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ON MACROSCOPIC DIMENSION OF UNIVERSAL COVERINGS OF CLOSED MANIFOLDS

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ABSTRACT. We give a homological characterization of n-manifolds whose universal covering $\widetilde{M}$ has Gromov's macroscopic dimension $\dim_{mc}\widetilde{M} < n$. As a result, we distinguish $\dim_{mc}$ from the macroscopic dimension $\dim_{MC}$ defined by the author in an earlier paper. We prove the inequality $\dim_{mc}\widetilde{M} < \dim_{MC}\widetilde{M} = n$ for every closed n-manifold M whose fundamental group π is a geometrically finite amenable duality group with the cohomological dimension $cd(\pi) > n$.

§1. INTRODUCTION

Gromov introduced the macroscopic dimension $\dim_{mc}$ to study manifolds with positive scalar curvature [11]. He noticed that the universal covering $\widetilde{M}$ of a Riemannian n-manifold with positive scalar curvature is dimensionally deficient on large scales. Gromov formulated his observation as the following conjecture: *For the universal covering $\widetilde{M}$ of an n-manifold with positive scalar curvature*

$$\dim_{mc}\widetilde{M} < n - 1.$$

The macroscopic dimension $\dim_{mc}$ was defined as follows.

Definition 1.1 ([11])**.** For a Riemannian manifold X its macroscopic dimension does not exceed n, $\dim_{mc} X \leq n$, if there is a continuous map $g\colon X \to K^n$ to an n-dimensional simplicial complex with a uniform upper bound on the size of the preimages: $diam\,(g^{-1}(y)) < b$ for all $y \in K^n$.

Gromov called such maps *uniformly cobounded.*

D. Bolotov proved this conjecture for 3-manifolds [3]. In [6] we proved Gromov's conjecture for spin manifolds with the fundamental groups π satisfying the analytic Novikov conjecture and the injectivity condition of the natural transformation $ko_*(B\pi) \to KO(B\pi)$ of the real connective K-theory to the periodic one.

The definition of $\dim_{mc} X$ is good for any metric space X. This concept differs drastically from the notion of the asymptotic dimension $\operatorname{asdim} X$ also defined by Gromov for all metric spaces. The former is an instrument to study open Riemannian manifolds and the later is the best for finitely generated groups taken with the word metric. We note that there is the inequality $\dim_{mc} X \leq \operatorname{asdim} X$ for all metric spaces.

We call an n-manifold M *md-small* if for its universal cover $\widetilde{M}$ we have the inequality $\dim_{mc}\widetilde{M} < n$. The Gromov conjecture states in particular that a closed Riemannian manifold with positive scalar curvature is md-small. The main source of md-small manifolds is inessential manifolds. In fact the md-smallness of manifolds with positive scalar curvature in [3] and [6] was obtained as a consequence of their inessentiality.

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We recall that an n-manifold M is *inessential* if a map $f\colon M \to B\pi$ that classifies its universal covering can be deformed to the $(n-1)$-skeleton $B\pi^{(n-1)}$. It is known that an orientable manifold is inessential if and only if $f_*([M]) = 0$ for the integral homology where $[M]$ is the fundamental class. An orientable manifold M is called *rationally inessential* if $f_*([M]) = 0$ for the rational homology. Gromov asked [11] whether every md-small manifold is rationally inessential. It turns out that the answer is negative [7]. In order to solve this problem I modified Gromov's definition of macroscopic dimension by introducing in [7] a more geometric invariant denoted as $\dim_{MC}$.

Definition 1.2 ([7])**.** For a metric space, $\dim_{MC} X \le n$ if there is a uniformly cobounded Lipschitz map $g\colon X \to K \subset \ell_2$ to an n-dimensional simplicial complex realized in the standard simplex of a Hilbert space.

We note that with the same result K can be equipped with the geodesic metric induced by the embedding $K \subset \ell_2$.

We recall that the asymptotic dimension is defined as follows ([12], [1]): $\operatorname{asdim} X \le n$ *if for every $\epsilon > 0$ there is a uniformly cobounded ϵ-Lipschitz map $g\colon X \to K \subset \ell_2$ to an n-dimensional simplicial complex realized in the standard simplex of a Hilbert space.*

It follows immediately from the definitions that

$$(*)\qquad \dim_{mc} X \le \dim_{MC} X \le \operatorname{asdim} X.$$

In the case when $X = \widetilde{M}$ is the universal cover of a closed manifold M with the metric lifted from M it is easy to see that all three dimensions do not depend on the choice of the metric on M. In that case the second inequality can be strict as can be seen from the following example: If M is a closed smooth 4-manifold with the fundamental group $\pi_1(M) = \mathbb{Z}^5$, then $\dim_{MC} \widetilde{M} \le 4$ and $\operatorname{asdim} \widetilde{M} = \operatorname{asdim} \mathbb{Z}^5 = 5$.

The first inequality in $(*)$ is a different story. It is not easy to come up with an example of a metric space X that distinguishes these two dimensions. Moreover, in [8] I have excluded the Lipschitz condition from the definition of $\dim_{MC}$ assuming that it is superfluous and that it can always be achieved for reasonable metric spaces like Riemannian manifolds. Recently I discovered to my surprise that these two concepts are different even for the universal coverings of closed manifolds.

In this paper we present an example of an n-manifold M with

$$\dim_{mc} \widetilde{M} < \dim_{MC} \widetilde{M} = n.$$

In fact we show that this condition holds true for every closed n-manifold M whose fundamental group π is a geometrically finite amenable duality group with the cohomological dimension $cd(\pi) > n$. This result (and the example) became possible after developing some (co)homological theory of md-small manifolds parallel to that for $\dim_{MC}$ from [7], [8]. It was done in §§ 3–5. In § 2 we present some general geometric and topological properties of md-small manifolds.

§2. md-Small manifolds

We recall that a cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in J}$ of a metric space X is called open if all U_α are open; it is uniformly bounded if there is an $a > 0$ such that $diam\, U_\alpha < a$ for all $\alpha \in J$. It has the order $\le n$ if every point of X is covered by no more than n elements of $\mathcal{U}$. A cover $\mathcal{U}$ of X is called irreducible if no subfamily of $\mathcal{U}$ covers X. A cover $\mathcal{W}$ of X refines a cover $\mathcal{U}$ if for every $W \in \mathcal{W}$ there is a $U \in \mathcal{U}$ such that $W \subset U$. We recall that $\epsilon > 0$ is a Lebesgue number for a cover $\mathcal{U}$ of X if for every subset $A \subset X$ of diameter $< \epsilon$ there is a $U \in \mathcal{U}$ such that $A \subset U$.

A metric space is called *proper* if every closed ball with respect to that metric is compact.

Proposition 2.1. *For a proper metric space X the following are equivalent:*

(1) $\dim_{mc} X \leq n$.
(2) *There is a uniformly bounded irreducible open cover $\mathcal{U}$ of X with the order $\leq n+1$.*
(3) *There is a continuous map $f\colon X \to K$ to a locally finite n-dimensional simplicial complex and a number $b > 0$ such that $f^{-1}(\Delta) \neq \emptyset$ and $diam\, f^{-1}(\Delta) < b$ for every simplex $\Delta \subset K$.*

Proof. (1) $\Rightarrow$ (2). Let $g\colon X \to N$ be a continuous map to an n-dimensional simplicial complex and let $a > 0$ be such that $diam\, g^{-1}(y) < a$. We may assume that N is countable. For every $y \in N$ there is a neighborhood V_y such that $diam\, g^{-1}(V_y) < a$. Consider a cover $\mathcal{V}$ of N. Since $\dim N \leq n$ there is an open locally finite cover $\mathcal{W}$ of N that refines V and has the order $\leq n+1$. It is easy to prove using Zorn's Lemma that every locally finite cover has an irreducible subcover. We define $\mathcal{U}$ to be an irreducible subcover of $g^{-1}(\mathcal{W}) = \{g^{-1}(W) \mid W \in \mathcal{W}\}$.

