

## SYMMETRIC BAND COMPLEXES OF THIN TYPE AND CHAOTIC SECTIONS WHICH ARE NOT QUITE CHAOTIC

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*On the occasion of Yu. Ilyashenko's 70th birthday*

**ABSTRACT.** In a recent paper we constructed a family of foliated 2-complexes of thin type whose typical leaves have two topological ends. Here we present simpler examples of such complexes that are, in addition, symmetric with respect to an involution and have the smallest possible rank. This allows for constructing a 3-periodic surface in the three-space with a plane direction such that the surface has a central symmetry, and the plane sections of the chosen direction are chaotic and consist of infinitely many connected components. Moreover, typical connected components of the sections have an asymptotic direction, which is due to the fact that the corresponding foliation on the surface in the 3-torus is not uniquely ergodic.

### 1. INTRODUCTION

Our motivation for this work came from the problem about the asymptotic behavior of plane sections of triply periodic surfaces in  $\mathbb{R}^3$  posed by S. P. Novikov in [18] in connection with conductivity theory in monocrystals. The physical model where such sections appeared was studied by I. M. Lifshitz and his school in the 1950–60s. The surface in the model is the Fermi surface of a normal metal and is defined as the level surface of the dispersion law in the space of quasimomenta, which topologically is a 3-torus. The Fermi surface of a metal can also be considered as a 3-periodic surface in the 3-space.

The model is designed to study the conductivity in a monocrystal at low temperature under the influence of a constant and uniform magnetic field  $H$ . According to the model the trajectories of electron's quasimomentum are connected components of the sections of the Fermi surface by planes perpendicular to  $H$ .

Novikov suggested studying plane sections of general null-homologous surfaces in the 3-torus and asked what asymptotic properties the unbounded connected components of such sections may have. The problem can be considered as one about a foliation induced by a closed 1-form on a closed oriented surface, but as such it is very specific, as there are serious restrictions on the cohomology class of the 1-form.

The first result in this area was obtained by A. Zorich, who discovered what is now called the integrable case [23]. It was shown later by I. Dynnikov that generically either the integrable case occurs or there are no open trajectories (trivial case) [7].

For non-generic vectors  $H$  whose components are dependent over  $\mathbb{Z}$ , S. Tsarev constructed examples that do not fit into the trivial or integrable case, though minimal

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components of the induced foliation on the Fermi surface were of genus 1; see [8]. A situation in which the foliation has a single minimal component of genus 3 and  $H$  is completely irrational was discovered by I. Dynnikov in [8]. Such examples are now referred to as chaotic.

Physical implications from different types of dynamics of the trajectories for the conductivity tensor are discussed in [15, 16].

After the work [9], where the construction of [8] was reformulated in different terms, it became clear that the main instrument for studying chaotic examples coincided with a particular case of an object that was well known in the geometric group theory and in the theory of dynamical systems under the name of band complex, which is a measured foliated 2-complex of certain type. The theory of such complexes was developed by E. Rips; see [3]. In a sense, constructing examples with chaotic dynamics in Novikov's problem is equivalent to constructing band complexes of thin type consisting of three bands.

Several years ago A. Maltsev drew the first author's attention to the fact that a Fermi surface of any monocrystal is always centrally symmetric. So, it is natural to single out the case when our surface has such a symmetry. For the corresponding band complexes this means that they must be invariant under an involution flipping the transverse orientation of the foliation. Symmetric band complexes of thin type, which give examples of chaotic dynamic on a centrally symmetric surface, are constructed by A. Skripchenko in [19].

The behavior of chaotic trajectories in Novikov's problem is not well understood in general. One of the interesting questions is how many trajectories may lie in a single plane. In the theory of band complexes of thin type this is related to the question about the number of topological ends of a typical leaf. A single topological end would imply a single connected component of a typical chaotic section, and two topological ends would imply infinitely many components (see [9], [20]). This question about possible typical leaf structure of thin type band complexes is also interesting on its own.

Before recently only examples of thin type band complexes had been known in which almost all leaves had exactly one topological end [3, 4, 20]. In [10] we described the reason for that, which was the self-similarity of the known examples, and constructed examples of thin type band complexes having two-ended typical leaves. Those examples did not obey any symmetry, and it was not clear for a while whether additional symmetry would be an obstruction for a band complex to have two-ended typical leaves.

Here we show that not only symmetry but also a certain degeneracy is not an obstruction (see Theorem 2.12). Quite surprisingly, the phenomenon can be observed for band complexes that are related to the so-called regular skew polyhedron  $\{4, 6 \mid 4\}$ , a surface for which the set of all chaotic regimes was explicitly described by I. Dynnikov and R. de Leo in [6]. Our construction here appears to be even simpler than in [10].

We also analyze the corresponding chaotic dynamics on the surface in the 3-torus. We show that the induced flow, though being minimal, decomposes into two ergodic components (see Proposition 3.4). This appears to be a reason for the existence of an asymptotic direction of the trajectories in  $\mathbb{R}^3$ . In principle, the proofs of these facts, which are given in Section 3, are self-contained and do not use any band complexes. However, band complexes provide for a more intuitive way to understand the origin of the construction, and we start the exposition by introducing them.

## 2. BAND COMPLEXES

We start by recalling basic definitions.

**Definition 2.1.** A *band* is a (possibly degenerate) rectangular

$$\mathcal{B} = [a, b] \times [0, 1] \subset \mathbb{R}^2, \quad a \leq b,$$

endowed with the 1-form  $dx$ , where  $x$  is the first coordinate in the plane  $\mathbb{R}^2$ . The horizontal sides  $[a, b] \times \{0\}$  and  $[a, b] \times \{1\}$  are called *the bases* of the band; the band is *degenerate* if  $a = b$ . The value  $(b - a)$  is called *the width* of the band.

**Definition 2.2.** A *band complex* is a 2-complex  $X$  endowed with a closed 1-form  $\omega$  obtained from a union  $D$  of pairwise disjoint closed (possibly degenerate to a point) intervals of  $\mathbb{R}$ , called *the support multi-interval of  $X$* , and several pairwise disjoint bands  $\mathcal{B}_i = [a_i, b_i] \times [0, 1]$  by gluing each base of every band isometrically and preserving the orientation to a closed subinterval of  $D$ . The form  $\omega$  is the one whose restriction to each band and to  $D$  is  $dx$ , so we keep using the notation  $dx$  for it.

The 1-form  $dx$  defines a *singular foliation*  $\mathcal{F}_X$  on  $X$  whose *leaves* are maximal path connected subsets of  $X$ , to which the restriction of  $dx$  vanishes. *Singularities* of  $\mathcal{F}_X$  are such points  $p \in X$  that the restriction of  $\mathcal{F}_X$  to any open neighborhood of  $p$  is not a fibration over an open interval. It is easy to see the set of singular points is the union of vertical sides of all the bands. Leaves containing a singularity are called *singular* and otherwise *regular*.

A band complex  $Y$  is called *annulus free* if all regular leaves are simply connected.

**Definition 2.3.** The dimension

$$\dim_{\mathbb{Q}} \left\{ \int_c dx; c \in H_1(X, \text{sing}(X); \mathbb{Z}) \right\},$$

where  $\text{sing}(X)$  is the set of all singularities of  $\mathcal{F}_X$ , is called the *rank* of a band complex  $X$  and is denoted  $\text{rank}(X)$ .

*Remark 2.4.* Our definition of a band complex is less general than appears in geometric group theory as an instrument for describing actions of free groups on  $\mathbb{R}$ -trees (see [3] for details). Band complexes also appear as suspension complexes for a generalization of interval exchange transformations (more precisely, this is an analogue of Veech’s construction of zippered rectangles; see [21]).

**Definition 2.5.** Let  $Y_1$  and  $Y_2$  be band complexes with support multi-intervals  $D_1$  and  $D_2$ , respectively. We say that they are *isomorphic* if there is a homeomorphism  $f: Y_1 \rightarrow Y_2$  (called then *an isomorphism from  $Y_1$  to  $Y_2$* ) such that we have  $f^*(dx) = dx$ . If, additionally,  $Y_1$  has minimal possible number of bands among all band complexes isomorphic to  $Y_2$  and we have  $f(D_1) \subset D_2$ , then the image  $f(\mathcal{B})$  of any band  $\mathcal{B}$  of  $Y_1$  is called a *long band of  $Y_2$* .

