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REAL-ANALYTIC SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We establish that the Riemann problem on the factorization of formal matrix-valued Laurent series subject to unitary symmetry has a solution. As an application, we show that any local real-analytic solution (in x and t) of the focusing nonlinear Schrödinger equation has a real-analytic extension to some strip parallel to the x-axis and that in each such strip there exists a solution that cannot be extended further.

§ 1. Introduction and formulation of the results

In this article, we take the term *nonlinear Schrödinger equation* (NSE) to mean a partial differential equation of the form

$$iu_t = Au_{xx} + (B|u|^2 + C)u, \tag{1}$$

where $A, B, C \in \mathbb{R}$ are given constants, $AB \neq 0$, and $u\colon D \to \mathbb{C}$ is a smooth function of the variables x and t, which is to be determined in some open subset D of $\mathbb{R}^2$. This equation is integrable by the inverse scattering method (the discovery of this fact in [1] was an important step in the development of the theory of solitons) and has wide applications in nonlinear optics and the theory of wave propagation. From the vast quantity of literature on this equation, we note only the monographs [2], [3] and the recent surveys [4], [5] as evidence of sustained interest in the special classes of real-analytic solutions. In accordance with the physical meaning of the solution, we distinguish the *focusing case* $AB > 0$ and the *defocusing case* $AB < 0$ of equation (1). The following result relates to the first of these.

Theorem 1. *Every real-analytic solution* $u : \Pi \to \mathbb{C}$ *of equation* (1) *with* $AB > 0$, *defined in the rectangle* $\Pi := \{(x,t) \in \mathbb{R}^2 \mid |x - x_0| < \varepsilon_1, |t - t_0| < \varepsilon_2\}$, *extends to a real-analytic solution* $\tilde{u}\colon \Sigma \to \mathbb{C}$ *in the strip* $\Sigma := \{(x,t) \in \mathbb{R}^2 \mid |t - t_0| < \varepsilon_2\}$. *Furthermore, in any strip* Σ *of the given form there exists a real-analytic solution that admits no real-analytic extension through any point of the boundary* $\partial\Sigma$.

Thus, every real-analytic solution of the focusing NSE is naturally defined in some strip[1] parallel to the x-axis. In the defocusing case (that is, when $AB < 0$), a weakened version of the first statement of Theorem 1 holds: we have to replace Σ by $\Sigma \setminus \Gamma(u)$, where $\Gamma(u)$, which depends on the initial solution $u(x,t)$, can be either the empty set or

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[1] This strip can be a half-plane or the whole plane $\mathbb{R}^2$. The latter occurs, for example, for all finite-gap solutions and their limiting cases, as studied in [4], [5], and also for a more general class of solutions with convergent Baker–Akhiezer function (see [6]).

a collection of real-analytic curves,[2] for which $|\tilde{u}(x,t)|$ tends to ∞ as we approach their common point. A detailed description of the possible nature of this limiting process, and also the structure of $\Gamma(u)$ (for example, whether $\Gamma(u)$ has isolated points and whether $\Sigma\setminus\Gamma(u)$ has bounded connected components), for arbitrary real-analytic solutions $u(x,t)$ is an interesting, and in many respects open, question.

We prove Theorem 1 by means of a locally holomorphic variant of the inverse scattering method for soliton equations with a unitary reality condition (for another way of deducing Theorem 1 from the results in [7], see Remark 2 below). In connection with this, we note that it has long been known that the solution of the focusing NSE constructed by means of a rapidly decreasing and finite-gap version of the inverse scattering method has no real singularities: see, for example, [2, Part I, Chapter 2, § 2] and [8, Theorem 1]. The key feature in the arguments they use is the solvability of the Riemann problem, as the inverse scattering transformation is constructed using its solution. In the first case, this was deduced from a theorem of Gokhberg and Krein [9, Theorem 8.1]; in the second case it can be deduced from a sufficient condition based on the symmetry principle [6, Theorem 1]. We establish a sufficient condition of this type, which we need for our purposes, in Theorem 3(A) below by means of the Fredholm alternative for operators in a Banach space isomorphic to l_1. This implies Theorem 3(B), which is a generalization of Theorem 1 to all soliton equations of parabolic type, obtained by the unitary reduction of an equation of zero curvature. We recall the relevant definitions here.

Let $\mathrm{gl}(n,\mathbb{C})$ be the set of all complex $(n\times n)$-matrices, $\mathscr{R}(x_0)$ the set of germs of all holomorphic $\mathrm{gl}(n,\mathbb{C})$-valued mappings at the point $x_0\in\mathbb{C}$, and $\mathscr{R}(x_0)^{\mathrm{od}}$ the set of all off-diagonal germs $q\in\mathscr{R}(x_0)$, that is, germs such that $q_{ll}(x)\equiv 0$ for $l=1,\dots,n$. A mapping $F\colon\mathscr{R}(x_0)\to\mathscr{R}(x_0)$ is called a *differential polynomial* if, for every germ $\varkappa\in\mathscr{R}(x_0)$, the function $F\varkappa$ is an ordinary polynomial (the same for all $\varkappa$) in $\varkappa$ and its derivatives $\varkappa_x,\varkappa_{xx},\dots$.

