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THE ASYMPTOTICS OF A SOLUTION OF A SECOND ORDER ELLIPTIC EQUATION WITH A SMALL PARAMETER MULTIPLYING ONE OF THE HIGHEST ORDER DERIVATIVES

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Abstract. The asymptotic behaviour of a solution of the first boundary value problem for a second order elliptic equation is analysed in the case where a small parameter is involved as a factor multiplying only one of the highest order derivatives and the limit equation is an ordinary differential equation. In spite of the fact that the order of the limit equation is the same as that of the original equation, the problem under consideration is bisingular. The asymptotic behaviour of a solution of this problem is analysed using the method of matching asymptotic expansions.

1. Introduction

In this paper we analyse the behaviour of the solution of the first boundary-value problem

$$\mathrm{L}_\varepsilon u = \varepsilon u_{xx} + u_{yy} + b(x,y)u_y + a(x,y)u = f(x,y), \quad (x,y) \in D \subset \mathbb{R}^2, \tag{1.1}$$

$$u(x,y) = h(x,y), \quad (x,y) \in \Gamma, \tag{1.2}$$

as $\varepsilon \to 0$, in a bounded domain D with a piecewise smooth boundary Γ. We assume that the parameter $\varepsilon > 0$ and that the coefficients and the right-hand side of equation (1.1) are sufficiently smooth functions. We also assume that there exists a bounded solution of problem (1.1), (1.2), which we denote by $u_\varepsilon(x,y)$, and that it satisfies the estimate

$$|u_\varepsilon(x,y)| \le M\Big(\max_{(x,y)\in D} |f(x,y)| + \sup_{(x,y)\in\Gamma} |h(x,y)|\Big), \tag{1.3}$$

where the constant M is independent of ε. (This condition holds, for example, if $a(x,y) \le \alpha < 0$.)

Problems for elliptic equations with a small parameter multiplying the highest order derivatives have been studied by many authors, and the bibliography relating to this problem is extensive and fairly well known. Therefore it seems appropriate only to mention a few papers, [1]–[6], which are closely related to this paper both in their contents and in the method they apply to analyse the asymptotic behaviour of the solution.

The special feature of the problem under investigation here is that the small parameter is involved as a factor multiplying only one of the highest order derivatives, so that the order of the limit equation is the same as that of the original equation. Furthermore, we consider the case where the limit equation is an ordinary differential equation.

To describe the results we have obtained we introduce the notation l_M for the straight line constructed through a point $M(x,y) \in \overline{D}$ parallel to the y-axis. On the straight line

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l_M we define some direction, for example, the direction of increasing y. We let Γ^+ denote the set of points on the boundary Γ at which these straight lines leave the domain D and are not tangent to the boundary of the domain. Similarly, Γ^- denotes the set of points on the boundary Γ at which these straight lines enter the domain D and are not tangent to the boundary of the domain. Finally, Γ_0 denotes the set of points of the boundary Γ at which the straight lines l_M are tangent to the boundary of the domain. In addition we assume that in a small neighbourhood of a point of tangency $M_0 \in \Gamma_0$ either the straight line l_{M_0} lies entirely outside the domain (outer tangency), or some segment of this straight line lies inside the domain (inner tangency). This implies that we are assuming that the order of tangency is odd, and we will denote it by $s_0 = 2n+1$, $n \geq 0$. The set Γ_1 denotes the corner points M_1 of the boundary Γ. The set Γ_2 denotes the part of the boundary Γ that coincides with segments of straight lines parallel to the y-axis.

We set

$$\mathrm{L}_0 = \frac{\partial^2}{\partial y^2} + b(x,y)\frac{\partial}{\partial y} + a(x,y).$$

The limit problem for the original problem (1.1), (1.2) takes the following form: for the straight line l_M constructed through an interior point M of the domain D, along the segment of l_M contained in the domain D, we seek the solution of the equation

$$\mathrm{L}_0 u = f(x,y), \quad y \in l_M, \tag{1.4}$$

satisfying the boundary condition (1.2) of the original problem at the intersection points of this segment with the parts $\Gamma^\pm$ of the boundary Γ:

$$u(x,y) = h(x,y), \quad (x,y) \in \Gamma^\pm. \tag{1.5}$$

(We note that the straight line l_M passing through an interior point of D can intersect the boundary of the domain in more than two points, so for one straight line there may be several such segments.)

We assume that the coefficients of equation (1.1) and the boundary Γ of the domain D are such that for any $f(x,y)$ and $\phi(x,y)$ for a fixed straight line l_M there exists a unique solution of problem (1.4), (1.5) and it satisfies the estimate

$$|u(x,y)| \leq C\Big(\max_{y\in l_M}|f(x,y)| + \max_{y\in\Gamma^\pm}|h(x,y)|\Big). \tag{1.6}$$

(This condition holds, for example, if $a(x,y) \leq \alpha < 0$.)

The standard (outer) asymptotic expansion will be constructed in the form

$$U(x,y,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(x,y). \tag{1.7}$$

Substituting the series (1.7) into equation (1.1) and equating the coefficients of the same powers of ε, we obtain the recurrence relations

$$\mathrm{L}_0 u_0 = f(x,y), \quad \mathrm{L}_0 u_k = -\frac{\partial^2 u_{k-1}}{\partial x^2}, \quad k \geq 1. \tag{1.8}$$

Given a straight line l_M constructed through an interior point M of D, along each of its segments lying in the domain D we define a function $u_k(x,y)$ to be the solution of equation (1.8) satisfying the boundary conditions

$$\begin{aligned} &u_0(x,y) = \phi(x,y), \quad (x,y) \in \Gamma^\pm, \\ &u_k(x,y) = 0, \quad k > 0, \quad (x,y) \in \Gamma^\pm \end{aligned} \tag{1.9}$$

at the intersection points of this segment with the nearest parts $\Gamma^\pm$ of the boundary Γ.

The functions $u_k(x,y)$ are solutions of the boundary-value problems (1.8), (1.9) for ordinary differential equations in the variable y, which are infinitely differentiable with respect to the variable y and depend on the variable x as a parameter. To decide whether the outer expansion (1.7) is suitable everywhere in $\overline{D}$ we need to analyse the behaviour of the solutions we have constructed with respect to the variable x everywhere in this domain.

It turns out that the coefficients of the outer expansion $u_k(x,y)$ have singularities at points of some subsets of the domain $\overline{D}$, and the order of these singularities, generally speaking, increases as k increases. Because of the singularities of the functions $u_k(x,y)$, the outer expansion (1.7) becomes unsuitable everywhere in the domain $\overline{D}$.

Such 'singular' subsets include

a) points $M_0 \in \Gamma_0$ where the straight lines l_{M_0} are tangent to the boundary Γ from outside D;
b) points $M_1 \in \Gamma_1$ which are corner points such that segments of the straight lines l_{M_1} lie outside D;
c) segments of the straight lines l_{M_0} lying in the domain D that are tangent to the boundary Γ at the point $M_0 \in \Gamma_0$ from inside D;
d) segments of the straight lines l_{M_1} lying in D that pass through corner points of the boundary $M_1 \in \Gamma_1$;
e) parts of the boundary Γ_2 that coincide with segments of straight lines parallel to the y-axis.

In [6] the behaviour of the solution $u_\varepsilon(x,y)$ as $\varepsilon \to 0$ was analysed in neighbourhoods of 'singular' sets of the forms a) and b). In this paper, as in [6], the method of matched asymptotic expansions [1, 2] is used to construct and justify an asymptotic expansion of the solution $u_\varepsilon(x,y)$ as $\varepsilon \to 0$ in a neighbourhood of a 'singular' subset of the form c), that is, in a neighbourhood of a segment of the straight line l_{M_1} that is tangent to the boundary Γ of the domain D from the inside.

The matching method consists of two parts, which are, generally speaking, independent of each other. In the first part, we construct formal asymptotic solutions (FAS) of the original problem in various subdomains of the domain where the independent variables take their values. That is, we construct matched asymptotic series from certain sequences of functions of the parameter ε such that partial sums of these series satisfy the original equation and the boundary condition in these subdomains with a sufficiently high degree of accuracy (with respect to ε). Then, from partial sums of these FAS, we construct a composite expansion that is an FAS of the original problem everywhere in the domain of the variables under consideration. In the second part of the matching method we justify our construction of the expansion, that is, we prove that the FAS we have constructed is an asymptotic representation of the solution $u_\varepsilon(x,y)$ as $\varepsilon \to 0$. For similar problems to ours, in which the corresponding estimate (for example, the estimate (1.3)) holds for the inverse operator of the original problem, it is not hard to justify the FAS, and the method is described in a fairly detailed way for similar situations in [2]. Therefore in this paper we deal mainly with the construction of FAS in neighbourhoods of the singular set under consideration.

In § 2 we carry out the analysis of the asymptotics of the coefficients $u_k(x,y)$ of the outer expansion (1.7) in a neighbourhood of a segment of the straight line l_{M_1} that is tangent to the boundary Γ of the domain D from the inside. In §§ 3, 4 we carry out the formal construction of the inner asymptotic expansions, in §§ 5, 6 we make some auxiliary constructions that we need to match the inner expansions, and in § 7 we use the matching method to construct the inner asymptotic expansions.

2. The outer expansion and its singularities

We take M_0 to be a point at which the straight line l_{M_0} is tangent to the smooth boundary Γ of the domain D from the inside of this domain. We assume that the point M_0 coincides with the origin $(0,0)$, the order of tangency is 1, and the boundary Γ of the domain D in some fixed neighbourhood of the origin $D_\delta : (x,y) \in D$, $-\delta \le x \le \delta$, $\delta > 0$, coincides with the curve $x = y^2$ and two smooth curves $y = \gamma^-(x)$ and $y = \gamma^+(x)$ contained in the lower and upper half-planes, respectively ($\gamma^-(x) < -\sqrt{x}$, $\gamma^+(x) > \sqrt{x}$). Thus, the domain D_δ is bounded by the parabola $x = y^2$, the curves $y = \gamma^\pm(x)$, and segments of the straight lines $x = -\delta$, $x = \delta$. We construct an asymptotic expansion of the solution $u_\varepsilon(x,y)$ as $\varepsilon \to 0$ in this domain D_δ. We assume without loss of generality that the boundary function is $h(x,y) \equiv 0$.

The general case, that is, when the tangency has any odd order $s_0 = 2n+1$ $(n > 0)$, is not fundamentally different from the special case we consider but the calculations are more cumbersome.

For $x < 0$ the straight lines l_M constructed through points of the domain D_δ intersect the boundary of the domain only in two points $\gamma^-(x) \in \Gamma^-$ and $\gamma^+(x) \in \Gamma^+$. In this part of the domain D_δ we define the coefficients of the outer expansion $u_k(x,y)$ as the solutions of the system of recurrence equations (1.8) for $\gamma^-(x) < y < \gamma^+(x)$ satisfying the boundary conditions (1.2); in the domain under consideration these take the form

$$u_k(x,\gamma^-(x)) = 0, \quad u_k(x,\gamma^+(x)) = 0. \tag{2.1}$$

For $x > 0$ the straight line l_M constructed through an interior point of the domain D_δ intersects the boundary of the domain in four points and consists of two segments, one of which lies between the curves $y = \sqrt{x} \in \Gamma^-$ and $\gamma^+(x) \in \Gamma^+$, and the second between the curves $\gamma^-(x) \in \Gamma^-$ and $y = -\sqrt{x} \in \Gamma^+$. For $\sqrt{x} \le y \le \gamma^+(x)$ (on the first of the segments) we define the coefficients $u_k(x,y)$ as the solutions of the recurrence system (1.8) satisfying the conditions

$$u_k(x,\sqrt{x}) = 0, \quad u_k(x,\gamma^+(x)) = 0. \tag{2.2}$$

For $\gamma^-(x) \le y \le -\sqrt{x}$ (on the second of the segments) we define the coefficients $u_k(x,y)$ as the solutions of the recurrence system (1.8) satisfying the conditions

$$u_k(x,\gamma^-(x)) = 0, \quad u_k(x,-\sqrt{x}) = 0. \tag{2.3}$$

The outer asymptotic expansion thus constructed is in essence three different asymptotic series whose coefficients are defined as solutions of three different boundary-value problems: (1.8), (2.1); (1.8), (2.2); and (1.8), (2.3). Obviously, the functions $u_k(x,y)$, generally speaking, are discontinuous on the straight line $x = 0$.

The functions $u_k(x,y)$ are solutions of ordinary differential equations in the variable y, which are infinitely differentiable with respect to the variable y in the entire domain D_δ. Furthermore, they depend on the parameter x. By assumption the functions $\gamma^\pm(x)$ are infinitely differentiable in a neighbourhood of the origin and, consequently, the functions $u_k(x,y)$ are also infinitely differentiable with respect to the parameter x to the left of the straight line $x = 0$ (that is, for $x < 0$). For $x > 0$ the functions $u_k(x,y)$ also depend on the parameter $\sqrt{x}$, and it is easy to see that they may have singularities with respect to the variable x as $x \to +0$. The following theorem holds.

Theorem 2.1. *For $\sqrt{x} \le y \le \gamma^+(x)$, as $x \to +0$ the functions $u_k(x,y)$ have the asymptotic series expansion*

$$u_0(x,y) = \sum_{j=0}^{\infty} x^{\frac{j}{2}} u_{0j}^+(y), \quad u_k(x,y) = x^{-2k} \sum_{j=1}^{\infty} x^{\frac{j}{2}} u_{kj}^+(y), \quad k \ge 1. \tag{2.4}$$

The series (2.4) *admit term-by-term differentiation of any order.*

Proof. Consider the function $u_0(x,y)$. This function is a solution of the ordinary differential equation (1.8) with respect to the variable y: $\mathrm{L}_0 u_0 = f(x,y)$. We seek its asymptotic representation as $x \to +0$ in the form of a series (2.4). By substituting this series into the equation (1.8), expanding the coefficients $b(x,y)$, $a(x,y)$ and the right-hand side $f(x,y)$ as a Taylor series in a neighbourhood of the straight line $x=0$, and equating the coefficients of like powers of x, we obtain a system of recurrence relations for the functions $u^+_{0j}(y)$:

$$
\text{(2.5)}\quad \begin{cases}
\mathrm{L}_0^{(0)} u^+_{00} = f(0,y), \quad \mathrm{L}_0^{(0)} u^+_{01} = 0, \\
\mathrm{L}_0^{(0)} u^+_{02} = \dfrac{1}{2!}\dfrac{\partial f}{\partial x}(0,y) - \dfrac{\partial b}{\partial x}(0,y)\dfrac{du^+_{00}}{dy} - \dfrac{\partial a}{\partial x}(0,y)u^+_{00}, \\
\mathrm{L}_0^{(0)} u^+_{03} = -\dfrac{\partial b}{\partial x}(0,y)\dfrac{du^+_{01}}{dy} - \dfrac{\partial a}{\partial x}(0,y)u^+_{01}, \\
\qquad \dots \\
\mathrm{L}_0^{(0)} u^+_{0,2j} = \dfrac{1}{j!}\dfrac{\partial^j f}{\partial x^j}(0,y) - \displaystyle\sum_{i=1}^{j}\dfrac{1}{i!}\left[\dfrac{\partial^i b}{\partial x^i}(0,y)\dfrac{du^+_{0,2(j-i)}}{dy} - \dfrac{\partial^i a}{\partial x^i}(0,y)u^+_{0,2(j-i)}\right], \\
\mathrm{L}_0^{(0)} u^+_{0,2j+1} = -\displaystyle\sum_{i=1}^{j}\dfrac{1}{i!}\left[\dfrac{\partial^i b}{\partial x^i}(0,y)\dfrac{du^+_{0,2(j-i)+1}}{dy} - \dfrac{\partial^i a}{\partial x^i}(0,y)u^+_{0,2(j-i)+1}\right].
\end{cases}
$$

Here we have set

$$
\mathrm{L}_0^{(0)} = \frac{d^2}{dy^2} + b(0,y)\frac{d}{dy} + a(0,y).
$$

By construction the function $u_0(x,y)$ satisfies the boundary conditions (2.2); that is, it vanishes for $y=\sqrt{x}$ and $y=\gamma^+(x)$. We require that as $x \to +0$ the asymptotic series (2.4) also satisfies these conditions.

First consider the boundary $y=\sqrt{x}$:

$$
\begin{aligned}
0 &= \sum_{i=0}^{\infty} x^{\frac{i}{2}} u^+_{0i}(\sqrt{x}) = \sum_{i=0}^{\infty} x^{\frac{i}{2}}\left[\sum_{l=0}^{\infty}\frac{d^l u^+_{0i}}{dy^l}(0)\frac{x^{l/2}}{l!}\right] \\
&= \sum_{m=0}^{\infty} x^{\frac{m}{2}}\left[\sum_{j=0}^{m}\frac{d^{m-j}u^+_{0j}}{dy^{m-j}}(0)\frac{1}{(m-j)!}\right] \equiv \sum_{i=0}^{\infty} x^{\frac{i}{2}} B_i^{(0)}.
\end{aligned}
$$

By equating the coefficients of $x^{i/2}$ to zero we obtain $B_i^{(0)}=0$ or, equivalently,

$$
\text{(2.6)}\quad \begin{cases}
u^+_{00}(0) = 0, \quad u^+_{01}(0) = -\dfrac{du^+_{00}}{dy}(0), \dots, \\
u^+_{0i}(0) = -\left[\dfrac{du^+_{0,i-1}}{dy}(0) + \dfrac{1}{2!}\dfrac{d^2u^+_{0,i-2}}{dy^2}(0) + \dots + \dfrac{1}{i!}\dfrac{d^i u^+_{0,0}}{dy^i}(0)\right].
\end{cases}
$$

We treat the second boundary $y=\gamma^+(x)$ in a similar way; that is, we require that as $x \to +0$ the series (2.4) vanishes on this boundary:

$$
\begin{aligned}
0 &= \sum_{i=0}^{\infty} x^{\frac{i}{2}} u^+_{0i}(\gamma^+(x)) = \sum_{i=0}^{\infty} x^{\frac{i}{2}} u^+_{0i}\left(\gamma^+(0) + \sum_{j=1}^{\infty}\gamma^+_j x^j\right) \\
&= \sum_{i=0}^{\infty} x^{\frac{i}{2}}\Big(\sum_{l=0}^{\infty}\frac{1}{l!}\big(\sum_{j=1}^{\infty}\gamma^+_j x^j\big)^l\Big) \equiv \sum_{i=0}^{\infty} x^{\frac{i}{2}} A_i^{(0)}.
\end{aligned}
$$

Here the $\gamma_j^+ = (j!)^{-1}(\gamma_j^+)^{(j)}(0)$ are the Taylor coefficients in the expansion of the function $\gamma^+(x)$ in a neighbourhood of the origin.

