomorphism $R\{t;\alpha\}\to\widehat{T}_R(R_\alpha)$ sending t to $1\in R_\alpha$. Consequently, if R is an Arens–Michael algebra, then so is $R\{t;\alpha\}$.*

Proof. It is easy to see that for any $n\in\mathbb{N}$ we have an isomorphism of R-$\widehat{\otimes}$-bimodules

$$\varphi_n\colon (R_\alpha)^{\widehat{\otimes}n}\to R_{\alpha^n},\quad r_0\otimes\cdots\otimes r_{n-1}\mapsto r_0\,\alpha(r_1)\cdots\alpha^{n-1}(r_{n-1}).$$

Moreover,

$$\|\varphi_n(u)\|_\nu^{(n)}=\|u\|_\nu^{\otimes n}$$

for any $\nu\in\Lambda$ and any $u\in(R_\alpha)^{\widehat{\otimes}n}$. Therefore, there is an isomorphism of LCS,

$$R\{t;\alpha\}\to\widehat{T}_R(R_\alpha),\quad \sum_{n=0}^{\infty}r_nt^n\mapsto x=(x_n)_{n=0}^\infty,$$
$$\text{where } x_n=r_n\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-1}\in(R_\alpha)^{\widehat{\otimes}n}.$$

It remains to carry the structure of a $\widehat{\otimes}$-algebra from $\widehat{T}_R(R_\alpha)$ to $R\{t;\alpha\}$. Under this construction, if $R[t;\alpha]$ is identified with $T_R(R_\alpha)$ (see Example 4.2), then the homomorphism $i\colon T_R(R_\alpha)\to\widehat{T}_R(R_\alpha)$ corresponds to $R[t;\alpha]\hookrightarrow R\{t;\alpha\}$ (see Proposition 4.7). Therefore, the above embedding is an algebra homomorphism. The assertion about uniqueness obviously follows from the density of $R[t;\alpha]$ in $R\{t;\alpha\}$. Finally, if R is an Arens–Michael algebra, then so is $\widehat{T}_R(R_\alpha)$ (see Proposition 4.7) and therefore $R\{t;\alpha\}$. □

Combining the preceding proposition with Proposition 4.8, we have the following property of $R\{t;\alpha\}$.

Corollary 4.10. *Suppose R is a $\widehat{\otimes}$-algebra and $\alpha\colon R\to R$ is an endomorphism. Then for any R-$\widehat{\otimes}$-algebra A which is Arens–Michael and any $x\in A$ such that $x\cdot r=\alpha(r)\cdot x$ for all $r\in R$ there is a unique continuous R-homomorphism $\psi\colon R\{t;\alpha\}\to A$ such that $\psi(t)=x$.*

Remark 4.10. If R is an Arens–Michael algebra and M is an R-$\widehat{\otimes}$-bimodule, then the property of $\widehat{T}_R(M)$ established in Proposition 4.8 is obviously universal and completely characterizes the pair $(\widehat{T}_R(M),i_1)$. In other words, there is a bijection

$$\mathrm{Hom}_{R\text{-}\mathsf{AM}}(\widehat{T}_R(M),A)\cong {}_R\mathbf{h}_R(M,A),$$

giving rise to an isomorphism of functors of A. In particular, if α is an endomorphism of R, then

$$\operatorname{Hom}_{R\text{-}\mathsf{AM}}(R\{t;\alpha\}, A) \cong \bigl\{x \in A : x \cdot r = \alpha(r) \cdot x \ \forall r \in R\bigr\}$$

(cf. Remark 4.6).

Proposition 4.11. *Assume, in addition to the conditions of Corollary* 4.10 *that R is an Arens–Michael algebra and α is m-localizable. Then there exists a unique R-isomorphism*

$$R\{t;\alpha\} \to \mathcal{O}(\mathbb{C}, R; \alpha), \quad t \mapsto (z \mapsto z \cdot 1).$$

Proof. This is a consequence of the universal properties of the above algebras (see Remarks 4.6 and 4.10). $\square$

Example 4.3. We shall now give an example of an endomorphism α of an Arens–Michael algebra R such that the underlying LCS of $R\{t;\alpha\}$ coincides with the space of formal power series $R[[t]]$. Of course, in view of the preceding proposition, such an endomorphism cannot be m-localizable.

Set $R = \mathcal{O}(\mathbb{C})$ and define $\alpha\colon R \to R$ by setting $\alpha(f)(z) = f(z-1)$ $(z \in \mathbb{C})$. For the defining system of prenorms on R take $\{\|\cdot\|_\rho : \rho > 0\}$, where $\|f\|_\rho = \max_{|z|\le\rho} |f(z)|$. We shall show that for any $\rho > 0$ there is an $N \in \mathbb{N}$ such that $\|\cdot\|_\rho^{(n)} \equiv 0$ for all $n > N$. Henceforth, for $z \in \mathbb{C}$ and $r > 0$, the open disc of radius r and center z will be denoted $U_r(z)$.

Fix an arbitrary $\rho > 0$ and choose an $r > \rho$ such that $2r < [2\rho]+1$. Then for any integer $n \ge [2\rho] + 1$ we have $U_r(-n) \cap U_r(0) = \varnothing$. Set $U = U_r(-n) \cup U_r(0)$, take an arbitrary $f \in \mathcal{O}(\mathbb{C})$ and $\varepsilon > 0$, and define a function $h \in \mathcal{O}(U)$ by setting $h(z) = 0$ for $z \in U_r(0)$ and $h(z) = f(z+n)$ for $z \in U_r(-n)$. By Runge's theorem (see, for example, [42, Ch. 4, Section 2]), the restriction homomorphism $\mathcal{O}(\mathbb{C}) \to \mathcal{O}(U)$ has a dense image. Hence there is a $g \in \mathcal{O}(\mathbb{C})$ such that $|g(z) - h(z)| < \varepsilon/2$ for any $z \in \overline{U_\rho(-n)} \cup \overline{U_\rho(0)}$. It follows from the equality $f = \alpha^n(g) + (f - \alpha^n(g))$ and the definition of the prenorms $\|\cdot\|_\rho^{(n)}$ (see (4.21) and (4.22)) that

$$\|f\|_\rho^{(n+1)} \le \|g\|_\rho + \|f - \alpha^n(g)\|_\rho. \tag{4.23}$$

By the choice of h and g, we have $\|g\|_\rho < \varepsilon/2$. Furthermore,

$$\begin{aligned}
\|f - \alpha^n(g)\|_\rho &= \max\bigl\{|f(z) - g(z-n)| : z \in \overline{U_\rho(0)}\bigr\} \\
&= \max\bigl\{|f(z+n) - g(z)| : z \in \overline{U_\rho(-n)}\bigr\} \\
&= \max\bigl\{|h(z) - g(z)| : z \in \overline{U_\rho(-n)}\bigr\} < \frac{\varepsilon}{2}.
\end{aligned}$$

Comparing this with (4.23), we have an estimate $\|f\|_\rho^{(n+1)} < \varepsilon$. As $\varepsilon > 0$ and $f \in \mathcal{O}(\mathbb{C})$ were arbitrary, this means that $\|\cdot\|_\rho^{(n+1)} \equiv 0$ for all $n \ge [2\rho] + 1$. Therefore it follows from the definition of $R\{t;\alpha\}$ that, as a locally convex space, it coincides with the space of formal power series $R[[t]]$.

4.3. **Analytic skew Laurent extensions.** Suppose R is an arbitrary associative algebra (without topology) and $\alpha\colon R \to R$ is an automorphism. Recall that the *skew Laurent extension* $R[t, t^{-1}; \alpha]$, as a left R-module, consists of all Laurent polynomials $\sum_{i=-m}^n r_i t^i$ in t with coefficients in R; the multiplication on $R[t, t^{-1}; \alpha]$ is uniquely determined by the relation $tr = \alpha(r)t$ for $r \in R$ and the requirement that the natural embeddings $R \hookrightarrow R[t, t^{-1}; \alpha]$ and $\mathbb{C}[t, t^{-1}] \hookrightarrow R[t, t^{-1}; \alpha]$ be algebra homomorphisms. Notice that the former embedding makes $R[t, t^{-1}; \alpha]$ an R-algebra. The algebra $R[t, t^{-1}; \alpha]$ and its element t have the following universal property, which completely characterizes them: For any R-algebra A and any invertible element $x \in A$ such that $x \cdot r = \alpha(r) \cdot x$ $(r \in R)$, there exists a unique R-homomorphism $f\colon R[t, t^{-1}; \alpha] \to A$ such that $f(t) = x$. Clearly, $R[t, t^{-1}; \alpha]$ contains $R[t; \alpha]$ as a subalgebra.

Now we shall construct a locally convex analog of the algebra $R[t, t^{-1}; \alpha]$. Suppose R is a $\widehat{\otimes}$-algebra. Set $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and recall that for any system of prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on R, the topology on $\mathcal{O}(\mathbb{C}^\times, R)$ is defined by all prenorms of the form

$$\|f\|_{\nu,\rho} = \sum_{n=-\infty}^{\infty} \|c_n\|_\nu \rho^n \quad (\nu \in \Lambda,\ \rho > 0), \tag{4.24}$$

where $f(z) = \sum_n c_n z^n$ is the Laurent expansion of $f \in \mathcal{O}(\mathbb{C}^\times, R)$.

Lemma 4.12. *Suppose R is a $\widehat{\otimes}$-algebra and $\alpha\colon R \to R$ is an automorphism. If α and α^{-1} are localizable, then there is a unique continuous linear map*

$$\tau\colon \mathcal{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} R \to R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}^\times),$$

such that

$$\tau(z^n \otimes r) = \alpha^n(r) \otimes z^n \tag{4.25}$$

for all $r \in R$ and $n \in \mathbb{Z}$.

The proof of this lemma is similar and in fact simpler than that of Lemma 4.2 and is therefore omitted. Notice that the description of τ given in Remark 4.3 carries over in an obvious way to the case in question.

Now let $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}^\times)$ and define a multiplication $m_A\colon A \mathbin{\widehat{\otimes}} A \to A$ by the formula obtained from (4.8) by replacing $\mathbb{C}$ with $\mathbb{C}^\times$. It is easy to see that we have the following analog of Proposition 4.3.

Proposition 4.13. *The operator $m_A\colon A \mathbin{\widehat{\otimes}} A \to A$ thus defined makes A a $\widehat{\otimes}$-algebra. Moreover the tautological embedding $i\colon R[z, z^{-1}; \alpha] \hookrightarrow A$ is a homomorphism.*

Definition 4.7. The algebra $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}^\times)$ with the above multiplication will be denoted $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ and called the *analytic Laurent extension* of R.

Notice that the algebra $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ contains R as a closed subalgebra and is therefore an R-$\widehat{\otimes}$-algebra. As in the case of analytic Ore extensions (see Remark 4.4), this algebra can be described as the analytic smash product of R and the $\widehat{\otimes}$-bialgebra $(\mathcal{O}(\mathbb{C}^\times), \Delta')$ acting on it (the latter, as is not difficult to see, is the Arens–Michael envelope of the group bialgebra $\mathbb{C}\mathbb{Z}$). Finally, similarly to the argument in Remark 4.5, one shows that α and α^{-1} being localizable is not only a sufficient but also a necessary condition for the multiplication on $R[z, z^{-1}; \alpha]$ to extend to $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$.

The next three propositions are proved similarly to Propositions 4.4, 4.5, and 4.6. We omit their proofs.

Proposition 4.14. *Let R be a $\widehat{\otimes}$-algebra and $\alpha\colon R \to R$ an automorphism such that α and α^{-1} are localizable. Suppose that B is an R-$\widehat{\otimes}$-algebra which is Arens–Michael and $x \in B$ is an invertible element such that $x \cdot r = \alpha(r) \cdot x$ for any $r \in R$. Then there exists a unique continuous R-homomorphism*

$$f\colon \mathcal{O}(\mathbb{C}^\times, R; \alpha) \to B,$$

such that $f(z) = x$.

Proposition 4.15. *Let R be an Arens–Michael algebra and $\alpha\colon R \to R$ an automorphism. Suppose that the family $\{\alpha, \alpha^{-1}\}$ is m-localizable. Then $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ is an Arens–Michael algebra.*

Proposition 4.16. *Let R be an Arens–Michael algebra and $\alpha\colon R \to R$ an m-isometric automorphism. Fix a system of submultiplicative prenorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ defining the topology on R and such that $\|\alpha(r)\|_\nu = \|r\|_\nu$ for all $r \in R$ and $\nu \in \Lambda$. Then each of the prenorms (4.24) on $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ is submultiplicative.*

Notice that the equality $\alpha^{\pm 1} = \mathrm{Ad}_{z^{\pm 1}}$ immediately implies that $\{\alpha, \alpha^{-1}\}$ being m-localizable is not only sufficient but also necessary for $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ to be Arens–Michael. If $\|1\|_\nu = 1$ for all ν and each of the prenorms (4.24) on $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ is submultiplicative, then α must be m-isometric (cf. Remark 4.8).

In the case of the analytic Laurent extension, the universal property of the analytic Ore extension mentioned in Remark 4.6 becomes

$$\mathrm{Hom}_{R\text{-}\mathsf{AM}}(\mathcal{O}(\mathbb{C}^\times, R; \alpha), B) \cong \bigl\{x \in B : x \text{ is invertible},\ x \cdot r = \alpha(r) \cdot x\ \forall r \in R\bigr\}. \tag{4.26}$$

5. Examples of Arens–Michael envelopes

5.1. **Ore extensions: the case $\alpha = \mathbf{1}_R$. Universal enveloping algebra.** Suppose R is an algebra (without topology) and $\delta\colon R \to R$ is a derivation. Let R_δ be the Arens–Michael algebra obtained by completing R with respect to the system of all δ-stable submultiplicative prenorms. The canonical homomorphism $R \to R_\delta$ will be denoted j. Clearly, δ uniquely defines an m-localizable derivation $\hat\delta$ of the algebra R_δ such that $\hat\delta \circ j = j \circ \delta$. Moreover we have homomorphisms

$$R[z;\delta] \to R_\delta[z;\hat\delta] \hookrightarrow \mathcal{O}(\mathbb{C}, R_\delta; \hat\delta)\,,$$

where the former coincides with j on R and sends z to z, and the latter is the canonical embedding (see above). The composition of these homomorphisms will be denoted $\iota_{R[z;\delta]}$.

Theorem 5.1. *The pair $(\mathcal{O}(\mathbb{C}, R_\delta; \hat\delta), \iota)$ is the Arens–Michael envelope of $R[z;\delta]$.*

Proof. Let $\varphi\colon R[z;\delta] \to B$ be a homomorphism from $R[z;\delta]$ to an Arens–Michael algebra B. Set $f = \varphi|_R$ and $x = \varphi(z)$. Fix a submultiplicative prenorm $\|\cdot\|$ on B and define a prenorm $\|\cdot\|^f$ on R by the formula $\|r\|^f = \|f(r)\|$. Clearly, this prenorm is submultiplicative and it is not difficult to check that it is δ-stable. Indeed,

$$\|\delta(r)\|^f = \bigl\|f(\delta(r))\bigr\| = \bigl\|\varphi([z,r])\bigr\| = \bigl\|[x, f(r)]\bigr\| \le 2\|x\| \cdot \|f(r)\| = 2\|x\| \cdot \|r\|^f$$

for any $r \in R$. It now follows that f is continuous in the topology on R defined by the system of all δ-stable submultiplicative prenorms. Therefore, there exists a unique continuous homomorphism $\eta_B\colon R_\delta \to B$ such that $\eta_B j = f$. Thus B becomes an R_δ-$\widehat{\otimes}$-algebra and φ an R-R_δ-homomorphism. This implies that the relation $x \cdot r = r \cdot x + \eta_B(\hat\delta(r))$ holds for any $r \in \operatorname{Im} j$ and hence for any $r \in R_\delta$. Therefore, by Proposition 4.4, there exists a unique continuous R_δ-homomorphism $\widehat\varphi\colon \mathcal{O}(\mathbb{C}, R_\delta; \hat\delta) \to B$ such that $\widehat\varphi(z) = x$. Therefore, the composition $\widehat\varphi \circ \iota_{R[z;\delta]}\colon R[z;\delta] \to B$ is an R-homomorphism and sends z to x. The uniqueness of an R-homomorphism with this property implies that $\widehat\varphi \circ \iota_{R[z;\delta]} = \varphi$, i.e., $\widehat\varphi$ extends φ in the sense of Definition 3.1. The uniqueness of such an extension is clear because the image of $\iota_{R[z;\delta]}$ is dense in $\mathcal{O}(\mathbb{C}, R_\delta; \hat\delta)$. The rest is clear. ∎

Remark 5.1. A similar result can be proved for more general smash products (see [49], Theorem 2.2).

The above construction of the Arens–Michael envelope of $R[t;\delta]$ can be used, in particular, for an inductive description of the Arens–Michael envelopes of universal enveloping algebras of solvable Lie algebras. If $\mathfrak{g}$ is a solvable finite-dimensional Lie algebra, then it has an ideal $\mathfrak{h}$ of codimension 1. Choose an arbitrary $x \in \mathfrak{g} \setminus \mathfrak{h}$ and set $R = U(\mathfrak{h})$. It is easy to see that $U(\mathfrak{g}) \cong R[x;\delta]$, where $\delta = \mathrm{ad}_x$. Therefore to describe the Arens–Michael envelope of $U(\mathfrak{g})$ it suffices to know the system of all submultiplicative δ-stable prenorms on $U(\mathfrak{h})$. We shall illustrate this approach by two very simple examples.

Example 5.1. Suppose $\mathfrak{g}$ is a two-dimensional Lie algebra with basis $\{x,y\}$ and commutator $[x,y]=y$. Set $\mathfrak{h}=\mathbb{C}y$ and $R=U(\mathfrak{h})=\mathbb{C}[y]$. Consider the derivation $\delta=\mathrm{ad}_x = y\frac{d}{dy}$ of R; by the above, $U(\mathfrak{g}) = R[x;\delta]$ and our goal is to describe all submultiplicative δ-stable prenorms on R.

For any $n \in \mathbb{Z}_+$ and any $r = \sum_i c_i y^i \in R$ we set $\|r\|_n = \sum_{i=0}^n |c_i|$. A standard calculation (which we omit) shows that $\|\cdot\|_n$ is a submultiplicative prenorm on R. Since $\delta(r) = \sum_i i c_i y^i$, we have $\|\delta(r)\|_n \le n\|r\|_n$ for any $r\in R$, so that $\|\cdot\|_n$ is δ-stable. Moreover, any δ-stable prenorm on R is majorized by some $\|\cdot\|_n$. Indeed, suppose $\|\cdot\|$ is a δ-stable prenorm and let a constant $C>0$ be such that $\|\delta(r)\| \le C\|r\|$ for any $r \in R$. Then for $r = y^n$ we have $\delta(y^n) = ny^n$, whence $n\|y^n\| \le C\|y^n\|$. Therefore, $\|y^n\|=0$ for sufficiently large n, which easily implies that $\|\cdot\| \le M\|\cdot\|_n$ for some $n\in\mathbb{Z}_+$ and some constant $M>0$. Using the notation introduced above, we have that R_δ coincides with the completion of R with respect to the system of prenorms $\{\|\cdot\|_n : n \in \mathbb{Z}_+\}$, i.e., with the algebra of formal power series $\mathbb{C}[[y]]$. Applying Theorem 5.1, we have the following

Proposition 5.2. *Suppose $\mathfrak{g}$ is a Lie algebra with basis $\{x,y\}$ and commutator $[x,y]=y$. The Arens–Michael envelope $\widehat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$ consists of all power series $a = \sum_{i,j} c_{ij} y^i x^j$ such that $\sum_j |c_{ij}|\rho^j < \infty$ for any $i \in \mathbb{Z}_+$ and any $\rho > 0$. The topology on $\widehat{U}(\mathfrak{g})$ is defined by the system of prenorms*

$$\|a\|_{n,\rho} = \sum_{j=0}^\infty \sum_{i=0}^n |c_{ij}|\rho^j \quad (n\in\mathbb{Z}_+,\ \rho>0).$$

In other words, the underlying LCS of $\widehat{U}(\mathfrak{g})$ coincides with $\mathcal{O}(\mathbb{C})[[y]] = \mathbb{C}[[y]] \widehat{\otimes} \mathcal{O}(\mathbb{C})$, and the multiplication is uniquely determined by the relation $[x,y]=y$ (where $x \in \mathcal{O}(\mathbb{C})$ denotes the identity map on $\mathbb{C}$) and by the condition that the embeddings $\mathbb{C}[[y]] \hookrightarrow \widehat{U}(\mathfrak{g})$ and $\mathcal{O}(\mathbb{C}) \hookrightarrow \widehat{U}(\mathfrak{g})$ be homomorphisms.

Example 5.2. Suppose $\mathfrak{g}$ is a three-dimensional Lie algebra with basis $\{x,y,z\}$ and commutator $[x,y]=z$, $[x,z]=[y,z]=0$ (i.e., the Heisenberg algebra). Set $\mathfrak{h}=\mathrm{span}\{y,z\}$ and $R=U(\mathfrak{h})=\mathbb{C}[y,z]$. Consider the derivation $\delta=\mathrm{ad}_x = z\frac{\partial}{\partial y}$ of R. As in the preceding example, we need to describe all submultiplicative δ-stable prenorms on R.

For any $\rho>0$ and any $a=\sum_{i,j} a_{ij} y^i z^j \in R$ we set

$$\|a\|_\rho = \sum_{i,j} |a_{ij}| \frac{i!}{(i+j)!}\rho^{i+j}. \tag{5.1}$$

Clearly, $\|\cdot\|_\rho$ is a norm on R.

Lemma 5.3. *The norms* (5.1) *are submultiplicative and δ-stable. Moreover, any submultiplicative δ-stable prenorm on R is majorized by some norm* (5.1).

Proof. Fix an arbitrary $\rho>0$ and for each pair of indices (i,j) set $p_{ij} = \frac{i!}{(i+j)!}\rho^{i+j}$. Obviously, to prove that $\|\cdot\|_\rho$ is submultiplicative it suffices to show that

$$p_{i+k,j+l} \le p_{ij}p_{kl} \quad \text{for any} \quad i,j,k,l \in \mathbb{Z}_+.$$

After canceling $\rho^{i+j+k+l}$, the last inequality becomes

$$\frac{(i+k)!}{(i+k+j+l)!} \le \frac{i!k!}{(i+j)!(k+l)!}$$

or, equivalently,

$$\binom{i+k}{i} \le \binom{i+k+j+l}{i+j}.$$

This formula can easily be deduced by induction on j and l from the obvious inequalities

$$\binom{n}{m} \le \binom{n+1}{m+1} \quad \text{and} \quad \binom{n}{m} \le \binom{n+1}{m}.$$

Thus the norms (5.1) are submultiplicative.

Furthermore, for any $a = \sum_{i,j} a_{ij} y^i z^j \in R$ we have

$$\|\delta(a)\|_\rho = \Big\| \sum_{\substack{i,j \\ i\ge 1}} a_{ij} i y^{i-1} z^{j+1} \Big\|_\rho = \sum_{\substack{i,j \\ i\ge 1}} |a_{ij}| i \frac{(i-1)!}{(i+j)!} \rho^{i+j} \le \sum_{i,j} |a_{ij}| \frac{i!}{(i+j)!} \rho^{i+j} = \|a\|_\rho.$$

Therefore, the norms (5.1) are δ-stable.

Now suppose that $\|\cdot\|$ is a submultiplicative δ-stable prenorm on R. Fix a constant $C \ge 1$ such that $\|\delta(a)\| \le C\|a\|$ for any $a \in R$ and notice that for any $i, j \in \mathbb{Z}_+$,

$$\delta^j(y^{i+j}) = \frac{(i+j)!}{i!} y^i z^j.$$

Therefore, substituting $a = y^{i+j}$ in $\|\delta^j(a)\| \le C^j \|a\|$, we have

$$\frac{(i+j)!}{i!} \|y^i z^j\| \le C^j \|y^{i+j}\| \le C^j r^{i+j},$$

where $r = \|y\|$. Hence

$$\|y^i z^j\| \le C^j \frac{i!}{(i+j)!} r^{i+j} \le \frac{i!}{(i+j)!} (Cr)^{i+j} = \|y^i z^j\|_{Cr}.$$

Therefore, setting $\rho = Cr$, for any $a = \sum_{i,j} a_{ij} y^i z^j$ we have an estimate

$$\|a\| = \Big\| \sum_{i,j} a_{ij} y^i z^j \Big\| \le \sum_{i,j} |a_{ij}| \, \|y^i z^j\| \le \sum_{i,j} |a_{ij}| \, \|y^i z^j\|_\rho = \|a\|_\rho.$$

Thus $\|a\| \le \|a\|_\rho$ for any $a \in R$. ☐

Combining the last lemma with Theorem 5.1, we have

Proposition 5.4. *Suppose $\mathfrak{g}$ is a Lie algebra with basis $\{x, y, z\}$ and commutator $[x, y] = z$, $[x, z] = [y, z] = 0$ (i.e., the Heisenberg algebra). The Arens–Michael envelope $\widehat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$ consists of all power series*

$$a = \sum_{i,j,k} a_{ijk} y^i z^j x^k,$$

such that for any $\rho > 0$,

$$\|a\|_\rho = \sum_{i,j,k} |a_{ijk}| \frac{i!}{(i+j)!} \rho^{i+j+k} < \infty.$$

The topology on $\widehat{U}(\mathfrak{g})$ is defined by the prenorms $\{\|\cdot\|_\rho : \rho > 0\}$.

5.2. **Tensor algebras.** Suppose R is an algebra and M is an R-bimodule. Using Proposition 4.8, it is not difficult to describe the Arens–Michael envelope of the tensor algebra $T_R(M)$. For this, we first notice that the homomorphism $R \xrightarrow{\iota_R} \widehat{R} \xrightarrow{i_0} \widehat{T}_{\widehat{R}}(\widehat{M})$ and the R-bimodule morphism $M \xrightarrow{\iota_M} \widehat{M} \xrightarrow{i_1} \widehat{T}_{\widehat{R}}(\widehat{M})$ uniquely determine (by the universal property of $T_R(M)$) an R-homomorphism $\iota_T \colon T_R(M) \to \widehat{T}_{\widehat{R}}(\widehat{M})$.

