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COMPARISON OF THE SINGULAR NUMBERS OF CORRECT RESTRICTIONS OF ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. The paper is dedicated to finding the asymptotics of singular numbers of a correct restriction of a uniformly elliptic differential operator of order $2l$ defined on a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, which is in general a non-selfadjoint operator. Conditions are established on a correct restriction, ensuring that its singular numbers s_k are of order $k^{2l/n}$ as $k \to \infty$. As an application of this result certain estimates are obtained for the deviation upon domain perturbation of singular numbers of such correct restrictions.

§ 1. Introduction

Let $l, n \in \mathbb{N}$ and $\mathcal{L}$ be an elliptic differential expression of the following form: for $u \in C^\infty(\mathbb{R}^n)$

$$(\mathcal{L}u)(x) = \sum_{|\alpha|,|\beta|\le l} (-1)^{|\alpha|+|\beta|} D^\alpha (A_{\alpha\beta}(x) D^\beta u), \quad x \in \mathbb{R}^n,$$

where $A_{\alpha\beta} \in C^l(\mathbb{R}^n)$ are real-valued functions for all multi-indices α, β satisfying $|\alpha|, |\beta| \le l$. Moreover, for any bounded domain $\Omega \subseteq \mathbb{R}^n$ with sufficiently smooth boundary let L_Ω be a linear operator closed in $L_2(\Omega)$ generated by $\mathcal{L}$ on Ω.

Our main aim is finding a family of correct restrictions B of the operator L_Ω having the same asymptotics of singular numbers $s_k(B)$.

This is done by comparing $s_k(B)$ with singular numbers $s_k(A)$ of a correct restriction A of the operator L_Ω whose asymptotics of singular numbers are known and represent the inverse B^{-1} in the form $B^{-1} = A^{-1} + K$, where A^{-1} and K are a leading operator and a non-leading operator respectively.

As an application we establish certain estimates for the deviation of singular numbers $s_k(B)$ of correct restrictions of this family upon domain perturbation. This is done by considering a correct restriction A for which such estimates are known and the fact that $s_k(B)$ and $s_k(A)$ have the same asymptotics which are uniform with respect to a certain family of correct restrictions and a certain class of domains Ω.

The paper is organized in the following way. In Section 2 we recall the notion of a correct restriction of a closed linear operator in a Hilbert space and related facts. In Section 3 the definition of leading and non-leading compact operators is given and, under certain assumptions, it is proved that the singular numbers of the sum of a leading and a non-leading operator have the same asymptotics as the singular numbers of the leading operator (Theorem 3.1). Section 4 is dedicated to some auxiliary estimates of the singular numbers of correct restrictions of elliptic differential operators. In Section 5

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a precise description of the class of domains under consideration is given. The core result on the representation of the inverse of a correct restriction as the sum of a leading and a non-leading operator is proved in Section 6 (Theorem 6.1). In Section 7 on the basis of Theorems 3.1 and 6.1 it is proved that the singular numbers of all correct restrictions of a certain family are of the same order $k^{2l/n}$ and have the same asymptotics as $k \to \infty$ (Theorem 7.1). Section 8 is dedicated to applying Corollary 7.1 (a "uniform" version of Theorem 7.1) to obtain spectral stability estimates for correct restrictions belonging to a certain family (Theorem 8.1).

Most of the results of this paper were formulated, without proofs, in [4].

§ 2. Preliminaries on correct restrictions of linear operators

Let $L\colon D(L) \to H$ be a closed linear operator in a Hilbert space H. It will always be assumed that the domain $D(L)$ is dense in H and that the equation

$$Lu = f \tag{1}$$

has a solution for any $f \in H$ (in general non-unique).

Recall that an operator $A\colon D(A) \to H$ is called a restriction of L, briefly $A \subseteq L$, if $D(A) \subseteq D(L)$ and $Au = Lu$ for all $u \in D(A)$.

Definition 2.1. A restriction A of L is called *correct* if the equation

$$Au = f \tag{2}$$

has a unique solution $u \in D(A)$ for any $f \in H$ and the corresponding inverse operator $A^{-1}\colon H \to D(A)$ is bounded, i. e.

$$\|A^{-1}\| = \sup_{f\in H,\, f\neq 0} \frac{\|A^{-1}f\|}{\|f\|} = \sup_{u\in D(A),\, u\neq 0} \frac{\|u\|}{\|Au\|} < \infty.$$

Remark 2.1. Clearly $LA^{-1} = I$, where I is the identity operator on H. Moreover, if $B\colon H \to D(L)$ is a bounded operator such that $LB = I$, then the operator A which is the restriction of L to $D(A) = B(H)$ is a correct restriction of L and $B = A^{-1}$.

Let L have a correct restriction A. Then it is possible to describe all correct restrictions of L. Namely, the following statement holds. See [9, 11].

Lemma 2.1. *Let A be a correct restriction of L.*

a) *If an operator $K\colon H \to \operatorname{Ker} L = \{u \in D(L)\colon Lu = 0\}$ is bounded, then the operator $A^{-1} + K$ is invertible and the operator*

$$B = (A^{-1} + K)^{-1} \tag{3}$$

is a correct restriction of L.

b) *If B is a correct restriction of L, then there exists a bounded operator*

$$K\colon H \to \operatorname{Ker} L,$$

namely $K = B^{-1} - A^{-1}$, such that equality (3) *holds.*

Remark 2.2. Under the assumptions of part a)

$$B^{-1} = A^{-1} + K \tag{4}$$

and

$$D(B) = R(A^{-1} + K) \subseteq D(A) + R(K) \subseteq D(A) + \operatorname{Ker} L. \tag{5}$$

Remark 2.3. In Lemma 2.1 the operators K can be nonlinear, which allows us to consider boundary value problems with nonlinear boundary conditions. However, in this paper we restrict ourselves to considering the case of linear operators K.

Example 2.1. Let B be an open ball in $\mathbb{R}^n$ and L be the closure of the Laplacian $-\Delta$ with the domain $C^\infty(\mathbb{R}^n)$ in $H = L_2(B)$; hence $L\colon D(L) \to L_2(B)$,

$$D(L) = \{u \in L_2(B)\colon\ -\Delta_w u \in L_2(B)\}, \quad Lu = -\Delta_w u \quad \text{for } u \in D(L),$$

where $-\Delta_w u$ is the weak Laplacian.

Furthermore, in the capacity of a restriction A of L, we consider the Dirichlet Laplacian $-\Delta_{\mathrm{D}}$ with the domain

$$D(-\Delta_{\mathrm{D}}) = \{u \in D(L),\ \operatorname{tr}_{\partial B} u = 0\}.$$

It is well known that for each $f \in L_2(B)$ the equation $-\Delta_{\mathrm{D}} u = f$ has a unique solution in $D(-\Delta_{\mathrm{D}})$ which continuously depends on f. Hence $-\Delta_{\mathrm{D}}$ is a correct restriction of L in the sense of Definition 1.

