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NONCOMMUTATIVE GEOMETRY AND THE TOMOGRAPHY OF MANIFOLDS

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Abstract. The tomography of manifolds describes a range of inverse problems in which we seek to reconstruct a Riemannian manifold from its boundary data (the "Dirichlet–Neumann" mapping, the reaction operator, and others). Different types of data correspond to physically different situations: the manifold is probed by electric currents or by acoustic or electromagnetic waves. In our paper we suggest a unified approach to these problems, using the ideas of noncommutative geometry. Within the framework of this approach, the underlying manifold for the reconstruction is obtained as the spectrum of an adequate Banach algebra determined by the boundary data.

§ 0. Introduction

This article is an expanded version of a lecture delivered at a conference dedicated to the 100th anniversary of the birth of Boris Moiseevich Levitan.

In inverse problems on manifolds, we are required to reconstruct a Riemannian manifold with boundary from its boundary data. Here are some examples of applications which motivate such situations.

Impedance tomography (IT). There exists a thin shell of complex form (a Riemann surface with boundary) which conducts current. A potential is applied to its boundary, inducing a stationary current through the shell. An external observer, operating at the boundary, has the ability to vary the potential and measure the current flowing through the boundary each time this is done. The measurements are formalised by the assignment of a *reaction operator*, which transforms the boundary "Dirichlet data" (connected with the elliptic problem on the surface) into the corresponding "Neumann data". This is the so-called DN-operator; it is determined by the shell: its topology, conductivity and so forth. We assume that the shell is inaccessible (invisible) to the observer, and by probing it using currents from the boundary, he is trying to determine its shape and parameters. In other words, we pose the *inverse problem*: to what extent does the DN-operator define the shell, and, if it is defined, how can we reconstruct it?

Acoustic tomography (AT). Suppose that a shell (membrane) is made of an elastic material and that the displacement of the points of its boundary gives rise to a wave which propagates along the shell with finite velocity. The wave is scattered by inhomogeneities, which gives rise to secondary waves, which return to the boundary and interact with the latter. The effects of the interaction are recorded by the external observer. The

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measurements are formalised by the assignment of a reaction operator which transforms displacements of the boundary (functions of the boundary points and time) into forces on the boundary (functions of the same variables). The inverse problem consists of reconstructing the shell by means of the reaction operator. This situation also makes sense in the multidimensional case and has significant applications in three dimensions (defectoscopy, geophysics, ultrasound diagnostics in medicine and others).

Electromagnetic tomography (EMT). A region in curved space (a three-dimensional Riemannian manifold with boundary) is illuminated by electromagnetic waves initiated by boundary sources. The waves interact with the internal structure and carry information on it to the boundary. The observer seeks to recover the structure: the topology and the metric.

The principal difference between these similar situations and the traditional coefficient inverse problem consists of the following. In the latter, the support of the required coefficients (the domain space or the manifold; in IT, the shell) is assumed to be known: we have the ability to fix a point of the support and consider how to recover the unknown function at it (the conductivity of the shell, the density of the medium, the speed of the wave, etc.). In our settings, *the support itself* is subject to definition. An additional complication is that the correspondence "manifold $\mapsto$ reaction operator" is not injective: two isometric manifolds with common boundary have identical reaction operators. For the observer, such manifolds respond identically to external stimuli; in principle, they are indistinguishable and it is unclear which of them to recover.

In this situation, the only adequate interpretation of the term "recovery" is as follows. Suppose that we are given a reaction operator of some manifold (the *original*); we are required to construct a Riemannian manifold using it, whose reaction operator coincides with that given. Such a manifold would be a *copy*, responding to external stimuli in the same way as the original, and, in this sense, would be indistinguishable from it. In this connection, the question arises: from which material can the external observer construct such a copy?

Approach. According to one of the main tenets of noncommutative geometry, any topological space corresponds to an algebra, by which the space is characterised. Applied to inverse problems, it would be more expressive to say that a space can be encoded by an adequate algebra. If the external observer is able to extract the *model* (representation) of this algebra from the reaction operator, then, decoding the model, he recovers the space. The mechanism for "coding–decoding" is as follows.

According to Gelfand, every uniform commutative Banach algebra has a canonical model in which elements of the algebra are realised as functions onto its *spectrum*, the compact set consisting of its characters. The transition to this model is referred to as the geometrisation of the original algebra.

Let (Ω, g) be the manifold subject to recovery, and let $\mathfrak{A}$ be an algebra whose spectrum $\widehat{\mathfrak{A}}$ is identical (homeomorphic) to Ω: in this sense, it encodes Ω. Suppose that the reaction operator defines some model $\mathfrak{A}^{\rm mod}$ isometrically isomorphic to $\mathfrak{A}$. Then the observer can construct this model and carry out its geometrisation. By defining the spectrum of the model $\widehat{\mathfrak{A}^{\rm mod}} =: \widetilde{\Omega}$, the observer obtains a *homeomorphic* copy of the original Ω. He thus decodes the model algebra and obtains the "material", about which we asked the question given above. In addition to this, the geometrisation allows us to distinguish a family of functions in $\widetilde{\Omega}$ which specify the Riemannian structure, transforming $\widetilde{\Omega}$ into an *isometric* copy $(\widetilde{\Omega}, \tilde{g})$ of the original (Ω, g). The Riemannian manifold $(\widetilde{\Omega}, \tilde{g})$ furnishes the solution of the tomography problem in an adequate sense of the word.

All three problems posed above can be solved using this scheme. In AT and EMT, which can be united under the common name "wave tomography", the path from the reaction operator to the model algebra is longer and more complicated than in IT. In IT, the model is a functional algebra, and in the wave problem, the $\mathfrak{A}^{\rm mod}$ are operator algebras;[1] their construction uses the specific physical properties of the corresponding dynamical systems. A key role is played by the *local completeness of waves*.

The approach here is an algebraic version of the *method of boundary control* (the BC-method: [1], [3]), which exploits the connection between inverse problems and the theory of systems.

§ 1. Impedance tomography

1.1. **The setting.** ***Direct problem.*** Let Ω be a compact smooth *two-dimensional* oriented manifold with boundary Γ, let g be a metric tensor, let $d\Omega$, be a 2-form of Riemann volume and let Δ_g be the Beltrami–Laplace operator on functions.

We consider the boundary problem

$$\Delta_g u = 0 \quad \text{in } \Omega \setminus \Gamma, \tag{1.1}$$

$$u|_\Gamma = f \tag{1.2}$$

with a real smooth[2] function $f \in L_2(\Gamma)$; let $u = u^f(x)$ be its solution. It is a formalisation of the (direct) EMT problem: Ω plays the role of the conducting shell, f is the potential applied to the boundary and u^f is the potential induced in Ω. The reaction of the shell to an effect is described by the *Calderón operator*[3] $\Lambda\colon C^\infty(\Gamma) \to C^\infty(\Gamma)$,

$$\Lambda f := u^f_\nu,$$

where $(\cdot)_\nu$ is the derivative with respect to the outward normal ν to Γ. From the physical point of view, Λf is the current flowing through the boundary and measured by the external observer.

The ***inverse problem*** consists of recovering the shell from its Calderón operator. In a strict sense, described in the introduction, it is required that we construct a copy of the shell using Λ: the Riemannian manifold $(\widetilde\Omega, \tilde g)$ with boundary $\widetilde\Gamma = \Gamma$ and operator $\widetilde\Lambda$ such that $\widetilde\Lambda = \Lambda$.

1.2. **Operators.** The ***operator*** $\boldsymbol{\Lambda}$, which plays the role of the data of the inverse problem, possesses the following well-known properties.

• Let $d\Gamma = d\Omega(\nu, \cdot)$ be a 1-form of the length on Γ, and let $|d\Gamma|$ be an (unoriented) length element. Let θ, $|\theta(\cdot)| = 1$, be a smooth field of tangent vectors on the boundary, understood as differentiation of functions from $C^\infty(\Gamma)$. Choosing θ means orienting Γ. We set $\dot C^\infty(\Gamma) := \theta C^\infty(\Gamma)$. We have the relations

$$\operatorname{Ker} \Lambda = \{\text{const}\}, \quad \operatorname{Ran} \Lambda = \dot C^\infty(\Gamma). \tag{1.3}$$

The inclusion $\operatorname{Ran} \Lambda \subset \dot C^\infty(\Gamma)$ follows from the relation

$$\int_\Gamma u^f_\nu \, |d\Gamma| = 0,$$

and the reverse inclusion follows because the boundary value problem with Neumann boundary condition, $\Delta_g u = 0$, $u^f_\nu = h \in \dot C^\infty(\Gamma)$, is well-posed.

• Let (Ω, g) and (Ω', g') be two manifolds with common boundary Γ, connected by the diffeomorphism $j\colon \Omega' \to \Omega$, such that $g' = \rho j^* g$, where ρ is a smooth positive function.[4] Then $\Lambda' = \rho^{-1/2}\Lambda$, and the additional condition $\rho|_\Gamma = 1$ leads to the equality $\Lambda' = \Lambda$.