Proposition 5.5. *Let R be an algebra and M an R-bimodule. Then the pair $(\widehat{T}_{\widehat{R}}(\widehat{M}), \iota_T)$ is the Arens–Michael envelope of $T_R(M)$.*

Proof. Suppose $\varphi \colon T_R(M) \to A$ is a homomorphism with values in an Arens–Michael algebra A. The homomorphism $\theta = \varphi i_0 \colon R \to A$ extends to a homomorphism $\eta_A = \widehat{\theta} \colon \widehat{R} \to A$, which makes A an $\widehat{R}$-$\widehat{\otimes}$-algebra and, in particular, an $\widehat{R}$-$\widehat{\otimes}$-bimodule. Therefore the R-bimodule morphism $\alpha = \varphi i_1 \colon M \to A$ extends to an $\widehat{R}$-bimodule morphism $\widehat{\alpha} \colon \widehat{M} \to A$, which, by Proposition 4.8, yields an $\widehat{R}$-homomorphism $\widehat{\varphi} \colon \widehat{T}_{\widehat{R}}(\widehat{M}) \to A$ coinciding with $\widehat{\alpha}$ on $\widehat{M}$. It is now easy to see that $\widehat{\varphi}\iota_T = \varphi$, because both R-homomorphisms coincide with α on M. Thus $\widehat{\varphi}$ is a homomorphism extending φ in the sense of Definition 3.1. Its uniqueness is obvious because the image of ι_T is dense in $\widehat{T}_{\widehat{R}}(\widehat{M})$. □

5.3. **Ore extensions: the case $\delta = 0$. Quantum affine spaces.** As a consequence of Proposition 5.5, we can easily describe the Arens–Michael envelope of the algebra $R[t; \alpha]$, where R is an algebra and α is an endomorphism of it. Notice that the definition of the Arens–Michael envelope implies that there exists a unique continuous endomorphism $\widehat{\alpha}$ of $\widehat{R}$ such that $\widehat{\alpha}\iota_R = \iota_R\alpha$. Moreover, we have homomorphisms

$$R[t; \alpha] \to \widehat{R}[t; \widehat{\alpha}] \hookrightarrow \widehat{R}\{t; \widehat{\alpha}\},$$

where the former coincides with ι_R on R and sends t to t, and the latter is the canonical embedding (see Proposition 4.9). The composition of these homomorphisms will be denoted $\iota_{R[t;\alpha]}$.

Corollary 5.6. *Suppose R is an algebra and α is an endomorphism of it. Then the pair $(\widehat{R}\{t; \widehat{\alpha}\}, \iota_{R[t;\alpha]})$ is the Arens–Michael envelope of $R[t; \alpha]$.*

Proof. We shall show that the pair $(\widehat{R}_{\widehat{\alpha}}, \iota_R)$ is the Arens–Michael envelope of the bimodule R in the sense of Definition 3.2 (see also Remark 3.8). Indeed, that $\iota_R \colon R_\alpha \to \widehat{R}_{\widehat{\alpha}}$ is an R-bimodule morphism easily follows from the relation $\iota_R\alpha = \widehat{\alpha}\iota_R$. Suppose X is an $\widehat{R}$-$\widehat{\otimes}$-bimodule and $\varphi \colon R_\alpha \to X$ a morphism of R-bimodules. Set $x = \varphi(1)$; then, clearly, $x \cdot r = \alpha(r) \cdot x$ for any $r \in R$, and therefore $x \cdot r = \widehat{\alpha}(r) \cdot x$ for any $r \in \widehat{R}$. This, in turn, implies that $\widehat{\varphi} \colon \widehat{R}_{\widehat{\alpha}} \to X$, $\widehat{\varphi}(r) = r \cdot x$ is a morphism of $\widehat{R}$-bimodules extending φ. The uniqueness of such extension is clear because the image of $\operatorname{Im} \iota_R$ is dense in $\widehat{R}$. Thus, $\widehat{R_\alpha} = \widehat{R}_{\widehat{\alpha}}$ as $\widehat{R}$-$\widehat{\otimes}$-bimodules. It remains to combine this fact with Propositions 5.5 and 4.9. □

Example 5.3. Set $R = \mathbb{C}[x]$ and define an endomorphism $\alpha \colon R \to R$ by the formula $\alpha(x) = x - 1$. Then it is easy to see that the algebra $R[y; \alpha]$ is isomorphic to the universal enveloping algebra of $U(\mathfrak{g})$, where $\mathfrak{g}$ is the two-dimensional Lie algebra with basis $\{x, y\}$ and commutator $[x, y] = y$ (see Example 5.1). By Corollary 5.6, we have an isomorphism of topological algebras $\widehat{U}(\mathfrak{g}) \cong \widehat{R}\{y; \widehat{\alpha}\}$. On the other hand, $\widehat{R} = \mathcal{O}(\mathbb{C})$, and it is not difficult to see that the endomorphism $\widehat{\alpha} \colon \widehat{R} \to \widehat{R}$ acts by the formula $\widehat{\alpha}(f)(z) = f(z-1)$. Therefore, by Example 4.3, the underlying LCS of $\widehat{U}(\mathfrak{g}) = \widehat{R}\{y; \widehat{\alpha}\}$ is isomorphic to $\mathcal{O}(\mathbb{C})[[y]]$; under this isomorphism the coordinate function $z \mapsto z$ from $\mathcal{O}(\mathbb{C})$ corresponds to $x \in \mathfrak{g}$. Thus we have a second proof of Proposition 5.2, which is independent of the results of 5.1.

Example 5.4. Fix a number $q \in \mathbb{C}$, $q \neq 0$. Recall (see [40]) that the *quantum plane* is an algebra with two generators x, y, subject to the relation $xy = qyx$. This algebra will be denoted $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^2)$.[5]

More generally, let $\mathbf{q} = (q_{ij})_{i,j=1}^n$ be a complex $(n \times n)$-matrix such that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for all i, j (such matrices are usually called *multiplicatively antisymmetric*). A *multiparameter quantum affine space* (see, for example, [1, 20]) is an algebra $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ with generators $x_1, \ldots, x_n$ and relations $x_i x_j = q_{ij} x_j x_i$ $(i, j = 1, \ldots, n)$. In the one-parameter case, i.e., when $q_{ij} = q$ for all $i < j$, this algebra is denoted $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$.

It is easy to see that $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ is an iterated Ore extension

$$\mathbb{C}[x_1][x_2; \alpha_2] \cdots [x_n; \alpha_n],$$

where for each $i = 2, \ldots, n$, the endomorphism α_i is uniquely determined by the condition $\alpha_i(x_j) = q_{ij} x_j$ $(j < i)$. Therefore to describe the Arens–Michael envelope of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ we could use Corollary 5.6. However we prefer a different approach, reflecting the specific features of the given algebra and leading to a more transparent description of its Arens–Michael envelope, which is not based on the "nonconstructive" formula (4.21).

First, we recall a convenient formula for the multiplication on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$. For each $\alpha \in \mathbb{Z}_+^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Define a function $c \colon \mathbb{Z}_+^n \times \mathbb{Z}_+^n \to \mathbb{C}^\times$ by setting

$$c(\alpha, \beta) = \prod_{i>j} q_{ij}^{\alpha_i \beta_j}. \tag{5.2}$$

Then in $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ we have

$$x^\alpha x^\beta = c(\alpha, \beta) x^{\alpha+\beta} \tag{5.3}$$

(see, for example, [1, 21]).

Let $w \colon \mathbb{Z}_+^n \to \mathbb{R}_+$ be an arbitrary function. For each $\rho > 0$ define a prenorm $\| \cdot \|_\rho^w$ on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ by the formula

$$\|a\|_\rho^w = \sum_\alpha |c_\alpha| w(\alpha) \rho^{|\alpha|} \quad \text{for} \quad a = \sum_\alpha c_\alpha x^\alpha. \tag{5.4}$$

Lemma 5.7. *The prenorm $\| \cdot \|_\rho^w$ is submultiplicative if and only if*

$$|c(\alpha, \beta)| w(\alpha + \beta) \le w(\alpha) w(\beta) \quad (\alpha, \beta \in \mathbb{Z}_+^n). \tag{5.5}$$

Proof. This follows easily from relations (5.3). □

Consider the free algebra F_n with generators $\xi_1, \ldots, \xi_n$. Obviously, there exists a unique homomorphism $F_n \to \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ such that $\xi_i \mapsto x_i$ for each $-1, \ldots, n$. The image of an arbitrary element $u \in F_n$ under the action of this homomorphism will be denoted $\bar{u}$.

Fix an $N \in \mathbb{Z}_+$ and let $X_N \subset F_n$ denote the set of all words in $\xi_1, \ldots, \xi_n$ of length N. The symmetric group S_N acts on X_N in an obvious way. Moreover, it follows from the definition of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ that for any $u \in X_N$ and $\sigma \in S_N$ there is a unique $\lambda(\sigma, u) \in \mathbb{C}^\times$ such that $\bar{u} = \lambda(\sigma, u) \overline{\sigma(u)}$.

Lemma 5.8. *The function $\lambda \colon S_N \times X_N \to \mathbb{C}^\times$ satisfies the relation*

$$\lambda(\sigma\tau, u) = \lambda(\sigma, \tau(u)) \lambda(\tau, u) \quad (\sigma, \tau \in S_N,\ u \in X_N).$$

[5] In the literature this algebra is usually denoted $\mathcal{O}_q(\mathbb{C}^2)$ or simply A_q; we have added the superscript "alg" in order to use the notation $\mathcal{O}_q(\mathbb{C}^2)$ for an analytic analog of this algebra.

Proof. It follows from the definition of λ that

$$\bar{u} = \lambda(\tau, u)\overline{\tau(u)} \quad \text{and} \quad \overline{\tau(u)} = \lambda(\sigma, \tau(u))\overline{\sigma\tau(u)},$$

whence

$$\bar{u} = \lambda(\tau, u)\lambda(\sigma, \tau(u))\overline{\sigma\tau(u)}. \tag{5.6}$$

On the other hand,

$$\bar{u} = \lambda(\sigma\tau, u)\overline{\sigma\tau(u)}. \tag{5.7}$$

It remains to combine (5.6) and (5.7). $\square$

Fix an $\alpha \in \mathbb{Z}_+^n$ and consider the element $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \in X_N$, where $N = |\alpha|$. Set

$$w_{\mathbf{q}}(\alpha) = \min\bigl\{|\lambda(\sigma, \xi^\alpha)| : \sigma \in S_N\bigr\}.$$

Since $\lambda(e, u) = 1$, where e is the identity permutation, we have $w_{\mathbf{q}}(\alpha) \le 1$ for any $\alpha \in \mathbb{Z}_n^+$.

Lemma 5.9. *The function* $w_{\mathbf{q}} \colon \mathbb{Z}_+^n \to \mathbb{R}_+$ *satisfies* (5.5).

Proof. Set $N_1 = |\alpha|$, $N_2 = |\beta|$, and $N = N_1 + N_2$. Clearly, there is a $\tau \in S_N$ such that $\tau(\xi^\alpha\xi^\beta) = \xi^{\alpha+\beta}$. Moreover, $x^\alpha x^\beta = \lambda(\tau, \xi^\alpha\xi^\beta)x^{\alpha+\beta}$, whence $\lambda(\tau, \xi^\alpha\xi^\beta) = c(\alpha, \beta)$. Therefore, in view of Lemma 5.8,

$$\begin{aligned} \min\bigl\{|\lambda(\sigma, \xi^\alpha\xi^\beta)| : \sigma \in S_N\bigr\} &= \min\bigl\{|\lambda(\sigma\tau, \xi^\alpha\xi^\beta)| : \sigma \in S_N\bigr\} \\ &= \min\bigl\{|\lambda(\sigma, \xi^{\alpha+\beta})| \cdot |\lambda(\tau, \xi^\alpha\xi^\beta)| : \sigma \in S_N\bigr\} = w_{\mathbf{q}}(\alpha+\beta)|c(\alpha, \beta)|. \end{aligned} \tag{5.8}$$

Suppose now that $\sigma_1 \in S_{N_1}$ and $\sigma_2 \in S_{N_2}$ are such that

$$|\lambda(\sigma_1, \xi^\alpha)| = w_{\mathbf{q}}(\alpha) \quad \text{and} \quad |\lambda(\sigma_2, \xi^\beta)| = w_{\mathbf{q}}(\beta).$$

Let $\sigma_1 \times \sigma_2 \in S_N$ denote the permutation

$$(\sigma_1 \times \sigma_2)(i) = \begin{cases} \sigma_1(i), & i \le N_1, \\ N_1 + \sigma_2(i - N_1), & i > N_1. \end{cases}$$

In other words, $\sigma_1 \times \sigma_2$ acts on the first N_1 elements as σ_1 and on the remaining elements as σ_2. Clearly, $(\sigma_1 \times \sigma_2)(\xi^\alpha\xi^\beta) = \sigma_1(\xi^\alpha)\sigma_2(\xi^\beta)$, and therefore

$$x^\alpha x^\beta = \lambda(\sigma_1, \xi^\alpha)\overline{\sigma_1(\xi^\alpha)}\lambda(\sigma_2, \xi^\beta)\overline{\sigma_2(\xi^\beta)} = \lambda(\sigma_1, \xi^\alpha)\lambda(\sigma_2, \xi^\beta)\overline{(\sigma_1 \times \sigma_2)(\xi^\alpha\xi^\beta)}.$$

On the other hand,

$$x^\alpha x^\beta = \lambda(\sigma_1 \times \sigma_2, \xi^\alpha\xi^\beta)\overline{(\sigma_1 \times \sigma_2)(\xi^\alpha\xi^\beta)},$$

whence $\lambda(\sigma_1 \times \sigma_2, \xi^\alpha\xi^\beta) = \lambda(\sigma_1, \xi^\alpha)\lambda(\sigma_2, \xi^\beta)$. Taking account of (5.8) we see that

$$\begin{aligned} w_{\mathbf{q}}(\alpha)w_{\mathbf{q}}(\beta) &= \bigl|\lambda(\sigma_1 \times \sigma_2, \xi^\alpha\xi^\beta)\bigr| \\ &\ge \min\bigl\{|\lambda(\sigma, \xi^\alpha\xi^\beta)| : \sigma \in S_N\bigr\} = w_{\mathbf{q}}(\alpha+\beta)|c(\alpha, \beta)|, \end{aligned}$$ $\square$

Henceforth the prenorm $\|\cdot\|_\rho^{w_{\mathbf{q}}}$ will be denoted $\|\cdot\|_\rho^{\mathbf{q}}$.

Lemma 5.10. *The prenorm* $\|\cdot\|_\rho^{\mathbf{q}}$ *is submultiplicative and is the largest of all the submultiplicative prenorms* $\|\cdot\|$ *on* $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ *such that* $\|x_i\| \le \rho$ *for all* $i = 1, \dots, n$.

Proof. First, $\|\cdot\|_\rho^{\mathbf{q}}$ is submultiplicative by Lemmas 5.9 and 5.7. It is also clear that $\|x_i\|_\rho^{\mathbf{q}} = \rho$ for all $i = 1, \dots, n$. Suppose $\|\cdot\|$ is some other prenorm on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ satisfying the assumptions of the lemma. Clearly, $\|\bar{u}\| \le \rho^N$ for any $u \in X_N$ and any $N \in \mathbb{Z}_+$. Take an arbitrary $\alpha \in \mathbb{Z}_+^n$, set $N = |\alpha|$, and choose $\sigma \in S_N$ such that $w_{\mathbf{q}}(\alpha) = |\lambda(\sigma, \xi^\alpha)|$. Since $x^\alpha = \lambda(\sigma, \xi^\alpha)\overline{\sigma(\xi^\alpha)}$, we have

$$\|x^\alpha\| = w_{\mathbf{q}}(\alpha)\|\overline{\sigma(\xi^\alpha)}\| \le w_{\mathbf{q}}(\alpha)\rho^{|\alpha|}.$$

This, together with the explicit description of $\|\cdot\|_\rho^{\mathbf{q}}$, implies that $\|a\| \le \|a\|_\rho^{\mathbf{q}}$ for any $a \in \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$. $\square$

Suppose, as before, that $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric complex $(n \times n)$-matrix. Set

$$\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n) = \Big\{ a = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha : \|a\|_\rho^{\mathbf{q}} = \sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha| w_{\mathbf{q}}(\alpha) \rho^{|\alpha|} < \infty \ \forall \rho > 0 \Big\}.$$

If $q_{ij} = q$ for all $i < j$, then the corresponding space will be denoted $\mathcal{O}_q(\mathbb{C}^n)$, the function $w_{\mathbf{q}}$ will be denoted w_q, and the prenorm $\|\cdot\|_\rho^{\mathbf{q}}$ will be denoted $\|\cdot\|_\rho^q$.

Clearly, $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ is a Fréchet space with respect to the topology defined by the system of prenorms $\{\|\cdot\|_\rho^{\mathbf{q}} : \rho > 0\}$. Notice that assigning to each $a = \sum_\alpha c_\alpha x^\alpha \in \mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ the function $f\colon \mathbb{C}^n \to \mathbb{C}$, $f(z) = \sum_\alpha c_\alpha w_{\mathbf{q}}(\alpha) z^\alpha$ we have an LCS isomorphism between $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ and the space of entire functions $\mathcal{O}(\mathbb{C}^n)$.

Theorem 5.11. *There is a unique multiplication on $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ making it a topological algebra and such that $x_i x_j = q_{ij} x_j x_i$ for all $i, j = 1, \dots, n$. Algebra thus obtained, together with the tautological embedding of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ in it, is the Arens–Michael envelope of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$. Moreover, each prenorm of the system $\{\|\cdot\|_\rho^{\mathbf{q}} : \rho > 0\}$ defining the topology on $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ is submultiplicative.*

Proof. By Lemma 5.10, the prenorm $\|\cdot\|_\rho^{\mathbf{q}}$ is submultiplicative on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$. Since $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ is, obviously, a completion of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ with respect to the system $\{\|\cdot\|_\rho^{\mathbf{q}} : \rho > 0\}$, we see that the multiplication on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ extends uniquely to a continuous multiplication on $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ and makes the latter an Arens–Michael algebra.

Suppose now that A is an Arens–Michael algebra and $\varphi\colon \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n) \to A$ is a homomorphism. Fix a continuous submultiplicative prenorm $\|\cdot\|$ on A and set $\rho = \max_{1 \le i \le n} \|\varphi(x_i)\|$. Applying Lemma 5.10, we have $\|\varphi(a)\| \le \|a\|_\rho^{\mathbf{q}}$ for any $a \in \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$. Therefore φ is continuous in the topology defined by the system $\{\|\cdot\|_\rho^{\mathbf{q}} : \rho > 0\}$. The rest is clear. $\square$

We shall now examine the two simplest cases when $w_{\mathbf{q}}$ can be computed explicitly.

Proposition 5.12. (i) *If $|q_{ij}| \ge 1$ for all $i < j$, then $w_{\mathbf{q}}(\alpha) = 1$ for all $\alpha \in \mathbb{Z}_+^n$.*
(ii) *If $|q_{ij}| \le 1$ for all $i < j$, then $w_{\mathbf{q}}(\alpha) = |c(\alpha, \alpha)|^{-1}$ for all $\alpha \in \mathbb{Z}_+^n$.*

Proof. (i) It follows from the assumption and (5.2) that $|c(\alpha, \beta)| \le 1$ for all $\alpha, \beta \in \mathbb{Z}_+^n$. Therefore the function $\varepsilon\colon \mathbb{Z}_+^n \to \mathbb{Z}_+$, $\varepsilon(\alpha) \equiv 1$, satisfies the condition of Lemma 5.7. Therefore, $\|\cdot\|_1^\varepsilon$ is submultiplicative on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$. Since $w_{\mathbf{q}}(\alpha) \le 1$ for any $\alpha \in \mathbb{Z}_+^n$, we have $\|\cdot\|_1^{\mathbf{q}} \le \|\cdot\|_1^\varepsilon$. On the other hand, Lemma 5.10 yields the opposite inequality. Thus, $\|\cdot\|_1^{\mathbf{q}} = \|\cdot\|_1^\varepsilon$, which, in turn, is equivalent to the equality $w_{\mathbf{q}}(\alpha) = 1$ for all $\alpha \in \mathbb{Z}_+^n$.

(ii) As before, consider the free algebra F_n with generators $\xi_1, \dots, \xi_n$. For any $i = 1, \dots, n$ set $\eta_i = \xi_{n-i}$. Define also a multiplicatively antisymmetric $(n \times n)$-matrix $\check{\mathbf{q}} = (\check{q}_{ij})$ by setting $\check{q}_{ij} = q_{n-i,n-j}$. Clearly, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = N$ we have

$$\min\big\{ |\lambda(\sigma, \eta^\alpha)| : \sigma \in S_N \big\} = w_{\check{\mathbf{q}}}(\alpha). \tag{5.9}$$

Define a permutation $\tau \in S_N$ by the formula $\tau(i) = N - i$ $(i = 1, \dots, N)$. Clearly, $\tau(\xi^\alpha) = \eta^{\check{\alpha}}$, where $\check{\alpha} = (\alpha_n, \dots, \alpha_1)$. Furthermore, the multiplication law on $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ shows that

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \prod_{i<j} q_{ij}^{\alpha_i \alpha_j} x_n^{\alpha_n} \cdots x_1^{\alpha_1} = c(\alpha, \alpha)^{-1} x_n^{\alpha_n} \cdots x_1^{\alpha_1}, \tag{5.10}$$

or, equivalently, $\lambda(\tau, \xi^\alpha) = c(\alpha,\alpha)^{-1}$. Therefore, in view of Lemma 5.8 and equality (5.9), we have

$$\begin{aligned} w_{\mathbf{q}}(\alpha) &= \min\bigl\{|\lambda(\sigma,\xi^\alpha)| : \sigma \in S_N\bigr\} = \min\bigl\{|\lambda(\sigma\tau,\xi^\alpha)| : \sigma \in S_N\bigr\} \\ &= \min\bigl\{|\lambda(\sigma,\eta^{\check\alpha})| : \sigma \in S_N\bigr\} \cdot |\lambda(\tau,\xi^\alpha)| = w_{\check{\mathbf{q}}}(\check\alpha)|c(\alpha,\alpha)|^{-1}. \end{aligned}$$

But $w_{\check{\mathbf{q}}} \equiv 1$ by (i) and therefore $w_{\mathbf{q}}(\alpha) = |c(\alpha,\alpha)|^{-1}$. □

Corollary 5.13. (i) *If $|q_{ij}| \ge 1$ for $i < j$, then*

$$\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n) = \Bigl\{a = \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha : \|a\|_\rho^{\mathbf{q}} = \sum_{\alpha\in\mathbb{Z}_+^n} |c_\alpha|\rho^{|\alpha|} < \infty \ \forall \rho > 0\Bigr\}. \tag{5.11}$$

(ii) *If $|q_{ij}| \le 1$ for $i < j$, then*

$$\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n) = \Bigl\{a = \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha : \|a\|_\rho^{\mathbf{q}} = \sum_{\alpha\in\mathbb{Z}_+^n} |c_\alpha| \cdot |c(\alpha,\alpha)|^{-1}\rho^{|\alpha|} < \infty \ \forall \rho > 0\Bigr\}.$$

In both cases the topology on $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^n)$ is defined by the system of submultiplicative prenorms $\{\|\cdot\|_\rho^{\mathbf{q}} : \rho > 0\}$ and the multiplication is uniquely determined by the relations

$$x_i x_j = q_{ij} x_j x_i \quad (i,j = 1,\dots,n).$$

Remark 5.2. Notice that assertion (i) of the preceding corollary can be deduced by induction from Corollary 5.6, Proposition 4.11, and Proposition 4.6.

In the case of the quantum plane, Corollary 5.13 becomes especially simple.

Corollary 5.14. *Suppose $q \in \mathbb{C}\setminus\{0\}$.*

(i) *If $|q| \ge 1$, then*

$$\mathcal{O}_q(\mathbb{C}^2) = \Bigl\{a = \sum_{i,j=0}^\infty c_{ij}x^i y^j : \|a\|_\rho^q = \sum_{i,j=0}^\infty |c_{ij}|\rho^{i+j} < \infty \ \forall \rho > 0\Bigr\}.$$

(ii) *If $|q| \le 1$, then*

$$\mathcal{O}_q(\mathbb{C}^2) = \Bigl\{a = \sum_{i,j=0}^\infty c_{ij}x^i y^j : \|a\|_\rho^q = \sum_{i,j=0}^\infty |c_{ij}||q|^{ij}\rho^{i+j} < \infty \ \forall \rho > 0\Bigr\}.$$

In both cases the topology on $\mathcal{O}_q(\mathbb{C}^2)$ is defined by the system of submultiplicative prenorms $\{\|\cdot\|_\rho^q : \rho > 0\}$ and the multiplication is uniquely determined by the relation $xy = qyx$.

5.4. **Connections with the algebra of holomorphic functions on the quantum ball.** Let $\overline{\mathbb{B}}_\rho \subset \mathbb{C}^n$ be the closed ball in $\mathbb{C}^n$ of radius $\rho > 0$ and center at the origin. Let $A(\overline{\mathbb{B}}_\rho)$ denote the Banach algebra consisting of all continuous functions on $\overline{\mathbb{B}}_\rho$ which are holomorphic on its interior. Fix an arbitrary $R > 1$ and consider the inverse system of Banach algebras

$$A(\overline{\mathbb{B}}_1) \leftarrow A(\overline{\mathbb{B}}_R) \leftarrow A(\overline{\mathbb{B}}_{R^2}) \leftarrow A(\overline{\mathbb{B}}_{R^3}) \leftarrow \cdots, \tag{5.12}$$

where the arrows denote the restriction homomorphisms. It is easy to see that the inverse limit of (5.12) is, up to isomorphism of topological algebras, the algebra $\mathcal{O}(\mathbb{C}^n)$ of entire functions.