In what follows, $a,b,c_1,c_2,\ldots\in\mathrm{gl}(n,\mathbb{C})$ are arbitrary diagonal matrices; in addition, a and b have simple spectra (so that all of their eigenvalues are distinct). For any such collection $a,b,c_1,c_2,\dots$, there exists a unique sequence of differential polynomials $F_0,F_1,F_2,\dots$ such that $F_0(\varkappa)\equiv b$, $F_k(0)\equiv c_k$, for all $k\ge 1$, and the formal power series

$$F(\varkappa,z):=\sum_{k=0}^{\infty}F_k(\varkappa)z^{-k}$$

in the new variable z^{-1} satisfies the differential equation

$$\partial_x F(\varkappa,z)=[az+\varkappa,F(\varkappa,z)]$$

identically for $\varkappa\in\mathscr{R}(x_0)^{\mathrm{od}}$; see, for example, [10, Theorem 1] (here and later, $[A,B]:=AB-BA$ denotes the commutator of matrices). The partial differential equation

$$q_t=[a,F_{m+1}(q)], \tag{2}$$

where $m\ge 2$ is a fixed integer and $q(x,t)$ is the unknown off-diagonal holomorphic function in a neighborhood of the point $(x_0,t_0)\in\mathbb{C}^2$, is equivalent (identically in z) to the *zero curvature condition*

$$U_t-V_x+[U,V]=0$$

[2] Generally speaking, with singularities. In addition, $\Sigma\setminus\Gamma(u)$ may turn out to be disconnected. In this case, the analytic extension may be understood thus: for some connected neighborhood $V\subset\mathbb{C}^1_t$ of the interval $(t_0-\varepsilon_2,\,t_0+\varepsilon_2)$, we may specify in the domain $S:=\mathbb{C}^1_x\times V\subset\mathbb{C}^2_{xt}$ a meromorphic function of two complex variables with set P of poles, which coincides with $u(x,t)$ on Π and with $\tilde{u}(x,t)$ on $\Sigma\setminus\Gamma(u)=(S\setminus P)\cap\mathbb{R}^2$.

holding for the connection

$$U(x,t,z)\,dx + V(x,t,z)\,dt,$$

where

$$U(x,t,z) = az + q(x,t) \quad \text{and} \quad V(x,t,z) = \sum_{j=0}^{m} F_{m-j}(q)(x,t)z^j$$

are polynomials in z of degrees 1 and m, respectively.

Soliton equations of parabolic type, of which (1) is an example, can be obtained from equations of the form (2) by imposing additional algebraic conditions on the components of $q(x,t)$, called *reductions* or *reality conditions* (see, for example, [2, Part II, Chapter 1, § 3]. Thus, if we take the matrices c_k to be null when $k \neq 2$ and set

$$a = b = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad c_2 = Ca, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ -\overline{u(\bar{x},\bar{t})} & 0 \end{pmatrix}, \tag{3}$$

from (2) we obtain equation (1), with $A = -1$ and $B = -2$; by scaling the real variables x and t it is easy to reduce the general NSE (1) with $AB > 0$ to this. In this paper we shall only study *unitary reductions*, when all matrices $a, b, c_1, c_2, \dots$ are taken to be skew-Hermitian and the solution $q(x,t)$ of equation (2) must be skew-Hermitian for all real values x, t. In addition to the focusing NSE corresponding to (3), such a reduction is useful in applications to the Hirota equations (see [11, p. 62]) and Sasa-Satsuma equations, and also to higher equations in the corresponding hierarchies.

A direct scattering transformation (in the terminology of [7]) associates to each off-diagonal holomorphic germ $q \in \mathscr{R}(x_0)^{\mathrm{od}}$ an off-diagonal formal power series

$$Lq(z) := \mu(x_0, z) - I, \tag{4}$$

where

$$\mu(x,z) = I + \sum_{k=0}^{\infty} m_k(x) z^{-(k+1)}$$

is the unique solution of the differential equation

$$\mu_x = (az + q(x))\mu - \mu a z$$

in the class of formal power series of the given form with $m_k \in \mathscr{R}(x_0)$, $k \geq 0$, such that $\mu(x_0, z) - I$ is off-diagonal (the existence and uniqueness of such a solution was proved in [7, § 6]). Here and later, I denotes the $n \times n$ identity matrix.

For each $\alpha \geq 0$, we define the *Gevrey class* Gev_α as the set of all formal power series of the form

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-(k+1)},$$

where $\varphi_k \in \mathrm{gl}(n, \mathbb{C})$, such that the power series

$$\sum_{k=0}^{\infty} (k!)^{-\alpha} |\varphi_k| x^k$$

has nonzero radius of convergence, where $|\cdot|$ is any norm on $\mathrm{gl}(n, \mathbb{C})$ with the property $|AB| \leq |A| \cdot |B|$. We denote the set of all off-diagonal $\varphi \in \mathrm{Gev}_\alpha$ by $\mathrm{Gev}_\alpha^{\mathrm{od}}$. We also set $\mathrm{Gev}_{\alpha-0} := \bigcup_{0 \leq s < \alpha} \mathrm{Gev}_s$. The following statement combines Theorems 1 and 2(A) from [7].

Theorem 2. (A) *The mapping $q \mapsto Lq$ is a bijection of $\mathscr{R}(x_0)^{\mathrm{od}}$ onto* $\mathrm{Gev}_1^{\mathrm{od}}$.