By equating to zero the coefficients of $x^{i/2}$ in the resulting relation, we obtain $A_i^{(0)} = 0$ or, which is the same,

$$\begin{cases} u_{00}^+(\gamma^+(0)) = 0, \quad u_{01}^+(\gamma^+(0)) = 0, \\ u_{02}^+(\gamma^+(0)) = -\dfrac{du_{00}^+}{dy}(\gamma^+(0)), \quad u_{03}^+(\gamma^+(0)) = -\dfrac{du_{01}^+}{dy}(\gamma^+(0)), \\ \qquad \dots \\ u_{0q}^+(\gamma^+(0)) = \displaystyle\sum_{i=1}^{[\frac{q}{2}]} \alpha_i^{(q)} \frac{d^i u_{0,q-2i}^+}{dy^i}(\gamma^+(0)). \end{cases} \tag{2.7}$$

Here the $\alpha_i^{(q)}$ are some constants that depend recurrently on the values of the derivatives of the function $\gamma^+(x)$ at the point $\gamma^+(0)$.

Thus, we define the coefficients $u_{0j}^+(y)$ of the series (2.4) as the solutions of equations (2.5) in the domain $0 < y < \gamma^+(0)$ satisfying the boundary conditions (2.6), (2.7) at $y = 0$, $y = \gamma^+(0)$. By the assumption of the original problem, such functions exist and are infinitely differentiable. Consider the partial sum of the series (2.4),

$$S_N(x,y) = \sum_{j=0}^{2N} x^{\frac{j}{2}} u_{0j}^+(y),$$

where N is sufficiently large. By construction, the difference $\sigma_N = u_0(x,y) - S_N(x,y)$ satisfies the relations

$$\mathrm{L}_0\sigma_N = O(x^{N+1/2}), \quad \sqrt{x} < y < \gamma^+(x),$$
$$\sigma_N(x,\gamma^+(x)) = O(x^{N+1/2}), \quad \sigma_N(x,\sqrt{x}) = O(x^{N+1/2}).$$

By (1.6) in the domain $\sqrt{x} \le y \le \gamma^+(x)$ we have the relation

$$u_0(x,y) - S_N(x,y) = O(x^{N+1/2});$$

that is, the series (2.4) whose coefficients are defined as the solutions of (2.5)–(2.7) is an asymptotic expansion of the function $u_0(x,y)$ as $x \to +0$. One can prove that similar estimates are also valid for the derivatives of the function $\sigma_N(x,y)$. The theorem is proved for the function $u_0(x,y)$.

The fact that the asymptotic expansion (2.4) holds for the functions $u_k(x,y)$, $k \ge 1$, is proved similarly. Here it is easy to see that all the successive functions $u_k(x,y)$, $k \ge 1$, have singularities as $x \to +0$. Indeed, the function $u_1(x,y)$ has a singularity of order $x^{-\frac{3}{2}}$ as $x \to +0$, since this function is a solution of the equation $\mathrm{L}_0 u_1 = -(u_0)_{xx}$, and by what we have just proved, as $x \to +0$, the asymptotic expansion of the right-hand side of this equation starts with a term that has order $x^{-\frac{3}{2}}$. □

The behaviour of the coefficients of the outer expansion in the domain $x > 0$, $\gamma^-(x) \le y \le -\sqrt{x}$ is analysed in quite a similar way. The following theorem holds.

Theorem 2.2. *For $\gamma^-(x) \le y \le -\sqrt{x}$, as $x \to +0$ the functions $u_k(x,y)$ expand into the asymptotic series*

$$u_0(x,y) = \sum_{j=0}^{\infty} x^{\frac{j}{2}} u_{0j}^-(y), \quad u_k(x,y) = x^{-2k} \sum_{j=1}^{\infty} x^{\frac{j}{2}} u_{kj}^-(y), \quad k \ge 1. \tag{2.8}$$

The series (2.8) admit term-by-term differentiation of any order.

Proof. The proof of this theorem is a repetition of the proof of the preceding theorem. The functions u^-_{kj} satisfy the same system of equations (2.5) as the functions u^+_{kj}, and the boundary conditions for them are given at the points $y = 0$ and $y = \gamma^-(0)$ and are obtained similarly to the way the boundary conditions (2.6), (2.7) were obtained. □

3. The inner expansion in a neighbourhood of the straight line $x = 0$

In a neighbourhood of $x = 0$ we go over from the variable x to new inner variable $\zeta = x\varepsilon^{-\frac{1}{2}}$ and construct the inner asymptotic expansion of the solution $u_\varepsilon(x, y)$ as $\varepsilon \to 0$ in the form

$$V(\zeta, y, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k(\zeta, y). \tag{3.1}$$

In the usual way, that is, by going over to the variable ζ in the original equation (1.1), expanding the coefficients $b(x, y)$, $a(x, y)$ and the right-hand side $f(x, y)$ in Taylor series in a neighbourhood of the straight line $x = 0$, replacing x by $\sqrt{\varepsilon}\zeta$ in the resulting expansion, and equating the coefficients of like powers of ε, we arrive at the system of recurrence relations

$$\begin{cases} \mathrm{L}_1 v_0 = f(0, y), \qquad \mathrm{L}_1 v_1 = 0, \\ \mathrm{L}_1 v_2 = \zeta \dfrac{\partial f}{\partial x}(0, y) - \zeta \dfrac{\partial b}{\partial x}(0, y) \dfrac{\partial v_0}{\partial y} - \zeta \dfrac{\partial a}{\partial x}(0, y) v_0, \\ \mathrm{L}_1 v_3 = -\zeta \dfrac{\partial b}{\partial x}(0, y) \dfrac{\partial v_1}{\partial y} - \zeta \dfrac{\partial a}{\partial x}(0, y) v_1, \\ \mathrm{L}_1 v_{2j} = \dfrac{\zeta^j}{j!} \dfrac{\partial^j f}{\partial x^j}(0, y) - \displaystyle\sum_{i=1}^{j} \dfrac{\zeta^i}{i!} \left[\dfrac{\partial^i b}{\partial x^i}(0, y) \dfrac{\partial v_{2j-2i}}{\partial y} + \dfrac{\partial^i a}{\partial x^i}(0, y) v_{2j-2i} \right], \\ \mathrm{L}_1 v_{2j+1} = -\displaystyle\sum_{i=1}^{j} \dfrac{\zeta^i}{i!} \left[\dfrac{\partial^i b}{\partial x^i}(0, y) \dfrac{\partial v_{2j+1-2i}}{\partial y} + \dfrac{\partial^i a}{\partial x^i}(0, y) v_{2j+1-2i} \right]. \end{cases} \tag{3.2}$$

Here we have set

$$\mathrm{L}_1 = \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial y^2} + b(0, y) \frac{\partial}{\partial y} + a(0, y) = \Delta_{\zeta, y} + b(0, y) \frac{\partial}{\partial y} + a(0, y).$$

In the variables ζ, y the equations of the boundary $y = \gamma^\pm(x)$ take the form

$$y = \gamma^\pm(x) = \gamma^\pm(0) + \sum_{j=1}^{\infty} \gamma^\pm_j \varepsilon^{\frac{j}{2}} \zeta^j,$$

and the equation of the parabola $y = \pm\sqrt{x}$ takes the form

$$y = \pm\varepsilon^{\frac{1}{4}} \sqrt{\zeta},$$

and thus, in the variables ζ, y, as $\varepsilon \to 0$, a neighbourhood of the straight line $x = 0$ becomes the infinite strip with a cut along the positive half-axis

$$\Omega = (\zeta, y : \gamma^-(0) < y < \gamma^+(0),\ 0 < \theta < 2\pi), \tag{3.3}$$

where θ is the polar angle on the plane (ζ, y).

In order to write out the boundary conditions for the coefficients $v_k(\zeta, y)$, we proceed in the same way as we did in constructing the functions $u^+_{kj}(y)$ in Theorem 2.1. By

requiring that the asymptotic series (3.1) satisfies the boundary condition of the original problem formally on the branches of the parabolas $y = \varepsilon^{\frac{1}{4}}\sqrt{\zeta}$ and $y = -\varepsilon^{\frac{1}{4}}\sqrt{\zeta}$, we obtain

$$0 = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k(\zeta, \pm\varepsilon^{\frac{1}{4}}\sqrt{\zeta}) = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} \left[\sum_{i=0}^{\infty} \frac{1}{i!} \varepsilon^{\frac{i}{4}} \frac{\partial^i v_k}{\partial y^i}(\zeta, \pm 0) \varepsilon^{\frac{i}{4}} (\pm\sqrt{\zeta})^i \right].$$

Equating the coefficients of the powers of ε to zero on the upper side of the cut we obtain

$$\begin{cases} v_0(\zeta, +0) = 0, \quad v_1(\zeta, +0) = -\dfrac{\partial v_0}{\partial y}(\zeta, +0)\sqrt{\zeta}, \\ v_2(\zeta, +0) = -\dfrac{\partial v_1}{\partial y}(\zeta, +0)\sqrt{\zeta} - \dfrac{1}{2!}\dfrac{\partial^2 v_0}{\partial y^2}(\zeta, +0)\zeta, \\ \qquad \dots \\ v_k(\zeta, +0) = -\displaystyle\sum_{l=1}^{k} \frac{1}{l!} \frac{\partial^l v_{k-l}}{\partial y^l}(\zeta, +0)\zeta^{\frac{l}{2}}, \end{cases} \tag{3.4}$$

and on the lower side of the cut

$$\begin{cases} v_0(\zeta, -0) = 0, \quad v_1(\zeta, -0) = \dfrac{\partial v_0}{\partial y}(\zeta, -0)\sqrt{\zeta}, \\ v_2(\zeta, -0) = -\dfrac{\partial v_1}{\partial y}(\zeta, -0)(-\sqrt{\zeta}) - \dfrac{1}{2!}\dfrac{\partial^2 v_0}{\partial y^2}(\zeta, -0)\zeta, \\ \qquad \dots \\ v_k(\zeta, -0) = -\displaystyle\sum_{l=1}^{k} \frac{1}{l!} \frac{\partial^l v_{k-l}}{\partial y^l}(\zeta, -0)(-1)^l\zeta^{\frac{l}{2}}. \end{cases} \tag{3.5}$$

Next, requiring that the asymptotic series (3.1) satisfies the boundary condition of the original problem formally on the boundaries $y = \pm\gamma(x)$:

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k(\zeta, \gamma^{\pm}(x)) = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k\left(\zeta, \gamma^{\pm}(0) + \sum_{j=1}^{\infty} \gamma_j^{\pm} \varepsilon^{\frac{j}{2}} \zeta^j\right) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m v_k}{\partial y^m}(\zeta, \gamma^{\pm}(0)) \left[\sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}} \zeta^{\frac{i}{2}} \gamma_i^{\pm} \right]^m \end{aligned}$$

gives the boundary conditions for the coefficients $v_k(\zeta, y)$ on the straight lines $y = \gamma^{\pm}(0)$:

$$\begin{cases} v_0(\zeta, \gamma^{\pm}(0)) = 0, \quad v_1(\zeta, \gamma^{\pm}(0)) = 0, \\ v_2(\zeta, \gamma^{\pm}(0)) = -\zeta\gamma_1^{\pm} \dfrac{\partial v_0}{\partial y}(\zeta, \gamma^{\pm}(0), \quad v_3(\zeta, \gamma^{\pm}(0)) = -\zeta\gamma_1^{\pm} \dfrac{\partial v_1}{\partial y}(\zeta, \gamma^{\pm}(0)), \\ \qquad \dots \\ v_k(\zeta, \gamma^{\pm}(0)) = \phi_k^{\pm}(\zeta), \end{cases} \tag{3.6}$$

where the boundary function $\phi_k^{\pm}(\zeta)$ has the form

$$\phi_k^{\pm}(\zeta) = \sum \beta_{ljs} \frac{\partial^l v_{k-2j}}{\partial y^l}(\zeta, \gamma^{\pm}(0))\zeta^s, \quad j \le \left[\frac{k}{2}\right], \quad l + s \le k - 2.$$

Thus, we need to construct the functions $v_k(\zeta, y)$; these are the solutions of the elliptic equations (3.2) in the strip Ω satisfying conditions (3.4)–(3.6) on the boundaries of this strip.

Boundary-value problems for elliptic equations in domains with corner points have been looked at by many authors. The most complete study of this type of problem was carried out in [7], where, apart from questions about the existence of solutions in

special classes of functions, the asymptotic behaviour of solutions was analysed in detail in neighbourhoods of the corner points themselves.

Using the results in [7] it is easy to establish that, starting from some k, the solutions $v_k(\zeta, y)$ of the problems (3.2), (3.4)–(3.6) have singularities at the origin. Indeed, according to that paper, there exists a function $v_0(\zeta, y)$ that is a solution of the equation $\mathrm{L}_1 v_0 = f(0, y)$ in the strip Ω, satisfies the conditions $v_0(\zeta, \gamma^{\pm}(0)) = 0$ on the boundaries $y = \gamma^{\pm}(0)$, and the condition $v_0(\zeta, \pm 0) = 0$ on the edges of the cut $y = \pm 0$. This function has the asymptotic representation

$$v_0(\zeta, y) \approx C r^{\frac{1}{2}} \sin \frac{\theta}{2}$$

as $r \to 0$, which admits differentiation. (Here r, θ are polar coordinates on the plane (ζ, y).) The function $v_0(\zeta, y)$ itself is bounded, but its derivatives have singularities, whose order increases as the order of the derivatives increases. It is easy to verify that

$$\frac{\partial^{2l} v_0}{\partial y^{2l}} \approx C \beta_l r^{-\frac{4l-1}{2}} \sin \frac{\theta}{2},$$
$$\frac{\partial^{2l+1} v_0}{\partial y^{2l+1}} \approx C \beta_l r^{-\frac{4l+1}{2}} \cos \frac{\theta}{2},$$

and consequently, derivatives of odd order satisfy the relations

$$\frac{\partial^{2l+1} v_0}{\partial y^{2l+1}}(\zeta, \pm 0) \approx C \beta(l) (\pm \sqrt{\zeta})^{-(4l+1)}.$$

Using these relations and the recurrence formulae (3.4), (3.5), it is easy to establish that the boundary functions for the solutions $v_1(\zeta, y)$, $v_2(\zeta, y)$ still remain bounded, but the boundary function $v_3(\zeta, \pm 0)$ no longer satisfies this condition:

$$v_3(\zeta, \pm 0) \approx -C \frac{\partial^3 v_0}{\partial y^3}(\zeta, \pm 0) \frac{1}{3!} (\pm \sqrt{\zeta})^3 = C \frac{1}{2^4} \zeta^{-1}.$$

Consequently, the solution $v_3(\zeta, y)$ has to be constructed in the class of functions that are unbounded at the origin.

It is easy to verify that all the subsequent boundary functions $v_k(\zeta, \pm 0)$ also have singularities at the origin, and the order of these singularities increases as the number k increases. Thus, formally, starting from $k = 3$ the coefficients $v_k(\zeta, y)$ of the asymptotic expansion (3.1) have singularities at the origin and, consequently, this asymptotic expansion becomes unsuitable in a neighbourhood of the origin, and it is necessary to construct another asymptotic expansion there. Furthermore, as with similar problems (see, for example, [2]–[6]), solutions of the problems (3.2), (3.4)–(3.6) are not uniquely determined in the class of unbounded functions, and therefore the asymptotic expansion (3.1) can only be constructed after we have analysed the asymptotic behaviour of the solution $u_\varepsilon(x, y)$ in a neighbourhood of the origin. Furthermore, it will become clear that we will even have to construct the solution $v_2(\zeta, y)$ in the class of functions that are unbounded at the origin.

4. The inner asymptotic expansion in a neighbourhood of the origin

In a neighbourhood of the origin we go from the variables x, y to the new inner variables $\xi = x\varepsilon^{-1}$, $\eta = y\varepsilon^{-\frac{1}{2}}$ and construct another inner asymptotic expansion of the solution $u_\varepsilon(x, y)$ as $\varepsilon \to 0$ in the form

$$W(\xi, \eta, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^{\frac{k}{4}} w_k(\xi, \eta). \tag{4.1}$$

In the new variables the parabola $x = y^2$ takes the form $\xi = \eta^2$, and in a neighbourhood of the origin the domain D_δ becomes the unbounded domain $\Omega_1 = \mathbb{R}^2 \setminus (\xi, \eta : \xi > 0, \eta^2 < \xi)$; it is the exterior of the domain situated in the half-plane $\xi > 0$ and bounded by the parabola $\xi = \eta^2$.