**Definition 2.6.** A band complex  $X$  is *symmetric* if there exists an involution  $\tau: X \rightarrow X$  such that it takes bands to bands and we have  $\tau^*(dx) = -dx$ .

**Definition 2.7.** An *enhanced band complex* is a band complex  $Y$  together with an assignment of a positive real number to each band. This number is called *the length of the band*.

A band of width  $w$  and length  $\ell$  is said to *have dimensions  $w \times \ell$* . The product  $w\ell$  will be referred to as *the area* of the band. *The length* of a long band  $\mathcal{B}$  is the sum of the lengths of all bands contained in  $\mathcal{B}$ .

Each band  $\mathcal{B}$  of an enhanced band complex  $Y$  will be endowed with the measure  $\mu_Y$  obtained from the standard Lebesgue measure on  $\mathcal{B} \subset \mathbb{R}^2$  by a rescaling so as to have the total measure of  $\mathcal{B}$  equal to its area.

Two enhanced band complexes  $Y_1$  and  $Y_2$  are *isomorphic* if there exists an isomorphism  $Y_1 \rightarrow Y_2$  that preserves the lengths of long bands.

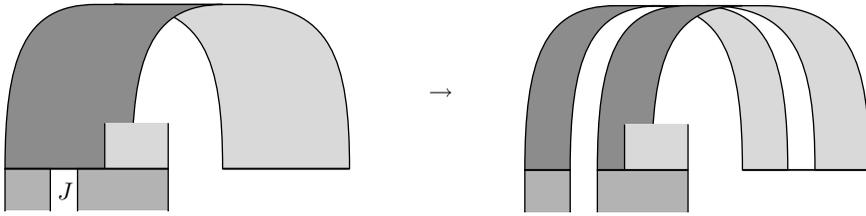


FIGURE 1. Collapse from a free arc

**Definition 2.8.** Let  $Y$  be an enhanced band complex with support multi-interval  $D$ . A *free arc* of  $Y$  is a maximal open interval  $J \subset D$  such that it is covered by one of the bases of bands, and all other bases are disjoint from  $J$ .

Let  $J$  be a free arc and  $\mathcal{B} = [a, b] \times [0, 1]$  be the band one of whose bases covers  $J$  under the attaching map. Let  $(c, d) \subset [a, b]$  be the subinterval such that  $(c, d) \times \{0\}$  or  $(c, d) \times \{1\}$  is identified with  $J$  in  $Y$ . Let  $Y'$  be the band complex obtained from  $Y$  by removing  $J$  from  $D$  and  $(c, d) \times [0, 1]$  from  $\mathcal{B}$ , thus replacing  $\mathcal{B}$  with two smaller bands,  $\mathcal{B}' = [a, c] \times [0, 1]$  and  $\mathcal{B}'' = [d, b] \times [0, 1]$ , whose bases are attached to  $D$  by the restriction of the attaching maps for the bases of  $\mathcal{B}$ . If this produces an isolated point of  $D$  such that only one degenerate band is attached to it (which may occur if  $a = c$  or  $b = d$ ), the point and the band are removed. We then say that  $Y'$  is obtained from  $Y$  by a *collapse from a free arc*; see Figure 1.

If  $Y$  is an enhanced band complex, then the lengths of  $\mathcal{B}'$  and  $\mathcal{B}''$  are set to that of  $\mathcal{B}$ .

**Definition 2.9.** An annulus free band complex  $Y$  is said to be *of thin type* if the following two conditions hold:

- (1) every leaf of the foliation  $\mathcal{F}_Y$  is everywhere dense in  $Y$ ;
- (2) there is an infinite sequence  $Y_0 = Y, Y_1, Y_2, \dots$  in which every  $Y_i, i \geq 1$ , is a band complex obtained from  $Y_{i-1}$  by a collapse from a free arc (such a sequence is said to be *produced by the Rips machine*).

*Remark 2.10.* Again, we use a particular case of a more general notion of a band complex of thin type, which need not necessarily be annulus free. For a full description of the Rips machine see [3].

From the general theory of the Rips machine [3] one can extract the following.

**Proposition 2.11.** *Let  $Y$  be a band complex made of three bands. Then the following conditions are equivalent:*

- (1)  $Y$  is of thin type;
- (2) all leaves of  $\mathcal{F}_Y$  are infinite trees that are not quasi-isometric to a straight line;
- (3) there are uncountably many leaves of  $\mathcal{F}_Y$  that are not quasi-isometric to a point, to a straight line, or to a plane.

The first example of a band complex of thin type was constructed by Levitt [14].

In [11] D. Gaboriau asked a question about the possible number of topological ends of orbits (or, equivalently, leaves) in the thin case. It was noted by M. Bestvina and M. Feighn in [3] and D. Gaboriau in [11] that all but finitely many leaves of a band complex of thin type are quasi-isometric to infinite trees with at most two topological ends, and shown that one-ended and two-ended leaves are always present and, moreover, there are uncountably many leaves of both kinds.

In [10] we constructed the first example when almost all orbits are trees with exactly two topological ends. However, due to the physical origin of our problem we are also interested to see if such band complexes exist among symmetric ones.



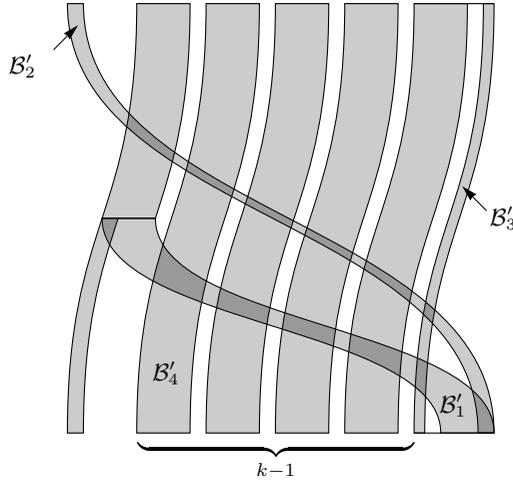


FIGURE 3. Running the Rips machine

**Lemma 2.13.** *Let  $\vec{\ell}, \vec{\ell}' \in (\mathbb{R}_+)^4$ ,  $\vec{w}, \vec{w}' \in (\mathbb{R}_+)^3$  be related as follows:*

$$\vec{\ell}A(k) = \vec{\ell}', \quad \vec{w} = B(k)\vec{w}',$$

where  $k$  is a natural number. Then the enhanced band complex  $Z(\vec{w}', \vec{\ell}')$  is isomorphic to one obtained from  $Z(\vec{w}, \vec{\ell})$  by several collapses from a free arc.

*Proof.* It is illustrated in Figure 3, where the result of the collapses is shown. One can see that the obtained band complex is isomorphic to  $Z(\vec{w}', \vec{\ell}')$ , and  $B'_i$ ,  $i = 1, 2, 3, 4$ , are the new bands. □

**Lemma 2.14.** *Let  $k_0, k_1, k_2, \dots$  be an arbitrary infinite sequence of natural numbers. Then there exists an infinite sequence  $\vec{w}_0, \vec{w}_1, \vec{w}_2, \dots$  of points from  $(\mathbb{R}_+)^3$  such that*

$$\vec{w}_i = B(k_i)\vec{w}_{i+1}.$$

Such a sequence is unique up to scale.

*Proof.* Let  $K = \mathbb{R}_+^3$  be the positive cone in the 3-space and let  $K' = \{\vec{w} \in K; w_3 < w_1 + w_2\}$ . For any  $k, l, m \in \mathbb{N}$  we have

$$B(k)B(l)B(m) = B'(k, l, m)B'',$$

where

$$B'(k, l, m) = \begin{pmatrix} k(l-1)+1 & k(l(m-1)+m) & 2k-1 \\ l-1 & l(m-1)+1 & 1 \\ 0 & m-1 & 1 \end{pmatrix}$$

and

$$B'' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

is a constant matrix. One can verify that

$$B'(\bar{K}) \subset \bar{K}', \quad B''(\bar{K}') \subset K', \quad B'(k, l, m)(K') \subset K'$$

for any  $k, l, m \in \mathbb{N}$ . It follows that the linear map  $B''$  restricted to  $K'$  is a contraction in the Hilbert projective metric (e. g., see [17] for the definition and basic properties), and

the linear map defined by  $B'(k, l, m)$  does not expand in this metric for any  $k, l, m \in \mathbb{N}$ . Therefore, the intersection

$$\bigcap_{i=1}^{\infty} B(k_1) \dots B(k_{3i})(K)$$

is a single open ray in  $K'$ . The claim follows. □

*Remark 2.15.* In [10] a flaw occurs in the proof of Lemma 14, where a similar argument is used. A  $6 \times 6$  matrix  $B(m, n)$  depending on two parameters arises there. The decomposition of  $B(m, n)$  that is given there does not work as proposed. One should use the following decomposition instead:

$$B(m_1, n_1)B(m_2, n_2) = B'(m_1, n_1, m_2, n_2)B'',$$

where

$$B'' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 \\ 2 & 4 & 5 & 2 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{pmatrix}.$$

The matrix  $B''$  has only positive entries, so it defines a contraction of the positive cone  $(\mathbb{R}_+)^5$  with respect to the Hilbert projective metric. It is a direct check that the matrix  $B'(m_1, n_1, m_2, n_2)$  has only non-negative entries, so the corresponding linear map does not expand the Hilbert metric.