[1] Although the IT-model can also be realised as an operator algebra; see [4].

[2] Throughout this paper, "smooth" means C^∞-smooth.

[3] Also known as the DN-operator; in our terminology, this is the reaction operator.

[4] That is, (Ω, g) and (Ω', g') are conformally equivalent.

The Hilbert transform. A *Hilbert transform*

$$\dot{C}^\infty(\Gamma) \xrightarrow{H} \dot{C}^\infty(\Gamma), \quad H := \Lambda J$$

acts in the linear manifold $\dot{C}^\infty(\Gamma) := \theta C^\infty(\Gamma)$, where J is integration: $J\theta f = f$ (mod const); H is well-defined, in view of (1.3).

1.3. **Algebras.** Let d be the exterior derivative, and $\star$ the Hodge operator corresponding to the form $d\Omega$. We denote the spaces of continuous real and complex functions by $C(\Omega)$ and $C_{\rm c}(\Omega)$, respectively; $C_{\rm c}(\Omega)$ is a commutative Banach algebra with norm $\|w\| := \sup_\Omega |w(\cdot)|$.

We recall that a *character* of a commutative Banach algebra $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}$ into $\mathbb{C}$. The *spectrum* of an algebra is the set $\widehat{\mathfrak{A}}$ of its characters, equipped with the weak-$*$ topology of the dual space $C'_{\rm c}(\Omega)$. The spectrum is compact. The geometrisation of $\mathfrak{A}$ is realised by the *Gelfand transform* [12]

$$\mathfrak{A} \xrightarrow{G} C(\widehat{\mathfrak{A}}), \quad (Ga)(\chi) := \chi(a).$$

In what follows, when speaking of *isometries* of algebras, we mean the isometric isomorphisms connecting them.

A complex analytic structure is defined in Ω which is compatible with the Riemannian metric and transforms Ω into a *Riemann surface* [10]. An algebra of analytic functions is defined on the latter:

$$\mathscr{A}(\Omega) := \big\{a = u + u_*{\rm i} \mid a \in C_{\rm c}(\Omega),\ du_* = \star du\big\}. \tag{1.4}$$

In solving the inverse problem, this plays the role of the basic algebra $\mathfrak{A}$ from the introduction. We list its known properties.

- The smooth subalgebra $\mathscr{A}^\infty(\Omega) := \mathscr{A}(\Omega) \cap C^\infty_{\rm c}(\Omega)$ is dense in $\mathscr{A}(\Omega)$.
- The spectrum $\widehat{\mathscr{A}(\Omega)}$ of $\mathscr{A}(\Omega)$ is exhausted by the Dirac measures

$$\delta_x \in C'_{\rm c}(\Omega)\colon \delta_x(a) = a(x).$$

The correspondence $\Omega \ni x \overset{s}{\longmapsto} \delta_x \in \widehat{\mathscr{A}(\Omega)}$ is a homeomorphism between topological spaces.[5]

- The equality $Ga = a \circ s^{-1}$ holds, thus establishing an isometry between $\mathscr{A}(\Omega)$ and its image $G\mathscr{A}(\Omega)$ under the Gelfand transform

$$\mathscr{A}(\Omega) \xrightarrow{G} C_{\rm c}(\widehat{\mathscr{A}(\Omega)}).$$

In view of the uniqueness theorem and the maximum principle for analytic functions, the correspondence $a \overset{t}{\longmapsto} a|_\Gamma$ is an isometry between $\mathscr{A}(\Omega)$ and the *algebra of boundary values*

$$\mathscr{A}(\Gamma) := \{a|_\Gamma \mid a \in \mathscr{A}(\Omega)\} \subset C_{\rm c}(\Gamma).$$

This algebra will play the role of the model $\mathfrak{A}^{\rm mod}$ from the introduction. The correspondence t^{-1} is an analytic extension from Γ into Ω. The subalgebra

$$\mathscr{A}^\infty(\Gamma) := t\mathscr{A}^\infty(\Omega) = \mathscr{A}(\Gamma) \cap C^\infty_{\rm c}(\Gamma)$$

is dense in $\mathscr{A}(\Gamma)$. The algebra isometries $\mathscr{A}(\Omega) \xrightarrow{t} \mathscr{A}(\Gamma)$ give rise to conjugate isometries (of Banach spaces) $\mathscr{A}'(\Gamma) \xrightarrow{t^*} \mathscr{A}'(\Omega)$, connecting their spectra:

$$t^*\widehat{\mathscr{A}(\Gamma)} = \widehat{\mathscr{A}(\Omega)} = s\,\Omega.$$

- The algebras $\mathscr{A}(\Omega)$ and $G\mathscr{A}(\Gamma)$ are isometric, and, for $b = a|_\Gamma \in \mathscr{A}(\Gamma)$, we have

$$(Gb)(p) = (t^{-1}b)(x) = a(x), \quad \text{where } x = s^{-1}t^*p.$$

[5]In the terminology of the introduction, this also means that $\mathscr{A}(\Omega)$ encodes Ω.

Consequently, by finding the spectrum $\widehat{\mathscr{A}(\Gamma)} =: \widetilde\Omega$ and the functions from $G\mathscr{A}(\Gamma) := \mathscr{A}(\widetilde\Omega)$ on it, we obtain a *homeomorphic copy* of Ω and an *isometric copy* of $\mathscr{A}(\Omega)$, respectively. The copies and originals are connected by the mappings

$$\widetilde\Omega \xrightarrow{j} \Omega, \quad \mathscr{A}(\Omega) \ni a \mapsto \tilde a = a \circ j \in \mathscr{A}(\widetilde\Omega), \tag{1.5}$$

where $j := s^{-1}t^*$ is a homeomorphism, and $a \mapsto \tilde a$ is an algebra isometry. The copies $\widetilde\gamma = j^{-1}\gamma$ of boundary points $\gamma \in \Gamma$ admit the following characterisation:

$$\big\{\widetilde\Omega \ni \widetilde\gamma \xrightarrow{j} \gamma \in \Gamma\big\} \ \Leftrightarrow\ \big\{(Gb)(\xi) = b(\gamma)\ \ \forall b \in \mathscr{A}(\Gamma)\big\}. \tag{1.6}$$

These considerations are illustrated in Figure 1; objects that are used in solving the inverse problem are enclosed in squares. The wavy arrows are transitions from an algebra to its spectrum. Objects that define the copy $(\widetilde\Omega, \tilde g)$ appear in double frames.

$$\begin{tikzcd}
\Omega \arrow[r, "s"] & \widehat{\mathscr{A}(\widetilde\Omega)} & \boxed{\boxed{\widetilde\Omega}} \arrow[l, "t^*"'] & \\
& \mathscr{A}(\Omega) \arrow[u, rightsquigarrow] \arrow[r, "t"] & \boxed{\mathscr{A}(\Gamma)} \arrow[u, rightsquigarrow] \arrow[r, "G"] & \boxed{\mathscr{A}(\widetilde\Omega)} \\
& & \boxed{\mathscr{A}^\infty(\Gamma)} \arrow[u] & \\
& & \boxed{\Lambda} \arrow[u] &
\end{tikzcd}$$

FIGURE 1. Reconstruction in IT

• Analysis of the Cauchy–Riemann conditions $du_* = \star du$ from (1.4) at points of Γ leads to the following characterisation of boundary values of analytic functions (see [2]):

$$\mathscr{A}^\infty(\Gamma) = \left\{f + f_*\mathrm{i} \in C^\infty_{\mathrm{c}}(\Gamma) \mid \theta f,\ \theta f_* \in \operatorname{Ker}(\mathbb{I} + H^2),\ \theta f_* = H\theta f\right\}, \tag{1.7}$$

where $\mathbb{I}$ is the identity operator. The set of pairs $\{f, f_*\}$ specified by the conditions on the right-hand side is the same for θ and $-\theta$; that is, it does not depend on the orientation of the boundary.

1.4. **Reconstruction of Ω.** As a result of measurements at the boundary, the external observer obtains Λ. For simplicity, we assume that the length element $|d\Gamma|$ is known. From these data, the observer can recover Ω using the following scheme.

1. We find $H = \Lambda J$ and choose a field θ. Using (1.7), we recover the smooth algebra $\mathscr{A}^\infty(\Gamma)$. Taking its closure in $C_{\mathrm{c}}(\Gamma)$, we obtain $\mathscr{A}(\Gamma)$.

2. We find the spectrum $\widehat{\mathscr{A}(\Gamma)} =: \widetilde\Omega$ and define the algebra $\mathscr{A}(\widetilde\Omega) = G\mathscr{A}(\Gamma) \subset C_{\mathrm{c}}(\widetilde\Omega)$. The latter induces the analytic structure of a Riemann surface on $\widetilde\Omega$.[6] This structure defines a family of conformally equivalent metrics compatible with it.[7] We choose one of these, g'. As a result, we have a Riemannian manifold $(\widetilde\Omega, g')$ with boundary $\widetilde\Gamma$.