In order to obtain recurrence relations for finding the coefficients $w_k(\xi, \eta)$, we expand the coefficients $b(x, y)$, $a(x, y)$ and the right-hand side $f(x, y)$ of the original equation (1.1) as Taylor series in a neighbourhood of the origin and, in the resulting expansions, we go over to the inner variables ξ, η:

$$
\begin{aligned}
b(x,y) &= \sum_{i,j\geq 0} b_{ij} x^i y^j = \sum_{i,j\geq 0} b_{ij} \varepsilon^i \xi^i \varepsilon^{\frac{j}{2}} \eta^j = \sum_{j=0}^{\infty} \varepsilon^{\frac{j}{2}} g_j^{(1)}(\xi,\eta),\\
a(x,y) &= \sum_{i,j\geq 0} a_{ij} x^i y^j = \sum_{j=0}^{\infty} \varepsilon^{\frac{j}{2}} g_j^{(2)}(\xi,\eta),\\
f(x,y) &= \sum_{i,j\geq 0} f_{ij} x^i y^j = \sum_{j=0}^{\infty} \varepsilon^{\frac{j}{2}} g_j^{(3)}(\xi,\eta).
\end{aligned}
$$

It is easy to verify that $g_j^{(p)}(\xi, \eta)$ is an inhomogeneous polynomial of degree j of the form

$$
g_j^{(p)}(\xi,\eta) = \sum_l \alpha_{jl}^{(p)} \eta^{j-2l} \xi^l, \quad 0 \leq l \leq \left[\frac{j}{2}\right], \tag{4.2}
$$

with leading term $\alpha_{j0}^{(p)} \eta^j$, where

$$
\begin{aligned}
\alpha_{j0}^{(1)} &= \frac{1}{j!} \frac{\partial^j b}{\partial y^j}(0,0),\\
\alpha_{j0}^{(2)} &= \frac{1}{j!} \frac{\partial^j a}{\partial y^j}(0,0),\\
\alpha_{j0}^{(3)} &= \frac{1}{j!} \frac{\partial^j f}{\partial y^j}(0,0).
\end{aligned}
$$

In a standard fashion, that is, by passing to the inner variables ξ, η in the original equation (1.1), replacing the coefficients $b(x, y)$, $a(x, y)$ and the right-hand side $f(x, y)$ by the expansions in a neighbourhood of the origin detailed above, and equating the coefficients of like powers of ε, we obtain that in the domain Ω_1 the coefficients $w_k(\xi, \eta)$ of the inner expansion (4.1) must satisfy the system of recurrence relations

$$
\begin{cases}
\Delta w_1 = 0, \quad \Delta w_2 = 0, \quad \Delta w_3 = -b(0,0)\dfrac{\partial w_1}{\partial \eta}, \quad \Delta w_4 = -b(0,0)\dfrac{\partial w_2}{\partial \eta},\\
\Delta w_5 = -b(0,0)\dfrac{\partial w_3}{\partial \eta} - \dfrac{\partial b}{\partial y}(0,0)\eta \dfrac{\partial w_1}{\partial \eta} - a(0,0) w_1,\\
\Delta w_6 = -b(0,0)\dfrac{\partial w_4}{\partial \eta} - \dfrac{\partial b}{\partial y}(0,0)\eta \dfrac{\partial w_2}{\partial \eta} - a(0,0) w_2,\\
\dots\\
\Delta w_{2m} = -\displaystyle\sum_{j=0}^{m-2} g_j^{(1)}(\xi,\eta)\frac{\partial w_{2m-2-2j}}{\partial \eta} - \sum_{j=0}^{m-3} g_j^{(2)}(\xi,\eta) w_{2m-4-2j} + g_{m-2}^{(3)}(\xi,\eta),\\
\Delta w_{2m+1} = -\displaystyle\sum_{j=0}^{m-1} g_j^{(1)}(\xi,\eta)\frac{\partial w_{2m-1-2j}}{\partial \eta} - \sum_{j=0}^{m-2} g_j^{(2)}(\xi,\eta) w_{2m-3-2j}.
\end{cases} \tag{4.3}
$$

Furthermore, by requiring that the asymptotic series (4.1) satisfies the boundary conditions of the original problem, we obtain the boundary conditions for the functions $w_k(\xi,\eta)$:

$$w_k(\eta^2,\eta)=0,\quad k\geq 0. \tag{4.4}$$

In constructing the solutions of (4.3), (4.4), it turns out to be convenient (see [8, p. 152]) to go from the variables ξ, η to the variables s, t defined by the formulae

$$\xi=s^2-t^2-t,\quad \eta=s+2st. \tag{4.5}$$

In the new variables the domain Ω_1 becomes the half-plane $t>0$, and its boundary, the parabola $\xi=\eta^2$, the s-axis. It can be verified that

$$\frac{\partial}{\partial\eta}=\frac{1}{J}\left[\frac{\partial}{\partial s}+2s\frac{\partial}{\partial t}\right],\quad \Delta_{\xi,\eta}=\frac{1}{J}\Delta_{s,t},$$

where $J=\xi_s\eta_t-\xi_t\eta_s=(2s)^2+(2t+1)^2$.

We set $w_k(s^2-t^2-t,s+2st)=z_k(s,t)$, $g_j^{(p)}(s^2-t^2-t,s+2st)=G_{2j}^{(p)}(s,t)$. We rewrite the series (4.1) in the form

$$Z(s,t,\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^{\frac{k}{4}}z_k(s,t). \tag{4.6}$$

We rewrite the system (4.3) in the form

$$\begin{cases}\Delta z_{2m+1}=H_{2m+1}(s,t),\quad m\geq 0,\\ \Delta z_{2m}=H_{2m}(s,t)+F_{2m}(s,t),\quad m\geq 1,\end{cases} \tag{4.7}$$

where we use the notation $l=\left[\dfrac{j-1}{2}\right]$,

$$\begin{aligned}H_j(s,t)&=H_j(z_{j-2},z_{j-4},\dots,z_{j-2l})\\ &=-\sum_{n=0}^{l-1}G_{2n}^{(1)}(s,t)\left[(2t+1)\frac{\partial}{\partial s}+2s\frac{\partial}{\partial t}\right]z_{j-2-2n}-\sum_{n=0}^{l-2}J(s,t)G_{2n}^{(2)}(s,t)z_{j-4-2n},\end{aligned}$$

$$F_{2m}(s,t)=J(s,t)G_{2m-4}^{(3)}(s,t),\quad F_k\equiv 0 \text{ for } k<0;$$

that is, the function $H_j(s,t)$ depends on the 'preceding' functions $z_i(s,t)$ $(i<j)$, and the function $F_{2m}(s,t)$ is determined only by the right-hand side $f(x,y)$ of the original equation (1.1).

We observe that by relation (4.2) the functions $G_{2j}^{(p)}(s,t)$ are inhomogeneous polynomials in s,t of degree $2j$, with leading term $\alpha_{j0}^{(p)}(st)^j$, and the function $F_{2m}(s,t)$ is a polynomial in s,t of degree $2m-2$.

The boundary conditions (4.4) become the boundary conditions

$$z_k(s,0)=0,\quad k\geq 0. \tag{4.8}$$

The functions $w_k(\xi,\eta)$ that are the solutions of the problems in the unbounded domain Ω_1 (or, equivalently, the functions $z_k(s,t)$ that are the solutions of the problems in the unbounded domain $t>0$) are, generally speaking, not uniquely determined by relations (4.3), (4.4). We also need to impose some additional conditions on these functions as $\xi^2+\eta^2\to\infty$. These conditions must be obtained from the conditions of matched asymptotic expansions (3.1) and (4.1). Recall that we have not yet constructed the asymptotic expansion (3.1), since we have still not resolved the question of how to choose the solutions $v_k(\zeta,y)$ in the class of functions which are unbounded as $\zeta^2+y^2\to 0$.

This situation is typical for singularly perturbed problems (see [2]–[6]), and the functions $z_k(\xi,\eta)$, $v_k(\zeta,y)$ will be constructed simultaneously; but before doing this, we will make some auxiliary constructions.

5. Auxiliary constructions

Let r, θ be polar coordinates on the plane $(\zeta,\ y)$. For an integer k we consider the functions

$$U_k^{(1)}(\zeta,y)=r^{\frac{k}{2}}\sin\frac{k\theta}{2},\quad U_k^{(2)}(\zeta,y)=r^{\frac{k}{2}}\cos\frac{k\theta}{2}. \tag{5.1}$$

For even $k\geq 0$ the functions $U_k^{(j)}$ are harmonic polynomials. In what follows we will denote them by $P_n^{(j)}(\zeta,y)$, that is,

$$P_n^{(1)}(\zeta,y)=U_{2n}^{(1)}(\zeta,y),\quad P_n^{(2)}(\zeta,y)=U_{2n}^{(2)}(\zeta,y),\quad n\geq 0.$$

The functions $U_k^{(j)}(\zeta,y)$ are conjugate harmonic functions in the plane (ζ,y) with a cut along the positive half-axis ζ:

$$\Delta U_k^{(j)}=0,\quad \frac{\partial U_k^{(2)}}{\partial\zeta}=\frac{\partial U_k^{(1)}}{\partial y},\quad \frac{\partial U_k^{(2)}}{\partial y}=-\frac{\partial U_k^{(1)}}{\partial\zeta}.$$

For all k the functions $U_k^{(1)}(\zeta,y)$ vanish on both sides of the cut, that is, at $y=\pm 0$ or, equivalently, at $\theta=0$, $\theta=2\pi$. As for the functions $U_k^{(2)}(\zeta,y)$, they satisfy the equations

$$U_k^{(2)}(\zeta,\pm 0)=(\pm\sqrt{\zeta})^k=(-1)^k\zeta^{\frac{k}{2}}. \tag{5.2}$$

From the explicit formulae (5.1) it is easy to obtain the relations

$$\frac{\partial U_k^{(2)}}{\partial\zeta}=\frac{\partial U_k^{(1)}}{\partial y}=\frac{k}{2}U_{k-2}^{(2)},\quad -\frac{\partial U_k^{(2)}}{\partial y}=\frac{\partial U_k^{(1)}}{\partial\zeta}=\frac{k}{2}U_{k-2}^{(1)}, \tag{5.3}$$

as well as the relations

$$\zeta U_k^{(1)}=\frac{1}{2}U_{k+2}^{(1)}+\frac{1}{2}r^2U_{k-2}^{(1)},\quad \zeta U_k^{(2)}=\frac{1}{2}U_{k+2}^{(2)}+\frac{1}{2}r^2U_{k-2}^{(2)},$$
$$yU_k^{(1)}=\frac{1}{2}U_{k+2}^{(2)}+\frac{1}{2}r^2U_{k-2}^{(2)},\quad yU_k^{(2)}=\frac{1}{2}U_{k+2}^{(1)}-\frac{1}{2}r^2U_{k-2}^{(1)},$$

which in the general form can be written as follows:

$$\zeta^m y^p U_k^{(j)}=\sum_{s=0}^{m+p}\alpha_s^{(m,p)}U_{k+2m+2p-4s}^{(l)}r^{2s}, \tag{5.4}$$

where $l=j$ for even p, and $l\neq j$ for odd p. Furthermore, when $U_k^{(j)}$ is a harmonic polynomial, that is, when $k=2n$, $n\geq 0$, we can assume that the coefficients $\alpha_s^{(m,p)}$ in relation (5.4) for which $k+2m+2p-4s<0$ vanish.

We say that a function $v(\zeta,y)$ of the form $v(\zeta,y)=r^\alpha\Phi(\theta)$ has order α. The set of linear combinations of functions of the form $v(\zeta,y)=\zeta^m y^p U_k^{(j)}$, where m and p are non-negative integers, is denoted by $\mathcal{W}$. The subset of the set $\mathcal{W}$, all of whose elements have fixed order $s/2$, is denoted by $\mathcal{W}_s$. Since $\zeta=U_2^{(2)}(\zeta,y)$ and $y=U_2^{(1)}(\zeta,y)$, by relations (5.4) any polynomial $Q(\zeta,y)$ belongs to $\mathcal{W}$, and any homogeneous polynomial $Q_n(\zeta,y)$ of degree n is an element of the set $\mathcal{W}_{2n}$.

It follows from relation (5.4) that a function $v(\zeta,y)$ belonging to the set $\mathcal{W}_q$ has the form

$$v(\zeta,y)=\sum_{m,k,j}\beta_{m,k,j}r^{2m}U_k^{(j)}, \tag{5.5}$$

where $m\geq 0$, $1\leq j\leq 2$, $k+4m=q$.

Next, it follows from relations (5.3), (5.4) that if $v(\zeta, y) \in \mathcal{W}_q$, then

$$\frac{\partial v}{\partial \zeta} \in \mathcal{W}_{q-2}, \quad \frac{\partial v}{\partial y} \in \mathcal{W}_{q-2}, \quad \zeta^m y^p v(\zeta, y) \in \mathcal{W}_{q+2m+2p}. \tag{5.6}$$

Lemma 5.1. *Suppose that m and k are integers such that $m \geq 0$ and $k \neq -2(m+1)$. Then there exists a function $z(\zeta, y) \in \mathcal{W}_{k+4m+4}$ that is a solution of the equation*

$$\Delta z = r^{2m} U_k^{(j)} \tag{5.7}$$

and vanishes on the edges of the cut:

$$z(\zeta, \pm 0) = 0, \quad \zeta > 0. \tag{5.8}$$

Proof. For any α and any harmonic function $v(\zeta, y)$ we have the relation

$$\Delta(r^\alpha v) = \alpha(\alpha - 1) r^{\alpha-2} v + 2\alpha r^{\alpha-1} \frac{\partial v}{\partial r} + \alpha r^{\alpha-2} v.$$

We set $v(\zeta, y) = U_k^{(j)}(\zeta, y)$ in this relation. (Recall that $U_k^{(j)}(\zeta, y)$ is a harmonic function.) From the explicit formulae (5.1) we obtain that

$$\frac{\partial U_k^{(j)}}{\partial r} = \frac{k}{2} r^{-1} U_k^{(j)},$$

and consequently,

$$\Delta(r^\alpha U_k^{(j)}) = \alpha(\alpha + k) U_k^{(j)}.$$

Using the equation we have obtained and the condition that $k \neq -(2m+2)$, we see that the function

$$z^*(\zeta, y) = \frac{1}{(2m+2)(2m+2+k)} r^{2m+2} U_k^{(j)} \tag{5.9}$$

satisfies equation (5.7).

We set $[(2m+2)(2m+2+k)]^{-1} = \beta$ and consider the function $z^* = \beta r^{2m+2} U_k^{(j)}$ that we have constructed. For $j = 1$ the boundary condition (5.8) holds for this function automatically, since $U_k^{(1)}(\zeta, \pm 0) = 0$ by definition; that is, in this case it suffices to put $z = z^*$. For $j = 2$, by (5.2), z^* satisfies the relation

$$z^*(\zeta, \pm 0) = \beta \zeta^{2m+2} (\pm\sqrt{\zeta})^k = \beta(\pm\sqrt{\zeta})^{k+4m+4} = \beta U_{k+4m+4}^{(2)}(\zeta, \pm 0).$$

In this case we set $z(\zeta, y) = z^*(\zeta, y) - \beta U_{k+4m+4}^{(2)}(\zeta, y)$. □

Lemma 5.2. *Suppose that a function $z(\zeta, y)$ has the form $z(\zeta, y) = r^{\frac{k}{2}} \Phi(\theta)$, where $\Phi(\theta)$ is a linear combination of trigonometric functions:*

$$\Phi(\theta) = \sum_{p,q} [\alpha_p \sin(p\theta) + \beta_q \cos(q\theta)]. \tag{5.10}$$

If the function $z(\zeta, y)$ satisfies the equation

$$\Delta z = v(\zeta, y) \tag{5.11}$$

for some function $v \in \mathcal{W}_m$, then $k = m + 4$, the function $z(\zeta, y)$ belongs to $\mathcal{W}_{m+4}$, and it has the representation

$$z = z^*(\zeta, y) + C_1 U_{m+4}^{(1)}(\zeta, y) + C_2 U_{m+4}^{(2)}(\zeta, y), \tag{5.12}$$

where $z^*(\zeta, y) \in \mathcal{W}_{m+4}$ is the solution of the inhomogeneous equation (5.11) *constructed according to Lemma* 5.1.

Proof. First of all we point out that the equation $k = m + 4$ is obvious. Next, by the hypothesis of the lemma, the function v belongs to $\mathcal{W}_m$ and, consequently, we can represent it in the form (5.5):

$$v(\zeta, y) = \sum_{n,l,j} \beta_{n,l,j} r^{2n} U_l^{(j)}(\zeta, y),$$

where $n \geq 0$, $1 \leq j \leq 2$, $4n + l = m$. Replacing the functions $U_l^{(j)}(\zeta, y)$ by their analytic expressions (5.1) we rewrite this representation in the form

$$v(\zeta, y) = \sum_{n,l} r^{\frac{4n+l}{2}} \left[\alpha_{n,l}^{(1)} \sin \frac{l\theta}{2} + \alpha_{n,l}^{(2)} \cos \frac{l\theta}{2} \right]. \tag{5.13}$$

On the other hand, equation (5.11) is solvable by the hypothesis of the lemma, but then the equation

$$\frac{d^2}{d\theta^2}\Phi + \left(\frac{m+4}{2}\right)^2 \Phi = \sum_{n,l} \left[\alpha_{n,l}^{(1)} \sin \frac{l\theta}{2} + \alpha_{n,l}^{(2)} \cos \frac{l\theta}{2} \right]$$

is also solvable, and its solution has the form (5.10). Consequently, $\pm(m+4) \neq l$ for any of the indices l in the representation (5.13) or, equivalently (as $m = 4n + l$), we have $l \neq -2(n+1)$. Consequently, all the summands $\beta_{n,l,j} r^{2n} U_l^{(j)}(\zeta, y)$ in the representation (5.13) satisfy the conditions of Lemma 5.1, and the function $z(\zeta, y)$ can indeed be represented in the form (5.12). The lemma is proved. □

We considered the Cartesian coordinates (ζ, y) and introduced the sets $\mathcal{W}$, $\mathcal{W}_q$ of functions 'generated' by the harmonic functions $U_k^{(j)}(\zeta, y)$. In the subsequent exposition it will be necessary to consider other pairs of Cartesian coordinates (ξ, η), (s, t) and the corresponding sets 'generated' by the harmonic functions $U_k^{(j)}(\xi, \eta)$, $U_k^{(j)}(s, t)$. Therefore sometimes we shall use the notation $\mathcal{W}(\zeta, y)$, $\mathcal{W}(\xi, \eta)$, $\mathcal{W}(s, t)$ to indicate the coordinates we are using at a given moment. All the assertions proved above for elements of the sets $\mathcal{W}(\zeta, y)$, $\mathcal{W}_q(\zeta, y)$ carry over with the corresponding notation to elements of the sets $\mathcal{W}_q(\xi, \eta)$, $\mathcal{W}_q(s, t)$.

We now consider the coordinates s, t introduced in the preceding section, the corresponding polar coordinates λ (polar radius), ω (polar angle), and the harmonic polynomials $P_k^{(j)}(s, t)$:

$$P_k^{(1)}(s, t) = \lambda^k \sin(k\omega), \quad P_k^{(2)}(s, t) = \lambda^k \cos(k\omega).$$

By (5.1), $P_k^{(j)}(s, t) = U_{2k}^{(j)}(s, t)$, and therefore the polynomials $P_k^{(j)}(s, t)$ satisfy the following relations similar to relation (5.4):

$$s^m t^p P_k^{(j)}(s, t) = \sum_{q=0}^{m+p} \alpha_q^{(m,p)} \lambda^{2q} P_{k+m+p-2q}^{(l)}(s, t),$$

which, in particular, imply that any homogeneous polynomial $Q_n(s, t)$ of degree n can be represented in the form

$$Q_n(s, t) = \sum \alpha_{ql} \lambda^{2q_1} P_{q_2}^{(l)}(s, t),$$

where $q_1 \geq 0$, $2q_1 + q_2 = n$, $l = 1, 2$. Consequently, the following lemma holds.

