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# SUBSTITUTIONS OF POLYTOPES AND OF SIMPLICIAL COMPLEXES, AND MULTIGRADED BETTI NUMBERS

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Abstract. For a simplicial complex $K$ on $m$ vertices and simplicial complexes $K_1, \dots, K_m$, we introduce a new simplicial complex $K(K_1, \dots, K_m)$, called a substitution complex. This construction is a generalization of the iterated simplicial wedge studied by A. Bari, M. Bendersky, F. R. Cohen, and S. Gitler. In a number of cases it allows us to describe the combinatorics of generalized joins of polytopes $P(P_1, \dots, P_m)$, as introduced by G. Agnarsson. The substitution gives rise to an operad structure on the set of finite simplicial complexes in which a simplicial complex on $m$ vertices is considered as an $m$-ary operation. We prove the following main results: (1) the complex $K(K_1, \dots, K_m)$ is a simplicial sphere if and only if $K$ is a simplicial sphere and the $K_i$ are the boundaries of simplices, (2) the class of spherical nerve-complexes is closed under substitution, (3) multigraded betti numbers of $K(K_1, \dots, K_m)$ are expressed in terms of those of the original complexes $K, K_1, \dots, K_m$. We also describe connections between the obtained results and the known results of other authors.

## § 1. Introduction

In the emerging discipline of toric topology, one studies connections between convex polytopes, simplicial complexes, topological spaces, and Stanley-Reisner algebras. For a given simple polytope $P$, one can construct the so-called moment-angle manifold $\mathcal{Z}_P$ with a torus action, whose orbit space is $P$ itself. On the other hand, a simplicial sphere $\partial P^*$ – the boundary of the dual polytope – gives rise to the moment-angle complex $\mathcal{Z}_{\partial P^*}(D^2, S^1)$. This complex is homeomorphic to $\mathcal{Z}_P$ and has a natural cellular structure which allows us to describe its cohomology ring $H^*(\mathcal{Z}_P; \Bbbk) \cong \operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[\partial P^*], \Bbbk)$. This observation allows us to translate topological problems into the language of Stanley-Reisner algebras and vice versa. Moreover, the cohomology ring $H^*(\mathcal{Z}_P; \Bbbk)$ provides information on the combinatorial structure of the polytope $P$. As an example, we mention that the well-known Dehn-Sommerville relations for simple polytopes follow from the bigraded Poincaré duality for $\mathcal{Z}_P$ [8, § 8.6].

With suitable modifications, the above construction carries over to arbitrary, i.e., not necessarily simple, polytopes. For a convex polytope $P$ one can define the moment-angle space $\mathcal{Z}_P$ as the intersection of real quadrics of special form (see [11], [3]). However, for a non-simple polytope $P$, the space $\mathcal{Z}_P$ is not a manifold. Moreover, with each polytope $P$ one can associate a simplicial complex $K_P$, called the nerve-complex. If $P$ is not simple, then $K_P$ is not a simplicial sphere. The complex $K_P$ carries complete information on the combinatorics of $P$ and has properties similar to those of simplicial spheres [3]. For an arbitrary polytope $P$ there is a homotopy equivalence $\mathcal{Z}_P \simeq \mathcal{Z}_{K_P}(D^2, S^1)$. An open

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question is how to describe the properties of the Stanley–Reisner algebra $\Bbbk[K_P]$ and the cohomology ring $H^*(\mathcal{Z}_P;\Bbbk) \cong \operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K_P],\Bbbk)$ for arbitrary convex polytopes.

In the work of A. Bahri, M. Bendersky, F. R Cohen, and S. Gitler [5] one finds a construction that allows, for a given simple polytope $P$ with $m$ facets and a set of natural numbers $(l_1,\dots,l_m)$, the building of a new simple polytope $P(l_1,\dots,l_m)$. The simplicial complex $\partial P(l_1,\dots,l_m)^*$ can be described combinatorially in terms of minimal non-simplices. This approach allows us to describe the moment-angle complex $\mathcal{Z}_{\partial P(l_1,\dots,l_m)^*}(D^2,S^1)$ as the polyhedral product $\mathcal{Z}_{P^*}(\underline{(D^{2l_i},S^{2l_i-1})})$, which yields a more efficient description of the cohomology ring $H^*(\mathcal{Z}_{P(l_1,\dots,l_m)})$, as well as of the cohomology rings of some quasitoric manifolds over such polytopes.

The use of arbitrary (i.e., not necessarily simple) polytopes in toric topology yields a wider class of examples and more general constructions. One such construction is known in convex geometry (see, for example, [1]) and, for a given a polytope $P \subset \mathbb{R}^m_{\geqslant}$ and polytopes $P_1,\dots,P_m$, produces a new polytope $P(P_1,\dots,P_m)$. In general, the combinatorial structure of $P(P_1,\dots,P_m)$ depends on the chosen geometric representation of $P \subset \mathbb{R}^m_{\geqslant}$. However, under some conditions, the construction can be made combinatorial, i.e., we may assume that the face poset of $P(P_1,\dots,P_m)$ depends only on the face posets of the original polytopes $P,P_1,\dots,P_m$. In the particular case when $P_i = \triangle_{l_i}$ is a simplex on $l_i$ vertices, the polytope $P(P_1,\dots,P_m)$ coincides with the polytope $P(l_1,\dots,l_m)$ from [5]. Notice that $P(P_1,\dots,P_m)$ need not be simple, even if the polytopes $P,P_1,\dots,P_m$ are simple.

Thus, the use of arbitrary convex polytopes yields many more interesting examples, as compared with simple polytopes (the simplicity of the polytope is a traditional restriction in toric topology).

In the present paper, we introduce a new operation on the set of abstract simplicial complexes:

$$K,K_1,\dots,K_m \mapsto K(K_1,\dots,K_m).$$

This operation, called substitution, corresponds to the substitution of polytopes, completely describes the properties of the polytopes $P(P_1,\dots,P_m)$, and naturally generalizes some constructions from [5]. The paper is organized as follows:

§ 1. Introduction, conventions, and notation.

§ 2. We construct the simplicial complex $K_P$ and define the abstract spherical nerve-complex of [3].

§ 3. We define a substitution polytope $P(P_1,\dots,P_m)$ and give several equivalent descriptions of it. We give conditions guaranteeing that the substitution operation is well-defined on the class of combinatorial polytopes.

§ 4. For a given simplicial complex $K$ on $m$ vertices and simplicial complexes $K_1,\dots,K_m$, we construct a simplicial complex $K(K_1,\dots,K_m)$, which is the central object of our study. Two definitions are given: one is combinatorial and the other uses the notion of a polyhedral join, which is a natural analog of the polyhedral product. We show that $K(\partial\Delta_{[l_1]},\dots,\partial\Delta_{[l_m]}) = K(l_1,\dots,l_m)$ is the iterated simplicial wedge of [5]. We also prove that $K_{P(P_1,\dots,P_m)} = K_P(K_{P_1},\dots,K_{P_m})$.

§ 5. Polyhedral products defined by substitution simplicial complexes are described. The proposed description generalizes some results of [5].

§ 6. We study the combinatorial and topological structures of substitution simplicial complexes. First, we describe the homotopy type of the complex $K(K_1,\dots,K_m)$. As it turns out, $K(K_1,\dots,K_m) \simeq K*K_1*\dots*K_m$. Then we look at the problem of describing simplicial complexes $K,K_1,\dots,K_m$ such that the substitution complex $K(K_1,\dots,K_m)$ is a sphere. This happens only when $K$ is a sphere and the $K_i$ are boundaries of simplices. Therefore, the class of simplicial spheres is not closed

under substitution. Nevertheless, if $K, K_1, \ldots, K_m$ are spherical nerve-complexes, then so is $K(K_1, \ldots, K_m)$.

§ 7. The multigraded betti numbers of $K(K_1, \ldots, K_m)$ are described. There is a simple formula expressing those numbers in terms of the multigraded betti numbers of the complexes $K, K_1, \ldots, K_m$. Applying that formula to $\partial\Delta_{[2]}(K_1, K_2)$ and $o_2(K_1, K_2)$, where $o_2$ is a complex with two ghost vertices, we reprove a result from [3].

§ 8. Using the known relations between bigraded betti numbers and the $h$-polynomial, we obtain formulas for the $h$-polynomials of substitution complexes in some particular cases.

§ 9. We give an example of computing the multigraded betti numbers for spheres with a small number of vertices.

§ 10. We describe the relations between the substitution operation, the Alexander duality, and the polarization of ideals in commutative algebra.

The following notation and conventions are used throughout this paper. A simplicial complex $K$ on the set of vertices $[m] = \{1, 2, \ldots, m\}$ is a family of subsets $K \subseteq 2^{[m]}$ closed under inclusion, i.e., $J \in K$ whenever $I \in K$ and $J \subset I$. $i \in [m]$ is called a ghost vertex if $\{i\} \notin K$. If $I \in K$ is a simplex, then its link, $\operatorname{link}_K I$, is the simplicial complex on the set $[m] \setminus I$ whose simplices are defined by the condition $J \in \operatorname{link}_K I \Leftrightarrow J \sqcup I \in K$. Notice that the link of $K$ may have ghost vertices even when $K$ has none. From the geometric point of view, a simplicial complex does not change upon removal of the ghost vertices. We use the same symbol $K$ for both the simplicial complex and its geometric realization, hoping that this would not lead to confusion. In particular, a homotopy equivalence between two complexes is understood as a homotopy equivalence between their geometric realizations. The complex $K$ is called a simplicial sphere if it is PL-homeomorphic to the boundary of a simplex (disregarding, if necessary, all ghost vertices). A simplicial sphere $K$ is called a polytopal sphere if $K$ is isomorphic to the boundary of a convex simplicial polytope. The simplicial complex $K$ is called a generalized homological sphere (or Gorenstein* complex) if $K$ and all of its links have the homology of spheres of the corresponding dimensions. If $K$ is a polytopal (respectively, simplicial, generalized homological) sphere, then the same property is shared by the links of all of its simplices.

For an arbitrary subset $A \subseteq [m]$ of vertices, one can define the full subcomplex $K_A$ on $A$, where $J \in K_A \Leftrightarrow J \in K$. Sometimes we will also denote that full subcomplex by $K|_A$, especially when the notation for the complex includes a lower index. The full simplex on the set $A$ of vertices will be denoted by $\Delta_A$. Its dimension equals $|A| - 1$. The boundary $\partial\Delta_A$ is the complex on $A$ consisting of all proper subsets of $A$. The symbol $o_k$ denotes the simplicial complex on $k$ vertices whose only simplex is the empty set. Thus, all vertices of $o_k$ are ghost vertices. Formally, $\partial\Delta_{[1]} = o_1$. A simplex viewed as a convex polytope will be denoted by $\triangle$.

For a simplicial complex $K$, the symbol $M(K)$ denotes the set of maximal by inclusion simplices and $N(K)$ denotes the set of minimal non-simplices, i.e., the sets of vertices $A \subseteq [m]$ such that $A \notin K$, but $B \in K$ for any proper subset $B \subset A$.

The symbol $\bar{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ denotes a row vector, and $\langle \bar{x}, \bar{y} \rangle$ denotes the sum $x_1y_1 + x_2y_2 + \ldots + x_my_m$. We will often build a row from other rows:

$$\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) = (x_{11}, \ldots, x_{1l_1}, \ldots, x_{m1}, \ldots, x_{ml_m}).$$

## § 2. Polytopes and nerve-complexes

Let $P$ be an $n$-dimensional convex polytope and $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ the set of all of its facets (i.e., proper faces of maximal dimension). Consider the simplicial complex $K_P$ on

the set of vertices $[m] = \{1, \dots, m\}$, whose simplices are defined by

$$I = \{i_1, \dots, i_k\} \in K_P \Leftrightarrow \mathcal{F}_{i_1} \cap \dots \cap \mathcal{F}_{i_k} \neq \varnothing.$$

Thus, $K_P$ is the nerve of the closed covering of the polytope boundary $\partial P$ by the facets. The complex $K_P$ is called the nerve-complex of the polytope. For proofs of all the facts on nerve-complexes from this section, see [3].

**Example 2.1.** If $P$ is a simple polytope, then $K_P$ coincides with the boundary of the dual simplicial polytope: $K_P = \partial P^*$. In this case, $K_P$ is a polytopal simplicial sphere. It is not difficult to show that when $P$ is not simple, $K_P$ is not a sphere.

Nerve-complexes allow us to study non-simple polytopes using the well-developed techniques of simplicial complexes. In particular, the moment-angle space $\mathcal{Z}_P$ of a convex polytope $P$ is homotopy equivalent to the moment-angle complex $\mathcal{Z}_{K_P}(D^2, S^1)$, the Buchstaber numbers $s(P)$ and $s(K_P)$ are equal, etc.

There are necessary conditions for a complex to be a nerve-complex. Those conditions are captured in the general notion of a *spherical nerve-complex*.

Let $K$ be a simplicial complex and $M(K)$ the set of its maximal by inclusion simplices. Define a set of simplices $F(K) = \{I \in K \mid I = \bigcap J_i, \text{ where } J_i \in M(K)\}$. It is partially ordered by inclusion.

*Remark* 2.2. For each simplex $I \notin F(K)$, the complex $\operatorname{link}_K I$ is a contractible space.

**Definition 2.3** (Spherical nerve-complex)**.** The simplicial complex $K$ is called a spherical nerve-complex of rank $n$ if the following conditions are satisfied:

- $\varnothing \in F(K)$, i.e., the intersection of all maximal simplices of $K$ is empty.
- $F(K)$ is a graded poset of rank $n$ (this means that all of its saturated, i.e., maximal by inclusion, chains have cardinality $n+1$). In this case, one can define a function rank: $F(K) \to \mathbb{Z}_{\geqslant}$, $\operatorname{rank}(I) =$ the cardinality of a saturated chain from $\varnothing$ to $I$ minus 1.
- For any simplex $I \in F(K)$, the simplicial complex $\operatorname{link}_K I$ is homotopy equivalent to the sphere $S^{n-\operatorname{rank}(I)-1}$. Here, by definition, $\operatorname{link}_K \varnothing = K$ and $S^{-1} = \varnothing$.

*Claim* 2.4. If $P$ is an $n$-dimensional polytope, then $K_P$ is a spherical nerve-complex of rank $n$ and, moreover, the poset $F(K_P)$ is isomorphic to the face poset of the polytope $P$ ordered by reverse inclusion.

As a consequence, the face poset of $P$ can be recovered from the complex $K_P$, i.e., $K_P$ is a complete invariant of the combinatorial polytope $P$.

## § 3. Substitution of polytopes

Let $[m] = \{1, \dots, m\}$ be a finite set and $\triangle_{[m]}$ the standard $(m-1)$-dimensional simplex in $\mathbb{R}^m$ given by

$$\Big\{\bar{x} = (x_1, \dots, x_m) \in \mathbb{R}^m \,\Big|\, x_i \geqslant 0;\ \sum x_i = 1\Big\}.$$

An affine embedding of a convex polytope $P \subset \mathbb{R}^m$ is said to be stochastic if $P \subseteq \triangle_{[m]}$. The following definition is taken from [1, Definition 4.5].