By Lemma 2.1 for each bounded operator $K\colon L_2(B) \to \operatorname{Ker} L$, the space of all solutions $u \in D(L)$ of the equation $\Delta_w u = 0$, and the operator $((-\Delta_{\mathrm{D}})^{-1} + K)^{-1}$ is a correct restriction of L. This implies that for each $f \in L_2(B)$ the equation

$$((-\Delta_{\mathrm{D}})^{-1} + K)^{-1} u = f \tag{6}$$

has a unique solution

$$\begin{aligned} u = ((-\Delta_{\mathrm{D}})^{-1} + K) f \in D(((-\Delta_{\mathrm{D}})^{-1} + K)^{-1}) &= R((-\Delta_{\mathrm{D}})^{-1}) + R(K) \\ &\subseteq D(-\Delta_{\mathrm{D}}) + R(K) \subseteq D(-\Delta_{\mathrm{D}}) + \operatorname{Ker} L. \end{aligned} \tag{7}$$

Moreover,

$$\begin{cases} u \in D(((-\Delta_{\mathrm{D}})^{-1} + K)^{-1}), \\ ((-\Delta_{\mathrm{D}})^{-1} + K)^{-1} u = f \end{cases} \Leftrightarrow \begin{cases} u \in L_2(B), -\Delta_w u \in L_2(B), \\ -\Delta_w u = f \text{ on } B, \\ \operatorname{tr}_{\partial B}(u + K(\Delta_w u)) = 0. \end{cases}$$

The implication $\Rightarrow$ follows since by (6) and (7) $-\Delta_w u = f$ on Ω and

$$\operatorname{tr}_{\partial B} u = \operatorname{tr}_{\partial B} (-\Delta_{\mathrm{D}})^{-1} f + \operatorname{tr}_{\partial B} Kf = \operatorname{tr}_{\partial B} Kf.$$

Conversely, assume that $u \in L_2(B), -\Delta_w u \in L_2(B), -\Delta_w u = f$ on $B, \operatorname{tr}_{\partial B} u = \operatorname{tr}_{\partial B} Kf$, and $v = u - Kf$. Then $v \in D(L)$ (since $Kf \in D(L)$), $-\Delta_w v = f$ on B, and $\operatorname{tr}_{\partial B} v = 0$. Therefore, $v = (-\Delta_{\mathrm{D}})^{-1} f$. Consequently,

$$u = (-\Delta_{\mathrm{D}})^{-1} f + Kf \ \Leftrightarrow\ u \in R((-\Delta_{\mathrm{D}})^{-1} + K) = D(((-\Delta_{\mathrm{D}})^{-1} + K)^{-1})$$

and $((-\Delta_{\mathrm{D}})^{-1} + K)^{-1} u = f$.

Example 2.2. Let H, L, A be the same as in Example 2.1, $n \geq 2$ and $\varphi_1, \varphi_2 \in L_2(B)$. Assume that φ_1 is a harmonic function on B and that for all $f \in L_2(B)$

$$(Kf)(x) = \varphi_1(x) \int_B \varphi_2(y) f(y)\, dy, \quad x \in B. \tag{8}$$

Then K is bounded from $L_2(B)$ to $\operatorname{Ker} L$, and all the statements of Example 2.1 are applicable to K. In particular this implies that the following problem

$$\begin{cases} u \in L_2(B), -\Delta_w u \in L_2(B), \\ -\Delta_w u = f \text{ on } B, \\ \operatorname{tr}_{\partial B} \left(u(x) + \varphi_1(x) \displaystyle\int_B \varphi_2(y) \Delta_w u(y)\, dy \right) = 0 \end{cases}$$

with a non-local boundary condition is well posed.

§ 3. Some estimates for singular numbers of general compact linear operators

Let A be a compact linear operator in a Hilbert space H. Recall that the eigenvalues of the self-adjoint operator $(A^*A)^{1/2}$ are called the singular numbers of the operator A. As usual all singular numbers of A will be written in the form of the sequence $\{s_n(A)\}_{n=1}^{\infty}$, where

$$s_1(A) \geq s_2(A) \geq \dots \geq s_n(A) \geq \dots$$

and each singular number is repeated as many times as its multiplicity. Note that $\lim_{n\to\infty} s_n(A) = 0$.

Lemma 3.1. *Let A and B be compact linear operators in a Hilbert space H, $0 < \alpha < \beta$ and $\alpha/\beta < \theta < 1$.*

Moreover, let for some $c_1, c_2 > 0$ for all $k \in \mathbb{N}$

$$c_1 k^{-\alpha} \leq s_k(A), \quad s_k(B) \leq c_2 k^{-\beta}. \tag{9}$$

Then for all $k \in \mathbb{N}$

$$s_k(A+B) \leq s_{k-[k^\theta]+1}(A)(1 + c_3 k^{\alpha-\theta\beta}), \tag{10}$$

where $c_3 = c_1^{-1} c_2 2^\beta$ and $[a]$ denotes the integer part of a real number a.

Proof. We apply the inequality

$$s_k(A+B) \leq \min_{\substack{m+n=k+1\\ m,n\in\mathbb{N}}} (s_m(A) + s_n(B)). \tag{11}$$

([5, 6, 7]; see also [8, Chapter 2, Corollary 2.2].)

Taking into account that $\frac{\mu}{[\mu]} \leq 2$ for each real number $\mu \geq 1$, we have

$$\begin{aligned} s_k(A+B) &\leq s_{k-[k^\theta]+1}(A) + s_{[k^\theta]}(B) \\ &\leq s_{k-[k^\theta]}(A) + c_2[k^\theta]^{-\beta} \frac{s_{k-[k^\theta]+1}(A)}{c_1(k-[k^\theta]+1)^{-\alpha}} \leq s_{k-[k^\theta]}(A)(1 + c_1^{-1} c_2 [k^\theta]^{-\beta} k^\alpha) \\ &\leq s_{k-[k^\theta]+1}(A)\Big(1 + c_1^{-1} c_2 \Big(\frac{k^\theta}{[k^\theta]}\Big)^\beta k^{\alpha-\theta\beta}\Big) \leq s_{k-[k^\theta]}(A)(1 + c_1^{-1} c_2 2^\beta k^{\alpha-\theta\beta}). \end{aligned}$$

Hence, inequality (10) follows. □

Remark 3.1. Assume that for some $k_1 \in \mathbb{N}$ inequalities (9) are satisfied with the same c_1, c_2 only for natural $k \geq k_1$. Since $[k^\theta], k - [k^\theta] \to \infty$ as $k \to \infty$, there exists $k_2 \in \mathbb{N}$, depending only on k_1 and θ, such that $[k^\theta], k - [k^\theta] + 1 \geq k_1$ for all $k \geq k_2$. Hence, by the above proof, it follows that inequality (10) is satisfied with the same c_3 for natural natural $k \geq k_2$.

Lemma 3.2. *Under the assumptions of Lemma 3.1 there exists $c_4 > 0$, depending only on $\alpha, \beta, \theta, c_1, c_2$ and for each $\gamma > 1$ there exists $\varkappa_\gamma \in \mathbb{N}$, depending only on $\gamma, \alpha, \beta, \theta, c_1, c_2$ such that*

$$s_{k+[\gamma k^\theta]}(A)(1 - c_4 k^{\alpha-\theta\beta}) \leq s_k(A+B) \leq s_{k-[k^\theta]}(A)(1 + c_4 k^{\alpha-\theta\beta}) \tag{12}$$

for all $k \geq \varkappa_\gamma$.

Proof. 1. By inequality (11), with B replaced by $-B$, A replaced by $A+B$, and k replaced by $2k$, we have

$$s_{2k}(A) \leq s_k(A+B) + s_{k+1}(B).$$

Hence, by assumption (9)

$$s_k(A+B) \geq s_{2k}(A) - s_{k+1}(B) \geq c_1(2k)^{-\alpha} - c_2(k+1)^{-\beta} \geq k^{-\alpha}(c_1 2^{-\alpha} - c_2 k^{\alpha-\beta}) \geq c_5 k^{-\alpha},$$

where $c_5 = c_1 2^{-\alpha-1}$, if $k \geq k_1 = [(c_1^{-1} c_2 2^{\alpha+1})^{1/(\beta-\alpha)}] + 1$.

Therefore by Lemma 3.1 and Remark 3.1 it follows that there exists $k_2 \in \mathbb{N}$, depending only on $\alpha, \beta, \theta, c_1, c_2$, such that inequality (10) holds, with B replaced by $-B$, A replaced by $A+B$, k replaced by m, and c_1 replaced by c_5, for all $m \geq k_2$. So

$$s_m(A) \leq s_{m-[m^\theta]+1}(A+B)(1 + c_4 m^{\alpha-\theta\beta}), \tag{13}$$

where $c_4 = c_5^{-1} c_2 2^\beta > c_3$, for all $m \geq k_2$.