Lemma 5.3. *For any polynomial $Q_n(s, t)$ of degree n there exists a polynomial $R_{n+2}(s, t)$ of degree $n + 2$ that is a solution of the equation*

$$\Delta_{st} u = Q_n(s, t)$$

for $t > 0$ and vanishes at $t = 0$.

Proof. The proof of the lemma is obvious. It suffices to represent the polynomial $Q_n(s,t)$ in the form of a sum of homogeneous polynomials $Q_j^{(n)}(s,t)$ of degree j, $0 \le j \le n$, and, by applying Lemma 5.1 to each term in the sum, construct homogeneous polynomials $R_{j+2}^{(n)}(s,t)$. We obtain a solution of the form

$$R_{n+2}(s,t) = \sum_{j=0}^{n} [R_{j+2}^{(n)}(s,t) + C_j P_{j+2}^{(2)}(s,t)]. \qquad \square$$

To proceed further we need to establish a connection between the variables s, t and ξ, η as $\xi^2+\eta^2 \to \infty$. In [2] it was proved that, as $\xi^2+\eta^2 \to \infty$, the polar coordinates ω and λ (in the (s,t)-plane) have the asymptotic representations

$$\begin{aligned} \lambda &= \rho^{\frac{1}{2}} + \sum_{j=0}^{\infty} A_j(\phi)\rho^{-\frac{j}{2}}, \\ \omega &= \frac{\phi}{2} + \sum_{j=1}^{\infty} B_j(\phi)\rho^{-\frac{j}{2}}, \end{aligned} \tag{5.14}$$

where ρ, ϕ are the polar coordinates in the plane (ξ,η).

Consider the harmonic polynomials $P_k^{(1)}(s,t)$, $P_k^{(2)}(s,t)$. Using the representations (5.14) it is easy to show that as $\xi^2+\eta^2 \to \infty$,

$$\begin{aligned} P_k^{(1)}(s,t) &= \lambda^k \sin(k\omega) = \rho^{\frac{k}{2}} \sin\frac{k\phi}{2} + \sum_{j=1}^{\infty} \rho^{\frac{k-j}{2}} \Psi_{k-j}^{(1)}(\phi) = U_k^{(1)}(\xi,\eta) + \sum_{j=1}^{\infty} \rho^{\frac{k-j}{2}} \Psi_{k-j}^{(1)}(\phi), \\ P_k^{(2)}(s,t) &= \lambda^k \cos(k\omega) = \rho^{\frac{k}{2}} \cos\frac{k\phi}{2} + \sum_{j=1}^{\infty} \rho^{\frac{k-j}{2}} \Psi_{k-j}^{(2)}(\phi) = U_k^{(2)}(\xi,\eta) + \sum_{j=1}^{\infty} \rho^{\frac{k-j}{2}} \Psi_{k-j}^{(2)}(\phi). \end{aligned} \tag{5.15}$$

We set

$$U_k^{(i)}(\xi,\eta) = \mu_k^{(i)}(\xi,\eta), \quad \rho^{\frac{k-j}{2}} \Psi_{k-j}^{(i)}(\phi) = \mu_{k-j}^{(i)}(\xi,\eta), \quad i=1,2, \quad j \ge 1;$$

that is, we rewrite the representation (5.15) in the form

$$P_k^{(i)}(s,t) = \sum_{j=0}^{\infty} \mu_{k-j}^{(i)}(\xi,\eta).$$

The functions $P_k^{(i)}(s,t)$ are harmonic; consequently, all the functions $\mu_{k-j}^{(i)}(\xi,\eta)$ for $j \ge 1$ are also harmonic. But then the functions $\Psi_{k-j}^{(i)}(\phi)$ $(j \ge 1)$ satisfy the equations

$$\left[\frac{d^2}{d\theta^2} + \left(\frac{k-j}{2}\right)^2\right] \Psi_{k-j}^{(i)} = 0$$

and, consequently, have the form

$$\Psi_{k-j}^{(i)} = C_{1,j}^{(i} \sin\frac{(k-j)\theta}{2} + C_{2,j}^{(i)} \cos\frac{(k-j)\theta}{2}, \quad i=1,2.$$

Therefore,

$$\mu_{k-j}^{(i)}(\xi,\eta) = C_{1,j}^{(i} U_{k-j}^{(1)}(\xi,\eta) + C_{2,j}^{(i)} U_{k-j}^{(2)}(\xi,\eta), \quad j \ge 1,$$

and the representation (5.15) can also be rewritten in the form

$$P_k^{(i)}(s,t) = \sum_{j=0}^{n} \mu_{k-j}^{(i)}(\xi,\eta) + O\big(\rho^{\frac{k-n-1}{2}}\big), \tag{5.16}$$

where the harmonic functions $\mu_{k-j}^{(i)}(\xi,\eta)$ belong to $\mathcal{W}_{k-j}(\xi,\eta)$ and, as we already mentioned, the first term of this asymptotic representation is $\mu_k^{(i)}(\xi,\eta) = U_k^{(i)}(\xi,\eta)$.

6. Asymptotics of the solution of the 'auxiliary' problem

In this section we consider the asymptotic behaviour as $r \to 0$ of the solution of the 'inner' problem of the form (3.2), (3.4)–(3.6) when the right-hand side and the boundary function $v(\zeta, \pm 0)$ decrease rapidly as $r \to 0$.

The asymptotics of the solution of an elliptic equation of the form (3.2) in a neighbourhood of the vertex of the cut was given in general form in [7], and in this section we give a description of these asymptotics in the 'language' of the sets $\mathcal{W}_n(\zeta, y)$ introduced in the preceding section.

Lemma 6.1. *Let $p_0^{(n)}(\zeta,y) = U_n^{(1)}(\zeta,y) = r^{\frac{n}{2}} \sin\frac{n\theta}{2}$, $n \geq 1$. There exist functions $p_j^{(n)}(\zeta,y) \in \mathcal{W}_{n+2j}(\zeta,y)$, $j \geq 0$, such that*

$$p_j^{(n)}(\zeta, \pm 0) = 0,$$

and the series

$$\Pi_n = p_0^{(n)}(\zeta,y) + \sum_{j=1}^{\infty} p_j^{(n)}(\zeta,y) = U_n^{(1)}(\zeta,y) + \sum_{j=1}^{\infty} p_j^{(n)}(\zeta,y) \tag{6.1}$$

is a formal asymptotic solution as $r \to 0$ of the equation

$$\mathrm{L}_1 u \equiv [\Delta_{\zeta,y} + b(0,y)\frac{\partial}{\partial y} + a(0,y)]u = 0.$$

Proof. In order to obtain recurrence relations to find the functions $p_j^{(n)}(\zeta,y)$, we substitute the series (6.1) into the equation $\mathrm{L}_1 u = 0$, after we have expanded the coefficients of this equation in a Taylor series as $y \to 0$. We obtain

$$\left[\Delta + \sum_{s=0}^{\infty} \frac{y^s}{s!}\frac{\partial^s b}{\partial y^s}(0,0)\frac{\partial}{\partial y} + \sum_{s=0}^{\infty} \frac{y^s}{s!}\frac{\partial^s a}{\partial y^s}(0,0)\right]\left[\sum_{j=0}^{\infty} p_j^{(n)}\right] = 0.$$

Next we use the fact that by (5.6), if a function $v(\zeta,y)$ has order α, the functions $v_1 = y^m v$, $v_2 = v_y$, $v_3 = \Delta v$ have orders $\alpha + m$, $\alpha - 1$, $\alpha - 2$, respectively, and equate the sum of terms with the same order to zero. As a result we arrive at the relations

$$\begin{cases} \Delta p_0^{(n)} = 0, \quad \Delta p_1^{(n)} = -b(0,0)\dfrac{\partial p_0^{(n)}}{\partial y}, \\ \Delta p_2^{(n)} = -b(0,0)\dfrac{\partial p_1^{(n)}}{\partial y} - \dfrac{\partial b}{\partial y}(0,0)y\dfrac{\partial p_0^{(n)}}{\partial y} - a(0,0)p_0^{(n)}, \\ \quad \dots \\ \Delta p_j^{(n)} = \displaystyle\sum_{s=0}^{j-1} \frac{y^s}{s!}\frac{\partial^s b}{\partial y^s}(0,0)\frac{\partial p_{j-1-s}^{(n)}}{\partial y} - \sum_{s=0}^{j-2} \frac{y^s}{s!}\frac{\partial^s a}{\partial y^s}(0,0)p_{j-2-s}^{(n)} \equiv T_j(\zeta,y). \end{cases} \tag{6.2}$$

By the hypothesis of the lemma, $p_0^{(n)} = U_n^{(1)}(\zeta,y)$, and by formula (5.3) the right-hand side $T_1(\zeta,y)$ has the form

$$T_1(\zeta,y) = -b(0,0)\frac{n}{2}U_{n-2}^{(2)}(\zeta,y) = -b(0,0)\frac{n}{2}r^0 U_{n-2}^{(2)}(\zeta,y) \in \mathcal{W}_{n-2}(\zeta,y).$$

We can apply Lemma 5.1 to the equation $\Delta p_1^{(n)} = T_1$. Indeed, the hypotheses of the lemma hold: $k = n-2$, $m = 0$, and $k \neq -2(m+1)$ for $n \geq 1$. The function $p_1^* = Ar^2 U_{n-2}^{(2)}$

constructed in accordance with Lemma 5.1 is a solution of the equation $\Delta p_1^{(n)} = T_1$. It does not satisfy the zero boundary conditions at $y = \pm 0$:

$$p_1^*(\zeta, +0) = A\zeta^2\zeta^{\frac{n-2}{2}}\cos 0 = A\zeta^{\frac{n}{2}+1},$$
$$p_1^*(\zeta, -0) = A\zeta^2\zeta^{\frac{n-2}{2}}\cos(n\pi) = A(-1)^n\zeta^{\frac{n}{2}+1}.$$

By adding the solution $-AU_{n+2}^{(2)}$ of the homogeneous equation $\Delta U = 0$ to the function p_1^* we obtain the function

$$p_1^n = Ar^2U_{n-2}^{(2)} - AU_{n+2}^{(2)},$$

which satisfies the zero boundary conditions. We have constructed the function $p_1^n \in \mathcal{W}_{n+2}(\zeta, y)$.

We next look at equation (6.2) with a view to finding the function $p_2^n(\zeta, y)$. Using relations (5.3), (5.6) and the form of the function $p_1^n(\zeta, y)$, we write the right-hand side $T_2(\zeta, y)$ in the form

$$\begin{aligned} T_2 &= -b(0,0)\left(Ar^2\frac{n-2}{2}U_{n-4}^{(1)} + 2AyU_{n-2}^{(2)}\right) - b_y(0,0)\frac{n}{2}yU_{n-2}^{(2)} - a(0,0)U_n^{(1)} \\ &= -b(0,0)A\frac{n-2}{2}r^2U_{n-4}^{(1)} - \left[b(0,0)2A + b_y(0,0)\frac{n}{2}\right]\frac{U_n^{(1)} + r^2U_{n-4}^{(1)}}{2} - a(0,0)U_n^{(1)} \\ &= f_1r^2U_{n-4}^{(1)} + f_2U_n^{(1)}. \end{aligned}$$

Both terms on the right-hand side of T_2 satisfy the hypotheses of Lemma 5.1: for the first term, $k = n-4$, $m = 1$, and $k \neq -2(m+1)$ for $n \geq 1$; for the second term, $k = n$, $m = 0$, and $k \neq -2(m+1)$ for $n \geq 1$. Consequently, by Lemma 5.1, there exists a function

$$p_2^{(n)}(\zeta, y) = A_1r^4U_{n-4}^{(1)} + A_2r^2U_n^{(1)} \in \mathcal{W}_{n+4}$$

which is a solution of the equation $\Delta p = T_2$. In this case the function constructed satisfies the zero boundary conditions and it is not necessary to add the harmonic function $\widetilde{A}U_{n+4}^{(2)}$.

The rest of the proof can be carried out by induction. Suppose that in accordance with Lemma 5.1 we have constructed functions $p_1^{(n)}, p_2^{(n)}, \ldots, p_{j-1}^{(n)}$ with the form

$$p_q^{(n)} = \sum_{i=0}^{q-1} r^{2q-2i}\left[A_{iq}^{(1)}U_{n-2q+4i}^{(1)} + A_{iq}^{(2)}U_{n-2q+4i}^{(2)}\right] + D_0^{(q)}U_{n+2q}^{(2)}.$$

Using relations (5.3) and (5.6) we can verify that the right-hand side $T_j(\zeta, y)$ belongs to $\mathcal{W}_{n+2j-4}(\zeta, y)$ and can be written in the form

$$T_j = \sum_{i=0}^{j-1} r^{2(j-1)-2i}\left[\widetilde{A}_{iq}^{(1)}U_{n-2j+4i}^{(1)} + \widetilde{A}_{iq}^{(2)}U_{n-2j+4i}^{(2)}\right].$$

Lemma 4.1 can be applied to each of the terms on the right-hand side. Indeed, $k = n - 2j + 4i$, $m = j - 1 - i$, and the condition $k \neq -2(m+1)$ turns into the condition $n - 2j + 4i = n + 2i - 2j + 2i \neq -2(j-i) = -2j + 2i$, which holds for $i \geq 0$, $n \geq 1$. Consequently, by Lemma 5.1, we can construct a function $p_j^{(n)}(\zeta, y) \in \mathcal{W}_{n+2j}(\zeta, y)$ of the 'special' form

$$p_j^{(n)} = \sum_{i=0}^{j-1} r^{2j-2i}\left[A_{ij}^{(1)}U_{n-2j+4i}^{(1)} + A_{ij}^{(2)}U_{n-2j+4i}^{(2)}\right] + D_0^{(j)}U_{n+2j}^{(2)}, \tag{6.3}$$

which is defined uniquely and satisfies all the requirements of Lemma 6.1. □

Lemma 6.2. *Suppose that the functions $h^{\pm}(\zeta)$ belong to $C^{\infty}(R^1)$, the functions $g(\zeta,y)$, $\psi(\zeta)$ are infinitely differentiable everywhere, apart from on a neighbourhood of the origin, and the asymptotic representations*

$$g(\zeta,y)=O\big(r^{\frac{N}{2}}\big),\quad r\to 0;\quad \psi(\zeta)=O\big(\zeta^{\frac{N+4}{2}}\big),\quad \zeta\to 0,\quad N\geq 0, \tag{6.4}$$

hold, as well as analogous asymptotic representations for the derivatives of these functions.

Then the solution $\widehat{v}(\zeta,y)$ of the problem

$$\begin{gathered}\mathrm{L}_1 v\equiv\left[\Delta+b(0,y)\frac{\partial}{\partial y}+a(0,y)\right]v=g(\zeta,y),\quad (\zeta,y)\in\Omega,\\ v(\zeta,\gamma^{\pm}(0))=h^{\pm}(\zeta),\\ v(\zeta,\pm 0)=\psi(\zeta)\end{gathered} \tag{6.5}$$

has the asymptotic representation

$$\widehat{v}(\zeta,y)=\sum_{j=1}^{N+3}\omega_j(\zeta,y)+\sigma_{N+4}(\zeta,y), \tag{6.6}$$

as $r\to 0$, where $\omega_j(\zeta,y)\in\mathcal{W}_j$ and the remainder term satisfies $\sigma_{N+4}(\zeta,y)=O\big(r^{\frac{N+4}{2}}\big)$. The representation (6.6) *admits term-by-term differentiation.*

Proof. As we said above, the asymptotic behaviour of solutions of boundary-value problems for elliptic equations in a neighbourhood of corner points of the boundary was investigated in [7]. According to this paper, an asymptotic expansion of the solution of the problem (6.5) can be written in the form

$$\widehat{v}(\zeta,y)=\sum_{j=1}^{\infty}r^{\frac{j}{2}}Q_{j-1}^{(j)}(\ln r,\theta), \tag{6.7}$$

where the $Q_{j-1}^{(j)}(\ln r,\theta)$ are polynomials in $\ln r$ of degree at most $n(j)=[\frac{j-1}{2}]$ of the form

$$Q_{j-1}^{(j)}=\sum_{l=0}^{n(j)}(\ln r)^{j-1-l}\Phi_{j-1-l}^{(j)}(\theta),\quad Q_0^{(0)}=C_1^1\sin\frac{\theta}{2}.$$

Consequently, the proof of the lemma reduces to verifying that in fact all the polynomials $Q_{j-1}^{(j)}$ are polynomials of degree zero, that is, $\Phi_{j-1}^{(j)}(\theta)=\Phi_{j-2}^{(j)}(\theta)=\dots=\Phi_1^{(j)}(\theta)\equiv 0$ for any j and thus $Q_{j-1}^{(j)}=\Phi_0^{(j)}(\theta)$, and verifying that $\omega_j=r^{\frac{j}{2}}\Phi_0^{(j)}(\theta)\in\mathcal{W}_j(\zeta,y)$.

It is easy to verify the validity of this assertion if we assume that in some way a formal asymptotic solution of the problem (6.5) has already been constructed having the form (6.6), that is, assume that an algorithm has been indicated for constructing an asymptotic series

$$\Pi(\zeta,y)=\sum_{j=1}^{\infty}r^{\frac{j}{2}}\Psi_j(\theta)=\sum_{j=1}^{\infty}\widetilde{\omega}_j(\zeta,y),\quad \widetilde{\omega}_j(\zeta,y)\in\mathcal{W}_j(\zeta,y),$$

that satisfies the homogeneous equation $\mathrm{L}_1 u=0$ formally as $r\to 0$ and vanishes at $y=\pm 0$. Then if the system of recurrence ordinary differential equations which are satisfied by the functions $\Psi_j(\theta)$ is solvable, the coefficients of the powers of $\ln r$ in the expansion (6.7) are equal to zero.