**Definition 3.1.** Let $P \subseteq \mathbb{R}^m$ and $P_i \subseteq \mathbb{R}^{l_i}$, $i \in [m]$, be stochastic embeddings of polytopes. The polytope (or, rather, embedding)

$$P(P_1, \dots, P_m) = \Big\{(t_1\bar{x}_1, t_2\bar{x}_2, \dots, t_m\bar{x}_m) \in \mathbb{R}^{\sum l_i} \,\Big|\, \bar{t} = (t_1, \dots, t_m) \in P,\ \bar{x}_i \in P_i \ \text{ for all } \ i\Big\} \subset \mathbb{R}^{\sum l_i} \tag{3.1}$$

is called the substitution of the polytopes $\{P_i\}$ and $P$.

In [1], this operation was called the action of $P$.

**Example 3.2.** $\triangle_{[m]}(P_1, \dots, P_m) = P_1 * \dots * P_m$ is the join of polytopes.

We see that Definition 3.1 is an extension of the join of polytopes to more general systems of parameters $t_i$.

*Remark* 3.3. Definition 3.1 takes into account the embedding of the polytopes, not just their combinatorial type.

**Definition 3.4.** Let $L \subseteq \mathbb{R}^m$ be an affine $n$-dimensional subspace such that $P = L \cap \mathbb{R}^m_{\geqslant}$ is a non-empty bounded set (hence, a polytope). If $P \subset \mathbb{R}^m$ is a stochastic embedding and each of its facets is uniquely defined as $\mathcal{F}_i = P \cap \{x_i = 0\}$, we shall call $P$ a *natural (stochastic) embedding.*

**Example 3.5.** Consider the embedding $I = \operatorname{conv}((1,0,0),(0,1/2,1/2)) \subset \mathbb{R}^3$. It is clear that this embedding is stochastic, but it is not natural, as the facet $(1,0,0)$ can be described as $I \cap \{x_2 = 0\}$ and as $I \cap \{x_3 = 0\}$, i.e., in two different ways. The embedding $\operatorname{conv}((1,0),(0,1)) \subset \mathbb{R}^2$ of the same polytope is natural stochastic.

*Remark* 3.6. Clearly, a natural embedding of $P$ in $\mathbb{R}^m$ has exactly $m$ facets.

For $\bar{x} \in \mathbb{R}^m_{\geqslant}$ we define a function $\widehat{\sigma}(\bar{x}) = \{i \in [m] \mid x_i = 0\}$ with values in the power set of the finite set $[m]$.

*Remark* 3.7. If $P \subseteq \mathbb{R}^m_{\geqslant}$ is a natural stochastic embedding, then its nerve-complex can be defined as follows: $I \in K_P$ if and only if there exists a point $\bar{x} \in P$ such that $I \subseteq \widehat{\sigma}(\bar{x})$. Indeed, since $I \in K_P$, we have that $\bigcap_{i \in I} \mathcal{F}_i \neq \varnothing$. Let $\bar{x} \in \bigcap_{i \in I} \mathcal{F}_i$. Then $x_i = 0$ for all $i \in I$, and therefore $I \in \widehat{\sigma}(\bar{x})$.

**Observation 3.8.** Each polytope $P$ has a natural stochastic embedding.

**Observation 3.9.** The space $L \subseteq \mathbb{R}^m$ in Definition 3.4 can be given by a system of affine relations

$$L = \left\{ \bar{x} \in \mathbb{R}^m \mid \sum_j c_i^j x_j + d_i = 0 \ \text{ for } \ i = 1, \dots, m-n \right\},$$

where all coefficients $c_i^j$ are positive and $d_i = -1$.

*Proof of both observations.* Let

$$P = \left\{ \bar{y} \in \mathbb{R}^n \mid \langle \bar{a}_i, \bar{y} \rangle + b_i \geqslant 0,\ i \in [m] \right\} \tag{3.2}$$

be a representation of $P$ as an intersection of half-spaces, where $\bar{a}_i$ is an interior normal vector to the $i$th facet (we assume that (3.2) has no redundant inequalities and $|\bar{a}_i| = 1$).

Consider the affine map $j_P \colon \mathbb{R}^n \to \mathbb{R}^m$ given by

$$j_P(\bar{y}) = \big( \langle \bar{a}_1, \bar{y} \rangle + b_1, \dots, \langle \bar{a}_m, \bar{y} \rangle + b_m \big).$$

It is not difficult to show that the rank of this map equals $n$ (normals to facets converging to a common vertex generate $\mathbb{R}^n$), and therefore $j_P$ is an embedding. Clearly, $j_P(P) \subseteq \mathbb{R}^m_{\geqslant}$ and, moreover, $j_P(P) = j_P(\mathbb{R}^n) \cap \mathbb{R}^m_{\geqslant}$. Denote the affine subspace $j_P(\mathbb{R}^n)$ by $L$, and describe $L$ by a system of affine relations:

$$L = \left\{ \bar{x} \in \mathbb{R}^m \mid \langle \bar{c}_i, \bar{x} \rangle + d_i = 0 \ \text{ for } \ i = 1, \dots, m-n \right\}.$$

The sets $j_P(P) \cap \{x_i = 0\}$ are the facets of the polytope $j_P(P)$.

By a theorem of Minkowski, we have a relation $\sum S_i \bar{a}_i = 0$, where the $S_i > 0$ are the $(n-1)$-dimensional volumes of the facets. Hence, one of the affine relations for $L$ is of the form

$$\sum_i S_i x_i = d,$$

where all coefficients $S_i$ are strictly positive. Adding this relation, multiplied by a large enough number, to the other relations, we may assume that, for each relation, the coefficients of the linear part are strictly positive.

Now, dividing each relation by $d > 0$, we have $\sum c_i^j x_j = 1$ with $c_i^j > 0$. The affine change of variables $x'_j = c_1^j x_j$ turns the first relation into $\sum x_j = 1$. Thus, the embedding $j_P \colon P \to \mathbb{R}^m$ is stochastic. Both observations are proved. □

**Proposition 3.10.** *Let $P \in \mathbb{R}^m$ be a natural stochastic embedding given by*

$$P = \mathbb{R}^m_{\geqslant} \cap \left\{\langle \bar{c}_i, \bar{x}\rangle = 1,\ i = 1, \dots, m-n\right\}, \quad \bar{c}_i = (c_i^1, \dots, c_i^m).$$

*Assume also that, for each $i \in [m]$, a natural stochastic embedding $P_i \in \mathbb{R}^{l_i}$ is given by*

$$P_i = \mathbb{R}^{l_i}_{\geqslant} \cap \left\{\langle \bar{c}_{ij_i}, \bar{x}_i\rangle = 1,\ j_i = 1, \dots, l_i - n_i\right\}, \quad \bar{c}_{ij_i} = \left(c_{ij_i}^1, \dots, c_{ij_i}^{l_i}\right).$$

*Then $P(P_1, \dots, P_m) \subset \mathbb{R}^{\sum l_i}$ is a natural stochastic embedding given by the relations*

$$P(P_1, \dots, P_m) = \Big\{(\bar{x}_1, \dots, \bar{x}_m) \in \mathbb{R}^{l_1}_{\geqslant} \times \ldots \times \mathbb{R}^{l_m}_{\geqslant} = \mathbb{R}^{\sum l_i}_{\geqslant} \mid \\ c_i^1 \langle \bar{c}_{1\,j_1}, \bar{x}_1\rangle + c_i^2 \langle \bar{c}_{2\,j_2}, \bar{x}_2\rangle + \ldots + c_i^m \langle \bar{c}_{m\,j_m}, \bar{x}_m\rangle = 1\Big\}. \tag{3.3}$$

*Proof.* By direct substitution, one can see that the polytope $P(P_1, \dots, P_m)$ satisfies the required affine relations. On the other hand, suppose that $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m) \in \mathbb{R}^{\sum l_i}_{\geqslant}$ satisfies (3.3) for all $i, j_1, \dots, j_m$. Denote $\langle \bar{c}_{ij_i}, \bar{x}_i\rangle \in \mathbb{R}$ by $t_i(j_i)$. Then $t_i(j_i) \geqslant 0$ since the coefficients in the affine relations are non-negative, and

$$c_i^1 t_1(j_1) + c_i^2 t_2(j_2) + \ldots + c_i^m t_m(j_m) = 1$$

for all $i$. Hence, $\bar{t}(\bar{j}) = (t_1(j_1), \dots, t_m(j_m)) \in P$.

We want to show that $t_i(j_i)$ does not actually depend on $j_i$. For simplicity, consider the index $j_1$. Let $j_1$ and $j'_1$ be distinct indices. The point $\bar{x}$ satisfies the relations

$$c_i^1 \langle \bar{c}_{1\,j_1}, \bar{x}_1\rangle + c_i^2 \langle \bar{c}_{2\,j_2}, \bar{x}_2\rangle + \ldots + c_i^m \langle \bar{c}_{m\,j_m}, \bar{x}_m\rangle = 1$$

and

$$c_i^1 \langle \bar{c}_{1\,j'_1}, \bar{x}_1\rangle + c_i^2 \langle \bar{c}_{2\,j_2}, \bar{x}_2\rangle + \ldots + c_i^m \langle \bar{c}_{m\,j_m}, \bar{x}_m\rangle = 1.$$

Subtracting one from the other, we have

$$c_i^1 t_1(j_1) = c_i^1 \langle \bar{c}_{1\,j_1}, \bar{x}_1\rangle = c_i^1 \langle \bar{c}_{1\,j_1}, \bar{x}_1\rangle = c_i^1 t_1(j'_1).$$

Since $c_i^1 \neq 0$, we have $t_1(j_1) = t_1(j'_1)$.

By the above, henceforth we can and shall write $t_i$ instead of $t_i(j_i)$, $\bar{t} = (t_1, \dots, t_m) \in P$. As a consequence, $\langle \bar{c}_{ij_i}, \bar{x}_i/t_i\rangle = 1$ for all $i$ and $j_i$. Then

$$\bar{x} = \left(t_1 \frac{\bar{x}_1}{t_1}, t_2 \frac{\bar{x}_2}{t_2}, \dots, t_m \frac{\bar{x}_m}{t_m}\right),$$

where $\bar{t} \in P$ and $\bar{x}_i/t_i \in P_i$. By the definition of polytope substitution, this means that $\bar{x} \in P(P_1, \dots, P_m)$. □

**Example 3.11.** Let $P \subseteq \mathbb{R}^m_{\geqslant}$ be the natural stochastic polytope embedding defined by the relations $\{\langle \bar{c}_i, \bar{x}\rangle = 1\}$, and $\triangle_{[l_i]} \subseteq \mathbb{R}^{l_i}$ be the standard simplex defined by the relation $\{x_1 + \ldots + x_{l_i} = 1\}$. Consider the polytope $P(l_1, \ldots, l_m) = P(\triangle_{[l_1]}, \ldots, \triangle_{[l_m]}) \subseteq \mathbb{R}^{\sum l_i}$. In $\mathbb{R}^{\sum l_i}_{\geqslant}$ it is defined by the relations

$$c_i^1(x_{11} + \ldots + x_{1l_1}) + c_i^2(x_{21} + \ldots + x_{2l_2}) + \ldots + c_i^m(x_{m1} + \ldots + x_{ml_m}) = 1. \tag{3.4}$$

If $P$ is simple, then so is $P(l_1, \ldots, l_m)$ (see §6 or [5]). Such polytopes, their quasitoric manifolds, and moment-angle manifolds were studied in [5].

*Remark* 3.12. In §6, we shall show that, for natural stochastic embeddings $P, P_1, \ldots, P_m$, the combinatorial substitution polytope $P(P_1, \ldots, P_m)$ depends only on the combinatorial types of the original polytopes. Since each polytope admits a natural stochastic embedding, one can consider the operation of substitution as an operation on combinatorial polytopes.

**Proposition 3.13** (Associativity for polytope substitution)**.** *Let $P \subset \mathbb{R}^m_{\geqslant}$ be a stochastic polytope embedding, $P_1, \ldots, P_m$ stochastic polytope embeddings in $\mathbb{R}^{l_1}_{\geqslant}, \ldots, \mathbb{R}^{l_m}_{\geqslant}$, respectively, and*

$$P_{11}, \ldots, P_{1l_1}, P_{21}, \ldots, P_{2l_2}, \ldots, P_{m1}, \ldots, P_{ml_m}$$

*other stochastic polytope embeddings. Then*

$$\begin{aligned} P\big(P_1(P_{11}, \ldots, P_{1l_1}), \ldots, P_m(P_{m1}, \ldots, P_{ml_m})\big) \\ = P(P_1, \ldots, P_m)(P_{11}, \ldots, P_{1l_1}, \ldots, P_{m1}, \ldots, P_{ml_m}). \end{aligned} \tag{3.5}$$

The proof follows from Definition 3.1.

*Remark* 3.14. It is clear that $P(\mathrm{pt}, \ldots, \mathrm{pt}) = \mathrm{pt}(P) = P$, where $\mathrm{pt} = \triangle_{[1]}$ is a point. Thus, on the set of all stochastic polytopes one has an operad structure, whereby a polytope in $\mathbb{R}^m_{\geqslant}$ is considered as an $m$-ary operation, and the substitution of operations is the substitution of polytopes defined above. Proposition 3.13 then expresses the associativity condition for this operad, and the polytope pt becomes the "unit element". By Proposition 3.10, natural stochastic polytopes form a suboperad.

## § 4. Substitution of simplicial complexes

Consider a simplicial complex $K$ on the set $[m]$ and a set of topological pairs $\{(X_i, A_i)\}_{i\in[m]}$, $A_i \subseteq X_i$. For a simplex $I \in K$, consider the subspace $V_I$ of $X_1 \times \ldots \times X_m$ given by $V_I = C_1 \times \ldots \times C_m$, where $C_i = X_i$ if $i \in I$, and $C_i = A_i$ if $i \notin I$. The space

$$\mathcal{Z}_K(\underline{(X_i, A_i)}) = \bigcup_{I\in K} V_I \subseteq \prod_i X_i$$

is called the *polyhedral product* of the pairs $(X_i, A_i)$ defined by the simplicial complex $K$.

**Example 4.1.** The main motivating examples for the definition of a polyhedral product are moment-angle complexes $\mathcal{Z}_K(D^2, S^1)$, real moment-angle complexes $\mathcal{Z}_K(D^1, S^0)$ and the Davis-Januszkiewicz spaces $DJ(K) = \mathcal{Z}_K(CP^\infty, \mathrm{pt})$; see [9].

Another series of examples is given by the wedges

$$\bigvee_\alpha X_\alpha \cong \mathcal{Z}_{\Delta^{(0)}_{[m]}}(\underline{(X_\alpha, \mathrm{pt})}),$$

fat wedges $\mathcal{Z}_{\partial\Delta_{[m]}}(\underline{(X_\alpha, \mathrm{pt})})$, and generalized fat wedges $\mathcal{Z}_{\Delta^{(k)}_{[m]}}(\underline{(X_\alpha, \mathrm{pt})})$. The spaces of the form $\mathcal{Z}_K(\underline{(X_\alpha, \mathrm{pt})})$ were studied in [2]. The spaces of the form $(X, A)^K = \mathcal{Z}_K(\underline{(X, A)})$ were studied in [7] under the name of K-powers. In the most general form, the spaces $\mathcal{Z}_K(\underline{(X_i, A_i)})$ were defined and studied from the point of view of homotopy theory in [4].