(B) *If $q \in \mathscr{R}(x_0)^{\mathrm{od}}$ and $Lq \in \mathrm{Gev}_{1-0}$, then $q(x)$ extends analytically to a meromorphic function on the whole of the complex plane $\mathbb{C}^1_x$.*

(C) *The Cauchy problem $q(x,t_0) = q_0(x)$ for equation* (2) *has a locally holomorphic solution $q(x,t)$ in a neighborhood of the point $(x_0,t_0) \in \mathbb{C}^2$ if and only if $Lq_0 \in \mathrm{Gev}_{1/m}$.*

In connection with point (A) of Theorem 2, we note that the mapping $B\colon \mathrm{Gev}_1^{\mathrm{od}} \to \mathscr{R}(x_0)^{\mathrm{od}}$ inverse to L (the inverse scattering transformation, in the terminology of [7]) is defined on the whole space Gev_1 by the following formulae (see [7, § 7, formulae (18)–(20)]:

$$B\varphi(x) = [g_0(x), a] = \partial_x \gamma_+(x,z)\gamma_+^{-1}(x,z) - az, \tag{5}$$

by expressing $B\varphi \in \mathscr{R}(x_0)^{\mathrm{od}}$ through any component

$$\gamma_-(x,z) = I + \sum_{k=0}^{\infty} g_k(x) z^{-(k+1)}$$

or $\gamma_+(x,z)$ of a solution of the Riemann problem

$$e^{a(x-x_0)z}(I+\varphi(z))^{-1} = \gamma_-^{-1}(x,z)\gamma_+(x,z). \tag{6}$$

We shall now describe a general setting of the Riemann problem that is of interest to us; for any fixed $x \in \mathbb{C}$ equation (6) is a special case of this.

For any given numbers $\alpha \geq 0$, $m \geq 1$, and $A, B > 0$, we consider the set $G_\alpha(A)$ of all formal series

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-(k+1)} \in \mathrm{Gev}_\alpha,$$

such that

$$\|\varphi\|_{\alpha,\,A} := \sum_{k=0}^{\infty} (k!)^{-\alpha} |\varphi_k| A^k < \infty,$$

and the set $E_m(B)$ of all entire $\mathrm{gl}(n,\mathbb{C})$-valued functions

$$\varepsilon(z) = \sum_{l=0}^{\infty} \varepsilon_l z^l$$

whose coefficients satisfy the inequality $|\varepsilon_l| \leq C(l!)^{-1/m} B^l$ for all l and some constant C. Thus, the elements of $E_m(B)$ are entire functions of order not greater than m and finite type not greater than $B^{1/m}$ for order m. It is clear that $G_\alpha(A)$ is a Banach space with norm $\|\cdot\|_{\alpha,\,A}$, isometrically isomorphic to l_1, and that $E_m(B)$ is a Banach space with norm

$$\|\varepsilon\|_{m,B} := \sup_{l\geq 0} |\varepsilon_l| (l!)^{1/m} B^{-l},$$

isometrically isomorphic to l_∞. An important property of $G_\alpha(A)$ and $E_m(B)$ is that it is possible to give a well-defined multiplication of their elements as formal Laurent series for $\alpha m \leq 1$ and $B < A$ (see [7, Lemma 1]). Moreover, these inequalities cannot in general be improved.

Now let $m \geq 1$ and $A_0 > B_0 > 0$. The Riemann problem is that we are given an entire function $E_0 \in E_m(B_0)$ and a formal series $\varphi \in G_{1/m}(A_0)$ (called the *data of the Riemann problem*) and are required to find an entire function $\gamma_+ \in E_m(B)$ and a formal series $\gamma_- \in I + G_{1/m}(A)$ (the *solution of the Riemann problem*) for some $A > B > 0$ such that we have the following equality of formal Laurent series:

$$E_0(z)(I+\varphi(z))^{-1} = \gamma_-^{-1}(z)\gamma_+(z). \tag{7}$$

The following theorem, of which Theorem 1 is a highly special case, is the main result of the paper. To formulate, we call an entire function $E\colon \mathbb{C} \to \mathrm{gl}(n,\mathbb{C})$ *symmetric* if $E(\bar{z})^*E(z) = I$ for all $z \in \mathbb{C}$. It is clear that all values of such a function are invertible matrices; that is, we automatically have a mapping $E\colon \mathbb{C} \to \mathrm{GL}(n,\mathbb{C})$. A formal series $f \in I+\mathrm{Gev}_1$ is called *symmetric* if the equality $f(\bar{z})^*f(z) \equiv I$ holds in the sense of formal power series. Finally, we say that a germ $q \in \mathscr{R}(x_0)^{\mathrm{od}}$ at a real point $x_0 \in \mathbb{R}$ is *symmetric* if $q(\bar{x})^* = -q(x)$ for all complex x in a neighborhood of x_0, or, equivalently, if the matrix $q(x)$ is skew-Hermitian for all real x in a neighborhood of x_0. The first part of the theorem is a result on the solvability of the Riemann problem (7) with symmetric data; the second part describes an application of this result to analytic extensions of symmetric (so that they give solutions of physically interesting equations such as (1) for reductions of type (3)) holomorphic solutions of equations of the form (2). In the theorem, we use the notation $\mathscr{R}(x_0,t_0)^{\mathrm{od}}$ for the set of germs of all holomorphic off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued functions at an arbitrary point $(x_0,t_0) \in \mathbb{C}^2$.