Let $z_1,\dots,z_n$ be the coordinates on $\mathbb{C}^n$. Notice that (5.12) is isomorphic to the inverse system

$$A(\overline{\mathbb{B}}) \xleftarrow{\gamma_r} A(\overline{\mathbb{B}}) \xleftarrow{\gamma_r} A(\overline{\mathbb{B}}) \xleftarrow{\gamma_r} A(\overline{\mathbb{B}}) \xleftarrow{\gamma_r} \cdots, \tag{5.13}$$

where $\overline{\mathbb{B}} = \overline{\mathbb{B}}_1$, $r = R^{-1}$, and γ_r denotes the unique continuous endomorphism of $A(\overline{\mathbb{B}})$ such that $\gamma_r(z_i) = rz_i$ for all $i = 1, \dots, n$. Thus

$$\mathcal{O}(\mathbb{C}^n) \cong \varprojlim \bigl(A(\overline{\mathbb{B}}), \gamma_r\bigr). \tag{5.14}$$

Below we shall show that (5.14) has a quantum analog, in which $\mathcal{O}(\mathbb{C}^n)$ is replaced by $\mathcal{O}_q(\mathbb{C}^n)$ (see above) and the role of the algebra $A(\overline{\mathbb{B}})$ is played by the algebra of holomorphic functions on the quantum ball, introduced by Vaksman [70].

First, we recall some definitions from [55] and [70]. Fix $q \in (0,1)$ and let $\mathcal{A}_q$ denote the algebra with generators $a_1, \dots, a_n, a_1^+, \dots, a_n^+$ and relations

$$\begin{aligned} a_i^+ a_j^+ &= q a_j^+ a_i^+ && (i < j),\\ a_j a_i^+ &= q a_i^+ a_j && (i \neq j),\\ a_j a_i &= q a_i a_j && (i < j),\\ a_i a_i^+ &= 1 + q^2 a_i^+ a_i - (1 - q^2) \sum_{k > i} a_k^+ a_k. \end{aligned}$$

This algebra was introduced by W. Pusz and S. Woronowicz [55]; this is a q-analog of the algebra of canonical commutation relations. There is a unique involution $a \mapsto a^*$ on $\mathcal{A}_q$ such that $a_i^* = a_i^+$ for all $i = 1, \dots, n$.

Notice that $\mathcal{A}_q$ is a particular case of the quantum Weyl algebra (see [14, 29]; in the simplest case of $n = 1$ the quantum Weyl algebra is discussed below in Example 5.5). This algebra can be viewed as an iterated Ore extension of the base field $\mathbb{C}$; as a consequence, the ordered monomials in $a_1^\pm, \dots, a_n^\pm$ form a basis of $\mathcal{A}_q$ (ibid.). This, in turn, implies that $a_1^+, \dots, a_n^+$ generate a subalgebra of $\mathcal{A}_q$ isomorphic to $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$.

Let H be a Hilbert space with orthonormal basis $\{e_\alpha : \alpha \in \mathbb{Z}_+^n\}$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ the basis element e_α will also be denoted $|\alpha_1, \dots, \alpha_n\rangle$. For any $k \in \mathbb{Z}_+$ we set

$$[k] = \frac{1 - q^{2k}}{1 - q^2}, \quad [k]! = [1] \cdot [2] \cdots [k].$$

Finally, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ set

$$[\alpha]! = [\alpha_1]! \cdots [\alpha_n]!\,.$$

By [55], there is an irreducible $*$-representation $\pi \colon \mathcal{A}_q \to \mathcal{B}(H)$, uniquely determined by the formulas

$$\pi(a_j)|\alpha_1, \dots, \alpha_n\rangle = \sqrt{[\alpha_j]}\, q^{\sum_{k>j} \alpha_k}\, |\alpha_1, \dots, \alpha_j - 1, \dots, \alpha_n\rangle, \tag{5.15}$$

$$\pi(a_j^+)|\alpha_1, \dots, \alpha_n\rangle = \sqrt{[\alpha_j + 1]}\, q^{\sum_{k>j} \alpha_k}\, |\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_n\rangle. \tag{5.16}$$

This is a q-analog of the classical Fock representation. Notice that, unlike the classical case, all the operators $\pi(a)$ with $a \in \mathcal{A}_q$ are bounded and $\|\pi(a_j^\pm)\| \le (1 - q^2)^{-1/2}$ for all $j = 1, \dots, n$. (It is easy to check that the last inequality is in fact an equality, but we will not need this fact.)

D. P. Proskurin and Yu. S. Samoĭlenko [52] showed that π is faithful and that the C^*-algebra $\overline{\pi(\mathcal{A}_q)} \subset \mathcal{B}(H)$ is the universal enveloping C^*-algebra of $\mathcal{A}_q$ in the sense that any $*$-homomorphism from $\mathcal{A}_q$ to a C^*-algebra uniquely factors through $\overline{\pi(\mathcal{A}_q)}$. As was noticed by L. L. Vaksman [70], the algebra $C(\overline{\mathbb{B}})_q = \overline{\pi(\mathcal{A}_q)}$ can be viewed as a q-analog of the algebra $C(\overline{\mathbb{B}})$ of continuous functions on the closed ball $\overline{\mathbb{B}}$, whereas its closed subalgebra $A(\overline{\mathbb{B}})_q$ generated by the operators $\pi(a_k^+)$ $(k = 1, \dots, n)$ can be viewed as a q-analog of the algebra $A(\overline{\mathbb{B}})$.

Henceforth we shall use the notation introduced in the preceding section. Recall, in particular, that for any $\alpha \in \mathbb{Z}_+$ we have

$$w_q(\alpha) = |c(\alpha,\alpha)|^{-1} = q^{\sum_{i<j} \alpha_i\alpha_j} \tag{5.17}$$

(see Proposition 5.12 (ii)).

Define a homomorphism

$$\varphi_0 \colon \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n) \to A(\overline{\mathbb{B}})_q, \quad x_i \mapsto \sqrt{1-q^2}\,\pi(a_i^+) \quad (i = 1,\dots,n).$$

As follows from the above remarks, it is injective. Henceforth we shall identify $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ with a dense subalgebra of $A(\overline{\mathbb{B}})_q$ via φ_0. The corresponding norm on $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ will be denoted $\|\cdot\|$; the norm $\|\cdot\|_\rho^q$ on $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$, introduced in the previous section for any $\rho > 0$ (see Corollary 5.13 (ii)) will be denoted $\|\cdot\|_\rho$.

Lemma 5.15. (i) *$\|a\| \le \|a\|_1$ for any $a \in \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$.*

(ii) *For any $r < \sqrt{1-q^2}$ there is a $C_r > 0$ such that $\|a\|_r \le C_r\|a\|$ for any $a \in \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$.*

Proof. Since $\|\pi(a_i^+)\| \le (1-q^2)^{-1/2}$ (see above), we have $\|x_i\| \le 1$ for any $i = 1,\dots,n$. In view of Lemma 5.10, this yields (i).

To prove (ii), notice that by induction one easily deduces from (5.16) that

$$\pi(a_1^+)^{\alpha_1}\cdots\pi(a_n^+)^{\alpha_n} e_0 = \sqrt{[\alpha]!}\; q^{\sum_{i<j}\alpha_i\alpha_j} e_\alpha = \sqrt{[\alpha]!}\; w_q(\alpha) e_\alpha.$$

Therefore,

$$x^\alpha e_0 = \sqrt{[\alpha]!}\;(1-q^2)^{|\alpha|/2} w_q(\alpha) e_\alpha,$$

and for any $a = \sum_\alpha c_\alpha x^\alpha \in \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ we have an estimate

$$\|a\|^2 \ge \|a e_0\|^2 = \sum_\alpha |c_\alpha|^2 (1-q^2)^{|\alpha|} w_q^2(\alpha)[\alpha]! \ge \sup_\alpha |c_\alpha|^2(1-q^2)^{|\alpha|} w_q^2(\alpha).$$

It now follows that

$$\|a\|_r = \sum_\alpha |c_\alpha| w_q(\alpha) r^{|\alpha|} = \sum_\alpha \Bigl(\frac{r}{\sqrt{1-q^2}}\Bigr)^{|\alpha|} |c_\alpha| (1-q^2)^{|\alpha|/2} w_q(\alpha) \le C_r\|a\|,$$

where

$$C_r = \sum_\alpha \Bigl(\frac{r}{\sqrt{1-q^2}}\Bigr)^{|\alpha|} = \Bigl(\frac{1}{1 - r/\sqrt{1-q^2}}\Bigr)^n.$$

This proves (ii). ☐

Theorem 5.16. *For any $r < \sqrt{1-q^2}$ there exists a unique continuous homomorphism $\gamma_r \colon A(\overline{\mathbb{B}})_q \to A(\overline{\mathbb{B}})_q$ such that $\gamma_r(x_i) = r x_i$ for all $i = 1,\dots,n$. The inverse limit of the system of Banach algebras*

$$A(\overline{\mathbb{B}})_q \xleftarrow{\gamma_r} A(\overline{\mathbb{B}})_q \xleftarrow{\gamma_r} A(\overline{\mathbb{B}})_q \xleftarrow{\gamma_r} A(\overline{\mathbb{B}})_q \xleftarrow{\gamma_r} \cdots \tag{5.18}$$

is isomorphic to $\mathcal{O}_q(\mathbb{C}^n)$.

Proof. For any $s > 0$ consider a homomorphism

$$\tilde\gamma_s \colon \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n) \to \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n), \quad x_i \mapsto s x_i \quad (i = 1,\dots,n).$$

It is easy to see that

$$\|\tilde\gamma_s(a)\|_t = \|a\|_{st} \quad \bigl(a \in \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n),\ s,t > 0\bigr). \tag{5.19}$$

Fix an arbitrary $r < \sqrt{1-q^2}$. It follows from Lemma 5.15 and (5.19) that

$$\|\tilde\gamma_r(a)\| \le \|\tilde\gamma_r(a)\|_1 = \|a\|_r \le C_r\|a\|$$

for any $a \in \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$. This and the fact that $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ is dense in $A(\overline{\mathbb{B}})_q$ imply the existence and uniqueness of the homomorphism γ_r with the required properties.

For the sake of convenience, label the terms of (5.18) by nonnegative integers. More precisely, for any $k \in \mathbb{Z}_+$ set $A_k = A(\overline{\mathbb{B}})_q$ and define homomorphisms $\tau_k^{k+1} \colon A_{k+1} \to A_k$ by setting $\tau_k^{k+1} = \gamma_r$ for all $k \in \mathbb{Z}_+$. Set $A = \varprojlim(A_k, \tau_k^{k+1})$, and let τ_k denote the canonical morphism from A to A_k. Define a prenorm $\|\cdot\|_k$ on A by setting $\|a\|_k = \|\tau_k(a)\|$ for any $a \in A$. Notice that the sequence of prenorms $\{\|\cdot\|_k : k \in \mathbb{Z}_+\}$ defines the standard inverse limit topology on A.

Set $R = r^{-1}$ and for any $k \in \mathbb{Z}_+$ define a homomorphism

$$\varphi_k \colon \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n) \to A_k, \quad x_i \mapsto R^k x_i \quad (i = 1, \dots, n).$$

Clearly, $\varphi_k = \tau_k^{k+1}\varphi_{k+1}$ for any $k \in \mathbb{Z}_+$. Therefore, there exists a unique homomorphism $\varphi \colon \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n) \to A$ such that $\varphi_k = \tau_k \varphi$ for all $k \in \mathbb{Z}_+$. In turn, it gives rise to a homomorphism $\widehat{\varphi} \colon \mathcal{O}_q(\mathbb{C}^n) \to A$.

We shall show that $\widehat{\varphi}$ is an isomorphism. For this, we first notice that for any $k \in \mathbb{Z}_+$ the image of φ_k is dense; hence the same is true for φ. Therefore, to prove that $\widehat{\varphi}$ is an isomorphism, it suffices to check that φ is topologically injective.

Using (5.19) and Lemma 5.15, we have

$$\|a\|_{R^k} = \|\tilde{\gamma}_{R^{k+1}}(a)\|_r \le C_r \|\tilde{\gamma}_{R^{k+1}}(a)\| = C_r \|\varphi_{k+1}(a)\| = C_r \|\tau_{k+1}\varphi(a)\| = C_r \|\varphi(a)\|_{k+1}$$

for any $k \in \mathbb{Z}_+$ and any $a \in \mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$. Since the topology on $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ induced from $\mathcal{O}_q(\mathbb{C}^n)$ is defined by $\{\|\cdot\|_{R^k} : k \in \mathbb{Z}_+\}$, it follows from the last computation that φ is topologically injective. As we mentioned before, this implies that $\widehat{\varphi}$ is an isomorphism. $\square$

5.5. **Ore extensions: the "general" case. Quantum Weyl algebras and quantum matrices.** In the general case, i.e., when $\alpha \ne \mathbf{1}_R$ and $\delta \ne 0$, there does not seem to be a satisfactory description of the Arens–Michael envelope of the Ore extension $R[t; \alpha, \delta]$ as an algebra of "skew power series" (cf. Example 3.4). Below we shall describe a special but often encountered situation when such a description is in fact possible and we shall also illustrate it with several examples.

Suppose R is an algebra, $\alpha \colon R \to R$ is an endomorphism, and $\delta \colon R \to R$ is an α-derivation. As before (see Section 5.3), $\widehat{\alpha}$ will denote the (unique) endomorphism of $\widehat{R}$ such that

$$\widehat{\alpha}\iota_R = \iota_R \alpha. \tag{5.20}$$

Clearly, ι_R can be viewed as an R-bimodule morphism ${}_\alpha R \to {}_{\widehat{\alpha}}\widehat{R}$. Therefore, $\iota_R\delta \colon R \to {}_{\widehat{\alpha}}\widehat{R}$ is a derivation. It follows from Corollary 3.9 that there is a unique derivation $\widehat{\delta} \colon \widehat{R} \to {}_{\widehat{\alpha}}\widehat{R}$, i.e., a unique $\widehat{\alpha}$-derivation of $\widehat{R}$ such that

$$\widehat{\delta}\iota_R = \iota_R \delta. \tag{5.21}$$

In the next theorem, as in Section 4.1, $z \in \mathcal{O}(\mathbb{C}, \widehat{R})$ denotes the function sending $\lambda \in \mathbb{C}$ to $\lambda 1_{\widehat{R}} \in \widehat{R}$.

Theorem 5.17. *Suppose that the family $\{\widehat{\alpha}, \widehat{\delta}\}$ is m-localizable. Then there exists a unique R-homomorphism*

$$\iota_{R[t;\alpha,\delta]} \colon R[t; \alpha, \delta] \to \mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta}) \quad \text{such that} \quad t \mapsto z. \tag{5.22}$$

The algebra $\mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta})$ with the homomorphism (5.22) is the Arens–Michael envelope of $R[t; \alpha, \delta]$.

Proof. The existence and uniqueness of the R-homomorphism (5.22) directly follows from the universal property of Ore extensions and relations (5.20) and (5.21). Furthermore, the algebra $\mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta})$ is Arens–Michael by Proposition 4.5. Suppose now that $\varphi\colon R[t;\alpha,\delta] \to A$ is a homomorphism to some Arens–Michael algebra A. Set $\psi = \varphi|_R$ and $x = \varphi(t)$. The homomorphism $\widehat{\psi}\colon \widehat{R} \to A$ extending ψ makes A an $\widehat{R}$-$\widehat{\otimes}$-algebra; moreover, since φ is a homomorphism, it follows from (5.20) and (5.21) that

$$x \cdot r = \widehat{\alpha}(r) \cdot x + \widehat{\psi}(\widehat{\delta}(r))$$

for any $r \in \operatorname{Im} \iota_R$. Since $\operatorname{Im} \iota_R$ is dense in $\widehat{R}$, the latter equality holds for any $r \in \widehat{R}$. Therefore, by Proposition 4.4 (see also Remark 4.6), there exists a unique $\widehat{R}$-homomorphism (or, equivalently, R-homomorphism) $\widehat{\varphi}\colon \mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta}) \to A$ such that $\widehat{\varphi}(z) = x$. This equality and the fact that $\widehat{\varphi}$ is an R-homomorphism is equivalent to the relation $\widehat{\varphi} \circ \iota_{R[t;\alpha,\delta]} = \varphi$. The rest is clear. $\square$

To apply the preceding theorem in concrete situations it is useful to have a purely algebraic condition guaranteeing that the family $\{\widehat{\alpha}, \widehat{\delta}\}$ is m-localizable. The next theorem gives one such possible condition of that kind.

Theorem 5.18. *Suppose A is an algebra with generators $x_1, \ldots, x_{n+1}$ and relations*

$$\begin{cases} x_i x_j = q_{ij} x_j x_i \quad (i, j \le n), \\ x_{n+1} x_i = \lambda_i x_i x_{n+1} \quad (2 \le i \le n), \\ x_{n+1} x_1 = \lambda_1 x_1 x_{n+1} + \mu x_1^{\gamma_1} \cdots x_n^{\gamma_n}, \end{cases} \tag{5.23}$$

where $(q_{ij})_{i,j=1}^n = \mathbf{q}$ is a multiplicatively antisymmetric $(n \times n)$-matrix, $\lambda_1, \ldots, \lambda_n, \mu \in \mathbb{C}$, and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n$. For any $j = 1, \ldots, n$ let $e_j \in \mathbb{Z}_+^n$ denote the element with 1 at the j-th place and 0 elsewhere. Define a function $c\colon \mathbb{Z}_+^n \times \mathbb{Z}_+^n \to \mathbb{C}^\times$ by (5.2) and assume that

$$\lambda_1 \ne c(\gamma, e_1), \quad \lambda_j = \frac{c(\gamma, e_j)}{q_{1j} c(e_j, \gamma)} \quad (j \ge 2). \tag{5.24}$$

Moreover, assume that

$$|\lambda_i| \le 1 \quad (i = 1, \ldots, n), \quad |q_{ij}| \ge 1 \quad (i < j). \tag{5.25}$$

Then the Arens–Michael envelope of A is isomorphic, as a locally convex space, to the space of entire functions

$$\mathcal{O}(\mathbb{C}^{n+1}) = \Big\{ a = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} c_\alpha x^\alpha : \|a\|_\rho = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} |c_\alpha| \rho^{|\alpha|} < \infty \ \forall \rho > 0 \Big\},$$

and the multiplication on it is uniquely determined by (5.23).

Proof. Suppose F is the free algebra with generators $\xi_1, \ldots, \xi_n$, and let $I \subset F$ be the two-sided ideal generated by the relations $\xi_i \xi_j = q_{ij} \xi_j \xi_i$ $(i, j = 1, \ldots, n)$. Set $R = F/I$, let π denote the quotient homomorphism from F to R, and set $y_i = \pi(\xi_i)$ $(i = 1, \ldots, n)$. As $R = \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$, the multiplication law (5.3) (with x replaced by y) holds on R.

It is easy to see that there are a unique endomorphism $\tilde{\sigma}$ and a unique $\tilde{\sigma}$-derivation $\tilde{\delta}$ of F such that

$$\tilde{\sigma}(\xi_i) = \lambda_i \xi_i \quad (1 \le i \le n), \quad \tilde{\delta}(\xi_1) = \mu \xi^\gamma, \quad \tilde{\delta}(\xi_k) = 0 \quad (k \ge 2).$$

Clearly, $\tilde{\sigma}(I) \subset I$. We shall show that $\tilde{\delta}(I) \subset I$. Indeed,

$$\tilde{\delta}(\xi_i \xi_j - q_{ij} \xi_j \xi_i) = 0 \quad \text{when} \quad i, j \ge 2,$$

and

$$\pi\tilde{\delta}(\xi_1\xi_j - q_{1j}\xi_j\xi_1) = \pi\big(\tilde{\delta}(\xi_1)\xi_j - q_{1j}\tilde{\sigma}(\xi_j)\tilde{\delta}(\xi_1)\big) = \mu\pi\big(\xi^\gamma\xi_j - q_{1j}\lambda_j\xi_j\xi^\gamma\big)$$
$$= \mu\big(c(\gamma, e_j) - q_{1j}\lambda_j c(e_j, \gamma)\big)y^{e_j+\gamma}$$

for any $j \geq 2$. But the latter expression is zero by (5.24). In view of the relation $q_{j1} = q_{1j}^{-1}$, this yields $\tilde{\delta}(I) \subset I$.

Thus $\tilde{\sigma}$ and $\tilde{\delta}$ fix I and therefore give rise to linear maps $\sigma, \delta\colon R \to R$. Moreover, it is easy to see that σ is an endomorphism and δ is a σ-derivation of R.

Consider the Ore extension $B = R[y_{n+1}; \sigma, \delta]$. The constructions of R, σ, and δ show that $y_1, \ldots, y_{n+1}$ satisfy the same relations (5.23) as the x_i. Therefore, there exists a unique homomorphism $A \to B$ such that $x_i \mapsto y_i$ $(i = 1, \ldots, n+1)$. On the other hand, the universal property of Ore extensions yields a unique homomorphism $B \to A$ such that $y_i \mapsto x_i$ $(i = 1, \ldots, n+1)$. This implies that A and B are isomorphic. Henceforth we shall assume that $A = B$ and $x_i = y_i$ $(i = 1, \ldots, n+1)$.

We shall show that σ and δ satisfy the conditions of Theorem 5.17. Since $R = \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$ and $|q_{ij}| \geq 1$ for $i < j$, it follows from Corollary 5.13 that

$$\widehat{R} = \Big\{ r = \sum_{\beta \in \mathbb{Z}_+^n} c_\beta x^\beta : \|r\|_\rho = \sum_{\beta \in \mathbb{Z}_+^n} |c_\beta| \rho^{|\beta|} < \infty \ \forall \rho > 0 \Big\},$$

and all the prenorms $\|\cdot\|_\rho$ $(\rho > 0)$ are submultiplicative. Since $\widehat{\sigma}(x_i) = \lambda_i x_i$ and $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$, we have

$$\|\widehat{\sigma}(r)\|_\rho \leq \|r\|_\rho \tag{5.26}$$

for all $r \in \widehat{R}$ and all $\rho > 0$.

It is easily checked by induction that

$$\delta(x_1^m) = \mu \frac{c(\gamma, e_1)^m - \lambda_1^m}{c(\gamma, e_1) - \lambda_1} x^{(m-1)e_1 + \gamma}$$

for all $m \in \mathbb{Z}_+$. This and the fact that $\delta(x_k) = 0$ for $k > 1$ imply that

$$\delta(x^\beta) = \delta(x_1^{\beta_1})x_2^{\beta_2} \cdots x_n^{\beta_n} = \mu \frac{c(\gamma, e_1)^{\beta_1} - \lambda_1^{\beta_1}}{c(\gamma, e_1) - \lambda_1} x^{(\beta_1 - 1)e_1 + \gamma} x_2^{\beta_2} \cdots x_n^{\beta_n} \tag{5.27}$$

for all $\beta \in \mathbb{Z}_+^n$, $\beta_1 > 0$, and

$$\delta(x^\beta) = 0 \quad \text{when} \quad \beta_1 = 0. \tag{5.28}$$

Furthermore, the condition $|q_{ij}| \geq 1$ for $i > j$ and (5.2) imply that $|c(\gamma, e_1)| \leq 1$. Since $|\lambda_1| \leq 1$, it follows from (5.27) and (5.28) that

$$\|\delta(x^\beta)\|_\rho \leq C_\rho \rho^{|\beta|}, \quad \text{where} \quad C_\rho = \frac{2\mu\rho^{|\gamma|-1}}{|c(\gamma, e_1) - \lambda_1|}\rho^{|\beta|},$$

for all $\beta \in \mathbb{Z}_+^n$ and all $\rho > 0$. This and the explicit description of $\|\cdot\|_\rho$ imply that

$$\|\widehat{\delta}(r)\|_\rho \leq C_\rho \|r\|_\rho$$

for all $r \in \widehat{R}$ and all $\rho > 0$. Comparing this with (5.26), we have that the family $\{\widehat{\sigma}, \widehat{\delta}\}$ is m-localizable. To finish the proof, apply Theorem 5.17 and the canonical isomorphism of locally convex spaces

$$\mathcal{O}(\mathbb{C}, \widehat{R}) \cong \mathcal{O}(\mathbb{C}, \mathcal{O}(\mathbb{C}^n)) \cong \mathcal{O}(\mathbb{C} \times \mathbb{C}^n) = \mathcal{O}(\mathbb{C}^{n+1}). \qquad \square$$

Example 5.5 (quantum Weyl algebras)**.** Recall (see, for example, [20]) that the *quantum Weyl algebra* (with parameter $q \in \mathbb{C} \setminus \{0\}$) is the algebra $A_1(q)$ with generators x, ∂ and relation $\partial x - qx\partial = 1$. When $q = 1$ we have the usual Weyl algebra (see Example 3.4).

Suppose that $|q| \le 1$ and $q \ne 1$. Then it is not difficult to see that $A_1(q)$ satisfies the conditions of Theorem 5.18; moreover, $n = 1$ (and therefore $c \equiv 1$), $\lambda_1 = q$, $\mu = 1$, and $\gamma = 0$. If $|q| \ge 1$ but still $q \ne 1$, then it is convenient for our purposes to represent $A_1(q)$ as the algebra with generators ∂, x and relation $x\partial - q^{-1}\partial x = -q^{-1}$. It also satisfies the conditions of Theorem 5.18 with $n = 1$, $\lambda_1 = q^{-1}$, $\mu = -q^{-1}$, and $\gamma = 0$. Applying this theorem, we have the following description of the Arens–Michael envelope of the quantum Weyl algebra.

Corollary 5.19. *Suppose $q \in \mathbb{C} \setminus \{0\}$ and $q \ne 1$.*

(i) *If $|q| \le 1$, then*

$$A_1(q)\widehat{\ } = \left\{ a = \sum_{i,j=0}^{\infty} c_{ij} x^i \partial^j : \|a\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^{i+j} < \infty \ \forall \rho > 0 \right\}.$$

(ii) *If $|q| \ge 1$, then*

$$A_1(q)\widehat{\ } = \left\{ a = \sum_{i,j=0}^{\infty} c_{ij} \partial^i x^j : \|a\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^{i+j} < \infty \ \forall \rho > 0 \right\}.$$

In both cases the topology on $A_1(q)\widehat{\ }$ is defined by the system of prenorms $\{\|\cdot\|_\rho : \rho > 0\}$, and the multiplication is uniquely determined by the relation $\partial x - qx\partial = 1$.

Remark 5.3. Thus we see that for all $q \ne 1$, the Arens–Michael envelopes of the corresponding quantum Weyl algebras are similar. In particular, their underlying LCS are isomorphic to the space of entire functions $\mathcal{O}(\mathbb{C}^2)$, and the canonical morphism from $A_1(q)$ to $A_1(q)\widehat{\ }$ is injective. The case $q = 1$ is thus degenerate: recall that $A_1(1)\widehat{\ } = \{0\}$ (see Example 3.4).