Let  $\vec{\ell}_0 = (1, 1, 1, 1)$ , and let  $\vec{w}_0$  be as in Lemma 2.14. Define recursively

$$(2) \quad \vec{\ell}_{i+1} = \vec{\ell}_i \cdot A(k_i).$$

**Proposition 2.16.** *For any sequence  $k_0, k_1, k_2, \dots$  of natural numbers the band complex  $Z(\vec{w}_0, \vec{\ell}_0)$  defined above is annulus free and of thin type.*

*If, in addition, for all  $i \geq 0$ , we have  $k_{i+1} \geq 2k_i$ , then the union of leaves in  $Z(\vec{w}_0, \vec{\ell}_0)$  that are not two-ended trees has zero measure.*

*Proof.* First, we show that  $Z(\vec{w}_0, \vec{\ell}_0)$  is annulus free. One can see from (1) and (2) that all entries of  $\vec{\ell}_i$  grow without bound with  $i$ . On the other hand, the length of any loop contained in a leaf of  $\mathcal{F}_{Z(\vec{w}_0, \vec{\ell}_0)}$  is preserved by the Rips machine and should remain fixed. Therefore, all the leaves of  $\mathcal{F}_{Z(\vec{w}_0, \vec{\ell}_0)}$  are simply connected.

Now verify that  $Z(\vec{w}_0, \vec{\ell}_0)$  is of thin type. The condition (2) of Definition 2.9 is satisfied by Lemma 2.13 and by construction of  $\vec{w}_0$ , so we need only check that any leaf of  $\mathcal{F}_{Z(\vec{w}_0, \vec{\ell}_0)}$  is everywhere dense. By Imanishi’s theorem (see [13] and [12]) the converse would imply the existence of an arc connecting two singularities of  $\mathcal{F}_Y$  through the regular part of a singular leaf. Such an arc can get only shorter under a collapse from a free arc, which is inconsistent with the infinite growth of all band lengths.

Now we prove the last claim of the proposition. Denote for short:

$$A_i = A(k_i), \quad B_i = B(k_i), \quad Z_i = Z(\vec{w}_i, \vec{\ell}_i).$$

It follows from Lemma 2.13 that  $Z_{i+1}$  can be identified with an enhanced band complex  $Z_i$  obtained from  $Z_i$  by a few collapses from a free arc. So we think of  $Z_{i+1}$  as a subset of  $Z_i$  and, hence, of  $Z_0$ .

Denote by  $S_k$  the total area of  $Z_i$ :

$$S_i = \vec{\ell}_i \cdot C \cdot \vec{w}_i,$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We claim that under the assumptions of the proposition we have

$$(3) \quad \lim_{i \rightarrow \infty} S_i > 0.$$

Indeed, it can be checked directly that the matrix

$$\left( A_i C - \left( 1 - \frac{2}{k_i} \right) C B_i \right) B_{i+1}$$

has only positive entries for all  $i \geq 0$  since they can be expressed as polynomials in  $k_i$  and  $(k_{i+1} - 2k_i)$  with positive coefficients. Therefore,

$$S_{i+1} - \left( 1 - \frac{2}{k_i} \right) S_i = \vec{\ell}_i \left( A_i C - \left( 1 - \frac{2}{k_i} \right) C B_i \right) B_{i+1} \vec{w}_{i+2} > 0,$$

which can be rewritten as

$$S_{i+1} > \left( 1 - \frac{2}{k_i} \right) S_i.$$

Since  $k_i$  grows exponentially fast with  $i$ , we have  $\sum_{i=0}^{\infty} \frac{2}{k_i} < \infty$ , which implies (3).

By definition of a collapse from a free arc the measure  $\mu_{Z_{i+1}}$  (see Definition 2.9) coincides with the restriction of  $\mu_{Z_i}$ , and hence of  $\mu_{Z_0}$ , to  $Z_{i+1}$ . So  $\lim_{i \rightarrow \infty} S_i$  equals  $\mu_{Z_0}(\bigcap_i Z_i)$ . By general theory of band complexes (see [3]) the subset  $\bigcap_i Z_i \subset Z_0$  has an empty intersection with one-ended leaves of  $\mathcal{F}_{Z_0}$ . Therefore, the union of two-ended leaves of  $\mathcal{F}_{Z_0}$  has positive measure. Lemma 2.14 implies “a unique ergodicity” for  $\mathcal{F}_{Z_0}$ , and hence ergodicity of  $\mathcal{F}_{Z_0}$  with respect to the transverse measure  $|dx|$ . This means that any measurable union of leaves of  $\mathcal{F}_{Z_0}$  has either zero or full measure with respect to  $\mu_{Z_0}$ . We conclude that the union of two-ended leaves has full measure.  $\square$

*Proof of Theorem 2.12.* Let  $Z(\vec{w})$  be a band complex with support interval  $D = [0, w_1 + w_2 + w_3]$  and three bands  $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$  whose bases are glued to the following subintervals of  $D$ :

$$\begin{aligned} \mathcal{B}_1 &: \text{ to } [0, w_1] \text{ and } [w_2 + w_3, w_1 + w_2 + w_3], \\ \mathcal{B}_2 &: \text{ to } [0, w_2] \text{ and } [w_1 + w_3, w_1 + w_2 + w_3], \\ \mathcal{B}_3 &: \text{ to } [0, w_3] \text{ and } [w_1 + w_2, w_1 + w_2 + w_3]. \end{aligned}$$

So the band complex  $Z(\vec{w})$  can be obtained from the enhanced band complex  $Z(\vec{w}, \vec{\ell})$  by collapsing the band  $\mathcal{B}_4$  and forgetting the lengths of the bands. More precisely, there is a continuous map  $\psi: Z(\vec{w}, \vec{\ell}) \rightarrow Z(\vec{w})$  that preserves the 1-form  $dx$  and takes the bands  $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$  of  $Z(\vec{w}, \vec{\ell})$  to the respective bands of  $Z(\vec{w})$  and takes  $\mathcal{B}_4$  to a subinterval of  $D$ . Clearly the map  $\psi$  takes leaves to leaves and preserves the quasi-isometry and homotopy class of each leaf. It is also clear that  $Z(\vec{w})$  is symmetric with respect to the involution that flips the support interval  $D$ .

It follows from Proposition 2.16 that there are uncountably many choices of parameters  $\vec{w}$  for which almost all leaves of  $Z(\vec{w})$  are two-ended trees.  $\square$

3. PLANE SECTIONS OF THE REGULAR SKEW POLYHEDRON  $\{4, 6 \mid 4\}$

We recall briefly the formulation of Novikov’s problem on plane sections of 3-periodic surfaces. Let  $M$  be a closed null-homologous surface in the 3-torus  $\mathbb{T}^3 = \mathbb{R}^3/L$ , where  $L \cong \mathbb{Z}^3$  is a lattice, and let  $H = (H_1, H_2, H_3) \in \mathbb{R}^3$  be a non-zero vector. We denote by  $p$  the projection  $\mathbb{R}^3 \rightarrow \mathbb{T}^3$ , and by  $\widehat{M} \subset \mathbb{R}^3$  the  $\mathbb{Z}^3$ -covering  $p^{-1}(M)$  of  $M$ . We also fix a smooth function  $f: \mathbb{T}^3 \rightarrow \mathbb{R}$  of which  $M$  is a level surface,  $M = \{x \in \mathbb{T}^3; f(x) = c\}$ .

