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ERGODIC HOMOCLINIC GROUPS, SIDON CONSTRUCTIONS AND POISSON SUSPENSIONS

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Abstract. We give some new examples of mixing transformations on a space with infinite measure: the so-called Sidon constructions of rank 1. We obtain rapid decay of correlations for a class of infinite transformations; this was recently discovered by Prikhod'ko for dynamical systems with simple spectrum acting on a probability space. We obtain an affirmative answer to Gordin's question about the existence of transformations with zero entropy and an ergodic homoclinic flow. We consider modifications of Sidon constructions inducing Poisson suspensions with simple singular spectrum and a homoclinic Bernoulli flow. We give a new proof of Roy's theorem on multiple mixing of Poisson suspensions.

§ 1. Introduction

Ergodic theory of measure-preserving transformations studies actions on a probability space and actions on a space with a sigma-finite invariant measure. The latter for brevity are said to be infinite and the phase space (X, μ), $\mu(X) = \infty$, is assumed to be isomorphic to the real line with Lebesgue measure.

The Poisson measure μ_* on the configuration space X_* (see [4, 6, 10]) induces a continuous embedding of the group $\mathrm{Aut}(\mu)$ of all invertible transformations preserving the measure μ into the group $\mathrm{Aut}(\mu_*)$. The mixing property for an infinite transformation T means that

$$\mu(T^n A \cap B) \to 0, \quad n \to \infty,$$

for any sets A, B of finite measure. This property implies the mixing property of the Poisson suspension T_*:

$$\mu_*(T_*^n V \cap W) \to \mu_*(V)\mu_*(W), \quad n \to \infty,$$

for measurable sets $V, W \subset X_*$.

As we shall see, among Poisson suspensions only the mixing suspensions have so-called homoclinic ergodic groups. Following Gordin [2] we define the homoclinic group $H(T)$ of a transformation T as follows:

$$H(T) = \{S\colon T^n S T^{-n} \to Id,\ n \to \infty\}.$$

If the phase space X has infinite measure and a transformation $T\colon X \to X$ is mixing, then the group F of all measure-preserving transformations S with support of finite measure, $\mu(\operatorname{supp} S) < \infty$, is contained in the group $H(T)$. The group F_* is contained in $H(T_*)$ and is ergodic.

The following remarkable fact was established in [2]: if the homoclinic group of an automorphism of a probability space is ergodic, then it has the mixing property. We generalize this result to the multiple mixing property. This makes it possible to give a

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new proof of Roy's theorem on multiple mixing of mixing Poisson suspensions. Thus, in the class of Poisson suspensions, we establish that the properties of mixing, of multiple mixing of all orders, and of ergodicity of the homoclinic group are equivalent. Recall that mixing of multiplicity 2 means that

$$\mu_*(U \cap T_*^m V \cap T_*^{m+n} W) \to \mu_*(U)\mu_*(V)\mu_*(W)$$

as $m, n \to \infty$.

The question arises: which mixing actions do not have ergodic homoclinic groups? By what was said above, an example is given by the mixing but not multiply mixing action of Ledrappier [7]. An oricyclic flow does not have homoclinic elements (apart from the identity). This can be derived from Ratner's theorem on the uniform distribution of orbits [9]. It would be interesting to find out whether there exist transformations that are not elements of some homoclinic group (as a candidate we would suggest an ergodic rotation of a circle).

It can be shown that *for mixing transformations of a probability space that have positive local rank* (see the definition in [11]) *the homoclinic group is finite*, *and so it cannot be ergodic. For mixing transformations of rank* 1 *the homoclinic group consists of the identity transformation.* It can be seen that any infinite Cartesian product is an element of the homoclinic group of a Bernoulli automorphism with infinite entropy. However, it is not obvious that there exists a transformation with zero entropy for which the Bernoulli automorphism is a homoclinic element.

Gordin posed the question whether a transformation with zero entropy and an ergodic homoclinic group exists. An affirmative answer was given by King in [5]. This paper contains different examples: it turns out that all mixing Poisson suspensions, including those with zero entropy, have an ergodic homoclinic group.

Can a transformation with zero entropy possess an ergodic homoclinic flow? King's example did not give an answer to this question. We shall show that there exists a Poisson suspension T_* with simple (and, consequently, singular) spectrum and with homoclinic group $H(T_*)$ containing a Bernoulli flow.