Theorem 3. (A) *Let $m \geq 1$ and $A_0 > 12B_0 > 0$. If the entire function $E_0 \in E_m(B_0)$ and the formal series $f \in I + G_{1/m}(A_0)$ are symmetric, then the Riemann problem $E_0 f^{-1} = \gamma_-^{-1}\gamma_+$ is solvable; that is, for some $A > B > 0$, there exists a symmetric entire function $\gamma_+ \in E_m(B)$ and a symmetric formal series $\gamma_- \in I + G_{1/m}(A)$ such that the equality $E_0 f^{-1} = \gamma_-^{-1}\gamma_+$ holds in the sense of formal Laurent series.*

(B) *Let $m \geq 2$ be an integer, and let $a, b, c_1, c_2, \ldots \in \mathrm{gl}(n,\mathbb{C})$ be diagonal skew-Hermitian matrices. Suppose further that a and b have simple spectra and that $(x_0,t_0) \in \mathbb{R}^2$. Then every locally holomorphic solution $q \in \mathscr{R}(x_0,t_0)^{\mathrm{od}}$ of equation (2), for which $q(x,t_0) \in \mathscr{R}(x_0)^{\mathrm{od}}$ is symmetric, can be extended analytically to a meromorphic off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued function $q(x,t)$ on the domain $S_\varepsilon = \{(x,t) \in \mathbb{C}^2 \mid |t-t_0| < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, this extension is holomorphic on $S_\varepsilon \cap \mathbb{R}^2_{xt}$.*

Remark 1. Probably Theorem 3(A) still holds when $A_0 > B_0 > 0$, that is, under the minimal assumption which, by [7, Lemma 1], guarantees that we can multiply the series $E_0(z)$ and $f^{-1}(z)$. But such a refinement of Theorem 3(A) is of no great value to us since in the Riemann problem of type (6) which is of interest to us (more precisely, see (8)) it is always possible to reduce B_0 by confining one's attention to a small neighborhood of the point under consideration in the parameter space.

Remark 2. We can prove Theorem 1 in a different way, which does not rely on Theorem 3, and only uses the fact that any local real-analytic solution $u(x,t)$ of equation (1) with $AB \neq 0$ is globally meromorphic in x (this comes from statements (B) and (C) in Theorem 2, in view of the reduction (3)). To show that there are no real poles we argue by contradiction, as suggested, for example, in [12, Assertion 1.1] or [13, § 2.1]. Suppose that for some real $t = t_0$ the function $u(x,t_0)$ has a real pole $x = \gamma(t_0)$ of order n. Differentiating the principal part of the Laurent expansion of $u(x,t)$ by t and x at this pole, we find that the left-hand side of (1) has order n, but the terms on the right-hand side have order $n+2$ and $3n$, respectively, and hence $n = 1$ and the residue $c(t_0)$ of $u(x,t_0)$ at this pole must satisfy the equality

$$2Ac(t_0) + B|c(t_0)|^2 c(t_0) = 0,$$

which implies that $c(t_0) = 0$ in the focusing case. To bring this argument to an end, we must establish that $\gamma(t)$ is real for real t, close to t_0, and also that the function $\gamma(t)$ and the Laurent coefficients are holomorphic in t, but we shall not go into detail. The second method of proving that $u(x,t)$ has no real poles is based on the fact that the matrix $q(x,t)$, defined in (3), satisfies the Picard condition[3] and the necessary condition for (2×2)-matrices to have this property, which is given in Theorem 1 in [15]. Under this condition, all poles of the function $-u(x,t_0)\overline{u(\bar{x},t_0)}$ must be poles of second order with positive leading Laurent coefficient, which is impossible in the case of a real pole $x=x_0\in\mathbb{R}$. We note that we cannot currently see how the first method described in the proof of Theorem 1 might be extended, if only to a higher equation in the hierarchy of the NSE, nor the second to equations of the form (2) for $(n\times n)$-matrices for $n\geq 3$. In this regard, an approach based on Theorem 3 has much to give both in conceptualization and generality.

§ 2. Auxiliary results

In this section we pave the way for the proof of Theorem 3. We begin with an assertion about the uniqueness of the solution of the Riemann problem (7) and with a description (similar to Theorem 4(A) in [6]) of the precise degree of nonuniqueness in the recovery of the formal series $\varphi\in\mathrm{Gev}_1$ by means of the off-diagonal holomorphic germ $B\varphi\in\mathscr{R}(x_0)^{\mathrm{od}}$ generated by it (see (5)).

Lemma 1. *If a solution $\gamma_\pm$ of the Riemann problem* (7) *exists, then it is unique.*

Proof. Suppose that $\delta_\pm$ is another solution. Then, for suitable $m\geq 1$ and $A>B>0$, the formal series $\gamma_-,\delta_-\in I+G_{1/m}(A)$ and the entire functions $\gamma_+,\delta_+\in E_m(B)$ are such that the equality $\gamma_-^{-1}\gamma_+=\delta_-^{-1}\delta_+$ holds for the formal Laurent series. This can be rewritten in the form $\gamma_-\delta_-^{-1}=\gamma_+\delta_+^{-1}$, where the left-hand side contains only nonpositive powers of z (moreover, the coefficient for the zero power is equal to I) and the right-hand side contains only nonnegative powers of z. Both sides are therefore identically equal to I. □