Non-singular connected components of the intersection of  $\widehat{M}$  with a plane of the form

$$(4) \quad \Pi_a = \{x \in \mathbb{R}^3; \langle H, x \rangle = a\},$$

where  $\langle \cdot, \cdot \rangle$  stands for the Euclidean scalar product, are trajectories of the following ODE:

$$(5) \quad \dot{x} = \nabla \widehat{f}(x) \times H,$$

where  $\widehat{f} = f \circ p$ . Their image in  $\mathbb{T}^3$  under  $p$  are leaves of the foliation  $\mathcal{F}_M$  on  $M$  defined by the kernel of the closed 1-form

$$(6) \quad \eta = (H_1 dx_1 + H_2 dx_2 + H_3 dx_3)|_M.$$

Novikov’s question was about the existence of an asymptotic direction of open trajectories defined by (5). As shown in [7] the foliation  $\mathcal{F}_M$  typically does not have minimal components of genus larger than one. For open trajectories this implies that they are typically either not present (in which case we call the pair  $(M, H)$  *trivial*) or have a *strong asymptotic direction* (then the pair  $(M, H)$  is called *integrable*), which means that, for a certain parametrization (not related to the one prescribed by (5)), they have the form

$$(7) \quad x(s) = sv + O(1),$$

where  $v \in \mathbb{R}^3$  is a constant vector. There is also a special case discovered by S. Tsarev (see [8]) when minimal components of  $\mathcal{F}_M$  have genus one but the trajectories have an asymptotic direction only in the usual, not the strong, sense, i. e. with  $o(s)$  instead of  $O(1)$  in (7). In Tsarev’s case, the vector  $H$  is not “maximally irrational”, i. e.  $\dim_{\mathbb{Q}} \langle H_1, H_2, H_3 \rangle = 2$ .

It is, however, possible that  $\mathcal{F}_M$  has a minimal component of genus  $> 1$  (as shown in [8] the genus cannot be equal to 2, so “ $> 1$ ” actually means “ $\geq 3$ ” here); see [8]. In this case, the pair  $(M, H)$  is called *chaotic* since there is a priori no reason for open trajectories to have an asymptotic direction. If the system is chaotic and uniquely ergodic, then, as A. Zorich notes in [24], trajectories, indeed, cannot have an asymptotic direction. Particular chaotic examples [8, 9, 19] are known in which almost all planes of the form (4) intersect  $\widehat{M}$  in a single open trajectory, which, in a sense, wanders around the whole plane [20].

Chaotic pairs  $(M, H)$  can be characterized in terms of any of the foliations  $\mathcal{F}_{N_-}, \mathcal{F}_{N_+}$  induced by the 1-form  $\omega = H_1 dx_1 + H_2 dx_2 + H_3 dx_3$  on the submanifolds  $N_- = \{x \in \mathbb{T}^3; f(x) \leq c\}$ ,  $N_+ = \{x \in \mathbb{T}^3; f(x) \geq c\}$ , of which  $M$  is the boundary. Namely, the following can be extracted from [8]:

**Proposition 3.1.** *A pair  $(M, H)$  is chaotic if and only if  $\mathcal{F}_{N_-}$  (or, equivalently,  $\mathcal{F}_{N_+}$ ) has uncountably many leaves that are not quasi-isometric (in the induced intrinsic metric) to a point, to a straight line, or to a plane.*

Since only the quasi-isometry class of the leaves matters, one can replace  $N_-$  by a foliated 2-complex  $Z$  embedded in  $N_-$  so that every leaf of  $Z$  embeds in a leaf of  $N_-$  quasi-isometrically. In the genus 3 case, such a 2-complex can be chosen among band complexes made of 3 bands.

This is how band complexes are related to Novikov’s problem in general. Below we demonstrate this relation explicitly in very detail for a single surface, which was also the main subject of [6], where the set of all  $H$ ’s giving rise to the chaotic case was described.

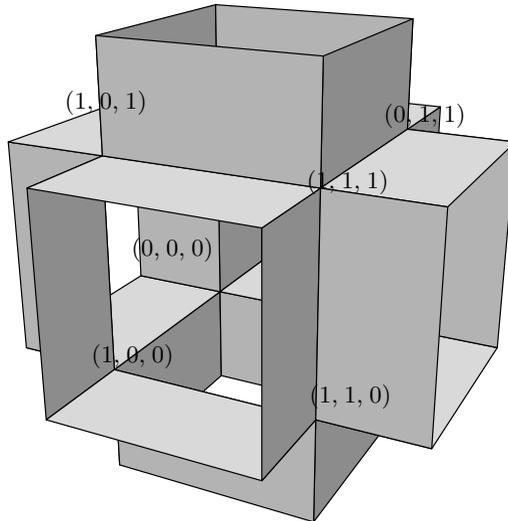


FIGURE 4. A fundamental domain of  $\widehat{M}$

It appeared to be a fractal set discovered earlier by G. Levitt [14] in connection with pseudogroups of rotations and arose also in symbolic dynamics (see [2]). It is shown by A. Avila, A. Skripchenko, and P. Hubert in [1] that the Hausdorff dimension of this set is strictly less than two.

Our 3-periodic surface  $\widehat{M}$  is going to be the one consisting of all squares of the form

$$\begin{aligned} & \{i\} \times [j, j + 1] \times [k, k + 1], \\ & [j, j + 1] \times \{i\} \times [k, k + 1], \\ & [j, j + 1] \times [k, k + 1] \times \{i\} \end{aligned}$$

with  $i, j, k \in \mathbb{Z}$ ,  $j + k \equiv 1 \pmod{2}$ .

The fundamental domain of  $M$  is shown in Figure 4. The lattice  $L$  is set to  $2\mathbb{Z}^3$ . One readily checks that  $M = \widehat{M}/L$  has genus 3. The surface  $\widehat{M}$  is known in the literature as the regular skew polyhedron  $\{4, 6 | 4\}$ ; see [5].

The reader may protest here since the surface  $M$  is PL but not smooth. However, for any fixed  $H$ , one can smooth it out so as to keep the topology of the foliation  $\mathcal{F}_M$  unchanged. In order to do so it suffices to  $C^0$ -approximate  $M$  so as to keep the positions of the two monkey saddle singularities of  $\mathcal{F}_M$  fixed (if  $H_1, H_2, H_3 > 0$ , they occur at points  $(0, 0, 0)$  and  $(1, 1, 1) \pmod{L}$ ) and to avoid introducing new singularities.

*Remark 3.2.* Our settings here are in a sense opposite to those of [9], where the vector  $H$  is fixed and the lattice  $L$  and the surface  $M$  are being varied.

**Proposition 3.3.** *The band complex  $Z(\vec{w})$  introduced in the proof of Theorem 2.12 is of thin type if and only if the pair  $(M, H)$  is chaotic, where*

$$(8) \quad 2H = (w_2 + w_3, w_1 + w_3, w_1 + w_2).$$

Note that due to the cubic symmetry of the surface  $M$  the pair  $(M, H)$  is chaotic if and only if so is  $(M, (|H_1|, |H_2|, |H_3|))$ . If all  $H_i$ 's are positive but don't have the form (8) with positive  $w_i$ 's, i. e. don't satisfy the triangle inequalities, then the pair  $(M, H)$  is integrable (see [6]).

*Proof.* For  $n \in \mathbb{Z}^3$  we denote:

by  $D(n)$  the straight line segment connecting  $n$  with  $n + (1, 1, 1)$ ;