(2) $\Rightarrow$ (3). Let $\mathcal{U}$ be a uniformly bounded irreducible open cover of X of order $\leq n+1$.

Claim. Every closed ball $B(x,r)$ in X is covered by finitely many elements of $\mathcal{U}$.

Assume the contrary. Consider the subfamily $\mathcal{A} = \{U \in \mathcal{U} \mid U \cap B(x, r+a) \neq \emptyset\}$ where a is an upper bound on the size of U. Since the ball $B(x, r+a)$ is compact, there is a finite subcover $\mathcal{B} \subset \mathcal{A}$ of $B(x, r+a)$. Let $\mathcal{C} = \{U \in \mathcal{U} \mid U \cap (X \setminus B(x, r+a)) \neq \emptyset\}$. Then every $U \in \mathcal{C}$ does not intersect $B(r,a)$. Therefore, $\mathcal{V} = \mathcal{B} \cup \mathcal{C}$ is a proper subcover of $\mathcal{U}$, which contradicts the irreducibility condition.

Using a partition of unity subordinated by $\mathcal{U}$ we construct a projection $g\colon X \to K$ of X to the nerve $N(\mathcal{U}) = K$ of $\mathcal{U}$. The irreducibility of $\mathcal{U}$ implies that $f^{-1}(v) \neq \emptyset$ for every vertex $v \in K$. Since the order of $\mathcal{U}$ does not exceed $n+1$, we obtain $\dim K \leq n$. The above claim implies that every $U \in \mathcal{U}$ intersects only finitely many other elements of $\mathcal{U}$. Hence every vertex in K has bounded valency. Thus, K is locally finite.

Clearly, (3) $\Rightarrow$ (1). ☐

We recall that a real-valued function $f\colon X \to \mathbb{R}$ is called upper semicontinuous if for every $x \in X$ and $\epsilon > 0$, there is a neighborhood U of x such that $f(y) \leq f(x) + \epsilon$ for all $y \in U$. Note that for a proper continuous map $g\colon X \to Y$ of a metric space the function $\psi\colon Y \to \mathbb{R}$ defined as $\psi(y) = diam\, g^{-1}(y)$ is upper semicontinuous.

A homotopy $H\colon X \times I \to Y$ in a metric space Y is called *bounded* if there is a uniform upper bound on the diameter of paths $H(x \times I)$, $x \in X$.

For a finitely presented group π we can choose an Eilenberg-McLane complex $K(\pi,1) = B\pi$ to be locally finite and hence metrizable. We consider a proper geodesic metric on $B\pi$ and lift it to the universal cover $E\pi$. Thus, every closed ball in $B\pi$ is compact.

We recall that a metric space Y is *uniformly contractible* if for every $R > 0$ there is an $S > 0$ such that every R-ball $B(y,R)$ in Y can be contracted to a point in the ball $B(y,S)$. Note that in the case of the finite complex $B\pi$ the space $E\pi$ is uniformly contractible. Generally, $E\pi$ is uniformly contractible over any compact set $C \subset B\pi$: *For every $R > 0$ there is an $S > 0$ such that every R-ball $B(y,R)$ in Y with $p(y) \in C$ can be contracted to a point in the ball $B(y,S)$ where $p\colon E\pi \to B\pi$ is the universal covering.*

Theorem 2.2. *For a closed n-manifold M the following conditions are equivalent:*

(1) *M is md-small.*
(2) *A lift $\widetilde{f}\colon \widetilde{M} \to E\pi$ of the classifying map $f\colon M \to B\pi$ to the universal coverings can be deformed to a map $q\colon \widetilde{M} \to E\pi^{(n-1)}$ by a bounded homotopy $H\colon \widetilde{M} \times [0,1] \to E\pi$ for any choice of proper geodesic metric on $B\pi$.*

The homotopy H can be taken to be cellular for any choice of CW structure on M and the product CW structure on $\widetilde{M} \times [0,1]$.

Proof. (1) $\Rightarrow$ (2). Let $g\colon \widetilde{M} \to K$ be a continuous map to an $(n-1)$-dimensional locally finite simplicial complex from Proposition 2.1. We recall that $g^{-1}(v) \neq \emptyset$ for all vertices $v \in K$ and $diam\, g^{-1}(\Delta) < b$ for every simplex Δ. Let $\widetilde{f}\colon \widetilde{M} \to E\pi$ be a map of the universal covers induced by $f\colon M \to B\pi$. We may assume that f is Lipschitz. Then so is $\widetilde{f}$. Using the uniform contractibility of $E\pi$ over compact sets by induction on the dimension of the skeleton of K we construct a map $h\colon K \to E\pi$ as follows. Since $g^{-1}(v) \neq \emptyset$ for each vertex $v \in K$, we can fix $h(v) \in \widetilde{f}(g^{-1}(v))$. For every edge $[v, v'] \subset K$ we have

$$d(h(v), h(v')) = d\big(\widetilde{f}(u), \widetilde{f}(u')\big) \leq \lambda d_{\widetilde{M}}(u, u') \leq \lambda b$$

where $u \in g^{-1}(v)$, $u' \in g^{-1}(v')$, and λ is a fixed Lipschitz constant for $\widetilde{f}$. Since $E\pi$ is uniformly contractible over $f(M)$, there is a $d_1 > 0$ the same for all edges such that we can join $h(v)$ and $h(v')$ by a path of diam eter $\leq d_1$. Since the covering map $p\colon E\pi \to B\pi$ is 1-Lipschitz, the image $p(h(K^{(1)}))$ lies in the d_1-neighborhood of $f(M)$. In view of the properness of the metric on $B\pi$ the set $p(h(K^{(1)})) \subset B\pi$ is compact. Hence we can use the uniform contractibility of $E\pi$ over $p(h(K^{(1)}))$ to define h on $K^{(2)}$ and so on. As a result, we construct a map $h\colon K \to E\pi$ such that $h \circ g$ is a finite distance from $\widetilde{f}$.

Then we take a cellular approximation $q\colon \widetilde{M} \to E\pi^{(n-1)}$ of $h \circ g$. Again, the map $q\colon \widetilde{M} \to E\pi$ is a finite distance from $\widetilde{f}$. Therefore it can be joined with $\widetilde{f}$ by a bounded homotopy $H\colon \widetilde{M} \times [0,1] \to E\pi$. Then we take a cellular approximation of H rel $M \times \{0,1\}$.

(2) $\Rightarrow$ (1). Without loss of generality we may assume that $B\pi$ and hence $E\pi$ are simplicial complexes. Let a map $g\colon \widetilde{M} \to E\pi^{(n-1)}$ be a distance $< a$ from $\widetilde{f}$. We show that g is uniformly cobounded. Let $N = N_a(f(M))$ be the closed a-neighborhood of $f(M)$ in $B\pi$. We consider a function $\psi\colon N \to \mathbb{R}$ defined as $\psi(z) = diam\, \widetilde{f}^{-1}(B(y, a))$ where $y \in p^{-1}(z)$. Clearly ψ is well-defined. Since ψ is upper semicontinuous, the compactness of N implies that it is bounded from above.

Thus, there is a $b > 0$ such that $diam\, \widetilde{f}^{-1}(B(x, a)) < b$ for all $x \in N$. If $x_1, x_2 \in g^{-1}(y)$, then $\widetilde{f}(x_1), \widetilde{f}(x_2) \in B(y, a)$. Therefore, $d_{\widetilde{M}}(x_1, x_2) < b$. □

Remark. In Theorem 2.2, M can be taken to be any finite complex.