2. For a given $k \in \mathbb{N}$, let a natural number m be such that

$$m - [m^\theta] + 1 = k. \tag{14}$$

2a. Note that the function f defined by $f(m) = m - [m^\theta] + 1$ for $m \in \mathbb{N}$ is non-decreasing on $\mathbb{N}$ and $f(\mathbb{N}) = \mathbb{N}$. Therefore this equation has a solution for each $k \in \mathbb{N}$, although the solution, in general, is non unique.

Indeed, first note that for all $m, n \in \mathbb{N}$ the inequalities

$$n - 1 < m^\theta \leq n, \quad (n-1)^{1/\theta} < m^\theta \leq n^{1/\theta} \quad \text{and} \quad [(n-1)^{1/\theta}] + 1 \leq m \leq [n^{1/\theta}]$$

are equivalent. Hence

$$\begin{aligned} f(\mathbb{N}) &= \bigcup_{n=1}^{\infty} f\big(\{m \in \mathbb{N} \colon n - 1 < m^\theta \leq n\}\big) \\ &= \bigcup_{n=1}^{\infty} f\big(\{m \in \mathbb{N} \colon [(n-1)^{1/\theta}] + 1 \leq m \leq [n^{1/\theta}]\}\big). \end{aligned}$$

Assume that n is fixed and $[(n-1)^{1/\theta}] + 1 < [n^{1/\theta}]$. If $[(n-1)^{1/\theta}] + 1 \leq m^\theta \leq [n^{1/\theta}] - 1$, then $n - 1 < m^\theta < n$; hence $[m^\theta] = n - 1$ and

$$f(m) = m - [m^\theta] + 1 = m - n + 2.$$

If $n^{1/\theta} \notin \mathbb{N}$, then $n - 1 \leq (n^{1/\theta} - 1)^\theta < [n^{1/\theta}]^\theta < n$ hence $[[n^{1/\theta}]^\theta] = n - 1$, and this equality also holds for $m = [n^{1/\theta}]$. If $n^{1/\theta} \notin \mathbb{N}$, then $[n^{1/\theta}]^\theta = n$ and

$$f([n^{1/\theta}]) = [n^{1/\theta}] - n + 1 = f([n^{1/\theta}] - 1).$$

In both cases

$$f\big(\{m \in \mathbb{N} \colon [(n-1)^{1/\theta}] + 1 \leq m \leq [n^{1/\theta}]\}\big) = \big[f([(n-1)^{1/\theta}] + 1), f([n^{1/\theta}])\big] \cap \mathbb{N}.$$

Clearly this equality is also satisfied if $[(n-1)^{1/\theta}] + 1 = [n^{1/\theta}]$.

Therefore

$$\begin{aligned} f(\mathbb{N}) &= \bigcup_{n=1}^{\infty} \Big(\big[f([(n-1)^{1/\theta}] + 1), f([n^{1/\theta}])\big] \cap \mathbb{N}\Big) \\ &= \Bigg(\bigcup_{n=1}^{\infty} \big[f([(n-1)^{1/\theta}] + 1), f([n^{1/\theta}])\big]\Bigg) \cap \mathbb{N} = \mathbb{N}, \end{aligned}$$

because $f(1) = 1$ and for each $n \in \mathbb{N}$

$$f([n^{1/\theta}] + 1) = [n^{1/\theta}] - [([n^{1/\theta}] + 1)^\theta] + 2 \leq [n^{1/\theta}] - [([n^{1/\theta}])^\theta] + 2 = f([n^{1/\theta}]) + 1.$$

Also note that

$$f([n^{1/\theta}] + 1) \geq [n^{1/\theta}] - [([n^{1/\theta}])^\theta + 1] + 2 = f([n^{1/\theta}]).$$

The above argument also implies that the function f is non-decreasing on $\mathbb{N}$, but is not strictly increasing.

2b. Assume that $m \in \mathbb{N}$ satisfies equation (14). Since $m = k + [m^\theta] - 1 \geq k$ it also follows that

$$m \geq k + [k^\theta] - 1.$$

Next we claim that for each $\gamma > 1$ there exists $\nu_\gamma > 0$, depending only on γ and θ such that

$$m \le k + [\gamma k^\theta] \tag{15}$$

for all natural $k \ge \nu_\gamma$.

Indeed, equality (14) implies that for each $0 < \delta < 1$

$$k \ge m - m^\theta = m(1 - m^{\theta-1}) \ge \delta m \quad \text{if } m \ge (1-\delta)^{1/(\theta-1)}.$$

Hence for all natural $k \ge (1-\delta)^{1/(\theta-1)}$ ($\Rightarrow m \ge (1-\delta)^{1/(\theta-1)}$)

$$k \ge m - [m^\theta] \ge m - [\delta^{-\theta} k^\theta] \;\Rightarrow\; m \le k + [\delta^{-\theta} k^\theta] = k + [\gamma k^\theta]$$

if $\delta^{-\theta} = \gamma$. So inequality (15) holds for $k \ge \nu_\gamma = (1 - \gamma^{-1/\theta})^{1/(\theta-1)}$.

3. Inequalities (13) and (15) imply that for natural $k \ge \max\{k_2, \nu_\gamma\}$

$$s_{k+[\gamma k^\theta]}(A) \le s_k(A+B)(1 + c_4(k + [\gamma k^\theta])^{\alpha-\theta\beta}) \le s_k(A+B)(1 + c_4 k^{\alpha-\theta\beta}).$$

Therefore

$$s_k(A+B) \ge s_{k+[\gamma k^\theta]}(A)(1 - c_4 k^{\alpha-\theta\beta}).$$

If $k \ge (2c_4)^{1/(\theta\beta-\alpha)}$, then $1 - c_4 k^{\alpha-\theta\beta} \ge 1/2$.

Also by (10) for $k \ge 2$

$$s_k(A+B) \le s_{k-[k^\theta]}(A)(1 + c_4 k^{\alpha-\theta\beta}),$$

since $c_4 > c_3$.

Hence, the statement follows for

$$k \ge \varkappa_\gamma = \left[\max\{k_2, \nu_\gamma, (2c_4)^{1/(\theta\beta-\alpha)}, 2\}\right] + 1. \qquad \square$$

Remark 3.2. Applying more delicate estimates one can prove that inequality (15) follows from the inequality $x \le \gamma((x+1)^{1/\theta} - x)^\theta$, $x \ge x_\gamma$, which implies that for $0 < \theta \le 1/2$ inequality (15) and hence inequality (12) also hold with $\gamma = 1$.

Definition 3.1. Let A and B be compact linear operators in a Hilbert space H.

If $c_1, c_2 > 0$, $0 < \alpha < \beta$, and condition (9) holds, we say that in the representation $C = A + B$ the operator A is a *leading* operator and the operator B is a *non-leading* operator with the parameters $c_1, c_2 > 0$, α, and β.

If $0 < \alpha < \beta$ and there exist $c_1, c_2 > 0$ such that condition (9) holds, we say that in the representation $C = A + B$ the operator A is a *leading* operator and the operator B is a *non-leading* operator with the parameters α and β.

If there exist $c_1, c_2 > 0$ and $0 < \alpha < \beta$ such that condition (9) holds, we say that in the representation $C = A + B$ the operator A is a *leading* operator and the operator B is a *non-leading* operator.

Theorem 3.1. *Let A and B be compact linear operators in a Hilbert space H. If in the representation $C = A + B$ the operator A is a leading operator and the operator B is a non-leading operator and for all $0 < \sigma < 1$*

$$\lim_{k\to\infty} \frac{s_{k+[k^\sigma]}(A)}{s_k(A)} = 1, \tag{16}$$

then

$$\lim_{k\to\infty} \frac{s_k(A+B)}{s_k(A)} = 1. \tag{17}$$

Proof. Let A be a leading operator and B a non-leading operator with the parameters α, β and in Lemma 3.2 let $\gamma = 2$ and $\theta = \frac{1}{2}\left(\frac{\alpha}{\beta} + 1\right)$. Moreover, let $\sigma = \frac{1}{2}(\theta + 1)$.