3. We identify $\widetilde\Gamma$ with Γ according to (1.6). Now $(\widetilde\Omega, g')$ and the (unknown) original (Ω, g) are two Riemannian manifolds with common boundary Γ, connected by the (unknown) diffeomorphism $j := s^{-1}t^*$; see (1.5). In addition, the metrics are connected by the relation $g' = \rho j^* g$ with an (unknown) smooth function $\rho > 0$.

[6]Functions from $\mathscr{A}(\widetilde\Omega)$ play the role of local complex coordinates.

[7]These are metrics in which the real and imaginary parts of functions from $\mathscr{A}(\widetilde\Omega)$ are harmonic. Therefore, to find such metrics, that is, to define the component g_{ij} of the metric tensor, we can use the equation $\Delta_g u = 0$ in local coordinates. The components are defined by this up to an arbitrary factor which is a positive function; see [2].

4. We will find the operator Λ' of $(\widetilde{\Omega}, g')$. It is connected with the original Λ by the relation $\Lambda' = \rho^{-1/2}\Lambda$ (see §1.2). Comparing these operators, we define $\rho|_\Gamma$. We then "adjust" the metric g' by the factor $\widetilde{\rho}^{1/2}$, where $\widetilde{\rho} > 0$ is any smooth extension of $\rho|_\Gamma$ from Γ onto $\widetilde{\Omega}$. The manifold $(\widetilde{\Omega}, \tilde{g})$ with metric $\tilde{g} := \widetilde{\rho}^{1/2} g'$ has DN-operator $\widetilde{\Lambda}$, which coincides with Λ *by construction*. It also solves the problem.

Without carrying out the full recovery scheme 1–4, we can use Λ to define some topological invariants of Ω: the Betti numbers and others; see [2], [7].

The IT problem in $n \geq 3$ dimensions has still not been properly solved; see the discussion in [4]. The difficulty lies in the lack of an adequate analogue of $\mathscr{A}(\Omega)$: it is likely that in higher dimensions Ω is identical to some spectrum, but the question "the spectrum *of what*?" remains open.

§ 2. Acoustic tomography

In this section, the manifold Ω, all functions on Ω, and the spaces and algebras are *real*. The manifold Ω itself is a smooth compact Riemannian manifold of dimension $n \geq 2$ with boundary Γ, and g is a metric tensor. We do not require that Ω be orientable. We denote a metric neighbourhood of the set $A \subset \Omega$ by

$$\Omega^r[A] := \{x \in \Omega \mid \operatorname{dist}(x, A) < r\}.$$

Since Ω is compact

$$\Omega^r[A] = \Omega, \quad r > \operatorname{diam}\Omega, \tag{2.1}$$

for any A.

2.1. **The settings.** ***The direct problem.*** The propagation of sound in Ω is described by the initial-boundary problem

$$u_{tt} - \Delta_g u = 0 \qquad \text{in } (\Omega\backslash\Gamma) \times (0, T), \tag{2.2}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$u|_{\Gamma\times[0,T]} = f, \tag{2.4}$$

in which $t = T > 0$ is the final moment of time, f is the *boundary control*, and $u = u^f(x, t)$ is the solution. From the point of view of physics, $u^f(x, t)$ is the pressure at the point x at time t, and u^f is a sound *wave*, initiated by the boundary source f and propagating into the domain Ω, which is made up of an inhomogeneous medium.

The response of the medium to the effect of the control is described by the *reaction operator*, which acts according to the rule

$$R^T f := u^f_\nu|_{\Gamma\times[0,T]},$$

where $(\cdot)_\nu$ is the derivative with respect to the outward normal to Γ. The quantity u^f_ν describes the forces arising at the boundary on account of the effect of the waves, as measured by an external observer on Γ for times $0 \leq t \leq T$.

The ***inverse problem*** consists of using the reaction operator to recover the manifold (Ω, g). The precise setting is as follows: using the operator R^{2T}, given for fixed $T > \operatorname{diam}\Omega$, is it required to construct the Riemannian manifold $(\widetilde{\Omega}, \tilde{g})$ with boundary $\widetilde{\Gamma} = \Gamma$ such that its reaction operator $\widetilde{R}^{2T}$satisfies $\widetilde{R}^{2T} = R^{2T}$.

We clarify the meaning of the condition on T and the doubling of the time of observation. The manifold is probed by waves initiated at the boundary. The waves propagate with finite (unit) speed. Therefore the observation time must be sufficiently large for them to fill the *whole* of Ω and have time to return to the boundary, carrying information on the internal structure.

2.2. **Eikonals.** ***Scalar eikonals.*** We say that a subset $\sigma \subset \Gamma$ is *regular* (and write $\sigma \in \mathscr{R}$) if it is diffeomorphic to an $(n-1)$-dimensional ball $\{p \in \mathbb{R}^{n-1} \mid |p| \leq 1\}$. Functions on Ω of the form

$$\tau_\sigma(x) := \operatorname{dist}(x, \sigma), \quad \sigma \in \mathscr{R},$$

are called *eikonals*. We present those of their properties that are used to solve the inverse problem.

1. Eikonals are continuous: $\tau_\sigma \in C(\Omega)$ is smooth almost everywhere in Ω and

$$|\nabla_g \tau_\sigma(x)| = 1 \quad \text{almost everywhere in } x \in \Omega. \tag{2.5}$$

2. Eikonals can be used as local coordinates: for any $x \in \Omega$, we can specify a neighbourhood $\Omega^\varepsilon[x]$ and a collection $\sigma_1, \ldots, \sigma_n \in \mathscr{R}$ such that $\tau_{\sigma_1}, \ldots, \tau_{\sigma_n}$ constitutes a chart on $\Omega^\varepsilon[x]$.

3. Boundary points are characterised in terms of eikonals by the equivalence

$$\{\gamma \in \Gamma\} \;\Leftrightarrow\; \{\exists \sigma \in \mathscr{R} \colon \tau_\sigma(\gamma) = 0\}. \tag{2.6}$$

The copy $(\widetilde{\Omega}, \tilde{g})$***.*** Eikonals give a Riemannian structure in Ω in the following sense. Suppose that we have a topological space $\widetilde{\Omega}$, connected with Ω by the homeomorphism $\Omega \xrightarrow{s} \widetilde{\Omega}$, and let $\widetilde{\tau}_\sigma := \tau_\sigma \circ s^{-1}$. We suppose that the homeomorphism is unknown but that we are given a mapping

$$\mathscr{R} \ni \sigma \mapsto \widetilde{\tau}_\sigma \in C(\widetilde{\Omega}). \tag{2.7}$$

This is sufficient to equip $\widetilde{\Omega}$ with a metric, turning it into a Riemannian manifold *isometric to* Ω. Briefly, the method consists of the following.[8] We take a point $p \in \widetilde{\Omega}$; in a neighbourhood of it, we choose coordinates $\widetilde{\tau}_{\sigma_1}, \ldots, \widetilde{\tau}_{\sigma_n}$. Written in these coordinates, the equation (2.5) contains components $\tilde{g}^{ij}$ of the metric tensor and can be used to find the latter. In addition to this, as we have (2.7), we can distinguish the image of the boundary $\widetilde{\Gamma} := s\Gamma$ in $\widetilde{\Omega}$, using the characterisation

$$\{\widetilde{\gamma} \in \widetilde{\Gamma}\} \overset{2.6}{\Longleftrightarrow} \{\exists \sigma \in \mathscr{R}(\Gamma) \colon \widetilde{\tau}_\sigma(\widetilde{\gamma}) = 0\}.$$

Moreover, the family $\{\widetilde{\tau}_\sigma\}$ gives a natural way of identifying the boundaries:

$$\{\Gamma \ni \gamma \equiv \widetilde{\gamma} \in \widetilde{\Gamma}\} \;\Leftrightarrow\; \{\widetilde{\tau}_\sigma(\widetilde{\gamma}) = 0 \;\; \forall \sigma \ni \gamma\}.$$

Summarising the arguments given, we can say that the mapping (2.7) permits the construction of an *isometric copy* $(\widetilde{\Omega}, \tilde{g})$ of the original manifold (Ω, g), having common boundary $\widetilde{\Gamma} = \Gamma$ with the original.

Operator eikonals. A space $\mathscr{H} := L_2(\Omega)$ with scalar product

$$(u, v)_{\mathscr{H}} = \int_\Omega u(x) v(x) \, |d\Omega|$$

is defined on Ω ($|d\Omega|$ is an element of Riemann volume). Let $A \subset \Omega$ be a measurable set, and $\chi_A(\,\cdot\,)$ be its indicator (characteristic function). We denote the subspace of functions with support in A by

$$\mathscr{H}\langle A \rangle := \{\chi_A y \mid y \in \mathscr{H}\}. \tag{2.8}$$

The projector X_A from $\mathscr{H}$ onto $\mathscr{H}\langle A \rangle$ cuts off functions on A, which is equivalent to multiplying by χ_A.