Lemma 2. *The series $\varphi,\psi\in\mathrm{Gev}_1$ satisfy $B\varphi=B\psi$ if and only if the formal power series $(I+\varphi)^{-1}(I+\psi)\in I+\mathrm{Gev}_1$ is diagonal.*

Proof. We assume that the series $f(z):=I+\varphi(z)$ and $g(z):=I+\psi(z)$, which, by hypothesis, belong to the class $I+\mathrm{Gev}_1$, satisfy $g(z)=f(z)r(z)$ for some *diagonal* series $r\in I+\mathrm{Gev}_1$. We denote the solutions of the Riemann problem (6) corresponding to f and g by $\gamma_\pm^f(x,z)$ and $\gamma_\pm^g(x,z)$. By Lemma 1, since r and $e^{a(x-x_0)z}$ are diagonal, we obtain $\gamma_-^g=\gamma_-^f r$ and $\gamma_+^g=\gamma_+^f$. Then the second equality in (5) shows that the germs $q_f:=B\varphi$ and $q_g:=B\psi$ coincide.

Conversely, suppose that q_f and q_g coincide: $q_f=q_g=q$. Then, from the second equality in (5), together with the initial condition $\gamma_+(x_0,z)=I$ (resulting from Lemma 1), by the uniqueness theorem for solutions of linear differential equations, we obtain $\gamma_+^f(x,z)=\gamma_+^g(x,z)$ for all x in a neighborhood of x_0 and all $z\in\mathbb{C}$. Setting $E_0(x,z):=\exp\{a(x-x_0)z\}$, we rewrite this equality in the form $\gamma_-^f E_0 f^{-1}=\gamma_-^g E_0 g^{-1}$, that is, $E_0 f^{-1} g E_0^{-1}=(\gamma_-^f)^{-1}\gamma_-^g$. Consider the formal power series $r:=f^{-1}g\in I+\mathrm{Gev}_1$. It follows from the last equality that all the off-diagonal matrix elements on its left-hand side, $r_{kl}(z)\exp\{(a_{kk}-a_{ll})(x-x_0)z\}$, $k\neq l$, considered as formal Laurent series, contain only negative powers of z. If $r_{kl}(z)\not\equiv 0$, then we can write $r_{kl}(z)=z^{-M}(c_0+c_1z^{-1}+\cdots)$, $c_0\in\mathbb{C}\setminus\{0\}$, for some integer $M\geq 1$. We conclude from this that the formal Laurent

[3]Meaning, in addition to Theorem 2(B), that a linear system of ordinary differential equations $E_x=(az+q(x,t))E$ has a fundamental system of solutions, globally meromorphic in x, for each value of the parameter $z\in\mathbb{C}$. That all holomorphic solutions of equations of the form (2) possess this property was observed in [7, § 9(B)] and developed in more detail in § 3(B) in [14]. We take this opportunity to correct an inaccuracy in [14]: in the first line of the statement of the main theorem in § 1 and in Remark 4 from § 3(A), the condition "r is not divisible by n" should be replaced by "r and n are relatively prime".

series $z^{-M}\exp\{(a_{kk}-a_{ll})(x-x_0)z\}$ contains only negative powers of z. However, this conclusion is untrue; since we have assumed that the spectrum of a is simple, we have $a_{kk}-a_{ll}\neq 0$ for $k\neq l$. The contradiction thus obtained shows that $r_{kl}(z)\equiv 0$ for all $k\neq l$. □

Until the end of the section, we assume that *the initial point* x_0 *is real*: $x_0\in\mathbb{R}$, and that *the diagonal matrix* $a\in\mathrm{gl}(n,\mathbb{C})$ *is skew-Hermitian.* We recall that a germ $q\in\mathscr{R}(x_0)^{\mathrm{od}}$ is called *symmetric* if $q(\bar{x})^*=-q(x)$ for all complex x in a neighborhood of x_0, and a formal series $f\in I+\mathrm{Gev}_1$ is called *symmetric* if $f(\bar{z})^*f(z)\equiv I$. We then have the following extended analogue of Theorem 4(B) of [6].

Lemma 3. (A) *A germ* $q\in\mathscr{R}(x_0)^{\mathrm{od}}$ *is symmetric if and only if the formal power series* $(I+Lq(\bar{z}))^*(I+Lq(z))\in I+\mathrm{Gev}_1$ *is diagonal.*

(B) *For any* α, $0\le\alpha\le 1$, *the series* $f\in I+\mathrm{Gev}_\alpha$ *is such that* $f(\bar{z})^*f(z)$ *is diagonal if and only if* $f=hr$ *for some symmetric series* $h\in I+\mathrm{Gev}_\alpha$ *and some diagonal series* $r\in I+\mathrm{Gev}_\alpha$.

(C) *A germ* $q\in\mathscr{R}(x_0)^{\mathrm{od}}$ *is symmetric if and only if it has the form* $q=B\varphi$ *for some symmetric series* $f=I+\varphi\in I+\mathrm{Gev}_1$. *For any* α, $0\le\alpha\le 1$, *a germ* $q\in\mathscr{R}(x_0)^{\mathrm{od}}$, *satisfying* $Lq\in\mathrm{Gev}_\alpha$, *is symmetric if and only if* $q=B\varphi$ *for some symmetric series* $f=I+\varphi\in I+\mathrm{Gev}_\alpha$.