Given that $k \in \mathbb{N}$, let $m = k - [k^\theta]$. By the proof of Lemma 3.2 (see (14) and (15)) $k \le m + 1 + [2(m+1)^\theta]$ for all $k \ge (1 - 2^{-1/\theta})^{1/(\theta-1)}$.

Hence there exists $\varkappa \in \mathbb{N}$, depending only on α and β such that for all $k \ge \varkappa$

$$k + [2k^\theta] \le k + [k^\sigma], \quad k \le m + [m^\sigma].$$

Therefore

$$\frac{s_{k+[k^\sigma]}(A)}{s_k(A)} \le \frac{s_{k+[2k^\theta]}(A)}{s_k(A)} \le 1 \le \frac{s_{k-[k^\sigma]}(A)}{s_k(A)} = \frac{s_m(A)}{s_k(A)} \le \frac{s_m(A)}{s_{m+[m^\sigma]}(A)}.$$

Consequently

$$\lim_{k\to\infty} \frac{s_{k+[2k^\theta]}(A)}{s_k(A)} = \lim_{k\to\infty} \frac{s_{k-[k^\theta]}(A)}{s_k(A)} = 1,$$

and the statement follows by inequality (12). □

Remark 3.3. Assume that, in addition to assumption (9)

$$\lim_{k\to\infty} k^\alpha s_k(A) = a > 0 \tag{18}$$

or, more generally,

$$\lim_{k\to\infty} k^\alpha L(k) s_k(A) = a > 0, \tag{19}$$

where L is an arbitrary positive function on $[1, \infty)$ for which for any $0 < b < \infty$

$$\lim_{k\to\infty} \frac{L(bk)}{L(k)} = 1.$$

Then the statement of Theorem 3.1 follows from Theorem 2.3 in [8], Chapter 2, according to which if (18) or (19) is satisfied and

$$\lim_{k\to\infty} \frac{s_k(B)}{s_k(A)} = 0,$$

then equality (17) holds.

Corollary 3.1. *Let $\mathfrak{A}, \mathfrak{B}$ be families of compact linear operators A, B respectively, in a Hilbert space H. If there exist $c_1, c_2 > 0$ and $0 < \alpha < \beta$ such that in the representations $C = A + B$ for all $A \in \mathfrak{A}, B \in \mathfrak{B}$ the operators A are leading operators and the operators B are non-leading operators with the parameters $c_1, c_2 > 0$ and $0 < \alpha < \beta$ and, for all $0 < \sigma < 1$, in* (16) *convergence is uniform with respect to $A \in \mathfrak{A}$, then in* (17) *convergence is uniform with respect to $A \in \mathfrak{A}$, $B \in \mathfrak{B}$.*

Proof. This proof follows from the proof of Theorem 3.1 and from Lemma 3.2, because the parameter $\varkappa$ in the above proof depends only on α and β, and the parameter c_4 in inequality (10) depends only on $c_1, c_2 > 0$, α, and β. □

§ 4. Some estimates for singular numbers of elliptic operators

Let $l \in \mathbb{N}$ and for $u \in C^\infty(\mathbb{R}^n)$

$$(\mathcal{L}u)(x) = \sum_{|\alpha|,|\beta|\le l} (-1)^{|\alpha|+|\beta|} D^\alpha (A_{\alpha\beta}(x) D^\beta u), \quad x \in \mathbb{R}^n, \tag{20}$$

where $A_{\alpha\beta} \in C^l(\mathbb{R}^n)$ are real-valued functions for all multi-indices α, β satisfying $|\alpha|, |\beta| \le l$. We assume the uniform ellipticity condition is satisfied, i. e. for some $\nu > 0$

$$\sum_{|\alpha|=|\beta|=l} A_{\alpha\beta}(x) \xi^\alpha \xi^\beta \ge \nu |\xi|^{2l} \tag{21}$$

for all $x, \xi \in \mathbb{R}^n$.

We shall consider the closures of $\mathcal{L}$ in $L_2(\Omega)$ for bounded domains $\Omega \subseteq \mathbb{R}^n$ which will be denoted by L_Ω.

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain.*

1. The singular numbers[1] *$s_k(A)$ of any correct restriction A of the operator L_Ω defined above whose inverse A^{-1} is compact, satisfy the following inequality: there exists $c_1 > 0$ such that for all $k \in \mathbb{N}$*

$$s_k(A) \le c_1 k^{2l/n}. \tag{22}$$

2. There exists a correct restriction A of the operator L_Ω, $k_0 \in \mathbb{N}$, and $c_2, c_3 > 0$ such that for all $k \in \mathbb{N}$, $k \ge k_0$

$$c_2 k^{2l/n} \le s_k(A) \le c_3 k^{2l/n}. \tag{23}$$

Proof. 1. Let $B_1 \subseteq \Omega$ be an open ball. Moreover, let $P_1 \colon D(P_1) \to L_2(\Omega)$ be the operator generated by L_Ω with the homogeneous Dirichlet boundary conditions, i. e.

$$D(P_1) = \left\{ u \in W_2^{2l}(B_1) \colon \operatorname{tr}_{\partial B_1} \left(\frac{\partial^m f}{\partial n^m} \right) = 0,\ m = 0, 1, \dots, l-1 \right\}$$

and $P_1 u = L_\Omega u$ for each $u \in D(P_1)$. It is well known that the operator P_1 has discrete spectrum and and its singular numbers satisfy the following inequality: there exist $k_0 \in \mathbb{N}$ and $c_4, c_5 > 0$ such that for all $k \in \mathbb{N}, k \ge k_0$

$$c_4 k^{2l/n} \le s_k(P_1) \le c_5 k^{2l/n}. \tag{24}$$

2. Let $T \colon W_2^{2l}(B_1) \to W_2^{2l}(\mathbb{R}^n)$ be a bounded linear extension operator. Then for any correct restriction A of L_Ω and for any $u \in D(P_1)$

$$\|ATu\|_{L_2(\Omega)} \le c_6 \|Tu\|_{W_2^{2l}(\Omega)} \le c_6 c_7 \|u\|_{W_2^{2l}(B_1)} \le c_6 c_7 c_8 \|P_1 u\|_{L_2(B_1)}.$$

Here

$$c_6 = \widetilde{c}_6 \max_{|\alpha|,|\beta| \le l} \max_{|\gamma| \le l} \sup_{x \in \Omega} |(D^\gamma A_{\alpha\beta})(x)|,$$

where $\widetilde{c}_6$ depends only on n and l, c_7 is the norm of the extension operator T, and c_8 is the norm of the inverse[2] operator $P_1^{-1} \colon L_2(B_1) \to W_2^{2l}(B_1)$.

By the Min-Max principle it follows that for all $k \in \mathbb{N}$

$$\begin{aligned} s_k(A) &= \inf_{\substack{M \subseteq D(A) \\ \dim M = k}} \sup_{v \in M,\, v \not\sim 0} \frac{\|Av\|_{L_2(\Omega)}}{\|v\|_{L_2(\Omega)}} \le \inf_{\substack{M \subseteq D(P_1) \\ \dim M = k}} \sup_{u \in M,\, u \not\sim 0} \frac{\|ATu\|_{L_2(\Omega)}}{\|Tu\|_{L_2(\Omega)}} \\ &\le c_6 c_7 c_8 \inf_{\substack{M \subseteq D(P_1) \\ \dim M = k}} \sup_{u \in M,\, u \not\sim 0} \frac{\|P_1 u\|_{L_2(B_1)}}{\|u\|_{L_2(B_1)}} = c_6 c_7 c_8 s_k(P_1) \le c_5 c_6 c_7 c_8 k^{2l/n}. \end{aligned}$$

3. Let B_2 be an open ball such that $\Omega \subseteq B_2$ and let P_2 be the operator similar to P_1 but defined on B_2. Moreover, let $T_0 \colon L_2(\Omega) \to L_2(B_2)$ be the extension-by-zero operator and $R_\Omega \colon L_2(B_2) \to L_2(\Omega)$ be the restriction operator. We define

$$B = R_\Omega P_2^{-1} T_0. \tag{25}$$

Then B is a bounded linear operator for which

$$L_\Omega B = L_\Omega R_\Omega P_2^{-1} T_0 = R_\Omega P_2 P_2^{-1} T_0 = R_\Omega T_0 = I$$

[1] As usual it is assumed that all singular numbers of A are written in the form of a sequence $\{s_k(A)\}_{k=1}^\infty$, where $s_1(A) \le s_2(A) \le \dots \le s_k(A) \le \dots$ and each singular number is repeated as many times as its multiplicity.