Let $\mathfrak{B}(\mathscr{H})$ be a normed algebra of bounded operators in $\mathscr{H}$. We compare each scalar eikonal τ_σ to the multiplication operator $\check{\tau}_\sigma \in \mathfrak{B}(\mathscr{H})$ which acts in $\mathscr{H}$ according to the rule

$$(\check{\tau}_\sigma y)(x) := \tau_\sigma(x) \, y(x), \quad x \in \Omega.$$

[8]For details, see [5].

As Ω is compact, each such operator is bounded and satisfies the relations

$$\|\check{\tau}_\sigma\|_{\mathfrak{B}(\mathscr{H})} = \max_{x\in\Omega} \tau_\sigma(x) = \|\tau_\sigma\|_{C(\Omega)} \leq \operatorname{diam}\Omega. \tag{2.9}$$

We use the same term *eikonal* for the $\check{\tau}_\sigma$.

Each eikonal is a selfadjoint positive operator, and we have the well-known representation in terms of the spectral theorem:

$$\check{\tau}_\sigma = \int_0^\infty r\, dX^r_\sigma, \tag{2.10}$$

in which the projectors $X^r_\sigma := X_{\Omega^r[\sigma]}$ cut off functions in neighbourhoods of the sets σ. The interval of integration is in fact finite, since, for $r > \max_{x\in\Omega}\tau_\sigma(x)$, the neighbourhood $\Omega^r[\sigma]$ exhausts the whole of Ω and the projectors X^r_σ are equal to the identity operator.

Eikonals corresponding to different σ commute. This follows from the commutativity of X^r_σ with $X^{r'}_{\sigma'}$ for all σ, σ' and r, r'.

2.3. **Algebras.** 1. The space $C(\Omega)$ with standard sup-norm is a commutative Banach algebra. It is the canonical example of an algebra encoding a space: its spectrum $\widehat{C(\Omega)}$ is exhausted by the Dirac measures $\{\delta_x \mid x \in \Omega\}$, and it is homeomorphic to Ω [12]. Traditionally, Ω and $\widehat{C(\Omega)}$ are identified by the rule $\Omega \ni x \equiv \delta_x \in \widehat{C(\Omega)}$. Thus, $C(\Omega)$ turns out to be identical to its Gelfand transform.

2. For a subset S of a Banach algebra, we denote the minimal closed algebra (by norm) which contains S by $\vee S$.

We next define the *eikonal algebra*

$$\mathfrak{T} := \bigvee\{\tau_\sigma \mid \sigma \in \mathscr{R}\} \subset C(\Omega).$$

From §2.2, property 2, it is clear that eikonals separate the points of Ω and do not have zeroes that are common for all τ_σ. As a consequence, the Stone–Weierstrass Theorem [12] gives $\mathfrak{T} = C(\Omega)$. From this, we have $\widehat{\mathfrak{T}} = \Omega$. Consequently, the eikonal algebra encodes Ω and can play the role of the basic algebra $\mathfrak{A}$ in the introduction.

3. We define the *algebra of operator eikonals*:

$$\mathfrak{T}_{\rm op} := \bigvee\{\check{\tau}_\sigma \mid \sigma \in \mathscr{R}\} \subset \mathfrak{B}(\mathscr{H}).$$

As we can see from (2.9), the correspondence $\tau_\sigma \mapsto \check{\tau}_\sigma$ between the generators of $\mathfrak{T}$ and $\mathfrak{T}_{\rm op}$ extends in a canonical way to an isometric isomorphism $\mathfrak{T} \xrightarrow{i} \mathfrak{T}_{\rm op}$.

4. Let $\mathscr{H}^{\rm mod}$ be some Hilbert space,[9] $U\colon \mathscr{H}^{\rm mod} \to \mathscr{H}$ be a unitary operator, and $\mathfrak{B}(\mathscr{H}) \ni a \xmapsto{u} U^*aU \in \mathfrak{B}(\mathscr{H}^{\rm mod})$ be the corresponding algebra isometry. We call the algebra

$$\mathfrak{T}^{\rm mod}_{\rm op} := u\,\mathfrak{T}_{\rm op}$$

the *model* of $\mathfrak{T}_{\rm op}$, and the operators $\check{\tau}^{\rm mod}_\sigma := u\check{\tau}_\sigma$ the model eikonals. The composition $u\,i$ establishes an isometric isomorphism between $\mathfrak{T} = C(\Omega)$ and $\mathfrak{T}^{\rm mod}_{\rm op}$. It follows from the isometry of the algebras that there exists a homeomorphism s connecting their spectra and the functions on the spectra:

$$\Omega \xrightarrow{s} \widetilde{\Omega} := \widehat{\mathfrak{T}^{\rm mod}_{\rm op}}, \quad G\mathfrak{T}^{\rm mod}_{\rm op} = C(\widetilde{\Omega}) \ni G\check{\tau}^{\rm mod}_\sigma =: \widetilde{\tau}_\sigma = \tau_\sigma \circ s^{-1} = Gui\,\tau_\sigma.$$

These considerations are illustrated in Figure 2. Double arrows indicate the definition of an algebra in terms of its generators.

5. Suppose that, by means of measurements at the boundary, the external observer was able to determine the pair $\{\mathscr{H}^{\rm mod}, m\}$, where m is the mapping

$$\mathscr{R} \ni \sigma \xmapsto{m} \check{\tau}^{\rm mod}_\sigma \in \mathfrak{B}(\mathscr{H}^{\rm mod}).$$

[9]The choice of $\mathscr{H}^{\rm mod}$ will be specified in §2.5.

$$
\begin{tikzcd}
\Omega \arrow[rr, "s"] & & \boxed{\boxed{\widetilde{\Omega}}} & \\
C(\Omega) = \mathfrak{T} \arrow[u, squiggly] \arrow[r, "i"] & \mathfrak{T}_{\mathrm{op}} \arrow[r, "u"] & \boxed{\mathfrak{T}^{\mathrm{mod}}} \arrow[u, squiggly] \arrow[r, "G"] & \boxed{\boxed{C(\widetilde{\Omega})}} \\
\tau_\sigma \arrow[u, Rightarrow] \arrow[r, "i"] & \check{\tau}_\sigma \arrow[u, Rightarrow] \arrow[r, "u"] & \boxed{\check{\tau}_\sigma^{\mathrm{mod}}} \arrow[u, Rightarrow] \arrow[r, "G"] & \boxed{\boxed{\widetilde{\tau}_\sigma}} \\
 & & \boxed{\sigma} \arrow[u, "m"'] \arrow[ur, "(2.7)"'] &
\end{tikzcd}
$$

FIGURE 2. Reconstruction in AT

This is sufficient to recover the manifold. In fact, the observer can:

- construct $\mathfrak{T}^{\mathrm{mod}}_{\mathrm{op}} = \underset{\sim}{\bigvee}\{\check{\tau}^{\mathrm{mod}}_\sigma \mid \sigma \in \mathscr{R}\} \subset \mathfrak{B}(\mathscr{H}^{\mathrm{mod}})$;
- find its spectrum $\widetilde{\Omega}$ and its canonical Gelfand realisation, namely, the algebra $G\mathfrak{T}^{\mathrm{mod}}_{\mathrm{op}} = C(\widetilde{\Omega})$;
- determine the mapping $\sigma \overset{Gm}{\longmapsto} \widetilde{\tau}_\sigma$, a version of (2.7). *This defines a copy* $(\widetilde{\Omega}, \tilde{g})$.

We will show later that the data of the inverse problem determine an adequate pair $\{\mathscr{H}^{\mathrm{mod}}, m\}$.

2.4. **The system α^T. *Attributes.*** We associate the problem (2.2)–(2.4) with the dynamical system α^T and equip it with the corresponding attributes, spaces and operators.

• The space of controls (inputs) $\mathscr{F}^T := L_2([0,T]; L_2(\Gamma))$ is called the *outer* space of α^T. The class of smooth controls

$$\mathscr{M}^T := \big\{ f \in C^\infty([0,T]; C^\infty(\Gamma)) \mid \operatorname{supp} f \subset (0,T] \big\}$$

is dense in $\mathscr{F}^T$. For $f \in \mathscr{M}^T$, the problem has a unique smooth classical solution. We set $L_2(\sigma) := \{ y \in L_2(\Gamma) \mid \operatorname{supp} y \subseteq \sigma \}$. The outer space contains the subspaces

$$\mathscr{F}^{T,r}_\sigma := \big\{ f \in L_2([0,T]; L_2(\sigma)) \mid \operatorname{supp} f \subseteq [T-r, T] \big\}, \quad 0 < r \le T, \ \sigma \in \mathscr{R},$$

generated by controls, acting from σ with delay $T - r$ (r is the time of action).

• The *inner* space is the space $\mathscr{H} = L_2(\Omega)$. Waves (states) $u^f(\cdot, t)$ are elements of $\mathscr{H}$ that depend on time.

• The correspondence "input $\mapsto$ state" is realised by the *control operator*

$$W^T \colon \mathscr{F}^T \to \mathscr{H}, \quad \operatorname{Dom} W^T = \mathscr{M}^T, \quad W^T f := u^f(\cdot, T).$$

The specific characteristic of the acoustic system is the fact that W^T is bounded. Therefore it can be extended from $\mathscr{M}^T$ to the whole of $\mathscr{F}^T$ by continuity, and in what follows we assume that this extension has been implemented. The images of controls under the action of the extended W^T are, by definition, generalised solutions of the problem (2.2)–(2.4).