The spectrum of the Poisson suspension T_* is completely determined by the spectrum of the transformation T, since T_* as a unitary operator is isomorphic to the operator

$$\exp(T) = \bigoplus_{n=0}^{\infty} T^{\odot n},$$

where $T^{\odot 0} = 1$ is the one-dimensional identity operator and $T^{\odot n}$ is the symmetric tensor power of the operator T. If the spectrum of T_* is simple, then it is singular. The entropy of such suspensions is equal to zero, since a transformation with positive entropy has a countably multiple Lebesgue component in the spectrum.

In constructing a suitable infinite transformation T using the method in [11] we ensure that the spectrum of $\exp(T)$ is simple and that there exists a dissipative element S in the group $H(T)$. In this case the Poisson suspension T_* has a Bernoulli homoclinic element S_*.

In the paper we use the so-called Sidon constructions of infinite transformations of rank 1. That they have the mixing property follows from the definition, and the details of the constructions make it possible to construct dissipative homoclinic elements. Sidon constructions are also interesting in that they make it possible to obtain infinite transformations with simple spectrum and rapid decay of correlations. A similar phenomenon in the case of a probability phase space was first discovered by Prikhod'ko [8]. In connection with the new theory of generic mixing transformations (see the recent papers [1, 12]) it

would be interesting to find out which generic properties are enjoyed by infinite mixing transformations.

§ 2. Constructions of rank-one transformations

A measure-preserving invertible transformation $T\colon X \to X$ of a Lebesgue space (X, μ) has rank 1 if there exists a sequence ξ_j of measurable partitions of the space X such that

$$\xi_j = \{E_j,\ TE_j,\ T^2E_j, \dots,\ T^{h_j}E_j, \tilde{E}_j\}$$

and ξ_j tends to the partition into points. The tuple of 'storeys' (or 'blocks')

$$E_j, TE_j,\ T^2E_j, \dots,\ T^{h_j}E_j$$

is called a tower. The disjunct union

$$X_j = \bigsqcup_{z=0}^{h_j} T^z E_j$$

is also called a tower. In this definition the measure of the space can be either finite or infinite. The measure of the set $\tilde{E}_j = X \setminus X_j$ tends to 0 in the case of finite measure and is equal to infinity in the case $\mu(X) = \infty$.

Construction of a transformation of rank 1. We fix $h_1 \geq 1$ (height of the tower at stage 1), a sequence $r_j \to \infty$ (the number of columns; in the general case it is only required that $r_j \geq 2$), and a sequence of integer vectors

$$\bar{s}_j = \bigl(s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)\bigr).$$

Below we give a description of a construction which is completely determined by the parameters h_1, r_j, $\bar{s}_j$.

At step $j = 1$ we are given an interval E_1. Suppose that at step j a system (tower) of disjoint intervals

$$E_j,\ TE_j,\ T^2E_j, \dots,\ T^{h_j-1}E_j$$

is defined, and the transformation T is a parallel translation on $E_j, TE_j, \dots, T^{h_j-2}E_j$.

We now pass to step $j + 1$. We represent the interval E_j as a disjunct union of r_j intervals of the same measures, that is,

$$E_j = E_j^1 \sqcup E_j^2 \sqcup E_j^3 \sqcup \dots \sqcup E_j^{r_j}.$$

For $i = 1, 2, \dots, r_j$ we consider the tuple (column) $E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j-1}E_j^i$ (the ith column at stage j). Above every column with number i, we add $s_j(i)$ intervals of measure $\mu(E_j^i)$ (new blocks in the tower at stage $j + 1$) and obtain the tuple

$$E_j^i,\ TE_j^i,\ T^2E_j^i, \dots,\ T^{h_j-1}E_j^i,\ T^{h_j}E_j^i,\ T^{h_j+1}E_j^i, \dots,\ T^{h_j+s_j(i)-1}E_j^i$$