Remark 5.4. It is not difficult to check that if q is not a root of unity, then $A_1(q)$ can be realized as the algebra of q-differentiable operators on $\mathbb{C}[x]$ with polynomial coefficients:

$$A_1(q) = \left\{ a = \sum_{i=0}^{N} a_n(x) D_q^n : a_n \in \mathbb{C}[x],\ N \in \mathbb{Z}_+ \right\},$$

where the q-derivative D_q is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{qx - x} \quad (f \in \mathbb{C}[x]).$$

On the other hand, it is easy to check that when $|q| \le 1$, $q \ne 1$, the operator D_q can be viewed as a bounded operator on the Hardy space H^2. Therefore, if $|q| \le 1$ and is not a root of unity, then the quantum Weyl algebra $A_1(q)$ has a faithful representation $\pi \colon A_1(q) \to \mathcal{B}(H^2)$, which is uniquely determined by the conditions

$$(\pi(x)f)(z) = zf(z), \quad (\pi(\partial)f)(z) = (D_q f)(z) \quad (f \in H^2,\ |z| < 1).$$

In particular, for the above values of q we have a simple proof that the Arens–Michael envelope of $A_1(q)$ is not trivial, which is not based on Theorem 5.18.

Example 5.6 (quantum matrices)**.** Suppose $q \in \mathbb{C} \setminus \{0\}$. Recall (see, for example, [31]), that the *algebra of quantum* (2×2)*-matrices* $M_q(2)$ has generators a, b, c, d and relations

$$\begin{aligned} &ab = qba, && ac = qca, && bc = cb, \\ &bd = qdb, && cd = qdc, && ad - da = (q - q^{-1})bc. \end{aligned} \tag{5.29}$$

Set $x_1 = a$, $x_2 = b$, $x_3 = c$, and $x_4 = d$. Then it is not difficult to see that $M_q(2)$ belongs to the type of algebras considered in Theorem 5.18. Moreover, $n = 3$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = q^{-1}$, $\mu = q^{-1} - q$, $\gamma = (0, 1, 1)$, and

$$\mathbf{q} = \begin{pmatrix} 1 & q & q \\ q^{-1} & 1 & 1 \\ q^{-1} & 1 & 1 \end{pmatrix}.$$

It is easy to see that

$$c(\gamma, e_1) = q^{-2}, \quad c(\gamma, e_2) = c(e_2, \gamma) = c(\gamma, e_3) = c(e_3, \gamma) = 1,$$

so that conditions (5.24) hold if and only if $q \neq \pm 1$. It is also clear that conditions (5.25) hold if and only if $|q| \geq 1$.

To cover the case $|q| \leq 1$, consider the algebra $M_{q^{-1}}(2)$. To distinguish its standard generators from the generators a, b, c, d of $M_q(2)$, denote them a', b', c', d', respectively. Then it is easy to see that there is an isomorphism $M_q(2) \to M_{q^{-1}}(2)$ such that $a \mapsto d'$, $d \mapsto a'$, $b \mapsto b'$, $c \mapsto c'$.

Corollary 5.20. *Suppose $q \in \mathbb{C} \setminus \{0\}$.*

(i) *If $|q| \geq 1$, then*

$$M_q(2)\widehat{\ } = \left\{ u = \sum_{i,j,k,l=0}^{\infty} c_{ijkl} a^i b^j c^k d^l : \|u\|_\rho = \sum_{i,j,k,l=0}^{\infty} |c_{ijkl}| \rho^{i+j+k+l} < \infty \ \forall \rho > 0 \right\}.$$

(ii) *If $|q| \leq 1$, then*

$$M_q(2)\widehat{\ } = \left\{ u = \sum_{i,j,k,l=0}^{\infty} c_{ijkl} d^i b^j c^k a^l : \|u\|_\rho = \sum_{i,j,k,l=0}^{\infty} |c_{ijkl}| \rho^{i+j+k+l} < \infty \ \forall \rho > 0 \right\}.$$

In both cases the topology on $M_q(2)\widehat{\ }$ is defined by the system of prenorms $\{\|\cdot\|_\rho : \rho > 0\}$, and the multiplication is uniquely determined by relations (5.29).

Proof. In view of the above, in the case $q \neq \pm 1$ the proof reduces to applying Theorem 5.18. If $q = 1$, then $M_q(2) = \mathbb{C}[a, b, c, d]$, and there is nothing to prove (see Example 3.1). If $q = -1$, then it is not difficult to see that $M_q(2) = \mathcal{O}^{\mathrm{alg}}_{\mathbf{q}'}(\mathbb{C}^4)$, where

$$\mathbf{q}' = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{5.30}$$

Thus in the case $q = -1$ the proof reduces to applying Corollary 5.13. $\square$

5.6. **Skew Laurent extensions. Quantum tori.** Let R be an algebra, $\alpha \colon R \to R$ an automorphism, and $\widehat{\alpha}$ its canonical extension to $\widehat{R}$ (see (5.20)). Clearly, $\widehat{\alpha}$ is an automorphism of $\widehat{R}$ and $\widehat{\alpha}^{-1} = (\alpha^{-1})\widehat{\ }$.

Theorem 5.21. *Suppose that the family $\{\widehat{\alpha}, \widehat{\alpha}^{-1}\}$ is m-localizable. Then there exists a unique R-homomorphism*

$$\iota_{R[t,t^{-1};\alpha]} \colon R[t, t^{-1}; \alpha] \to \mathcal{O}(\mathbb{C}^\times, \widehat{R}; \widehat{\alpha}) \quad \text{such that} \quad t \mapsto z. \tag{5.31}$$

The algebra $\mathcal{O}(\mathbb{C}^\times, \widehat{R}; \widehat{\alpha})$ with the homomorphism (5.31) *is the Arens–Michael envelope of $R[t, t^{-1}; \alpha]$.*

In view of Propositions 4.15 and 4.14, the proof of this theorem is similar to the proof of Theorem 5.17.

Example 5.7. Suppose $\mathbf{q} = (q_{ij})_{i,j=1}^n$ is a multiplicatively antisymmetric $(n\times n)$-matrix. Recall that the *multiparameter quantum torus* (see, for example, [41, 20]) is the algebra $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)$ with generators $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$ and relations $x_i x_j = q_{ij} x_j x_i$ and $x_i x_i^{-1} = x_i^{-1} x_i = 1$ $(i, j = 1, \dots, n)$.

Corollary 5.22. *Let $\mathbf{q} = (q_{ij})_{i,j=1}^n$ be a multiplicatively antisymmetric $(n \times n)$-matrix such that $|q_{ij}| = 1$ for all i, j. Let $x_1, \dots, x_n$ denote the coordinates on $\mathbb{C}^n$. Then $\mathcal{O}\big((\mathbb{C}^\times)^n\big)$ admits a unique multiplication making it a $\widehat{\otimes}$-algebra such that $x_i x_j = q_{ij} x_j x_i$ and $x_i x_i^{-1} = x_i^{-1} x_i = 1$ for all $i, j = 1, \dots, n$. The algebra thus obtained, together with the tautological embedding of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)$ in it, is the Arens–Michael envelope of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)$. Its topology is defined by the family of submultiplicative prenorms*

$$\|a\|_\rho = \sum_{\alpha \in \mathbb{Z}^n} |c_\alpha| \rho^{|\alpha|} \quad (\rho > 0), \tag{5.32}$$

where $a = \sum_\alpha c_\alpha x^\alpha$ is the Laurent expansion of $a \in \mathcal{O}\big((\mathbb{C}^\times)^n\big)$.

Proof. We induct on n. When $n = 0$, the assertion is obvious. Suppose that it is true for matrices of order $n-1$. Set $\bar{\mathbf{q}} = (q_{ij})_{i,j=1}^{n-1}$ and $R = \mathcal{O}_{\bar{\mathbf{q}}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^{n-1}\big)$. It is not difficult to see that $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big) = R[x_n, x_n^{-1}; \alpha]$, where the automorphism $\alpha\colon R \to R$ is uniquely determined by the conditions $\alpha(x_i) = q_{ni} x_i$ $(i = 1, \dots, n-1)$. The induction assumption and the condition $|q_{ni}| = 1$ imply that $\|\widehat{\alpha}(a)\|_\rho = \|a\|_\rho$ for any $a \in \widehat{R}$ and any $\rho > 0$. It remains to apply Theorem 5.21, Proposition 4.16 and the canonical isomorphism of locally convex spaces

$$\mathcal{O}(\mathbb{C}^\times, \widehat{R}) \cong \mathcal{O}\big(\mathbb{C}^\times, \mathcal{O}\big((\mathbb{C}^\times)^{n-1}\big)\big) \cong \mathcal{O}\big(\mathbb{C}^\times \times (\mathbb{C}^\times)^{n-1}\big) = \mathcal{O}\big((\mathbb{C}^\times)^n\big). \qquad \square$$

Definition 5.1. The Arens–Michael envelope of $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)$ described in the preceding proposition will be denoted $\mathcal{O}_{\mathbf{q}}\big((\mathbb{C}^\times)^n\big)$.

When the condition $|q_{ij}| = 1$ fails for at least one pair of indices i, j, the Arens–Michael envelope of the quantum torus degenerates:

Proposition 5.23. *Suppose that $|q_{ij}| \neq 1$ for some i, j. Then $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)^\frown = 0$.*

Proof. It suffices to show that there are no homomorphisms from $\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)$ to nonzero Banach algebras. Suppose that $\varphi\colon \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big) \to A$ is such a homomorphism. Set $u = \varphi(x_i)$, $v = \varphi(x_j)$, and $q = q_{ij}$. The spectrum of an arbitrary element $a \in A$ will be denoted $\sigma(a)$. We have that $u, v \in A$ are invertible and $uv = qvu$. Therefore, setting $K = \sigma(uv)$ and using the fact that $\sigma(uv) = \sigma(vu)$ (see, for example, [24, 2.1.8]), we have that $K = qK$. Since K is a nonempty compact and $|q| \neq 1$, the last relation is only possible when $K = \{0\}$. But this is also impossible since uv is invertible. The contradiction obtained shows that $A = 0$. $\square$

6. Homological epimorphisms

Let $\mathcal{A}, \mathcal{B}$ be categories and $F\colon \mathcal{A} \to \mathcal{B}$ a covariant functor. Recall that F is said to be *fully faithful* if it is full and faithful (see [39] for terminology), i.e., if it gives rise to a bijection between $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$ for any $X, Y \in \mathrm{Ob}\,(\mathcal{A})$. In other words, F is fully faithful if and only if it gives rise to an equivalence between $\mathcal{A}$ and a full subcategory of $\mathcal{B}$.

Proposition 6.1. *Let $f\colon A \to B$ be a homomorphism of $\widehat{\otimes}$-algebras. The following conditions are equivalent:*

(i) *f is an epimorphism in the category of $\widehat{\otimes}$-algebras;*

(ii) *the functor* $f^\bullet\colon B\text{-}\mathsf{mod}\to A\text{-}\mathsf{mod}$ *is fully faithful;*
(iii) *the canonical morphism* $B\mathbin{\widehat{\otimes}}_A B\to B$ *is an isomorphism of LCS;*
(iv) *for any* $X\in\mathrm{Ob}\,(\mathsf{mod}\text{-}B)$ *and any* $Y\in\mathrm{Ob}\,(B\text{-}\mathsf{mod})$ *the canonical morphism* $X\mathbin{\widehat{\otimes}}_A Y\to X\mathbin{\widehat{\otimes}}_B Y$ *is an isomorphism of LCS.*

Proof. The proof that (i)$\iff$(ii)$\iff$(iii) is a verbatim repetition of the proof given in [64, XI.1] for the category of rings. The implication (iv)$\implies$(iii) is obvious, and (iv) can be obtained from (iii) by tensoring B with X on the left and with Y on the right. □

Remark 6.1. In [67], $\widehat{\otimes}$-algebra epimorphisms are called *pseudo-quotient* homomorphisms.

Lemma 6.2. *Suppose* $\mathcal{A},\mathcal{B}$ *are exact categories and* $F\colon\mathcal{A}\to\mathcal{B}$ *is an exact additive covariant functor. The following conditions are equivalent:*
(i) $\mathrm{R}F\colon\mathsf{D}^b(\mathcal{A})\to\mathsf{D}^b(\mathcal{B})$ *is fully faithful;*
(ii) *for any* $X,Y\in\mathrm{Ob}\,(\mathcal{A})$ *and any* $n\in\mathbb{Z}_+$ *the canonical morphism* $\mathrm{Ext}^n_{\mathcal{A}}(X,Y)\to\mathrm{Ext}^n_{\mathcal{B}}(F(X),F(Y))$ *is a bijection.*

The proof of this lemma given in [17] for abelian categories carries over verbatim to exact categories.

Theorem 6.3. *Suppose* $f\colon A\to B$ *is an* R-S*-homomorphism from an* R-$\widehat{\otimes}$*-algebra* A *to an* S-$\widehat{\otimes}$*-algebra* B*. The following conditions are equivalent:*
(1) $\mathrm{R}f^\bullet\colon\mathsf{D}^b((B,S)\text{-}\mathsf{mod})\to\mathsf{D}^b((A,R)\text{-}\mathsf{mod})$ *is fully faithful;*
(2) *for any* $X,Y\in\mathrm{Ob}\,((B,S)\text{-}\mathsf{mod})$ *and any* $n\in\mathbb{Z}_+$ *the canonical morphism* $\mathrm{Ext}^n_{B,S}(X,Y)\to\mathrm{Ext}^n_{A,R}(X,Y)$ *is bijective;*
(3) *for any* $X\in\mathrm{Ob}\,(\mathsf{D}^b((B,S)\text{-}\mathsf{mod}))$ *the canonical morphism*

$$B\mathbin{\widehat{\otimes}}^{\mathrm{L}}_A(\mathrm{R}f^\bullet(X))\to X$$

is an isomorphism in $\mathsf{D}^b((B,S)\text{-}\mathsf{mod})$*;*
(4) f *is an epimorphism of* $\widehat{\otimes}$*-algebras and, moreover, any* $X\in\mathrm{Ob}\,((B,S)\text{-}\mathsf{mod})$*, viewed as an object of* $(A,R)\text{-}\mathsf{mod}$*, is acyclic relative to the functor*

$$B\mathbin{\widehat{\otimes}}_A(\,\cdot\,)\colon(A,R)\text{-}\mathsf{mod}\to(B,S)\text{-}\mathsf{mod}\,.$$

Proof. (1)$\iff$(2): This is a particular case of Lemma 6.2.

(1)$\iff$(3): Since $B\mathbin{\widehat{\otimes}}_A(\,\cdot\,)$ is left adjoint to $f^\bullet\colon(B,S)\text{-}\mathsf{mod}\to(A,R)\text{-}\mathsf{mod}$, the same is true for their derived functors [32, Lemma 15.6]. It remains to apply Theorem 4.3.1 from [39].

(3)$\implies$(4): Taking into account that $f^\bullet$ is exact, it follows from (3) that for any $X\in\mathrm{Ob}\,((B,S)\text{-}\mathsf{mod})$, the canonical morphism $B\mathbin{\widehat{\otimes}}^{\mathrm{L}}_A X\to X$ is an isomorphism in $\mathsf{D}^b((B,S)\text{-}\mathsf{mod})$ and therefore in $\mathsf{D}^b(\mathsf{LCS})$. Applying to it the functor H^0, we have that the composition of the canonical morphisms

$$\mathrm{Tor}^{A,R}_0(B,X)\to B\mathbin{\widehat{\otimes}}_A X\to X$$

is an isomorphism in LCS. Since the first morphism in this composition is the canonical map from an LCS to its completion (see Section 2), we conclude that both morphisms are isomorphisms in LCS. Therefore, $B\mathbin{\widehat{\otimes}}_A X\to X$ is an isomorphism in B-mod. The rest is clear.

(4)$\implies$(2): Suppose $P_\bullet$ is a projective resolution of X in (A,R)-mod. By (4), $B\mathbin{\widehat{\otimes}}_A P_\bullet$ is a projective resolution of X in (B,S)-mod. Therefore

$$\mathrm{Ext}^n_{A,R}(X,Y)\cong H^n({}_A\mathbf{h}(P_\bullet,Y))\cong H^n\big({}_B\mathbf{h}(B\mathbin{\widehat{\otimes}}_A P_\bullet,Y)\big)\cong\mathrm{Ext}^n_{B,S}(X,Y).\qquad\square$$

Definition 6.1. If the homomorphism $f\colon A \to B$ satisfies the conditions of the preceding theorem, then we shall call it a *left relative homological epimorphism.* The homomorphism $f\colon A \to B$ is called a *right relative homological epimorphism* if $f\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ is a left relative homological epimorphism.

Lemma 6.4. *Suppose $f\colon A \to B$ is an R-S-homomorphism from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B which is an epimorphism of $\widehat{\otimes}$-algebras. Let $\mathcal{A}$ denote any of the exact categories*

$$(A,R)\text{-}\mathsf{mod}\text{-}(A,R),\ (A,A)\text{-}\mathsf{mod}\text{-}(A,R),\ (A,R)\text{-}\mathsf{mod}\text{-}(A,A),$$

and $\mathcal{B}$ any of the exact categories

$$(B,S)\text{-}\mathsf{mod}\text{-}(B,S),\ (B,B)\text{-}\mathsf{mod}\text{-}(B,S),\ (B,S)\text{-}\mathsf{mod}\text{-}(B,B).$$

Consider the functor

$$T = B \mathbin{\widehat{\otimes}_A} (\,\cdot\,) \mathbin{\widehat{\otimes}_A} B\colon \mathcal{A} \to \mathcal{B}. \tag{6.1}$$

Then, if A is T-acyclic for some choice of the exact categories $\mathcal{A}$and $\mathcal{B}$, then it is also T-acyclic for any other choice of those categories.

Proof. Suppose that (2.5) is a projective resolution of A in (A,R)-mod-(A,R). In view of Proposition 2.6, it is also a projective resolution of A in (A,R)-mod-(A,A) and in (A,A)-mod-(A,R). Since f is an epimorphism, it is easy to see that $T(A) \cong B$. Moreover, objects of $T(P_\bullet)$ are projective in $\mathcal{B}$ for any choice of $\mathcal{B}$. Therefore, again applying Proposition 2.6, we have that the admissibility of the complex

$$0 \leftarrow B = T(B) \leftarrow T(P_\bullet)$$

in $\mathcal{B}$ does not depend on the choice of $\mathcal{B}$. It remains to notice that, in view of the above, the latter complex is admissible in $\mathcal{B}$ if and only if A is T-acyclic. $\square$

Definition 6.2. An R-S-homomorphism $f\colon A \to B$ from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B is called a *two-sided relative homological epimorphism* if f is an epimorphism of $\widehat{\otimes}$-algebras and A is acyclic relative to functor (6.1) from Lemma 6.4.

Proposition 6.5. *Any two-sided relative homological epimorphism $f\colon A \to B$ from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B is a left and a right relative homological epimorphism.*

Proof. Fix a projective resolution

$$0 \leftarrow A \leftarrow P_\bullet \tag{6.2}$$

in (A,R)-mod-(A,R). In view of Proposition 2.6, the augmented complex (6.2) is split in R-mod-A. Therefore, for any $X \in \mathrm{Ob}\,((B,S)\text{-}\mathsf{mod})$

$$0 \leftarrow X \leftarrow P_\bullet \mathbin{\widehat{\otimes}_A} X \tag{6.3}$$

is a projective resolution of X in (A,R)-mod. On the other hand, it follows from the assumption that the complex

$$0 \leftarrow B \leftarrow Q_\bullet = B \mathbin{\widehat{\otimes}_A} P_\bullet \mathbin{\widehat{\otimes}_A} B$$

is split in S-mod-B, and therefore the complex

$$0 \leftarrow X \leftarrow Q_\bullet \mathbin{\widehat{\otimes}_B} X$$

is split in S-mod. Taking account of the isomorphism $B \mathbin{\widehat{\otimes}_A} X \cong X$, we have that the latter complex is isomorphic to the complex $B \mathbin{\widehat{\otimes}_A} (6.3)$. Thus $B \mathbin{\widehat{\otimes}_A} (6.3)$ is split in S-mod, and therefore the conditions of Theorem 6.3, (4) hold. This shows that f is a left relative homological epimorphism. A similar argument shows that f is a right relative homological epimorphism. $\square$

Example 6.1. Fix a $\widehat{\otimes}$-algebra A and set $B = S = A$, $R = \mathbb{C}$, and $f = \mathbf{1}_A$. Then it is easy to see that f is a left (resp., right) relative homological epimorphism if and only if $\operatorname{dg} A = 0$ (resp., $\operatorname{dg} A^{\mathrm{op}} = 0$). On the other hand, f is a two-sided relative homological epimorphism if and only if $\operatorname{db} A = 0$.

The next proposition shows that one-sided relative homological epimorphisms do inherit some of the properties of two-sided ones.

Proposition 6.6. *Suppose $f\colon A \to B$ is an R-S-homomorphism from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B and that f is a left or right relative homological epimorphism. Then A is acyclic relative to the functors*

$$\begin{aligned} T_\ell &= B \mathbin{\widehat{\otimes}}_A (\,\cdot\,)\colon (A,R)\text{-}\mathsf{mod}\text{-}(A,R) \to \mathsf{LCS}_{\mathrm{spl}},\\ T_r &= (\,\cdot\,) \mathbin{\widehat{\otimes}}_A B\colon (A,R)\text{-}\mathsf{mod}\text{-}(A,R) \to \mathsf{LCS}_{\mathrm{spl}},\\ T &= B \mathbin{\widehat{\otimes}}_A (\,\cdot\,) \mathbin{\widehat{\otimes}}_A B\colon (A,R)\text{-}\mathsf{mod}\text{-}(A,R) \to \mathsf{LCS}_{\mathrm{spl}}. \end{aligned}$$

Proof. Fix a projective resolution $P_\bullet$ of A in (A,R)-mod-(A,R). By Proposition 2.6, augmented complex (2.5) is split in R-mod-A. Therefore the complex

$$0 \leftarrow B \leftarrow P_\bullet \mathbin{\widehat{\otimes}}_A B \tag{6.4}$$

is split in R-mod-B and, in particular, in LCS. This proves that A is T_r-acyclic. A similar argument shows that the complex

$$0 \leftarrow B \leftarrow B \mathbin{\widehat{\otimes}}_A P_\bullet \tag{6.5}$$

is split in B-mod-R and, in particular, in LCS, whence the T_ℓ-acyclicity of A. Furthermore, notice that (6.4) (resp., (6.5)) without the term on the left is a projective resolution of B in (A,R)-mod (resp., in mod-(A,R)). Therefore if f is a left relative homological epimorphism, then, applying the functor $B \mathbin{\widehat{\otimes}}_A (\,\cdot\,)$ to (6.4), we have a complex

$$0 \leftarrow B \leftarrow B \mathbin{\widehat{\otimes}}_A P_\bullet \mathbin{\widehat{\otimes}}_A B, \tag{6.6}$$

which is split in S-mod. If f is a right relative homological epimorphism, then, applying the functor $(\,\cdot\,) \mathbin{\widehat{\otimes}}_A B$ to (6.5), we have that (6.6) is split in mod-S. Thus in both cases (6.6) is split in LCS, which means that A is T-acyclic. ◻

Remark 6.2. In fact, the proof of the preceding proposition reduces to the analysis of the diagram

$$\begin{tikzcd} (A,A)\text{-}\mathsf{mod}\text{-}(A,R) \arrow[r, "T_\ell", "\text{(exact)}"'] & \mathsf{mod}\text{-}(A,R) \arrow[r, "T_r"] & \mathsf{mod}\text{-}(B,S) \arrow[d, "\text{(exact)}"] \\ (A,R)\text{-}\mathsf{mod}\text{-}(A,R) \arrow[u, "\text{(A-acyclic)}"] \arrow[d, "\text{(A-acyclic)}"'] \arrow[rr, "T"] & & \mathsf{LCS}_{\mathrm{spl}} \\ (A,R)\text{-}\mathsf{mod}\text{-}(A,A) \arrow[r, "T_r", "\text{(exact)}"'] & (A,R)\text{-}\mathsf{mod} \arrow[r, "T_\ell"] & (B,S)\text{-}\mathsf{mod} \arrow[u, "\text{(exact)}"'] \end{tikzcd}$$

Here the vertical arrows are functors acting identically on objects and morphisms. Starting with an object $A \in (A,R)$-mod-(A,R) and moving along the arrows of the diagram, at each step we have an object which is acyclic relative to the preceding arrow. Moreover, the right half of the top row corresponds to the case when f is a right relative homological epimorphism, and the right half of the bottom row, to the case when f is a left relative homological epimorphism.

The next theorem shows that in the case $R = S = \mathbb{C}$ the possible distinctions between left, right, and two-sided homological epimorphisms disappear.