[2] Here and in similar situations in the sequel we assume, without loss of generality, that 0 does not belong to the spectrum of P_1. Otherwise, we may consider the operator $P_1 + \mu I$ with the appropriately chosen $\mu \in \mathbb{R}$.

and

$$R(B) \subseteq R_\Omega(D(P_2)) \subseteq W_2^{2l}(\Omega) \subseteq D(L_\Omega).$$

By Remark 2.1 B is the inverse of a certain correct restriction A of the operator L_Ω. So

$$A^{-1} = R_\Omega P_2^{-1} T_0. \tag{26}$$

This implies that there exist $c_9, c_{10} > 0$ such that $c_9 s_k(P_2) \le s_k(A) \le c_{10} s_k(P_2)$ for all $k \in \mathbb{N}$. Hence, by (24), where the operator P_1 and the ball B_1 are replaced by P_2 and B_2, the statement follows. □

Remark 4.1. Let Ω contain an open ball $B(x,r)$ of radius $r > 0$ for some $x \in \Omega$ and be contained in the ball $B(0,R)$. Moreover, let

$$\tau = \max_{|\alpha|,|\beta| \le l} \max_{|\gamma| \le l} \sup_{x \in B(0,R)} |(D^\gamma A_{\alpha,\beta})(x)|.$$

Then the above proof implies that it may be assumed that in inequality (22) c_1 depends only on n, l, r, R, τ, and the ellipticity constant ν.

§ 5. Classes of domains

For any set V in $\mathbb{R}^n$ and $\delta > 0$ we denote by V_δ the set $\{x \in V : d(x, \partial\Omega) > \delta\}$ and by V^δ the δ-neighbourhood of V.

Let $n \in \mathbb{N}$, $n \ge 2$, $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \le s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^n$.

We say that that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^n$ with the parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^n$.

Let $l \in \mathbb{N}$ and $M > 0$. We denote by $C_M^l(\mathcal{A})$ the family of all bounded domains Ω in $\mathbb{R}^n$ satisfying the following properties:

(i) $\Omega \subseteq \bigcup_{j=1}^s (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \ne \emptyset$;

(ii) $V_j \cap \partial\Omega \ne \emptyset$ for $j = 1, \dots, s'$, $V_j \cap \partial\Omega = \emptyset$ for $s' < j \le s$;

(iii) for $j = 1, \dots, s$

$$r_j(V_j) = \{x \in \mathbb{R}^n : a_{ij} < x_i < b_{ij},\ i = 1, \dots, n\}$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^n : a_{nj} < x_n < g_j(\bar{x}),\ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, \dots, x_{n-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{n-1} : a_{ij} < x_i < b_{ij},\ i = 1, \dots, n-1\}$ and $g_j \in C^l(\bar{W}_j)$ (it is meant that if $s' < j \le s$, then $g_j(\bar{x}) = b_{nj}$ for all $\bar{x} \in \bar{W}_j$); moreover, for $j = 1, \dots, s'$

$$a_{nj} + \rho \le g_j(\bar{x}) \le b_{nj} - \rho$$

and

$$|D^\alpha g_j(\bar{x})| \le M, \quad 1 \le |\alpha| \le l,$$

for all $\bar{x} \in \bar{W}_j$.

We say that a domain Ω in $\mathbb{R}^n$ is of class C^l (or with boundary of class C^l) if $\Omega \in C_M^l(\mathcal{A})$ for some atlas $\mathcal{A}$ and $M > 0$.

§ 6. Representation of the inverse of a correct restriction as the sum of a leading and a non-leading operator

In this section we prove the core result of the paper.

Theorem 6.1. *Let $l, n \in \mathbb{N}$, $n \geq 2$ and Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^{2l}.*

Let A and B be correct restrictions of the operator L_Ω, defined in Section 4, such that

$$D(A) \subseteq W_2^{2l}(\Omega), \tag{27}$$

$$D(B) \subseteq W_2^{s}(\Omega), \tag{28}$$

where[3]

$$2l\Big(1 - \frac{1}{n}\Big) < s \leq 2l, \tag{29}$$

and the operators

$$A^{-1}\colon L_2(\Omega) \to W_2^{2l}(\Omega), \tag{30}$$

$$B^{-1}\colon L_2(\Omega) \to W_2^{s}(\Omega) \tag{31}$$

are bounded.

Then in the representation

$$B^{-1} = A^{-1} + K, \tag{32}$$

the operator A^{-1} is a leading operator and the operator $K = B^{-1} - A^{-1}$ is a non-leading operator with the parameters $\alpha = 2l/n$ and any $2l/n < \beta < s/(n-1)$.

Proof. 1. We start by stating several corollaries of the general results on the solvability of the boundary value problems proved in [10]. Here we use the definitions of Sobolev spaces $W_2^r(\partial\Omega)$ for arbitrary real r given in that book.

Let

$$B_m u = \operatorname{tr}_{\partial\Omega}\Big(\frac{\partial^m u}{\partial n^m}\Big), \quad m = 0, 1, \dots, l-1$$

and $0 < \sigma \leq 2l$. Then the following Dirichlet problem

$$\begin{cases} u \in W_2^{\sigma}(\Omega), \\ L_\Omega u = 0 \text{ on } \Omega, \\ B_m u = h_m \text{ on } \partial\Omega,\ m = 0, 1, \dots, l-1, \\ h_m \in W_2^{\sigma - m - 1/2}(\partial\Omega) \end{cases} \tag{33}$$

has a unique solution[4]. Moreover, there exists $c_1 > 0$ such that

$$\|u\|_{W_2^{\sigma}(\Omega)} \leq c_1 \sum_{m=0}^{l-1} \|h_m\|_{W_2^{\sigma-m-1/2}(\partial\Omega)} \tag{34}$$

for all collections of functions $h_m \in W_2^{\sigma-m-1/2}(\partial\Omega)$, $m = 0, 1, \dots, l-1$. Note that, for sufficiently small positive σ, $\sigma - m - 1/2 < 0$.

---

[3]For $s > 0$, $s \notin \mathbb{N}$, $W_2^s(\Omega)$ is the Sobolev space of fractional order with the norm

$$\|u\|_{W_2^s(\Omega)} = \Bigg(\sum_{|\alpha| \leq [s]} \int_\Omega |D_w^\alpha u(x)|^2\, dx + \sum_{|\alpha| = [s]} \int_\Omega \int_\Omega \frac{|D_w^\alpha u(x) - D_w^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}}\, dx\, dy \Bigg)^{1/2},$$

where $D_w^\alpha u$ denotes the weak derivative of u of order α on Ω.

[4]We assume that 0 does not belong to the spectrum of the corresponding Dirichlet operator. See footnote 2.

Assume that $0 < \mu < s$. We shall use the above statement for $\sigma = s$ and $\sigma = \mu$. If $\sigma = \mu$ this implies, in particular, that for each $j = 0, 1, \dots, l-1$ the Dirichlet problem

$$\begin{cases} u \in W_2^{\mu}(\Omega), \\ L_\Omega u = 0 \text{ on } \Omega, \\ B_m u = 0 \text{ on } \partial\Omega, \ m = 0, 1, \dots, j-1, j+1, \dots, l-1, \\ B_j u = h \text{ on } \partial\Omega, \\ h \in W_2^{\mu-j-1/2}(\partial\Omega) \end{cases} \tag{35}$$

has a unique solution and the corresponding inverse operator

$$R_j \colon W_2^{\mu-j-1/2}(\partial\Omega) \to W_2^{\mu}(\Omega)$$

is bounded.