• The mapping "input $\mapsto$ output" is realised by the *reaction operator*

$$R^T \colon \mathscr{F}^T \to \mathscr{F}^T, \quad \operatorname{Dom} R^T = \mathscr{M}^T, \quad R^T f := u^f_\nu|_{\Gamma \times [0,T]},$$

where $(\cdot)_\nu$ is the derivative with respect to the outward normal. We recall here that the reaction operators of (Ω, g) and its isometric copy $(\widetilde{\Omega}, \tilde{g})$ coincide: $R^T = \widetilde{R}^T$, for any $T > 0$.

• We define the *connecting operator* $C^T \colon \mathscr{F}^T \to \mathscr{F}^T$ by

$$C^T := (W^T)^* W^T. \tag{2.11}$$

It follows that

$$(C^T f, g)_{\mathscr{F}^T} = (W^T f, W^T g)_{\mathscr{H}} = (u^f(\cdot, T), u^g(\cdot, T))_{\mathscr{H}};$$

that is, C^T connects the metrics of the inner and outer spaces. It is important that the connecting operator is expressed clearly and simply in terms of the reaction operator, namely, that we have the relation

$$C^T = 2^{-1}(S^T)^* R^{2T} J^{2T} S^T, \tag{2.12}$$

in which $S^T\colon \mathscr{F}^T \to \mathscr{F}^{2T}$ is the extension of controls for times t from $\Gamma \times [0,T]$ onto $\Gamma \times [0, 2T]$ as odd functions with respect to $t = T$; $J^{2T}\colon \mathscr{F}^{2T} \to \mathscr{F}^{2T}$ is given by (see [1], [3])

$$(J^{2T} f)(\cdot, t) = \int_0^t f(\cdot, s)\, ds.$$

• The operator Δ_g, determined by the evolution of α^T, does not depend on time; that is, the system is *stationary.* As a consequence, a delay in the control causes the same delay in the waves, expressed by the relation

$$u^{f_{T-r}}(\cdot, T) = u^f(\cdot, r), \quad 0 < r \le T, \tag{2.13}$$

in which

$$f_{T-r}(\cdot, t) := \begin{cases} 0, & 0 \le t < T - r, \\ f(\cdot, t - (T - r)), & T - r \le t < T \end{cases}$$

is the delayed control.

Local completeness of waves. A set of waves

$$\mathscr{U}_\sigma^r := \{u^f(\cdot, r) \mid f \in \mathscr{F}_\sigma^T\} \subset \mathscr{H}$$

is called *accessible* (from the part of the boundary σ, up to the moment $t = r$). From (2.13), we have

$$\mathscr{U}_\sigma^r = W^T \mathscr{F}_\sigma^{T,r}, \quad 0 < r \le T. \tag{2.14}$$

Problem (2.2)–(2.4) is hyperbolic and, as such, has a *finite domain of influence.* A consequence of this property, and one way of describing it, is the relation

$$\operatorname{supp} u^f(\cdot, r) \subset \overline{\Omega^r[\sigma]}, \quad f \in \mathscr{F}_\sigma^T.$$

This relation means that acoustic waves propagate in Ω with finite (unit) speed and, up to time $t = r$, fill a subdomain in Ω consisting of points r-close to σ. As a result, we have the following inclusion:[10] $\mathscr{U}_\sigma^r \subset \mathscr{H}\langle \Omega^r[\sigma]\rangle$. Another fundamentally important characteristic of this inclusion is that it turns out to be dense, such that the following equality holds:

$$\overline{\mathscr{U}_\sigma^r} \overset{(2.14)}{=} \overline{W^T \mathscr{F}_\sigma^{T,r}} = \mathscr{H}\langle \Omega^r[\sigma]\rangle, \quad 0 < r \le T \tag{2.15}$$

(closure in $\mathscr{H}$). In particular, if $\operatorname{diam} \Omega < r \le T$, then, for any $\sigma \in \mathscr{R}$, we have $\overline{\mathscr{U}_\sigma^r} = \mathscr{H}\langle \Omega \rangle = \mathscr{H}$; see (2.1). Thus, since clearly $\mathscr{U}_\sigma^r \subset W^T \mathscr{F}^T =: \operatorname{Ran} W^T$, we obtain

$$\overline{\mathscr{U}_\sigma^r} = \overline{\operatorname{Ran} W^T} = \mathscr{H}, \quad r > \operatorname{diam} \Omega. \tag{2.16}$$

Property (2.15), that waves are dense in the domain filled by them, can be deduced from the fundamental Holmgren–John–Tatar uniqueness theorem; see [1], [3]. In control theory, it is interpreted as the *local boundary controllability* of α^T. This shows that accessible sets are sufficiently rich: any function with support in $\Omega^r[\sigma]$ can be approximated (in the $\mathscr{H}$-metric) by a wave $u^f(\cdot, T)$ for a suitable choice of control $f \in \mathscr{F}_\sigma^{T,r}$. From the point of view of the inverse problem, controllability is a positive characteristic.

[10] For the definition of $\mathscr{H}\langle A \rangle$, see (2.8).

According to a general principle of the theory of systems [11], the richer the set of states that an external observer can recognise in a system, the greater the range of information on its internal properties that he can extract from the mapping “input $\mapsto$ output”.

2.5. **The wave model.** In the acoustic tomography problem, the algebra $\mathfrak{T}_{\rm op}^{\rm mod}$, introduced below, plays the role of the model $\mathfrak{A}^{\rm mod}$ in the introduction.

• We call the projector P^r_σ from $\mathscr{H}$ onto a accessible subspace $\overline{\mathscr{U}^r_\sigma}$ a *wave projector.* We recall that X^r_σ is a projector from $\mathscr{H}$ onto $\mathscr{H}\langle\Omega^r[\sigma]\rangle$ which cuts off functions in a neighbourhood of $\Omega^r[\sigma]$. In view of property (2.15), that waves are dense, we have

$$P^r_\sigma = X^r_\sigma, \quad r>0,\ \sigma\in\mathscr{R}. \tag{2.17}$$

From this and (2.10), for $T > \operatorname{diam}\Omega$ we obtain the *wave representation* of an eikonal

$$\check{\tau}_\sigma \overset{2.16,\,(2.17)}{=} \int_0^\infty r\,dP^r_\sigma = \int_0^T r\,dP^r_\sigma, \quad \sigma\in\mathscr{R}. \tag{2.18}$$

• Let

$$W^T = U^T|W^T| \tag{2.19}$$

be a *polar decomposition* of the control operator, in which

$$|W^T| := [(W^T)^*W^T]^{1/2} \overset{(2.11)}{=} [C^T]^{1/2} \tag{2.20}$$

is an operator acting in $\mathscr{F}^T$, and $\mathscr{F}^T \xrightarrow{U^T} \mathscr{H}$ is an isometry from $\operatorname{Ran}|W^T|$ onto $\operatorname{Ran}W^T$.

In what follows, we shall assume that $T > \operatorname{diam}\Omega$, so that (2.16) holds.

• We call the subspace

$$\mathscr{H}^{\rm mod} := \overline{\operatorname{Ran}|W^T|} \overset{(2.20)}{=} \overline{\operatorname{Ran}[C^T]^{1/2}} \subset \mathscr{F}^T \tag{2.21}$$

the *model inner space* of α^T. In view of (2.16), U^T is an isometry from $\mathscr{H}^{\rm mod}$ onto $\mathscr{H}$ and can play the role of the unitary operator U from paragraph 4 of §2.3.

In $\mathscr{H}^{\rm mod}$, we choose *model accessible sets*

$$\mathscr{U}^{r\,\rm mod}_\sigma := |W^T|\mathscr{F}^{T,r}_\sigma \overset{(2.19)}{=} (U^T)^*W^T\mathscr{F}^{T,r}_\sigma \overset{(2.14)}{=} (U^T)^*\mathscr{U}^r_\sigma = [C^T]^{1/2}\mathscr{F}^{T,r}_\sigma, \tag{2.22}$$

for $0<r\le T$. Let $P^{r\,\rm mod}_\sigma$ be a projector from $\mathscr{H}^{\rm mod}$ onto $\overline{\mathscr{U}^{r\,\rm mod}_\sigma}$. The connection between projectors $P^{r\,\rm mod}_\sigma = (U^T)^*P^r_\sigma U^T$ follows from the relation $\mathscr{U}^{r\,\rm mod}_\sigma = (U^T)^*\mathscr{U}^r_\sigma$. This leads in turn to the representation

$$\begin{aligned}\check{\tau}^{\rm mod}_\sigma := (U^T)^*\check{\tau}_\sigma U^T \overset{(2.18)}{=} (U^T)^*\left[\int_0^T r\,dP^r_\sigma\right]U^T \\ = \int_0^T r\,d[(U^T)^*P^r_\sigma U^T] = \int_0^T r\,dP^{r\,\rm mod}_\sigma\end{aligned} \tag{2.23}$$

for *model eikonals* $\check{\tau}^{\rm mod}_\sigma \in \mathfrak{B}(\mathscr{H}^{\rm mod})$.