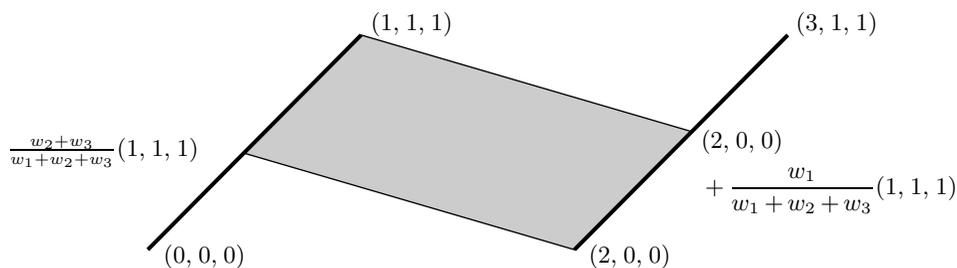


FIGURE 5. The strip  $S_1(0, 0, 0)$

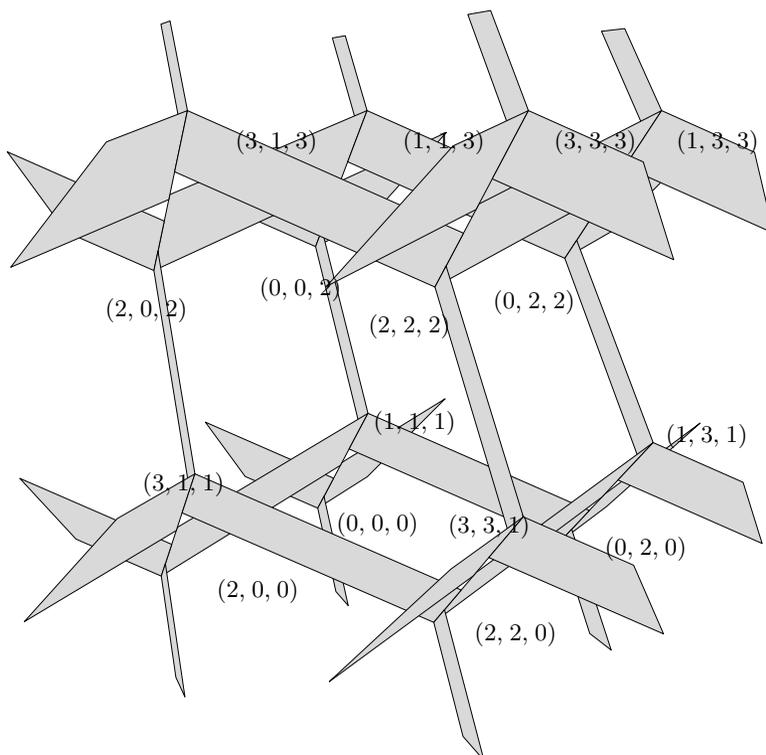


FIGURE 6. The 2-complex  $\widehat{Z}$

by  $e_1, e_2, e_3$  the standard basis of  $\mathbb{Z}^3$ ;  
 by  $S_i(n)$ ,  $i = 1, 2, 3$ , the parallelogram with vertices

$$2n + \left(1 - \frac{w_i}{w_1 + w_2 + w_3}\right) (1, 1, 1), \quad 2n + (1, 1, 1),$$

$$2n + 2e_i + \frac{w_i}{w_1 + w_2 + w_3} (1, 1, 1), \quad 2n + 2e_i$$

(see Figure 5);  
 by  $\widehat{Z}$  the union

$$\bigcup_{n \in \mathbb{Z}^3} (D(n) \cup S_1(n) \cup S_2(n) \cup S_3(n))$$

(see Figures 6, 7);

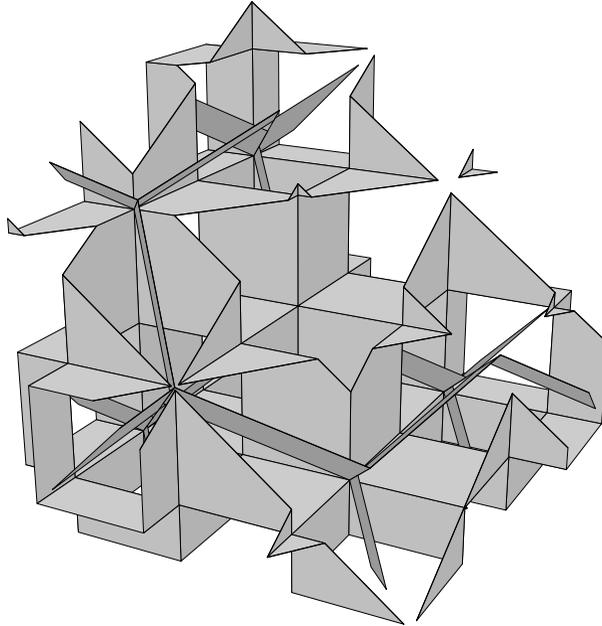


FIGURE 7. The surface  $\widehat{M}$  and the 2-complex  $\widehat{Z}$  cut by a plane  $\Pi_a$

- by  $Z$  the projection  $p(\widehat{Z}) \subset \mathbb{T}^3$ ;
- by  $\mathfrak{C}_0$  the unit cube  $[0, 1] \times [0, 1] \times [0, 1]$ ;
- by  $\mathfrak{C}_0(n)$  the cube  $\mathfrak{C}_0 + 2n$ ;
- by  $\mathfrak{C}_i(n)$ ,  $i = 1, 2, 3$ , the cube  $\mathfrak{C}_0 + e_i + 2n$ ;
- by  $\widehat{N}_-$  the union

$$\bigcup_{n \in \mathbb{Z}^3} (\mathfrak{C}_0(n) \cup \mathfrak{C}_1(n) \cup \mathfrak{C}_2(n) \cup \mathfrak{C}_3(n));$$

- by  $\widehat{N}_+$  the subset  $\widehat{N}_+$  shifted by the vector  $(1, 1, 1)$ ;
- and by  $N_-$  (respectively,  $N_+$ ) the projection  $p(\widehat{N}_-) \subset \mathbb{T}^3$  (respectively,  $p(\widehat{N}_+)$ ).

One can readily check the following:

$$N_- \cap N_+ = \partial N_- = \partial N_+ = M, \quad N_- \cup N_+ = \mathbb{T}^3;$$

$$D(n) \subset \mathfrak{C}_0(n) \text{ and } \mathcal{S}_i(n) \subset \mathfrak{C}_0(n) \cup \mathfrak{C}_i(n) \cup \mathfrak{C}_0(n + e_i) \text{ for all } n \in \mathbb{Z}^3;$$

the two sides of each  $\mathcal{S}_i(n)$ ,  $i = 1, 2, 3$ ,  $n \in \mathbb{Z}^3$ , that are not parallel to  $(1, 1, 1)$  are orthogonal to  $H$ ;

the intersection  $\Pi_a \cap \mathfrak{C}_0(n)$  is non-empty if and only if so is  $\Pi_a \cap D(n)$ ;

the interiors of all the cubes  $\mathfrak{C}_i(n)$ ,  $i \in \{0, 1, 2, 3\}$ ,  $n \in \mathbb{Z}^3$ , are pairwise disjoint;

each  $\mathfrak{C}_i(n)$ ,  $i = 1, 2, 3$ ,  $n \in \mathbb{Z}^3$ , shares a face with  $\mathfrak{C}_0(n)$  and with  $\mathfrak{C}_0(n + e_i)$ , and the rest of the boundary of  $\mathfrak{C}_i(n)$  is disjoint from all other cubes  $\mathfrak{C}_j(m)$ ,  $j \in \{0, 1, 2, 3\}$ ,  $m \in \mathbb{Z}^3$ ;

the boundary of the polygon  $\Pi_a \cap \mathfrak{C}_i(n)$ ,  $i = 1, 2, 3$ , has non-empty intersection with those of  $\Pi_a \cap \mathfrak{C}_0(n)$  and  $\Pi_a \cap \mathfrak{C}_0(n + e_i)$  which are not empty;

the intersection  $\Pi_a \cap \mathcal{S}_i(n)$ ,  $i = 1, 2, 3$ , is a straight line segment connecting  $\Pi_a \cap D(n)$  and  $\Pi_a \cap D(n + e_i)$  if  $\Pi_a \cap D(n) \neq \emptyset \neq \Pi_a \cap D(n + e_i)$ , and is otherwise empty.