It seems quite reasonable to try and replace the product of topological spaces in the definition of polyhedral degree by some other operation defined on topological spaces. This way one can obtain the polyhedral smash-product $\mathcal{Z}_K^{\wedge}(\underline{(X_i, A_i)})$ [4], and the polyhedral join $\mathcal{Z}_K^{*}(\underline{(X_i, A_i)})$ as defined below.

**Definition 4.2.** Let $\{(X_i, A_i)\}_{i\in[m]}$ be given topological pairs and $K$ a simplicial complex on the set $[m]$. For each simplex $I \in K$, consider the subspace $U_I \subseteq X_1 * \ldots * X_m$ of the form $U_I = C_1 * \ldots * C_m$, where $C_i = X_i$ if $i \in I$, and $C_i = A_i$ if $i \notin I$. The space

$$\mathcal{Z}_K^{*}(\underline{(X_i, A_i)}) = \bigcup_{I\in K} U_I \subseteq \divideontimes X_i$$

is called the polyhedral join of the pairs $(X_i, A_i)$.

**Observation 4.3.** If $X_i$ are simplicial complexes and $A_i$ are their simplicial subcomplexes, then the space $\mathcal{Z}_K^{*}(\underline{(X_i, A_i)})$ is a simplicial subcomplex of the simplicial complex $\divideontimes X_i$, i.e., it has a canonical simplicial structure. Thus, the polyhedral join is well-defined, unlike the polyhedral product, on the category of simplicial complexes.

Consider an arbitrary simplicial complex $K$ on the set $[m]$. Then $K$ may be viewed as a subcomplex of the simplex $\Delta_{[m]}$, the full complex on the set $[m]$.

**Definition 4.4.** Let $K$ be a simplicial complex on the set $[m]$ and $K_i$ a simplicial complex on the set $[l_i]$ for all $i \in [m]$. The simplicial complex $K(\underline{K_\alpha}) = K(K_1, \ldots, K_m) = \mathcal{Z}_K^{*}(\underline{(\Delta_{[l_i]}, K_i)})$ is called the *substitution* of the complexes $K_i, i \in [m]$, in $K$.

Let us describe the operation of substitution of simplicial complexes in combinatorial terms. Let $K$ be a simplicial complex on $m$ vertices, and $K_1, \ldots, K_m$ be simplicial complexes on the sets $[l_1], \ldots, [l_m]$. All of these complexes may have ghost vertices. Then $K(\underline{K_i})$ is a simplicial complex on the set $[l_1] \sqcup \ldots \sqcup [l_m]$, defined as follows: the set $I = I_1 \sqcup \ldots \sqcup I_m, I_i \subseteq [l_i]$ is a simplex in the complex $K(K_1, \ldots, K_m)$ if and only if $\{i \in [m] \mid I_i \notin K_i\} \in K$.

The process of constructing the complex $K(\underline{K_\alpha}) = K(K_1, \ldots, K_m)$ is shown in Figure 1. The set of vertices of $K(\underline{K_\alpha})$ is the union of the sets of vertices of the complexes $K_i$, which is schematically shown in the left part of Figure 1. To construct a simplex of $K(\underline{K_\alpha})$, we fix an arbitrary simplex $J \in K$ and take the full complex $\Delta_{[l_i]}$ (or any of its faces) for $i \in J$, and any simplex $I_i \in K_i$ for $i \notin J$. The union of all these sets yields a simplex of $K(\underline{K_\alpha})$ (Figure 1, right). All simplices $I \in K(\underline{K_\alpha})$ can be obtained this way.

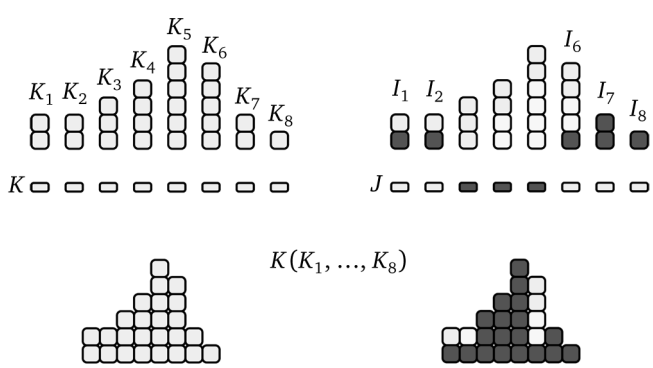


FIGURE 1. The vertices (left) and a simplex (right) of the complex $K(K_1, \ldots, K_m)$

Recall that $o_l$ is a simplicial complex on $l > 0$ ghost vertices. By Remark 3.7, we can set $K_{\mathrm{pt}} = o_1$ since the polytope $\mathrm{pt} = \triangle_{[1]}$ is defined as the set $\mathbb{R}_{\geqslant} \cap \{x \in \mathbb{R} \mid x = 1\}$ and is disjoint from the hyperplane $\{x = 0\}$.

**Example 4.5.** By definition, $K(o_1, \ldots, o_1) = o_1(K) = K$.

**Example 4.6.** Let us consider the structure of $K(o_l, o_1, \ldots, o_1)$ in more detail. Let $v_1$ be the first vertex of $K$. Replace $v_1$ by the simplex $I_1 = \{v_1^1, \ldots, v_1^l\}$. Then each simplex $I \in K$ containing $v_1$ will be replaced by the simplex $(I \setminus \{v_1\}) \sqcup I_1$. Thus,

$$K(o_l, o, \ldots, o) = K_{[m]\setminus v_1} \cup (\mathrm{link}_K\, v_1) * I_1$$

is the substitution in the first vertex.

**Example 4.7.** $o_m(K_1, \ldots, K_m) = K_1 * \ldots * K_m$.

The next result relates the substitution of natural stochastic embeddings of polytopes, as introduced in the previous section, and the substitution of simplicial complexes.

**Proposition 4.8.** *Let $P \subseteq \mathbb{R}^m_{\geqslant}$, $P_1 \subseteq \mathbb{R}^{l_1}_{\geqslant}$, $\ldots$, $P_m \subseteq \mathbb{R}^{l_m}_{\geqslant}$ be natural stochastic embeddings. Then $K_{P(P_1,\ldots,P_m)} = K_P(K_{P_1}, \ldots, K_{P_m})$.*

*Proof.* We will need a technical lemma. Recall that, for $\bar{x} \in \mathbb{R}^m_{\geqslant}$, $\widehat{\sigma}(\bar{x}) = \{i \in [m] \mid x_i = 0\}$ (see §3).

**Lemma 4.9.** *Let $Q = \mathbb{R}^m_{\geqslant} \cap \left\{\sum c_i^j x_j = 1,\ i = 1, \ldots, m-n\right\}$ be a polytope embedding and $c_i^j > 0$. Fix a number $y \in \mathbb{R}$. If $\bar{x} \in \mathbb{R}^m_{\geqslant}$ is a solution of the system $\sum c_i^j x_j = y$, then $y \geqslant 0$, and either $\widehat{\sigma}(\bar{x}) \in K_Q$ when $y > 0$ or $\bar{x} = \bar{0}$ when $y = 0$.*

*Proof.* If $y = 0$, the assertion is obvious since $c_i^j > 0$ and $\bar{x}$ has non-negative coordinates. If $y > 0$, consider the point $\bar{x}/y$. By definition, $\bar{x}/y \in Q$ and $\widehat{\sigma}(\bar{x}/y) = \widehat{\sigma}(\bar{x})$. Hence, $\widehat{\sigma}(\bar{x}) \in K_Q$. □

Now consider a row vector $\bar{x} = (x_1, \ldots, x_l) \in \mathbb{R}^l$. If $\bar{x} \in P$, then $\widehat{\sigma}(\bar{x}) \in K_P$. Conversely, if $I \in K_P$, then there exists $\bar{x} \in P$ such that $I \subseteq \widehat{\sigma}(\bar{x})$ (Remark 3.7).

Notice that $K_{P(P_1,\ldots,P_m)}$ and $K_P(K_{P_1}, \ldots, K_{P_m})$ share the same set of vertices: $[l_1] \sqcup \ldots \sqcup [l_m]$. Denote $l_1 + \ldots + l_m$ by $\Sigma$. Let

$$\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) = (x_{11}, \ldots, x_{1l_1}, x_{21}, \ldots, x_{2l_2}, \ldots, x_{m1}, \ldots, x_{ml_m}) \in \mathbb{R}^{\Sigma}_{\geqslant}$$

be a point of $P(P_1, \ldots, P_m)$. Then, for $\bar{x} \in P(P_1, \ldots, P_m)$,

$$\left\langle \bar{c}_i, \left(\langle \bar{c}_{1\,j_1}, \bar{x}_1\rangle, \langle \bar{c}_{2\,j_2}, \bar{x}_2\rangle, \ldots, \langle \bar{c}_{m\,j_m}, \bar{x}_m\rangle\right)\right\rangle = 1.$$

Denote $\langle \bar{c}_{sj_s}, \bar{x}_s\rangle$ by $t_s$ (this number does not depend on $j_1, \ldots, j_m$ – a similar fact was used in the proof of Proposition 3.10) and set $\bar{t} = (t_1, \ldots, t_m)$. Then, by definition, $\bar{t} \in P$. By Observation 3.9, we may assume that $t_s \geqslant 0$. Hence, $\widehat{\sigma}(\bar{t}) \in K_P$. For each $s \in [m]$, we have the following alternative:

- If $s \in \widehat{\sigma}(\bar{t})$, then $t_s = 0$ and $\langle \bar{c}_{sj_s}, \bar{x}_s\rangle = 0$. Hence, $\bar{x}_s = \bar{0}$, by Lemma 4.9.
- If $s \notin \widehat{\sigma}(\bar{t})$, then $t_s \neq 0$ and $\langle \bar{c}_{sj_s}, \bar{x}_s\rangle = t_s > 0$. Then, by Lemma 4.9, $\widehat{\sigma}(\bar{x}_s) \in K_{P_s}$.

Therefore, $\widehat{\sigma}(\bar{x}) \in K_P(K_{P_1}, \ldots, K_{P_m})$. This argument shows, that if $I \in K_{P(P_1,\ldots,P_m)}$, then $I \in K_P(K_{P_1}, \ldots, K_{P_m})$. Let us prove the reverse inclusion.

Let $J \in K_P(K_{P_1}, \ldots, K_{P_m})$ and $J = A_1 \sqcup \ldots \sqcup A_m$, where $A_s \subseteq [l_s]$. We need to show that there is a point $\bar{x} \in P(P_1, \ldots, P_m)$ such that $J \in \widehat{\sigma}(\bar{x})$.

By the definition of a substitution complex, there is a simplex $I \in K_P$ such that $A_s \in K_{P_s}$ for all $s \notin I$. There is a point $\bar{t} = (t_1, \ldots, t_m) \in P$ with $I \subseteq \widehat{\sigma}(\bar{t})$. Moreover, for each $s$ there exists a solution of the system $\{\langle \bar{c}_{s,\,j_s}, \bar{x}_s\rangle = t_s\}_{j_s=1,\ldots,l_s}$ such that

$A_s \subseteq \widehat{\sigma}(\bar{x}_s)$ if $t_s \neq 0$, and $\bar{x}_s = \bar{0}$ if $t_s = 0$ (or, equivalently, $\widehat{\sigma}(\bar{x}_s) = [l_s] \supseteq A_s$). Then the row vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m)$ is a non-negative solution of the system

$$\big\langle \bar{c}_i, \big(\langle \bar{c}_{1\,j_1}, \bar{x}_1\rangle, \langle \bar{c}_{2\,j_2}, \bar{x}_2\rangle, \dots, \langle \bar{c}_{m\,j_m}, \bar{x}_m\rangle\big)\big\rangle = 1,$$

with $\widehat{\sigma}(\bar{x}) = \widehat{\sigma}(\bar{x}_1) \sqcup \dots \sqcup \widehat{\sigma}(\bar{x}_m) \supseteq J$. This concludes the proof. □

**Corollary 4.10.** *If the polytopes $P, P_1, \dots, P_m$ are combinatorially equivalent to the polytopes $Q, Q_1, \dots, Q_m$, respectively, and all the polytopes are naturally stochastically embedded, then the polytope $P(P_1, \dots, P_m)$ is combinatorially equivalent to the polytope $Q(Q_1, \dots, Q_m)$. Hence, the substitution operation is well-defined on combinatorial polytopes.*

*Proof.* We have

$$K_{P(P_1,\dots,P_m)} = K_P(K_{P_1}, \dots, K_{P_m}) = K_Q(K_{Q_1}, \dots, K_{Q_m}) = K_{Q(Q_1,\dots,Q_m)}.$$

It now follows, by Claim 2.4, that $P(P_1, \dots, P_m)$ and $Q(Q_1, \dots, Q_m)$ are combinatorially equivalent. □

**Example 4.11.** A non-trivial example of substitution for complexes is given by the *iterated simplicial wedge*, as defined in [5] (this cumbersome term originated in [20], where a simplicial wedge was defined). Consider a simplicial complex $K$ on $m$ vertices and a set of natural numbers $(l_1, \dots, l_m)$. The iterated simplicial wedge is the complex $K(l_1, \dots, l_m) = K(\partial\Delta_{[l_1]}, \dots, \partial\Delta_{[l_m]})$.

If $P \subset \mathbb{R}^m_{\geqslant}$ is a natural stochastic polytope embedding, then, by Proposition 4.8,

$$\begin{aligned} K_P(l_1, \dots, l_m) &= K_P\big(\partial\Delta_{[l_1]}, \dots, \partial\Delta_{[l_m]}\big) \\ &= K_P\big(K_{\triangle_{[l_1]}}, \dots, K_{\triangle_{[l_m]}}\big) = K_{P(\triangle_{[l_1]},\dots,\triangle_{[l_m]})} = K_{P(l_1,\dots,l_m)}. \end{aligned}$$

In §6, we shall show that, for any set $(l_1, \dots, l_m)$, the simplicial complex $K$ is a simplicial sphere if and only if $K(l_1, \dots, l_m)$ is a simplicial sphere. Then $P$ is simple if and only if $P(l_1, \dots, l_m)$ is simple (see Example 2.1).