We recall first that a map between metric spaces $f\colon X \to Y$ is called *coarsely Lipschitz* if there are positive constants (λ, b) such that

$$d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + b$$

for all $x, x' \in X$.

Corollary 2.3. *A closed n-manifold M is md-small if and only if for any choice of a locally finite CW complex $B\pi$ and a proper geodesic metric on it the universal cover $\widetilde{M}$ admits a uniformly cobounded coarsely Lipschitz cellular map $g\colon \widetilde{M} \to E\pi^{(n-1)}$ into the $(n-1)$-skeleton of $E\pi$ with the compact image $pg(\widetilde{M})$ where the metric on $E\pi$ is lifted from $B\pi$ and $p\colon E\pi \to B\pi$ is a universal covering map.*

Proof. The backward implication is trivial. For the forward implication we consider a map $g\colon \widetilde{M} \to E\pi^{(n-1)}$ from Theorem 2.2(2). In the proof of the implication (2) $\Rightarrow$ (1), there it was shown that g is uniformly cobounded. It is coarsely Lipschitz since it is a finite distance from a Lipschitz map $\widetilde{f}$. The last condition of the corollary is satisfied for the same reason and by the fact that $B\pi$ is locally compact. □

Similarly n-manifolds M with $\dim_{MC} \widetilde{M} < n$ will be called *MD-small.* A similar result holds for MD-small manifolds.

Theorem 2.4 ([7])**.** *For a closed n-manifold M the following conditions are equivalent:*

(1) *M is MD-small.*
(2) *A lift $\widetilde{f}\colon \widetilde{M} \to E\pi$ of the classifying map $f\colon M \to B\pi$ to the universal coverings can be deformed to the $(n-1)$-dimensional skeleton $E\pi^{(n-1)}$ of $E\pi$ by a Lipschitz homotopy where the metric on $E\pi$ is lifted from the metric on $B\pi$ for any choice of a locally finite complex $B\pi$ and a proper geodesic metric on it.*

The dimension $\dim_{MC}$ can be defined in terms of open covers as follows.

Proposition 2.5. *For a proper metric space X the following are equivalent:*
(1) $\dim_{MC} X \le n$.
(2) There is a uniformly bounded irreducible open cover $\mathcal{U}$ of X of the order $\le n+1$ and with a Lebesque number > 0.

Thus the concepts $\dim_{mc}$ and $\dim_{MC}$ stated in the language of covers differ only by the Lebesgue number condition.

§3. Coarsely equivariant cohomology

Let X be a CW complex and let $E_n(X)$ denote the set of its n-dimensional cells. We recall that the (co)homology of a CW complex X with coefficients in an abelian group G are defined by means of the cellular chain complex $C_*(X) = \{C_n(X), \partial_n\}$ where $C_n(X)$ is the free abelian group generated by the set $E_n(X)$. The resulting groups $H_*(X;G)$ and $H^*(X;G)$ do not depend on the choice of the CW structure on X. The proof of this fact appeals to singular (co)homology theory and it is a part of all textbooks on algebraic topology. The same holds true for (co)homology groups with locally finite coefficients, i. e., for coefficients in a π-module L where $\pi = \pi_1(X)$. The chain complex defining the homology groups $H_*(X;L)$ is $\{C_n(\widetilde{X}) \otimes_\pi L\}$ and the cochain complex defining the cohomology $H^*(X;L)$ is $Hom_\pi(C_n(\widetilde{X}), L)$ where $\widetilde{X}$ is the universal cover of X with the cellular structure induced from X. The resulting groups $H_*(X;L)$ and $H^*(X;L)$ do not depend on the CW structure on X.

These groups can be interpreted as the equivariant (co)homology:

$$H_*(X;L) = H_*^{lf,\pi}(\widetilde{X};L) \quad \text{and} \quad H^*(X;L) = H^*_\pi(\widetilde{X};L).$$

The last equality is obvious since the equivariant cohomology groups $H^*_\pi(\widetilde{X};L)$ are defined by equivariant cochains $C^n_\pi(\widetilde{X}, L)$, i. e., homomorphisms $\phi\colon C_n(\widetilde{X}) \to L$ such that the set

$$S_{\phi,\,c} = \{\gamma^{-1}\phi(\gamma c) \mid \gamma \in \pi\}$$

consists of one element for every $c \in C_n(\widetilde{X})$. In [7] we defined the group of the almost equivariant cochains $C^n_{ae}(\widetilde{X}, L)$ that consist of homomorphisms $\phi\colon C_n(\widetilde{X}) \to L$ for which the set $S_{\phi,\,c}$ is finite for every chain $c \in C_n(\widetilde{X})$. Here we consider the group of coarsely equivariant cochains $C^n_{ce}(\widetilde{X}, L)$ that consist of homomorphisms $\phi\colon C_n(\widetilde{X}) \to L$ such that the set $S_{\phi,\,c} \subset L$ generates a finitely generated abelian group for every $c \in C_n(\widetilde{X})$. Thus we have a string of the inclusions of cochain complexes

$$C^*_\pi(\widetilde{X}, L) \subset C^*_{ae}(\widetilde{X}, L) \subset C^*_{ce}(\widetilde{X}, L) \subset C^*(\widetilde{X}, L)$$

which induces a chain of homomorphisms of corresponding cohomology groups

$$H^n(X;L) \xrightarrow{\beta} H^n_{ae}(\widetilde{X};L) \xrightarrow{\alpha} H^n_{ce}(\widetilde{X};L) \to H^n(\widetilde{X};L).$$

The first homomorphism was denoted in [7] as $\beta = pert^*_X$. We call $ec^*_X = \alpha \circ pert^*_X$ the *equivariant coarsening* homomorphism. Note that the last arrow is an isomorphism for L finitely generated as abelian groups. The cohomology groups $H^*_{ae}(\widetilde{X};L)$ are called the *almost equivariant cohomology* of $\widetilde{X}$ with coefficients in a π-module L [7] and the cohomology groups $H^*_{ce}(\widetilde{X};L)$ are called the *coarsely equivariant cohomology* of $\widetilde{X}$ with coefficients in a π-module L.

One can define singular coarsely equivariant cohomology by replacing cellular n-chains c in the above definition by singular n-chains. The standard argument shows that the singular version of coarsely equivariant cohomology coincides with the cellular. Thus the group $H^*_{ce}(\widetilde{X};L)$ does not depend on the choice of a CW complex structure on X.

Remark. In the case when $L=\mathbb{Z}$ is a trivial module the cohomology theory $H^*_{ae}(\widetilde{X};\mathbb{Z})$ coincides with the ℓ_∞-cohomology. In particular, when $X=E\pi$, it is the ℓ_∞-cohomology of a group $H^*_{(\infty)}(\pi;\mathbb{Z})$ in the sense of [10], whereas the coarsely equivariant cohomology $H^*_{ce}(\widetilde{X};\mathbb{Z})$ equals the standard (cellular) cohomology of the universal cover $H^*(\widetilde{X};\mathbb{Z})$.