Next, by writing out the recurrence relations satisfied by the functions $\omega_j(\zeta,y)$, taking into account that

$$\omega_0(\zeta,y)=C_1^1 r^{\frac{1}{2}}\sin\frac{\theta}{2}=C_1^1U_1^{(1)}(\zeta,y)\in\mathcal{W}_1(\zeta,y),$$

and using Lemma 5.2, we arrive at the conclusion that $\omega_j(\zeta, y) \in \mathcal{W}_j(\zeta, y)$.

In order to complete the proof of the lemma, it remains to verify that the series $\Pi(\zeta, y)$ exists. Obviously, we can take the series

$$\Pi = \sum_{n=1}^{\infty} D_n \Pi_n = \sum_{n=1}^{\infty} D_n \left[\sum_{j=0}^{\infty} p_j^{(n)} \right] = \sum_{k=1}^{\infty} \sum_{n+2j=k} D_n p_j^{(n)} \tag{6.8}$$

for this series, where the D_n are any constants, $D_1 \neq 0$, and the Π_n are the asymptotic series constructed in Lemma 6.1, while the functions $p_j^{(n)}(\zeta, y)$ are given by formulae (6.3). □

Remark. We point out that there are infinitely many formal asymptotic solutions of the form (6.8), since the D_n are any constants, but for the concrete problem (6.5) the constants D_n are uniquely determined.

Let $\mathcal{A} = \{A_1, A_2, \dots, A_k, \dots\}$ be some sequence of numbers. For integers $k \geq 1$ we define a sequence of functions $X_k(s, t)$ as follows. For $k = 1, 2$ we set

$$X_1(s, t) = A_1 P_1^{(1)}(s, t), \quad X_2(s, t) = A_2 P_2^{(1)}(s, t). \tag{6.9}$$

For $k > 2$ the function $X_k(s, t)$ is the sum

$$X_k(s, t) = \widetilde{X}_k(s, t) + A_k P_k^{(1)}(s, t), \tag{6.10}$$

where the $\widetilde{X}_k(s, t)$ are the solutions of the homogeneous recurrence system (4.7),

$$\Delta \widetilde{X}_k = H_k(X_{k-2}, X_{k-4}, \dots, X_{k-2l}), \qquad l = \left[\frac{k-1}{2} \right], \tag{6.11}$$

vanishing at $t = 0$ (on the s-axis) and constructed in accordance with the algorithm proposed in Lemma 5.1.

We can obtain 'explicit' representations for the functions $\widetilde{X}_k(s, t)$. To do this we define a sequence of polynomials $P_{k,j}(s, t)$ as follows: for $j = 0$ we set $P_{k,0}(s, t) = P_k^{(1)}(s, t)$, for $j > 0$ the functions $P_{k,j}(s, t)$ are the (inhomogeneous) polynomials of degree $k + 2j$ that are the solutions of the recurrence system (6.11),

$$\Delta P_{k,j} = H_j(P_{k,j-1}, P_{k,j-2}, \dots, P_{k,0}),$$

vanishing at $t = 0$ (on the s-axis) and constructed in accordance with the algorithm proposed in Lemma 5.1. The polynomial $P_{k,j}$ can be viewed as the jth iteration of the harmonic polynomial $P_k^{(1)}(s, t)$, which is an eigenfunction of the problem: $\Delta u = 0$ for $t > 0$, $u(s, 0) = 0$. Using the notation we have introduced, we can write

$$\begin{aligned} \widetilde{X}_3 &= A_1 P_{1,1}, \quad \widetilde{X}_5 = A_1 P_{1,2} + A_3 P_{3,1}, \dots, \\ \widetilde{X}_{2j+1} &= A_1 P_{1,j} + A_3 P_{3,j-1} + \dots + A_{2j-1} P_{2j-1,1}, \\ \widetilde{X}_4 &= A_2 P_{2,1}, \quad \widetilde{X}_6 = A_2 P_{2,2} + A_4 P_{4,1}, \dots, \\ \widetilde{X}_{2j} &= A_2 P_{2,j} + A_4 P_{4,j-1} + \dots + A_{2j-2} P_{2j-2,1}. \end{aligned} \tag{6.12}$$

Lemma 6.3. *As $\xi^2 + \eta^2 \to \infty$, the polynomial $X_k(s, t)$ decomposes into an asymptotic series of the form*

$$X_k(s, t) = \sum_{l=0}^{n} \sigma_l^{(k)}(\xi, \eta) + \nu_{k,n}, \tag{6.13}$$

where the functions $\sigma_l^{(k)}(\xi,\eta)$ *belong to* $\mathcal{W}_{k-l}(\xi,\eta)$*, and the remainder term satisfies the estimate*

$$|\nu_{k,n}(\xi,\eta)| \le M\rho^{\frac{k-n-1}{2}}. \tag{6.14}$$

Proof. The asymptotic representation (6.13) is given above for the harmonic polynomials $P_k^{(1)}(s,t)$ (see (5.16)), and thus the lemma holds for the functions $X_1(s,t)$, $X_2(s,t)$. As for the polynomials X_k for $k>2$, using the asymptotic representations (5.14) for the polar coordinates λ and ω it is easy to see that, as $\xi^2+\eta^2\to\infty$, these polynomials have representations of the form

$$X_k(s,t) = \sum_{l=0}^{\infty} \rho^{\frac{k-l}{2}} \Phi_l^{(k)}(\phi), \tag{6.15}$$

where the functions $\Phi_l^{(k)}(\phi)$ are linear combinations of the trigonometric functions $\sin(m_1\phi)$, $\cos(m_2\phi)$. (Recall that ρ, ϕ are polar coordinates on the plane (ξ,η).)

We set

$$\rho^{k-l}\Phi_l^{(k)}(\phi) = \sigma_l^{(k)}(\xi,\eta);$$

that is, we rewrite the representation (6.15) in the form

$$X_k(s,t) = \sum_{l=0}^{\infty} \sigma_l^{(k)}(\xi,\eta).$$

Next we use the fact that the resulting series are formal asymptotic solutions (as $\xi^2+\eta^2\to\infty$) in the variables ξ,η of the homogeneous ($f\equiv 0$) recurrence system (4.3). Substituting these series into the homogeneous system (4.3), taking account of the form (4.2) of the polynomials $g_n^{(1)}(\xi,\eta)$, $g_n^{(2)}(\xi,\eta)$, and equating terms of the same order, we arrive at a system of recurrence relations satisfied by the functions $\sigma_l^{(k)}(\xi,\eta)$. This system has the form

$$\Delta\sigma_l^{(k)} = \sum_{m,n} \alpha_{mn}\xi^{m_1}\eta^{m_2}\frac{\partial}{\partial\eta}\sigma_{m_3}^{(n)} + \sum_{m,n}\beta_{mn}\xi^{m_1}\eta^{m_2}\sigma_{m_3}^{(n)},$$

where $n<k$, $m=(m_1,m_2,m_3)$. (The indices m_1,m_2,m_3,n are connected by certain relations, but in this case we need not write them out.) Since we have already established that $\sigma_l^{(1)}(\xi,\eta)\in\mathcal{W}_{1-l}(\xi,\eta)$ and $\sigma_l^{(2)}(\xi,\eta)\in\mathcal{W}_{2-l}(\xi,\eta)$, by applying Lemma 5.2 consecutively to the equations written out above we arrive at the conclusion that for any $k>0$ the functions $\sigma_l^{(k)}(\xi,\eta)$ belong to $\mathcal{W}_{k-l}(\xi,\eta)$. The lemma is proved. □

Consider the series

$$X(\mathcal{A},s,t,\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^{\frac{k}{4}} X_k(s,t). \tag{6.16}$$

We replace the functions $X_k(s,t)$ by their asymptotic expansions as $\xi^2+\eta^2\to\infty$ (see (6.13)) and rewrite the resulting representations in terms of the variables ζ, y. Now,

$$\xi = x\varepsilon^{-1} = \zeta\varepsilon^{-\frac{1}{2}}, \quad \eta = y\varepsilon^{-\frac{1}{2}}, \quad \sqrt{\xi^2+\eta^2} = \rho = \varepsilon^{-\frac{1}{2}}\sqrt{\zeta^2+y^2} = \varepsilon^{-\frac{1}{2}}r, \quad \phi=\theta,$$

and so

$$\sigma_l^{(k)}(\xi,\eta) = \varepsilon^{-\frac{k-l}{4}}\sigma_l^{(k)}(\zeta,y),$$

and thus we obtain

$$X(\mathcal{A},s,t,\varepsilon) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\varepsilon^{\frac{l}{4}}\sigma_l^{(k)}(\zeta,y) = \sum_{l=0}^{\infty}\varepsilon^{\frac{l}{4}}\sum_{k=1}^{\infty}\sigma_l^{(k)}(\zeta,y).$$

We set

$$\sum_{k=1}^{\infty} \sigma_l^{(k)}(\zeta, y) = R_l(\zeta, y) \tag{6.17}$$

and rewrite the relation obtained above in the form

$$X(\mathcal{A}, s, t, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{\frac{l}{4}} R_l(\mathcal{A}, \zeta, y). \tag{6.18}$$

We point out the following obvious properties of the series $R_l(\mathcal{A}, \zeta, y)$. Since for fixed l the orders of the functions $\sigma_l^{(k)}(\zeta, y)$ increase as k increases (they are equal to $(k-l)/2$), these series are asymptotic as $r \to 0$. Next, due to the fact that by construction the series $X(\mathcal{A}, s, t, \varepsilon)$ is a formal asymptotic solution (as $\varepsilon \to 0$) of the homogeneous equation $\mathrm{L}_\varepsilon u = 0$ that vanishes at $t = 0$ (or, in the variables ξ, η, on the parabola $\eta = \pm\sqrt{\xi}$), the series $R_l(\mathcal{A}, \zeta, y)$ are formal asymptotic solutions (as $r \to 0$) of the homogeneous recurrence system (3.2) and satisfy the boundary conditions (3.4), (3.5) on the sides of the cut $y = \pm 0$:

$$\begin{cases} \mathrm{L}_1 R_0 = 0, \quad \mathrm{L}_1 R_1 = 0, \\ \mathrm{L}_1 R_2 = -\zeta \dfrac{\partial b}{\partial x}(0, y) \dfrac{\partial R_0}{\partial y} - \zeta \dfrac{\partial a}{\partial x}(0, y) R_0, \\ \mathrm{L}_1 R_l = -\displaystyle\sum_{i=1}^{[\frac{l}{2}]} \frac{\zeta^i}{i!} \left[\frac{\partial^i b}{\partial x^i}(0, y) \frac{\partial R_{l-2i}}{\partial y} + \frac{\partial^i a}{\partial x^i}(0, y) R_{l-2i} \right]; \end{cases} \tag{6.19}$$

$$\begin{cases} R_0(\zeta, \pm 0) = 0, \quad R_1(\zeta, \pm 0) = -(\pm\sqrt{\zeta}) \dfrac{\partial R_0}{\partial y}(\zeta, \pm 0), \\ R_l(\zeta, \pm 0) = -\displaystyle\sum_{m=1}^{l} \frac{1}{m!} \frac{\partial^m R_{l-m}}{\partial y^m}(\zeta, \pm 0)(\pm\sqrt{\zeta})^m. \end{cases} \tag{6.20}$$

We let $R_l^{(n)}(\mathcal{A}, \zeta, y)$ denote the partial sum of the series $R_l(\zeta, y)$ (see (6.17)):

$$R_l^{(n)}(\mathcal{A}, \zeta, y) = \sum_{k=1}^{n} \sigma_l^{(k)}(\zeta, y),$$

and $X^{(n)}(\mathcal{A}, s, t, \varepsilon)$ the partial sum of the series $X(\mathcal{A}, s, t, \varepsilon)$ (see (6.16)):

$$X^{(n)}(\mathcal{A}, s, t, \varepsilon) = \sum_{k=1}^{n} \varepsilon^{\frac{k}{4}} X_k(s, t).$$

Obviously, both $R_l^{(n)}(\mathcal{A}, \zeta, y)$ and $X^{(n)}(\mathcal{A}, s, t, \varepsilon)$ depend only on the 'first' n coefficients A_k or, to put it simply, on the set $\mathcal{A}^{(n)} = \{A_1, A_2, \dots, A_n\}$. Sometimes, in order to emphasize this dependence, we use the notation $R_l^{(n)}(\mathcal{A}^{(n)}, \zeta, y)$, $X^{(n)}(\mathcal{A}^{(n)}, s, t, \varepsilon)$.

We point out that for $l \geq 2$ the functions $R_l^{(n)}(\mathcal{A}^{(n)}, \zeta, y)$ have a singularity at the origin:

$$R_l^{(n)}(\mathcal{A}^{(n)}, \zeta, y) = O(r^{\frac{1-l}{2}}). \tag{6.21}$$

The partial sum $X^{(n)}(\mathcal{A}^{(n)}, s, t, \varepsilon)$ has the following representation similar to the representation (6.18):

$$X^{(n)}(\mathcal{A}^{(n)}, s, t, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{\frac{l}{4}} R_l^{(n)}(\mathcal{A}^{(n)}, \zeta, y). \tag{6.22}$$

We denote the partial sum of this representation by $X^{(n,m)}$:

$$X^{(n,m)}(\mathcal{A}^{(n)},\zeta,y,\varepsilon)=\sum_{l=0}^{m}\varepsilon^{\frac{l}{4}}R_l^{(n)}(\mathcal{A}^{(n)},\zeta,y)=\sum_{l=0}^{m}\varepsilon^{\frac{l}{4}}\sum_{k=1}^{n}\sigma_l^{(k)}(\zeta,y). \tag{6.23}$$

Using relations (6.13) it is easy to see that

$$X^{(n)}(\mathcal{A}^{(n)},s,t,\varepsilon)=X^{(n,m)}(\mathcal{A}^{(n)},\zeta,y,\varepsilon)+\theta_{n,m}(\zeta,y,\varepsilon), \tag{6.24}$$

where

$$\theta_{n,m}(\zeta,y,\varepsilon)=\sum_{l=1}^{n}\nu_{l,m}(\xi,\eta).$$

Taking the estimates (6.14) into account we arrive at the following estimate for the remainder term $\theta_{n,m}$:

$$|\theta_{n,m}(\zeta,y)|\le M\varepsilon^{\frac{m+1}{4}}r^{-\frac{m}{2}}\big(1+r^{\frac{n-1}{2}}\big). \tag{6.25}$$

We now consider problem (6.4), (6.5) and denote the partial sum of the asymptotic expansion (6.6) constructed in Lemma 6.2 by $S_{N+3}(\zeta,y)$:

$$S_{N+3}(\zeta,y)=\sum_{m=1}^{N+3}\omega_m(\zeta,y). \tag{6.26}$$

Lemma 6.4. *Suppose that a function $v(\zeta,y)$ is a solution of problem* (6.4), (6.5). *There exist constants $A_1,A_2,\dots,A_{N+3}$ such that the principal term $R_0^{(N+3)}(\mathcal{A}^{(N+3)},\zeta,y)$ in the representation* (6.22) *of the partial sum $X^{(N+3)}(\mathcal{A}^{(N+3)},s,t,\varepsilon)$ of the series X defined on the set $\mathcal{A}$ coincides with the partial sum $S_{N+3}(\zeta,y)$ of the asymptotic expansion* (6.6) *of the function $v(\zeta,y)$: $R_0^{(N+3)}(\mathcal{A}^{(N+3)},\zeta,y)=S_{N+3}(\zeta,y)$.*

Proof. We consider the asymptotic series (6.6) and write out a recurrence system (similar to the system (6.2)) that its terms, the functions $\omega_m(\zeta,y)$, satisfy:

$$\begin{aligned}
&\Delta\omega_1=0;\quad \Delta\omega_2=0;\\
&\dots\\
&\Delta\omega_{2j}=-\sum_{s=0}^{j-2}\frac{y^s}{s!}\frac{\partial^s b}{\partial y^s}(0,0)\frac{\partial\omega_{2j-2-2s}}{\partial y}-\sum_{s=0}^{j-3}\frac{y^s}{s!}\frac{\partial^s a}{\partial y^s}(0,0)\omega_{2j-4-2s}\equiv T_{2j}(\zeta,y);\\
&\Delta\omega_{2j+1}=-\sum_{s=0}^{j-1}\frac{y^s}{s!}\frac{\partial^s b}{\partial y^s}(0,0)\frac{\partial\omega_{2j-1-2s}}{\partial y}-\sum_{s=0}^{j-2}\frac{y^s}{s!}\frac{\partial^s a}{\partial y^s}(0,0)\omega_{2j-3-2s}\equiv T_{2j+1}(\zeta,y).
\end{aligned} \tag{6.27}$$

Furthermore, the functions $\omega_m(\zeta,y)$ satisfy the boundary conditions

$$\omega_m(\zeta,\pm 0)=0. \tag{6.28}$$

We also consider the asymptotic series $R_0(\zeta,y)$ formed in accordance with relation (6.17) by the principal terms $\sigma_0^{(k)}(\zeta,y)$ of the asymptotic expansions (6.13) of the functions $X_k(s,t)$. By relations (6.9), (6.10), (6.12) for the polynomials $X_k(s,t)$ and using (5.16), which gives the asymptotic representations for the harmonic polynomials $P_k^{(1)}(s,t)$, the functions $\sigma_0^{(k)}(\zeta,y)$ have the form

$$\begin{aligned}
&\sigma_0^{(1)}=A_1U_1^{(1)}(\zeta,y);\quad \sigma_0^{(2)}=A_2U_2^{(1)}(\zeta,y);\\
&\dots\\
&\sigma_0^{(2j)}=\widetilde{\sigma}_0^{(2j)}(A_2,A_4,\dots,A_{2j-2})+A_{2j}U_{2j}^{(1)}(\zeta,y);\\
&\sigma_0^{(2j+1)}=\widetilde{\sigma}_0^{(2j+1)}(A_1,A_3,\dots,A_{2j-1})+A_{2j+1}U_{2j+1}^{(1)}(\zeta,y).
\end{aligned} \tag{6.29}$$

As we pointed out above (see (6.19), (6.20)), the asymptotic series $R_0(\zeta, y)$ is a formal asymptotic solution (as $r \to 0$) of the homogeneous equation $\mathrm{L}_1 R_0 = 0$ and satisfies the boundary conditions $R_0(\zeta, \pm 0) = 0$. Consequently, its terms $\sigma_0^{(m)}$ satisfy the same recurrence relations (6.27) and boundary conditions (6.28) as the functions $\omega_m(\zeta, y)$.