**Proposition 4.12** (Associativity for the substitution of simplicial complexes)**.** *Let $K$ be a simplicial complex on $m$ vertices, $K_1, \dots, \dots, K_m$ simplicial complexes on $l_1, \dots, l_m$ vertices, respectively, and $K_{11}, \dots, K_{1l_1}, K_{21}, \dots, K_{2l_2}, \dots, K_{m1}, \dots, K_{ml_m}$ simplicial complexes on the sets $[r_{sj_s}]$. Then*

$$\begin{aligned} K\big(K_1(K_{11}, \dots, K_{1l_1}), \dots, K_m(K_{m1}, \dots, K_{ml_m})\big) \\ = K(K_1, \dots, K_m)(K_{11}, \dots, K_{1l_1}, \dots, K_{m1}, \dots, K_{ml_m}) \end{aligned} \tag{4.1}$$

*when both sides are viewed as simplicial complexes on $\bigsqcup_{s,\,j_s}[r_{sj_s}]$.*

*Proof.* It is clear that both complexes have the same vertex set:

$$V = \big([r_{11}] \sqcup \dots \sqcup [r_{1l_1}]\big) \sqcup \dots \sqcup \big([r_{m1}] \sqcup \dots \sqcup [r_{ml_m}]\big).$$

Let $A \subseteq V$, i.e., $A = (A_{11} \sqcup \dots \sqcup A_{1l_1}) \sqcup \dots \sqcup (A_{m1} \sqcup \dots \sqcup A_{ml_m})$, where $A_{sj_s} \subseteq [r_{sj_s}]$. Now we have a chain of equivalent conditions:

$$\begin{aligned} A \in K\big(K_1(K_{11}, \dots, K_{1l_1}), \dots, K_m(K_{m1}, \dots, K_{ml_m})\big) \\ \Leftrightarrow \exists I \in K \ \ \forall s \notin I\colon (A_{s1} \sqcup \dots \sqcup A_{sl_s}) \in K_s(K_{s1}, \dots, K_{sl_s}) \\ \Leftrightarrow \exists I \in K \ \ \forall s \notin I \ \ \exists I_s \in K_s \ \ \forall i_s \notin I_s\colon A_{si_s} \in K_{si_s} \\ \Leftrightarrow \exists J \in K(K_1, \dots, K_m) \ \ \forall s \ \ \forall i_s \in [l_s] \setminus J\colon A_{si_s} \in K_{si_s} \\ \Leftrightarrow A \in K(K_1, \dots, K_m)(K_{11}, \dots, K_{1l_1}, \dots, K_{m1}, \dots, K_{ml_m}), \end{aligned} \tag{4.2}$$

finishing the proof. □

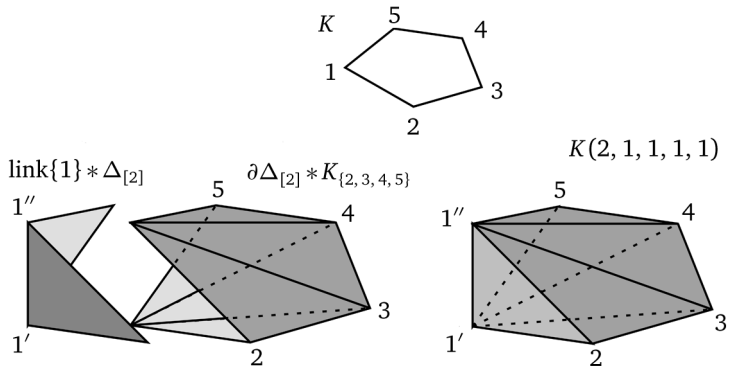


FIGURE 2. Doubling of a vertex in the boundary of a pentagon

*Remark* 4.13. As is in the case for polytopes, simplicial complexes form an operad. A simplicial complex $K$ on $m$ vertices can be viewed as an $m$-ary operation. The unit operation is the complex $o_1$, since $K(o_1, \dots, o_1) = o_1(K) = K$.

**Corollary 4.14.** *A substitution can be performed in steps. Namely, let $K_i$ be a complex on the set $[l_i]$. Then*

$$K(K_1, \dots, K_m) = K(o_1, \dots, K_i, \dots, o_1)\big(K_1, \dots, K_{i-1}, \underbrace{o_1, \dots, o_1}_{l_i}, K_{i+1}, \dots, K_m\big).$$

**Corollary 4.15** ([5, Section 2])**.** *Let $l_i$ be natural numbers. Then*

$$K(l_1, \dots, l_m) = K(1, \dots, l_i, \dots, 1)\big(l_1, \dots, l_{i-1}, \underbrace{1, \dots, 1}_{l_i}, l_{i+1}, \dots, l_m\big).$$

Thus the simplicial wedge can indeed be iterated, which explains its name. The simplicial complex $K(l_1, 1, 1, \dots, 1)$ can be described geometrically [5], [20]:

$$K(l, 1, 1, \dots, 1) = K_{[m]\setminus\{1\}} * \partial\Delta_{[l]} \cup (\mathrm{link}_K\{1\}) * \Delta_{[l]}.$$

Figure 2 illustrates this operation when $K$ is the boundary of a pentagon and $l = 2$.

One can check that $K(l, 1, 1, \dots, 1) \cong_{PL} \Sigma^{l-1}K \cong_{PL} K * \partial\Delta_{[l]}$. In the case when $K$ is the boundary of a simplicial polytope, the complex $K(l, 1, 1, \dots, 1)$ is also the boundary of a simplicial polytope [5, Th. 2.3]. Indeed, if $K = \partial Q$, then $K = K_{Q^*}$ for the dual $Q^*$ of a simple polytope, and $K(l_1, \dots, l_m) = K_{Q^*(l_1, \dots, l_m)} = \partial(Q^*(l_1, \dots, l_m))^*$.

Using Corollary 4.15 and an inductive argument, one can prove the following fact.

**Corollary 4.16.** *If $K$ is a simplicial sphere (the boundary of a simplicial polytope, a triangulated topological sphere, or a homological sphere), then so is the simplicial complex $K(l_1, \dots, l_m)$ for any set $l_1, \dots, l_m$ of natural numbers.*

It is natural to ask if the converse is true: for which complexes $K, K_1, \dots, K_m$ is the result of the substitution $K(K_1, \dots, K_m)$ a sphere (in any of the above cases)? We shall deal with this question in §6.

## § 5. POLYHEDRAL PRODUCTS DETERMINED BY SUBSTITUTIONS OF SIMPLICIAL COMPLEXES

**Proposition 5.1.** *Let, as before, $K$ be a simplicial complex on $m$ vertices, and $\{K_i\}_{i\in[m]}$ simplicial complexes on $l_i$ vertices. Consider a set of topological pairs indexed by the*

*elements of the set* $\bigsqcup_i [l_i]$*:*

$$(X_{11}, A_{11}), \ldots, (X_{1l_1}, A_{1l_1}), \ldots, (X_{m1}, A_{m1}), \ldots, (X_{ml_m}, A_{ml_m}).$$

*For each* $i \in [m]$ *consider the spaces*

$$Y_i = X_{i1} \times X_{i2} \times \ldots \times X_{il_i}$$

*and*

$$Z_i = \mathcal{Z}_{K_i}\big(\underline{(X_{ij}, A_{ij})}_{j\in[l_i]}\big) \subseteq Y_i.$$

*Then the spaces*

$$\mathcal{Z}_{K(\underline{K_\alpha})}\Big(\underline{(X_{ij}, A_{ij})}_{i\in[m],\ j\in[l_i]}\Big) \quad \textit{and} \quad \mathcal{Z}_K\Big(\underline{(Y_i, Z_i)}_{i\in[m]}\Big)$$

*coincide as subsets of* $\prod_{i,\,j} X_{ij} = \prod_i Y_i$*.*

The proof is straightforward (similar to Proposition 4.12).

**Example 5.2.** When $K_i = \partial\Delta_{[l_i]}$, Proposition 5.1 reduces to [5, Th. 7.2].

**Example 5.3.** Let, as before, $K, K_1, \ldots, K_m$ be simplicial complexes. Consider a set of pairs $\{(X_{ij}, A_{ij})\}_{i\in[m],\, j\in[l_i]}$, where $(X_{ij}, A_{ij}) = \{(D^2, S^1)\}$ for all $i \in [m]$ and $j \in [l_i]$. Then

$$\mathcal{Z}_{K(\underline{K_\alpha})}(D^2, S^1) = \mathcal{Z}_K\Big(\underline{(D^{2l_i}, \mathcal{Z}_{K_i})}_{i\in[m]}\Big).$$

**Example 5.4.** As a particular case of the previous example, let $K_i = \partial\Delta_{[l_i]}$. Then

$$\mathcal{Z}_{K(l_1,\ldots,l_m)}(D^2, S^1) = \mathcal{Z}_K\Big(\underline{(D^{2l_i}, S^{2l_i-1})}_{i\in[m]}\Big),$$

since $\mathcal{Z}_{\partial\Delta_{[l_i]}}(D^2, S^1) = S^{2l_i-1}$, which is [5, Cor. 7.3]. A similar result holds for real moment-angle complexes

$$\mathcal{Z}_{K(l_1,\ldots,l_m)}(D^1, S^0) = \mathcal{Z}_K\Big(\underline{(D^{l_i}, S^{l_i-1})}_{i\in[m]}\Big).$$

In particular,

$$\mathcal{Z}_{K(2,\ldots,2)}(D^1, S^0) = \mathcal{Z}_{K(2,\ldots,2)}(D^1, S^0) = \mathcal{Z}_K(D^2, S^1).$$

This observation was used by Yu. Ustinovsky in [23] in his proof of the toral rank conjecture for moment-angle manifolds.

## § 6. Combinatorial and topological properties of substitution complexes

It was shown in [4] that $\mathcal{Z}_K^\wedge(\underline{(X_i, A_i)}) \simeq \Sigma(K \wedge A_1 \wedge \ldots \wedge A_m)$ if, for all $i \in [m]$, the space $X_i$ is contractible. The next result is proved by a similar argument.

**Proposition 6.1.** *Let* $(X_i, A_i)$ *be topological pairs such that the space* $X_i$ *is contractible for all* $i \in [m], m \neq 0$*. Then, for any simplicial complex* $K$ *on the set* $[m]$*, the space* $\mathcal{Z}_K^*(\underline{(X_i, A_i)})$ *is homotopy equivalent to* $A_1 * \ldots * A_m * K$*.*

*Proof.* Consider a small category $\mathrm{CAT}(K)$ associated with the simplicial complex $K$. The objects of $\mathrm{CAT}(K)$ are simplices of $K$, and the morphisms are inclusions of simplices (thus, there is at most one morphism between any two objects).

Now let us define a diagram $\Psi\colon \mathrm{CAT}(K) \to \mathrm{TOP}$. For each simplex $I \in \operatorname{Ob}\mathrm{CAT}(K)$, set $\Psi(I) = U_I = B_1 * B_2 * \ldots * B_m$, where $B_i = X_i$ if $i \in I$, and $B_i = A_i$ otherwise (see Definition 4.2). The morphism $\Psi(I \hookrightarrow I')$ is given by the natural inclusion $U_I \hookrightarrow U_{I'}$.

Then $\operatorname{colim}\Psi \cong \mathcal{Z}_K^*(\underline{(X_i, A_i)})$. All maps in $\Psi$ are closed cofibrations. Therefore, by the projection lemma (see, for example, [24, Prop. 3.1]), $\operatorname{colim}\Psi \simeq \operatorname{hocolim}\Psi$.

Consider the diagram $\Phi\colon \mathrm{CAT}(K) \to \mathrm{TOP}$: $\Phi(\varnothing) = A_1 * \ldots * A_m$, and $\Phi(I) = \mathrm{pt}$ if $I \neq \varnothing$. The values of $\Phi$ on morphisms are defined in the natural way.

1) For each simplex $I \in \operatorname{Ob}\mathrm{CAT}(K)$, we have a homotopy equivalence $h_I\colon \Psi(I) \to \Phi(I)$. Indeed, for $I = \varnothing$ we have $\Psi(I) = A_1 * \ldots * A_m = \Phi(I)$, so for $h_\varnothing$ we can take the identity map. When $I \neq \varnothing$ we have $\Psi(I) = B_1 * \ldots * B_m$, where at least one of the $B_i$ coincides with $X_i$ and is therefore contractible. Hence, the entire join $B_1 * \ldots * B_m$ is contractible. Therefore, the natural map $h_I\colon \Psi(I) \to \Phi(I) = \mathrm{pt}$ is a homotopy equivalence.

2) The map $h_I\colon \Psi(I) \to \Phi(I)$ forms a morphism of diagrams, hence $\operatorname{hocolim}\Psi \simeq \operatorname{hocolim}\Phi$.

3) $\operatorname{hocolim}\Phi \simeq \Phi(\varnothing) * K \cong K * A_1 * \ldots * A_m$ (see [24, Lemma 3.4]). This fact follows easily from the explicit definition of a homotopy colimit.

4) The sequence of homotopy equivalences

$$\mathcal{Z}^*_K(\underline{(X_i, A_i)}) \cong \operatorname{colim}\Psi \simeq \operatorname{hocolim}\Psi \simeq \operatorname{hocolim}\Phi = K * A_1 * \ldots * A_m$$

now concludes the proof. □

**Corollary 6.2.** *For all simplicial complexes $K, K_1, \ldots, K_m$ with non-empty vertex sets, there is a homotopy equivalence $K(\underline{K_\alpha}) \simeq K * K_1 * K_2 * \ldots * K_m$.*

**Corollary 6.3.** *If $K \simeq S^{n-1}$, $K_i \simeq S^{n_i-1}$, then $K(\underline{K_\alpha}) \simeq S^{n+n_1+\ldots+n_m-1}$.*

**Corollary 6.4.** *Let $P, P_1, \ldots, P_m$ be polytopes of dimensions $n, n_1, \ldots, n_m$ respectively. Then $\dim P(\underline{P_\alpha}) = n + n_1 + \ldots + n_m$.*

*Proof.* If $\dim Q = n$, then $K_Q \simeq S^{n-1}$ for any convex polytope $Q$ (see §2). Hence,

$$S^{\dim P(\underline{P_\alpha})-1} \simeq K_{P(\underline{P_\alpha})} = K_P(K_{P_1}, \ldots, K_{P_m}) \simeq K_P * K_{P_1} * \ldots * K_{P_m} \simeq S^{n+n_1+\ldots+n_m-1}.$$

Therefore, $\dim P(\underline{P_\alpha}) = n + n_1 + \ldots + n_m$. □

**Example 6.5.** Consider the substitution of "ghost complexes" $K(o_{l_1}, \ldots, o_{l_m})$ in $K$. By Corollary 6.2, $K(o_{l_1}, \ldots, o_{l_m}) \simeq K$. This homotopy equivalence can be observed directly from Example 4.6, where an explicit description of this complex was given.

**Theorem 6.6.** *Suppose the complex $K(\underline{K_\alpha})$ is a simplicial sphere (respectively, the boundary of a simplicial polytope or a generalized homological sphere) and that the complex $K$ has no ghost vertices. Then $K$ is a simplicial sphere (respectively, the boundary of a simplicial polytope or a generalized homological sphere), and for each $i \in [m]$ there is $l_i > 0$ such that $K_i = \partial\Delta_{[l_i]}$.*

*Proof.* For the proof we will need a technical lemma describing links of simplices in the substitution complex $K(\underline{K_\alpha})$.