Proof. (A) Suppose that $q\in\mathscr{R}(x_0)^{\mathrm{od}}$ is symmetric. We apply the operation of Hermitian conjugation to the differential equation $\mu_x=(az+q(x))\mu-\mu az$, which arose in the definition of the series $Lq(z)=\mu(x_0,z)-I$ given in (4) . We then replace the variables x and z in the equality thus obtained by $\bar{x}$ and $\bar{z}$, respectively. Because the matrix a is skew-Hermitian and $q(x)$ is symmetric, the equality we obtain can be rewritten in the form $\nu_x=az\nu-(az+q(x))\nu$, meaning that the series $\nu(x,z):=\mu(\bar{x},\bar{z})^*$ is a formal gauge transformation (in the sense of § 6 in [7]) of the connection $U(x,z)=az+q(x)$ into the zero connection $U_0(x,z)\equiv az$. Then, by the group property, the gauge transformation $\nu(x,z)\mu(x,z)$ translates U_0 into itself; that is, it is diagonal (this follows from equation (14) in [7] with $q\equiv 0$). For $x=x_0$, this gives the required conclusion, that the series

$$\nu(x_0,z)\mu(x_0,z)=(I+Lq(\bar{z}))^*(I+Lq(z))$$

is diagonal. We note that this series always belongs to $I+\mathrm{Gev}_1$, by [7, Lemma 2], since both factors lie in $I+\mathrm{Gev}_1$ (see Theorem 2(A)).

Conversely, suppose that $q\in\mathscr{R}(x_0)^{\mathrm{od}}$ is such that the series

$$r(z)=(I+Lq(\bar{z}))^*(I+Lq(z))$$

is diagonal. We put

$$f(z):=I+Lq(z),\quad g(z):=((I+Lq(\bar{z}))^*)^{-1}$$

and write $f=I+\varphi$, $g=I+\psi$ for some $\varphi,\psi\in\mathrm{Gev}_1$. Then, since $r(z)$ is diagonal, it follows by Lemma 2 that $B\varphi=B\psi$. But $B\varphi=q$ since B and L are mutually inverse. In order to find $B\psi$, we use the fact that the mapping S of the space $I+\mathrm{Gev}_1$ into itself, given by the formula $Sf(z):=(f(\bar{z})^*)^{-1}$, possesses the following properties:

$$S(f_1f_2)=S(f_1)S(f_2),\quad S(Sf)=f,\quad S(f)=f\ \Leftrightarrow\ f(\bar{z})^*f(z)\equiv I.$$

Applying S to equality (6), substituting $\bar{x}$ for x, and using Lemma 1 and the above properties of S, we obtain successively that $\gamma^g_\pm(x,z)=(\gamma^f_\pm(\bar{x},\bar{z})^*)^{-1}$ and $B\psi(x)=-q^*(\bar{x})$. Therefore, the equality $B\varphi=B\psi$ is reduced to the symmetry condition $q(x)=-q(\bar{x})^*$ on $q(x)$.

(B) That the condition $f = hr$ is sufficient for the series $f(\bar z)^* f(z)$ to be diagonal is clear from the equalities

$$f(\bar z)^* f(z) = r(\bar z)^* h(\bar z)^* h(z) r(z) = r(\bar z)^* r(z),$$

as $r(z)$ is diagonal. We prove the necessity of this condition. Suppose that $f \in I + \mathrm{Gev}_\alpha$ and that the series $\delta(z) := f(\bar z)^* f(z) \in I + \mathrm{Gev}_\alpha$ is diagonal. We see from the definition of $\delta(z)$ that $\delta(\bar z)^* = \delta(z)$. Therefore all elements of the diagonal matrix $\delta(z)$ are formal power series in z^{-1} with real coefficients and with free term 1. By [7, Lemma 2], the set $I + G_\alpha(A)$ is a Banach algebra for every $A > 0$. We choose $A > 0$ sufficiently small that $\|\delta - I\|_{\alpha, A} < 1$ and, using the holomorphic functional calculus in Banach algebras (see, for example, [16, paragraph 10.21]), we apply the holomorphic function

$$F(w) = (1 + w)^{1/2} = 1 + \sum_{k=1}^{\infty} C_{1/2}^k w^k$$

to each diagonal element of the matrix $\delta(z) - I$, where $|w| < 1$ and $C_\beta^k := \beta(\beta - 1)\cdots(\beta - k + 1)/k!$ (substituting $\delta(z) - I$ for w, the series converges in the norm of $I + G_\alpha(A)$). We obtain the diagonal formal series $r \in I + \mathrm{Gev}_\alpha$ with the properties $r(\bar z)^* = r(z)$ and $r(z)^2 = \delta(z)$. We rewrite the last equality as $\delta(z) = r(\bar z)^* r(z)$. Then the formula $f(\bar z)^* f(z) = \delta(z)$ takes the form $h(\bar z)^* h(z) = I$ for $h := f r^{-1} \in I + \mathrm{Gev}_\alpha$, as was required.