Thus, the functions $\sigma_0^{(m)}(\zeta, y)$, $\omega_m(\zeta, y)$ are elements of the set $\mathcal{W}_m(\zeta, y)$ and satisfy the same system of recurrence equations (6.27). Taking the form of these functions into account we easily arrive at the conclusion that they can only differ by a term that is a harmonic function belonging to the set $\mathcal{W}_m(\zeta, y)$. Now if we take (6.29) into account, which gives the form of the functions $\sigma_0^{(m)}(\zeta, y)$, we can verify that the constants $A_1, \dots, A_{N+3}$ can indeed be (uniquely) determined so that $\sigma_0^{(m)}(\zeta, y) = \omega_m(\zeta, y)$ for $1 \le m \le N+3$.

In fact, it is known that $\omega_1(\zeta, y) = C_1 U_1^{(1)}(\zeta, y)$. Setting $A_1 = C_1$ we obtain $\sigma_0^{(1)}(\zeta, y) = \omega_1(\zeta, y)$. It is also known that $\omega_2(\zeta, y) = C_2 U_2^{(1)}(\zeta, y)$, and setting $A_2 = C_2$ we obtain $\sigma_0^{(2)}(\zeta, y) = \omega_2(\zeta, y)$. Consider the function $\sigma_0^{(3)}(\zeta, y) = \widetilde{\sigma}_0^{(3)}(A_1, \zeta, y) + A_3 U_3^{(1)}(\zeta, y)$. The functions $\widetilde{\sigma}_0^{(3)}$ and $\omega_3(\zeta, y)$ satisfy the relation

$$\widetilde{\sigma}_0^{(3)}(\zeta, y) = \omega_3(\zeta, y) + C_3 U_3^{(1)}(\zeta, y),$$

where C_3 is some well-determined constant. By setting $A_3 = -C_3$ we make sure that the functions $\sigma_0^{(3)}(\zeta, y)$ and $\omega_3(\zeta, y)$ coincide.

Continuing this process further we may choose all the constants A_j in a unique way. The lemma is proved. □

7. The construction of inner expansions

In this section we finally construct the inner asymptotic expansions (3.1), (4.1), (4.6) introduced earlier. As we have already said, to construct both the functions $v_k(\zeta, y)$ and the functions $w_k(\xi, \eta)$ (or, equivalently, the functions $z_k(s, t)$), the recurrence relations (3.2), (3.4)–(3.6) and (4.3), (4.4) (or (4.7), (4.8)) alone are not enough to define these functions uniquely. So, to do this we will use the method of matched asymptotic expansions.

We define the polynomials $Q_{2m}(s, t)$ of degree $2m$ ($m \ge 2$) to be the solutions of the inhomogeneous recurrence system (4.7) vanishing at $t = 0$,

$$\Delta Q_{2m} = H_{2m}(Q_{2m-2}(s, t), Q_{2m-4}(s, t), \dots, Q_4(s, t)) + F_{2m}(s, t), \tag{7.1}$$

constructed according to Lemma 5.3.

Lemma 7.1. *The polynomial $Q_{2m}(s, t)$ decomposes, as $\xi^2 + \eta^2 \to \infty$, into an asymptotic series of the form*

$$Q_{2m} = \sum_{j=0}^{n} q_j^{(2m)}(\xi, \eta) + \tau_{m,n}(\xi, \eta), \tag{7.2}$$

where the functions $q_j^{(2m)}(\xi, \eta)$ belong to $\mathcal{W}_{2m-j}(\xi, \eta)$, and the remainder term $\tau_{m,n}(\xi, \eta)$ satisfies the estimate

$$|\tau_{m,n}(\xi, \eta)| \le M \rho^{\frac{2m-n-1}{2}}. \tag{7.3}$$

The representation (7.2) *admits term-by-term differentiation.*

Proof. The proof of this lemma is similar to the proof of Lemma 6.3. □

We now consider the series

$$Q(s, t, \varepsilon) = \sum_{m=2}^{\infty} \varepsilon^{\frac{m}{2}} Q_{2m}(s, t). \tag{7.4}$$

By proceeding in the same way as we did in § 6 for the series $X(\mathcal{A}, s, t, \varepsilon)$, that is, by replacing the polynomials $Q_{2m}(s,t)$ by their asymptotic expansions (7.2) and passing from the variables ξ, η to the variables ζ, y, we arrive at the representation

$$Q(s,t,\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{\frac{l}{4}} \sum_{m=2}^{\infty} q_l^{(2m)}(\zeta, y).$$

Setting

$$\sum_{m=2}^{\infty} q_l^{(2m)}(\zeta, y) = \Lambda_l(\zeta, y),$$

we rewrite the relation we have obtained in the form

$$Q(s,t,\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{\frac{l}{4}} \Lambda_l(\zeta, y). \tag{7.5}$$

The properties of the series $\Lambda_l(\zeta, y)$ are similar to the properties of the series $R_l(\zeta, y)$ we looked at in the preceding section: a) these series are asymptotic as $r \to 0$; b) these series are formal asymptotic solutions of the problem for the functions $v_l(\zeta, y)$; that is, they satisfy the inhomogeneous system (3.2) (see (7.1)) formally (as $r \to 0$) and the boundary conditions (3.4), (3.5) (at $y = \pm 0$).

We denote the partial sums of the series $Q(s,t,\varepsilon)$ and $\Lambda_l(\zeta, y)$ by $Q^{(n)}(s,t,\varepsilon)$ and $\Lambda_l^{(n)}(\zeta, y)$:

$$Q^{(n)}(s,t,\varepsilon) = \sum_{m=2}^{n} \varepsilon^{\frac{m}{2}} Q_m(s,t), \quad \Lambda_l^{(n)}(\zeta, y) = \sum_{m=2}^{n} q_l^{(2m)}(\zeta, y), \quad m \geq 2. \tag{7.6}$$

Obviously, these partial sums satisfy the following relation, similar to relation (7.5):

$$Q^{(n)}(s,t,\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{\frac{l}{4}} \Lambda_l^{(n)}(\zeta, y). \tag{7.7}$$

We denote the partial sum of the series (7.7) by $Q^{(n,p)}(s,t,\varepsilon)$:

$$Q^{(n,p)}(s,t,\varepsilon) = \sum_{l=0}^{p} \varepsilon^{\frac{l}{4}} \Lambda_l^{(n)}(\zeta, y). \tag{7.8}$$

Using the representation (7.2) it is easy to see that

$$Q^{(n)}(s,t,\varepsilon) = Q^{(n,p)}(\zeta, y) + \mu_{n,p}(\zeta, y, \varepsilon), \tag{7.9}$$

where

$$\mu_{n,p}(\zeta, y, \varepsilon) = \sum_{m=1}^{n} \varepsilon^{\frac{m}{2}} \tau_{m,p}(\xi, \eta).$$

By further using the estimates (7.3) for the remainder terms $\tau_{m,p}(\xi, \eta)$ (taking into account that $\rho = \varepsilon^{-\frac{1}{2}} r$), we easily obtain an estimate for the remainder term of the representation (7.9):

$$|\mu_{n,p}(\zeta, y, \varepsilon)| \leq M \varepsilon^{\frac{p+1}{4}} r^{-\frac{p}{2}} (r^{\frac{3}{2}} + r^{\frac{2n-1}{2}}). \tag{7.10}$$

We now turn to the construction of the coefficients $z_k(s,t)$. First of all we point out that in the class of functions of polynomial growth any harmonic polynomial $P_k^{(1)}(s,t)$ is an eigenfunction of the problem (4.7), (4.8). We define an eigenfunction $B_m(s,t)$ of order m as a linear combination of harmonic polynomials

$$B_m(s,t) = E_m^{(m)} P_m^{(1)}(s,t) + E_{m-1}^{(m)} P_{m-1}^{(1)}(s,t) + \cdots + E_1^{(m)} P_1^{(1)}(s,t), \tag{7.11}$$

where $E_{m-j}^{(m)}$ ($m \geq 1$, $0 \leq j \leq m-1$) are any constants, $E_m^{(m)} \neq 0$.

Let $E_{k-j}^{(k)}$, $k \geq 1$, $j = 0, 1, \dots, k-1$ be some sequence of numbers. We denote it by $\mathcal{E}$ and order it as follows:

$$\mathcal{E} = \{E_1^{(1)}; E_2^{(2)}, E_1^{(2)}; E_3^{(3)}, E_2^{(3)}, E_1^{(3)}; \dots; E_k^{(k)}, E_{k-1}^{(k)}, \dots, E_1^{(k)}; \dots\}.$$

It is convenient to depict the set $\mathcal{E}$ as an infinite triangular table

$$\begin{array}{lllll} E_1^{(1)} & & & & \\ E_2^{(2)} & E_1^{(2)} & & & \\ E_3^{(3)} & E_2^{(3)} & E_1^{(3)} & & \\ E_4^{(4)} & E_3^{(4)} & E_2^{(4)} & E_1^{(4)} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

The rows of this table are composed of the coefficients $E_{k-j}^{(k)}$ of the eigenfunctions $B_k(s,t)$. We denote by e_l ($l \geq 1$) the lth column of this table, that is,

$$e_l = \{E_1^{(l)}, E_2^{(l+1)}, E_3^{(l+2)}, \dots, E_{k-l+1}^{(k)}, \dots\}.$$

Thus, the column $e_1 = \{E_1^{(1)}, E_2^{(2)}, E_3^{(3)}, \dots, E_k^{(k)}, \dots\}$ is the set of the first (leading) coefficients of the eigenfunctions $B_k(s,t)$. Similarly, the column $e_2 = \{E_1^{(2)}, E_2^{(3)}, E_3^{(4)}, \dots\}$ consists of the second (with respect to order) coefficients of the eigenfunctions $B_k(s,t)$, and so on. The subset of the set e_l consisting of the first N elements of this set is denoted by $e_l^{(N)}$ in what follows.

For a fixed $N \geq 1$ the set of coefficients $E_{k-j}^{(k)}$, $k \leq N$, is denoted by $\mathcal{E}^{(N)}$. This set can be represented by the finite triangular table

$$\begin{array}{llllll} E_1^{(1)} & & & & & \\ E_2^{(2)} & E_1^{(2)} & & & & \\ E_3^{(3)} & E_2^{(3)} & E_1^{(3)} & & & \\ E_4^{(4)} & E_3^{(4)} & E_2^{(4)} & E_1^{(4)} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_N^{(N)} & E_{N-1}^{(N)} & E_{N-2}^{(N)} & E_{N-3}^{(N)} & \dots & E_1^{(N)}. \end{array}$$

We define functions $Y_k(s,t)$, which are polynomials of degree k (inhomogeneous for $k > 1$) as follows. For $k = 1, 2$ we set

$$\begin{aligned} Y_1(s,t) &= B_1(s,t) = E_1^{(1)} P_1^{(1)}(s,t), \\ Y_2(s,t) &= B_2(s,t) = E_2^{(2)} P_2^{(1)}(s,t) + E_1^{(2)} P_1^{(1)}(s,t). \end{aligned} \tag{7.12}$$

For $k > 2$ the functions $Y_k(s,t)$ are the sums

$$Y_k(s,t) = \widetilde{Y}_k(s,t) + B_k(s,t), \tag{7.13}$$

where the $\widetilde{Y}_k(s,t)$ are the solutions of the homogeneous recurrence system (4.7),

$$\Delta \widetilde{Y}_k = H_k(Y_{k-2}, Y_{k-4}, \dots, Y_{k-2l}), \quad l = \left[\frac{k-1}{2}\right], \tag{7.14}$$

vanishing at $t = 0$ (on the s-axis) and constructed in accordance with Lemma 5.3. The polynomial $\widetilde{Y}_k(s,t)$ depends on the preceding constants $E_{m-l}^{(m)}$, $m \leq k-1$, that is, on the set $\mathcal{E}^{(k-1)}$; the polynomial $Y_k(s,t)$ depends on the set $\mathcal{E}^{(k)}$. Sometimes, in order to emphasize this dependence, we use the notation $Y_k(\mathcal{E}^{(k)}, s, t)$.

We now consider the series

$$Y(\mathcal{E}, s, t, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^{\frac{k}{4}} Y_k(\mathcal{E}^{(k)}, s, t) \tag{7.15}$$

and construct the inner asymptotic expansion $Z(s, t, \varepsilon)$ (see (4.6)) in the form

$$Z(s, t, \varepsilon) = Q(s, t, \varepsilon) + Y(\mathcal{E}, s, t, \varepsilon), \tag{7.16}$$

or, equivalently, the functions $z_k(s, t)$ (solutions of the problems (4.7), (4.8)) in the form

$$z_k(s, t) = \begin{cases} Q_{2m}(s, t) + Y_{2m}(\mathcal{E}^{(2m)}, s, t), & k = 2m, \quad m \geq 1, \\ Y_{2m+1}(\mathcal{E}^{(2m+1)}, s, t), & k = 2m+1, \quad m \geq 0, \end{cases} \tag{7.17}$$

where $Q_{2m}(s, t) \equiv 0$ for $m < 2$.

We transform the series $Y(\mathcal{E}, s, t, \varepsilon)$ to a form that is convenient for what we need later. First of all we observe that by construction every function $Y_j(s, t)$ is the sum of polynomials of the form $X_m(s, t)$, $m < j$. Since it is awkward to verify this assertion in its general form (although the assertion itself is obvious), we confine ourselves to considering several initial functions $Y_j(s, t)$. To do this we turn to relations (7.11)–(7.13), which define the functions $Y_k(s, t)$, and to relations (6.9)–(6.11), which define the functions $X_m(s, t)$. It follows from relations (6.9), (7.11) that

$$\begin{aligned} Y_1 &= B_1 = E_1^{(1)} P_1^{(1)} = X_1(E_1^{(1)}), \\ Y_2 &= B_2 = E_2^{(2)} P_2^{(1)} + E_1^{(2)} P_1^{(1)} \\ &= X_2(E_2^{(2)}) + X_1(E_1^{(2)}), \end{aligned}$$

and it follows from relation (6.10) and the method of construction of the functions X_k, Y_k that

$$Y_3 = \widetilde{X}_3(E_1^{(1)}) + E_3^{(3)} P_3^{(1)} + E_2^{(3)} P_2^{(1)} + E_1^{(3)} P_1^{(1)}.$$

Combining the first and second terms in this sum we obtain the function $X_3(E_1^{(1)}, E_3^{(3)})$, and using our alternative notation for the last two terms we arrive at the relation

$$Y_3 = X_3(E_1^{(1)}, E_3^{(3)}) + X_2(E_2^{(3)}) + X_1(E_1^{(3)}).$$

In exactly the same fashion we can verify that

$$Y_4 = \widetilde{X}_4(E_2^{(2)}) + \widetilde{X}_3(E_1^{(2)}) + E_4^{(4)} P_4^{(1)} + E_3^{(4)} P_3^{(1)} + E_2^{(4)} P_2^{(1)} + E_1^{(4)} P_1^{(1)}.$$

Combining the first term with the third, and the second with the fourth, and changing the notation in the last two terms, we arrive at the relation

$$Y_4 = X_4(E_2^{(2)}, E_4^{(4)}) + X_3(E_1^{(2)}, E_3^{(4)}) + X_2(E_2^{(4)}) + X_1(E_1^{(4)}).$$

It is also easy to write the analogous relations in the general form:

$$\begin{aligned} Y_{2j} = {} & X_{2j}(E_2^{(2)}, E_4^{(4)}, \dots, E_{2j}^{(2j)}) + X_{2j-1}(E_1^{(2)}, E_3^{(4)}, \dots, E_{2j-1}^{(2j)}) \\ & + X_{2j-2}(E_2^{(4)}, E_4^{(6)}, \dots, E_{2j-2}^{(2j)}) + \dots + X_4(E_2^{(2j-2)}, E_4^{(2j)}) \\ & + X_3(E_1^{(2j-2)}, E_3^{(2j)}) + X_2(E_2^{(2j)}) + X_1(E_1^{(2j)}), \\ Y_{2j+1} = {} & X_{2j+1}(E_1^{(1)}, E_3^{(3)}, \dots, E_{2j+1}^{(2j+1)}) + X_{2j}(E_2^{(3)}, E_4^{(5)}, \dots, E_{2j}^{(2j+1)}) \\ & + X_{2j-1}(E_1^{(3)}, E_3^{(5)}, \dots, E_{2j-1}^{(2j+1)}) + \dots + X_4(E_2^{(2j-1)}, E_4^{(2j+1)}) \\ & + X_3(E_1^{(2j-1)}, E_3^{(2j+1)}) + X_2(E_2^{(2j+1)}) + X_1(E_1^{(2j+1)}). \end{aligned}$$

Using these relations we rewrite the series (7.15):

$$\begin{aligned}
Y(\mathcal{E};s,t,\varepsilon) &= \varepsilon^{\frac{1}{4}}\big(X_1(E_1^{(1)})\big) + \varepsilon^{\frac{2}{4}}\big(X_2(E_2^{(2)}) + X_1(E_1^{(2)})\big)\\
&\quad + \varepsilon^{\frac{3}{4}}\big(X_3(E_1^{(1)},E_3^{(3)}) + X_2(E_2^{(3)}) + X_1(E_1^{(3)})\big)\\
&\quad + \varepsilon\big(X_4(E_2^{(2)},E_4^{(4)}) + X_3(E_1^{(2)},E_3^{(4)}) + X_2(E_2^{(4)}) + X_1(E_1^{(4)})\big) + \cdots\\
&= \big[\varepsilon^{\frac{1}{4}}X_1(E_1^{(1)}) + \varepsilon^{\frac{2}{4}}X_2(E_2^{(2)}) + \varepsilon^{\frac{3}{4}}X_3(E_1^{(1)},E_3^{(3)}) + \varepsilon X_4(E_2^{(2)},E_4^{(4)}) + \cdots\big]\\
&\quad + \varepsilon^{\frac{1}{4}}\big[\varepsilon^{\frac{1}{4}}X_1(E_1^{(2)}) + \varepsilon^{\frac{2}{4}}X_2(E_2^{(3)}) + \varepsilon^{\frac{3}{4}}X_3(E_1^{(2)},E_3^{4)}) + \cdots\big]\\
&\quad + \varepsilon^{\frac{2}{4}}\big[\varepsilon^{\frac{1}{4}}X_1(E_1^{(3)}) + \varepsilon^{\frac{2}{4}}X_2(E_2^{(4)}) + \cdots\big] + \varepsilon^{\frac{3}{4}}\big[\varepsilon^{\frac{1}{4}}X_1(E_1^{(4)}) + \cdots\big].
\end{aligned}$$

Turning to the definition of the series $X(\mathcal{A};s,t,\varepsilon)$ (see (6.9), (6.12)) and the sets e_m, it is easy to see that the first square bracket contains the series $X(e_1;s,t,\varepsilon)$, the second the series $X(e_2;s,t,\varepsilon)$, and so on. Thus, the series (7.15) can be rewritten in the form

$$Y(\mathcal{E};s,t,\varepsilon) = \sum_{k=1}^{\infty}\varepsilon^{\frac{k}{4}}Y_k(\mathcal{E}^{(k)},s,t) = \sum_{m=0}^{\infty}\varepsilon^{\frac{m}{4}}X(e_{m+1};s,t,\varepsilon); \tag{7.18}$$

that is, only the column e_{m+1} of $\mathcal{E}$ is used in constructing the term $\varepsilon^{\frac{m}{4}}X(e_{m+1};s,t,\varepsilon)$.