(all the sets are disjoint). Denoting $E_{j+1} = E_j^1$, we set $T^{h_j+s_j(i)}E_j^i = E_j^{i+1}$ for all $i < r_j$. Thus, the columns are stacked up into a new tower

$$E_{j+1},\ TE_{j+1},\ T^2E_{j+1},\ \dots,\ T^{h_{j+1}}E_{j+1},\ T^{h_{j+1}-1}E_{j+1},$$

where

$$h_{j+1} = h_j r_j + \sum_{i=1}^{r_j} s_j(i).$$

Continuing the construction we obtain the phase space X as the union of all the intervals and an invertible transformation T on X. The measure of the space X is infinite if the series

$$\sum_j \frac{s_j(1) + s_j(2) + \dots + s_j(r_j)}{h_j r_j}$$

diverges. It is well known that a transformation of rank 1 is ergodic and has simple spectrum. (In the general case, if an infinite transformation has simple spectrum this does not imply it is ergodic.)

§ 3. Sidon constructions and rapid decay of correlations

In this section we consider constructions of rank 1 with special suspensions. These transformations are remarkable because their mixing property follows easily from the definition and, moreover, mixing can be rapid.

Sidon constructions. Let $r_j \to \infty$ and suppose that at every step j we have

$$(*) \qquad h_j \ll s_j(1) \ll s_j(2) \ll \cdots \ll s_j(r_j - 1) \ll s_j(r_j).$$

Then for fixed ξ_{j_0}-measurable sets $A, B \subset X_{j_0}$ we have

$$\mu(A \cap T^m B) \le \frac{\mu(A)}{r_j}$$

for all $m \in [h_j, h_{j+1}]$, $j > j_0$. Thus, for all sets A, B of finite measure we have

$$\mu(A \cap T^m B) \to 0.$$

Such a construction has the *Sidon property*: *for $h_j < m \le h_{j+1}$ the intersection $X_j \cap T^m X_j$ can be contained only in one column of the tower X_j* (we consider what was given as the definition of a Sidon construction). Obviously, a Sidon construction is mixing as $r_j \to \infty$.

Optimal Sidon constructions. Suppose that we are given a sequence $N_j \to \infty$. (Note that the sequence N_j need not be given beforehand; it can be defined step by step in the course of construction of a transformation.) As is well known, the integer interval $\{1, 2, \dots, N_j\}$ contains a Sidon set $\mathbf{S_j}$ of maximal cardinality close to $\sqrt{N_j}$. Recall that a Sidon set $\mathbf{S_j}$ by definition satisfies the following condition: for $m > 0$ the intersection $\mathbf{S} \cap (\mathbf{S} + m)$ contains at most one element. We denote elements of the set $\mathbf{S_j}$ by $S_j(0), S_j(1), \dots, S_j(r_j)$ assuming that $S_j(0) < S_j(1) < \cdots < S_j(r_j)$. We set $r_j = |\mathbf{S}_j| - 1$ and

$$s_j(i) = h_j(S_j(i) - S_j(i-1) - 1), \quad i = 1, 2, \dots, r_j.$$

This is a Sidon construction, and it has the feature that the suspensions are minimal.

Theorem 3.1. *For any function $\psi(m) \to \infty$ (its growth may be arbitrarily slow) such that the sequence $\psi(m)/\sqrt{m}$ tends monotonically to 0, for some optimal Sidon construction T for a dense family of sets A (dense in the class of all sets of finite measure), the condition for a rapid decay of correlations holds*:

$$\mu(A \cap T^m A) \le C \frac{\psi(m)}{\sqrt{m}},$$

where the constant C depends on the set A.

Proof. At step j we define r_j by the condition $\psi(h) \ge \sqrt{h_j}$ for all $h > r_j^2$. We set $N_j = r_j^2$. We find $s_j(i)$, $i = 1, 2, \dots, r_j$, as described above, corresponding to an optimal Sidon construction. We obtain

$$h_{j+1} \sim h_j N_j = h_j r_j^2, \quad \frac{\sqrt{h_j}}{\psi(h_{j+1})} \le 1.$$

Since $\frac{\psi(m)}{\sqrt{m}}$ is monotonic for $m \in [h_j + 1, h_{j+1}]$ we have

$$\frac{\psi(h_{j+1})}{\sqrt{h_{j+1}}} \le \frac{\psi(m)}{\sqrt{m}}.$$

Then for all $m \in [h_j + 1, h_{j+1}]$ and for the set A consisting of the tuple of storeys of some tower X_{j_0} we have

$$\mu(A \cap T^m A) \le \frac{\mu(A)}{r_j} \le C \frac{\sqrt{h_j}}{\sqrt{h_{j+1}}} = C \frac{\psi(h_{j+1})}{\sqrt{h_{j+1}}} \frac{\sqrt{h_j}}{\psi(h_{j+1})} \le C \frac{\psi(m)}{\sqrt{m}}. \qquad \square$$

It remains to observe that such sets A are dense in the class of all sets of finite measure.