**Lemma 6.7.** *Let $K, K_1, \ldots, K_m$ be simplicial complexes on the sets $[m], [l_1], \ldots, [l_m]$. Let $A \in K(\underline{K_\alpha})$, $A = A_1 \sqcup \ldots \sqcup A_m$, $A_i \subseteq [l_i]$, and $J = \{i \in [m] \mid A_i \notin K_i\} \in K$. Consider the set of indices $[m] \setminus J = \{i_1, \ldots, i_k\}$. For each index $i \in J$ consider set $M_i = [l_i] \setminus A_i$ and the simplex $\Delta_{M_i}$ spanned by that set. Then*

$$\operatorname{link}_{K(\underline{K_\alpha})} A = \operatorname{link}_K J\big(\operatorname{link}_{K_{i_1}} A_{i_1}, \ldots, \operatorname{link}_{K_{i_k}} A_{i_k}\big) * (\divideontimes_{i\in J}\Delta_{M_i}).$$

*Proof.* It is clear that both complexes are defined on the same set $\bigsqcup_{i=1}^m([l_i]\setminus A_i)$. Consider $I = I_1 \sqcup \ldots \sqcup I_m \in \operatorname{link}_{K(\underline{K_\alpha})} A$. By definition, $I \sqcup A \in K(\underline{K_\alpha})$. Equivalently,

$$B' = \big\{i \in [m] \mid I_i \sqcup A_i \notin K_i\big\} \in K.$$

Equivalently,

$$B = \big\{i \in [m] \setminus J \mid I_i \sqcup A_i \notin K_i\big\} \sqcup J \in K,$$

since the condition $A_i \notin K_i$ implies that $A_i \sqcup I_i \notin K_i$ and, therefore, $J \subseteq B'$. Equivalently,

$$B = \{i \in [m] \setminus J \mid I_i \sqcup A_i \notin K_i\} \in \operatorname{link}_K J.$$

Equivalently,

$$B = \{i \in [m] \setminus J \mid I_i \notin \operatorname{link}_{K_i} A_i\} \in \operatorname{link}_K J. \tag{6.1}$$

Thus,

$$I = \bigsqcup_{i \notin J} I_i \sqcup \bigsqcup_{i \in J} I_i,$$

where the set $\bigsqcup_{i \notin J} I_i$ satisfies condition (6.1) and, therefore,

$$\bigsqcup_{i \notin J} I_i \in \operatorname{link}_K J\big(\operatorname{link}_{K_{i_1}} A_{i_1}, \dots, \operatorname{link}_{K_{i_k}} A_{i_k}\big).$$

Since there are no restrictions on $\bigsqcup_{i \in J} I_i$, we have the desired formula. □

Now let $K(\underline{K_\alpha})$ be a simplicial (respectively, homological) sphere. Then, for all $A \in K(\underline{K_\alpha})$, the complex $\operatorname{link}_{K(\underline{K_\alpha})} A$ is also a simplicial (respectively, homological) sphere. First of all, notice that $K_i \neq \Delta_{[l_i]}$. Otherwise, we would have that

$$K(\underline{K_\alpha}) \simeq K * K_1 * \ldots * K_m$$

is a contractible complex, contrary to the assumption of the theorem.

Henceforth, we shall use the same notation as in the proof of Lemma 6.7. Suppose that there exists an index $j \in [m]$ such that there is a non-simplex $A_j \notin K_j$ in $K_j$ with $A_j \neq [l_j]$. Consider the subset

$$A = \varnothing \sqcup \ldots \sqcup A_j \sqcup \ldots \sqcup \varnothing \subseteq [l_1] \sqcup \ldots \sqcup [l_m].$$

Since $J = \{i \in [m] \mid A_i \notin K_i\} = \{j\} \in K$, we have $A \in K(\underline{K_\alpha})$. By Lemma 6.7, $\operatorname{link}_{K(\underline{K_\alpha})} A = X * \Delta_{M_j}$, where $X$ is a complex and $\Delta_{M_j}$ is a simplex on the set $[l_j] \setminus A_j \neq \varnothing$. Hence, $\operatorname{link}_{K(\underline{K_\alpha})} A$ is contractible. Since the link of a simplex of a sphere is again a sphere, we have a contradiction.

Therefore, for each index $i$, the only non-simplices of $K_i$ are the $[l_i]$. Thus, we have shown that $K_i = \partial\Delta_{[l_i]}$.

For each $i \in [m]$, consider an arbitrary maximal simplex (facet) $I_i^{\max} \in K_i = \partial\Delta_{[l_i]}$. As was shown before, $l_i - |I_i^{\max}| = 1$ and $\operatorname{link}_{K_i} I_i^{\max} = o_1$ is a complex on a single ghost vertex.

Consider the simplex $A = I_1^{\max} \sqcup \ldots \sqcup I_m^{\max} \in K(\underline{K_\alpha})$. Since

$$J = \{i \in [m] \mid I_i \notin K_i\} = \varnothing \in K,$$

we have $\operatorname{link}_K J = \operatorname{link}_K \varnothing = K$. Applying Lemma 6.7 to the simplex $A$, we have

$$\operatorname{link}_{K(\underline{K_\alpha})} A = \operatorname{link}_K J(\operatorname{link}_{K_1} I_1^{\max}, \dots, \operatorname{link}_{K_m} I_m^{\max}) = K(o, \dots, o) = K.$$

Since $K(\underline{K_\alpha})$ is a simplicial sphere (respectively, a homological sphere, the boundary of a simplicial polytope), so is the complex $\operatorname{link}_{K(\underline{K_\alpha})} A = K$. □

*Remark* 6.8. The restriction that the complex $K$ have no ghost vertices in Theorem 6.6 is not essential. The assertion can be stated and proved in a more general form. Let $K(\underline{K_\alpha})$ be a simplicial (homological) sphere. Then:

1) $K$ is a simplicial (homological) sphere;
2) if $i$ is not a ghost vertex of $K$, then $K_i = \partial\Delta_{[l_i]}$;
3) if $i$ is a ghost vertex of $K$, then $K_i$ is a simplicial (homological) sphere.

**Proposition 6.9.** *Suppose that $K(\underline{K_\alpha}) = K_Q$ for some simple polytope $Q$. Then there are a simple polytope $P$ and numbers $l_i > 0$ such that $K = K_P$ and $K_i = \partial\Delta_{[l_i]}$. Moreover, the polytopes $Q$ and $P(l_1, \dots, l_m)$ are combinatorially equivalent.*

*Proof.* If the polytope $Q$ is simple, then $K_Q = \partial Q^*$. Hence, if $K(\underline{K_\alpha}) = K_Q$, then $K(\underline{K_\alpha})$ is the boundary of a simplicial polytope. By Theorem 6.6, $K_i = \partial\Delta_{[l_1]}$ and $K$ is the boundary of a simplicial polytope. Then $K = K_P$ for some simple polytope $P$ and, by Proposition 4.8,

$$K_Q = K(\underline{K_\alpha}) = K_P(\partial\Delta_{[l_1]}, \dots, \partial\Delta_{[l_m]}) = K_{P(l_1,\dots,l_m)}.$$

This means that $Q$ and $P(l_1, \dots, l_m)$ are combinatorially equivalent (see § 2). □

*Remark* 6.10. The following fact follows easily from Lemma 6.7. Suppose a set of natural numbers $(l_1, \dots, l_m)$ is bounded above by another set of natural numbers $(s_1, \dots, s_m)$, i.e., $l_i \leqslant s_i$ for all $i \in [m]$. Then the complex $K(l_1, \dots, l_m)$ is isomorphic to the link of a simplex of the complex $K(s_1, \dots, s_m)$. The fact that $K = K(1, \dots, 1)$ is the link of $K(l_1, \dots, l_m)$ was essential in the proof of Theorem 6.6.

This observation can be used for constructing simplicial spheres with a small number of vertices and without a convex realization. Let $V(S)$ denote the number of vertices of a simplicial sphere $S$.

*Claim* 6.11 ([16, Section 5]). For any $d \geqslant 3$ there exists a simplicial sphere $S$ of dimension $d$ which is not isomorphic to the boundary of a convex polytope and such that $V(S) - d = 5$.

*Proof.* In principle, the proof is the same as in [16]. In the case $\dim S = 3$ we can take one of the known examples, say the Barnette sphere $X$ on 8 vertices [6]. For non-polytopal spheres in large dimensions, consider $X(l_1, \dots, l_8)$ for an arbitrary set $l_1, \dots, l_8$. First, as can easily be seen, for an any complex $K$,

$$V(K(l_1, \dots, l_m)) - \dim K(l_1, \dots, l_m) = V(K) - \dim K.$$

In particular, the number of vertices of the sphere $X(l_1, \dots, l_8)$ is 5 more than its dimension. Second, the sphere $X(l_1, \dots, l_8)$ is non-polytopal, since otherwise all of its links, including $X$, would be polyhedral, and that is not true. □

*Remark* 6.12. Without the condition $V(S) - \dim S = 5$, the solution would have been even simpler: just take the suspension of the Barnette sphere. However, the suspension operation increases the number of vertices of the sphere by 2, whereas its dimension goes up by 1. Hence the suspension would not work, and we have to use a more economical construction.

Proposition 4.8 indicates that the class of spherical nerve-complexes is closed under substitution, in contrast with the more narrow class of simplicial spheres.

**Theorem 6.13.** *Let $K$ be a spherical nerve-complex of rank $n$ on $m$ vertices and $K_1, \dots, K_m$ be spherical nerve-complexes of ranks $n_1, \dots, n_m$ on the sets of vertices $[l_1]$, $\dots, [l_m]$, respectively, Then $K(\underline{K_\alpha})$ is a spherical nerve-complex of rank $n+n_1+\ldots+n_m$.*

*Proof.* We shall use the notation introduced in §2. Let us describe the set of maximal simplices $M(K(\underline{K_\alpha}))$ and the set of all of their intersections $F(K(\underline{K_\alpha}))$. We have that $I_1 \sqcup \ldots \sqcup I_m \in M(K(\underline{K_\alpha}))$ if and only if there exists a simplex $I \in M(K)$ such that $I_j = [l_j]$ for $j \in I$ and $I_j \in M(K_j)$ for $j \notin I$. Then $I_1 \sqcup \ldots \sqcup I_m \in F(K(\underline{K_\alpha}))$ if and only if there exists $I \in F(K)$ such that $I_j = [l_j]$ for $j \in I$ and $I_j \in F(K_j)$ for $j \notin I$. In this case we shall say that $I$ is the support of the simplex $I_1 \sqcup \ldots \sqcup I_m$. Clearly, $\varnothing \in F(K(\underline{K_\alpha}))$.

The poset $F(K(\underline{K_\alpha}))$ is graded by the rank function

$$\operatorname{rank}_{F(K(\underline{K_\alpha}))}(I_1 \sqcup \ldots \sqcup I_m) = \operatorname{rank}'_{F(K_1)} I_1 + \ldots + \operatorname{rank}'_{F(K_m)} I_m + \operatorname{rank}_{F(K)} I,$$

where $\operatorname{rank}'_{F(K_j)} I_j = \operatorname{rank}_{F(K_j)} I_j$ if $I_j \in F(K_j)$ (i.e., $j \notin I$), and $\operatorname{rank}'_{F(K_j)} I_j = n_j$, the rank of the nerve-complex $K_j$, if $I_j = [l_j]$ (i.e., in the case $j \in I$).

It follows from Lemma 6.7 that the link of the simplex $I_1 \sqcup \ldots \sqcup I_m$ with support $I$ is of the form

$$\operatorname{link}_{K(\underline{K_\alpha})}(I_1 \sqcup \ldots \sqcup I_m) = \operatorname{link}_K I\Big(\underline{\big\{\operatorname{link}_{K_j} I_j\big\}}_{j\notin I}\Big).$$

By Corollary 6.2,

$$\operatorname{link}_K I\Big(\underline{\big\{\operatorname{link}_{K_j} I_j\big\}}_{j\notin I}\Big) \simeq \operatorname{link}_K I * \big(\ast_{j\notin I} \operatorname{link}_{K_j} I_j\big) \tag{6.2}$$

$$\simeq S^{n-\operatorname{rank}_{F(K)} I-1} * \big(\ast_{j\notin I} S^{n_j-\operatorname{rank}_{F(K_j)} I_j-1}\big) \cong S^{n+\sum\limits_{j\notin I} n_j-\operatorname{rank}_{F(K)} I-\sum\limits_{j\notin I}\operatorname{rank}_{F(K_j)} I_j-1}.$$

Since $\operatorname{rank}'_{F(K_j)} I_j = n_j$ for $j \in I$, by adding and subtracting $\sum\limits_{j\in I} \operatorname{rank}'_{F(K_j)} I_j$ from the dimension of the sphere in the last expression, we have

$$\begin{aligned} n + \sum_{j\notin I} n_j - \operatorname{rank}_{F(K)} I - \sum_{j\notin I} \operatorname{rank}_{F(K_j)} I_j - 1 \\ = n + \sum_{j\in[m]} n_j - \sum_{j\in[m]} \operatorname{rank}'_{F(K_j)} I_j - 1 \\ = n + \sum_{j\in[m]} n_j - \operatorname{rank}_{F(K(\underline{K_\alpha}))}(I_1 \sqcup \ldots \sqcup I_m) - 1, \end{aligned} \tag{6.3}$$

which concludes the proof. □

## § 7. Multigraded betti numbers of substitution complexes

In this section, we shall recall the known definition of multigraded betti numbers of the simplicial complex $K$ and also recall the Hochster formula, expressing the multigraded betti numbers via the ranks of cohomology groups of full subcomplexes of $K$. Corollary 6.2 and the Hochster formula yield an exact formula expressing the multigraded betti numbers of the substitution complex $K(\underline{K_\alpha})$ via the multigraded betti numbers of the complexes $K, K_1, \ldots, K_m$.

Fix a ground field $\Bbbk$ and consider the polynomial ring in $m$ variables $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$. The ring $k[m]$ carries a natural $\mathbb{Z}^m$-grading: $\deg v_i = (0, \ldots, 2, \ldots, 0)$, with the 2 at the $i$s place. The epimorphism $\Bbbk[m] \to \Bbbk$, $v_i \mapsto 0$ gives rise to a $\Bbbk[m]$-module structure on the field $\Bbbk$.

Let $K$ be a simplicial complex on $m$ vertices. Its Stanley-Reisner algebra $\Bbbk[K]$ is called the quotient algebra $\Bbbk[m]/I_{SR}$, where the Stanley-Reisner ideal $I_{SR}$ is generated by the square-free monomials $v_{\alpha_1} \ldots v_{\alpha_k}$ corresponding to non-simplices $\{\alpha_1, \ldots, \alpha_m\} \notin K$. The algebra $\Bbbk[K]$ has a natural structure of a $\Bbbk[m]$-module defined by the epimorphism $\Bbbk[m] \to \Bbbk[m]/I_{SR}$. Since $I_{SR}$ is a homogeneous ideal, the module $\Bbbk[K]$ is $\mathbb{Z}^m$-graded.