A proper cellular map $f\colon X\to Y$ that induces an isomorphism of the fundamental groups lifts to a proper cellular map of the universal covering spaces $\bar{f}\colon\widetilde{X}\to\widetilde{Y}$. The lifting $\bar{f}$ defines a chain homomorphism $\bar{f}_*\colon C_n(\widetilde{X})\to C_n(\widetilde{Y})$ and a cochain homomorphism $\bar{f}^*\colon Hom_{ce}(C_n(\widetilde{Y}),L)\to Hom_{ce}(C_n(\widetilde{X}),L)$. The latter defines a homomorphisms of the coarsely equivariant cohomology groups

$$\bar{f}^*_{ce}\colon H^*_{ce}(\widetilde{Y};L)\to H^*_{ce}(\widetilde{X};L).$$

Suppose that π acts freely on CW complexes $\widetilde{X}$ and $\widetilde{Y}$ such that the actions preserve the CW complex structures. We call a cellular map $g\colon\widetilde{X}\to\widetilde{Y}$ *coarsely equivariant* if the set

$$\bigcup_{\gamma\in\pi}\{\gamma^{-1}g_*(\gamma e)\}\subset C_*(\widetilde{Y})$$

generates a finitely generated subgroup for every cell e in $\widetilde{X}$ where $g_*\colon C_*(\widetilde{X})\to C_*(\widetilde{Y})$ is the induced chain map. In other words g is coarsely equivariant if for every cell e the union $\bigcup_{\gamma\in\pi}\{\gamma^{-1}g_*(\gamma e)\}$ is contained in a finite subcomplex.

On a locally finite simplicial complex we consider the geodesic metric in which every simplex is isometric to the standard.

Proposition 3.1. *Let $f\colon X\to Y$ be a proper coarsely equivariant cellular map. Then the induced homomorphism on cochains takes the coarsely equivariant cochains to coarsely equivariant cochains.*

Proof. Let $\phi\colon C_n(Y)\to L$ be a coarsely equivariant cochain. Let e' be an n-cell in X. There are finitely many cells $c_1,\dots,c_m\in\widetilde{Y}$ such that $f(\gamma e')\subset\bigcup\gamma\{c_1,\dots,c_m\}$ for all $\gamma\in\pi$. Then the group

$$\left\langle\bigcup_{\gamma\in\pi}\{\gamma^{-1}\phi(f(\gamma e'))\}\right\rangle\subset\left\langle\bigcup_{\gamma\in\pi}\{\gamma^{-1}\phi(\gamma\{c_1,\dots,c_m\})\}\right\rangle\subset L$$

is finitely generated since the later group is generated by finitely many finitely generated subgroups. $\square$

For every CW complex X we consider the product CW complex structure on $X\times[0,1]$ with the standard cellular structure on $[0,1]$.

Proposition 3.1 and the standard facts about cellular chain complexes imply the following.

Proposition 3.2. *Let X and Y be complexes with free cellular actions of a group π.*

(A) *Then every coarsely equivariant cellular map $f\colon X \to Y$ induces an homomorphism of the coarsely equivariant cohomology groups*

$$f^*\colon H^*_{ce}(Y;L) \to H^*_{ce}(X;L).$$

(B) *If two coarsely equivariant maps $f_1, f_2\colon X \to Y$ are homotopic by means of a cellular coarsely equivariant homotopy $H\colon X \times [0,1] \to Y$, then they induce the same homomorphism of the coarsely equivariant cohomology groups, $f_1^* = f_2^*$.*

A similar situation is true with homologies. We recall that the equivariant locally finite homology groups are defined by the complex of infinite *locally finite invariant chains*

$$C_n^{lf,\pi}(\widetilde{X};L) = \Big\{ \sum_{e \in E_n(\widetilde{X})} \lambda_e e \mid \lambda_{ge} = g\lambda_e,\ \lambda_e \in L \Big\}.$$

The local finiteness condition on a chain requires that for every $x \in \widetilde{X}$ there is a neighborhood U such that the number of n-cells e intersecting U for which $\lambda_e \neq 0$ is finite. This condition is satisfied automatically when X is a locally finite complex. Even in that case *lf* is part of the notation for the equivariant homology since it was inherited from singular theory. The following proposition implies the equality $H_*(X;L) = H_*^{lf,\pi}(\widetilde{X};L)$ for finite X.

Proposition 3.3. *For every finite CW complex X with the fundamental group π and a π-module L the chain complex $\{C_n(\widetilde{X}) \otimes_\pi L\}$ is isomorphic to the chain complex of locally finite equivariant chains $C_*^{lf,\pi}(\widetilde{X},L)$.*

For locally finite X one should take direct limits.

Similarly one can define the coarsely equivariant homology groups on a locally finite CW complex by considering infinite coarsely equivariant chains. Let X be a complex with the fundamental group π and the universal cover $\widetilde{X}$. We call an infinite chain $\sum\limits_{e \in E_n(\widetilde{X})} \lambda_e e$ *coarsely equivariant* if the group generated by the set $\{\gamma^{-1}\lambda_{\gamma e} \mid \gamma \in \pi\} \subset L$ is finitely generated for every cell e. As we have already mentioned, the complex of equivariant locally finite chains defines the equivariant locally finite homology $H_*^{lf,\pi}(\widetilde{X};L)$. The homology groups defined by the coarsely equivariant locally finite chains we call *the coarsely equivariant locally finite homologies.* We denote them by $H_*^{lf,ce}(\widetilde{X};L)$. We note that as in the case of cohomology this definition can be carried out for the singular homology and it gives the same groups. In particular the groups $H_*^{lf,ce}(\widetilde{X};L)$ do not depend on the choice of a CW complex structure on X

As in the case of the cohomology for any complex K there is an equivariant coarsening homomorphism

$$ec_*^K\colon H_*(K;L) = H_*^{lf,\pi}(\widetilde{K};L) \to H_*^{lf,ce}(\widetilde{K};L)$$

which factors through the perturbation homomorphism

$$pert_*^K\colon H_*(K;L) \to H_*^{lf,ae}(\widetilde{K};L)$$

for the almost equivariant homology defined in [7]. In the case when $L = \mathbb{Z}$ is a trivial π-module the homomorphism $ec_*^K\colon H_*(K;\mathbb{Z}) \to H_*^{lf}(\widetilde{K};\mathbb{Z})$ is the transfer generated by the inclusion of invariant infinite chains to locally finite chains.

There is an analog of Proposition 3.2 for the coarsely equivariant locally finite homology.

Proposition 3.4. *Let X and Y be complexes with free cellular actions of a group π.*

(A) *Then every coarsely equivariant cellular map $f\colon X \to Y$ induces a homomorphism of the coarsely equivariant homology groups*

$$f_*\colon H_*^{lf,ce}(X;L) \to H_*^{lf,ce}(Y;L).$$

(B) *If two coarsely equivariant maps $f_1, f_2\colon X \to Y$ are homotopic by means of a cellular coarsely equivariant homotopy, then they induce the same homomorphism of the coarsely equivariant cohomology groups, $(f_1)_* = (f_2)_*$.*

Let M be a closed oriented n-dimensional PL manifold with a fixed triangulation. Denote by M^* the dual complex. There is a bijection between k-simplices e and the dual $(n-k)$-cells e^* which defines the Poincaré duality isomorphism. This bijection extends to a similar bijection on the universal cover $\widetilde{M}$. Let $\pi = \pi_1(M)$. For any π-module L the Poincaré duality on M with coefficients in L is given on the cochain-chain level by isomorphisms

$$Hom_\pi(C_k(\widetilde{M^*}), L) \xrightarrow{PD_k} C_{n-k}^{lf,\pi}(\widetilde{M};L)$$

where PD_k takes a cochain $\phi\colon C_k(\widetilde{M^*}) \to L$ to the chain

$$\sum_{e\in E_{n-k}(\widetilde{M})} \phi(e^*)e.$$

The family PD_* is a chain isomorphism which is also known as the cap product

$$PD_k(\phi) = \phi \cap F$$

with the fundamental cycle $F \in C_n^{lf,\pi}(\widetilde{M})$, where

$$F = \sum_{e\in E_n(\widetilde{M})} e.$$

We note that the homomorphisms PD_k and PD_k^{-1} extend to the coarsely equivariant chains and cochains:

$$Hom_{ce}(C_k(\widetilde{M^*}), L) \xrightarrow{PD_k} C_{n-k}^{lf,ce}(\widetilde{M};L).$$