• We call the algebra

$$\begin{aligned}\mathfrak{T}_{\rm op}^{\rm mod} &:= \bigvee\{\check{\tau}^{\rm mod}_\sigma \mid \sigma\in\mathscr{R}\} = \bigvee\{(U^T)^*\check{\tau}_\sigma U^T \mid \sigma\in\mathscr{R}\} \\ &= (U^T)^*\left[\bigvee\{\check{\tau}_\sigma \mid \sigma\in\mathscr{R}\}\right]U^T = (U^T)^*\mathfrak{T}_{\rm op}U^T \subset \mathfrak{B}(\mathscr{H}^{\rm mod})\end{aligned}$$

a *wave model* of $\mathfrak{T}_{\rm op}$. As we see from its construction, the wave model is an inherent object of α^T, an attribute of it. Namely, it turns out to be that realisation of the general construction of paragraphs 4 and 5 of §2.3 which is adequate for the inverse problem.

2.6. **Reconstruction of Ω.** Suppose that the external observer has the reaction operator R^{2T} available to him, given for fixed $T > \operatorname{diam}\Omega$. He can recover the manifold Ω by the following scheme.

1. Using R^{2T}, we define C^T in accordance with (2.12) and find the positive square root $[C^T]^{1/2}$.

2. We form the space $\mathscr{H}^{\rm mod}$ (see (2.21)). Using (2.22), we define the sets $\{\mathscr{U}_\sigma^{r\,\rm mod}\}_{\sigma\in\mathscr{R}}$ in the latter and find the corresponding projectors $P_\sigma^{r\,\rm mod}$. Using the projectors, we find the model eikonals $\check\tau_\sigma^{\rm mod}$ in accordance with (2.23). In this way, we obtain a mapping $m\colon \sigma\mapsto\check\tau_\sigma^{\rm mod}$.

3. Taking the pair $\{\mathscr{H}^{\rm mod}, m\}$, we find a copy $(\widetilde\Omega,\tilde g)$ by the scheme of paragraph 5 of §2.3. It solves the problem posed.

§ 3. Electromagnetic tomography

3.1. **The settings.** Now suppose, in addition to the condition in (2.1), that (Ω, g) is an *orientable* manifold and that $\dim\Omega = 3$. The operations of vector analysis are defined in Ω: the scalar product $g(\cdot,\cdot)$, the vector product $\wedge$, the curl rot, and the divergence div (see, for example, [13]).

Direct problem. The propagation of electromagnetic waves in Ω is described by the Maxwell system

$$e_t = \operatorname{rot} h,\ \ h_t = -\operatorname{rot} e \quad \text{in } (\Omega\backslash\Gamma)\times(0,T), \tag{3.1}$$

$$e|_{t=0} = 0,\ \ h|_{t=0} = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$e_\theta = f,\ \ 0\le t\le T, \tag{3.3}$$

where $e_\theta := e - g(e,\nu)\nu$ is the tangential component of the vector e on the boundary, f is the *boundary control* (a time-dependent tangent field on Γ), and $e = e^f(x,t)$ and $h = h^f(x,t)$ are electric and magnetic fields, respectively.

The response of the system to the effect of a control is described by the *reaction operator*, which acts according to the rule

$$R^T f := \nu\wedge h^f|_{\Gamma\times[0,T]}.$$

The ***inverse problem*** consists of recovering (Ω, g) by means of its reaction operator. To give a precise formulation: using the operator R^{2T}, given for fixed $T > \operatorname{diam}\Omega$, it is required to construct the Riemannian manifold $(\widetilde\Omega,\tilde g)$ with boundary $\widetilde\Gamma = \Gamma$ such that $\widetilde R^{2T} = R^{2T}$ for its reaction operator $\widetilde R^{2T}$.

The meaning of the condition on T and the doubling of the time of observation is the same as in the inverse AT problem.

3.2. **The eikonal algebra.** ***The space $\mathscr{C}$.*** The space of vector fields $\mathscr{H} := \vec L_2(\Omega)$ with scalar product

$$(a,b)_{\mathscr{C}} := \int_\Omega a\cdot b\,|d\Omega|$$

contains the *space of rotors* $\mathscr{C} := \{\operatorname{rot} h \mid h, \operatorname{rot} h\in\mathscr{H}\}$, which plays a major role in what follows. Its elements are *solenoidal fields*: $y\in\mathscr{C}$ implies $\operatorname{div} y = 0$. Smooth fields in $\mathscr{C}\cap\vec C^\infty(\Omega)$ are dense in $\mathscr{C}$. The space $\mathscr{C}$ contains subspaces

$$\mathscr{C}\langle\Omega^r[\sigma]\rangle := \overline{\left\{\operatorname{rot} h \mid h\in\vec C^\infty(\Omega),\ \operatorname{supp} h\subset\Omega^r[\sigma]\right\}},\quad r>0,\ \sigma\in\mathscr{R}.$$

We denote the projector from $\mathscr{C}$ onto $\mathscr{C}\langle\Omega^r[\sigma]\rangle$ by Y^r_σ. Unlike its scalar analogue — X^r_σ from (2.10) — the action of Y^r_σ is not reduced to a section of the field in the subdomain $\Omega^r[\sigma]$. Moreover, Y^r_σ is not a local operator.[11] As a consequence, projectors Y^r_σ corresponding to different σ do not in general commute.

Eikonals. We adopt a definition that is inspired by (2.10): we call operators of the form

$$\varepsilon_\sigma := \int_0^\infty r\, dY^r_\sigma \in \mathfrak{B}(\mathscr{C}) \tag{3.4}$$

eikonals. Such an operator is bounded in view of the compactness of Ω, and the integration in (3.4) is in fact taken over the interval $0 \le r \le \operatorname{diam}\Omega$, since, for $r > \operatorname{diam}\Omega$, Y^r_σ is equal to the identity operator, for any σ. The eikonals ε_σ are selfadjoint positive operators in $\mathscr{C}$. Since projectors Y^r_σ corresponding to different σ do not, generally speaking, commute, the corresponding eikonals ε_σ may not commute either.

The key to using eikonals in the inverse problem is the following result [6]. We have the representation

$$\varepsilon_\sigma y = \tau_\sigma y + Ky, \tag{3.5}$$

in which τ_σ is understood as an operator from $\mathscr{C}$ into $\mathscr{H}$, multiplying fields pointwise by functions $\tau_\sigma := \operatorname{dist}(\cdot,\sigma)$, and the operator $K : \mathscr{C} \to \mathscr{H}$ is compact. We note that the terms on the right-hand side may not belong separately to $\mathscr{C}$.

The algebra $\mathfrak{E}$. Eikonals generate the algebra

$$\mathfrak{E} := \bigvee\{\varepsilon_\sigma \mid \sigma \in \mathscr{R}\} \subset \mathfrak{B}(\mathscr{C}).$$

Unlike the algebra of acoustic eikonals, this is no longer commutative. From the point of view of the inverse problem, noncommutativity is a complication.[12] We can get around this, however, thanks to the following result.

Let $\mathfrak{K} \subset \mathfrak{B}(\mathscr{C})$ be an ideal of compact operators. Let

$$\dot{\mathfrak{E}} := \mathfrak{E}/[\mathfrak{K} \cap \mathfrak{E}] = \bigvee\{\pi\varepsilon_\sigma \mid \sigma \in \mathscr{R}\} \tag{3.6}$$

be the quotient algebra, equipped with the canonical norm and algebraic operations, where π is the canonical projection of $\mathfrak{E}$ onto $\dot{\mathfrak{E}}$, associating to ε_σ the corresponding equivalence class. As shown in [6], $\dot{\mathfrak{E}}$ is a *commutative* Banach algebra, isometrically isomorphic to the algebra of scalar eikonals $\mathfrak{T}$. The isometry $\dot{\mathfrak{E}} \xrightarrow{D} \mathfrak{T}$ is given by the connection of generators

$$\dot{\mathfrak{E}} \ni \pi\varepsilon_\sigma \overset{D}{\longmapsto} \tau_\sigma \in \mathfrak{T}, \quad \sigma \in \mathscr{R},$$

where ε_σ and τ_σ are connected by the relation (3.5). It follows from the isometry of the algebras that the spectra $\widehat{\dot{\mathfrak{E}}}$ and $\widehat{\mathfrak{T}} = \Omega$ are homeomorphic. Consequently, $\widehat{\dot{\mathfrak{E}}}$ encodes Ω and is suitable for the role of the basic algebra $\mathfrak{A}$ in the introduction.