Consider a free resolution

$$\ldots \to R^{-i} \to R^{-i+1} \to \ldots \to R^{-1} \to R^0 \to \Bbbk[K]$$

of the module $\Bbbk[K]$ by $\mathbb{Z}^m$-graded $\Bbbk[m]$-modules $R^{-i}$. We have

$$R^{-i} = \bigoplus_{\bar{j}\in\mathbb{Z}^m} R^{-i,\bar{j}}.$$

Thus, the Tor-module of the complex $K$ has a natural $\mathbb{Z}^{m+1}$-grading:

$$H^*(R^* \otimes_{\Bbbk[m]} \Bbbk, d \otimes_{\Bbbk[m]} \Bbbk) = \operatorname{Tor}_{\Bbbk[m]}(\Bbbk[K], \Bbbk) = \bigoplus_{i \in \mathbb{Z}_{\geqslant};\ \bar{j} \in \mathbb{Z}^m} \operatorname{Tor}_{\Bbbk[m]}^{-i,\,2\bar{j}}(\Bbbk[K], \Bbbk).$$

The multigraded betti numbers of the complex $K$ are, by definition, the dimensions of the homogeneous components of the Tor-module:

$$\beta_{\Bbbk}^{-i,\,2\bar{j}}(K) = \dim_{\Bbbk} \operatorname{Tor}_{\Bbbk[m]}^{-i,\,2\bar{j}}(\Bbbk[K], \Bbbk).$$

In general, these numbers depend on the ground field, but, for brevity, we shall omit the index $\Bbbk$. A combinatorial description of multigraded betti numbers is given by the well-known Hochster formula ([15]; the topological meaning of that formula is explained in [8]).

**Theorem 7.1** (Hochster, [10, Th. 3.2.8]). *If $K$ is a simplicial complex on $m$ vertices and $\bar{j} = (j_1, \dots, j_m) \in \mathbb{Z}^m$, then $\beta^{-i,\,2\bar{j}}(K) = 0$ for $\bar{j} \notin \{0,1\}^m$. If $\bar{j} \in \{0,1\}^m$ and $A = \{i \in [m] \mid j_i = 1\}$, then*

$$\beta^{-i,\,2\bar{j}}(K) = \dim \widetilde{H}^{|A|-i-1}(K_A; \Bbbk), \tag{7.1}$$

*where $K_A$ is the full subcomplex of $K$ on the vertex set $A$. Here and later we set $\widetilde{H}^{-1}(\varnothing; \Bbbk) = \Bbbk$.*

For $A \subseteq [m]$, we shall write $\beta^{-i,\,2A}$ instead of $\beta^{-i,\,2\bar{j}}$, where $\bar{j} = (j_1, \dots, j_m)$, $j_i = 1$ if $i \in A$ and $j_i = 0$ otherwise.

In particular, the theorem of Hochster shows that the set of all multigraded betti numbers is a complete combinatorial invariant of a simplicial complex. Indeed, $J \subseteq [m]$ is a minimal non-simplex if and only if

$$\beta^{-1,2A}(K) = \dim \widetilde{H}^{|A|-2}(K_A; \Bbbk) = \dim \widetilde{H}^{|A|-2}(\partial \Delta_A; \Bbbk) = 1,$$

and the set of minimal non-simplices uniquely determines the simplicial complex.

Bigraded betti numbers are defined by the formula

$$\beta^{-i,\,2j}(K) = \sum_{|A|=j} \beta^{-i,\,2A}.$$

These numbers are the dimensions of the graded homogeneous components of the Tor-module $\operatorname{Tor}_{\Bbbk[m]}^{*,*}(\Bbbk[K], \Bbbk)$ when the $\mathbb{Z}^m$-grading $(j_1, \dots, j_m)$ is specialized to the $\mathbb{Z}$-grading $\sum j_i$.

To work with multigraded betti numbers we shall use their generating functions, called the beta-polynomials of $K$. Set

$$\beta_K(s, \bar{t}) = \beta_K(s, t_1, t_2, \dots, t_m) = \sum_{i \in \mathbb{Z},\ \bar{j} \in \mathbb{Z}^m} \beta^{-i,\,2\bar{j}}(K) s^i \bar{t}^{\bar{j}},$$

where $\bar{t}^{\bar{j}}$ denotes the monomial $t_1^{j_1} t_2^{j_2} \dots t_m^{j_m}$.

By the Hochster formula,

$$\beta_K(s, \bar{t}) = \sum_{i \in \mathbb{Z},\ A \subseteq [m]} \beta^{-i,\,2A}(K) s^i \bar{t}^A,$$

where $\bar{t}^A = \prod_{l \in A} t_l$. The constant term corresponding to the indices $A = \varnothing$, $i = 0$ equals 1 for all complexes $K$. We shall also need the reduced beta-polynomial,

$$\widetilde{\beta}_K(s, t) = \beta_K(s, t) - 1 = \sum_{\substack{i \in \mathbb{Z} \\ A \subseteq [m],\ A \neq \varnothing}} \beta^{-i,\,2A} s^i \bar{t}^A.$$

The two-parameter beta-polynomial (see [3, §8]) is defined similarly:

$$b_K(s,t) = \sum_{i,\,j\in\mathbb{Z}} \beta^{-i,\,2j} s^{-i} t^{2j} = \beta_K(s^{-1}, t^2, t^2, \dots, t^2)$$

and

$$\widetilde{b}_K(s,t) = \sum_{i,\,j\in\mathbb{Z},\,j\neq 0} \beta^{-i,\,2j} s^{-i} t^{2j} = \widetilde{\beta}_K(s^{-1}, t^2, t^2, \dots, t^2) = b_K(s,t) - 1.$$

**Example 7.2.** Let $K = \partial\Delta_{[m]}$. Then, by the Hochster formula,

$$\beta_{\partial\Delta_{[m]}}(s,\bar{t}) = 1 + st_1t_2\dots t_m,$$

since the only non-acyclic full subcomplexes of $K$ are $K_\varnothing$ and $K_{[m]} = K$. The non-trivial reduced cohomology of those complexes are contained in degrees $-1$ and $m-2$, respectively.

**Example 7.3.** Consider $K = o_m$. Then

$$\beta_{o_m}(s,\bar{t}) = \sum_{A\subseteq[m]} s^{|A|}\bar{t}^A,$$

since, for any subset $A \subseteq [m]$, the full subcomplex $o_m|_A$ is empty and its $(-1)$st cohomology group has rank 1. Hence,

$$\beta_{o_m}(s,\bar{t}) = (1+st_1)\cdot\ldots\cdot(1+st_m). \tag{7.2}$$

For the polytope $P$ define polynomials $\beta, \widetilde{\beta}, b, \widetilde{b}$ as the corresponding polynomials of its nerve-complex $K_P$:

$$\beta_P(s,\bar{t}) = \beta_{K_P}(s,\bar{t}), \quad \widetilde{\beta}_P(s,\bar{t}) = \widetilde{\beta}_{K_P}(s,\bar{t}), \tag{7.3}$$

$$b_P(s,t) = b_{K_P}(s,t), \quad \widetilde{b}_P(s,t) = \widetilde{b}_{K_P}(s,t). \tag{7.4}$$

To express $\beta_{K(\underline{K_\alpha})}(s,\bar{t})$ in terms of $\beta_K(s,\bar{t})$ and $\beta_{K_i}(s,\bar{t})$, we describe the structure of the full subcomplexes of $K(\underline{K_\alpha})$.

**Lemma 7.4.** *Consider a complex $K$ on $m$ vertices and complexes $K_\alpha$ on $l_\alpha$ vertices for $\alpha \in [m]$. Thus, the set of vertices $K(\underline{K_\alpha})$ is $[l_1]\sqcup\ldots\sqcup[l_m]$. Let $A \subseteq [l_1]\sqcup\ldots\sqcup[l_m]$ and $A = A_1\sqcup\ldots\sqcup A_m$, where $A_\alpha \subseteq [l_\alpha]$. Set $\nu = \{\alpha_1,\dots,\alpha_k\} = \{\alpha\in[m] \mid A_\alpha \neq \varnothing\}$. Then*

$$K(\underline{K_\alpha})_A = K|_\nu\big(K_{\alpha_1}|_{A_{\alpha_1}}, K_{\alpha_2}|_{A_{\alpha_2}}, \dots, K_{\alpha_k}|_{A_{\alpha_k}}\big).$$

The proof follows directly from the definitions.

**Theorem 7.5.** *Let, as before, $K, K_1, \dots, K_m$ be simplicial complexes on $m, l_1, \dots, l_m$ vertices, respectively. For each index $j \in [m]$ set*

$$\bar{t}_j = (t_{j1}, \dots, t_{jl_j}) \quad \textit{and} \quad \bar{t} = (t_{11}, \dots, t_{1l_1}, \dots, t_{m1}, \dots, t_{ml_m}) = (\bar{t}_1, \dots, \bar{t}_m).$$

*Then*

$$\beta_{K(\underline{K_\alpha})}(s,\bar{t}) = \beta_K\Big(s, s^{-1}\widetilde{\beta}_{K_1}(s,\bar{t}_1), s^{-1}\widetilde{\beta}_{K_2}(s,\bar{t}_2), \dots, s^{-1}\widetilde{\beta}_{K_m}(s,\bar{t}_m)\Big). \tag{7.5}$$

*Proof.* Using the Hochster formula (7.1), we have

$$\begin{aligned}\beta_{K(\underline{K_\alpha})}(s,\bar{t}) &= \sum_{A\subseteq[l_1]\sqcup\ldots\sqcup[l_m]} \sum_{i'} \dim \widetilde{H}^{|A|-i'-1}(K(\underline{K_\alpha})_A;\Bbbk) s^{i'}\bar{t}^A \\ &= \sum_{A\subseteq[l_1]\sqcup\ldots\sqcup[l_m]} \sum_{i'} H_{i',A} s^{i'}\bar{t}^A,\end{aligned} \tag{7.6}$$

where the $H_{i',A}$ denote the dimensions of the cohomology groups. Each subset $A \subseteq [l_1] \sqcup \ldots \sqcup [l_m]$ is given as $A = A_1 \sqcup \ldots \sqcup A_k$ for some $B = \{\alpha_1, \ldots, \alpha_k\} \subseteq [m]$ and $A_1 \subseteq [l_{\alpha_1}], \ldots, A_k \subseteq [l_{\alpha_k}]$, $A_i \neq \varnothing$. Expanding the sum in (7.6), we have

$$\sum_{A \subseteq [l_1] \sqcup \ldots \sqcup [l_m]} \sum_{i'} H_{i',A} s^{i'} \bar{t}^A = \sum_{i'} \sum_{B=\{\alpha_1,\ldots,\alpha_k\} \subseteq [m]} \left( \sum_{\substack{A_1 \subseteq [l_{\alpha_1}] \\ A_1 \neq \varnothing}} \cdots \sum_{\substack{A_k \subseteq [l_{\alpha_k}] \\ A_k \neq \varnothing}} H_{i',A} s^{i'} \bar{t}_{\alpha_1}^{A_1} \ldots \bar{t}_{\alpha_k}^{A_k} \right). \tag{7.7}$$

Using Lemma 7.4 and Corollary 6.2, the $H_{i',A}$ can be written as follows:

$$\begin{aligned} H_{i',A} &= \dim \widetilde{H}^{|A|-i'-1}(K(\underline{K_\alpha})_A; \Bbbk) \\ &= \dim \widetilde{H}^{|A_1|+\ldots+|A_k|-i'-1}\big(K|_B(K_{\alpha_1}|_{A_1}, \ldots, K_{\alpha_k}|_{A_k}); \Bbbk\big) \\ &= \dim \widetilde{H}^{|A_1|+\ldots+|A_k|-i'-1}\big(K|_B * K_{\alpha_1}|_{A_1} * \ldots * K_{\alpha_k}|_{A_k}; \Bbbk\big). \end{aligned} \tag{7.8}$$

For the cohomology of the join, we have

$$\begin{aligned} &\dim \widetilde{H}^{|A_1|+\ldots+|A_k|-i'-1}\big(K|_B * K_{\alpha_1}|_{A_1} * \ldots * K_{\alpha_k}|_{A_k}; \Bbbk\big) \\ &= \sum_{\substack{r, r_1, \ldots, r_k \\ r+r_1+\ldots+r_k = |A_1|+\ldots+|A_k|-i'-1-k}} \dim \widetilde{H}^r(K|_B; \Bbbk) \cdot \dim \widetilde{H}^{r_1}(K_{\alpha_1}|_{A_1}; \Bbbk) \\ &\quad \cdot \ldots \cdot \dim \widetilde{H}^{r_k}(K_{\alpha_k}|_{A_k}; \Bbbk). \end{aligned} \tag{7.9}$$

Consider the indices $i, i_1, \ldots, i_k$ satisfying the conditions

$$r = k - i - 1 = |B| - i - 1, \quad r_s = |A_s| - i_s - 1$$

for $s \in [k]$. Since

$$r + \sum_{s \in [k]} r_s = \left( \sum_{s \in [k]} |A_s| \right) - i' - 1 - k,$$

we have $i' = i - k + \sum\limits_{s \in [k]} i_s$. Then

$$\begin{aligned} &\sum_{r, r_1, \ldots, r_k} \dim \widetilde{H}^r(K|_B; \Bbbk) \cdot \left( \prod_{j=1}^k \dim \widetilde{H}^{r_j}(K_{\alpha_j}|_{A_j}; \Bbbk) \right) s^{i'} \bar{t}_{\alpha_1}^{A_1} \ldots \bar{t}_{\alpha_k}^{A_k} \\ &= \sum_{i_1, \ldots, i_k} \dim \widetilde{H}^{k-i-1}(K|_B; \Bbbk) s^i \prod_{j=1}^k \left( s^{-1} \dim \widetilde{H}^{|A_j|-i_j-1}(K_{\alpha_j}|_{A_j}; \Bbbk) s^{i_j} \bar{t}_{\alpha_j}^{A_j} \right). \end{aligned} \tag{7.10}$$

Hence,

$$\begin{aligned} &\sum_{\substack{A_1 \subseteq [l_{\alpha_1}] \\ A_1 \neq \varnothing}} \cdots \sum_{\substack{A_k \subseteq [l_{\alpha_k}] \\ A_k \neq \varnothing}} \sum_{i_1, \ldots, i_k} \prod_{j=1}^k \left( s^{-1} \dim \widetilde{H}^{|A_j|-i_j-1}(K_{\alpha_j}|_{A_j}; \Bbbk) s^{i_j} \bar{t}_{\alpha_j}^{A_j} \right) \\ &= \prod_{j=1}^k \left( \sum_{\substack{A_j \subseteq [l_{\alpha_j}] \\ A_j \neq \varnothing}} \sum_{i_j} s^{-1} \dim \widetilde{H}^{|A_j|-i_j-1}(K_{\alpha_j}|_{A_j}; \Bbbk) s^{i_j} \bar{t}_{\alpha_j}^{A_j} \right) \\ &= \prod_{j=1}^k \left( \sum_{\substack{A_j \subseteq [l_{\alpha_j}] \\ A_j \neq \varnothing}} \sum_{i_j} s^{-1} \beta^{-i_j, 2A_j}(K_{\alpha_j}) s^{i_j} \bar{t}_{\alpha_j}^{A_j} \right) = \prod_{j=1}^k \big( s^{-1} \widetilde{\beta}_{K_{\alpha_j}}(s, \bar{t}_{\alpha_j}) \big). \end{aligned} \tag{7.11}$$

Substituting (7.11) in (7.7), we have

$$\beta_{K(\underline{K_\alpha})}(s,\bar{t}) = \sum_i \sum_{B=\{\alpha_1,\cdots,\alpha_k\}\subseteq[m]} \beta^{-i,\,2B}(K|_B)s^i \prod_{j=1}^{k}\big(s^{-1}\widetilde{\beta}_{K_{\alpha_j}}(s,\bar{t}_{\alpha_j})\big) \tag{7.12}$$
$$= \beta_K\big(s, s^{-1}\widetilde{\beta}_{K_1}(s,\bar{t}_1),\ldots,s^{-1}\widetilde{\beta}_{K_m}(s,\bar{t}_m)\big).$$

This concludes the proof. □

**Corollary 7.6.** $b_{K(\underline{K_\alpha})}(s,t) = \beta_K\big(s^{-1}, s\widetilde{b}_{K_1}(s,t),\ldots,s\widetilde{b}_{K_m}(s,t)\big)$.