Thus, the homomorphisms PD_* define the Poincaré duality isomorphisms PD_{ce} between the coarsely equivariant cohomology and homology. We summarize this in the following

Proposition 3.5. *For any closed oriented n-manifold M and any $\pi_1(M)$-module L the Poincaré duality forms the following commutative diagram:*

$$\begin{array}{ccc} H^k(M;L) & \xrightarrow{ec_M^*} & H^k_{ce}(\widetilde{M};L) \\ {\scriptstyle \cap[M]}\Big\downarrow & & {\scriptstyle PD_{ce}}\Big\downarrow \\ H_{n-k}(M;L) & \xrightarrow{ec_*^M} & H_{n-k}^{lf,ce}(\widetilde{M};L). \end{array}$$

We note that the operation of the cap product for equivariant homology and cohomology automatically extends on the chain-cochain level to the cap product on the coarsely equivariant homology and cohomology. Then the Poincaré duality isomorphism PD_{ce} for $\widetilde{M}$ can be described as the cap product with the homology class $ec_*^M([M]) = [\widetilde{M}] = [F]$.

§4. Obstruction to the inequality $\dim_{mc} \widetilde{X} < n$

We consider locally finite CW complexes supplied with a geodesic metric with finitely many isometry types of n-cells for each n. A typical example is a uniform simplicial complex, i.e., a simplicial complex supplied with the geodesic metric such that every simplex is isometric to the standard simplex. Let π be a finitely presented group. Then the classifying space $B\pi = K(\pi, 1)$ can be taken to be a locally finite simplicial complex. We fix a geodesic metric on $B\pi$. Let $p_\pi \colon E\pi \to B\pi$ denote the universal covering. We consider the induced CW complex structure and induced geodesic metric on $E\pi$.

We recall some notions from coarse geometry [13]. A subset $X' \subset X$ of a metric space is called *coarsely dense* if there is a $D > 0$ such that every D-ball $B(x, D)$ in X has a nonempty intersection with X'. A (not necessarily continuous) map $f \colon X \to Y$ between metric spaces is called a *coarse embedding* [13] if there are two nondecreasing functions $\rho_1, \rho_2 \colon \mathbb{R}_+ \to \mathbb{R}_+$ tending to infinity such that

$$\rho_1(d_X(x, x')) \le d_Y(f(x), f(x')) \le \rho_2(d_X(x, x'))$$

for all $x, x' \in X$.

If $f \colon K \to B\pi$ is a classifying map of the universal covering $\widetilde{K}$ of a finite complex K, then any lift $\widetilde{f} \colon \widetilde{K} \to E\pi$ is a coarse embedding. Note that for a geodesic metric space X every coarse embedding is coarsely Lipschitz. Also note that every coarse embedding $f \colon X \to Y$ is uniformly cobounded. Indeed, for each $a \in \mathbb{R}_+$ with $\rho_1(a) > 0$ we have $diam\,(f^{-1}(y)) \le a$ for all y.

A coarse embedding $f \colon X \to Y$ is called a *coarse equivalence* if the image $f(X)$ is coarsely dense in Y. Thus for metric CW complexes X that we deal with the inclusion $X^{(n-1)} \subset X^{(n)}$ is a coarse equivalence. For every coarse equivalence $f \colon X \to Y$ there is an *inverse* coarse equivalence $g \colon Y \to X$, i.e., a map g such that $f \circ g$ and $g \circ f$ are a bounded distance from the identities 1_Y and 1_X, respectively.

Let A be a subset of a CW complex X. The star neighborhood $St(A)$ of A is the closure of the union of all cells in X that have a nonempty intersection with A.

Proposition 4.1. *Let X and Y be universal covers of locally finite complexes with fundamental group π and let $X' \subset X$ be a coarsely dense π-invariant subset. Let $\Phi \colon X \to Y$ be a coarsely Lipschitz cellular map with the coarsely equivariant restriction $\Phi|_{X'}$. Then Φ is coarsely equivariant.*

Proof. The coarsely dense condition implies that there is a number $D > 0$ such that for every cell $b \subset X$ there is a cell $e \subset X'$ with $dist(e, b) < D$. Since $\Phi|_{X'}$ is coarsely equivariant, for every closed cell $e \subset X'$ the union

$$\bigcup_{\gamma \in \pi} \gamma^{-1}\Phi(\gamma\bar{e}) \subset \bar{\sigma}_1 \cup \ldots \cup \bar{\sigma}_k$$

lies in the finite union of closed cells. Then

$$\bigcup_{\gamma \in \pi} \gamma^{-1}\Phi(\gamma\bar{b}) \subset St^m(\bar{\sigma}_1 \cup \ldots \cup \bar{\sigma}_k).$$

The existence of m follows from the fact that Φ is coarsely Lipschitz and from the existence of universal D. The local finiteness of Y implies that the m-times iterated star neighborhood $St^m(\bar{\sigma}_1 \cup \ldots \cup \bar{\sigma}_k)$ lies in a finite subcomplex of Y. □

Corollary 4.2. *Let X and Y be universal covers of finite complexes with fundamental group π. Then a cellular coarsely Lipschitz homotopy $\Phi \colon X \times [0,1] \to Y$ of a coarsely equivariant map is coarsely equivariant.*

Proposition 4.3. *Let X and Y be metric CW complexes. A coarsely Lipschitz extension $\bar{f}: X^{(n)} \to Y$ of a coarse embedding $f: X^{(n-1)} \to Y$ is a coarse embedding. In particular, $\bar{f}$ is uniformly cobounded.*

Proof. As we noted above, the inclusion $i: X^{(n-1)} \to X^{(n)}$ is a coarse equivalence. Let $j: X^{(n)} \to X^{(n-1)}$ be a coarse inverse with $dist(j, 1_{X^{(n)}}) < D$. We show that $\bar{f}$ is a finite distance from the coarse embedding $f \circ j$. This would imply that $\bar{f}$ is a coarse embedding. Indeed,

$$d_Y\big(\bar{f}(x), fj(x)\big) = d_Y\big(\bar{f}(x), \bar{f}(j(x))\big) \le \lambda d_X(x, j(x)) + c \le \lambda D + c,$$

where λ and c are from the definition of the coarsly Lipschitz map $\bar{f}$. $\square$

Here we recall some basic facts of elementary obstruction theory. Let $f: X \to Y$ be a cellular map that induces an isomorphism of the fundamental groups. We want to deform the map f to a map to the $(n-1)$-skeleton $Y^{(n-1)}$. For that we consider the extension problem

$$X \supset X^{(n-1)} \xrightarrow{f} Y^{(n-1)},$$

i.e., the problem of extending $f: X^{(n-1)} \to Y^{(n-1)}$ continuously to a map $\bar{f}: X \to Y^{(n-1)}$. The primary obstruction for this problem o_f is defined by the cochain $C_f: C_n(X) \to \pi_{n-1}(Y^{(n-1)})$ which is defined as $C_f(e) = [f \circ \phi_e]$ where $\phi_e: \partial D^n \to X$ is the attaching map of the n-cell e^n. It turns out that C_f is a cocycle which defines the obstruction $o_f \in H^n(X; L)$ to extend f to the n-skeleton where $L = \pi_{n-1}(Y^{(n-1)})$ is the $(n-1)$-dimensional homotopy group considered as a π-module for $\pi = \pi_1(Y) = \pi_1(X)$. Obstruction theory says that a map $g: X \to Y^{(n-1)}$ that agrees with f on the $(n-2)$-skeleton $X^{(n-2)}$ exists if and only if $o_f = 0$. The primary obstruction is natural: If $g: Z \to X$ is a cellular map, then $o_{gf} = g^*(o_f)$. In particular, in our case $o_f = f^*(o_1)$ where $o_1 \in H^n(Y; L)$ is the primary obstruction to the retraction of Y to the $(n-1)$-skeleton.