Note that the solution of the problem (33) with $\sigma = \mu$ can be represented in the form

$$u = \sum_{j=0}^{l-1} R_j h_j.$$

Since for each function $f \in L_2(\Omega)$ by (30) and (31) we have $Kf = B^{-1}f - A^{-1}f \in W_2^s(\Omega)$, $L_\Omega Kf = 0$ on Ω and by the trace theorem for Sobolev spaces (note that by (29) $s > l - 1/2$)

$$B_m Kf \in W_2^{s-m-1/2}(\partial\Omega) \subseteq W_2^{\mu-m-1/2}(\partial\Omega), \quad m = 0, 1, \dots, l-1,$$

this implies that

$$Kf = \sum_{j=0}^{l-1} R_j B_j Kf, \quad f \in L_2(\Omega). \tag{36}$$

2. We shall make use of the following fact: for a compact linear operator $G \colon H_1 \to H_2$ acting from a Hilbert space H_1 to a Hilbert space H_2,

$$s_1(G) = \|G\|; \quad s_k(G) = \min_{\dim F = k-1} \|A - F\|, \quad k \geq 2,$$

where the minimum is taken with respect to all linear operators $F \colon H_1 \to H_2$ of dimension $k-1$. (See [1] and [8, Theorem 2.1].)

3. Consider the embedding operator

$$E_1 \colon W_2^{\mu}(\Omega) \to L_2(\Omega).$$

This operator is compact and its singular numbers $s_k(E_1)$ have the order $k^{-\mu/n}$. Hence, there exist linear operators $M_k \colon W_2^{\mu}(\Omega) \to L_2(\Omega)$ of dimension k, $k \in \mathbb{N}$, such that

$$\|(E_1 - M_k)g\|_{L_2(\Omega)} = s_{k+1}(E_1) \leq c_2\,(k+1)^{-\mu/n}\,\|g\|_{W_2^{\mu}(\Omega)}, \tag{37}$$

where $c_2 > 0$ is independent of k and $g \in W_2^{\mu}(\Omega)$.

Consider also the embedding operators

$$E_{2j} \colon W_2^{s-j-1/2}(\partial\Omega) \to W_2^{\mu-j-1/2}(\partial\Omega), \quad j = 0, 1, \dots, l-1.$$

These operators are compact and their singular numbers $s_k(E_{2j})$ have the same order $k^{-(s-\mu)/(n-1)}$. (See [12].) Hence, there exist linear operators

$$F_{kj} \colon W_2^{s-j-1/2}(\partial\Omega) \to W_2^{\mu-j-1/2}(\partial\Omega)$$

of dimension k, $k \in \mathbb{N}$, such that

$$\|(E_{2j} - F_{kj})g\|_{W_2^{s-j-1/2}(\partial\Omega)} = s_{k+1}(E_{2j}) \leq c_3\,(k+1)^{-(s-\mu)/(n-1)}\,\|g\|_{W_2^{\mu-j-1/2}(\partial\Omega)}, \tag{38}$$

where $c_3 > 0$ is independent of j, k and $g \in W_2^{\mu-j-1/2}(\partial\Omega)$.

4. Denote by T_k, $k \in \mathbb{N}$, the operators defined by

$$T_k f = Kf - \sum_{j=0}^{l-1} (E_1 - M_k) R_j (E_{2j} - F_{kj}) B_j K f, \quad f \in L_2(\Omega). \tag{39}$$

By (36)

$$\sum_{j=0}^{l-1} E_1 R_j E_{2j} B_j K f = \sum_{j=0}^{l-1} R_j B_j K f = Kf.$$

Therefore

$$T_k f = \sum_{j=0}^{l-1} E_1 R_j F_{kj} B_j K f + M_k \Big(\sum_{j=0}^{l-1} R_j (E_{2j} - F_{kj}) B_j K f \Big).$$

Hence, $T_k \colon L_2(\Omega) \to L_2(\Omega)$ are linear operators of dimension not exceeding $(l+1)k$.

Moreover, by (37)

$$\begin{aligned}
\|(K - T_k) f\|_{L_2(\Omega)} &= \Big\| \sum_{j=0}^{l-1} (E_1 - M_k) R_j (E_{2j} - F_{kj}) B_j K f \Big\|_{L_2(\Omega)} \\
&\le \sum_{j=0}^{l-1} \big\| (E_1 - M_k) R_j (E_{2j} - F_{kj}) B_j K f \big\|_{L_2(\Omega)} \\
&\le c_2 \, (k+1)^{-\mu/n} \sum_{j=0}^{l-1} \big\| R_j (E_{2j} - F_{kj}) B_j K f \big\|_{W_2^{\mu}(\Omega)} \\
&\le c_2 c_4 \, (k+1)^{-\mu/n} \sum_{j=0}^{l-1} \big\| (E_{2j} - F_{kj}) B_j K f \big\|_{W_2^{\mu-j-1/2}(\partial\Omega)},
\end{aligned}$$

where

$$c_4 = \max_{j=0,\dots,l-1} \|R_j\|_{W_2^{\mu-j-1/2}(\partial\Omega) \to W_2^{\mu}(\Omega)}.$$

Next, by (38) and the trace theorem for Sobolev spaces

$$\begin{aligned}
&\|(K - T_k) f\|_{L_2(\Omega)} \le c_2 c_3 c_4 \, (k+1)^{-\mu/n - (s-\mu)/(n-1)} \sum_{j=0}^{l-1} \|B_j K f\|_{W_2^{s-j-1/2}(\partial\Omega)} \\
&\le c_2 c_3 c_4 c_5 \, (k+1)^{-\frac{1}{n-1}(s-\mu/n)} \, \|Kf\|_{W_2^s(\Omega)} \le c_2 c_3 c_4 c_5 c_6 \, (k+1)^{-\frac{1}{n-1}(s-\mu/n)} \, \|f\|_{L_2(\Omega)},
\end{aligned}$$

where

$$\begin{aligned}
c_5 &= \sum_{j=0}^{l-1} \|B_j\|_{W_2^s(\Omega) \to W_2^{s-j-1/2}(\partial\Omega)}, \\
c_6 &= \|K\|_{L_2(\Omega) \to W_2^s(\Omega)} \le \|B^{-1}\|_{L_2(\Omega) \to W_2^s(\Omega)} + \|A^{-1}\|_{L_2(\Omega) \to W_2^s(\Omega)} \\
&\qquad \le \|B^{-1}\|_{L_2(\Omega) \to W_2^s(\Omega)} + c_7 \|A^{-1}\|_{L_2(\Omega) \to W_2^{2l}(\Omega)}
\end{aligned} \tag{40}$$

and c_7 is the norm of the embedding operator $E \colon W_2^{2l}(\Omega) \to W_2^s(\Omega)$.

5. Given $\frac{2l}{n} < \beta < \frac{s}{n-1}$, we choose $0 < \mu < s$ such that $\frac{1}{n-1}\Big(s - \frac{\mu}{n}\Big) = \beta$. This is possible if and only if $s > 2l\Big(1 - \frac{1}{n}\Big)$. So for all $\frac{2l}{n} < \beta < \frac{s}{n-1}$

$$\|(K - T_k) f\|_{L_2(\Omega)} \le c_8 \, (k+1)^{-\beta} \, \|f\|_{L_2(\Omega)},$$

where

$$c_8 = c_2c_3c_4c_5\big(\|B^{-1}\|_{L_2(\Omega)\to W_2^s(\Omega)} + c_7\|A^{-1}\|_{L_2(\Omega)\to W_2^{2l}(\Omega)}\big)$$

is independent of $k \in \mathbb{N}$ and $f \in L_2(\Omega)$.