If, instead of the whole series $Y(\mathcal{E};s,t,\varepsilon)$, we only consider its partial sum

$$Y^{(n)}(\mathcal{E};s,t,\varepsilon) = \sum_{m=0}^{n}\varepsilon^{\frac{m}{4}}Y_m(s,t,\varepsilon), \tag{7.19}$$

which obviously depends only on the set $\mathcal{E}^{(n)}$, then using the table for the set $\mathcal{E}^{(n)}$ it is easy to obtain the following relation similar to (7.18):

$$Y^{(n)}(\mathcal{E};s,t,\varepsilon) = Y^{(n)}(\mathcal{E}^{(n)};s,t,\varepsilon) = \sum_{m=0}^{n-1}\varepsilon^{\frac{m}{4}}X^{(n-m)}(e_{m+1}^{(n-m)},s,t,\varepsilon), \tag{7.20}$$

where, in accordance with the notation introduced in § 6, $X^{(n-m)}(e_{m+1}^{(n-m)},s,t,\varepsilon)$ is the sum of the first $n-m$ terms of the series X defined on the set (column) e_{m+1}.

Replacing the partial sums $X^{(n-m)}(e_{m+1}^{(n-m)},s,t,\varepsilon)$ in the representation (7.19) by their asymptotic representations (6.22) we arrive at the relation

$$Y^{(n)}(\mathcal{E}^{(n)};s,t,\varepsilon) = \sum_{m=0}^{n-1}\varepsilon^{\frac{m}{4}}\left[\sum_{l=0}^{\infty}\varepsilon^{\frac{l}{4}}R_l^{(n-m)}(e_{m+1}^{(n-m)},\zeta,y)\right] = \sum_{l=0}^{\infty}\varepsilon^{\frac{l}{4}}K_l(\mathcal{E}^{(n)},\zeta,y). \tag{7.21}$$

To avoid cumbersome notation when we write the expressions which determine the functions $K_l(\mathcal{E}^{(n)},\zeta,y)$, we will be leaving out some of the arguments of the functions $Y^{(n)}(\mathcal{E}^{(n)};s,t,\varepsilon)$, $R_l^{(N-m)}(e_{m+1}^{(n-m)},\zeta,y)$, $K_l(\mathcal{E}^{(n)},\zeta,y)$ in some cases.

It is easy to verify that the functions $K_l(\mathcal{E}^{(n)},\zeta,y)$ have the form

$$
\begin{aligned}
K_0(\mathcal{E}^{(n)}) &= K_0(e_1^{(n)}) = R_0^{(n)}(e_1^{(n)}),\\
K_1(\mathcal{E}^{(n)}) &= K_1(e_1^{(n)},e_2^{(n-1)}) = R_1^{(n)}(e_1^{(n)}) + R_0^{(n-1)}(e_2^{(n-1)}),\\
K_2(\mathcal{E}^{(n)}) &= K_2(e_1^{(n)},e_2^{(n-1)},e_3^{(n-2)})\\
&= R_2^{(n)}(e_1^{(n)}) + R_1^{(n-1)}(e_2^{(n-1)}) + R_0^{(n-2)}(e_3^{(n-2)}),\\
K_j(\mathcal{E}^{(n)}) &= K_j(e_1^{(n)},e_2^{(n-1)},\dots,e_{j+1}^{(n-j)})\\
&= R_j^{(n)}(e_1^{(n)}) + R_{j-1}^{(n-1)}(e_2^{(n-1)}) + R_{j-2}^{(n-2)}(e_3^{(n-2)}) + \cdots\\
&\quad + R_1^{(n-(j-1))}(e_j^{(n-(j-1))}) + R_0^{(n-j)}(e_{j+1}^{(n-j)}),\\
K_{n-1}(\mathcal{E}^{(n)}) &= K_{n-1}(e_1^{(n)},e_2^{(n-1)},\dots,e_{n-1}^{(2)},e_n^{(1)})\\
&= R_{n-1}^{(n)}(e_1^{(n)}) + R_{n-2}^{(n-1)}(e_2^{(n-1)}) + R_{n-3}^{(n-2)}(e_3^{(n-2)}) + \cdots\\
&\quad + R_1^{(2)}(e_{n-1}^{(2)}) + R_0^{(1)}(e_n^{(1)}),\\
K_{n+m}(\mathcal{E}^{(n)}) &= K_{n+m}(e_1^{(n)},e_2^{(n-1)},\dots,e_{n-1}^{(2)},e_n^{(1)})\\
&= R_{n+m}^{(n)}(e_1^{(n)}) + R_{n+m-1}^{(n-1)}(e_2^{(n-1)}) + \cdots\\
&\quad + R_{m+2}^{(2)}(e_{n-1}^{(2)}) + R_{m+1}^{(1)}(e_n^{(1)}), \quad m\ge 0.
\end{aligned}
\tag{7.22}
$$

We denote the partial sum of the series (7.21) by $Y^{(n,p)}(\mathcal{E}^{(n)};\zeta,y)$:

$$
Y^{(n,p)}(\mathcal{E}^{(n)};\zeta,y) = \sum_{l=0}^{p} \varepsilon^{\frac{l}{4}} K_l(\mathcal{E}^{(n)},\zeta,y). \tag{7.23}
$$

In order to obtain an estimate for the difference between the functions $Y^{(n)}(\mathcal{E}^{(n)};\zeta,y)$ and $Y^{(n,p)}(\mathcal{E}^{(n)};\zeta,y)$, we use relations (7.20), (6.24):

$$
\begin{aligned}
Y^{(n)}(\mathcal{E}^{(n)}) &= \sum_{m=0}^{n-1} \varepsilon^{\frac{m}{4}} X^{(n-m)}(e_{m+1}^{(n-m)})\\
&= \big[X^{(n,p)}(e_1^{(n)}) + \theta_{n,p}\big] + \varepsilon^{\frac{1}{4}}[X^{(n-1,p-1)}(e_2^{(n-1)}) + \theta_{n-1,p-1}] + \cdots\\
&\quad + \varepsilon^{\frac{l}{4}}\big[X^{(n-l,p-l)}(e_{l+1}^{(n-l)}) + \theta_{n-l,p-l}\big] + \cdots\\
&\quad + \varepsilon^{\frac{n-1}{4}}\big[X^{(1,p-n+1)}(e_n^{(1)}) + \theta_{1,p-n+1}\big].
\end{aligned}
$$

Taking the estimates (6.25) for the remainder terms $\theta_{k,m}$ into account, we rewrite the resulting relation in the form

$$
Y^{(n)}(\mathcal{E}^{(n)}) - \sum_{l=0}^{n-1} \varepsilon^{\frac{l}{4}} X^{(n-l,p-l)}(e_{l+1}^{(n-l)}) = Y^{(n)}(\mathcal{E}^{(n)}) - Y^{(n,p)}(\mathcal{E}^{(n)}) = \Theta_{n,p}(\zeta,y,\varepsilon), \tag{7.24}
$$

where the remainder term $\Theta_{n,p}$ satisfies the estimate

$$
|\Theta_{n,p}(\zeta,y,\varepsilon)| \le M\varepsilon^{\frac{p+1}{4}} r^{-\frac{p}{2}}\big(1 + r^{\frac{n-1}{2}}\big). \tag{7.25}
$$

Similar relations and estimates also hold for the derivatives of the difference $Y^{(n)}(\mathcal{E}^{(n)}) - Y^{(n,p)}(\mathcal{E}^{(n)})$.

Theorem 7.1. *For any sufficiently large positive integer N there exist functions $v_j(\zeta,y)$, $0\le j\le N-1$, that are solutions of the problems* (3.2), (3.4)–(3.6), *and constants $E_{k-m}^{(k)}$,*

$1 \le k \le N$, $0 \le m \le k-1$, *such that as* $r \to 0$ *the functions* $v_j(\zeta, y)$ *have the asymptotic representation*

$$v_j(\zeta, y) = \Lambda_j^{(N_1)}(\zeta, y) + K_j(\mathcal{E}^{(N)}, \zeta, y) + \sigma_{N+1-j}^{(j)}(\zeta, y), \tag{7.26}$$

where $N_1 = \left[\frac{N}{2}\right]$, $\mathcal{E}^{(N)} = \{E_{k-m}^{(k)}, 1 \le k \le N\}$, *and the remainder term* $\sigma_{N+1-j}^{(j)}(\zeta, y)$ *satisfies the estimate*

$$|\sigma_{N+1-j}^{(j)}(\zeta, y)| \le M r^{\frac{N+1-j}{2}}. \tag{7.27}$$

The representation (7.27) *admits term-by-term differentiation. The constants* $E_{k-m}^{(k)}$ *and functions* $v_j(\zeta, y)$ *are uniquely determined.*

Proof. Recall that $\Lambda_j^{(N_1)}(\zeta, y)$ and $K_j(\mathcal{E}^{(N)}, \zeta, y)$ are the partial sums of the corresponding series defined by relations (7.6), (7.22). We assume without loss of generality that N is odd: $N = 2n+1$ and, consequently, $N_1 = n$. We construct the function $v_0(\zeta, y)$ in the form of the sum

$$v_0(\zeta, y) = \Lambda_0^{(n)}(\zeta, y) + \delta_0(\zeta, y).$$

As mentioned above, the series $\Lambda_0(\zeta, y)$ is a formal asymptotic solution as $r \to 0$ of the equation $\mathrm{L}_1 z = f(0, y)$, which must also be satisfied by the function we seek, $v_0(\zeta, y)$. Consequently, its partial sum $\Lambda_0^{(n)}(\zeta, y)$ satisfies the same equation with a sufficiently high degree of accuracy. It is easy to verify that this partial sum satisfies the relation

$$\mathrm{L}_1 \Lambda_0^{(n)} = f(0, y) + O\big(r^{\frac{N-3}{2}}\big).$$

Furthermore, by construction, all the terms in this partial sum vanish at $y = \pm 0$.

Thus, in the strip Ω (see (3.3)) the function $\delta_0(\zeta, y)$ must be a solution of the problem

$$\mathrm{L}_1 \delta_0 = O\big(r^{\frac{N-3}{2}}\big), \quad r \to 0;$$
$$\delta_0(\zeta, \pm 0) = 0, \quad \delta_0(\zeta, \gamma^{\pm}(0)) = \widetilde{\phi}_0^{\pm},$$

where we have set $\widetilde{\phi}_0^{\pm} = \phi_0^{\pm} - \Lambda_0^{(n)}(\zeta, \gamma^{\pm}(0))$ (see the notation $\phi_k^{\pm}$ in (3.6)).

Such a solution exists in the class of functions bounded as $r \to 0$, it is unique, and by Lemma 6.2 this solution decomposes into the asymptotic series (6.6):

$$\delta_0(\zeta, y) = \sum_{l=1}^{N} \omega_l^{(0)}(\zeta, y) + \sigma_{N+1}^{(0)}(\zeta, y) = S^{(N,0)}(\zeta, y) + \sigma_{N+1}^{(0)}(\zeta, y), \tag{7.28}$$

where the functions $\omega_l^{(0)}(\xi, \eta)$ belong to $\mathcal{W}_l(\xi, \eta)$ and the remainder term $\sigma_{N+1}^{(0)}(\zeta, y)$ satisfies the estimate

$$|\sigma_{N+1}^{(0)}(\zeta, y)| \le M r^{\frac{N+1}{2}}.$$

Next, by Lemma 6.4, there exist uniquely determined constants $A_1^{(0)}, A_2^{(0)}, \dots, A_N^{(0)}$ such that the principal term $R_0^{(N)}(\mathcal{A}^{(N)}, \zeta, y)$ in the representation (6.22) of the partial sum $X(\mathcal{A}^{(N)}), \zeta, y)$ defined on the set $\mathcal{A}^{(N)}$ coincides with the partial sum $S^{(N,0)}(\zeta, y)$ in the representation (7.28): $R_0^{(N)}(\mathcal{A}^{(N)}) = S^{(N,0)}(\zeta, y)$. We set $A_1^{(0)} = E_1^{(1)}, A_2^{(0)} = E_2^{(2)}, \dots, A_N^{(0)} = E_N^{(N)}$, and thus define the first column $e_1^{(N)}$ of the table $\mathcal{E}^{(N)}$ consisting of the coefficients of the leading terms of the eigenfunctions $B_k(s, t)$, $1 \le k \le N$. By relations (7.22), $K_0^{(N)} = R_0^{(N)}(e_1^{(N)})$ and, consequently, we have constructed the function $v_0(\zeta, y)$ for which the representation (7.26) holds. The theorem is proved for the function $v_0(\zeta, y)$.

We now turn to the construction of the function $v_1(\zeta, y)$. Consider the function $K_1(\zeta, y)$. By (7.22) it is the sum of two summands, one of which, $R_1^{(N)}(e_1^{(N)})$, depends

on the first column $e_1^{(N)}$ in the table $\mathcal{E}^{(N)}$ already defined in the process of matching the function $v_0(\zeta, y)$ and the leading terms $E_m^{(m)} P_m^{(1)}(s,t)$ of the eigenfunctions $B_m(s,t)$ (see (7.10)) for all m such that $1 \le m \le N$, while the second term $R_0^{(N-1)}(e_2^{(N-1)})$ depends on the second column $e_2^{(N-1)}$, which is yet to be defined.

We construct the function $v_1(\zeta, y)$ as the sum

$$v_1(\zeta, y) = \Lambda_1^{(n)}(\zeta, y) + R_1^{(N)}(e_1^{(N)}, \zeta, y) + \delta_1(\zeta, y).$$

As in the construction of the function $v_0(\zeta, y)$, we use the fact that $\Lambda_1^{(N)}(\zeta, y)$ and $R_1^{(N)}(e_1^{(N)})$ are the partial sums of the asymptotic series $\Lambda_1(\zeta, y)$ and $R_1(e_1, \zeta, y)$, which are formal asymptotic solutions as $r \to 0$ of the problems (3.2), (3.4), (3.5) and (6.19), (6.20), respectively. We can verify that

$$\mathrm{L}_1(\Lambda_1^{(n)} + R_1^{(N)}(e_1^{(N)})) = O\big(r^{\frac{N-4}{2}}\big),$$
$$(\Lambda_1^{(N)} + R_1^{(N)}(e_1^{(N)}))(\zeta, \pm 0) = \pm\sqrt{\zeta}\frac{\partial\big(\Lambda_1^{(N)} + R_1^{(N)}(e_1^{(N)})\big)}{\partial y}(\zeta, \pm 0),$$

and, consequently, in the strip Ω the function $\delta_1(\zeta, y)$ must be a solution of the problem

$$\mathrm{L}_1\delta_1 = O\big(r^{\frac{N-4}{2}}\big), \quad r \to 0;$$
$$\delta_1(\zeta, \pm 0) = O\big(r^{\frac{N}{2}}\big),$$
$$\delta_1(\zeta, \gamma^{\pm}) = \widetilde{\phi}_1^{\pm},$$

where we have set $\widetilde{\phi}_1^{\pm} = \phi_1^{\pm} - \Lambda_1^{(n)}(\zeta, \gamma^{\pm}(0)) - R_1^{(N)}(e_1^{(N)}, \zeta, \gamma^{\pm}(0))$.

In the class of functions bounded as $r \to 0$, such a solution exists, it is unique, and by Lemma 6.2 an asymptotic representation of the form (6.6) holds for it:

$$\delta_1(\zeta, y) = \sum_{l=1}^{N-1} \omega_l^{(1)}(\zeta, y) + \sigma_N^{(1)}(\zeta, y) = S^{(N-1,1)}(\zeta, y) + \sigma_N^{(1)}(\zeta, y), \tag{7.29}$$

where the functions $\omega_l^{(1)}(\xi, \eta)$ belong to $\mathcal{W}_l(\xi, \eta)$ and the remainder term satisfies $\sigma_N^{(1)}(\zeta, y) = O(r^{\frac{N}{2}})$.

Next, by Lemma 6.4, there exist uniquely determined constants $A_1^{(1)}, A_2^{(1)}, \dots, A_{N-1}^{(1)}$ such that the partial sum $S^{(N-1,1)}(\zeta, y)$ of the representation (7.29) coincides with the partial sum $R_0^{(N-1)}(A_1^{(1)}, A_2^{(1)}, \dots, A_{N-1}^{(1)})$:

$$R_0^{(N-1)}(A_1^{(1)}, A_2^{(1)}, \dots, A_{N-1}^{(1)}) = S^{(N-1,1)}(\zeta, y).$$

We set $A_1^{(1)} = E_1^{(2)}, A_2^{(0)} = E_2^{(3)}, \dots, A_{N-1}^{(1)} = E_{N-1}^{(N)}$, and thus define the second column $e_2^{(N-1)}$ of the table $\mathcal{E}^{(N)}$.