(C) The assertion follows immediately from statements (A) and (B) and Lemma 2. $\square$

Remark 3. The condition $h(\bar z)^* h(z) = I$ on an element $h \in I + \mathrm{Gev}_1$ is equivalent to the following conditions (understood as equality between formal power series in z^{-1}) on matrix elements $h_{jk}(z)$, $j, k = 1, \dots, n$:

$$\sum_{l=1}^{n} \overline{h_{lj}(\bar z)} h_{lk}(z) = \delta_{jk} \quad \text{for all } j, k = 1, \dots, n.$$

In particular, it follows from these that

$$|h_{1j}(z)|^2 + |h_{2j}(z)|^2 + \dots + |h_{nj}(z)|^2 = 1$$

for all $j = 1, \dots, n$ and all $z = \bar z \in \mathbb{R}$. Therefore, it might seem that the formal power series $h_{1j}(z), \dots, h_{nj}(z)$ have to take finite values (that is, converge) for the given values of j and z. But this is an illusion. All of the above identities can also hold for series that diverge everywhere. For example, we put

$$\varphi(z) = \sum_{k=1}^{\infty} k! z^k \in \mathrm{Gev}_1 .$$

Then the formal series

$$h_{11}(z) := (1 + \varphi(z)^2)^{-1/2}, \quad h_{12}(z) := \varphi(z)(1 + \varphi(z)^2)^{-1/2},$$

formed by substituting series into series, also have zero radius of convergence (they belong to the spaces $I + \mathrm{Gev}_1$ and Gev_1, respectively, but do not belong to any Gevrey classes with smaller exponents) and satisfy the identity $|h_{11}(z)|^2 + |h_{12}(z)|^2 = 1$ for all $z = \bar z \in \mathbb{R}$. This example is not accidental. It is taken from the following criterion for the symmetry of (2×2)-matrices, easily verified by means of Lemma 3(B): a series $h \in I + \mathrm{Gev}_1$ satisfies the identity $h(\bar z)^* h(z) = I$ if and only if

$$h(z) = \frac{1}{\sqrt{1 + \varphi(z)\overline{\varphi(\bar z)}}} \begin{pmatrix} 1 & \varphi(z) \\ -\overline{\varphi(\bar z)} & 1 \end{pmatrix}$$

for some (arbitrary) scalar series $\varphi \in \mathrm{Gev}_1$ with complex coefficients, where the square root in the denominator of the fraction is defined by means of the holomorphic functional calculus in Banach algebras, just as in the proof of Lemma 3(B) above.

§ 3. The proof of Theorem 3

(A) We will denote by $\{\cdot\}_+$ and $\{\cdot\}_-$ the positive and negative parts of Laurent series:

$$\Big\{\sum_{n\in\mathbb{Z}} a_n z^n\Big\}_+ := \sum_{n=0}^{\infty} a_n z^n, \quad \Big\{\sum_{n\in\mathbb{Z}} a_n z^n\Big\}_- := \sum_{n=-\infty}^{-1} a_n z^n.$$

From the discussion in § 5 of [7], for the Riemann problem (7) under consideration to have a solution, it is *sufficient* that the operator $X := I + T^{-1}M^{-1}K$ on the left-hand side of equality (11) in [7] be invertible, where $T\varphi := \{\varphi E_0\}_-$, $M\varphi := \varphi f^{-1}$, $K\varphi := \{\{\varphi E_0\}_+ f^{-1}\}_-$. We note that, by [7, Lemma 6], X is a Fredholm operator of index 0 in the Banach space $G_{1/m}(A)$ for appropriate $A > 0$ (where we can assume that $2B_0 < A < A_0$: see the choice of B, A_1, A_2 immediately after the proof of Lemma 7 in [7]). According to the Fredholm alternative (see, for example, Theorem 4.25 in [16]), this operator is invertible if and only if it has null kernel. Therefore, it suffices to verify that if $\varphi \in G_{1/m}(A)$ and $X\varphi = 0$, then $\varphi = 0$.

In order to check this, we note that it follows from the equality $(I + T^{-1}M^{-1}K)\varphi = 0$ that $(MT + K)\varphi = 0$. But the expression

$$(MT + K)\varphi = \{\varphi E_0\}_- f^{-1} + \{\{\varphi E_0\}_+ f^{-1}\}_-$$

coincides with $\{\varphi E_0 f^{-1}\}_-$. Therefore, the equality $(MT + K)\varphi = 0$ means that the formal Laurent series

$$E(z) := \varphi(z)E_0(z)f^{-1}(z) \in G_{1/m}(A - 2B_0) \oplus E_m(B_0)$$

contains only nonnegative powers of z, so it is an entire function from $E_m(B_0)$. Hence, the conjugate series

$$E(\bar z)^* = f^{-1}(\bar z)^* E_0(\bar z)^* \varphi(\bar z)^*$$

is also an entire function of z, as is the product

$$E(z)E(\bar z)^* = \varphi(z)E_0(z)\{f(\bar z)^* f(z)\}^{-1} E_0(\bar z)^* \varphi(\bar z)^*,$$

which, by virtue of the symmetry of f and E_0, is equal to $\varphi(z)\varphi(\bar z)^*$ and thus contains only negative powers of z. Consequently, $\varphi(z)\varphi(\bar z)^* \equiv 0$. Substituting

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-k-1}, \quad \varphi_k \in \mathrm{gl}(n, \mathbb{C}),$$

and equating the coefficient of z^{-2} to zero, we obtain $\varphi_0\varphi_0^* = 0$, and so $\varphi_0 = 0$. Using the same reasoning, we deduce by induction that all the matrices φ_k are equal to 0. This completes the proof that X is injective.