Definition 4.4. Let $g: Y^{(n-1)} \to Z$ be a coarsely Lipschitz map of the $(n-1)$-skeleton of an n-dimensional complex to a metric space. We call the problem of extending g to a coarsely Lipschitz map $\bar{g}: Y \to Z$ a *coarsely Lipschitz extension problem.*

Definition 4.5. Let X be a finite n-complex, $n \ge 3$, with $\pi_1(X) = \pi$ and let $\widetilde{f}: \widetilde{X} \to \widetilde{Y}$ be a lift of a cellular map $f: X \to Y$ that induces an isomorphism of the fundamental groups. We note that the obstruction cocycle

$$C_{\widetilde{f}}: C_n(\widetilde{X}) \to \pi_{n-1}(\widetilde{Y}^{(n-1)}) = \pi_{n-1}(Y^{(n-1)})$$

being equivariant is coarsely equivariant. Thus, it defines an element

$$o_{\widetilde{f}} \in H^n_{ce}\big(\widetilde{X}; \pi_{n-1}(Y^{(n-1)})\big).$$

Since it also defines an element $\kappa_f \in H^n_\pi(\widetilde{X}; L) = H^n(X; L)$ of the equivariant cohomology with the π-module $L = \pi_{n-1}(Y)$, we have $o_{\widetilde{f}} = ec^*_X(\kappa_f)$.

We consider a metric CW complex with the induced metric on their universal coverings.

Proposition 4.6. *Let $\widetilde{f}: \widetilde{X} \to \widetilde{Y}$ be a lift of a Lipschitz cellular map $f: X \to Y$ of a finite complex to a locally finite complex that induces an isomorphism of the fundamental groups. Then the above cohomology class $o_{\widetilde{f}} \in H^n_{ce}\big(\widetilde{X}; \pi_{n-1}(Y^{(n-1)})\big)$ is the primary obstruction for the following coarsely Lipschitz extension problem*

$$\widetilde{X} \supset \widetilde{X}^{(n-1)} \xrightarrow{\widetilde{f}|} \widetilde{Y}^{(n-1)}.$$

Thus, $o_{\widetilde{f}} = 0$ if and only if there is a coarsely Lipschitz map $\bar{g}\colon \widetilde{X}^{(n)} \to \widetilde{Y}^{(n-1)}$ which agrees with $\widetilde{f}$ on $\widetilde{X}^{(n-2)}$.

Proof. The proof goes along the lines of a similar statement from classical obstruction theory. Let $C_{\widetilde{f}} = \delta\Psi$ where $\Psi\colon C_{n-1}(\widetilde{X}) \to \pi_{n-1}(\widetilde{Y}^{(n-1)})$ is a coarsely equivariant homomorphism. For each $(n-1)$-cell e of X we fix a section $\widetilde{e} \subset \widetilde{X}$, an $(n-1)$-cell in $\widetilde{X}$. Then the set $\{\gamma^{-1}\Psi(\gamma\widetilde{e}) \mid \gamma \in \pi\}$ spans a finitely generated subgroup in $\pi_{n-1}(\widetilde{Y}^{(n-1)})$. Thus this subgroup is contained in the image of $\pi_{n-1}(F_{\widetilde{e}})$ for some finite subcomplex $F_{\widetilde{e}} \subset \widetilde{Y}^{(n-1)}$.

As in classical obstruction theory we define a map $g_\gamma\colon \gamma\widetilde{e} \to \gamma F_{\widetilde{e}} \subset Y^{(n-1)}$, $\gamma \in \pi$, on cells $\gamma\widetilde{e}$ such that g_γ agrees with $\widetilde{f}$ outside a small $(n-1)$-ball $B_\gamma \subset \gamma\widetilde{e}$ and the difference of the restriction of $\widetilde{f}$ and g_γ to B_γ defines a map

$$d_{\widetilde{f},\, g_\gamma}\colon S^{n-1} = B_\gamma^+ \cup B_\gamma^- \to \gamma F_{\widetilde{e}} \subset Y^{(n-1)}$$

that represents the class $-\Psi(\gamma\widetilde{e})$.

The union of g_γ defines a bounded map $g\colon \widetilde{X}^{(n-1)} \to \widetilde{Y}^{(n-1)}$ in such a way that the difference map $d_{\widetilde{f},\, g}\colon S^{n-1} \to Y^{(n-1)}$ on the cell $\gamma\widetilde{e}$, $\gamma \in \pi$, represents the element $-\Psi(\gamma\widetilde{e})$. Then elementary obstruction theory implies that for every n-cell $\sigma' \subset \widetilde{X}$ there is an extension $\bar{g}_{\sigma'}\colon \overline{\sigma'} \to \widetilde{Y}^{(n-1)}$ of $g|_{\partial\sigma'}$. Since $\partial\sigma'$ is contained in finitely many $(n-1)$-cells, there are $e_1, \dots, e_k \subset X^{(n-1)}$ and $\gamma_1, \dots, \gamma_k \in \pi$ such that

$$g(\partial\sigma') \subset Z = \bigcup_{i=1}^{k} \gamma_i F_{\widetilde{e}_i}.$$

Since $\pi_{n-1}(Z)$ is finitely generated, there is a finite subcomplex $W \subset \widetilde{Y}^{(n-1)}$ containing Z such that

$$ker\big\{\pi_{n-1}(Z) \to \pi_{n-1}(W)\big\} = ker\big\{\pi_{n-1}(Z) \to \pi_{n-1}(\widetilde{Y}^{(n-1)})\big\}.$$

Thus, we may assume that $\bar{g}_{\sigma'}(\sigma') \subset W$. If $\gamma\sigma'$ is a translate of σ', we obtain $g(\partial\gamma\sigma') \subset \gamma Z$. Therefore we may assume that $\bar{g}_{\gamma\sigma'}(\gamma\sigma') \subset \gamma W$.

Thus, the resulting extension $\bar{g}\colon \widetilde{X} \to \widetilde{Y}^{(n-1)}$ is uniformly bounded and hence coarsely Lipschitz.

In the other direction, if there is a coarsely Lipschitz map $\bar{g}\colon \widetilde{X} \to Y^{(n-1)}$ that coincides with $\widetilde{f}$ on the $(n-2)$-dimensional skeleton, then the difference cochain $d_{\widetilde{f},\, \bar{g}}$ is coarsely equivariant. Indeed, for any $\lambda, b > 0$ only homotopy classes from a finitely generated subgroup of $\pi_{n-1}(Y)$ can be realized by coarsely Lipschitz maps preserving a base point. Then the formula $\delta d_{\widetilde{f},\, \bar{g}} = C_{\bar{g}} - C_{\widetilde{f}}$ and the fact that $o_{\bar{g}} = 0$ imply that $o_{\widetilde{f}} = 0$. □

Let $[e] \in \pi_{n-1}(\widetilde{Y}^{(n-1)})$ denote the element of the homotopy group defined by the attaching map of an n-cell e. Then the homomorphism $C_{\widetilde{1}}\colon C_n(\widetilde{Y}) \to \pi_{n-1}(\widetilde{Y}^{(n-1)})$ defined as $C_{\widetilde{1}}(e) = [e]$ is an equivariant cocycle with the cohomology class $o_{\widetilde{1}} \in H^n_{ce}\big(\widetilde{Y}; \pi_{n-1}(Y^{(n-1)})\big)$.

Proposition 4.7.