*Proof.* Substitute $s^{-1}$ and $t^2$ for $s$ and $t_{ji_j}$ in (7.5) and use the definition of the two-parameter beta-polynomial. □

**Corollary 7.7.** *Consider the stochastic polytope embeddings $P \subset \mathbb{R}^m$, $P_1 \subset \mathbb{R}^{l_1},\ldots,$ $P_m \subset \mathbb{R}^{l_m}$. For each index $j \in [m]$, set*

$$\bar{t}_j = (t_{j1},\ldots,t_{jl_j}) \quad \textit{and} \quad \bar{t} = (t_{11},\ldots,t_{1l_1},\ldots,t_{m1},\ldots,t_{ml_m}) = (\bar{t}_1,\ldots,\bar{t}_m).$$

*Then*

$$\beta_{P(P_1,\ldots,P_m)}(s,\bar{t}) = \beta_P\Big(s, s^{-1}\widetilde{\beta}_{P_1}(s,\bar{t}_1), s^{-1}\widetilde{\beta}_{P_2}(s,\bar{t}_2),\ldots,s^{-1}\widetilde{\beta}_{P_m}(s,\bar{t}_m)\Big); \tag{7.13}$$
$$b_{P(\underline{P_\alpha})}(s,t) = \beta\big(s^{-1}, s\widetilde{b}_{P_1}(s,t),\ldots,s\widetilde{b}_{P_m}(s,t)\big).$$

*Proof.* By Proposition 4.8,

$$\beta_{P(P_1,\ldots,P_m)}(s,\bar{t}) = \beta_{K_{P(P_1,\ldots,P_m)}}(s,\bar{t}) = \beta_{K_P(K_{P_1},\ldots,K_{P_m})}(s,\bar{t}).$$

The latter expression is rewritten by Theorem 7.5. □

**Corollary 7.8.** *Let $P_1$ and $P_2$ be convex polytopes, and $P_1 * P_2$ their join. Let $\bar{t}_i = (t_{i1},\ldots,t_{il_i})$ be formal variables corresponding to the facets of $P_i$ for $i = 1,2$ and $\bar{t} = (\bar{t}_1,\bar{t}_2)$. Then*

$$\beta_{P_1*P_2}(s,\bar{t}) = 1 + s^{-1}\widetilde{\beta}_{P_1}(s,\bar{t}_1)\widetilde{\beta}_{P_2}(s,\bar{t}_2),$$
$$b_{P_1*P_2}(s,t) = 1 + s\widetilde{b}_{P_1}(s,t)\widetilde{b}_{P_2}(s,t).$$

*Proof.* By Example 3.2, $P_1 * P_2 = \triangle_{[2]}(P_1,P_2)$. Now apply Corollary 7.7. In [3], one finds another proof of this result. □

**Corollary 7.9.** *Let $K$ be a simplicial complex on $m$ vertices and $(l_1,\ldots,l_m)$ a set of natural numbers. Then*

$$b_{K(l_1,\ldots,l_m)}(s,t) = \beta_K(s^{-1}, t^{2l_1}, t^{2l_2},\ldots,t^{2l_m}).$$

*In particular, if $l_1 = l_2 = \ldots = l_m = l$, then*

$$b_{K(\underline{l})}(s,t) = b_K(s,t^l).$$

*Proof.* By definition,

$$K(l_1,\ldots,l_m) = K(\partial\Delta_{[l_1]},\ldots,\partial\Delta_{[l_m]})$$

and $\widetilde{\beta}_{\partial\Delta_{[l_r]}}(s,\bar{t}_r) = st_{r1}\ldots t_{rl_r}$ (Example 7.2). Hence,

$$\beta_{K(l_1,\ldots,l_m)}(s,\bar{t}) = \beta_K\big(s, s^{-1}st_{11}\ldots t_{1l_1},\ldots,s^{-1}st_{m1}\ldots t_{ml_m}\big)$$
$$= \beta_K(s, t_{11}\cdot\ldots\cdot t_{1l_1},\ldots,t_{m1}\cdot\ldots\cdot t_{ml_m}).$$

Substituting $s^{-1}$ for $s$ and $t^2$ for $t_{rj}$, we have the desired formula. □

**Example 7.10.** Consider the complex $o_m(\underline{K_\alpha}) = K_1 * \ldots * K_m$. Using Theorem 7.5 and (7.2), we have

$$\begin{aligned}(7.14)\quad \beta_{K_1*\ldots*K_m}(s,\bar{t}) &= \beta_{K_1*\ldots*K_m}(s,\bar{t}) \\ &= \big(1+s\cdot s^{-1}\widetilde{\beta}_{K_1}(s,\bar{t}_1)\big)\cdot\ldots\cdot\big(1+s\cdot s^{-1}\widetilde{\beta}_{K_m}(s,\bar{t}_m)\big) \\ &= \beta_{K_1}(s,\bar{t}_1)\cdot\ldots\cdot\beta_{K_m}(s,\bar{t}_m).\end{aligned}$$

This can also be proved directly by using the algebra isomorphism

$$\Bbbk[K_1 * \ldots * K_m] \cong \Bbbk[K_1] \otimes \ldots \otimes \Bbbk[K_m]$$

and the definition of multigraded betti numbers.

## § 8. Enumerative polynomials

Let $K$ be a simplicial complex. For each $i \geqslant 0$, define the number

$$f_i = |\{I \in K \mid |I| = i\}|.$$

The polynomial

$$f_K(t) = \sum_i f_i t^i = \sum_{I\in K} t^{|I|}$$

is called the $f$-polynomial of $K$. If $\dim K = n-1$, then $\deg f_K(t) = n$. The numbers $h_i$ are defined by the relations

$$h_0 t^n + \ldots + h_{n-1}t + h_n = f_0(t-1)^n + f_1(t-1)^{n-1} + \ldots + f_n.$$

The polynomial $h_K(t) = h_0 + h_1 t + \ldots + h_n t^n$ is called the $h$-polynomial of $K$. The defining relations for the $h_i$ yield

$$(8.1)\qquad h_K(t) = (1-t)^n f_K\left(\frac{t}{1-t}\right).$$

Since the relation (8.1) is reversible, the $h$- and the $f$-polynomials encode the same combinatorial information. The $h$-polynomial is related to the Hilbert-Poincaré series of the $\mathbb{Z}$-graded algebra $\Bbbk[K]$ by the formula (see [21], [8])

$$(8.2)\qquad \operatorname{Hilb}(\Bbbk[K];t) = \frac{h_K(t^2)}{(1-t^2)^n}.$$

There is a formula connecting the $h$-polynomial of $K$ with its bigraded betti numbers. Consider the numbers

$$\chi_j(K) = \sum_{i=0}^{m} (-1)^i \beta^{-i,\,2j}(K)$$

and define the polynomial

$$\chi_K(t) = \sum_{j=0}^{m} \chi_j(K) t^{2j}.$$

Then, by [8, Theorem 8.14],

$$(8.3)\qquad \chi_K(t) = (1-t^2)^{m-n} h_K(t^2) = (1-t^2)^m \operatorname{Hilb}(\Bbbk[K];t).$$

Since $\chi_K(t) = b_K(-1,t)$,

$$(8.4)\qquad b_K(-1,t) = (1-t^2)^{m-n} h_K(t^2).$$

The relation (8.4) allows us to express the $h$-polynomial of $K(l,\ldots,l)$ in terms of the $h$-polynomial of $K$.

**Proposition 8.1.** *Let $K$ be an $(n-1)$-dimensional complex on $m$ vertices, $l>0$, and $K(\underline{l}) = K(l,\dots,l) = K(\partial\Delta_{[l]},\dots,\partial\Delta_{[l]})$. Then*

$$h_{K(\underline{l})}(t) = (1+t+\ldots+t^{l-1})^{m-n} h_K(t^l).$$

*Proof.* The complex $K(\underline{l})$ has $m' = ml$ vertices. It is not difficult to check that $\dim K(\underline{l})+1 = n' = nl+(m-n)(l-1)$. Hence, $m'-n' = m-n$. By (8.4), $b_{K(\underline{l})}(-1,t) = (1-t^2)^{m'-n'} h_{K(\underline{l})}(t^2)$. On the other hand, by Corollary 7.9, $b_{K(\underline{l})}(s,t) = b_K(s,t^l)$, and therefore $b_{K(\underline{l})}(-1,t) = b_K(-1,t^l)$. We now have

$$\begin{aligned}(1-t^2)^{m-n} h_{K(\underline{l})}(t^2) = (1-t^2)^{m'-n'} h_{K(\underline{l})}(t^2) &= b_{K(\underline{l})}(-1,t)\\ &= b_K(-1,t^l) = (1-t^{2l})^{m-n} h_K(t^{2l}).\end{aligned}$$

Hence,

$$h_{K(\underline{l})}(t^2) = \left(\frac{1-t^{2l}}{1-t^2}\right)^{m-n} \cdot h_K(t^{2l}) = (1+t^2+\ldots+t^{2(l-1)})^{m-n} h_K(t^{2l}). \qquad \square$$

In particular, when $l=2$, we have $h_{K(\underline{2})}(t) = (1+t)^{m-n} h_K(t^2)$. This result was first proved in [22] by a different method.

*Remark* 8.2. Proposition 8.1 can be proved by using (8.2) and the structure of the ring $\Bbbk[K(\underline{l})]$ (for details, see [5]).

To work with substitution complexes, it is convenient to introduce yet another enumerative polynomial, $q_K(t)$. For an $(n-1)$-dimensional complex $K$ on $m$ vertices, set

$$q_K(t) = 1-(1-t)^{m-n} h_K(t).$$

**Example 8.3.** It is known that $h_{\partial\Delta_{[m]}}(t) = 1+t+\ldots+t^{m-1}$. Then $q_{\partial\Delta_{[m]}}(t) = t^m$.

We have

$$\widetilde{b}_K(-1,t) = b_K(-1,t)-1 = (1-t^2)^{m-n} h_K(t)-1 = -q_K(t^2), \tag{8.5}$$

$$\widetilde{\beta}_K(-1,t,\dots,t) = -q_K(t). \tag{8.6}$$

**Proposition 8.4.** *Consider arbitrary complexes $K_1,\dots,K_m$. Then*

$$q_{\partial\Delta_{[m]}(\underline{K_\alpha})} = q_{K_1}(t)\cdot\ldots\cdot q_{K_m}(t).$$

*Proof.* By Corollary 7.6,

$$\begin{aligned}\widetilde{b}_{\partial\Delta_{[m]}(\underline{K_\alpha})}(s,t) &= \widetilde{\beta}_{\partial\Delta_{[m]}}\big(s^{-1}, s\widetilde{b}_{K_1}(s,t),\dots,s\widetilde{b}_{K_m}(s,t)\big)\\ &= s^{-1}\cdot\big(s\widetilde{b}_{K_1}(s,t)\big)\cdot\ldots\cdot\big(s\widetilde{b}_{K_m}(s,t)\big).\end{aligned} \tag{8.7}$$

Setting $s=-1$ and using (8.5), we have the desired equality. $\square$

*Remark* 8.5. In [3], the fat join operation

$$K_1 \circledast K_2 = \partial\Delta_{[2]}(K_1,K_2)$$

was considered. It is not difficult to check that it is associative and

$$\partial\Delta_{[m]}(\underline{K_\alpha}) = K_1 \circledast K_2 \circledast \ldots \circledast K_m.$$

**Proposition 8.6.** *For arbitrary non-empty complexes $K$ and $L$, we have $q_{K(L,\dots,L)}(t) = q_K(q_L(t))$.*

*Proof.* By Corollary 7.6,

$$\widetilde{b}_{K(L,\dots,L)}(s,t) = \widetilde{\beta}_K\big(s^{-1}, s\widetilde{b}_L(s,t), \dots, s\widetilde{b}_L(s,t)\big).$$

Setting $s = -1$, we have

$$-q_{K(L,\dots,L)}(t^2) = \widetilde{\beta}_K\big(-1, q_L(t^2), \dots, q_L(t^2)\big) = -q_K(q_L(t^2)),$$

which is the desired result. □

## § 9. An example of a beta-polynomial

Consider a regular convex $m$-gon $C_m$, where $m = 2n+1$, with a sequential order on its vertices. Also consider a simplicial complex $\Theta(m)$ on $m$ vertices whose simplices are defined by the condition $I \in \Theta(m)$ if and only if the convex hull of the vertices of $C_m$ with indices from $[m] \setminus I$ contains the center of $C_m$. By [14, 5.4 and 6.3], for all $m \geqslant 5$, the complex $\Theta(m)$ is isomorphic to the boundary of an $(m-3)$-dimensional simplicial polytope and, in particular, is a simplicial sphere. Figure 3 shows the simplest case $m = 5$.

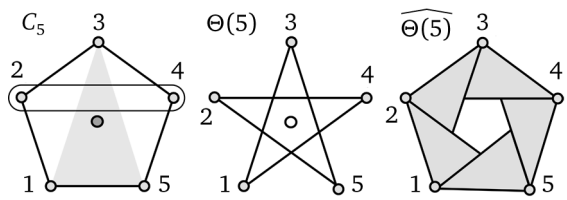


Figure 3. Constructing the complex $\Theta(5)$ and its Alexander-dual

The bigraded betti numbers of such complexes, as well as the multiplicative structure of the ring $\mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[\Theta(m)], \Bbbk)$, was studied in [12]. Here we give an example of computing the multigraded betti numbers of the sphere $\Theta(m)$ in order to show how Corollary 7.9 can be applied. For such computations, it is convenient to use the Alexander-dual complex $\widehat{\Theta(m)}$.

Recall that if $K \neq \Delta_{[m]}$ is a simplicial complex on the set $[m]$, then the dual complex is the complex on $[m]$ given by $\widehat{K} = \{J \subseteq [m] \mid [m] \setminus J \notin K\}$. A classical result in combinatorial topology says that $\widetilde{H}^j(K) \cong \widetilde{H}_{m-3-j}(\widehat{K})$ for (co)homology with coefficients in the field $\Bbbk$ or the ring $\mathbb{Z}$. Moreover, under this duality, full subcomplexes correspond links and vice versa. More precisely, suppose $I \notin K$, i.e., $K_I$ is different from the simplex $\Delta_I$. Then $\widehat{K_I} = \mathrm{link}_{\widehat{K}}([m] \setminus I)$ (notice that $[m] \setminus I \in \widehat{K}$ by the definition of the dual complex). It now follows that

$$\widetilde{H}^j(K_I) \cong \widetilde{H}_{|I|-3-j}\big(\mathrm{link}_{\widehat{K}}([m] \setminus I)\big)$$

for $I \notin K$ (see [8, Proposition 2.34]).