Theorem 6.7. *item Suppose $f\colon A\to B$ is a homomorphism of $\widehat{\otimes}$-algebras. The following conditions are equivalent:*

(1) $\mathrm{R}f^\bullet\colon \mathsf{D}^b(B\text{-}\mathsf{mod})\to \mathsf{D}^b(A\text{-}\mathsf{mod})$ *is fully faithful;*

(1′) $\mathrm{R}f^\bullet\colon \mathsf{D}^b(\mathsf{mod}\text{-}B)\to \mathsf{D}^b(\mathsf{mod}\text{-}A)$ *is fully faithful;*

(2) *for any $X,Y\in \mathrm{Ob}\,(B\text{-}\mathsf{mod})$ and any $n\in\mathbb{Z}_+$ the canonical morphism*

$$\mathrm{Ext}\,^n_B(X,Y)\to \mathrm{Ext}\,^n_A(X,Y)$$

is bijective;

(2′) *for any $X,Y\in \mathrm{Ob}\,(\mathsf{mod}\text{-}B)$ and any $n\in\mathbb{Z}_+$ the canonical morphism*

$$\mathrm{Ext}\,^n_B(X,Y)\to \mathrm{Ext}\,^n_A(X,Y)$$

is bijective;

(3) *for any $Y\in \mathrm{Ob}\,(B\text{-}\mathsf{mod})$ the canonical morphism $Y\to {}_A\mathbf{h}(B,Y)$ is bijective, and $\mathrm{Ext}\,^n_A(B,Y)=0$ for all $n\ge 1$;*

(3′) *for any $Y\in \mathrm{Ob}\,(\mathsf{mod}\text{-}B)$ the canonical morphism $Y\to \mathbf{h}_A(B,Y)$ is bijective, and $\mathrm{Ext}\,^n_A(B,Y)=0$ for all $n\ge 1$;*

(4) *for any $X\in \mathrm{Ob}\,(\mathsf{D}^b(B\text{-}\mathsf{mod}))$ the canonical morphism*

$$B\mathbin{\widehat{\otimes}}^{\mathrm{L}}_A(\mathrm{R}f^\bullet(X))\to X$$

is an isomorphism in $\mathsf{D}^b(B\text{-}\mathsf{mod})$;

(4′) *for any $X\in \mathrm{Ob}\,(\mathsf{D}^b(\mathsf{mod}\text{-}B))$ the canonical morphism*

$$(\mathrm{R}f^\bullet(X))\mathbin{\widehat{\otimes}}^{\mathrm{L}}_A B\to X$$

is an isomorphism in $\mathsf{D}^b(\mathsf{mod}\text{-}B)$;

(5) *f is an epimorphism of $\widehat{\otimes}$-algebras and, furthermore, any $X\in \mathrm{Ob}\,(B\text{-}\mathsf{mod})$, as an object of A-mod, is acyclic relative to the functor*

$$B\mathbin{\widehat{\otimes}}_A(\,\cdot\,)\colon A\text{-}\mathsf{mod}\to B\text{-}\mathsf{mod};$$

(5′) *f is an epimorphism of $\widehat{\otimes}$-algebras and, furthermore, any $X\in \mathrm{Ob}\,(\mathsf{mod}\text{-}B)$, as an object of mod-A, is acyclic relative to the functor*

$$(\,\cdot\,)\mathbin{\widehat{\otimes}}_A B\colon\ \mathsf{mod}\text{-}A\to \mathsf{mod}\text{-}B;$$

(6) *f is an epimorphism of $\widehat{\otimes}$-algebras and, furthermore, B, as an object of A-mod, is acyclic relative to the functor*

$$B\mathbin{\widehat{\otimes}}_A(\,\cdot\,)\colon A\text{-}\mathsf{mod}\to B\text{-}\mathsf{mod};$$

(6′) *f is an epimorphism of $\widehat{\otimes}$-algebras and, furthermore, B, as an object of mod-A, is acyclic relative to the functor*

$$(\,\cdot\,)\mathbin{\widehat{\otimes}}_A B\colon\ \mathsf{mod}\text{-}A\to \mathsf{mod}\text{-}B;$$

(7) *f is an epimorphism of $\widehat{\otimes}$-algebras and, furthermore, A, as an object of A-mod-A, is acyclic relative to the functor*

$$B\mathbin{\widehat{\otimes}}_A(\,\cdot\,)\mathbin{\widehat{\otimes}}_A B\colon A\text{-}\mathsf{mod}\text{-}A\to B\text{-}\mathsf{mod}\text{-}B.$$

Proof. The implications (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) and (1′) $\Longleftrightarrow$ (2′) $\Longleftrightarrow$ (4′) $\Longleftrightarrow$ (5′) follow from Theorem 6.3.

For (5) $\Longleftrightarrow$ (5′) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (6′) $\Longleftrightarrow$ (7), see [48, 3.2].

(2) $\Longrightarrow$ (3) and (2′) $\Longrightarrow$ (3′) are obvious.

(3) $\Longrightarrow$ (6). As the functors $B\mathbin{\widehat{\otimes}}_A(\,\cdot\,)$ and $f^\bullet$ are adjoint, it follows from (3), that for any $Y\in \mathrm{Ob}\,(B\text{-}\mathsf{mod})$ we have natural isomorphisms

$${}_B\mathbf{h}(B,Y)\cong Y\cong {}_A\mathbf{h}(B,Y)\cong {}_B\mathbf{h}(B\mathbin{\widehat{\otimes}}_A B,Y).$$

It is easy to see that the composition of these isomorphisms is induced by the canonical morphism $B \mathbin{\widehat{\otimes}}_A B \to B$. Therefore it is an isomorphism in B-mod, i.e., f is an epimorphism of $\widehat{\otimes}$-algebras. Now we fix a projective resolution

$$0 \leftarrow B \leftarrow P_\bullet$$

in A-mod. It follows from (3) that for any $Y \in \mathrm{Ob}\,(B\text{-}\mathsf{mod})$ the complex

$$0 \to {}_A\mathbf{h}(B, Y) \to {}_A\mathbf{h}(P_\bullet, Y)$$

is exact. On the other hand, it is isomorphic to the complex

$$0 \to {}_B\mathbf{h}\big(B \mathbin{\widehat{\otimes}}_A B, Y\big) \to {}_B\mathbf{h}\big(B \mathbin{\widehat{\otimes}}_A P_\bullet, Y\big),$$

which is exact for any $Y \in \mathrm{Ob}\,(B\text{-}\mathsf{mod})$ if and only if the complex

$$0 \leftarrow B \mathbin{\widehat{\otimes}}_A B \leftarrow B \mathbin{\widehat{\otimes}}_A P_\bullet$$

is split in B-mod. Therefore, (6) holds. The implication $(3)' \implies (6)'$ is proved similarly. $\square$

Definition 6.3. A $\widehat{\otimes}$-algebra homomorphism $f\colon A \to B$ is called a *homological epimorphism* if it satisfies the equivalent conditions of Theorem 6.7. In this case we shall say (following [46]) that B is *stably flat* over A.

Remark 6.3. Theorem 6.7 is a "locally convex analog" of a purely algebraic Theorem 4.4 from [17]. We decided to give a detailed proof, since the proof in [17] does not carry over verbatim to the case of $\widehat{\otimes}$-algebras. Moreover, [17] does not mention conditions (4), (4′) and (7), and conditions (5), (5′), (6), and (6′) are stated in terms of the functor Tor, which is not enough in the $\widehat{\otimes}$-case.

Remark 6.4. If we state condition (7) of Theorem 6.7 in an explicit form, then we arrive at the following definition: *a homomorphism $A \to B$ is a homological epimorphism if and only if for some (or, equivalently, for any) projective resolution*

$$0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

of the bimodule A in A-mod-A the complex

$$0 \leftarrow B \leftarrow B \mathbin{\widehat{\otimes}}_A P_0 \mathbin{\widehat{\otimes}}_A B \leftarrow B \mathbin{\widehat{\otimes}}_A P_1 \mathbin{\widehat{\otimes}}_A B \leftarrow \cdots$$

is split in LCS. Thus a homological epimorphism is the same thing as the *absolute localization* in Taylor's terminology [67].

For examples of homological epimorphisms the reader is referred to [67, 68, 37, 26, 17, 46, 16, 15, 48, 44] (see also the introduction to the present paper). A basic example, obtained by Taylor [67, 4.3], is the embedding of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ with the strongest locally convex topology in the Fréchet algebra $\mathcal{O}(\mathbb{C}^n)$ of entire functions. Other examples will be given in Section 9 of this paper (see Theorem 9.12).

7. Relatively quasi-free algebras

Throughout this section R is a fixed $\widehat{\otimes}$-algebra. Suppose A is an R-$\widehat{\otimes}$-algebra. By an *R-extension* of A we shall understand an arbitrary open homomorphism of R-$\widehat{\otimes}$-algebras $\sigma\colon \mathfrak{A} \to A$, where $\mathfrak{A}$ is an R-$\widehat{\otimes}$-algebra. As usual, an extension will written as a sequence

$$0 \to I \xrightarrow{i} \mathfrak{A} \xrightarrow{\sigma} A \to 0, \tag{7.1}$$

where $I = \operatorname{Ker}\sigma$ and i is the tautological embedding. Notice that (7.1) is a topologically exact (see the Appendix, Example 10.5) sequence of R-$\widehat{\otimes}$-bimodules. Extension (7.1) is said to be *R-admissible* (henceforth, simply *admissible*) if this sequence is split in R-mod-R, i.e., if σ has a right inverse morphism of R-$\widehat{\otimes}$-bimodules. If σ has a right inverse morphism of R-$\widehat{\otimes}$-algebras, then extension (7.1) is said to be R-*split* (henceforth, simply *split*).

Recall that extension (7.1) is said to be *singular* if $I^2 = 0$. In this case I becomes an A-$\widehat{\otimes}$-bimodule when, for $a \in A$ and $x \in I$, we set $a \cdot x = ux$ and $x \cdot a = xu$ for arbitrary $u \in \sigma^{-1}(a)$. If X is an A-$\widehat{\otimes}$-bimodule, then *an R-extension of A by X* is, by definition, a topologically exact sequence

$$0 \to X \xrightarrow{\varkappa} \mathfrak{A} \xrightarrow{\sigma} A \to 0, \tag{7.2}$$

in which $\mathfrak{A}$ is an R-$\widehat{\otimes}$-algebra, σ is a homomorphism of R-$\widehat{\otimes}$-algebras, and $\varkappa$ is a morphism of $\mathfrak{A}$-$\widehat{\otimes}$-bimodules. Clearly, each R-extension of A by X gives rise to a singular R-extension of the form (7.1), and conversely, as follows from the above, any singular extension can be obtained this way; moreover, $\varkappa$ establishes an isomorphism between X and I in A-mod-A.

Two R-extensions $0 \to X \to \mathfrak{A} \to A \to 0$ and $0 \to X \to \mathfrak{A}' \to A \to 0$ of A by X are said to be *equivalent* if there is an isomorphism of R-$\widehat{\otimes}$-algebras $\varphi \colon \mathfrak{A} \to \mathfrak{A}'$ such that the diagram

$$\begin{tikzcd}
 & \mathfrak{A} \arrow[dd, "\varphi"] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & A \\
 & \mathfrak{A}' \arrow[ur] &
\end{tikzcd}$$

is commutative. As in the case $R = \mathbb{C}$, it is not difficult to check that the set of equivalence classes of admissible R-extensions of A by X is in one-to-one correspondence with $\mathcal{H}^2(A, R; X)$; moreover the zero element of $\mathcal{H}^2(A, R; X)$ corresponds to the equivalence class of a split extension.

Proposition 7.1. *The following properties of the R-$\widehat{\otimes}$-algebra A are equivalent:*

(i) $\mathrm{db}_R\, A \le 1$*;*
(ii) $\mathcal{H}^2(A, R; X) = 0$ *for any* $X \in A$-mod-A*;*
(iii) *any admissible singular R-extension of A is split;*
(iv) *for any R-$\widehat{\otimes}$-algebra B, any admissible singular R-extension* $0 \to I \to \mathfrak{A} \to B \to 0$*, and any R-homomorphism $\varphi \colon A \to B$ there is an R-homomorphism $\psi \colon A \to \mathfrak{A}$ making the diagram*

$$\begin{tikzcd}
 & & & A \arrow[d, "\varphi"] \arrow[dl, dashed, "\psi"'] & \\
0 \arrow[r] & I \arrow[r] & \mathfrak{A} \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

commute.

Proof. Clearly, (i)$\Longleftrightarrow$(ii) and (iv)$\Longrightarrow$(iii); furthermore, it follows from the above that (ii)$\Longleftrightarrow$(iii). To prove the implication (iii)$\Longrightarrow$(iv), take an arbitrary admissible singular R-extension $0 \to I \xrightarrow{i} \mathfrak{A} \xrightarrow{\sigma} B \to 0$ and consider its preimage under φ:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i'"] \arrow[d, equal] & \mathfrak{A} \times_B A \arrow[r, "\sigma'"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & I \arrow[r, "i"] & \mathfrak{A} \arrow[r, "\sigma"] & B \arrow[r] & 0
\end{tikzcd}$$

Here $\mathfrak{A} \times_B A = \{(u, a) \in \mathfrak{A} \times A : \sigma(u) = \varphi(a)\}$ (it is easy to see that this is an R-$\widehat{\otimes}$-algebra), $i'(a) = (i(a), 0)$, $\sigma'(u, a) = a$ and $\pi(u, a) = u$. The top row of this diagram is an admissible singular R-extension of A (its admissibility follows from Axiom Ex2 applied to the exact category $(R\text{-}\mathsf{mod}\text{-}R)_{\mathrm{spl}}$, see the Appendix, but it is also easy to check it directly). By (iii), there is an R-homomorphism $\psi' \colon A \to \mathfrak{A} \times_B A$ which is a right inverse of σ'. It remains to set $\psi = \pi\psi'$. $\square$

Definition 7.1. An R-$\widehat{\otimes}$-algebra A, satisfying the equivalent conditions of Proposition 7.1 is said to be *relatively quasi-free*. If $R = \mathbb{C}$, then A is said to be *quasi-free*.

To check whether an algebra is quasi-free it is convenient to use the bimodule of noncommutative differential 1-forms [7]. It is defined as follows. Fix an R-$\widehat{\otimes}$-algebra A and consider the functor $\operatorname{Der}_R(A, \cdot\,)$ from $A\text{-}\mathsf{mod}\text{-}A$ to the category of sets. As was shown by Cuntz and Quillen [7] (in a purely algebraic situation), this functor is representable. Its representing object is denoted $\Omega^1_R A$ and is called the *bimodule of relative differential* 1*-forms* of A. In other words, the bimodule of relative differential 1-forms is a pair $(\Omega^1_R A, d_A)$ consisting of an A-$\widehat{\otimes}$-bimodule $\Omega^1_R A$ and an R-derivation $d_A \colon A \to \Omega^1_R A$ such that for any A-$\widehat{\otimes}$-bimodule X and any R-derivation $D \colon A \to X$ there exists a unique morphism of A-$\widehat{\otimes}$-bimodules $\varphi \colon \Omega^1_R A \to X$ making the diagram

$$\begin{tikzcd} & \Omega^1_R A \arrow[d, dashed, "\varphi"'] \\ A \arrow[ur, "d_A"] \arrow[r, "D"'] & X \end{tikzcd} \tag{7.3}$$

commute. The derivation d_A is called the *universal R-derivation* of A.

To construct $\Omega^1_R A$, denote the completion of the quotient space $A/\overline{\operatorname{Im}\eta_A}$ by $\overline{A}$; clearly, this is an R-$\widehat{\otimes}$-bimodule. The image of $a \in A$ under the canonical map $A \to \overline{A}$ will be denoted $\bar{a}$. As a left A-$\widehat{\otimes}$-module, $\Omega^1_R A$ coincides with $A \mathbin{\widehat{\otimes}}_R \overline{A}$. To define a structure of a right A-$\widehat{\otimes}$-module on it, denote the elementary tensor $a_0 \otimes \bar{a}_1 \in A \mathbin{\widehat{\otimes}}_R \overline{A}$ by $a_0 da_1$; the tensor $1da$ will be simply written as da. It is not difficult to see that there is a unique continuous bilinear map $\Omega^1_R A \times A \to \Omega^1_R A$, $(\omega, a) \mapsto \omega \cdot a$, such that

$$a_0 da_1 \cdot a = a_0 d(a_1 a) - a_0 a_1 da. \tag{7.4}$$

This map defines a right action of A on $\Omega^1_R A$ and makes it an A-$\widehat{\otimes}$-bimodule. It follows from (7.4) that the map $d_A \colon A \to \Omega^1_R A$, $a \mapsto da$, is an R-derivation. If $D \colon A \to X$ is an arbitrary R-derivation of A with values in a bimodule X, then the unique morphism φ making diagram (7.3) commute is given by $\varphi(a_0 da_1) = a_0 \cdot D(a_1)$; it is straightforward to check that φ is a morphism of bimodules. Thus, the pair $(\Omega^1_R A, d_A)$ is indeed the bimodule of relative differential 1-forms.

Consider the sequence of A-$\widehat{\otimes}$-bimodules

$$0 \leftarrow A \xleftarrow{m} A \mathbin{\widehat{\otimes}}_R A \xleftarrow{j} \Omega^1_R A \leftarrow 0. \tag{7.5}$$

Here m is the multiplication on A, and j is the morphism of bimodules corresponding to the inner derivation $-\operatorname{ad}_{1\otimes 1} \colon A \to A \mathbin{\widehat{\otimes}}_R A$, $a \mapsto [1 \otimes 1, a]$. In other words, j is uniquely determined by the formula $j(a_0 da_1) = a_0(1 \otimes a_1 - a_1 \otimes 1)$. Clearly, $mj = 0$, i.e., (7.5) is a complex of A-bimodules.

Proposition 7.2. *Sequence* (7.5) *is split in $A\text{-}\mathsf{mod}\text{-}R$ and $R\text{-}\mathsf{mod}\text{-}A$.*

Proof. Consider the morphisms of A-R-$\widehat{\otimes}$-bimodules,

$$s \colon A \to A \mathbin{\widehat{\otimes}}_R A, \quad a \mapsto a \otimes 1;$$
$$t \colon A \mathbin{\widehat{\otimes}}_R A \to \Omega^1_R A, \quad a_0 \otimes a_1 \mapsto a_0 da_1.$$

A direct calculation shows that s and t provide a contracting homotopy for (7.5) making it split in $A\text{-}\mathsf{mod}\text{-}R$. Furthermore, define a morphism of R-A-$\widehat{\otimes}$-bimodules $s' \colon A \to A \mathbin{\widehat{\otimes}}_R A$

by setting $s'(a) = 1 \otimes a$; then, since $ms' = \mathbf{1}_A$ and $j = \ker m$, we have that (7.5) is also split in R-mod-A. $\square$

Proposition 7.3. *The R-$\widehat{\otimes}$-algebra A is relatively quasi-free if and only if $\Omega^1_R A$ is projective in some (equivalently, in any) of the exact categories (A, R)-mod-(A, R), (A, A)-mod-(A, R), and (A, R)-mod-(A, A).*

Proof. By Proposition 7.2, the sequence (7.5) is admissible in the exact categories (A, R)-mod-(A, R), (A, A)-mod-(A, R), and (A, R)-mod-(A, A). Moreover, its middle term is projective in each of these categories. Hence, taking into account Corollary 2.8, for any A-$\widehat{\otimes}$-bimodule X we have isomorphisms

$$\mathcal{H}^2(A, R; X) \cong \operatorname{Ext}^1_{A^e,\, R^e}(\Omega^1_R A, X) \cong \operatorname{Ext}^1_{A^e,\, A \mathbin{\widehat{\otimes}} R^{\mathrm{op}}}(\Omega^1_R A, X) \cong \operatorname{Ext}^1_{A^e,\, R \mathbin{\widehat{\otimes}} A^{\mathrm{op}}}(\Omega^1_R A, X).$$

The rest is clear. $\square$

Remark 7.1. Notice that the notion of a relatively quasi-free algebra has an obvious purely algebraic analog (see [7] for $R = \mathbb{C}$). Everything that was said above about relatively quasi-free $\widehat{\otimes}$-algebras and noncommutative differential forms carries over easily to algebras without topology. In that regard we remark that if R is an algebra and A is an R-algebra having a countable dimension as a vector space, then its bimodule of relative differential 1-forms, constructed in a purely algebraic situation and then endowed with the strongest locally convex topology, is isomorphic to the bimodule of relative differential forms of the strongest locally convex algebra A_s over R_s (see Proposition 2.3). Combining this with Remark 2.2, we have that A_s is relatively quasi-free over R_s as a $\widehat{\otimes}$-algebra if and only if A is relatively quasi-free over R in the purely algebraic sense.

Remark 7.2. Quasi-free algebras admit some other equivalent definitions. In particular, they can be characterized in terms of connections. For details (when $R = \mathbb{C}$), see [7, 9] or [43].

Next, we want to determine the action of the Arens–Michael functor on the bimodule $\Omega^1_R A$ (see Remark 3.8). Consider first a more general situation. Suppose A is an R-$\widehat{\otimes}$-algebra, B is an S-$\widehat{\otimes}$-algebra, and (f, g) is an R-S-homomorphism from A to B. Fix an arbitrary B-$\widehat{\otimes}$-bimodule X and consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Der}_S(B, X) \arrow[r] \arrow[d, equal] & \operatorname{Der}_R(A, X) \arrow[d, equal] & \\
{}_B\mathbf{h}_B(\Omega^1_S B, X) \arrow[r] & {}_A\mathbf{h}_A(\Omega^1_R A, X) \arrow[r, equal] & {}_B\mathbf{h}_B(B \mathbin{\widehat{\otimes}}_A \Omega^1_R A \mathbin{\widehat{\otimes}}_A B, X)
\end{tikzcd}$$

Here the equalities denote canonical isomorphisms, the top arrow was defined in (3.5), and the bottom arrow is uniquely determined by the top one and the commutativity of the diagram. Substituting $\Omega^1_S B$ for X in the bottom row, we see that the morphism of B-$\widehat{\otimes}$-bimodules $B \mathbin{\widehat{\otimes}}_A \Omega^1_R A \mathbin{\widehat{\otimes}}_A B \to \Omega^1_S B$ (this is a noncommutative analog of the pullback operation of differential forms), denoted f_*, corresponds to the identity morphism of $\Omega^1_S B$.

Proposition 7.4. *Let A, R, S be $\widehat{\otimes}$-algebras and $g \colon R \to S$ an epimorphism of $\widehat{\otimes}$-algebras. Assume that A is an R-$\widehat{\otimes}$-algebra and that its Arens–Michael envelope $\widehat{A}$ is an S-$\widehat{\otimes}$-algebra such that the pair (ι_A, g) is an R-S-homomorphism from A to $\widehat{A}$. Then the morphism*

$$(\iota_A)_* \colon \widehat{A} \mathbin{\widehat{\otimes}}_A \Omega^1_R A \mathbin{\widehat{\otimes}}_A \widehat{A} \to \Omega^1_S \widehat{A}$$

is an isomorphism of $\widehat{A}$-$\widehat{\otimes}$-bimodules.

Proof. Use Proposition 3.8. □

Corollary 7.5. *For any R-$\widehat{\otimes}$-algebra A the morphism*

$$(\iota_A)_* \colon \widehat{A} \mathbin{\widehat{\otimes}}_A \Omega^1_R A \mathbin{\widehat{\otimes}}_A \widehat{A} \to \Omega^1_{\widehat{R}} \widehat{A}$$

is an isomorphism of $\widehat{A}$-$\widehat{\otimes}$-bimodules.

Theorem 7.6. *Let A, R, S be $\widehat{\otimes}$-algebras and $g \colon R \to S$ an epimorphism of $\widehat{\otimes}$-algebras. Suppose that A is an R-$\widehat{\otimes}$-algebra and its Arens–Michael envelope $\widehat{A}$ is an S-$\widehat{\otimes}$-algebra such that the pair (ι_A, g) is an R-S-homomorphism from A to $\widehat{A}$. Assume also that A is relatively quasi-free over R. Then:*

(i) *$\widehat{A}$ is relatively quasi-free over S;*
(ii) *$\iota_A \colon A \to \widehat{A}$ is a two-sided relative homological epimorphism.*

Proof. Assertion (i) follows from Propositions 7.3 and 7.4. The same propositions imply that (7.5) is a projective resolution of A in (A, R)-mod-(A, R). Applying to it the functor $\widehat{A} \mathbin{\widehat{\otimes}}_A (\,\cdot\,) \mathbin{\widehat{\otimes}}_A \widehat{A}$ and using Propositions 6.1 (iv) and 7.4, we have a sequence

$$0 \leftarrow \widehat{A} \xleftarrow{m} \widehat{A} \mathbin{\widehat{\otimes}}_S \widehat{A} \xleftarrow{j} \Omega^1_S \widehat{A} \leftarrow 0,$$

which is admissible in $(\widehat{A}, S)$-mod-$(\widehat{A}, S)$ by Proposition 7.2. The rest is clear. □

Corollary 7.7. *Suppose A is a relatively quasi-free R-$\widehat{\otimes}$-algebra. Then $\widehat{A}$ is relatively quasi-free over $\widehat{R}$ and $\iota_A \colon A \to \widehat{A}$ is a two-sided relative homological epimorphism.*

Remark 7.3. Setting in the preceding corollary $R = \mathbb{C}$ and $A = F_n$ we recover Taylor's result [67] that the embedding of the free algebra F_n in its Arens–Michael envelope is a homological epimorphism.

Now we want to give several examples of relatively quasi-free algebras.

Proposition 7.8. *Suppose R is an Arens–Michael algebra and A is any of the following two R-$\widehat{\otimes}$-algebras:*

(i) *$A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where $\alpha \colon R \to R$ is an endomorphism and $\delta \colon R \to R$ is an α-derivation such that the family $\{\alpha, \delta\}$ is m-localizable;*
(ii) *$A = \mathcal{O}(\mathbb{C}^\times, R; \alpha)$, where $\alpha \colon R \to R$ is an automorphism such that the family $\{\alpha, \alpha^{-1}\}$ is m-localizable.*

Then the bimodule $\Omega^1_R A$ is isomorphic to $A_\alpha \mathbin{\widehat{\otimes}}_R A$, and the universal derivation

$$d \colon A \to A_\alpha \mathbin{\widehat{\otimes}}_R A$$

is uniquely determined by the formula $d(z) = 1 \otimes 1$. Consequently, A is relatively quasi-free.

Proof. Consider first case (i). In view of (2.3) and Remark 4.6, for any A-$\widehat{\otimes}$-bimodule X we have the canonical bijections

$$\begin{aligned}
\operatorname{Der}_R(A, X) &\cong \bigl\{\varphi \in \operatorname{Hom}_{R\text{-}\mathsf{AM}}(A, A \times X) : p_1\varphi = \mathbf{1}_A\bigr\}\\
&\cong \bigl\{b \in A \times X : p_1(b) = z,\ b \cdot r = \alpha(r) \cdot b + \delta_{A\times X}(r)\ \forall r \in R\bigr\}\\
&\cong \bigl\{x \in X : (zr, x \cdot r) = (\alpha(r)z, \alpha(r) \cdot x) + (\delta(r), 0)\ \forall r \in R\bigr\}\\
&= \bigl\{x \in X : x \cdot r = \alpha(r) \cdot x\ \forall r \in R\bigr\} \cong {}_R\mathbf{h}_R(R_\alpha, X)\\
&\cong {}_A\mathbf{h}_A(A \mathbin{\widehat{\otimes}}_R R_\alpha \mathbin{\widehat{\otimes}}_R A, X) \cong {}_A\mathbf{h}_A(A_\alpha \mathbin{\widehat{\otimes}}_R A, X).
\end{aligned}$$

It is easy to see that these bijections give rise to an isomorphism of functors of X, whence the first assertion. To prove the second assertion, we set $X = A_\alpha \mathbin{\widehat{\otimes}}_R A$ and notice that the R-derivation d corresponding to the identity morphism $\mathbf{1}_X$ under the above bijections sends $z \in A$ to $1 \otimes 1 \in X$.