Thus the intersection  $\Pi_a \cap \widehat{Z}$  is a graph  $\Gamma_a$  with the set of vertices

$$\Pi_a \cap \left( \bigcup_{n \in \mathbb{Z}^3} D(n) \right).$$

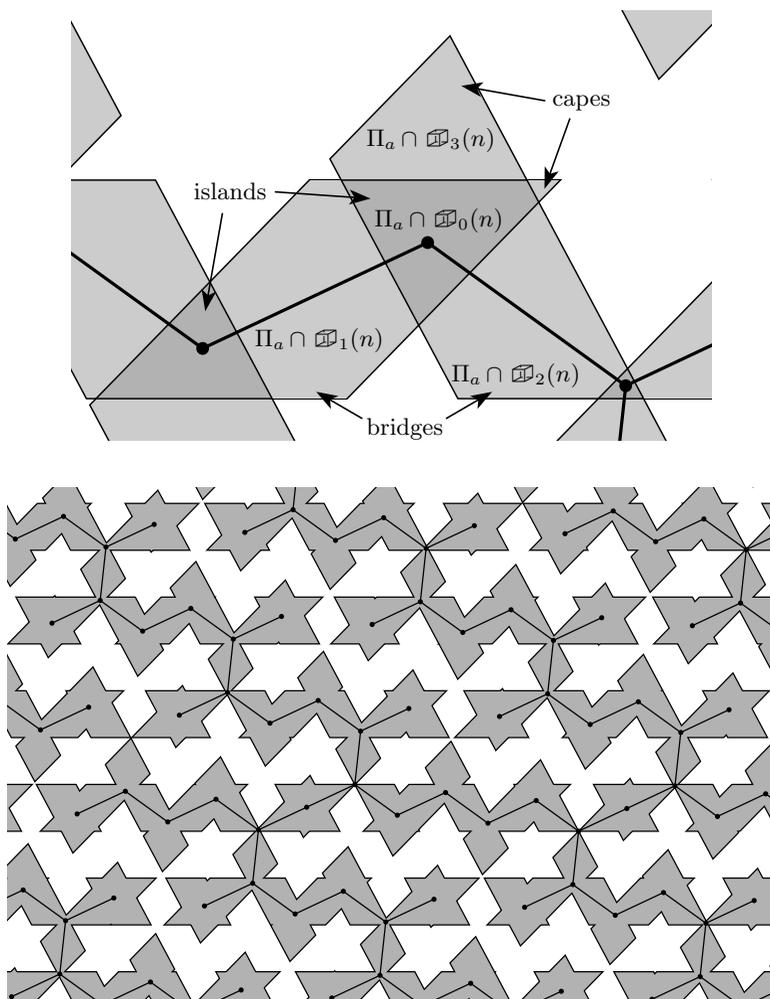


FIGURE 8. The intersections  $\Pi_a \cap \widehat{N}_-$  and  $\Pi_a \cap \widehat{Z}$

The intersection  $\Pi_a \cap \widehat{N}_-$  has the following structure. It contains the union of disjoint discs (some may be degenerate to a point)

$$\Pi_a \cap \left( \bigcup_{n \in \mathbb{Z}^3} \mathbb{D}_0(n) \right)$$

in each of which there is a single vertex of  $\Gamma_a$ . We call these discs *islands*.

The whole intersection  $\Pi_a \cap \widehat{N}_-$  is obtained from the union of islands by attaching disks of the form  $\Pi_a \cap \mathbb{D}_i(n)$ ,  $i = 1, 2, 3$ ,  $n \in \mathbb{Z}^3$ . Among such disks there are some whose boundary has a single connected component of intersection with an island. We call such disks *capes*. An island with all adjoint capes attached is still a disk containing a single vertex of  $\Gamma_a$ .

The boundary of any disk of the form  $\Pi_a \cap \mathbb{D}_i(n)$  that is not a cape has exactly two connected components in common with islands. We call such a disk a *bridge*.

One can see that two islands are connected by a bridge if and only if the corresponding vertices of  $\Gamma_a$  are connected by an edge; see Figure 8. Since all islands, capes, and

bridges have uniformly bounded diameter, the inclusion of any component of  $\Gamma_a$  into the corresponding component of  $\Pi_a \cap \widehat{N}_-$  is a quasi-isometry.

Connected components of  $\Pi_a \cap \widehat{N}_-$  and of  $\Gamma_a$  project under  $p$  onto leaves of  $\mathcal{F}_{N_-}$  and  $\mathcal{F}_Z$ , respectively. The claim now follows from Propositions 2.11 and 3.1.  $\square$

**Proposition 3.4.** *Let  $k_0, k_1, \dots$  be a sequence of natural numbers such that the series  $\sum_{i=0}^\infty \frac{1}{k_i}$  converges. Let  $\vec{w}_0, \vec{w}_1, \vec{w}_2, \dots$  be defined as in Lemma 2.14 and*

$$H = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{w}_0.$$

Then:

- (1) *the pair  $(M, H)$  is chaotic;*
- (2) *the foliation  $\mathcal{F}_M$  is not ergodic with respect to the transverse measure  $|\eta|$  defined by the 1-form (6); there are two ergodic components;*
- (3) *almost all connected components of the sections  $\Pi_a \cap \widehat{M}$  have an asymptotic direction, which is, up to sign, common for all of them.*

*Proof.* The first claim follows from Propositions 2.16 and 3.3. It is also a corollary to Lemma 3.5 below.

Let  $\xi$  be an oriented simple arc transverse to  $\mathcal{F}_M$  such that  $a = \int_\xi \eta > 0$ . For any  $b \in (0, a]$  we denote by  $\xi(b)$  the initial subarc of  $\xi$  such that  $\int_{\xi(b)} \eta = b$ .

Let  $\widehat{\xi}_1, \widehat{\xi}_2, \widehat{\xi}_3$  be the transversals of  $\mathcal{F}_{\widehat{M}}$ , starting at  $(0, 0, 0)$  composed of the following straight line segments:

$$\begin{aligned} \widehat{\xi}_1 &: [(0, 0, 0), (0, 0, 1)] \cup [(0, 0, 1), (0, 1, 1)], \\ \widehat{\xi}_2 &: [(0, 0, 0), (1, 0, 0)] \cup [(1, 0, 0), (1, 0, 1)] \cup [(1, 0, 1), (1, 1, 1)], \\ \widehat{\xi}_3 &: [(0, 0, 0), (0, 1, 0)] \cup [(0, 1, 0), (1, 1, 0)] \cup [(1, 1, 0), (1, 1, 1)], \end{aligned}$$

and let  $\xi_i = p(\widehat{\xi}_i)$ ,  $i = 1, 2, 3$ . We have

$$\int_{\xi_1} \eta = w_{10} + \frac{w_{20} + w_{30}}{2}, \quad \int_{\xi_2} \eta = \int_{\xi_3} \eta = w_{10} + w_{20} + w_{30},$$

so we have  $\int_{\xi_i} \eta > w_{10}$  for all  $i = 1, 2, 3$ .

Let  $R(k)$  be the matrix of the following linear transformation of  $\mathbb{R}^9$ :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} \mapsto \begin{pmatrix} (k-1)(x_2 + x_3 + x_7 + x_8 + x_9) + x_7 \\ x_8 \\ x_2 + x_9 \\ x_3 + x_7 \\ x_1 + x_2 + x_3 + x_4 \\ (k-1)(x_2 + x_3 + x_7 + x_8 + x_9) + x_2 + x_3 + x_7 \\ x_5 \\ x_2 + x_3 + x_6 \\ (k-1)(x_1 + x_2 + x_3 + x_4 + x_8) + x_4 \end{pmatrix}.$$

$\square$

**Lemma 3.5.** *The first return map defined on*

$$(9) \quad \xi_1(w_{1i}) \cup \xi_2(w_{1i}) \cup \xi_3(w_{1i})$$

by the foliation  $\mathcal{F}_M$  (for a proper orientation of leaves) endowed with the invariant measure  $|\eta|$  is an interval exchange map with permutation

$$(10) \quad \left( \begin{array}{ccc|cc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 3 & 7 & 6 & & 4 & 8 & 1 & & 9 & 2 & 5 \end{array} \right)$$

and vector of parameters

$$(11) \quad \vec{x}_i = (w_{1i} - w_{2i} - w_{3i}, w_{3i}, w_{2i} - w_{3i}, w_{3i}, w_{2i}, w_{1i} - w_{2i}, w_{3i}, w_{2i}, w_{1i} - w_{2i} - w_{3i})^T.$$

Let  $\mu$  be another invariant transverse measure for  $\mathcal{F}_M$ , and let  $\vec{y}_i$  be the vector of parameters of the corresponding interval exchange map induced on the union of transversals (9) (with the same numbering as for  $\vec{x}_i$ ). Then for all  $i = 0, 1, 2, 3, \dots$  we have

$$(12) \quad \vec{y}_{i+1} = R(k_i)\vec{y}_i, \quad \vec{y}_i \in V.$$

Any sequence  $\vec{y}_0, \vec{y}_1, \dots \in \mathbb{R}_+^9$  satisfying (12) defines an invariant transverse measure for  $\mathcal{F}_M$ .

We refer the reader to [22] for a detailed account on interval exchange transformations and on the Rauzy-Veech induction. Here we use a slightly modified version of the standard construction by taking a union of three transverse arcs instead of just one. That's why we subdivided each row in (10) into three blocks that correspond to  $\xi_1(w_{1k})$ ,  $\xi_2(w_{1k})$ , and  $\xi_3(w_{1k})$  (not in this order if  $k \not\equiv 0 \pmod{3}$ ).