Since $\dim T_{[\frac{k-1}{l+1}]} \le k-1$ we have

$$\begin{aligned} s_k(K) \le \big\|K - T_{[\frac{k-1}{l+1}]}\big\|_{L_2(\Omega)\to L_2(\Omega)} &\le c_8\Big(\Big[\frac{k-1}{l+1}\Big]+1\Big)^{-\beta} \\ &\le c_8(l+1)^\beta(k-1)^\beta \le c_8(2(l+1))^\beta k^{-\beta} \end{aligned} \tag{41}$$

for all $k \ge 2$. This inequality also holds for $k=1$ because without loss of generality one may assume that $c_2, c_3, c_4, c_5 \ge 1$, in which case $s_1(K) = \|K\| \le c_8$.

Finally, by Lemma 4.1 it follows that for all $k \in \mathbb{N}$

$$s_k(A^{-1}) \ge c_1^{-1}k^{-2l/n}, \tag{42}$$

where c_1 is from inequality (22), which implies the statement. □

Analysis of the proof. *Estimate* (37) *with* $\mu = s$ *implies immediately that for* $k \ge 2$

$$\begin{aligned} s_k(K) &\le \|K - M_{k-1}K\|_{L_2(\Omega)\to L_2(\Omega)} = \\ &= \|(E_1 - M_{k-1})K\|_{L_2(\Omega)\to L_2(\Omega)} \le c_2\|K\|_{W_2^s(\Omega)\to L_2(\Omega)}k^{-s/n}. \end{aligned} \tag{43}$$

This is a rather rough estimate and combined with estimate (42) *it does not allow us to claim that* K *is a non-leading operator. However, this estimate holds for any bounded linear operator* $K\colon W_2^s(\Omega) \to L_2(\Omega)$ *whilst the operator* $K = B^{-1} - A^{-1}$ *under consideration apart from that is such that* $L_\Omega K = 0$. *This essential fact enables us to improve significantly estimate* (43). *In the above proof this is done by considering* Kf *as a solution of the Dirichlet boundary value problem for the operator* L_Ω *with the boundary data* $B_mKf \in W_2^{s-m-1/2}(\partial\Omega)$, $m = 0, \dots, l-1$, *where* $s - m - 1/2 > 0$, *using estimate* (38) *with any* $0 < \mu < s$ *and deep results from* [10] *on solvability of this Dirichlet problem for the case in which the boundary data belongs to Sobolev spaces of negative order.*

Example 6.1. In Example 2.2 let $\varphi_1 \in W_2^s(B)$ where $2 - 2/n < s \le 2$. Consider the operator $L_s\colon D(L_s) \to L_2(B)$ where

$$D(L_s) = \Big\{u \in L_2(B)\colon -\Delta_w u \in L_2(B),\ \mathrm{tr}_{\partial B}\Big(u(x) + \varphi_1(x)\int\limits_B \varphi_2(y)\Delta_w u(y)\,dy\Big) = 0\Big\}$$

and $L_s u = -\Delta_w u$ for $u \in D(L_s)$. By Examples 2.1 and 2.2 L_s is a correct restriction of the operator L. Moreover, the operator $L_s^{-1}\colon L_2(B) \to W_2^s(B)$ is bounded. Therefore by Theorem 6.1 the singular numbers $s_k(L_s)$ have the same asymptotics as those of the Dirichlet Laplacian:

$$\lim_{k\to\infty} s_k(L_s)k^{-2/n} = \lim_{k\to\infty} s_k(-\Delta_{\mathrm{D}})k^{-2/n}.$$

Corollary 6.1. *Let* $l, n \in \mathbb{N}$, $n \ge 2$, $2l\Big(1 - \frac{1}{n}\Big) < s \le 2l$, $\alpha = \frac{2l}{n}$, *and* $\frac{2l}{n} < \beta < \frac{s}{n-1}$. *Moreover, let* $\mathcal{A}$ *be a fixed atlas in* $\mathbb{R}^n$ *and* $M > 0$.

Assume that $G, H \subseteq C_M^{2l}(\mathcal{A})$ *and* $\mathfrak{A}(\mathcal{A}) = \{A_\Omega\}_{\Omega\in G}$, $\mathfrak{B}(\mathcal{A}) = \{B_\Omega\}_{\Omega\in H}$ *are families of correct restrictions* A_Ω *and* B_Ω *of the operator* L_Ω *defined in Section* 4, *satisfying the conditions*

$$D(A_\Omega) \subseteq W_2^{2l}(\Omega), \quad A_\Omega \in \mathfrak{A}(\mathcal{A}), \tag{44}$$

$$D(B_\Omega) \subseteq W_2^{s}(\Omega), \quad B_\Omega \in \mathfrak{B}(\mathcal{A}), \tag{45}$$

and

$$\theta_{\mathfrak{A}(\mathcal{A})} = \sup_{A_\Omega \in \mathfrak{A}(\mathcal{A})} \|A_\Omega^{-1}\|_{L_2(\Omega)\to W_2^{2l}(\Omega)} < \infty, \tag{46}$$

$$\theta_{\mathfrak{B}(\mathcal{A})} = \sup_{B_\Omega \in \mathfrak{B}(\mathcal{A})} \|B_\Omega^{-1}\|_{L_2(\Omega)\to W_2^{s}(\Omega)} < \infty. \tag{47}$$

Then there exist $c_9, c_{10} > 0$ such that for all $A_\Omega \in \mathfrak{A}(\mathcal{A})$, $B_\Omega \in \mathfrak{B}(\mathcal{A})$ in the representations

$$B_\Omega^{-1} = A_\Omega^{-1} + K_\Omega$$

the operators A_Ω^{-1} are leading and the operators K_Ω are non-leading with the parameters c_9, c_{10}, α, and β.

Proof. By Remark 4.1 it follows that there exists $c_9 > 0$, depending only on n, l, the operator $\mathcal{L}$, the atlas $\mathcal{A}$, and M such that

$$s_k(A_\Omega^{-1}) = s_k(A_\Omega)^{-1} \geq c_9 k^{-2l/n}$$

for all $k \in \mathbb{N}$ and for all $A_\Omega \in \mathfrak{A}(\mathcal{A})$.

Tracing the proofs in [2], [10], and [12] and it follows that in the proof of Theorem 6.1 it may be assumed that c_2, c_3, c_5, c_7 depend only of n, l, s, β, the atlas $\mathcal{A}$, and M. As for c_1 and c_4, they also depend on the operator $\mathcal{L}$. Hence, by (40), (41), (46), and (47),

$$s_k(K) \leq c_{10} k^{-\beta},$$

where

$$c_{10} = c_2 c_3 c_4 c_5 (\theta_{\mathfrak{B}(\mathcal{A})} + c_7\, \theta_{\mathfrak{A}(\mathcal{A})})(2(l+1))^\beta$$

for all $k \in \mathbb{N}$ and for all $A_\Omega \in \mathfrak{A}(\mathcal{A})$ and $B_\Omega \in \mathfrak{B}(\mathcal{A})$. □

§ 7. Comparison of the asymptotics of singular numbers of correct restrictions of elliptic operators

Theorem 7.1. *Let $l, n \in \mathbb{N}$, $n \geq 2$, $2l(1-1/n) < s \leq 2l$ and Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^{2l}.*

Then there exists $b > 0$ such that for all correct restrictions B of the operator L_Ω, defined in Section 4, satisfying conditions (28) and (31)

$$\lim_{k\to\infty} s_k(B) k^{-2l/n} = b. \tag{48}$$

Proof. Let the correct restriction A of the operator L_Ω be defined by

$$D(A) = \left\{ u \in W_2^{2l}(\Omega) \colon \operatorname{tr}_{\partial\Omega}\left(\frac{\partial^m u}{\partial n^m}\right) = 0,\ m = 0, 1, \ldots, l-1 \right\}$$

and $Au = L_\Omega u$ for each $u \in D(A)$. Without loss of generality we assume that the operator A has compact inverse A^{-1} (see footnote 2). It is well known that its singular numbers $s_k(A^{-1})$ are such that

$$\lim_{k\to\infty} s_k(A^{-1}) k^{2l/n} = a$$

for some $a > 0$.