The model. Let $\mathscr{C}^{\mathrm{mod}}$ be some Hilbert space, $U\colon \mathscr{C}^{\mathrm{mod}} \to \mathscr{C}$ a unitary operator, and $\mathfrak{B}(\mathscr{C}) \ni a \overset{u}{\longmapsto} U^*aU \in \mathfrak{B}(\mathscr{C}^{\mathrm{mod}})$ the corresponding algebra isometry. We call the algebra

$$\mathfrak{E}^{\mathrm{mod}} := u\,\mathfrak{E}$$

the *model* of $\mathfrak{E}$, and the operators $\varepsilon^{\mathrm{mod}}_\sigma := u\varepsilon_\sigma$ model eikonals.

Let $\mathfrak{K}^{\mathrm{mod}} \subset \mathfrak{B}(\mathscr{C}^{\mathrm{mod}})$ be an ideal of compact operators; we introduce the quotient algebra

$$\dot{\mathfrak{E}}^{\mathrm{mod}} := \mathfrak{E}^{\mathrm{mod}}/[\mathfrak{K}^{\mathrm{mod}} \cap \mathfrak{E}^{\mathrm{mod}}] = \bigvee\{\pi\varepsilon^{\mathrm{mod}}_\sigma \mid \sigma \in \mathscr{R}\}. \tag{3.7}$$

---

[11] An operator L is *local* if $\operatorname{supp} Ly \subset \operatorname{supp} y$. The projectors Y^r_σ do not possess this property; see [6].

[12] Because of it, we cannot solve the graph reconstruction problem; see [8].

We see from (3.7) and (3.6) that $\dot{\mathfrak{E}}^{\rm mod}$ and $\dot{\mathfrak{E}}$ are isometric. The latter is isometric to the eikonal algebra $\mathfrak{T} = D\dot{\mathfrak{E}}$. Consequently, $\dot{\mathfrak{E}}^{\rm mod}$ and $\mathfrak{T}$ are isometric and there exists a homeomorphism s, connecting their spectra and functions of the spectra:

$$\widehat{\mathfrak{T}} = \Omega \xrightarrow{s} \widetilde{\Omega} := \widehat{\dot{\mathfrak{E}}^{\rm mod}}, \quad G\dot{\mathfrak{E}}^{\rm mod} = C(\widetilde{\Omega}) \ni G\pi\varepsilon_\sigma^{\rm mod} =: \widetilde{\tau}_\sigma = \tau_\sigma \circ s^{-1}, \tag{3.8}$$

where G is the Gelfand transform. These considerations are illustrated by the diagram in Figure 3.

$$\begin{tikzcd}
\Omega \arrow[rr, "s"] & & \boxed{\widetilde{\Omega}} & \\
C(\Omega) = \mathfrak{T} \arrow[u, squiggly] & \dot{\mathfrak{E}} \arrow[l, "D"'] \arrow[r] & \boxed{\dot{\mathfrak{E}}^{\rm mod}} \arrow[u, squiggly] \arrow[r, "G"] & \boxed{C(\widetilde{\Omega})} \\
& \mathfrak{E} \arrow[u, "\pi"] \arrow[r, "u"] & \boxed{\mathfrak{E}^{\rm mod}} \arrow[u, "\pi"] & \\
\tau_\sigma \arrow[uu, Rightarrow] \arrow[r, "i"] & \varepsilon_\sigma \arrow[u, Rightarrow] \arrow[r, "u"] & \boxed{\varepsilon_\sigma^{\rm mod}} \arrow[u, Rightarrow] \arrow[r, "G"] & \boxed{\widetilde{\tau}_\sigma} \\
& & \boxed{\sigma} \arrow[u, "m"] \arrow[ur, "(2.7)"'] &
\end{tikzcd}$$

FIGURE 3. Reconstruction in EMT

Scheme for reconstruction. Suppose that, by measurement at the boundary, the external observer has determined the pair $\{\mathscr{C}^{\rm mod}, m\}$, where m is the mapping

$$\mathscr{R} \ni \sigma \overset{m}{\longmapsto} \varepsilon_\sigma^{\rm mod} \in \mathfrak{B}(\mathscr{C}^{\rm mod}).$$

He can then recover the manifold by the following scheme:

- construct $\mathfrak{E}^{\rm mod} = \bigvee\{\varepsilon_\sigma^{\rm mod} \mid \sigma \in \mathscr{R}\} \subset \mathfrak{B}(\mathscr{C}^{\rm mod})$, then pass to the quotient algebra $\dot{\mathfrak{E}}^{\rm mod}$ in accordance with (3.7);
- find its spectrum $\widetilde{\Omega} := \widehat{\dot{\mathfrak{E}}^{\rm mod}}$ and its canonical Gelfand realisation, the algebra $G\dot{\mathfrak{E}}^{\rm mod} = C(\widetilde{\Omega})$;
- use (3.8) to define the mapping $\sigma \overset{G\pi m}{\longmapsto} \widetilde{\tau}_\sigma$, a version of (2.7). *This determines the copy* $(\widetilde{\Omega}, \tilde{g})$.

It remains to show that the reaction operator of the Maxwell system gives an adequate pair $\{\mathscr{C}^{\rm mod}, m\}$.

3.3. **The system μ^T.** We associate problem (3.1)–(3.3) with the dynamical system μ^T and endow it with the corresponding attributes.

• The control space $\mathscr{F}^T := L_2([0,T]; \vec{L}_2(\Gamma))$ is the *outer space* of μ^T. The smooth class

$$\mathscr{M}^T := \big\{f \in C^\infty([0,T]; \vec{C}^\infty(\Gamma)) \mid \operatorname{supp} f \subset (0,T]\big\}$$

is dense in $\mathscr{F}^T$. For $f \in \mathscr{F}^T$, problem (3.1)–(3.3) has a unique classical smooth solution. We set $\vec{L}_2(\sigma) := \{b \in \vec{L}_2(\Gamma) \mid \operatorname{supp} b \subseteq \sigma\}$. The outer space contains the subspaces

$$\mathscr{F}_\sigma^{T,r} := \big\{f \in L_2([0,T]; \vec{L}_2(\sigma)) \mid \operatorname{supp} f \subseteq [T-r, T]\big\}, \quad 0 < r \le T, \ \sigma \in \mathscr{R},$$

generated by the controls, acting from σ with delay $T - r$ (r is the time of action).

• *The outer space* is the space $\mathscr{C}\oplus\mathscr{C}$; the solutions of the problem $\{e^f(\cdot,t),h^f(\cdot,t)\}$ are those of its elements that depend on time. We select[13] the *electrical part* $\mathscr{C}\oplus\{0\}\ni e^f(\cdot,t)$.

• The correspondence "input $\mapsto$ state" is realised by the *control operator* $W^T_\mu\colon\mathscr{F}^T\to\mathscr{C}\oplus\mathscr{C}$, $\operatorname{Dom}W^T_\mu=\mathscr{M}^T$, $W^T_\mu f:=\{e^f(\cdot,T),\,h^f(\cdot,T)\}$. The operator $W^T\colon\mathscr{F}^T\to\mathscr{C}$, $W^Tf:=e^f(\cdot,T)$ corresponds to the electrical subsystem and is also defined on $\mathscr{M}^T$. In contrast to the acoustic system, W^T_μ and W^T are not bounded, but admit closure.

• The mapping "input $\mapsto$ output" in μ^T is realised by the *reaction operator* $R^T\colon\mathscr{F}^T\to\mathscr{F}^T$, $\operatorname{Dom}R^T=\mathscr{M}^T$, $R^Tf=\nu\wedge h^f\big|_{\Gamma\times[0,T]}$.

• We introduce the connecting operator of the electrical part $C^T\colon\mathscr{F}^T\to\mathscr{F}^T$ via the *connecting form* c^T, $\operatorname{Dom}c^T=\mathscr{M}^T\times\mathscr{M}^T$,

$$c^T[f,g]:=(e^f(\cdot,T),e^g(\cdot,T))_{\mathscr{C}}=(W^Tf,W^Tg)_{\mathscr{C}},$$

a nonnegative bilinear form. As such, it is closed, and the closure $\bar c^T$ is defined on $\mathscr{N}^T\times\mathscr{N}^T$, where $\mathscr{N}^T$ is a linear manifold in $\mathscr{F}^T$, $\mathscr{N}^T\supset\mathscr{M}^T$. There is a selfadjoint operator C^T associated with $\bar c^T$, defined by the relation

$$(C^Tf,g)_{\mathscr{F}^T}=\bar c^T[f,g],\quad f\in\operatorname{Dom}C^T,\ g\in\mathscr{N}^T;$$

see [9]. The closure of c^T is equivalent to the closure of W^T, so that $\mathscr{N}^T=\operatorname{Dom}\overline{W^T}=\operatorname{Dom}(C^T)^{1/2}$. Consequently, as we have c^T, we can extend W^T from $\mathscr{M}^T$ to $\mathscr{N}^T$. In what follows, this extension will be assumed to have been implemented, and we put $\operatorname{Dom}W^T=\mathscr{N}^T$.