*Proof.* Note that by definition of  $\vec{w}_k$  we always have  $w_{1k} > w_{2k} + w_{3k}$  and  $w_{2k} > w_{3k}$ .

For  $k = 0$  the claim of the lemma is obtained by a direct routine check. The surface  $M$  is cut into 9 strips each foliated by arcs. Preimages of the strips in  $\mathbb{R}^3$  are shown in Figure 9.

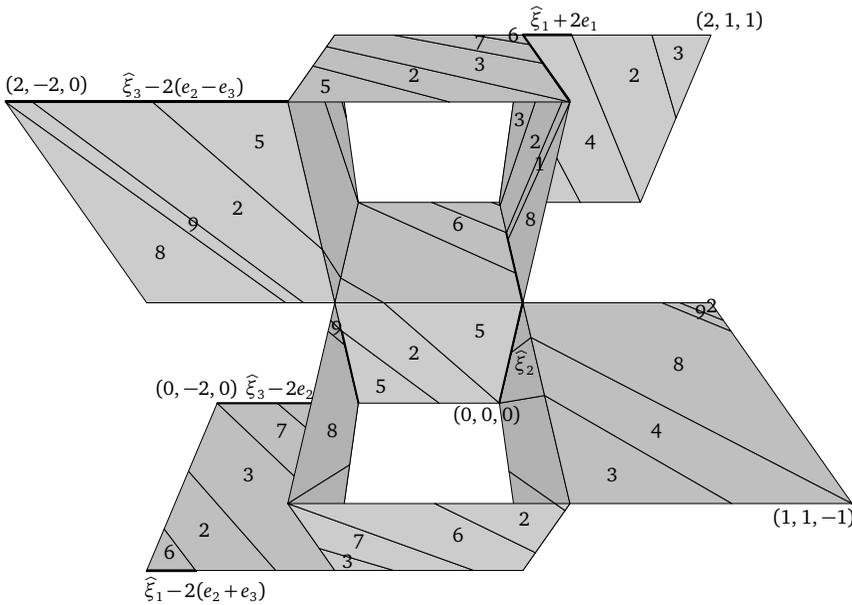


FIGURE 9. Cutting  $M$  into 9 strips

Let  $\vec{x} \in V$  and  $\vec{y} = R(k)\vec{x}$ ,  $k \in \mathbb{N}$ . We claim that we can run the Rauzy-Veech induction starting from the permutation (10) and the vector of parameters  $\vec{y}$  in order to obtain

after several steps an interval exchange map with the same permutation and the vector of parameters  $\vec{x}$ . Relations (14) guarantee that, for any  $i = 1, 2, 3$ , the sum of parameters corresponding to the  $i$ th block is the same for the top and bottom rows of the permutation.

The procedure will be slightly more general than it usually is since we are using three transversals instead of one. This simply means that we can exchange the blocks synchronously in both rows so the process is not uniquely defined by the initial data. Figure 10 shows how the Rauzy-Veech induction can be run. The transition between any two subsequent lines is the result of several steps of the ordinary Rauzy-Veech induction with the same winner or just a permutation of blocks. Each line displays the current permutation, the vector of parameters, and relations (if any) used to obtain the subsequent transition.

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 \end{pmatrix} \mid \begin{pmatrix} 5 & 6 \\ 4 & 8 & 1 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 9 \\ 9 & 2 & 5 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9)$	$y_9 = x_4 + (k-1)(y_2 + y_5)$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 \end{pmatrix} \mid \begin{pmatrix} 5 & 6 \\ 4 & 8 & 1 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 9 \\ 9 & 2 & 5 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_4)$	$y_5 = x_1 + x_2 + x_3 + x_4$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 \end{pmatrix} \mid \begin{pmatrix} 5 & 9 & 6 \\ 4 & 8 & 1 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 \\ 9 & 2 & 5 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, x_1 + x_2 + x_3, y_6, y_7, y_8, x_4)$	
$\begin{pmatrix} 7 & 8 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 \end{pmatrix} \mid \begin{pmatrix} 5 & 9 & 6 \\ 4 & 8 & 1 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, x_1 + x_2 + x_3, y_6, y_7, y_8, x_4)$	$y_6 = y_1 + x_2 + x_3$
$\begin{pmatrix} 7 & 8 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 & 1 \end{pmatrix} \mid \begin{pmatrix} 5 & 9 & 6 \\ 4 & 8 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, x_1 + x_2 + x_3, x_2 + x_3, y_7, y_8, x_4)$	$y_8 = x_2 + x_3 + x_6$
$\begin{pmatrix} 7 & 8 & 6 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 & 1 \end{pmatrix} \mid \begin{pmatrix} 5 & 9 \\ 4 & 8 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, x_1 + x_2 + x_3, x_2 + x_3, y_7, x_6, x_4)$	
$\begin{pmatrix} 5 & 9 \\ 4 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 & 1 \end{pmatrix}$	$(y_1, y_2, y_3, y_4, x_1 + x_2 + x_3, x_2 + x_3, y_7, x_6, x_4)$	$y_1 = (k-1)(y_2 + y_3 + y_4) + x_7$
$\begin{pmatrix} 5 & 9 \\ 4 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 & 1 \end{pmatrix}$	$(x_7, y_2, y_3, y_4, x_1 + x_2 + x_3, x_2 + x_3, y_7, x_6, x_4)$	$y_4 = x_3 + x_7$
$\begin{pmatrix} 5 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 \end{pmatrix}$	$(x_7, y_2, y_3, x_3, x_1 + x_2 + x_3, x_2 + x_3, y_7, x_6, x_4)$	$y_2 = x_8, y_7 = x_5$
$\begin{pmatrix} 5 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 & 4 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 \\ 3 & 7 & 6 \end{pmatrix}$	$(x_7, x_8, y_3, x_3, x_1 + x_2 + x_3, x_2, x_5, x_6, x_4)$	$y_3 = x_2 + x_9$
$\begin{pmatrix} 5 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 & 4 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix}$	$(x_7, x_8, x_9, x_3, x_1 + x_2 + x_3, x_2, x_5, x_6, x_4)$	
$\begin{pmatrix} 5 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 & 4 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix}$	$(x_7, x_8, x_9, x_3, x_1 + x_2 + x_3, x_2, x_5, x_6, x_4)$	
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix} \mid \begin{pmatrix} 5 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 & 6 & 4 \\ 9 & 2 & 5 \end{pmatrix}$	$(x_7, x_8, x_9, x_3, x_1 + x_2 + x_3, x_2, x_5, x_6, x_4)$	
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix} \mid \begin{pmatrix} 5 & 6 & 4 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 \\ 9 & 2 & 5 \end{pmatrix}$	$(x_7, x_8, x_9, x_3, x_1, x_2, x_5, x_6, x_4)$	
$\begin{pmatrix} 5 & 6 & 4 & 9 \\ 4 & 1 & 8 \end{pmatrix} \mid \begin{pmatrix} 7 & 8 \\ 9 & 2 & 5 \end{pmatrix} \mid \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix}$	$(x_7, x_8, x_9, x_3, x_1, x_2, x_5, x_6, x_4)$	

FIGURE 10. Running the Rauzy-Veech induction

After reordering parameters in the last line we get the original permutation with the same subdivision into blocks.

It remains to check that vectors  $\vec{x}_i$  defined by (11) (written as columns) satisfy

$$(13) \quad \vec{x}_i = R(k_i)\vec{x}_{i+1}$$

for all  $i \geq 0$ , which is straightforward.  $\square$

Let  $V$  be the subset of  $\mathbb{R}_+^9$  defined by the equations

$$(14) \quad x_1 + x_4 - x_6 = x_5 - x_8 = x_7 - x_2.$$

It is invariant under  $R(k)$  for any  $k > 0$ .

For  $i \geq 0$  denote

$$V_i = R(k_0)R(k_1) \dots R(k_i)(V), \quad V_\infty = \bigcap_{i=0}^\infty V_i.$$

We obviously have  $V \supset V_0 \supset V_1 \supset \dots$

**Lemma 3.6.** *The subset  $V_\infty$  has the form*

$$(15) \quad V_\infty = \{ \alpha \vec{u} + \beta \vec{v}; \alpha, \beta \geq 0, (\alpha, \beta) \neq (0, 0) \}$$

for some non-collinear  $\vec{u}, \vec{v} \in V$ . They can be chosen so as to have  $\vec{x}_0 = \vec{u} + \vec{v}$ .