Let B be any correct restriction of the operator L_Ω satisfying conditions (28) and (31). By Theorem 6.1 $B^{-1} = A^{-1} + K$, where A^{-1} is a leading operator and K is a non-leading operator. Since condition (16) is satisfied by Theorem 3.1

$$\lim_{k\to\infty} \frac{s_k(B^{-1})}{s_k(A^{-1})} = 1.$$

Consequently

$$\lim_{k\to\infty} s_k(B^{-1}) k^{2l/n} = \lim_{k\to\infty} s_k(A^{-1}) k^{2l/n} = a$$

and

$$\lim_{k\to\infty} s_k(B)k^{-2l/n} = b \equiv \frac{1}{a}. \qquad \square$$

Corollary 7.1. *Let $l, n \in \mathbb{N}$, $n \geq 2$, $2l(1-1/n) < s \leq 2l$, $\mathcal{A}$ be a fixed atlas in $\mathbb{R}^n$, and $M > 0$. Assume that $H \subseteq C_M^{2l}(\mathcal{A})$ and $\mathfrak{B}(\mathcal{A}) = \{B_\Omega\}_{\Omega\in H}$ is a family of correct restrictions B_Ω of the operator L_Ω defined in Section 4, satisfying conditions* (28) *and* (47).

Then the expressions $s_k(B_\Omega)k^{-2l/n}$ converge to b as $k \to \infty$ uniformly with respect to $B_\Omega \in \mathfrak{B}(\mathcal{A})$.

Proof. Let the family $\mathfrak{A}(\mathcal{A})$ consist of only one operator A defined above. Since condition (16) is satisfied the statement follows by Corollaries 3.1 and 6.1. $\square$

§ 8. Application to spectral stability estimates

In this section we restrict ourselves to considering the operator $\mathcal{L}$ for the case in which $l = 1$, $A_{\alpha\beta} = A_{\beta\alpha}$ and there are no lower terms. So for $u \in C^\infty(\mathbb{R}^n)$

$$\mathcal{L}u = -\sum_{i,\,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big), \quad x \in \mathbb{R}^n,$$

where $a_{ij} \in C^1(\mathbb{R}^n)$ are real-valued functions satisfying $a_{ij} = a_{ji}\ \forall\, i, j = 1, \dots, n$, and

$$\sum_{i,\,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$$

for all $\xi, x \in \mathbb{R}^n$. As in previous sections, for bounded domains $\Omega \subseteq \mathbb{R}^n$, L_Ω denotes the closure of $\mathcal{L}$ in $L_2(\Omega)$.

Theorem 8.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $2-2/n < s \leq 2$, $\mathcal{A}$ be a fixed atlas in $\mathbb{R}^n$, and $M > 0$.*

Moreover, let $\mathfrak{B}(\mathcal{A}) = \{B_\Omega\}_{\Omega\in C_M^2(\mathcal{A})}$ be a family of correct restrictions B_Ω of the operator L_Ω defined above, satisfying conditions (28) *and* (47).

Then there exist $\varepsilon_1, c_1 > 0$ and for each $\varepsilon \in (0, \varepsilon_1]$ there exists $k(\varepsilon) \in \mathbb{N}$ such that for all $k \geq k(\varepsilon)$

$$|s_k(B_{\Omega_1}) - s_k(B_{\Omega_2})| \leq c_1 k^{2/n}\varepsilon \tag{49}$$

for all $\Omega_1, \Omega_2 \in C_M^2(\mathcal{A})$ satisfying

$$(\Omega_1)_\varepsilon \subseteq \Omega_2 \subseteq (\Omega_1)^\varepsilon \quad \text{or} \quad (\Omega_2)_\varepsilon \subseteq \Omega_1 \subseteq (\Omega_2)^\varepsilon. \tag{50}$$

Proof. 1. Let $\mathfrak{A}(\mathcal{A}) = \{A_\Omega\}_{\Omega\in C_M^2(\mathcal{A})}$ be the family of the correct restrictions A_Ω of the operator L_Ω defined by

$$D(A_\Omega) = \{u \in W_2^2(\Omega)\colon\ \operatorname{tr}_{\partial\Omega} u = 0\}$$

and $A_\Omega u = L_\Omega u$ for each $u \in D(A_\Omega)$. By Theorem 7.15 in [3] there exist $\varepsilon_2, c_2 > 0$, depending only on $\mathcal{A}$ and L, such that the eigenvalues $\lambda_k(A_\Omega)$ satisfy the following estimate: for all $k \in \mathbb{N}$ for all $\varepsilon \in (0, \varepsilon_2]$

$$|\lambda_k(A_{\Omega_1}) - \lambda_k(A_{\Omega_2})| \leq c_2\, \lambda_k(A_{\Omega_1})\varepsilon$$

for all $\Omega_1, \Omega_2 \in C_M^2(\mathcal{A})$ satisfying condition (50).

Note also that there exists $c_3 > 0$, depending only on n, l, $\mathcal{A}$ and L, such that for all $k \in \mathbb{N}$

$$\lambda_k(A_\Omega) \leq c_3 k^{2/n} \tag{51}$$

for all $A_\Omega \in \mathfrak{A}(\mathcal{A})$.

This follows from Remark 4.1 since each $\Omega \in C_M^2(\mathcal{A})$ contains an open ball $B(x, \varrho/2)$ for some $x \in \Omega$ and is contained in the ball $B(0, R)$, where $R > 0$ depends only on $\{V_j\}_{j=1}^s$.

2. Next for all $k \in \mathbb{N}$ and for all $\Omega_1, \Omega_2 \in C_M^2(\mathcal{A})$

$$\begin{aligned}|s_k(B_{\Omega_1}) - s_k(B_{\Omega_2})| &\le |s_k(B_{\Omega_1}) - \lambda_k(A_{\Omega_1})| + |\lambda_k(A_{\Omega_1}) - \lambda_k(A_{\Omega_2})| + |s_k(B_{\Omega_2}) - \lambda_k(A_{\Omega_2})| \\ &\le \Big|\frac{s_k(B_{\Omega_1})}{\lambda_k(A_{\Omega_1})} - 1\Big|\, \lambda_k(A_{\Omega_1}) + c_2\, \lambda_k(A_{\Omega_1})\, \varepsilon + \Big|\frac{s_k(B_{\Omega_2})}{\lambda_k(A_{\Omega_2})} - 1\Big|\, \lambda_k(A_{\Omega_2}) \\ &\le c_6\Big(\Big|\frac{s_k(B_{\Omega_1})}{s_k(A_{\Omega_1})} - 1\Big| + \varepsilon + \Big|\frac{s_k(B_{\Omega_2})}{s_k(A_{\Omega_2})} - 1\Big|\Big) k^{2/n},\end{aligned}$$

where $c_6 = \max\{c_2, c_3\}$.

The statement follows since by Corollary 7.1

$$\frac{s_k(B_{\Omega_1})}{s_k(A_{\Omega_1})} = \frac{s_k(B_{\Omega_1})k^{-2/n}}{s_k(A_{\Omega_1})k^{-2/n}} \quad \text{and} \quad \frac{s_k(B_{\Omega_2})}{s_k(A_{\Omega_2})} = \frac{s_k(B_{\Omega_2})k^{-2/n}}{s_k(A_{\Omega_2})k^{-2/n}}$$

converge as $k \to \infty$ to 1 uniformly with respect to $A_{\Omega_1}, A_{\Omega_2} \in \mathfrak{A}(\mathcal{A})$ and $B_{\Omega_1}, B_{\Omega_2} \in \mathfrak{B}(\mathcal{A})$. □

Concluding remarks. A natural question arises related to the statements of Theorems 6.1 and 7.1. Is the assumption $s > 2l(1 - 1/n)$ sharp? We conjecture that it is.

In this paper we are dealing with linear operators in Hilbert spaces. It is of interest to extend the results of the paper to the non-Hilbert case and to some classes of nonlinear operators.

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