As a result, we have the relation

$$\bar c^T[f,g]=\big([C^T]^{1/2}f,[C^T]^{1/2}g\big)_{\mathscr{F}^T}=(W^Tf,W^Tg)_{\mathscr{C}},\quad f,g\in\mathscr{N}^T. \tag{3.9}$$

The key fact is that the connecting form is expressed in terms of the reaction operator in an explicit and simple way. We have the representation

$$c^T[f,g]=2^{-1}\big((S^T)^*R^{2T}J^{2T}S^Tf,g\big)_{\mathscr{F}^T},\quad f,g\in\mathscr{M}^T, \tag{3.10}$$

in which the operator $S^T\colon\mathscr{F}^T\to\mathscr{F}^{2T}$ extends the controls (with respect to time) from $\Gamma\times[0,T]$ onto $\Gamma\times[0,2T]$ as an odd function with respect to $t=T$; $J^{2T}\colon\mathscr{F}^{2T}\to\mathscr{F}^{2T}$ is given by

$$(J^{2T}f)(\cdot,t)=\int_0^t f(\cdot,s)\,ds\quad\text{(see [3]).}$$

Local completeness of waves. The set

$$\mathscr{E}^r_\sigma:=\big\{e^f(\cdot,r)\mid f\in\mathscr{F}^T_\sigma\cap\mathscr{N}^T\big\}=W^T[\mathscr{F}^{T,r}_\sigma\cap\mathscr{N}^T] \tag{3.11}$$

is said to be *accessible* (from the part of the boundary σ, in time $t=s$). The equality in (3.11) holds because the system μ^T is stationary.

For the Maxwell system, the principle of the finiteness of the domain of influence is valid, and the waves propagate in it with unit speed. Hence, we have

$$\operatorname{supp}e^f(\cdot,r)\subset\overline{\Omega^r[\sigma]},\quad f\in\mathscr{F}^T_\sigma\cap\mathscr{N}^T,$$

leading to the inclusion $\mathscr{E}^r_\sigma\subset\mathscr{C}\langle\Omega^r[\sigma]\rangle$. This inclusion turns out to be dense: we have the equality

$$\overline{\mathscr{E}^r_\sigma}\overset{(3.11)}{=}\overline{W^T[\mathscr{F}^{T,r}_\sigma\cap\mathscr{N}^T]}=\mathscr{C}\langle\Omega^r[\sigma]\rangle,\quad 0<r\le T. \tag{3.12}$$

[13]The selection of the electrical part is motivated by the character of the control; see (3.3).

In particular, if $\operatorname{diam}\Omega < r \le T$, then, for any $\sigma \in \mathscr{R}$, we have $\overline{\mathscr{E}^r_\sigma} = \mathscr{C}\langle\Omega\rangle = \mathscr{C}$; see (2.1). Whence, in view of the obvious $\mathscr{E}^r_\sigma \subset W^T\mathscr{F}^T =: \operatorname{Ran} W^T$, we obtain

$$\overline{\mathscr{E}^r_\sigma} = \overline{\operatorname{Ran} W^T} = \mathscr{C}, \quad \operatorname{diam}\Omega < r \le T. \tag{3.13}$$

We can deduce (3.12) from the vector variant of the Holmgren–John–Tatar uniqueness theorem; see [3]. This shows that the key property for our approach — *waves are complete in the domain they fill* — also holds in μ^T.

3.4. **The wave model.** Here we produce the pair $\{\mathscr{C}^{\mathrm{mod}}, m\}$, which we discussed at the end of §3.2.

• A projector P^r_σ from $\mathscr{C}$ onto an accessible subspace $\overline{\mathscr{E}^r_\sigma}$ is called a *wave projector.* We recall that Y^r_σ is a projector from $\mathscr{C}$ onto $\mathscr{C}\langle\Omega^r[\sigma]\rangle$. In view of the density of waves (3.12), we have the equality

$$P^r_\sigma = Y^r_\sigma, \quad r > 0,\ \sigma \in \mathscr{R}. \tag{3.14}$$

In particular, in view of (3.13) for $\operatorname{diam}\Omega < r \le T$, we have $P^r_\sigma = \mathbb{I}$, and the *wave representation* of an eikonal

$$\breve{\varepsilon}_\sigma \overset{(3.14)}{=} \int_0^\infty r\, dP^r_\sigma = \int_0^T r\, dP^r_\sigma, \quad \sigma \in \mathscr{R}, \tag{3.15}$$

follows from (3.4).

• Let

$$W^T = U^T|W^T| \tag{3.16}$$

be a polar decomposition of the control operator in which

$$|W^T| := [(W^T)^*W^T]^{1/2} \overset{(3.9)}{=} [C^T]^{1/2}$$

is an operator which acts from $\mathscr{F}^T$ onto the domain of definition $\operatorname{Dom}|W^T| = \mathscr{N}^T$, and $\mathscr{F}^T \xrightarrow{U^T} \mathscr{C}$ is an isometry from $\operatorname{Ran}|W^T|$ onto $\operatorname{Ran} W^T$.

The following arguments are carried out for $T > \operatorname{diam}\Omega$. Under this assumption, we have $\overline{\operatorname{Ran}|W^T|} \overset{(3.13)}{=} \mathscr{C}$.

• The subspace

$$\mathscr{C}^{\mathrm{mod}} := \overline{\operatorname{Ran}|W^T|} = \overline{\operatorname{Ran}[C^T]^{1/2}} \subset \mathscr{F}^T \tag{3.17}$$

is the model inner space of α^T_μ. In view of (3.13), U^T turns out to be an isometry from $\mathscr{C}^{\mathrm{mod}}$ onto $\mathscr{C}$.

In $\mathscr{C}^{\mathrm{mod}}$, we select model accessible sets

$$\begin{aligned}\mathscr{E}^{r\,\mathrm{mod}}_\sigma &:= |W^T|[\mathscr{F}^{T,r}_\sigma \cap \mathscr{N}^T] \overset{(3.16)}{=} (U^T)^*W^T[\mathscr{F}^{T,r}_\sigma \cap \mathscr{N}^T]\\ &= (U^T)^*\mathscr{E}^r_\sigma = [C^T]^{1/2}[\mathscr{F}^{T,r}_\sigma \cap \mathscr{N}^T], \quad 0 < r \le T.\end{aligned} \tag{3.18}$$

Let $P^{r\,\mathrm{mod}}_\sigma$ be a projector from $\mathscr{C}^{\mathrm{mod}}$ onto $\overline{\mathscr{E}^{r\,\mathrm{mod}}_\sigma}$. The connection between projectors $P^{r\,\mathrm{mod}}_\sigma = (U^T)^*P^r_\sigma U^T$ follows from the relation $\mathscr{E}^{r\,\mathrm{mod}}_\sigma = (U^T)^*\mathscr{E}^r_\sigma$. This leads in turn to the representation

$$\begin{aligned}\varepsilon^{\mathrm{mod}}_\sigma &:= (U^T)^*\varepsilon_\sigma U^T \overset{(3.15)}{=} (U^T)^*\left[\int_0^T r\, dP^r_\sigma\right]U^T\\ &= \int_0^T r\, d[(U^T)^*P^r_\sigma U^T] = \int_0^T r\, dP^{r\,\mathrm{mod}}_\sigma\end{aligned} \tag{3.19}$$

for model eikonals $\varepsilon^{\mathrm{mod}}_\sigma \in \mathfrak{B}(\mathscr{C}^{\mathrm{mod}})$.

• Let $\mathfrak{B}(\mathscr{C}) \ni a \overset{u}{\longmapsto} (U^T)^* a U^T \in \mathfrak{B}(\mathscr{C}^{\rm mod})$ be the algebra isometry induced by U^T. The algebra

$$\mathfrak{E}^{\rm mod} := \bigvee\{\varepsilon_\sigma^{\rm mod} \mid \sigma \in \mathscr{R}\} = \bigvee\{u\varepsilon_\sigma \mid \sigma \in \mathscr{R}\} = u \vee \{\varepsilon_\sigma \mid \sigma \in \mathscr{R}\} = u\,\mathfrak{E} \subset \mathfrak{B}(\mathscr{C}^{\rm mod})$$

is the *wave model* of $\mathfrak{E}$.

3.5. **Reconstruction of Ω.** Suppose that for fixed $T > \operatorname{diam}\Omega$, we are given the reaction operator R^{2T} of μ^T. The following procedure restores Ω.

1. Using R^{2T}, we define C^T via the connecting form (3.10), and find $[C^T]^{1/2}$.

2. We form $\mathscr{C}^{\rm mod}$ (see (3.17)). Using (3.18), we define the sets $\{\mathscr{E}_\sigma^{r\,\rm mod}\}_{\sigma\in\mathscr{R}}$ in the latter, and find the corresponding projectors $P_\sigma^{r\,\rm mod}$. Using these, we find the model eikonals $\varepsilon_\sigma^{\rm mod}$ in accordance with (3.19). Thus, we obtain the mapping $\sigma \overset{m}{\longmapsto} \varepsilon_\sigma^{\rm mod}$. We therefore have the pair $\{\mathscr{C}^{\rm mod}, m\}$.

3. Using the scheme described at the end of §3.2, the pair $\{\mathscr{C}^{\rm mod}, m\}$ defines a copy $(\widetilde{\Omega}, \tilde{g})$, thus solving the problem posed.

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