To prove (ii), notice that for any A-$\widehat{\otimes}$-bimodule X and any $x \in X$ the element $(z, x) \in A \times X$ is invertible and its inverse is $(z^{-1}, -z^{-1} \cdot x \cdot z^{-1})$. In view of this and (4.26), we can finish the argument as in case (i). $\square$

Proposition 7.9. *Suppose R is an algebra of countable dimension and A is any of the following two R-algebras:*

(i) *$A = R[t; \alpha, \delta]$, where $\alpha \colon R \to R$ is an endomorphism and $\delta \colon R \to R$ is an α-derivation;*

(ii) *$A = R[t, t^{-1}; \alpha]$, where $\alpha \colon R \to R$ is an automorphism.*

Endow R and A with the strongest locally convex topology. Then the bimodule $\Omega^1_R A$ is isomorphic to $A_\alpha \mathbin{\widehat{\otimes}_R} A$ (which, in turn, is isomorphic to $A_\alpha \otimes_R A$ endowed with the strongest locally convex topology), and the universal derivation $d \colon A \to A_\alpha \mathbin{\widehat{\otimes}_R} A$ is uniquely determined by the formula $d(t) = 1 \otimes 1$. Consequently, A is relatively quasi-free.

The proof of this proposition is similar to that of Proposition 7.8 and is therefore omitted. We also remark that the assumption that R is of countable dimension was needed only to guarantee that the multiplications on R and A are jointly continuous. A purely algebraic analog of the preceding proposition is, of course, true without any restrictions on the dimension of R. The same applies to Proposition 7.11 below.

The following two propositions are proved exactly the same way as Proposition 2.9 of [7]. We omit the proofs.

Proposition 7.10. *Suppose R is an Arens–Michael algebra, M is an R-$\widehat{\otimes}$-bimodule, and $A = \widehat{T}_R(M)$. Then the bimodule $\Omega^1_R A$ is isomorphic to $A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$, and the universal derivation $d \colon A \to A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$ is uniquely determined by the formula $d(m) = 1 \otimes m \otimes 1$ for $m \in M$. As a consequence, A is relatively quasi-free.*

Proposition 7.11. *Suppose R is a countably dimensional algebra, M is a countably dimensional R-bimodule, and $A = T_R(M)$. Endow R, M, and A with the strongest locally convex topology. Then the bimodule $\Omega^1_R A$ is isomorphic to $A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$ (which, in turn, is isomorphic to $A \otimes_R M \otimes_R A$ endowed with the strongest locally convex topology), and the universal derivation $d \colon A \to A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$ is uniquely determined by the formula $d(m) = 1 \otimes m \otimes 1$ for $m \in M$. Consequently, A is relatively quasi-free.*

8. Finiteness conditions

Definition 8.1. Suppose A is a topological algebra and X is a left topological A-module. We shall say that X is *strictly finitely generated* if for some $n \in \mathbb{N}$ there is an open morphism from A^n onto X.

Remark 8.1. It is not difficult to see that if A is locally convex and X is finitely generated and endowed with the strongest locally convex topology, then it is strictly finitely generated.

Remark 8.2. If A is a Fréchet algebra, then it follows easily from the Banach Open Mapping Theorem that a left topological A-module X is strictly finitely generated if and only if it is finitely generated and is Fréchet.

Remark 8.3. If $A \to B$ is a homomorphism of Fréchet algebras and X is a finitely generated left Fréchet A-module, then the left Fréchet B-module $B \mathbin{\widehat{\otimes}_A} X$ is also finitely generated. This follows from the preceding remark and the fact that the functor of taking projective tensor product with a Fréchet module sends open morphisms of Fréchet modules to open morphisms (see [25, II.4.12]).

Proposition 8.1. *Suppose A is a topological algebra and X is a strictly finitely generated left topological A-module. Suppose that $\widehat{A}$ is metrizable. Then the $\widehat{A}$-module $\widehat{X}$ is strictly finitely generated.*

Proof. We may assume that $X = A^n/Y$ for some submodule $Y \subset A^n$. It follows from Proposition 3.12 and Remark 3.7 that $\widehat{X} = \widehat{A}^n/\overline{\iota_A^n(Y)}$. The rest is clear. $\square$

Suppose A is an R-$\widehat{\otimes}$-algebra and P is an A-$\widehat{\otimes}$-bimodule. Recall that P is projective in (A,R)-mod-(A,R) if and only if P is a retract of a bimodule $A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$, where $M \in R$-mod-R.

Definition 8.2. We shall say that a projective (A,R)-$\widehat{\otimes}$-bimodule P *satisfies the finiteness condition* (f_1) if it is a retract of a bimodule $A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$, where $M \in R$-mod-R is strictly finitely generated as a left R-module.

Definition 8.3. We shall say that a projective (A,R)-$\widehat{\otimes}$-bimodule P *satisfies the finiteness condition* (f_2) if it is a retract of a bimodule $A \mathbin{\widehat{\otimes}_R} M \mathbin{\widehat{\otimes}_R} A$, where $M \in R$-mod-R is isomorphic to the R-$\widehat{\otimes}$-bimodule R^n for some n.

Remark 8.4. Clearly, condition (f_2) implies condition (f_1). If $R = \mathbb{C}$, then both conditions are equivalent to P being a retract of $(A \mathbin{\widehat{\otimes}} A)^n$ for some n. This means precisely that P is strictly projective (see [27]) and finitely generated in A-mod-A.

Definition 8.4. We shall say that an R-$\widehat{\otimes}$-algebra A is an algebra of (f_1)-*finite type* (resp., (f_2)-*finite type*) if in (A,R)-mod-(A,R) it has a finite projective resolution all of whose bimodules satisfy the finiteness condition (f_1) (resp., (f_2)).

When $R = \mathbb{C}$, the preceding definition means that A has a finite projective resolution all of whose bimodules are strictly projective and finitely generated. These conditions are satisfied, for example, by the algebra of polynomials $\mathbb{C}[t_1, \dots, t_n]$ with the strongest locally convex topology, and also by the algebra $\mathcal{O}(U)$ of holomorphic functions on a polydomain $U \subset \mathbb{C}^n$: in both cases the resolution can be chosen to be a Koszul complex [67]. Other examples will appear in the next section.

Definition 8.5. We shall say that an R-$\widehat{\otimes}$-algebra A is *relatively* (f_1)-*quasi-free* (resp., *relatively* (f_2)-*quasi-free*) if $\Omega^1_R A$ satisfies condition (f_1) (resp., (f_2)).

It follows immediately from Propositions 7.2 and 7.3 that a relatively (f_1)-quasi-free (resp., relatively (f_2)-quasi-free) algebra is quasi-free in the sense of Definition 7.1 and is an algebra of (f_1)-finite type (resp., (f_2)-finite type).

Remark 8.5. Notice that Definitions 8.3, 8.4, and 8.5 have obvious purely algebraic analogs (cf. Remark 7.1). For example, the R-algebra A is relatively (f_1)-quasi-free if the bimodule $\Omega^1_R A$ constructed in a purely algebraic situation is a retract of the A-bimodule $A \otimes_R M \otimes_R A$, where the R-bimodule M is finitely generated as a left R-module. It is not difficult to see that if we restrict ourselves to countably dimensional algebras and modules, then the "algebraic" finiteness conditions (i.e., conditions (f_1) and (f_2) for bimodules, the property of being an algebra of (f_1)- or (f_2)-finite type, or (f_1)-quasi-free, or (f_2)-quasi-free) are equivalent to the corresponding "topological" finiteness conditions for the same algebras and bimodules endowed with the strongest locally convex topology. We skip the verification of all these assertions since this is not difficult and reduces to applying Proposition 2.3.

Example 8.1. The algebras $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ and $\mathcal{O}(\mathbb{C}^\times, R; \alpha)$ of Proposition 7.8 are relatively (f_1)-quasi-free. The same is true for the algebras $R[t; \alpha, \delta]$ and $R[t, t^{-1}; \alpha]$ of Proposition 7.9. If $\alpha = \mathbf{1}_R$, then the above algebras are relatively (f_2)-quasi-free.

Example 8.2. The algebras $\widehat{T}_R(M)$ and $T_R(M)$ of Propositions 7.10 and 7.11 are relatively (f_1)-quasi-free if M is strictly finitely generated in R-mod, and relatively (f_2)-quasi-free if M is a retract of R^n in R-mod-R for some $n \in \mathbb{N}$.

Lemma 8.2. *Suppose B is an Arens–Michael algebra and $N \in B$-mod is a strictly finitely generated module. Then the algebra ${}_B\mathbf{h}(N)$ endowed with the topology of simple convergence is Arens–Michael.*

Proof. If $N = B^n$, then ${}_B\mathbf{h}(N)$ is topologically isomorphic to $M_n(B)$ and is therefore Arens–Michael (see Section 2). In the general case, for some $n \in \mathbb{N}$ there is a open morphism $\sigma\colon B^n \to N$. Set $K = \operatorname{Ker}\sigma$ and $C = \{\varphi \in {}_B\mathbf{h}(B^n) : \varphi(K) \subset K\}$. Clearly, C is a closed subalgebra of ${}_B\mathbf{h}(B^n)$ and is therefore Arens–Michael. It is also clear that to each $\varphi \in C$ there corresponds a unique $\psi \in {}_B\mathbf{h}(N)$, acting by the rule $\psi(\sigma(x)) = \sigma(\varphi(x))$ $(x \in B^n)$. Moreover, the map $\theta\colon C \to {}_B\mathbf{h}(N)$, $\varphi \mapsto \psi$, is a homomorphism of algebras.

The morphism σ induces two continuous linear maps

$$\sigma^*\colon {}_B\mathbf{h}(N,N) \to {}_B\mathbf{h}(B^n,N), \quad \psi \mapsto \psi\circ\sigma,$$
$$\sigma_*\colon {}_B\mathbf{h}(B^n,B^n) \to {}_B\mathbf{h}(B^n,N), \quad \varphi \mapsto \sigma\circ\varphi.$$

Obviously, σ^* is injective and

$$\operatorname{Im}\sigma^* = \bigl\{\varphi \in {}_B\mathbf{h}(B^n,N) : \varphi(K) = 0\bigr\}.$$

Therefore $\operatorname{Im}\sigma^*$ is closed. Moreover, a simple verification shows that for any finite set $D \subset B^n$ and any neighborhood of zero $U \subset N$ we have

$$\sigma^*\bigl(M(\sigma(D),U)\bigr) = M(D,U) \cap \operatorname{Im}\sigma^*$$

(see (2.1) for the notation). Since any finite subset in N is an image of some finite subset of B^n, we have that σ^* is topologically injective, i.e., it is a topological isomorphism between ${}_B\mathbf{h}(N,N)$ and $\operatorname{Im}\sigma^*$. The inverse isomorphism will be denoted $\varkappa$. Notice that the space ${}_B\mathbf{h}(N,N)$ is complete, because it is isomorphic to a closed subspace $\operatorname{Im}\sigma^*$ of the complete space ${}_B\mathbf{h}(B^n,N) \cong N^n$.

Furthermore, it is not difficult to see that σ_* is open. Indeed, under the natural isomorphism ${}_B\mathbf{h}(B^n,X) \cong X^n$ $(X \in B$-$\mathsf{mod})$ it is sent to σ^n, which is open as so is σ.

It follows from the definition of the algebra C that $C = \sigma_*^{-1}(\operatorname{Im}\sigma^*)$. Since the restriction of an open map to the preimage of any set is open, we have that $\sigma_*|_C\colon C \to \operatorname{Im}\sigma^*$ is open. It follows from the definition of the homomorphism θ that $\theta = \varkappa\circ\sigma_*|_C$; hence θ is continuous and open. Therefore, ${}_B\mathbf{h}(N)$ is isomorphic to the quotient algebra $C/\operatorname{Ker}\theta$ and is therefore locally m-convex. In view of the aforementioned completeness of ${}_B\mathbf{h}(N)$, this concludes the proof. $\square$

Lemma 8.3. *Suppose R is a $\widehat{\otimes}$-algebra whose Arens–Michael envelope S is metrizable. Let B be an Arens–Michael algebra and N a Fréchet B-R-$\widehat{\otimes}$-bimodule which is strictly finitely generated in B-mod. Then N admits a structure of a right S-module, which extends the original structure of a right R-module (i.e., such that $n\cdot\iota_R(r) = n\cdot r$ for all $n \in N$ and $r \in R$) and makes it a Fréchet B-S-$\widehat{\otimes}$-bimodule.*

Proof. Consider the algebra ${}_B\mathbf{h}(N)$ and endow it, as in the preceding lemma, with the topology of simple convergence. Clearly, the map

$$m\colon R \to {}_B\mathbf{h}(N)^{\mathrm{op}}, \quad m(r)(n) = n\cdot r$$

is an algebra homomorphism and it is not difficult to check that it is continuous. Indeed, for any neighborhood of zero $V \subset N$ and any finite subset $D \subset N$ there is a neighborhood of zero $U \subset R$ such that $D\cdot U \subset V$, which is equivalent to the inclusion $m(U) \subset M(D,V)$. Therefore m is continuous. By the preceding lemma, ${}_B\mathbf{h}(N)$ is Arens–Michael; hence there exists a unique continuous homomorphism $\widehat{m}\colon S \to {}_B\mathbf{h}(N)^{\mathrm{op}}$ such that $\widehat{m}\iota_R = m$. Setting $n \cdot s = \widehat{m}(s)(n)$ for $s \in S$, $n \in N$, we have a structure of a right S-module on N extending the original structure of a right R-module and making N a B-S-bimodule. Clearly, the constructed action $N \times S \to N$ is separately continuous. But S and N are Fréchet spaces; hence this action is jointly continuous. Therefore N is a Fréchet B-S-$\widehat{\otimes}$-bimodule. $\square$

Corollary 8.4. *Suppose R is a $\widehat{\otimes}$-algebra whose Arens–Michael envelope S is metrizable, and a bimodule $M \in R\text{-}\mathsf{mod}\text{-}R$ is strictly finitely generated as a left R-module. Then the S-$\widehat{\otimes}$-bimodule $\widehat{M} = S \mathbin{\widehat{\otimes}}_R M \mathbin{\widehat{\otimes}}_R S$ is isomorphic to $S \mathbin{\widehat{\otimes}}_R M$ in $S\text{-}\mathsf{mod}$ and, in particular, is strictly finitely generated in $S\text{-}\mathsf{mod}$.*

Proof. Consider the S-R-$\widehat{\otimes}$-bimodule $N = S \mathbin{\widehat{\otimes}}_R M$. It follows from Propositions 8.1 and 3.11 that it is strictly finitely generated in $S\text{-}\mathsf{mod}$ and, in particular, is a Fréchet module. On the other hand, it follows from Lemma 8.3 that on N there is a structure of a right S-$\widehat{\otimes}$-module extending the structure of a right R-$\widehat{\otimes}$-module and making it an S-$\widehat{\otimes}$-bimodule. As the canonical image of R in S is dense, we have the isomorphisms of S-$\widehat{\otimes}$-bimodules,

$$\widehat{M} = N \mathbin{\widehat{\otimes}}_R S \cong N \mathbin{\widehat{\otimes}}_S S \cong N.$$

But, as we mentioned before, N is strictly finitely generated in $S\text{-}\mathsf{mod}$. $\square$

Corollary 8.5. *Suppose A, R, S are $\widehat{\otimes}$-algebras and $g\colon R \to S$ is an epimorphism of $\widehat{\otimes}$-algebras. Suppose that on A we have a structure of an R-$\widehat{\otimes}$-algebra, and on its Arens–Michael $\widehat{A}$ envelope we have a structure of an S-$\widehat{\otimes}$-algebra such that the pair (ι_A, g) is an R-S-homomorphism from A to $\widehat{A}$. Assume also that A is relatively quasi-free over R. Then:*

(i) *If A is relatively (f_2)-quasi-free over R, then $\widehat{A}$ is relatively (f_2)-quasi-free over S;*
(ii) *If $(S, g) = (\widehat{R}, \iota_R)$, S is metrizable and A is relatively (f_1)-quasi-free over R, then $\widehat{A}$ is relatively (f_1)-quasi-free over S.*

Proof. Suppose $\Omega^1_R A$ is a retract of $A \mathbin{\widehat{\otimes}}_R M \mathbin{\widehat{\otimes}}_R A$ in $A\text{-}\mathsf{mod}\text{-}A$. By Proposition 7.4, the $\widehat{A}$-$\widehat{\otimes}$-bimodule $\Omega^1_S \widehat{A}$ is isomorphic to $\widehat{A} \mathbin{\widehat{\otimes}}_A \Omega^1_R A \mathbin{\widehat{\otimes}}_A \widehat{A}$ and is therefore a retract of $\widehat{A} \mathbin{\widehat{\otimes}}_R M \mathbin{\widehat{\otimes}}_R \widehat{A}$, which, in turn, is isomorphic to $\widehat{A} \mathbin{\widehat{\otimes}}_S \widehat{M} \mathbin{\widehat{\otimes}}_S \widehat{A}$.

If $M \cong R^n$ for some $n \in \mathbb{N}$, then $\widehat{M} \cong (S \mathbin{\widehat{\otimes}}_R S)^n \cong S^n$. In view of the above, this proves (i). To prove (ii), apply Corollary 8.4. $\square$

Corollary 8.6. *If A is a relatively (f_2)-quasi-free R-$\widehat{\otimes}$-algebra, then $\widehat{A}$ is relatively (f_2)-quasi-free over $\widehat{R}$. If A is a relatively (f_1)-quasi-free R-$\widehat{\otimes}$-algebra and $\widehat{R}$ is metrizable, then $\widehat{A}$ is relatively (f_1)-quasi-free over $\widehat{R}$.*

9. Theorems on stable flatness of Arens–Michael envelopes

In this section we shall show, for some finitely generated algebras A, that the canonical homomorphism ι_A from A to its Arens–Michael envelope $\widehat{A}$ is a homological epimorphism, i.e., $\widehat{A}$ is stably flat over A (see Definition 6.3). Our main technical tool will be the following theorem, establishing a connection between the notions of relative and "absolute" homological epimorphism.

Theorem 9.1. *Suppose $(f, g)\colon (A, R) \to (B, S)$ is an R-S-homomorphism from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B. Assume that g is a homological epimorphism and f is a left or right relative homological epimorphism. Also assume that A is projective* *in R-mod and B is projective in S-mod, and that one of the following two conditions is satisfied:*

(i) *S and B are Fréchet–Arens–Michael algebras such that $S = \widehat{R}$, $g = \iota_R$, and A is an algebra of (f_1)-finite type;*
(ii) *A is an algebra of (f_2)-finite type.*

Then f is a homological epimorphism.

For the proof we need the following lemma.

Lemma 9.2. *Let $\mathcal{A}, \mathcal{B}$ be exact categories and $F \colon \mathcal{A} \to \mathcal{B}$ an additive covariant functor having a left derived functor. Suppose $X \to Y \to Z$ is an admissible pair in $\mathcal{A}$ and that F sends it to an admissible pair in $\mathcal{B}$. Suppose also that two objects from X, Y, Z are F-acyclic. Then so is the third.*

Proof. The admissible pairs $X \to Y \to Z$ and $FX \to FY \to FZ$ give rise to distinguished triangles $X \to Y \to Z \to X[1]$ and $FX \to FY \to FZ \to (FX)[1]$ in $\mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^-(\mathcal{B})$, respectively (see [32, 11.6]). Applying the functor $\mathrm{L}F$ to the first one, we have a distinguished triangle in $\mathsf{D}^-(\mathcal{B})$ which maps to the second one:

$$\begin{tikzcd}
\mathrm{L}F(X) \arrow[r] \arrow[d] & \mathrm{L}F(Y) \arrow[r] \arrow[d] & \mathrm{L}F(Z) \arrow[r] \arrow[d] & \mathrm{L}F(X)[1] \arrow[d] \\
FX \arrow[r] & FY \arrow[r] & FZ \arrow[r] & (FX)[1]
\end{tikzcd}$$

By the assumptions, the first two vertical arrows are isomorphisms in $\mathsf{D}^-(\mathcal{B})$. Hence so is the third one (see Proposition 10.1). $\square$

Corollary 9.3. *Suppose $\mathcal{A}$, $\mathcal{B}$ and F are the same as in the preceding lemma. Assume furthermore that F sends admissible epimorphisms to epimorphisms. Let*

$$0 \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow 0$$

be an admissible complex in $\mathcal{A}$ which is sent by F to an admissible complex in $\mathcal{B}$ and whose objects $X_1, \dots, X_n$ are F-acyclic. Then X_0 is also F-acyclic.

Proof. When $n = 1$, there is nothing to prove, and when $n = 2$ everything follows from Lemma 9.2. Suppose that the assertion is true for complexes of length at most n, where $n \geq 2$. We shall show that it is true for complexes of length $n+1$. By the assumption, the complexes

$$0 \leftarrow X_0 \xleftarrow{d_0} \cdots \xleftarrow{d_{n-2}} X_{n-1} \xleftarrow{d_{n-1}} X_n \xleftarrow{d_n} X_{n+1} \leftarrow 0,$$
$$0 \leftarrow FX_0 \xleftarrow{Fd_0} \cdots \xleftarrow{Fd_{n-2}} FX_{n-1} \xleftarrow{Fd_{n-1}} FX_n \xleftarrow{Fd_n} FX_{n+1} \leftarrow 0$$

are admissible. Setting $K = \operatorname{Ker} d_{n-2}$ and $L = \operatorname{Ker} Fd_{n-2}$, we have admissible complexes

$$0 \leftarrow X_0 \xleftarrow{d_0} X_1 \xleftarrow{d_1} \cdots \xleftarrow{d_{n-2}} X_{n-1} \leftarrow K \leftarrow 0, \tag{9.1}$$
$$0 \leftarrow K \leftarrow X_n \xleftarrow{d_n} X_{n+1} \leftarrow 0, \tag{9.2}$$
$$0 \leftarrow FX_0 \xleftarrow{Fd_0} FX_1 \xleftarrow{Fd_1} \cdots \xleftarrow{Fd_{n-2}} FX_{n-1} \leftarrow L \leftarrow 0,$$
$$0 \leftarrow L \leftarrow FX_n \xleftarrow{Fd_n} FX_{n+1} \leftarrow 0.$$

Since they are admissible, we have, in particular, that $K = \operatorname{Coker} d_n$, and $L = \operatorname{Coker} Fd_n$.

Consider the diagram

$$\begin{tikzcd} \cdots & FX_{n-1} \arrow[l, "Fd_{n-2}"'] & FX_n \arrow[l, "Fd_{n-1}"'] \arrow[d, "\beta"] \arrow[dl, "\delta"] & \cdots \arrow[l, "Fd_n"'] \\ & L \arrow[u, "\alpha"] \arrow[r, shift right, "v"'] & FK \arrow[l, shift right, "u"'] \arrow[ul, "\gamma"'] & \end{tikzcd}$$

Here $\alpha = \ker Fd_{n-2}$, $\beta = F(\operatorname{coker} d_n)$, $\gamma = F(\ker d_{n-2})$, $\delta = \operatorname{coker} Fd_n$, and u and v are uniquely determined by the conditions $\alpha u = \gamma$ and $v\delta = \beta$.

We have

$$\alpha u v \delta = \gamma\beta = Fd_{n-1} = \alpha\delta.$$

Since α is a monomorphism and δ is an epimorphism, we also have $uv = \mathbf{1}_L$. Furthermore, the assumption on F and the admissibility of (9.2) imply that β is an epimorphism. As $v\delta = \beta$, we have that v is also an epimorphism. This and the equality $uv = \mathbf{1}_L$ show that u and v are inverses of each other and thus $FK \cong L$. To finish the proof, apply the induction assumption to (9.1) and (9.2). ☐

Proof of Theorem 9.1. Consider the commutative diagram

$$\begin{tikzcd} \mathsf{mod}\text{-}A \arrow[r, "(\,\cdot\,)\widehat{\otimes}_A B"] & \mathsf{mod}\text{-}B \\ \mathsf{mod}\text{-}R \arrow[u, "(\,\cdot\,)\widehat{\otimes}_R A"] \arrow[r, "(\,\cdot\,)\widehat{\otimes}_R S"'] & \mathsf{mod}\text{-}S \arrow[u, "(\,\cdot\,)\widehat{\otimes}_S B"'] \end{tikzcd} \tag{9.3}$$

We need to show that $B \in \mathsf{mod}\text{-}A$ is acyclic relative to $(\,\cdot\,)\widehat{\otimes}_A B$.

Choose a projective resolution

$$0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n \leftarrow 0 \tag{9.4}$$

of A in $(A,R)\text{-}\mathsf{mod}\text{-}(A,R)$ such that all the bimodules P_i satisfy condition (f_1) in case (i) (resp., condition (f_2) in case (ii)). By Proposition 6.6, the complexes

$$0 \leftarrow B \leftarrow B\widehat{\otimes}_A P_0 \leftarrow B\widehat{\otimes}_A P_1 \leftarrow \cdots \leftarrow B\widehat{\otimes}_A P_n \leftarrow 0, \tag{9.5}$$

$$0 \leftarrow B \leftarrow B\widehat{\otimes}_A P_0\widehat{\otimes}_A B \leftarrow B\widehat{\otimes}_A P_1\widehat{\otimes}_A B \leftarrow \cdots \leftarrow B\widehat{\otimes}_A P_n\widehat{\otimes}_A B \leftarrow 0 \tag{9.6}$$

are split in LCS. Notice that (9.6) can be obtained from (9.5) by applying the functor $(\,\cdot\,)\widehat{\otimes}_A B\colon \mathsf{mod}\text{-}A \to \mathsf{mod}\text{-}B$. Since the epimorphisms in these categories are exactly the morphisms with dense images, it is easy to see that this functor sends epimorphisms to epimorphisms. Therefore, in view of Corollary 9.3, to finish the proof it suffices to show that the modules $B\widehat{\otimes}_A P_i$ $(i = 0,\dots,n)$ are acyclic relative to this functor.