§ 4. Multiple mixing, homoclinic groups and Poisson suspensions

Following [2] we define the homoclinic group $H(T)$ of a transformation T by setting

$$H(T) = \{S \in \mathrm{Aut}(X, \mu) \colon T^n S T^{-n} \to Id,\ n \to \infty\}.$$

Gordin [2] proved the mixing property of an automorphism T of a probability space in the case where the group $H(T)$ is ergodic. We shall strengthen this result. An automorphism T of a probability space has mixing of multiplicity 2 if for any measurable sets A, B, C

$$\mu(A \cap T^m B \cap T^{m+n} C) \to \mu(A)\mu(B)\mu(C)$$

as $m, n \to \infty$. Mixing of multiplicity $n > 2$ is defined in similar fashion.

Theorem 4.1. *An automorphism of a probability space with ergodic homoclinic group has mixing of any multiplicity.*

This result is an obvious consequence of the following lemma.

Lemma 4.2. *Let R_i and T_i be sequences of automorphisms of a probability space. Suppose that an automorphism S satisfies*

$$R_i^{-1} S R_i \to Id, \quad T_i^{-1} S T_i \to Id.$$

(i) If for some measure ν on the product $X \times X \times X$ we have

$$\mu(A \cap R_i B \cap T_i C) \to \nu(A \times B \times C)$$

for all measurable sets A, B, C, then

$$\nu(SA \times B \times C) = \nu(A \times B \times C).$$

(ii) Suppose that $\nu(SA \times B \times C) = \nu(A \times B \times C)$ for all $S \in H(T)$, and the group $H(T)$ is ergodic. If $\nu(X \times B \times C) = \mu(B)\mu(C)$ for any measurable sets B and C, then $\nu = \mu \times \mu \times \mu$.

Proof. (i) We have the chain of equalities:

$$\begin{aligned} \nu(A \times B \times C) &= \lim_{i \to \infty} \mu(A \cap R_i B \cap T_i C) \\ &= \lim_{i \to \infty} \mu(SA \cap R_i R_i^{-1} S R_i B \cap T_i T_i^{-1} S T_i C) \\ &= \lim_{i \to \infty} \mu(SA \cap R_i B \cap T_i C) = \nu(SA \times B \times C). \end{aligned}$$

(ii) The measure ν is the joining of the ergodic action of $H(T)$ on the space $X_{(1)}$ and the identity action on the space $X_{(2)} \times X_{(3)}$. It is well known that in this case the measure ν is the direct product of the projections of ν onto these spaces, that is, $\nu = \mu \times (\mu \times \mu)$. $\square$

Lemma 4.3. *If the phase space has infinite measure and a transformation T is mixing, then the group F of all transformations S with support of finite measure, $\mu(\operatorname{supp} S) < \infty$, is contained in the group $H(T)$.*

Proof. For any set A of finite measure we have

$$\mu(A \cap \operatorname{supp}(T^{-n}ST^n)) \to 0 < \infty, \quad n \to \infty,$$

by the mixing property of the transformation T. Consequently,

$$T^{-n}ST^n \to Id. \qquad \square$$

Poisson suspensions. We fix a space (X, μ) with infinite measure. On the configuration space X_*, that is, the space of countable subsets of the space X, we introduce the Poisson measure μ_*. This measure is defined by the following condition: for any family of sets $A_i \subset X$, $i = 1, \dots, N$, of finite measure we have

$$\mu_*\big(\{x_* \colon |x_* \cap A_i| = n_i,\ i = 1, \dots, N\}\big) = \prod_{i=1}^{N} \frac{e^{-\mu(A_i)}\mu(A_i)^{n_i}}{(n_i)!}.$$

Let T be an automorphism of the space (X, μ). The Poisson suspension is defined to be the transformation

$$T_*(\{x_k\}) = \{(Tx)_i\}.$$

The suspension T_* preserves the probability measure μ_*. The association $T \to T_*$ effects a continuous embedding of $\operatorname{Aut}(X, \mu)$ into $\operatorname{Aut}(X_*, \mu_*)$.