(1) *The cohomology class $o_{\widetilde{f}}$ from the above proposition is the image under $\widetilde{f}^*$ of the class $o_{\widetilde{1}} \in H^n_{ce}\big(\widetilde{Y}; \pi_{n-1}(Y^{(n-1)})\big)$.*

(2) *The class $o_{\widetilde{1}}$ comes under the equivariant coarsening homomorphism ec^*_π from the primary obstruction $\kappa_1 \in H^n\big(Y; \pi_{n-1}(Y^{(n-1)})\big)$ to retract Y to the $(n-1)$-dimensional skeleton.*

Proof. The first part is the naturality of obstructions for coarsely Lipschitz extension problems with respect to coarsely Lipschitz maps. As in the case of classical obstruction theory, it follows from the definition.

The second part follows from definition (see Definition 4.5). □

Theorem 4.8. *Let X be a finite n-complex with $\pi_1(X) = \pi$ and let $f\colon X \to B\pi$ be a map that induces an isomorphism of the fundamental groups where $B\pi$ is a locally finite CW complex. Then $\dim_{mc} \widetilde{X} < n$ if and only if the above obstruction is trivial, $o_{\widetilde{f}} = 0$.*

Proof. If $\dim_{mc} \widetilde{X} < n$, then by Theorem 2.2 there is a coarsely Lipschitz cellular homotopy of $\widetilde{f}\colon \widetilde{X} \to E\pi$ to a map $g\colon \widetilde{X} \to E\pi^{(n-1)}$. By Corollary 4.2, the map g is coarsely equivariant. Then by Proposition 3.2, $o_{\widetilde{f}} = \widetilde{f}^*(o_1) = g^*i^*(o_1) = 0$ where $i\colon E\pi^{(n-1)} \subset E\pi$ is the inclusion.

We assume that $B\pi$ is a locally finite simplicial complex. If $o_{\widetilde{f}} = 0$, then by Proposition 4.6 there is a coarsely Lipschitz continuous map $g\colon \widetilde{X} \to E\pi^{(n-1)}$ which agrees with $\widetilde{f}$ on the $(n-2)$-skeleton $X^{(n-2)}$. Note that the restriction $\widetilde{f}|_{X^{(n-2)}}$ is a coarse embedding. By consecutive applications of Proposition 4.3 we obtain that g is uniformly cobounded. Therefore, $\dim_{mc} \widetilde{X} < n$. □

§5. Obstruction for a manifold to be md-small

We recall that the Berstein-Schwarz class $\beta \in H^1(\pi, I(\pi))$ of a group π is define by the cochain on the Cayley graph $\phi\colon G \to I(\pi)$ which takes an ordered edge $[g, g']$ between the vertices labeled by $g, g' \in G$ to $g' - g$. Here $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$. We use the notation $I(\pi)^k$ for the k-times tensor product $I(\pi)\otimes\ldots\otimes I(\pi)$ over $\mathbb{Z}$. Then the cup product $\beta \smile \ldots \smile \beta$ is defined as an element of $H^k(\pi, I(\pi)^k)$.

The Berstein-Schwarz class is universal in the following sense.

Theorem 5.1 ([14, 2, 9])**.** *For every π-module L and every element $\alpha \in H^k(\pi, L)$ there is a π-homomorphism $\xi\colon I(\pi)^k \to L$ such that $\xi^*(\beta^k) = \alpha$ where*

$$\xi^*\colon H^k(\pi, I(\pi)^k) \to H^k(\pi, L)$$

is the coefficient homomorphism.

Corollary 5.2. *The cohomological dimension of a group π can be computed by the formula $cd(\pi) = \max\{n \mid \beta^n \neq 0\}$ where β is the Berstein-Schwarz class of π.*

The following is well known (see [6], Proposition 3.2).

Theorem 5.3. *For a closed oriented n-manifold M with the classifying map $f\colon M \to B\pi$ the following are equivalent:*

1. *M is inessential.*
2. *$f_*([M]) = 0$ in $H_*(B\pi; \mathbb{Z})$.*
3. *$f^*(\beta^n) = 0$ in $H^*(M; I(\pi)^n)$ where β is the Berstein-Schwarz class of π.*

The following can be considered as a coarse analog of Theorem 5.3.

Theorem 5.4. *For a closed oriented n-manifold M with the classifying map $f\colon M \to B\pi$ and its lift to the universal covers $\widetilde{f}\colon \widetilde{M} \to E\pi$ the following are equivalent:*

1. *M is md-small.*
2. *$\widetilde{f}_*([\widetilde{M}]) = 0$ in $H_n^{lf}(E\pi; \mathbb{Z})$ where $[\widetilde{M}] \in H_n^{lf}(\widetilde{M}; \mathbb{Z})$ is the fundamental class of $\widetilde{M}$.*
3. *$f_*([M]) \in ker(ec_*^{\pi})$ where $[M]$ is the fundamental class of M.*
4. *$f^*(\beta^n) \in ker(ec_M^*)$ where β is the Berstein-Schwarz class of π.*

Proof. 1. $\Rightarrow$ 2. We may assume that $f\colon M \to B\pi$ is cellular and Lipschitz for some metric CW complex structure on $B\pi$. If $\dim_{mc} \widetilde{M} < n$, then by Corollary 2.3 there is a coarsely Lipschitz cellular homotopy of $\widetilde{f}\colon \widetilde{X} \to E\pi$ to a map $g\colon \widetilde{X} \to E\pi^{(n-1)}$ with a compact projection to $B\pi$. By Corollary 4.2, it is coarsely equivariant. Then by Proposition 3.2 it follows that $\widetilde{f}_*([\widetilde{M}]) = 0$.

2. $\Rightarrow$ 3. Note that

$$ec_*^{\pi}\big(f_*([M])\big) = \widetilde{f}_*\big(ec_*^M([M])\big) = \widetilde{f}_*([\widetilde{M}]) = 0$$

and hence $f_*([M]) \in ker(ec_*^{\pi})$.

3. $\Rightarrow$ 4. If $f_*([M]) \in ker(ec_*^{\pi})$, then $ec_*^{\pi}\big(f_*([M]) \cap \beta^n\big) = 0$. Since the commutative diagram

$$\begin{array}{ccc} H_0^{lf,ce}(\widetilde{M}; I(\pi)^n) & \xrightarrow{\bar{f}_*} & H_0^{lf,ce}(E\pi; I(\pi)^n) \\ {\scriptstyle ec_*^M}\uparrow & & {\scriptstyle ec_*^{\pi}}\uparrow \\ H_0(M; I(\pi)^n) & \xrightarrow{f_*} & H_0(B\pi; I(\pi)^n) \end{array}$$

has isomorphisms for horizontal arrows, $ec_*^M([M] \cap f^*(\beta^n)) = 0$. Thus,

$$ec_*^M([M]) \cap ec_M^* f^*(\beta^n) = 0.$$

By Poincaré duality, $ec_M^* f^*(\beta^n) = 0$.

4. $\Rightarrow$ 1. We show that the obstruction $o_{\widetilde{f}}$ to the inequality $\dim_{mc} \widetilde{M} < n$ is zero and apply Theorem 4.8. By Theorem 5.1 there is a π-homomorphism $\xi\colon I(\pi)^n \to L = \pi_{n-1}(B\pi^{(n-1)})$ such that $\xi^*(\beta^n) = \kappa_1$ and $\xi^*(f^*(\beta^n)) = o_f$ where κ_1 is the primary obstruction for retracting $B\pi$ onto $B\pi^{(n-1)}$ and o_f is the primary obstruction for deforming f into $B\pi^{(n-1)}$. Then $o_{\widetilde{f}} = ec_M^*(o_f) = \xi^* ec_M^*(\beta^n) = 0$. ☐

Corollary 5.5. *For every finitely presented group π and every n there is a subgroup $H_n^{sm}(B\pi) \subset H_n(B\pi; \mathbb{Z})$ of md-small classes such that:*

1. *If there is an md-small orientable manifold M with a classifying map $f\colon M \to B\pi$, then $f_*([M]) \in H_*^{sm}(B\pi)$.*
2. *If a class $\alpha \in H_*^{sm}(B\pi)$ is the image $\alpha = f_*([M])$ of the fundamental class of a manifold M for a classifying map $f\colon M \to B\pi$, then M is md-small.*

Proof. We define $H_*^{sm}(B\pi) = ker(ec_*^{\pi})$. ☐

For the class of MD-large manifolds a similar theorem was proven in [7].