*Proof.* The matrix  $R(k)$  can be written in the form  $R(k) = kR' + R''$  with  $R', R''$  not depending on  $k$  in a unique way. By explicit check we get  $(R')^4 = (R')^2$ . Since the series  $\sum_{i=0}^\infty \frac{1}{k_i}$  converges this implies that the limit

$$\begin{aligned} \tilde{R} &= \lim_{i \rightarrow \infty} \frac{R(k_0)}{k_0} \frac{R(k_1)}{k_1} \dots \frac{R(k_{2i-1})}{k_{2i-1}} \\ &= \lim_{i \rightarrow \infty} \left( R' + \frac{1}{k_0} R'' \right) \left( R' + \frac{1}{k_1} R'' \right) \dots \left( R' + \frac{1}{k_{2i-1}} R'' \right) \end{aligned}$$

exists and satisfies the relation

$$(16) \quad \tilde{R} \cdot (R')^2 = \tilde{R}.$$

The product of six matrices of the form  $R(k)$  with  $k > 0$  has only positive entries. This implies  $(V_i \cup \{0\}) \supset \bar{V}_{i+6}$  and

$$V_\infty \cup \{0\} = \bigcap_{i=0}^\infty \bar{V}_i = \bar{V}_\infty.$$

Together with (16) this gives

$$V_\infty \cup \{0\} = \tilde{R}(\bar{V}) = \tilde{R}R'(\bar{V}).$$

Denote

$$\vec{u}_\infty = (1, 0, 0, 0, 0, 1, 0, 0, 0)^\top, \quad \vec{v}_\infty = (0, 0, 0, 0, 0, 0, 0, 0, 1)^\top.$$

One easily checks the following:

$$R'(\bar{V}) = \{ \alpha \vec{u}_\infty + \beta \vec{v}_\infty; \alpha, \beta \geq 0 \}, \quad R' \vec{u}_\infty = \vec{v}_\infty, \quad R' \vec{v}_\infty = \vec{u}_\infty.$$

Thus, (15) holds for  $\vec{u} = \tilde{R} \vec{u}_\infty$ , and  $\vec{v} = \tilde{R} \vec{v}_\infty$ .

The matrix  $R(k)$  is invertible for all  $k$ , so we can set  $\vec{u}_0 = \vec{u}$ ,  $\vec{v}_0 = \vec{v}$ :

$$(17) \quad \vec{u}_{i+1} = k_i R(k_i)^{-1} \vec{u}_i, \quad \vec{v}_{i+1} = k_i R(k_i)^{-1} \vec{v}_i \quad \text{for } i \geq 0.$$

We will have

$$\lim_{i \rightarrow \infty} \vec{u}_{2i} = \lim_{i \rightarrow \infty} \vec{v}_{2i+1} = \vec{u}_\infty, \quad \lim_{i \rightarrow \infty} \vec{v}_{2i} = \lim_{i \rightarrow \infty} \vec{u}_{2i+1} = \vec{v}_\infty,$$

which implies that  $\vec{u}$  and  $\vec{v}$  are not collinear.

Now  $\vec{x}_0 \in V_\infty$ , so  $\vec{x}_0$  is a non-trivial linear combination  $\alpha \vec{u} + \beta \vec{v}$ . From (13) and (17) we have

$$\vec{x}_{2i} = \frac{1}{\prod_{j=0}^{2i-1} k_j} (\alpha \vec{u}_{2i} + \beta \vec{v}_{2i}).$$

From the definition of  $\vec{w}_i$  (see Lemma 2.14) it follows that

$$\lim_{i \rightarrow \infty} \frac{w_{2i}}{w_{1i}} = \lim_{i \rightarrow \infty} \frac{w_{3i}}{w_{1i}} = 0$$

if  $k_i \rightarrow \infty$ . Together with (11) this implies

$$\vec{x}_i = w_{1i} \left( (1, 0, 0, 0, 0, 1, 0, 0, 1)^\top + o(1) \right) = w_{1i} (\vec{u}_\infty + \vec{v}_\infty + o(1)), \quad i \rightarrow \infty;$$

hence  $\alpha = \beta$ , and by rescaling  $\vec{x}_0$  we can make  $\alpha = \beta = 1$ . □

We can now finalize the proof of Proposition 3.4. It follows from Lemmas 3.5 and 3.6 that  $\mathcal{F}_M$  admits two invariant ergodic transverse measures,  $\mu$  and  $\nu$ , say, that correspond to the vectors  $\vec{u}$  and  $\vec{v}$  from Lemma 3.6, and, for an appropriate normalization, we have  $|\eta| = \mu + \nu$ . Let  $c_\mu, c_\nu \in H_1(M; \mathbb{R})$  be the asymptotic cycles of  $\mu$  and  $\nu$ , respectively (see [25] for the definition), and  $\mathcal{E}_\mu, \mathcal{E}_\nu$  the respective ergodic components of  $\mathcal{F}_M$ .

Denote by  $\iota$  the inclusion  $M \hookrightarrow \mathbb{T}^3$ . Since  $\eta$  is a restriction of a closed 1-form in  $\mathbb{T}^3$ , we have

$$(18) \quad \iota_*(c_\mu + c_\nu) = \iota(\eta^*) = 0 \in H_1(\mathbb{T}^3),$$

where  $\eta^* \in H_1(M)$  is the Poincaré dual of the cohomology class of  $\eta$ . We claim that  $\iota_*(c_\mu) \neq 0 \neq \iota_*(c_\nu)$ , which implies the assertion of the proposition about the existence of an asymptotic direction.

Indeed, suppose on the contrary that  $\iota_*(c_\mu) = 0$ . Then for any 1-cycle  $c$  on  $M$  that is null-homologous in  $\mathbb{T}^3$  we must have  $c \cap c_\mu = 0$ . Let  $c = \xi_2 - \xi_3$ , where  $\xi_{2,3}$  are oriented transversals of  $\mathcal{F}$  introduced in the proof of Proposition 3.4 (see Figure 9, where initial portions of  $\xi_{2,3}$  are shown). The cycle  $c$  is homologous to zero in  $\mathbb{T}^3$ , so we must have  $\int_{\xi_2} \mu = \int_{\xi_3} \mu$ . Similarly, we must have  $\int_{\xi_2} \mu = \int_{\xi_4} \mu = \int_{\xi_5} \mu$ , where

$$\begin{aligned} \xi_4 &= p([(0, 0, 0), (0, 0, 1)] \cup [(0, 0, 1), (1, 0, 1)] \cup [(1, 0, 1) \cup (1, 1, 1)]), \\ \xi_5 &= p([(0, 0, 0), (0, 0, 1)] \cup [(0, 0, 1), (0, 1, 1)] \cup [(0, 1, 1) \cup (1, 1, 1)]). \end{aligned}$$

We have

$$(19) \quad \begin{aligned} \int_{\xi_2} \mu &= u_2 + u_3 + u_4 + u_5 + u_6, & \int_{\xi_3} \mu &= u_2 + u_5 + u_7 + u_8 + u_9, \\ \int_{\xi_4} \mu &= u_2 + 2u_3 + u_4 + u_6 + u_7, & \int_{\xi_5} \mu &= u_2 + 2u_3 + u_6 + 2u_7. \end{aligned}$$

So  $\vec{u}$  must satisfy the relations

$$(20) \quad u_3 + u_6 = u_8 + u_9, \quad u_5 = u_3 + u_7, \quad u_4 = u_7.$$

It must also satisfy (14) (with  $x_i$  replaced by  $u_i$ ). The subspace in  $\mathbb{R}^9$  defined by all these equations is invariant under  $R(k)^{\pm 1}$ . Therefore, they must hold true also for  $\vec{u}_\infty$ , but the first relation in (19) does not, a contradiction.

It follows from (18) that  $\iota_*(c_\mu) = -\iota_*(c_\nu)$ , which implies that the asymptotic direction of trajectories for  $\mathcal{E}_\mu$  will be opposite the one for  $\mathcal{E}_\nu$ .

The hypothesis on the sequence  $(k_i)$  in Proposition 3.4 is much weaker than in Proposition 2.16. One can show that it can be weakened in Proposition 2.16, too, by deducing it from Proposition 3.4, but the argument will be less straightforward.

We expect that all thin type band complexes with three bands give rise, through the construction of [9], to a chaotic dynamics in Novikov’s problem with almost all trajectories having an asymptotic direction, but we don’t see a rigorous proof of that so far.

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