Theorem 4.4 (see [10])**.** *Mixing Poisson suspensions have mixing of all orders.*

Proof. For an infinite mixing transformation T, the group $F = \{S \colon \mu(\operatorname{supp} S) < \infty\}$ is contained in the group $H(T)$ (Lemma 4.3). The Poisson suspension F_* is an ergodic group, since its closure contains all possible T_*. The latter is a consequence of the fact that the closure of F contains all automorphisms T of the space (X, μ). It remains to observe that F_* is homoclinic with respect to T_* and to apply Theorem 4.1. $\square$

§ 5. A Poisson suspension with singular spectrum and a homoclinic Bernoulli flow

We consider the following question: *Can an automorphism with zero entropy have a homoclinic ergodic flow*? We shall find a Poisson suspension with the required properties.

Assertion 5.1. *There exists a Poisson suspension with singular spectrum having a Bernoulli homoclinic flow.*

Consider a mixing automorphism R of an infinite space such that R_* has simple, and therefore also continuous, singular spectrum. There are examples in [3]. These examples can also be obtained using Sidon constructions. By adding a nonmixing special part to a Sidon construction, by analogy with [3, 11], we can ensure that the spectrum of the operator $\exp(R)$ is simple. Reducing the nonmixing part preserves the property that the spectrum of $\exp(R)$ is simple but gives the mixing property

$$\mu(A \cap R^m B) \le \varepsilon_j + \frac{\mu(A)}{r_j},$$

where the estimate $\varepsilon_j \to 0$ corresponds to the vanishing nonmixing part. The mixing property for such modified Sidon automorphisms R does not require a proof and follows from the definition.

On the space $X = \mathbf{R} \times \mathbf{R}^+$ we consider the transformation

$$T(x, y) = (x, R(y)).$$

Let S_t be the flow on X defined by the equation

$$S^t(x, y) = (x + \varphi(y)t, y),$$

where $\varphi > 0$ and $\varphi(y) \to 0$ as $y \to \infty$. The flow is dissipative and homoclinic with respect to T. The first is obvious, and the second is easily established. Indeed, for functions of the form $f = \chi_{[a,b]\times[c,d]}$ we have

$$\|f - T^{-n}S^tT^nf\|_2 \to 0, \quad n \to \infty,$$

since for a fixed t the transformation $T^nS^tT^{-n}$ on $X = \mathbf{R} \times [c,d]$ is close to the identity transformation. This is evident from the formula

$$T^{-n}S^tT^n(x,y) = (x + \varphi(R^n(y))t, y)$$

and from the fact that by the mixing property the quantity $R^n(y)$ is large for most $y \in [c,d]$, but $\varphi(y) \to 0$ as $y \to \infty$; therefore the quantity $\varphi(R^n(y))$ is small. It follows from what was said above that $T^{-n}S^tT^n \to Id$ as $n \to \infty$.

The Poisson suspension S_*^t is a Bernoulli flow (S^t is a dissipative flow) which is homoclinic with respect to T_*. The spectrum of the suspension T_* is singular: the measure of the maximal spectral type coincides with the spectrum of the transformation R_*, and the multiplicity of the spectrum of the suspension T_* is infinite. The assertion is proved.

In the next section we present another method for constructing homoclinic actions.

§ 6. Poisson suspensions with simple spectrum and a Bernoulli homoclinic flow

We now describe a construction T of rank 1 that has a dissipative transformation S as one of the homoclinic elements. Recall that S being dissipative means the existence of a measurable set Y such that

$$X = \bigsqcup_{z\in\mathbf{Z}} S^zY.$$

Lemma 6.1. *Suppose that for a construction T of rank 1 and some automorphism S we have*

$$\min\bigl\{\mu(ST^nT^kE_j \mid T^nT^kE_j)\colon\ 0 \le k \le h_j,\ h_j \le n \le h_{j+1}\bigr\} \to 1 \quad \text{as } j \to \infty.$$

Then the transformation S is homoclinic with respect to T.