Corollary 5.5 states that the property for manifolds to be md-small is a group homology property. This result is in the spirit of the results of Brunnbauer and Hanke [5] where the notion of the small homology subgroup $H_*^{sm}(\pi) \subset H_*(\pi)$ for a given class of large manifolds was first introduced. They proved similar results for several classes of large manifolds. The major difference is that all their results are rational and deal with the rational homology $H_*(\pi; \mathbb{Q})$. It is an open question if Corollary 5.5 holds true rationally. In particular, it is unknown whether all torsion classes in $H_*(\pi)$ are small. A related question is

Question 5.6. Suppose that the connected sum $M\#M$ of a manifold M with itself is md-small. Does it follow that M is md-small?

Theorem 5.7. *M is md-small if and only if $M \times S^1$ is md-small.*

Proof. Let $n = \dim M$ and let $\widetilde{f}\colon \widetilde{M} \to E\pi$ denote a lift to the universal covers of a classifying map $f\colon M \to B\pi$. In the commutative diagram

$$\begin{array}{ccc} H_n^{lf}(\widetilde{M};\mathbb{Z}) & \xrightarrow{\widetilde{f}_*} & H_n^{lf}(E\pi;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_{n+1}^{lf}(\widetilde{M}\times\mathbb{R};\mathbb{Z}) & \xrightarrow{(\widetilde{f}\times 1)_*} & H_{n+1}^{lf}(E\pi\times\mathbb{R};\mathbb{Z}) \end{array}$$

the vertical arrows are the suspension isomorphisms. Then $\widetilde{f}_* = 0$ if and only if $(\widetilde{f}\times 1)_* = 0$ and the result follows from Theorem 5.4 and the fact that

$$ec_*^{\pi} f_*([M]) = \widetilde{f}_* ec_*^M([M]).$$

□

§6. MD-Large manifolds that are md-small

A criterion similar to Theorem 5.4 for manifolds to be MD-small was proven in [7]:

Theorem 6.1. *For a closed oriented n-manifold M with a map $f\colon M \to B\pi$ classifying the universal cover the following are equivalent:*

1. *M is MD-small.*
2. *$f_*([M]) \in ker(pert_*^{\pi})$.*
3. *$f^*(\beta^n) \in ker(pert_M^*)$ where β is the Berstein-Schwarz class.*

A group π is called *geometrically finite* if there is a finite Eilenberg-MacLane complex $K(\pi,1)$. We recall that a group π is called a *duality group* [4] if there is a module D such that

$$H^i(\pi, M) \cong H_{m-i}(\pi, M \otimes D)$$

for all π-modules M and all i where $m = cd(\pi)$.

Theorem 6.2. *Let π be a geometrically finite duality group. Then $H_i^{lf}(E\pi;\mathbb{Z}) = 0$ for all $i \neq cd(\pi)$.*

Proof. From Theorem 10.1, Chapter 8 of [4] it follows that $H^i(\pi,\mathbb{Z}\pi) = 0$ for $i \neq m = cd(\pi)$ and $H^m(\pi,\mathbb{Z}\pi)$ is a free abelian group. In view of the equality $H^i(\pi,\mathbb{Z}\pi) = H_c^i(E\pi;\mathbb{Z})$ for geometrically finite groups (see [4], Theorem 7.5, Chapter 8) and the short exact sequence for the Steenrod homology of a compact metric space

$$0 \to Ext(H^{i+1}(X),\mathbb{Z}) \to H_i^s(X;\mathbb{Z}) \to Hom(H^i(X),\mathbb{Z}) \to 0$$

applied to the one point compactification $\alpha(E\pi)$ of $E\pi$, we obtain that

$$H_i^s(\alpha(E\pi);\mathbb{Z}) = 0$$

for $i < m$. The equality $H_i^{lf}(E\pi;\mathbb{Z}) = H_i^s(\alpha(E\pi);\mathbb{Z})$ completes the proof. □

Theorem 6.3. *Every rationally essential n-manifold M whose fundamental group π is a geometrically finite amenable duality group with $cd(\pi) > n$ is md-small and MD-large.*

Proof. It was proven in [8], Theorem 7.6, that a rationally essential manifold with an amenable fundamental group is MD-large. Here we present a sketch of an alternative proof of that theorem for geometrically finite groups suggested to me by M. Marcinkowski. He noticed that the perturbation homomorphism

$$pert_*^{\pi} : H_*(\pi,\mathbb{Z}) \to H_*^{lf,\,ae}(E\pi;\mathbb{Z})$$

can be identified with the coefficient homomorphism

$$i_*\colon H_*(\pi,\mathbb{Z}) \to H_*(\pi, \ell^\infty(\pi,\mathbb{Z}))$$

induced by the inclusion $i\colon \mathbb{Z} \to \ell^\infty(\pi, \mathbb{Z})$ of the constant functions into the π-module of all bounded functions. This homomorphism i_* and the similar homomorphism $\bar{i}_*\colon H_*(\pi, \mathbb{R}) \to H_*(\pi, \ell^\infty(\pi, \mathbb{R}))$ for coefficients in $\mathbb{R}$ form a commutative diagram

$$\begin{array}{ccc} H_*(\pi, \mathbb{Z}) & \xrightarrow{i_*} & H_*(\pi, \ell^\infty(\pi, \mathbb{Z})) \\ \downarrow & & \downarrow \\ H_*(\pi, \mathbb{R}) & \xrightarrow{\bar{i}_*} & H_*(\pi, \ell^\infty(\pi, \mathbb{R})). \end{array}$$

In view of the amenability of π, the homomorphism $\bar{i}\colon \mathbb{R} \to \ell^\infty(\pi, \mathbb{R})$ is a split injection. Therefore, $\bar{i}_*$ is an injection. Since M is rationally essential, $f_*([M]) \neq 0$ in $H_*(\pi, \mathbb{R})$. Therefore, $pert^\pi_*(f_*[M]) \neq 0$ and hence M is MD-large by Theorem 6.1.

Since π is a duality group, we obtain $H^{lf}_n(E\pi; \mathbb{Z}) = 0$ for $n < cd(\pi)$. The coarsely equivariant homology group $H^{lf,ce}_*(E\pi; \mathbb{Z})$ coincides with $H^{lf}_*(E\pi; \mathbb{Z})$. By Theorem 6.2, $H^{lf}_n(E\pi; \mathbb{Z}) = 0$; therefore, by Theorem 5.4, M is md-small. $\square$

Remark. In the definition of $\dim_{MC}$ in [8], Definition 1.1, the Lipschitz condition on f was omitted by mistake. It was used implicitly in the proof of Theorem 7.6 of [8].

Example. For $\pi = \mathbb{Z}^m$, $m > 5$, every homology class $\alpha \in H_n(B\pi; \mathbb{Q}) = \bigoplus \mathbb{Q}$, $n < m$, can be realized by a manifold $f\colon M \to B\pi$ (due to Thom's theorem). For $n > 4$ applying surgery in dimensions 0 and 1 we can achieve that f induces an isomorphism of the fundamental groups. By Theorem 6.3, M is md-small and MD-large simultaneously.

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