Now we describe the complex $\widehat{\Theta(m)}$. The minimal non-simplices of $\Theta(m)$ are the complements of the maximal sets of vertices of the polygon $C_m$ whose convex hulls do not contain the center of the polygon. It is clear that such maximal sets consist of contiguous sequences of $n+1 = (m+1)/2$ vertices: $\{i, i+1, \dots, i+n\}$, where the numbering is assumed to be cyclic; $m+1 = 1, m+2 = 2$, etc. Since maximal simplices of the dual complex $I \in M(\widehat{\Theta(m)})$ are also complements of minimal non-simplices of $\Theta(m)$, the complex $\widehat{\Theta(m)}$ is generated by the simplices of the form $\{1, 2, \dots, n+1\}$, $\{2, 3, \dots, n+2\}$, $\dots$, $\{m = 2n+1, 1, \dots, n\}$. When $n = 2$ (i.e., $m = 5$) we obtain the Möbius band

(Figure 3). In general, we have a cycle composed of $m = 2n+1$ simplices of dimension $n$ (Figure 4).

Our goal is to compute, for all indices $i$ and subsets $A \subseteq [m]$, the numbers $\beta^{-i,\,2A}(\Theta(m)) = \dim \widetilde{H}^{|A|-i-1}(\Theta(m)_A)$. If $A = \varnothing$, then $\beta^{0,2\varnothing}(\Theta(m)) = 1$. Otherwise, if $A \in \Theta(m)$, then the full subcomplex $\Theta(m)_A = \Delta_A$ is acyclic. Hence, without loss of generality, we may assume that $A \notin \Theta(m)$ and

$$\beta^{-i,\,2A}(\Theta(m)) = \dim \widetilde{H}^{|A|-i-1}(\Theta(m)_A) = \dim \widetilde{H}_{i-2}\big(\operatorname{link}_{\widehat{\Theta(m)}}([m] \setminus A)\big). \tag{9.1}$$

Thus, we need to find all acyclic links of $\widehat{\Theta(m)}$ and describe their homology.

The complex $\widehat{\Theta(m)}$ is homotopy equivalent to the circle $S^1$, and therefore,

$$\dim \widetilde{H}_1\big(\operatorname{link}_{\widehat{\Theta(m)}} \varnothing\big) = 1.$$

By Remark 2.2, if a simplex $I \in \widehat{\Theta(m)}$ is not an intersection of maximal simplices, then $\operatorname{link}_{\widehat{\Theta(m)}} I$ is contractible, and therefore such simplices can be omitted. Intersections of maximal simplices of $\widehat{\Theta(m)}$ are of the form $J_{j,k} = \{j, j+1, \dots, j+k\}$. When $k = n$, the simplex $J_{j,k}$ is maximal, and therefore,

$$\dim \widetilde{H}_{-1}\big(\operatorname{link}_{\widehat{\Theta(m)}} J_{j,n}\big) = \dim \widetilde{H}_{-1}(\varnothing) = 1.$$

Otherwise, the complex $\operatorname{link}_{\widehat{\Theta(m)}} J_{j,k}$ is generated by the maximal simplices of the form $J_{j-n+k,n} \setminus J_{j,k}$, $J_{j-n+k+1,n} \setminus J_{j,k}, \dots,$ $J_{j,n} \setminus J_{j,k}$ (Figure 4). Clearly, $\operatorname{link}_{\widehat{\Theta(m)}} J_{j,k}$ is contractible for $k < n-1$, and $\operatorname{link}_{\widehat{\Theta(m)}} J_{j,n-1} = S^0$ consists of two points. Hence, $\widetilde{H}_0(\operatorname{link}_{\widehat{\Theta(m)}} J_{j,n-1}) = 1$. Returning to the expression (9.1) and taking into account that $[m] \setminus J_{j,k} = J_{j+k+1,2n-k-1}$, we have

$$\beta^{-i,\,2A}(\Theta(m)) = \begin{cases} 1, & \text{if } i = 0, A = \varnothing; \\ 1, & \text{if } i = 1, A = J_{j,n-1} \text{ for some } j \in [m]; \\ 1, & \text{if } i = 2, A = J_{j,n} \text{ for some } j \in [m]; \\ 1, & \text{if } i = 3, A = [m]; \\ 0 & \text{in all other cases.} \end{cases} \tag{9.2}$$

We have thus proved

**Proposition 9.1.** *Suppose $m = 2n+1$, $m \geqslant 5$, and $\Theta(m)$ is the sphere constructed from $C_m$ as above. Then*

$$\beta_{\Theta(m)}(s, \bar{t}) = 1 + \sum_{j \in [m]} s \prod_{i=j}^{j+n-1} t_i + \sum_{j \in [m]} s^2 \prod_{i=j}^{j+n} t_i + s^3 \prod_{i \in [m]} t_i,$$

*where the numbering of the variables $t_i$ is assumed cyclic: $i = i + m$.*

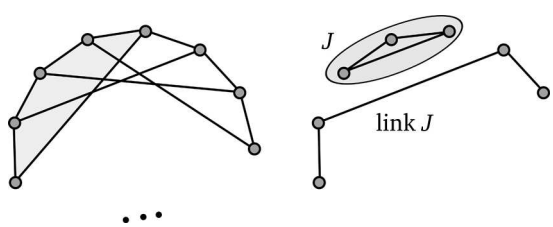


FIGURE 4. The complex $\widehat{\Theta(m)}$ and the link of the simplex $J(j,k)$

Using Corollary 7.9, we have

**Corollary 9.2.** *Let $m = 2n+1$ and $l_1, \dots, l_m$ be arbitrary natural numbers. For the sphere $\Theta(m)(l_1, \dots, l_m)$, we have*

$$b_{\Theta(m)(l_1,\dots,l_m)}(s,t) = 1 + s^{-1} \sum_{j\in[m]} t^{2\sum\limits_{i=j}^{j+n-1} l_i} + s^{-2} \sum_{j\in[m]} t^{2\sum\limits_{i=j}^{j+n} l_i} + s^{-3} t^{2\sum\limits_{i\in[m]} l_i}.$$

This result was first proved in N. Yu. Erokhovets' papers [12, 13]. In [13], it was also shown that the operation $P(l_1, \dots, l_m)$ on a polytope corresponds to assigning multiplicities $l_1, \dots, l_m$ to the points of the Gale diagram of the dual polytope $P^*$.

We want to comment on the importance of spheres of the form $\Theta(m)(l_1, \dots, l_m)$. It is easy to compute that such a sphere has dimension $\sum l_i - 4$, with $\sum l_i$ vertices. As was shown in [16], all $(n-1)$-dimensional triangulated topological spheres on $n+3$ vertices are polytopal, i.e., they realize as the boundaries of simplicial polytopes of dimension $n$. On the other hand, M. Perles (see [14, 6.3]) showed that the boundary of a simplicial polytope of dimension $n$ with $n+3$ vertices is isomorphic to either the complex $\partial\Delta^{n_1} * \partial\Delta^{n_2} * \partial\Delta^{n_3}$, $n_1 + n_2 + n_3 = n$ or the complex $\Theta(m)(l_1, \dots, l_m)$ for some odd $m \geqslant 5$ and $l_i > 0$ with $\sum l_i = n+3$. By Examples 7.2 and 7.10,

$$\begin{aligned}&\beta_{\partial\Delta^{n_1} * \partial\Delta^{n_2} * \partial\Delta^{n_3}}(s,\bar{t})\\ &\qquad = (1 + st_1 \dots t_{n_1+1})(1 + st_{n_1+2} \dots t_{n_1+n_2+2})(1 + st_{n_1+n_2+3} \dots t_{n_1+n_2+n_3+3}).\end{aligned}$$

Thus, all spheres whose number of vertices is 4 more than their dimension have been described, and for each of them their multigraded betti numbers have been determined.

It is known (see [14, Section 6]) that if the number of vertices of a sphere is 2 more than its dimension, then it is the boundary of a simplex $\partial\Delta_{[n]}$, and if the number of vertices of a sphere is 3 more than its dimension, then it is $\partial\Delta_{[n_1]} * \partial\Delta_{[n_2]}$, i.e., the multigraded betti numbers of such spheres are also known.

## § 10. The Alexander duality, minimal non-simplices and the weighted Stanley-Reisner ring

Now we describe a construction relating substitution of simplicial complexes and the Alexander duality.

**Proposition 10.1.** $\widehat{K(\underline{K_\alpha})} = \widehat{K}\big(\widehat{K}_1, \dots, \widehat{K}_m\big)$ *if $K \neq \Delta_{[m]}$ and $K_i \neq \Delta_{[l_i]}$ (i.e., all complexes on the right-hand side are well-defined).*

*Proof.* The proof is a sequence of equivalences:

$$\begin{aligned}&\bigsqcup_{i\in[m]} J_i \in \widehat{K(\underline{K_\alpha})} \Leftrightarrow \bigsqcup_{i\in[m]} ([l_i] \setminus J_i) \notin K(\underline{K_\alpha})\\ &\qquad \Leftrightarrow \big\{i \in [m] \mid [l_i] \setminus J_i \notin K_i\big\} \notin K \Leftrightarrow [m] \setminus \big\{i \in [m] \mid [l_i] \setminus J_i \notin K_i\big\} \in \widehat{K}\\ &\qquad \Leftrightarrow \big\{i \in [m] \mid J_i \notin \widehat{K_i}\big\} \in \widehat{K} \Leftrightarrow \bigsqcup_{i\in[m]} J_i \in \widehat{K}\big(\widehat{K}_1, \dots, \widehat{K}_m\big).\end{aligned} \tag{10.1}$$

□

**Corollary 10.2.** $K(l_1, \dots, l_m)\widehat{\ } = \widehat{K}(o_{l_1}, \dots, o_{l_m})$.

Recall that $N(K)$ denotes the set of minimal non-simplices of $K$. We have $J \in N(K) \Leftrightarrow [m] \setminus J \in M(\widehat{K})$.

**Corollary 10.3.** *All minimal non-simplices of $K(\underline{K_\alpha})$ are of the form $\bigsqcup_{i\in J} J_i$, where $J \in N(K)$, $J_i \in N(K_i)$.*

*Remark* 10.4. The complexes $K(l_1,\dots,l_m)$ can be described in terms of minimal non-simplices, which, in this case, are of the form $\bigsqcup_{i\in J}[l_i]$ for $J\in N(K)$. This description of the iterated simplicial wedge $K(l_1,\dots,l_m)$ is taken as a primary definition in [5] and [13]. The Stanley-Reisner algebra $\Bbbk[K(l_1,\dots,l_m)]$ is of the form

$$\frac{\Bbbk[v_{1,1},\dots,v_{1,l_1},\dots,v_{m,1},\dots,v_{m,l_m}]}{\big(v_{i_1,1}\dots v_{i_1,l_{i_1}}\cdot\dots\cdot v_{i_k,1}\dots v_{i_k,l_{i_k}}\ \text{for}\ \{i_1,\dots,i_k\}\notin K\big)}.$$

Finally we mention yet another relationship between the substitution of complexes and commutative algebra.

Consider a simplicial complex $K$ on the set $[m]$ and a set of natural numbers $\bar l=\{l_1,\dots,l_m\}$. Define a weighted Stanley-Reisner ideal $I^{\bar l}_{SR}\subset\Bbbk[v_1,\dots,v_m]$ as the ideal generated by the monomials $v_{i_1}^{l_{i_1}}\dots v_{i_k}^{l_{i_k}}$ with $\{i_1,\dots,i_k\}\notin K$. The algebra $\Bbbk[K]^{\bar j}=\Bbbk[m]/I^{\bar j}_{SR}$ is called the weighted Stanley-Reisner algebra [5, Def. 10.4]. The standard Stanley-Reisner algebra arises as a particular case: $\Bbbk[K]=\Bbbk[K]^{\bar 1}$.

It is natural to study the bigraded (or multigraded) betti numbers of a weighted Stanley-Reisner algebra, i.e., the numbers $\beta^{*,*}(\Bbbk[K]^{\bar l})=\dim_{\Bbbk}\operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K]^{\bar l},\Bbbk)$. For that purpose, in commutative algebra one uses the notion of *polarization*, which reduces the problem to the study of ideals generated by square-free monomials (i.e., the Stanley-Reisner ideals).

Consider a monomial $X=v_1^{s_1}\dots v_m^{s_m}\in\Bbbk[v_1,\dots,v_m]$. The monomial

$$P(X)=v_{1,1}\dots v_{1,s_1}\cdot v_{2,1}\dots v_{2,s_2}\cdot\dots\cdot v_{m,1}\dots v_{m,s_m}\in\Bbbk[v_{1,1},\dots,v_{1,s_1},\dots,v_{m,1},\dots,v_{m,s_m}]$$

is called the polarization of $X$. If the ideal $I\subset\Bbbk[v_1,\dots,v_m]$ is minimally generated by monomials $X_1,\dots,X_k$ and if $v_1^{l_1}\cdot\dots\cdot v_m^{l_m}$ is the smallest common divisor of $X_1,\dots,X_k$, then the ideal of $\Bbbk[v_{1,1},\dots,v_{1,l_1},\dots,v_{m,1},\dots,v_{m,l_m}]$ generated by the monomials $P(X_1),\dots,P(X_k)$ is called the polarization of $P(I)$.

The multigrading on $\Bbbk[v_{1,1},\dots,v_{1,l_1},\dots,v_{m,1},\dots,v_{m,l_m}]$ is given by the requirement $\operatorname{mdeg}(v_{i,k})=\operatorname{mdeg}(v_i)=(0,\dots,2,\dots,0)\in\mathbb Z^m$, where the 2 is at the $i$th spot.

It is known (see, for example, [17, pp. 59–60] [18]), that the multigraded betti numbers of $\Bbbk[m]/I$ and $\Bbbk[v_{1,1},\dots,v_{1,l_1},\dots,v_{m,1},\dots,v_{m,l_m}]/P(I)$ coincide.

Now notice that, by Remark 10.4, the polarization of the weighted Stanley-Reisner ideal $P(I^{\bar l}_{SR}(K))$ coincides with the standard Stanley-Reisner ideal of the complex $K(l_1,\dots,l_m)$.

Thus,

$$\beta^{-i,\,2\bar j}(\Bbbk[K]^{\bar l})=\beta^{-i,\,2\bar j}\big(\Bbbk[K(l_1,\dots,l_m)]\big).$$

By Corollary 7.9 (more precisely, by its multigraded version),

$$\beta^{-i,\,2(l_1\,j_1,\,l_2\,j_2,\,\dots,\,l_m\,j_m)}(\Bbbk[K]^{\bar l})=\beta^{-i,\,2(j_1,\,j_2,\,\dots,\,j_m)}(\Bbbk[K]).$$

This formula can also be easily deduced from an explicit description of a minimal free resolution of the module $\Bbbk[K]^{\bar l}$.

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