Proof. We fix a set A consisting of a union of some blocks in some tower in our construction at stage j_0. For all $j > j_0$ the set A is a union of blocks of the form T^kE_j. Let $h_j \le n \le h_{j+1}$. Since T^nA is a union of some sets of the form $T^nT^kE_j$, while the latter are little different from $ST^nT^kE_j$, we obtain that T^nA is little different from ST^nA. This means that $T^{-n}ST^nA$ asymptotically coincides with the set A. The automorphism S is homoclinic. □

Transformations S and T satisfying the hypotheses of Lemma 6.1. Let T be a Sidon construction. For $h_j \le n \le h_{j+1}$ the image of a block T^kE_j under the action of T^n mainly consists of suspensions of stages $j+1$ and $j+2$, except for the set getting into the stage j tower. The measure of this set does not exceed the quantity $\mu(E_j)/r_j$.

We choose a transformation for S such that for some sequence $\varepsilon_j \to 0$ we have

$$\mu(ST^mE_{j+1} \mid T^mE_{j+1}) > 1 - \varepsilon_j$$

provided that the block T^mE_{j+1} is not contained in the tower X_j. Then the image of the block T^kE_j under the action of T^n mainly consists of such (new) blocks by the Sidon property of our construction.

Construction of a dissipative transformation S. The space X is the disjunct union of X_1 and the sets $X_j \setminus X_{j+1}$. The latter consist of so-called new blocks. We number them in the order of construction and denote them by D_k.

We fix a sequence $s_k \to \infty$ such that

$$\sum_k \frac{\mu(D_k)}{s_k} = \infty.$$

We partition every block D_k into s_k intervals $B_k^1, B_k^2, \dots, B_k^{s_k}$ of the same measure. We define a transformation $\tilde{S}$ that permutes these intervals cyclically within the block, so that $\tilde{S}^{s_k}$ is the identity on D_k.

We consider some dissipative transformation P on the union $\bigsqcup_k B_k^1$, which has infinite measure. We extend P to the entire space X by regarding it as the identity outside the union $\bigsqcup_k B_k^1$. The transformation $S = P\tilde{S}$ is dissipative (the wandering set of the transformation P is wandering for S). By construction, the transformation S satisfies the property $\mu(SD_k \mid D_k) \to 1$ as $s \to \infty$. The image of the storey $T^k E_j$ $(0 \le k \le h_j)$ under the action of T^n $(n > h_j)$ mainly consists of the $D_{k'}$ such that the values of all such k' are increasing as $j \to \infty$. Now we can apply Lemma 6.1. We find that S is homoclinic with respect to the Sidon construction T.

We point out that the construction of the dissipative transformation S is independent of the properties of the construction T. But for a Sidon construction T such a transformation turns out to be homoclinic. In this case we can apply the lemma, since for $h_j \le n \le h_{j+1}$ the image of the block $T^k E_j$ under the action of T^n is contained in the union of some new blocks in $X \setminus X_j$, except for an asymptotically negligible set.

Modification of the Sidon construction. We consider a construction T_ε that differs from the Sidon one in that it is not required that the Sidon property holds on the union of columns for i such that $(1-\varepsilon)r_j < i \le r_j$. Here, it is possible to ensure that weak limits of the powers of the transformation exist, which in turn implies that the spectrum of the operator $\exp(T_\varepsilon)$ is simple. By the method described in [11] we can prove that there exists a mixing construction T that is the limit of such constructions T_{ε_k}, $\varepsilon_k \to 0$, and inherits the property that the spectrum of $\exp(T)$ is simple. A similar problem, but for a different class of infinite transformations, was solved in [3]. The transformation S constructed above is homoclinic with respect to T. Obviously, S can be embedded in a dissipative flow that is homoclinic with respect to T. Thus, the following assertion holds.

Theorem 6.2. *There exists a mixing Poisson suspension with simple singular spectrum that has a Bernoulli homoclinic flow.*

Remark. It would be interesting to know whether a mixing Poisson suspension always has a Bernoulli homoclinic element. It is probably possible to prove that *there exists a Poisson suspension T_* with simple spectrum such that for any Poisson suspension R_* the homoclinic group $H(T_*)$ contains an element conjugate to R_** (*universality of the group $H(T_*)$*). To do this it is sufficient to establish the universality of $H(T)$ for the modified Sidon construction T.

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