Fix an arbitrary $i \in \{0,\dots,n\}$. Without loss of generality we may assume that $P_i = A\widehat{\otimes}_R M\widehat{\otimes}_R A$, where $M \in R\text{-}\mathsf{mod}\text{-}R$ and, in addition, that M is strictly finitely generated in $R\text{-}\mathsf{mod}$ in case (i), and that M is isomorphic to R^m in $R\text{-}\mathsf{mod}\text{-}R$ in case (ii).

Now we shall show that the structure of a right R-$\widehat{\otimes}$-module on the B-R-$\widehat{\otimes}$-bimodule $N = B\widehat{\otimes}_R M$ extends to a structure of a right S-$\widehat{\otimes}$-module. Indeed, in case (ii), $N \cong B^m$ in $B\text{-}\mathsf{mod}\text{-}R$ and the assertion is true for obvious reasons. In case (i), M is strictly finitely generated in $R\text{-}\mathsf{mod}$; hence $S\widehat{\otimes}_R M$ is strictly finitely generated in $S\text{-}\mathsf{mod}$ (see Proposition 8.1) and therefore $N = B\widehat{\otimes}_R M = B\widehat{\otimes}_S (S\widehat{\otimes}_R M)$ is strictly finitely generated in $B\text{-}\mathsf{mod}$ (see Remark 8.3). Thus, applying Lemma 8.3, we have that the structure of a right R-$\widehat{\otimes}$-module on N extends to a structure of a right S-$\widehat{\otimes}$-module.

Combining this result with the fact that g is a homological epimorphism, we see that N is acyclic relative to the bottom arrow in (9.3). On the other hand, since $A \in R\text{-}\mathsf{mod}$

and $B \in S\text{-}\mathsf{mod}$ are projective, the vertical arrows in that diagram are exact functors. Therefore, by Corollary 2.5, the right A-$\widehat{\otimes}$-module $N \widehat{\otimes}_R A = B \widehat{\otimes}_A P_i$ is acyclic relative to the top arrow in (9.3). In view of the above, this implies that f is a homological epimorphism. □

Henceforth all finitely generated algebras and finitely generated modules over them will be assumed to be endowed with the strongest locally convex topology.

Corollary 9.4. *Let R be a finitely generated algebra and A a finitely generated relatively (f_1)-quasi-free R-algebra which is projective as a left R-module. Suppose that $\widehat{A}$ is projective in $\widehat{R}\text{-}\mathsf{mod}$. If $\iota_R \colon R \to \widehat{R}$ is a homological epimorphism, then $\iota_A \colon A \to \widehat{A}$ is also a homological epimorphism.*

Proof. Combine Corollary 7.7 with Theorem 9.1. □

Corollary 9.5. *Let R be a finitely generated algebra and M an R-bimodule which is finitely generated and projective as a left R-module. Suppose that $\widehat{T}_{\widehat{R}}(\widehat{M})$ is projective in $\widehat{R}\text{-}\mathsf{mod}$. If $\iota_R \colon R \to \widehat{R}$ is a homological epimorphism, then $\iota_{T_R(M)} \colon T_R(M) \to \widehat{T}_{\widehat{R}}(\widehat{M})$ is also a homological epimorphism.*

Proof. Notice that since M is projective as a left R-module, $T_R(M)$ is projective as a left R-module. Now use Corollary 9.4 and Example 8.2. □

Remark 9.1. When $R = \mathbb{C}$, the preceding corollary is Taylor's result [67] that the embedding of the free algebra F_n in its Arens–Michael envelope is a homological epimorphism (see Remark 4.9 and Remark 7.3).

Remark 9.2. In regard to the proof of the preceding corollary, we remark that the $\widehat{R}$-bimodule $\widehat{M} = \widehat{R} \widehat{\otimes}_R M \widehat{\otimes}_R \widehat{R}$ is isomorphic to $\widehat{R} \widehat{\otimes}_R M$ in $\widehat{R}\text{-}\mathsf{mod}$ (see Corollary 8.4) and, therefore, is projective in $\widehat{R}\text{-}\mathsf{mod}$. By induction, this easily implies that all its tensor powers $\widehat{M}^{\widehat{\otimes} n}$ (see Section 4.2) are also projective in $\widehat{R}\text{-}\mathsf{mod}$. Since the analytic tensor algebra $\widehat{T}_{\widehat{R}}(\widehat{M})$ is the completion of the direct sum of the bimodules $\widehat{M}^{\widehat{\otimes} n}$, it is natural to ask whether it is projective in $\widehat{R}\text{-}\mathsf{mod}$, i.e., whether one of the conditions of the preceding corollary is redundant. Unfortunately, the analytic tensor algebra has the following drawback, which is not present for the algebraic tensor algebra: if S is an Arens–Michael algebra and N is an S-$\widehat{\otimes}$-bimodule, then the projectivity of N in $S\text{-}\mathsf{mod}$ (or in $\mathsf{mod}\text{-}S$) does not, in general, imply that the analytic tensor algebra $\widehat{T}_S(N)$ is projective in $S\text{-}\mathsf{mod}$ (resp., in $\mathsf{mod}\text{-}S$). In particular, if α is an m-localizable automorphism S, then the analytic Ore extension $\mathcal{O}(\mathbb{C}, S; \alpha) = S\{t; \alpha\} = \widehat{T}_S(S_\alpha)$ (see Propositions 4.9 and 4.11), being a projective (and even free) left S-$\widehat{\otimes}$-module need not be a projective right S-$\widehat{\otimes}$-module. In fact, one can show that the algebra of holomorphic functions on the quantum plane $\mathcal{O}_q(\mathbb{C}^2)$ (see Corollary 5.14) with $|q| > 1$ is not projective as a right module over its subalgebra generated by the element x (in contrast with a purely algebraic situation). We will not prove this assertion, since we do not need it for our purposes.

Corollary 9.6. *Let R be a finitely generated algebra, $\alpha \colon R \to R$ an endomorphism, and $\delta \colon R \to R$ an α-derivation. Suppose that their canonical extensions $\widehat{\alpha}, \widehat{\delta}$ to $\widehat{R}$ form an m-localizable family. If $\iota_R \colon R \to \widehat{R}$ is a homological epimorphism, then $\iota_{R[t;\alpha,\delta]} \colon R[t; \alpha, \delta] \to \mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta})$ is also a homological epimorphism.*

Proof. Set $A = R[t; \alpha, \delta]$. By Theorem 5.17, we have

$$\widehat{A} \cong \mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta}).$$

This implies, in particular, that $\widehat{A}$ is projective (and even free) in $\widehat{R}$-mod. In view of Example 8.1, to finish the proof it suffices to apply Corollary 9.4. $\square$

Corollary 9.7. *Let R be a finitely generated algebra, $\alpha\colon R \to R$ an automorphism, and $\widehat{\alpha}$ its canonical extension to $\widehat{R}$. Suppose that the family $\{\widehat{\alpha}, \widehat{\alpha}^{-1}\}$ is m-localizable. If $\iota_R\colon R \to \widehat{R}$ is a homological epimorphism, then $\iota_{R[t,t^{-1};\alpha]}\colon R[t,t^{-1};\alpha] \to \mathcal{O}(\mathbb{C}^\times, \widehat{R}; \widehat{\alpha})$ is also a homological epimorphism.*

Proof. In view of Theorem 5.21, the proof is similar to that of the preceding corollary. $\square$

Notice that in Corollaries 9.4–9.7 we dealt with the situation described in Theorem 9.1, (i). We shall now examine part (ii).

Corollary 9.8. *Let R be a finitely generated algebra and $\delta\colon R \to R$ a derivation. If $j\colon R \to R_\delta$ is a homological epimorphism, then $\iota_{R[t;\delta]}\colon R[t;\delta] \to \mathcal{O}(\mathbb{C}, R_\delta; \widehat{\delta})$ is also a homological epimorphism.*

Proof. The proof is based on Theorem 5.1 and Theorem 9.1, (ii) and is similar to the proof of Corollary 9.6. $\square$

To apply these results to the concrete algebras from Section 5, we will need some facts about homological epimorphisms of polynomial algebras.

An *augmented $\widehat{\otimes}$-algebra* will be understood as a $\widehat{\otimes}$-algebra A with a homomorphism $\varepsilon_A\colon A \to \mathbb{C}$. Homomorphisms between augmented $\widehat{\otimes}$-algebras are defined in an obvious way. If A is an augmented $\widehat{\otimes}$-algebra, then we consider $\mathbb{C}$ as an A-$\widehat{\otimes}$-module obtained by a change of scalars along ε_A.

Definition 9.1. Suppose $f\colon A \to B$ is a homomorphism of augmented $\widehat{\otimes}$-algebras. We shall say that f is a *weak homological epimorphism* if the canonical morphism $B \mathbin{\widehat{\otimes}}{}^{\mathrm{L}}_A \mathbb{C} \to \mathbb{C}$ is an isomorphism in $\mathsf{D}^b(B\text{-}\mathsf{mod})$.

Remark 9.3. Weak homological epimorphisms were introduced in [48] under the name of “weak localization”. Notice that $f\colon A \to B$ is a weak homological epimorphism if and only if $\mathbb{C} \in A$-mod is acyclic relative to the functor $B \mathbin{\widehat{\otimes}}_A (\,\cdot\,)\colon A\text{-}\mathsf{mod} \to B\text{-}\mathsf{mod}$ and the canonical morphism $B \mathbin{\widehat{\otimes}}_A \mathbb{C} \to \mathbb{C}$ is an isomorphism in B-mod. This is proved by considering the morphisms

$$B \mathbin{\widehat{\otimes}}{}^{\mathrm{L}}_A \mathbb{C} \to B \mathbin{\widehat{\otimes}}_A \mathbb{C} \to \mathbb{C}$$

and using arguments similar to the ones used in the proof of the implication (3)$\Longrightarrow$(4) in Theorem 6.3.

Remark 9.4. In the “expanded” form, Definition 9.1 means that for some (or, equivalently, for any) projective resolution $0 \leftarrow \mathbb{C} \leftarrow P_\bullet$ in A-mod the complex $0 \leftarrow \mathbb{C} \leftarrow B \mathbin{\widehat{\otimes}}_A P_\bullet$ is admissible (cf. Remark 6.4).

It follows from Theorem 6.7 (4) (see also Proposition 3.6 of [48]) that any homological epimorphism of augmented $\widehat{\otimes}$-algebras is a weak homological epimorphism.

Henceforth we shall use the notion of *Hopf $\widehat{\otimes}$-algebra*. Formally, a Hopf $\widehat{\otimes}$-algebra is defined as a Hopf algebra in a symmetric monoidal category of complete LCS endowed with the functor $\widehat{\otimes}$ of completed projective tensor product. For details, see [36, 5, 47, 48]; we just remark that the definition of Hopf $\widehat{\otimes}$-algebra can be obtained from the usual algebraic definition of Hopf algebra by replacing vector spaces with complete LCS, and the symbol $\otimes$, with the symbol $\widehat{\otimes}$. We also remark that the Arens–Michael envelope of a Hopf $\widehat{\otimes}$-algebra H has a canonical structure of a Hopf $\widehat{\otimes}$-algebra, which is uniquely determined by the requirement that $\iota_H\colon H \to \widehat{H}$ be a morphism of Hopf $\widehat{\otimes}$-algebras; see [48, 6.7].

The next result is Proposition 3.7 from [48].

Proposition 9.9. *Suppose $f\colon U \to H$ is a morphism of Hopf $\widehat{\otimes}$-algebras with invertible antipodes. Then f is a homological epimorphism if and only if f is a weak homological epimorphism.*

Next, we recall the definition of the Koszul complex (see, for example, [6]). Suppose A is a commutative algebra and $\mathbf{x} = (x_1, \ldots, x_n)$ is a set of elements of A. For any $p = 0, \ldots, n$ set $K_p(\mathbf{x}, A) = A \otimes \bigwedge^p \mathbb{C}^n$. Choose a basis $e_1, \ldots, e_n$ in $\mathbb{C}^n$ and for any $p = 1, \ldots, n$ define a morphism $d_p\colon K_p(\mathbf{x}, A) \to K_{p-1}(\mathbf{x}, A)$ by the formula

$$d_p(a \otimes e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} x_{i_k} a \otimes e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_p}.$$

The sequence of A-modules $K_\bullet(\mathbf{x}, A) = (K_p(\mathbf{x}, A), d_p)_{0 \le p \le n}$ forms a chain complex, called the *Koszul complex* of $\mathbf{x} \in A^n$.

If A is an augmented commutative algebra and $x_i \in \operatorname{Ker} \varepsilon_A$ for all $i = 1, \ldots, n$, then we can also define the *augmented Koszul complex* $0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon_A} K_\bullet(\mathbf{x}, A)$.

Proposition 9.10. *Let A be an augmented $\widehat{\otimes}$-algebra containing the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ as a dense augmented subalgebra. Suppose that for any $p = 0, \ldots, n-1$ there exist continuous linear operators $T_1^{(p)}, \ldots, T_n^{(p)}\colon A \to A$ such that for any homogeneous polynomial f of degree m we have*

$$T_i^{(p)}(f) = \frac{1}{p+m} \frac{\partial f}{\partial x_i} \quad (i = 1, \ldots, n).$$

Then the embedding $\mathbb{C}[x_1, \ldots, x_n] \to A$ is a weak homological epimorphism.[6]

Proof. Set $A_0 = \mathbb{C}[x_1, \ldots, x_n]$, $\mathbf{x} = (x_1, \ldots, x_n)$ and consider the augmented Koszul complex

$$0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon} K_\bullet(\mathbf{x}, A_0). \tag{9.7}$$

It is well known (see, for example, [6]) that complex (9.7) is exact and is thus a projective resolution of $\mathbb{C}$ in A_0-mod. Notice that since A_0 is dense in A, the canonical morphism $A \mathbin{\widehat{\otimes}}_{A_0} \mathbb{C} \to \mathbb{C}$ is an isomorphism. Therefore, applying the functor $A \mathbin{\widehat{\otimes}}_{A_0} (\,\cdot\,)$ to (9.7), we have an augmented Koszul complex

$$0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon} K_\bullet(\mathbf{x}, A). \tag{9.8}$$

Notice that (9.7) is a dense subcomplex of (9.8).

For each $p = 0, \ldots, n-1$ define an operator $h_p\colon K_p(\mathbf{x}, A) \to K_{p+1}(\mathbf{x}, A)$ by the formula

$$h_p(a \otimes \omega) = \sum_{i=1}^{n} T_i^{(p)}(a) \otimes e_i \wedge \omega \quad (a \in A,\ \omega \in \textstyle\bigwedge^p \mathbb{C}^n).$$

We also define an operator $h_{-1}\colon \mathbb{C} \to A$, $1_{\mathbb{C}} \mapsto 1_A$. It is known (see, for example, [31, 18.7.6 (3)]) that the restrictions of the operators h_p to the subcomplex (9.7) yield a contracting homotopy of the latter. As h_p is continuous and (9.7) is dense in (9.8), it follows that $\{h_p\}$ is a contracting homotopy of the complex (9.8). Therefore, (9.8) is split in LCS, which means that $A_0 \to A$ is a weak homological epimorphism (see Remark 9.4). ∎

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an ideal of codimension 1. Fix some $x \in \mathfrak{g} \setminus \mathfrak{h}$ and set $R = U(\mathfrak{h})$. It is easy to see that there is an algebra isomorphism $U(\mathfrak{g}) \cong R[x; \delta]$, where $\delta = \mathrm{ad}_x$ (see the discussion preceding Example 5.1). Identifying $U(\mathfrak{g})$

[6]Of course, $\mathbb{C}[x_1, \ldots, x_n]$ is viewed here as a $\widehat{\otimes}$-algebra relative to the strongest locally convex topology.

with $R[x;\delta]$, and its Arens–Michael envelope $\widehat{U}(\mathfrak{g})$, with $\mathcal{O}(\mathbb{C}, R_\delta; \hat{\delta})$ (see Theorem 5.1), we have Hopf $\widehat{\otimes}$-algebra structures on $R[x;\delta]$ and $\mathcal{O}(\mathbb{C}, R_\delta; \hat{\delta})$; moreover, the canonical morphism $\iota_{R[x;\delta]} \colon R[x;\delta] \to \mathcal{O}(\mathbb{C}, R_\delta; \hat{\delta})$ becomes a morphism of Hopf $\widehat{\otimes}$-algebras (see the discussion preceding Proposition 9.9).

We shall consider R_δ as the closed subalgebra of $H = \mathcal{O}(\mathbb{C}, R_\delta; \hat{\delta})$ consisting of constant functions. Clearly, R_δ is complemented in H as a subspace; therefore $R_\delta \mathbin{\widehat{\otimes}} R_\delta$ is canonically identified with a closed subalgebra of $H \mathbin{\widehat{\otimes}} H$. Let Δ_H and S_H denote, respectively, the comultiplication and the antipode of H. Notice that $\Delta_H(R_\delta) \subset R_\delta \mathbin{\widehat{\otimes}} R_\delta$ and $S_H(R_\delta) \subset R_\delta$ (this immediately follows from the facts that R is a Hopf subalgebra of $R[x;\delta]$, $\iota\colon R[x;\delta] \to H$ is a morphism of Hopf $\widehat{\otimes}$-algebras, and $\overline{\iota(R)} = R_\delta$). Thus R_δ is a Hopf $\widehat{\otimes}$-algebra and the canonical homomorphism $j = \iota|_R \colon R \to R_\delta$ is a morphism of Hopf $\widehat{\otimes}$-algebras. Notice also that $S_H^2 = \mathbf{1}_H$ immediately implies that R_δ is a Hopf $\widehat{\otimes}$-algebra with an invertible antipode.

Corollary 9.11. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a codimension 1 ideal, and $R = U(\mathfrak{h})$. Fix some $x \in \mathfrak{g} \setminus \mathfrak{h}$ and set $\delta = \mathrm{ad}_x \colon R \to R$. Suppose that $j \colon R \to R_\delta$ is a weak homological epimorphism. Then $\iota_{U(\mathfrak{g})} \colon U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ is a homological epimorphism.*

Proof. Taking account of Proposition 9.9 and the above discussion, this is a particular case of Corollary 9.8. □

Theorem 9.12. *Suppose A is any of the following algebras:*

(i) *$A = \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$, the multiparameter quantum affine space (where $\mathbf{q} = (q_{ij})_{i,j=1}^n$ is a multiplicatively antisymmetric $(n \times n)$-matrix such that either $|q_{ij}| \geq 1$ for all $i < j$ or $|q_{ij}| \leq 1$ for all $i < j$);*
(ii) *$A = \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}\big((\mathbb{C}^\times)^n\big)$, the multiparameter quantum torus (where $\mathbf{q} = (q_{ij})_{i,j=1}^n$ is a multiplicatively antisymmetric $(n \times n)$-matrix);*
(iii) *A is an algebra satisfying the conditions of Theorem 5.18;*
(iv) *$A = A_1(q)$, the quantum Weyl algebra (where $q \in \mathbb{C}^\times$);*
(v) *$A = M_q(2)$, the algebra of quantum (2×2)-matrices (where $q \in \mathbb{C}^\times$);*
(vi) *$A = U(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra with basis $\{x, y\}$ and commutator $[x, y] = y$;*
(vii) *$A = U(\mathfrak{g})$, where $\mathfrak{g}$ is the three-dimensional Heisenberg Lie algebra.*

Then $\iota_A \colon A \to \widehat{A}$ is a homological epimorphism.

Proof. (i) Notice first that we have an obvious isomorphism

$$\mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n) \cong \mathcal{O}_{\check{\mathbf{q}}}^{\mathrm{alg}}(\mathbb{C}^n),$$

where $\check{q}_{ij} = q_{n-i,n-j}$, sending x_i to x_{n-i} (cf. the proof of Proposition 5.12 (ii)). Therefore, without loss of generality we may assume that $|q_{ij}| \geq 1$ for all $i < j$.

We now induct on n. When $n = 0$ the assertion is obvious. Suppose that it is true for matrices of order $n-1$. Set $\bar{\mathbf{q}} = (q_{ij})_{i,j=1}^{n-1}$ and consider the algebra $R = \mathcal{O}_{\bar{\mathbf{q}}}^{\mathrm{alg}}(\mathbb{C}^{n-1})$. By Corollary 5.13 (i), for its Arens–Michael envelope $\widehat{R} = \mathcal{O}_{\bar{\mathbf{q}}}(\mathbb{C}^{n-1})$, representation (5.11) holds (with n replaced by $n-1$). As we said before, there is an isomorphism $A = R[x_n; \alpha]$, where the endomorphism $\alpha \colon R \to R$ is uniquely determined by the conditions $\alpha(x_i) = q_{ni} x_i$ $(i = 1, \ldots, n-1)$. Since $|q_{ni}| \leq 1$, it follows from (5.11) that $\|\widehat{\alpha}(a)\|_\rho \leq \|a\|_\rho$ for any $a \in \widehat{R}$ and any $\rho > 0$. Therefore, $\widehat{\alpha}$ is m-localizable, and to complete the proof, apply Corollary 9.6.

(ii) If $|q_{ij}| \neq 1$ at least for one pair i, j, then $\widehat{A} = 0$ (see Proposition 5.23), and there is nothing to prove. If $|q_{ij}| = 1$ for all i, j, then, using Corollaries 5.22 and 9.7, we can argue as in the proof of (i).

(iii) Set $R = \mathcal{O}_{\mathbf{q}}^{\mathrm{alg}}(\mathbb{C}^n)$; then $A = R[x_{n+1}, \sigma, \delta]$, where the endomorphism $\sigma\colon R \to R$ and the σ-derivation $\delta\colon R \to R$ are uniquely determined by the conditions

$$\sigma(x_i) = \lambda_i x_i \quad (1 \le i \le n), \quad \delta(x_1) = \mu x^\gamma, \quad \delta(x_k) = 0 \quad (k \ge 2)$$

(see the proof of Theorem 5.18). Furthermore, the canonical extension of σ and δ to $\widehat{R}$ form an m-localizable family $\{\widehat{\sigma}, \widehat{\delta}\}$ (ibid.). Now use (i) and Corollary 9.6.

(iv) If $q = 1$, then $\widehat{A} = 0$ (see Example 3.4), and there is nothing to prove. If $q \ne 1$, then A satisfies the conditions of Theorem 5.18 (see Example 5.5), and everything reduces to (iii).

(v) If $q \ne \pm 1$, then A satisfies the conditions of Theorem 5.18 (see Example 5.6) and the desired assertion follows from (iii). If $q = -1$, then $A = \mathcal{O}_{\mathbf{q}'}(\mathbb{C}^4)$ (see (5.30)) and everything reduces to (i). Finally, when $q = 1$ we have $A = \mathbb{C}[a, b, c, d]$. Now apply Proposition 4.3 from [67] (or use (i) with the matrix $\mathbf{q}$ consisting of ones).

(vi) Set $\mathfrak{h} = \mathbb{C}y$, $R = U(\mathfrak{h}) = \mathbb{C}[y]$, and $\delta = \mathrm{ad}_x\colon R \to R$. Then $R_\delta = \mathbb{C}[[y]]$ (see Example 5.1). Consider the linear operator

$$T_1^{(0)}\colon \mathbb{C}[[y]] \to \mathbb{C}[[y]], \quad \sum_n c_n y^n \mapsto \sum_{n \ge 1} c_n y^{n-1}.$$

Clearly, $T_1^{(0)}$ is continuous and satisfies the conditions of Proposition 9.10. Therefore, $j\colon R \to R_\delta$ is a weak homological epimorphism, and the desired assertion follows from Corollary 9.11.

(vii) Set $\mathfrak{h} = \mathrm{span}\{y, z\}$, $R = U(\mathfrak{h}) = \mathbb{C}[y, z]$, and $\delta = \mathrm{ad}_x\colon R \to R$. Then

$$R_\delta = \left\{ a = \sum_{i,j=0}^\infty c_{ij} y^i z^j : \|a\|_\rho = \sum_{i,j=0}^\infty |c_{ij}| \frac{i!}{(i+j)!} \rho^{i+j} < \infty \; \forall \rho > 0 \right\}$$

(see Example 5.2). For $p = 0, 1$ consider the linear operators

$$T_1^{(p)}\colon R_\delta \to R_\delta, \quad \sum_{i,j \ge 0} c_{ij} y^i z^j \mapsto \sum_{i \ge 1,\, j \ge 0} c_{ij} \, \frac{i}{i+j+p} \, y^{i-1} z^j,$$

$$T_2^{(p)}\colon R_\delta \to R_\delta, \quad \sum_{i,j \ge 0} c_{ij} y^i z^j \mapsto \sum_{i \ge 0,\, j \ge 1} c_{ij} \, \frac{j}{i+j+p} \, y^i z^{j-1}.$$

It is not difficult to check that these operators are well-defined and continuous. Indeed, for any $a = \sum_{i,j} c_{ij} y^i z^j \in R_\delta$ and any $\rho > 0$ we have

$$\sum_{i \ge 1,\, j \ge 0} |c_{ij}| \, \frac{i}{i+j+p} \, \frac{(i-1)!}{(i+j-1)!} \, \rho^{i+j-1} \le \sum_{i \ge 1,\, j \ge 0} |c_{ij}| \, \frac{i!}{(i+j)!} \, \rho^{i+j-1} \le \rho^{-1} \|a\|_\rho,$$

whence $\|T_1^{(p)}(a)\|_\rho \le \rho^{-1} \|a\|_\rho$. Furthermore,

$$\sum_{i \ge 0,\, j \ge 1} |c_{ij}| \, \frac{j}{i+j+p} \, \frac{i!}{(i+j-1)!} \, \rho^{i+j-1} \le \sum_{i \ge 0,\, j \ge 1} |c_{ij}| \, \frac{2^{j-1} i!}{(i+j)!} \, \rho^{i+j-1}$$
$$\le \sum_{i \ge 0,\, j \ge 1} |c_{ij}| \, \frac{i!}{(i+j)!} \, (2\rho)^{i+j-1} \le (2\rho)^{-1} \|a\|_{2\rho},$$

and therefore $\left\| T_2^{(p)}(a) \right\|_\rho \le (2\rho)^{-1} \|a\|_{2\rho}$. Thus the operators $T_1^{(p)}$ and $T_2^{(p)}$ are well-defined, continuous, and, clearly, satisfy the conditions of Proposition 9.10. Therefore $j\colon R \to R_\delta$ is a weak homological epimorphism, and the desired assertion follows from Corollary 9.11. $\square$

10. Appendix: Derived categories

For the convenience of the reader, in this appendix we mention some facts from the theory of derived categories. Notice that most of the standard references [23, 18, 30] only consider the derived categories of abelian categories, whereas the typical categories of functional analysis are not abelian. We shall consider the more general case of exact categories in the sense of Quillen [56]. We shall mostly follow the survey paper by B. Keller [32].