In order to establish the symmetry of the solution $\gamma_\pm(z)$, we multiply the equalities

$$E_0(z)f^{-1}(z) = \gamma_-^{-1}(z)\gamma_+(z) \quad \text{and} \quad f^{-1}(\bar z)^* E_0(\bar z)^* = \gamma_+(\bar z)^* \gamma_-^{-1}(\bar z)^*$$

and use the symmetry of f and E_0. We obtain

$$I = \gamma_-^{-1}(z)\gamma_+(z)\gamma_+(\bar z)^* \gamma_-^{-1}(\bar z)^*,$$

so that

$$\gamma_-(z)\gamma_-(\bar z)^* = \gamma_+(z)\gamma_+(\bar z)^*.$$

Since the right-hand side of the last equality contains only nonnegative powers of z, and the left-hand side only nonpositive, with free term I, both sides are equal to I, as required.

(B) The existence of a meromorphism extension of $q(x,t)$ in S_ε follows from parts (B) and (C) in Theorem 2. In order to prove that the extension thus obtained is holomorphic at all real points of this strip (perhaps with smaller $\varepsilon > 0$), we set $q_0(x) := q(x,t_0) \in \mathscr{R}(x_0)^{\mathrm{od}}$ and note that $Lq_0 \in \mathrm{Gev}_{1/m}$, by Theorem 2(C). By virtue of the symmetry of q_0, Lemma 3(C) enables us to write $q_0 = B\varphi$ for some symmetric series $f = I + \varphi \in I + G_{1/m}(A_0)$, $A_0 > 0$. We consider the Riemann problem

$$e^{P(\xi,z)} f^{-1}(z) = \gamma_-^{-1}(\xi,z)\gamma_+(\xi,z) \tag{8}$$

with parameter $\xi = (x,t) \in \mathbb{C}^2$, the polynomial

$$P(\xi,z) = az(x-x_0) + (bz^m + c_1 z^{m-1} + \dots + c_m)(t-t_0),$$

and the symmetric series $f(z)$ just constructed above. The conditions of statement (A) hold for this Riemann problem for all $(x,t) \in \mathbb{R}^2$ provided that $|t-t_0|$ is sufficiently small. Indeed, because the matrices $a, b, c_1, \dots, c_m$ are skew-Hermitian, x and t being real guarantees the symmetry of the entire function $E_0(z) = \exp P(\xi,z)$, and, by [7, Lemma 5], as $|t-t_0|$ is small, E_0 belongs to some space $E_m(B_0)$ with some positive $B_0 < A_0/12$. It follows from the solvability of this Riemann problem for all given values of the parameters x,t that the extended solution $q \in \mathscr{M}(S_\varepsilon)$ of equation (2) (given by any of the formulae (5) with x replaced by $\xi = (x,t)$) is holomorphic at this point.

§ 4. The proof of Theorem 1

We construct a matrix $q(x,t)$ for the solution $u(x,t)$, in accordance with formula (3). Since u is real-analytic, the function $q(x,t)$ is holomorphic in some neighborhood of Π in $\mathbb{C}^2_{xt}$. We apply Theorem 3(B), taking as t_0 any fixed point on the vertical side of Π. This gives us an analytic extension of $q(x,t)$ to a holomorphic, off-diagonal, $\mathrm{gl}(2,\mathbb{C})$-valued function in some neighborhood of Σ in $\mathbb{C}^2_{xt}$. By restricting ourselves to Σ, we obtain the required real-analytic extension of $u(x,t)$. Thus, the first assertion of Theorem 1 is proved for the case $A = -1$, $B = -2$. The transition to the general case of the focusing NSE is carried out by scaling of the variables x,t.

To construct a solution on Σ which cannot be extended further, we write equation (1) in the form of the equivalent system in the real and imaginary parts of $u(x,t) = k(x,t) + il(x,t)$:

$$\begin{aligned} k_t &= Al_{xx} + B(k^2+l^2)l + Cl, \\ -l_t &= Ak_{xx} + B(k^2+l^2)k + Ck. \end{aligned} \tag{9}$$

Noting that $A \neq 0$, we apply the Cauchy–Kovalevskaya Theorem, given in Ch. I, § 7 in [17]. Thus, for any real-analytic functions $\varkappa_0(t)$, $\varkappa_1(t)$, $\lambda_0(t)$, $\lambda_1(t)$ defined on the interval $J := \{t_0 - \varepsilon_2 < t < t_0 + \varepsilon_2\}$ of the t-axis, this system has a real-analytic solution in some open neighborhood D of the interval $\{x_0\} \times J$ in the plane $\mathbb{R}^2_{xt}$ satisfying the initial conditions

$$\begin{aligned} k(x_0,t) &= \varkappa_0(t), \quad \partial_x k(x_0,t) = \varkappa_1(t), \\ l(x_0,t) &= \lambda_0(t), \quad \partial_x l(x_0,t) = \lambda_1(t), \end{aligned} \tag{10}$$

for all $t \in J$. By the part of Theorem 1 already proved, this solution is defined and real-analytic on the whole strip Σ. If at least one of the four initial functions given in (10) does not extend real-analytically at the endpoints of J, then the solution $u(x,t)$ of equation (1) we have constructed admits no real-analytic extension at any boundary point of Σ. Indeed, the existence of such an extension would imply, by the part of Theorem 1 already proved, that $u(x,t)$ is analytic in some strip wider than Σ, and this, for $x = x_0$, contradicts our choice of initial conditions (10).

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