Thus, we have constructed the function $v_1(\zeta, y)$ such that as $r \to 0$,

$$\begin{aligned} v_1(\zeta, y) &= \Lambda_1^{(n)} + R_1^{(N)}\big(e_1^{(N)}\big) + R_0^{(N-1)}\big(e_2^{(N-1)}\big) + O\big(r^{\frac{N}{2}}\big) \\ &= \Lambda_1^{(n)} + K_1\big(e_1^{(N)}, e_2^{(N-1)}\big) + O\big(r^{\frac{N}{2}}\big), \end{aligned}$$

and the theorem holds for it.

For the subsequent indices $j \ge 2$ the construction is carried out in a similar fashion. Suppose that we have constructed the functions $v_0(\zeta, y)$, $v_1(\zeta, y), \dots, v_{j-1}(\zeta, y)$ and defined the columns $e_1^{(N)}$, $e_2^{(N-1)}, \dots, e_j^{(N+1-j)}$ of the table $\mathcal{E}_N$. We consider the function $K_j(\mathcal{E}_N)$ given by relation (7.22) and represent it in the form of the sum of two terms:

$$K_j = \widetilde{K}_j(e_1^{(N)}, e_2^{(N-1)}, \dots, e_j^{(N+1-j)}) + R_0^{(N-j)}(e_{j+1}^{(N-j)}),$$

where we have set

$$\widetilde{K}_j\big(e_1^{(N)}, e_2^{(N-1)}, \dots, e_j^{(N+1-j)}\big) = R_j^{(N)}\big(e_1^{(N)}\big) + R_{j-1}^{(N-1)}\big(e_2^{(N-1)}\big) + R_{j-2}^{(N-2)}\big(e_3^{(N-2)}\big) + \dots + R_1^{(N-(j-1))}\big(e_j^{(N-(j-1))}\big).$$

The first term $\widetilde{K}_j$ is determined by the columns $e_1^{(N)}, e_2^{(N-1)}, \dots, e_j^{(N+1-j)}$ obtained in the matching process at the preceding steps, that is, in the construction of the functions $v_0, v_1, \dots, v_{j-1}$. We point out that the function $\widetilde{K}_j$ for $j \geq 2$, generally speaking, has a singularity at the origin: $\widetilde{K}_j = O\big(r^{\frac{1-j}{2}}\big)$. This follows from relations (6.21) for the functions $R_l^{(N)}$. The second term $R_0^{(N-j)}\big(e_{j+1}^{(N-j)}\big)$ depends on the column $e_{j+1}^{(N-j)}$, which has to be defined, and has no singularity at the origin.

We have already described the algorithm for constructing the function $v_j(\zeta, y)$ above. The function $v_j(\zeta, y)$ is constructed as the sum

$$v_j(\zeta, y) = \Lambda_j^{(n)}(\zeta, y) + \widetilde{K}_j(\zeta, y) + \delta_j(\zeta, y),$$

and since the functions $\Lambda_j^{(n)}(\zeta, y)$ and $\widetilde{K}_j(\zeta, y)$ are the partial sums of asymptotic series that are formal asymptotic solutions as $r \to 0$ of the problems (3.2), (3.4), (3.5) and (6.19), (6.20), respectively, the construction of the function $v_j(\zeta, y)$ which has a singularity at the origin for $j \geq 2$ reduces to constructing a solution of the corresponding problem where the right-hand side and the boundary function decrease rapidly as $r \to 0$. We can verify that

$$\mathrm{L}_1\delta_j = O\big(r^{\frac{N-3-j}{2}}\big), \quad \delta_j(\zeta, \pm 0) = O\big(r^{\frac{N+1-j}{2}}\big), \quad r \to 0.$$

In the class of functions bounded as $r \to 0$, a solution of such a problem exists, it is unique, and by Lemma 6.2 an asymptotic representation of the form (6.6) holds for it:

$$\delta_j(\zeta, y) = \sum_{l=1}^{N-j} \omega_l^{(j)}(\zeta, y) + \sigma_{N+1-j}^{(j)}(\zeta, y) = S^{(N+1-j,j)}(\zeta, y) + \sigma_{N+1-j}^{(j)}(\zeta, y),$$

where

$$\sigma_{N+1-j}^{(j)}(\zeta, y) = O\big(r^{\frac{N+1-j}{2}}\big).$$

Making further use of Lemma 6.4 we define the column $e_{j+1}^{(N-j)}$ of the table $\mathcal{E}_N$ so that

$$S^{(N+1-j,j)}(\zeta, y) \equiv R_0^{(N-j)}\big(e_{j+1}^{(N-j)}\big),$$

and thus make sure that the function $v_j(\zeta, y)$ we have constructed has the asymptotic representation (7.26) as $r \to 0$.

At the last step, that is, for $j = N - 1$, we construct the function $v_{N-1}(\zeta, y)$ and define the last column $e_N^{(1)}$ of the table $\mathcal{E}_N$ consisting of only one constant $E_1^{(N)}$.

Thus, for all $0 \leq j \leq N - 1$, we have constructed the functions $v_j(\zeta, y)$, defined the constants $E_{k-m}^{(k)}$, $1 \leq k \leq N$, and, consequently, we have constructed the functions $z_k(s, t)$, $1 \leq k \leq N$, or, equivalently, constructed the functions $w_k(\xi, \eta)$. The theorem is proved. □

Thus, in the matching process, we have constructed the partial sums $Q^{(N_1(N))}(s, t, \varepsilon)$ and $Y^{(N)}(\mathcal{E}^{(N)}; s, t, \varepsilon)$ of the series (7.4) and (7.15) or, equivalently, the partial sum $Z^{(N)}(\mathcal{E}^{(N)}; s, t, \varepsilon)$ of the series (7.16):

$$Z^{(N)}(\mathcal{E}^{(N)}) = Q^{(N_1(N))} + Y^{(N)}(\mathcal{E}^{(N)}), \tag{7.30}$$

and the partial sum $V^{(N-1)}(\mathcal{E}^{(N)};\zeta,y,\varepsilon)$ of the series (3.1):

$$V^{(N-1)}(\mathcal{E}^{(N)})=\sum_{k=0}^{N-1}\varepsilon^{\frac{k}{4}}v_k(\zeta,y). \tag{7.31}$$

We set

$$Z^{(N,N-1)}(\mathcal{E}^{(N)})=Q^{(N_1(N),N-1)}+Y^{(N,N-1)}(\mathcal{E}^{(N)}), \tag{7.32}$$

where the partial sums $Q^{(N_1(N),N-1)}$ and $Y^{(N,N-1)}(\mathcal{E}^{(N)})$ are defined by relations (7.8) and (7.23), respectively. By construction, the function $Z^{(N,N-1)}(\mathcal{E}^{(N)})$ is a 'common' part of the asymptotic representation of the function $Z^{(N)}(\mathcal{E}^{(N)};s,t,\varepsilon)$ as $\lambda\to\infty$ and the asymptotic representation of the function $V^{(N-1)}(\mathcal{E}^{(N)};\zeta,y,\varepsilon)$ as $r\to 0$.

Lemma 7.2. *The differences $Z^{(N)}-Z^{(N,N-1)}$ and $V^{(N-1)}-Z^{(N,N-1)}$ satisfy the inequalities*

$$\begin{gathered}\bigl|Z^{(N)}(\mathcal{E}^{(N)})-Z^{(N,N-1)}(\mathcal{E}^{(N)})\bigr|\le M\varepsilon^{\frac{N}{4}}r^{-\frac{N-1}{2}}(1+r^{\frac{N-2}{2}}),\\ \bigl|V^{(N-1)}(\mathcal{E}^{(N)})-Z^{(N,N-1)}(\mathcal{E}^{(N)})\bigr|\le Mr^{\frac{N+1}{2}}\left(1+\left(\frac{\sqrt{\varepsilon}}{r}\right)^{\frac{N-1}{2}}\right),\end{gathered} \tag{7.33}$$

and similar inequalities also hold for the derivatives of these differences.

Proof. The proof of these assertions obviously follows from the estimates (7.10), (7.25), (7.27). □

We further state assertions concerning the behaviour as $\zeta\to\pm\infty$ of the functions $v_k(\zeta,y)$ we have constructed. To do this we consider the coefficients of the outer expansion $u_k(x,y)$ constructed in § 2. As already mentioned above, the functions $u_k(x,y)$ take a different form in different subdomains of the domain D_δ under consideration. First we consider that part of the domain D_δ where $x>0$ and $0\le y\le\gamma^{(+)}(x)$. We consider the partial sum

$$U_+^{(N)}(x,y,\varepsilon)=\sum_{k=0}^{N}\varepsilon^{\frac{k}{2}}u_k(x,y) \tag{7.34}$$

and according to Theorem 2.1 replace the functions $u_k(x,y)$ by their asymptotic representations (2.4):

$$U_+^{(N)}(x,y,\varepsilon)=\sum_{j=0}^{N-1}x^{\frac{j}{2}}u_{0j}^+(y)+O\bigl(x^{\frac{N}{2}}\bigr)+\sum_{k=1}^{N}\varepsilon^{\frac{k}{2}}\left[\sum_{j=0}^{N-1}x^{-2k+\frac{j}{2}}u_{kj}^+(y)+O\bigl(x^{-2k+\frac{N}{2}}\bigr)\right].$$

In the relation thus obtained we go from the variable x to the variable ζ: $x=\sqrt{\varepsilon}\zeta$. It is easy to verify that as a result of this we obtain

$$U_+^{(N)}(x,y,\varepsilon)=\sum_{j=0}^{N-1}\varepsilon^{\frac{j}{4}}R_{j,N}^+(\zeta,y)+O\bigl(\varepsilon^{\frac{N}{4}}\zeta^{\frac{N}{2}}(1+\zeta^{-2N})\bigr), \tag{7.35}$$

where we set

$$R_{0,N}^+(\zeta,y)=u_{00}^+(y),\quad R_{j,N}^+(\zeta,y)=\sum_{l=0}^{N}\zeta^{\frac{j}{2}-2l}u_{lj}^+(y),\quad j\ge 1.$$

We set

$$\sum_{j=0}^{N-1}\varepsilon^{\frac{j}{4}}R_{j,N}^+(\zeta,y)=U_+^{(N,N-1)}(\zeta,y,\varepsilon)$$

and rewrite equation (7.35) in the form

$$U_+^{(N)}(x,y,\varepsilon) - U_+^{(N,N-1)}(\zeta,y,\varepsilon) = O\big(\varepsilon^{\frac{N}{4}}\zeta^{\frac{N}{2}}(1+\zeta^{-2N})\big). \tag{7.36}$$

Relations similar to relations (7.34)–(7.36), with the functions $U_+^{(N)}(x,y,\varepsilon)$, $R_j^+(\zeta,y)$, $U_+^{(N,N-1)}(\zeta,y,\varepsilon)$ replaced by the functions $U_-^{(N)}(x,y,\varepsilon)$, $R_j^-(\zeta,y)$, $U_-^{(N,N-1)}(\zeta,y,\varepsilon)$, respectively, are also valid in that part of the domain D_δ where $x>0$ and $\gamma^{(-)}(x)\le y\le 0$:

$$U_-^{(N)}(x,y,\varepsilon) - U_-^{(N,N-1)}(\zeta,y,\varepsilon) = O\big(\varepsilon^{\frac{N}{4}}\zeta^{\frac{N}{2}}(1+\zeta^{-2N})\big). \tag{7.37}$$

It remains to consider the outer expansion for $x<0$. In this domain we denote it by $\widetilde{U}(x,y,\varepsilon)$. As already mentioned above, for $x<0$ the functions $u_k(x,y)$ are infinitely differentiable and, consequently, as $x\to 0$ they have a Taylor series representation, i.e.

$$\widetilde{U}(x,y,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k u_k(x,y) = \sum_{k=0}^{\infty}\varepsilon^{\frac{k}{2}}\tau_k(\zeta,y),$$

where $\tau_k(\zeta,y)$ is a polynomial in ζ of degree k with coefficients which are infinitely differentiable (with respect to y) for $\gamma^{(-)}(x)\le y\le\gamma^{(+)}(x)$.

Assuming for definiteness (as before) that N is odd, $N=2n+1$, we consider the partial sum $\widetilde{U}^{(n)}(x,y,\varepsilon)=\sum_{k=0}^{n}\varepsilon^{\frac{k}{2}}u_k(x,y)$. Obviously,

$$\widetilde{U}^{(n)}(x,y,\varepsilon) - \widetilde{U}^{(n,n)}(\zeta,y,\varepsilon) = O(\varepsilon^{\frac{n+1}{2}}(\zeta^{n+1}+1)), \tag{7.38}$$

where we have set $\widetilde{U}^{(n,n)}(\zeta,y,\varepsilon)=\sum_{k=0}^{n}\varepsilon^{\frac{k}{2}}\tau_k(\zeta,y)$, and similar relations are also valid for the derivatives of the difference $\widetilde{U}^{(n)}-\widetilde{U}^{(n,n)}$.

We set

$$U^{(N)}(x,y,\varepsilon) = \begin{cases} U_-^{(N)}(x,y,\varepsilon), & x\ge 0, \quad y\le 0;\\ U_+^{(N)}(x,y,\varepsilon), & x\ge 0, \quad y\ge 0;\\ \widetilde{U}^{n}(x,y,\varepsilon), & x\le 0. \end{cases} \tag{7.39}$$

Lemma 7.3. *The functions $v_k(\zeta,y)$, which are solutions of the problems* (3.2), (3.4)–(3.6)*, have the following asymptotic representations as $|\zeta|\to+\infty$:*

$$\begin{aligned} v_k(\zeta,y) &= R_{k,N}^-(\zeta,y)+\sigma_{k,N}^-(\zeta,y), && \zeta\to+\infty, \quad \gamma^{(-)}(x)\le y\le 0,\\ v_k(\zeta,y) &= R_{k,N}^+(\zeta,y)+\sigma_{k,N}^+(\zeta,y), && \zeta\to+\infty, \quad 0\le y\le\gamma^{(+)}(x),\\ v_{2k}(\zeta,y) &= \tau_k(\zeta,y)+\widetilde{\sigma}_{k,n}(\zeta,y), && \zeta\to-\infty, \quad \gamma^{(-)}(x)\le y\le\gamma^{(+)}(x),\\ v_{2k+1}(\zeta,y) &= \widetilde{\sigma}_{2k+1,n}(\zeta,y), && \zeta\to-\infty, \quad \gamma^{(-)}(x)\le y\le\gamma^{(+)}(x). \end{aligned} \tag{7.40}$$

The remainder terms $\sigma_{k,N}^{\pm}(\zeta,y)$ satisfy the estimates

$$\sigma_{k,N}^{\pm}(\zeta,y) = O(\zeta^{\frac{k}{2}-2(N+1)}),$$

and the remainder terms $\widetilde{\sigma}_{j,n}(\zeta,y)$ tend to zero as $\zeta\to-\infty$ faster than any power of ζ^{-1}.

Proof. The proof of this lemma is similar to the proof of analogous lemmas in [2, 4] and is based on the maximum principle. □

Corollary. *The partial sum $V^{(N-1)}(\zeta,y,\varepsilon)$ of the series* (3.1) *(see* (7.31)*) satisfies*

$$\begin{aligned} V^{(N-1)}(\zeta,y,\varepsilon) - U_\pm^{(N,N-1)}(\zeta,y,\varepsilon) &= O\big(\zeta^{-(2N+1)}(1+\varepsilon^{\frac{N-1}{4}}\zeta^{\frac{N-1}{2}})\big), \quad \zeta>0;\\ V^{(N-1)}(\zeta,y,\varepsilon) - \widetilde{U}^{(n,n)}(\zeta,y,\varepsilon) &= o\big(|\zeta|^{-N_2}\big), \quad \zeta<0, \quad N_2>0. \end{aligned} \tag{7.41}$$

Thus, we have constructed three asymptotic series: the outer expansion (1.7), the inner expansion (3.1) in a neighbourhood of the y-axis, and the inner expansion (4.6) (or (4.1))

in a neighbourhood of the origin. Each of these series is a formal asymptotic solution of the original problem (1.1), (1.2) in the corresponding subdomain of the domain D_δ under consideration. The asymptotic series we have constructed are matched: for the series (3.1), (4.6) this follows from relations (7.33); for the series (1.7), (3.1) this follows from relations (7.36)–(7.38), (7.41) which are given just above.

We will use the partial sums of these matched expansions $U^{(N)}(x,y,\varepsilon)$ (see (7.39)), $V^{(N-1)}(\zeta,y,\varepsilon)$ (see 7.31), $Z^{(N)}(s,t,\varepsilon)$ (see (7.30), to construct a composite expansion, which will be a formal asymptotic solution of the original problem (1.1), (1.2) in the entire domain D_δ.

Let $\chi(\omega)\in C^\infty(-\infty,+\infty)$ be such that $\chi(\omega)\equiv 1$ for $\omega\le 1$ and $\chi(\omega)\equiv 0$ for $\omega\ge 2$. We fix some ν such that $0<\nu<1$ and define a function $S^{(N)}(x,y,\varepsilon)$ as follows:

$$\begin{aligned}S^{(N)}(x,y,\varepsilon) &= Z^{(N)}(s,t,\varepsilon)\chi(x\varepsilon^{-\nu})\chi(-x\varepsilon^{-\nu})\chi(y\varepsilon^{\frac{\nu}{2}})\chi(-y\varepsilon^{\frac{\nu}{2}})\\ &\quad + V^{(N-1)}(\zeta,y,\varepsilon)\chi(x\varepsilon^{-\frac{\nu}{2}})\chi(-x\varepsilon^{-\frac{\nu}{2}})\\ &\qquad \times\big[1-\chi(x\varepsilon^{-\nu})\chi(-x\varepsilon^{-\nu})\chi(y\varepsilon^{\frac{\nu}{2}})\chi(-y\varepsilon^{\frac{\nu}{2}})\big]\\ &\quad + U^{(N)}(x,y,\varepsilon)\big[1-\chi(x\varepsilon^{-\frac{\nu}{2}})\chi(-x\varepsilon^{-\frac{\nu}{2}})\big].\end{aligned}$$

Theorem 7.2. *The estimate*

$$\big|u_\varepsilon(x,y)-S^{(N)}(x,y,\varepsilon)\big|\le M\varepsilon^{n\lambda} \tag{7.42}$$

holds for all points in the domain D_δ, where the constant M is independent of ε, while λ depends only on ν.

Proof. The proof of theorems of this type is described in detail in [2] and is not given here. □

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