Let $\mathcal{A}$ be an additive category. We shall always assume that idempotents split in $\mathcal{A}$; i.e., each morphism $p\colon X \to X$ in $\mathcal{A}$ such that $p^2 = p$ has a kernel. An *exact pair* in $\mathcal{A}$ is any sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$, such that $i = \ker p$ and $p = \operatorname{coker} i$. Suppose $\mathcal{E}$ is some class of exact pairs in $\mathcal{A}$, closed under isomorphisms. Elements of $\mathcal{E}$ will be called *admissible pairs*. A morphism $i\colon X \to Y$ (resp., $p\colon Y \to Z$) is said to be an *admissible monomorphism* (resp., *admissible epimorphism*) if it is part of an admissible pair $X \xrightarrow{i} Y \xrightarrow{p} Z$.

An *exact category* is an additive category $\mathcal{A}$ endowed with a class of exact pairs $\mathcal{E}$ satisfying the following axioms:

Ex0. The identity morphism of a zero object is an admissible epimorphism.

Ex1. The composition of two admissible epimorphisms is an admissible epimorphism.

Ex1°. The composition of two admissible monomorphisms is an admissible monomorphism.

Ex2. Each diagram

$$\begin{tikzcd} & Y' \arrow[d] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

where p is an admissible epimorphism, can be extended to a pull-back square

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

where p' is an admissible epimorphism.

Ex2°. Each diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \\ X' & \end{tikzcd}$$

where i is an admissible monomorphism, can be extended to a push-out square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "i'"] & Y' \end{tikzcd}$$

where i' is an admissible monomorphism.

If $\mathcal{A}$ is an exact category, then the dual category $\mathcal{A}^\circ$ is also exact, provided the admissible pairs in $\mathcal{A}^\circ$ are defined as the pairs $X \xrightarrow{i} Y \xrightarrow{p} Z$ such that the pair $Z \xrightarrow{p} Y \xrightarrow{i} X$ in $\mathcal{A}$ is admissible.

Example 10.1. Each additive category $\mathcal{A}$ can be made exact if $\mathcal{E}$ is defined as the class of all split exact sequences. The corresponding exact category will be denoted $\mathcal{A}_{\mathrm{spl}}$. We shall then say that $\mathcal{A}$ is endowed with the *split exact structure*.

Example 10.2 (quasi-abelian and abelian categories)**.** Suppose that $\mathcal{A}$ is an additive category with kernels and cokernels, and let $\mathcal{E}$ denote the class of all exact pairs in $\mathcal{A}$. If, under these assumptions, A satisfies axioms Ex2 and Ex2°, then $\mathcal{A}$ is said to be *quasi-abelian* [53, 61] (or, in the original terminology, *semi-abelian* [58]). Since axioms Ex0, Ex1, and Ex1° are automatically satisfied, any quasi-abelian category endowed with a class of all exact pairs becomes an exact category. Notice that many standard categories of functional analysis are quasi-abelian [54]. The examples include: the category LCS of locally convex spaces (LCS), its full subcategory consisting of Hausdorff LCS, the category of Fréchet spaces, the categories of all normed spaces and of all Banach spaces, and others. Notice, however, that the category of all complete LCS is not quasi-abelian [54].

Recall that $\mathcal{A}$ is said to be *abelian* if any monomorphism in it is a kernel, and any epimorphism is a cokernel (see, for example, [39]). Any abelian category is quasi-abelian and therefore is an exact category.

Example 10.3. Suppose $\mathcal{A}$ is an additive category with kernels and cokernels, $\mathcal{B}$ is an exact category, and $\square\colon \mathcal{A} \to \mathcal{B}$ is an additive functor preserving kernels and cokernels. Then it is easy to check that $\mathcal{A}$ becomes an exact category if one defines admissible pairs in $\mathcal{A}$ as the pairs whose images under $\square$ are admissible in $\mathcal{B}$. An important particular case of this construction is when $\mathcal{B} = \mathcal{B}_{\mathrm{spl}}$. In this case we denote the exact category obtained by $\mathcal{A}_{\mathcal{B}}$.

For a given additive category $\mathcal{A}$ the category of all chain complexes in $\mathcal{A}$ is denoted $\mathsf{C}(\mathcal{A})$. Its full subcategory consisting of complexes bounded on the right (resp., bounded on the left, bounded), i.e., complexes $X \in \mathrm{Ob}\,(\mathsf{C}(\mathcal{A}))$ such that $X^n = 0$ for all $n > N$ (resp., for all $n < N$, for all $|n| > N$) holds for some $N \in \mathbb{Z}$, is denoted $\mathsf{C}^-(\mathcal{A})$ (resp., $\mathsf{C}^+(\mathcal{A})$, $\mathsf{C}^b(\mathcal{A})$). The differential in a chain complex X mapping X^n to X^{n+1} will be denoted d_X^n. Recall that the homotopy category $\mathsf{H}(\mathcal{A})$ is defined as follows: it has the same objects as $\mathsf{C}(\mathcal{A})$, and the group of morphisms $\mathrm{Hom}_{\mathsf{H}(\mathcal{A})}(X,Y)$ is defined as the quotient of $\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X,Y)$ by the subgroup of all null-homotopic morphisms. The categories $\mathsf{H}^-(\mathcal{A})$, $\mathsf{H}^+(\mathcal{A})$, and $\mathsf{H}^b(\mathcal{A})$ are defined similarly.

Following the convention, we identify chain complexes with cochain complexes by viewing a chain complex (X_n, d_n) as a cochain complex (X^n, d^n), where $X^n = X_{-n}$ and $d^n = d_{-n}$. Notice also that there is an isomorphism $\mathsf{C}(\mathcal{A}^\circ) \cong \mathsf{C}(\mathcal{A})^\circ$, assigning to a complex (X^n, d^n) in $\mathcal{A}^\circ$ a complex $(\bar{X}^n, \bar{d}^n)$ in $\mathcal{A}$, such that $\bar{X}^n = X^{-n}$ and $\bar{d}^n = d^{-n-1}$. This isomorphism, in turn, gives rise to isomorphisms $\mathsf{C}^-(\mathcal{A}^\circ) \cong \mathsf{C}^+(\mathcal{A})^\circ$ and $\mathsf{H}^-(\mathcal{A}^\circ) \cong \mathsf{H}^+(\mathcal{A})^\circ$.

For any cochain complex X in $\mathcal{A}$, let $X[1]$ be the complex such that $X[1]^n = X^{n+1}$ and $d_{X[1]}^n = -d_X^{n+1}$ for all $n \in \mathbb{Z}$. If $f\colon X \to Y$ is a morphism in $\mathsf{C}(\mathcal{A})$, then the morphism $f[1]\colon X[1] \to Y[1]$ is defined by the formula $f[1]^n = f^{n+1}$. Thus we have a functor $X \mapsto X[1]$ from $\mathsf{C}(\mathcal{A})$ in $\mathsf{C}(\mathcal{A})$, called the *suspension functor* (or the *shift functor*). Similarly, one defines the suspension functor from $\mathsf{H}(\mathcal{A})$ to $\mathsf{H}(\mathcal{A})$.

Recall that for any morphism $f\colon X \to Y$ in $\mathsf{C}(\mathcal{A})$ its *cone* $M(f)$ is the complex defined as follows:

$$M(f)^n = X^{n+1} \oplus Y^n, \quad d^n_{M(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^n & d^n_Y \end{pmatrix}.$$

Each morphism $f\colon X \to Y$ is part of a sequence

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1], \tag{10.1}$$

where $\alpha(f)\colon y \mapsto (0,y)$ and $\beta(f)\colon (x,y) \mapsto x$.

A *triangle* in $\mathsf{H}(\mathcal{A})$ is any sequence of morphisms of the form $X \to Y \to Z \to X[1]$. A morphism from this triangle to a triangle $X' \to Y' \to Z' \to X'[1]$ is a triple of morphisms (u,v,w) which is part of a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & X[1] \arrow[d, "{u[1]}"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

A triangle in $\mathsf{H}(\mathcal{A})$ is said to be *distinguished* if it is isomorphic to a triangle of the form (10.1).

The category $\mathsf{H}(\mathcal{A})$ endowed with the functor $X \mapsto X[1]$ and the class of all distinguished triangles is a *triangulated category*. We do not give here the precise definition of a triangulated category (see, for example, [18] or [30]); we just remark that a triangulated category $(\mathcal{C}, T)$ is an additive category $\mathcal{C}$ endowed with an automorphism $T\colon \mathcal{C} \to \mathcal{C}$ and a class of sequences $X \to Y \to Z \to T(X)$, called distinguished triangles, satisfying certain axioms.

For the proof of the following standard fact, see, for example, [18] or [30].

Proposition 10.1. *Suppose (u,v,w) is a morphism of distinguished triangles in a triangulated category $(\mathcal{C}, T)$ such that two of the morphisms u, v, w are isomorphisms. Then so is the third.*

We continue to assume that $\mathcal{A}$ is an exact category. A complex X in $\mathcal{A}$ is said to be *admissible* if for any $n \in \mathbb{Z}$ the differential d^n has a kernel, and the canonical sequence $\operatorname{Ker} d^n \to X^n \to \operatorname{Ker} d^{n+1}$ is an admissible pair. A morphism $f\colon X \to Y$ in $\mathsf{C}(\mathcal{A})$ is called a *quasi-isomorphism* if its cone $M(f)$ is admissible.

Example 10.4. If $\mathcal{A}$ is abelian, then a complex in $\mathcal{A}$ is admissible if and only if it is exact, and a morphism $f\colon X \to Y$ in $\mathsf{C}(\mathcal{A})$ is a quasi-isomorphism if and only if it induces an isomorphism in cohomology.

Example 10.5. Suppose that $\mathcal{A}$ is an additive category with kernels and cokernels. Recall that a morphism $f\colon X \to Y$ is said to be *strict* if the induced morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism. Notice that $\mathcal{A}$ is abelian if and only if each morphism in it is strict. If $\mathcal{A}$ is quasi-abelian (see Example 10.2), then a complex in $\mathcal{A}$ is admissible if and only if it is *strictly exact*, i.e., when it is exact and each differential d^n is a strict morphism [53, 61]. In particular, a complex in the quasi-abelian category LCS is admissible if and only if it is *topologically exact* [25], i.e., when each d_n is an open map from X^n to $\operatorname{Ker} d^{n+1}$ [54].

Example 10.6. For an arbitrary additive category $\mathcal{B}$ a complex $X \in \operatorname{Ob}(\mathsf{C}(\mathcal{B}))$ is admissible in the exact category $\mathcal{B}_{\mathrm{spl}}$ if and only if it is split (i.e., null-homotopic). In this case quasi-isomorphisms are precisely homotopy equivalences.

Example 10.7. As an immediate consequence of the preceding example, we have that in the exact category $\mathcal{A}_{\mathcal{B}}$ (see Example 10.3) a complex X is admissible if and only if $\Box X$ is split in $\mathcal{B}$, and a morphism of complexes $f\colon X \to Y$ is a quasi-isomorphism if and only if $\Box f\colon \Box X \to \Box Y$ is a homotopy equivalence.

Suppose $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$ are triangulated categories. A *triangular functor* from $(\mathcal{C}, T)$ to $(\mathcal{C}', T')$ is a pair (F, α), where $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor and $\alpha\colon FT \to T'F$ is a natural isomorphism such that for any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ in $\mathcal{C}$ the triangle

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\alpha_{FT(X)} \circ Fh} T'(FX)$$

is distinguished in $\mathcal{C}'$. A morphism of triangular functors and a triangular equivalence of triangulated categories are defined in an obvious way.

The *derived category* of an exact category $\mathcal{A}$ is defined as the localization of the homotopy category $\mathsf{H}(\mathcal{A})$ by the system of all quasi-isomorphisms. In other words, the derived category is a pair $(\mathsf{D}(\mathcal{A}), Q_{\mathcal{A}})$ consisting of a category $\mathsf{D}(\mathcal{A})$ and a functor $Q_{\mathcal{A}}\colon \mathsf{H}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ sending quasi-isomorphisms to isomorphisms; also, for any category $\mathcal{C}$ and any functor $F\colon \mathsf{H}(\mathcal{A}) \to \mathcal{C}$ sending quasi-isomorphisms to isomorphisms there exists a unique functor $G\colon \mathsf{D}(\mathcal{A}) \to \mathcal{C}$ such that $F = G \circ Q_{\mathcal{A}}$. Moreover, $\mathsf{D}(\mathcal{A})$ has a canonical structure of a triangulated category and the functor $Q_{\mathcal{A}}$ is triangular. The derived category always exists and is unique up to a triangular equivalence. If $\mathcal{C}$ is a triangulated category and $F\colon \mathsf{H}(\mathcal{A}) \to \mathcal{C}$ is a triangular functor sending admissible objects to zero, then it sends quasi-isomorphisms to isomorphisms, and the functor $G\colon \mathsf{D}(\mathcal{A}) \to \mathcal{C}$, whose existence is postulated above, is also triangular.

In a similar way (replacing $\mathsf{H}(\mathcal{A})$ by $\mathsf{H}^-(\mathcal{A})$, $\mathsf{H}^+(\mathcal{A})$, or $\mathsf{H}^b(\mathcal{A})$) one defines derived categories $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^+(\mathcal{A})$, and $\mathsf{D}^b(\mathcal{A})$. Notice that there are isomorphisms $\mathsf{D}^-(\mathcal{A}^\circ) \cong \mathsf{D}^+(\mathcal{A})^\circ$ and $\mathsf{D}^+(\mathcal{A}^\circ) \cong \mathsf{D}^-(\mathcal{A})^\circ$. Moreover, $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^+(\mathcal{A})$, and $\mathsf{D}^b(\mathcal{A})$ can be identified with full subcategories of $\mathsf{D}(\mathcal{A})$. For more details, see [32] or (for the abelian case) [18, 30].

Associating with each $X \in \mathrm{Ob}\,(\mathcal{A})$ the complex whose degree zero term is X and all other terms are zero, we have a fully faithful additive functor $\mathcal{A} \to \mathsf{C}(\mathcal{A})$. Its compositions with the canonical functors $\mathsf{C}(\mathcal{A}) \to \mathsf{H}(\mathcal{A})$ and $\mathsf{C}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ are also fully faithful functors; for this reason $\mathcal{A}$ is customarily identified with the full subcategories of $\mathsf{C}(\mathcal{A})$, $\mathsf{H}(\mathcal{A})$, and $\mathsf{D}(\mathcal{A})$ consisting of complexes concentrated in degree zero. A similar convention is used for $\mathsf{C}^*(\mathcal{A})$, $\mathsf{H}^*(\mathcal{A})$, and $\mathsf{D}^*(\mathcal{A})$, where $* \in \{+, -, b\}$.

Let $\mathcal{A}, \mathcal{B}$ be exact categories and $F\colon \mathcal{A} \to \mathcal{B}$ an additive covariant functor. In an obvious way, F gives rise to a triangular functor $\mathsf{H}^-(F)\colon \mathsf{H}^-(\mathcal{A}) \to \mathsf{H}^-(\mathcal{B})$. The *left derived functor* of F is defined as a pair $(\mathrm{L}F, s)$, consisting of a triangular functor

$$\mathrm{L}F\colon \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}^-(\mathcal{B})$$

and a morphism of triangular functors

$$s_F\colon \mathrm{L}F \circ Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ \mathsf{H}^-(F), \tag{10.2}$$

such that for any other such pair (G, s) there exists a unique morphism of triangular functors $\alpha\colon G \to \mathrm{L}F$ with $s = s_F \circ (\alpha Q_{\mathcal{A}})$.

Right derived functors, as well as derived functors of contravariant functors can be defined by passing to the dual categories. For each functor $F\colon \mathcal{A} \to \mathcal{B}$ we have the functors $F^\circ\colon \mathcal{A}^\circ \to \mathcal{B}$, $F_\circ\colon \mathcal{A} \to \mathcal{B}^\circ$, and $F^\circ_\circ\colon \mathcal{A}^\circ \to \mathcal{B}^\circ$, acting the same way as F; moreover, if F is covariant (resp., contravariant), then so is $F^\circ_\circ$, whereas F° and $F_\circ$ are contravariant (resp., covariant).

If $F\colon \mathcal{A} \to \mathcal{B}$ is an additive covariant functor, then its *right derived* functor is defined as the functor $\mathrm{R}F = \mathrm{L}(F^\circ_\circ)^\circ_\circ\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$. If F is contravariant, then we set $\mathrm{R}F = \mathrm{L}(F_\circ)_\circ\colon \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ and $\mathrm{L}F = \mathrm{L}(F^\circ)^\circ\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^-(\mathcal{B})$.

A functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to be *exact* if it sends admissible complexes to admissible ones. If F is exact, then, as follows from the definition of the derived category, there exists a unique triangular functor $F'\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ such that $F' \circ Q_\mathcal{A} = Q_\mathcal{B} \circ \mathsf{H}(F)$. It is easy to see that the restriction of F' to $\mathsf{D}^-(\mathcal{A})$ (resp., $\mathsf{D}^+(\mathcal{A})$) is the left (resp., right) derived functor of F; moreover, (10.2) is the identity morphism.

Recall that an object $P \in \mathrm{Ob}\,(\mathcal{A})$ is said to be *projective* if for any admissible epimorphism $X \to Y$ in $\mathcal{A}$ the induced map $\mathrm{Hom}_\mathcal{A}(P, X) \to \mathrm{Hom}_\mathcal{A}(P, Y)$ is surjective. One says that $\mathcal{A}$ has *enough projectives* if for any $X \in \mathrm{Ob}\,(\mathcal{A})$ there are a projective object $P \in \mathrm{Ob}\,(\mathcal{A})$ and an admissible epimorphism $P \to X$.

Example 10.8. Under the assumptions of Example 10.3, assume in addition that the functor $\square\colon \mathcal{A} \to \mathcal{B}$ has a left adjoint $F\colon \mathcal{B} \to \mathcal{A}$ and that $\mathcal{B} = \mathcal{B}_{\mathrm{spl}}$. Then for any admissible epimorphism $Y \to Z$ in $\mathcal{A}_\mathcal{B}$ the morphism $\square Y \to \square Z$ is a retraction in $\mathcal{B}$, and therefore for any $E \in \mathrm{Ob}\,(\mathcal{B})$ the map $\mathrm{Hom}_\mathcal{B}(E, \square Y) \to \mathrm{Hom}_\mathcal{B}(E, \square Z)$ is surjective. Since F and $\square$ are adjoint, we conclude that $F(E)$ is projective in $\mathcal{A}_\mathcal{B}$. Furthermore, since the composition $\square \to \square F \square \to \square$ is the identity morphism of $\square$ (see, for example, [39, 4.1.1]), we have that for any $X \in \mathrm{Ob}\,(\mathcal{A})$ the morphism $\square F(\square X) \to \square X$ is a retraction in $\mathcal{B}$, or, equivalently, that $F(\square X) \to X$ is an admissible epimorphism. Therefore, $\mathcal{A}_\mathcal{B}$ has enough projectives. Moreover, an object X is projective if and only if the canonical morphism $F(\square X) \to X$ is a retraction.

Suppose that $\mathcal{A}$ is an exact category with enough projectives. Then for any complex $X \in \mathrm{Ob}\,(\mathsf{C}^-(\mathcal{A}))$ there are a complex $P \in \mathrm{Ob}\,(\mathsf{C}^-(\mathcal{A}))$ consisting of projectives and a quasi-isomorphism $P \to X$. Any two complexes P, P' with this property are homotopy equivalent (and therefore quasi-isomorphic). Any such complex P together with a quasi-isomorphism $P \to X$ is called a *projective resolution* of X. Let $\mathcal{P}$ be the full subcategory of $\mathcal{A}$ consisting of projectives. Then, as follows from the above discussion, the canonical functor $\mathsf{D}^-(\mathcal{P}) \to \mathsf{D}^-(\mathcal{A})$ is an equivalence of triangulated categories. On the other hand, any acyclic complex consisting of projectives is homotopy equivalent to a zero complex; hence $Q_\mathcal{P}\colon \mathsf{H}^-(\mathcal{P}) \to \mathsf{D}^-(\mathcal{P})$ is an equivalence. The left derived functor $\mathrm{L}F\colon \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}^-(\mathcal{B})$ can now be defined as the composition of the functors

$$\mathsf{D}^-(\mathcal{A}) \xleftarrow{\sim} \mathsf{D}^-(\mathcal{P}) \xleftarrow{\sim} \mathsf{H}^-(\mathcal{P}) \to \mathsf{H}^-(\mathcal{A}) \xrightarrow{\mathsf{H}^-(F)} \mathsf{H}^-(\mathcal{B}) \xrightarrow{Q_\mathcal{B}} \mathsf{D}^-(\mathcal{B}).$$

Notice that having enough projectives is a rather strict sufficient condition for the existence of a left derived functor. For much more general conditions, see [32] or (in the abelian case) [18, 30].

An object $I \in \mathrm{Ob}\,(\mathcal{A})$ is said to be *injective* if it is projective as an object of $\mathcal{A}^\circ$. As follows from the above, for the existence of the right derived functor of a covariant functor or of the left derived functor of a contravariant functor it suffices that $\mathcal{A}$ has enough injectives, i.e., for any $X \in \mathrm{Ob}\,(\mathcal{A})$ there is an admissible monomorphism $X \to I$, where I is injective. Similarly, for the existence of the right derived functor of a contravariant functor it suffices that $\mathcal{A}$ has enough projectives.

If the category $\mathcal{B}$ is quasi-abelian, then for any $n \in \mathbb{Z}$ the cohomology functor $H^n\colon \mathsf{C}^-(\mathcal{B}) \to \mathcal{B}$ sends quasi-isomorphisms to isomorphisms. This is proved the same way as for abelian categories, using the long cohomology exact sequence (see [34]).[7] Therefore, the functor H^n extends to a functor $\mathsf{D}^-(\mathcal{B}) \to \mathcal{B}$, also denoted H^n. The

[7] In fact, a simple argument shows that this is true in any exact category $\mathcal{B}$ with kernels and cokernels. We skip the proof, since we do not need such a level of generality.

composition $H^{-n} \circ \mathrm{L}F \colon \mathsf{D}^-(\mathcal{A}) \to \mathcal{B}$ (and its restriction to $\mathcal{A}$) is called the *n-th classical left derived functor* of F and is denoted $\mathrm{L}_n F$. Similarly, the *n-th classical right derived functor* $\mathrm{R}^n F \colon \mathsf{D}^+(\mathcal{A}) \to \mathcal{B}$ is defined as the composition $H^n \circ \mathrm{R}F$.

Example 10.9. Suppose that $\mathcal{A}$ has enough projectives. Fix some $Y \in \mathrm{Ob}\,(\mathcal{A})$ and consider the functor

$$\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, Y) \colon \mathcal{A} \to \mathcal{A}b,$$

where $\mathcal{A}b$ is the (abelian) category of abelian groups. Its right derived functor is denoted $\mathrm{RHom}_{\mathcal{A}}(\,\cdot\,, Y)$, and its n-th classical right derived functor is denoted $\mathrm{Ext}^n_{\mathcal{A}}(\,\cdot\,, Y)$. The functor $\mathrm{RHom}_{\mathcal{A}}(\,\cdot\,, Y)$ extends to a bifunctor

$$\mathrm{RHom}_{\mathcal{A}} \colon \mathsf{D}^-(\mathcal{A}) \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A}b)$$

(the proof is the same as in the case of abelian categories; see, for example, [23, I.6] or [30, 1.10]). Notice that we have a natural isomorphism

$$\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$$

(ibid. or [18, III.6.15]).

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are exact categories and $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{C}$ are additive covariant functors. Suppose that F, G, and GF have left derived functors. It follows from the definition of the derived functor that there is a morphism of functors $c \colon \mathrm{L}G \circ \mathrm{L}F \to \mathrm{L}(GF)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{L}G \circ \mathrm{L}F \circ Q_{\mathcal{A}} \arrow[r, "(\mathrm{L}G)s_F"] \arrow[d, "cQ_{\mathcal{A}}"'] & \mathrm{L}G \circ Q_{\mathcal{B}} \circ \mathsf{H}^-(F) \arrow[d, "s_G(\mathsf{H}^-(F))"] \\
\mathrm{L}(GF) \circ Q_{\mathcal{A}} \arrow[r, "s_{GF}"'] & Q_{\mathcal{C}} \circ \mathsf{H}^-(G) \circ \mathsf{H}^-(F)
\end{tikzcd}$$

If c is an isomorphism, then we say that the theorem on the derived functor of the composition holds for F and G. For some of the sufficient conditions guaranteeing this, see [32] or (in the abelian case) [18, 30]. In particular, this theorem holds if $\mathcal{A}$ and $\mathcal{B}$ have enough projectives, and F sends projectives to projectives.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are exact categories and $F \colon \mathcal{A} \to \mathcal{B}$ is an additive covariant functor having a derived functor. As before, we identify $\mathcal{A}$ and $\mathcal{B}$ with the full subcategories of $\mathsf{D}^-(\mathcal{A})$ and, respectively, $\mathsf{D}^-(\mathcal{B})$, and consider F and $\mathrm{L}F$ as functors from $\mathcal{A}$ to $\mathsf{D}^-(\mathcal{B})$. Then the morphism (10.2) gives rise to a morphism of functors $\mathrm{L}F \to F$. An object $X \in \mathrm{Ob}\,(\mathcal{A})$ is said to be (left) *F-acyclic* if $\mathrm{L}F(X) \to F(X)$ is an isomorphism in $\mathsf{D}^-(\mathcal{B})$.

Example 10.10. A projective object is acyclic relative to any functor (see [32]).

Example 10.11. If F is exact, then any object of $\mathcal{A}$ is F-acyclic.

Example 10.12. Suppose A is a $\widehat{\otimes}$-algebra and Y is a left A-$\widehat{\otimes}$-module. Let Vect denote the abelian category of vector spaces. Then Y is flat (in the sense of [27, 25]) if and only if Y is acyclic relative to the functor $X \mathbin{\widehat{\otimes}}_A (\,\cdot\,) \colon A\text{-}\mathsf{mod} \to \mathsf{Vect}$ for any right A-$\widehat{\otimes}$-module X.

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