The Cat in Numberland

Reviewed by James Propp

The Cat in Numberland
Ivar Ekeland, illustrated by John O'Brien
Cricket Books, 2006 (reading level ages 9–12)
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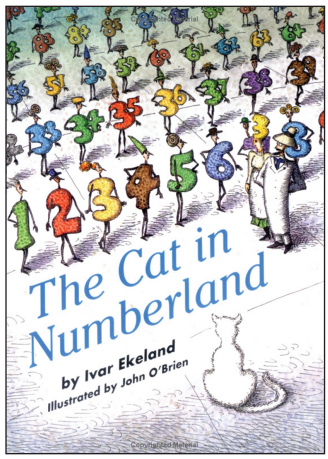
The Cat in Numberland is a well-thought-out and stylish attempt to present ideas about infinity to children who are ready to take a step beyond the notion of infinity as "the largest number". I found Ekeland's text engaging, with enough whimsy to keep the story from being dry but not so much as to be cutesy or condescending, and I thought O'Brien's charming black-and-white illustrations compensated for their lack of color through their loopy, nervy vigor.

There's something mind-numbing about the concept of infinity, and for many students, even the word itself invites a retreat from forward-moving thought into static wonder; so, when leading a first-timer on a trip to infinity, it's best to use the word as little as possible. Ekeland borrows a famous pedagogical device from David Hilbert's popularization of Cantor's ideas about infinity, namely the idea of a hotel with infinitely many rooms, but even though Ekeland's hotel is called "Hotel Infinity", you will not find any other occurrence of the words "infinity" or "infinite" in his book. Ekeland honors Hilbert by making him the proprietor of this hotel on the planet of Numberland, though the character's fastidiousness, quarrelsomeness, and lack of creativity make this homage a mixed compliment at best. Where Ekeland departs from Hilbert is his fancy that the guests in the hotel are not people but the actual numbers One, Two, Three, etc., personified. The number One starts out in room 1 of the hotel, the number Two starts out in room 2, and so on. The use of names for the numbers, and numerals for the rooms they occupy, at first struck me as strange, but I later realized that this is an astute authorial choice that wards off numerous potential confusions.

The plot is driven by the difference between the temperaments of Mr. Hilbert and his wife (Mr. Hilbert wants to keep all the rooms occupied, while Mrs. Hilbert wants to admit new guests), and all the puzzles that the Hilberts and their guests tackle are driven by the pursuit of marital harmony. The tension between Mr. and Mrs. Hilbert as described by Ekeland is just one instance of a fictionalizing touch that might at first seem to pull the story away from mathematical issues but actually plays a pedagogical role. Another example is the discussion in Chapter 1 of the "games" (addition, subtraction, multiplication, and division) that the numbers play with one another; this leads to a seemingly incidental discussion of odd and even numbers that lays the groundwork for the problem faced in Chapter 4 (how can you keep the hotel full when infinitely many guests leave?). Likewise, the discussion of how the letters A through Z attempt to participate in these games, while it plays no role in later developments in the book, serves as a nice preparation for the idea of using a letter as a place-holder, which the young reader will encounter when starting the study of algebra.



The climax of the book occurs in Chapter 5, when the hotel must be made to accommodate an infinite number of new guests, the Fractions, who arrive in an infinite rectangular two-dimensional array, each of whose rows is infinite. The solution to this problem comes from a change in perspective, quite literally: the number Zero, by looking out the high window of Room 1,234,566, is able to see his old hotel-mates and all the new arrivals as forming a triangular array each of whose rows is finite, which makes it possible to fit them into the hotel.

If you have a copy of the book available, jump immediately to page 55 for a masterly visual rendition of the key idea. The scene can be parsed in two different ways, and the viewer can go back and forth between them: now you see it, now you don't, now you do again. "I see it now!" says Mr. Hilbert. "But we could not see it from where we were standing." This is a fine motto for every

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stage of the process of learning mathematics, from pre-kindergarten to post-graduate. Each time we make a conceptual advance, we should jump back and forth across the divide we have crossed, to understand what made the leap so difficult the first time and so easy afterward, with the goal of enabling ourselves to make other jumps with less trouble in the future.

I have one mathematical quibble with this otherwise excellent little book, namely, the description of the layout of the hotel. We are told in Chapter 1 that there is a first room, which is Room 1; but we are also told that each odd-numbered room lies between two even-numbered rooms and vice versa (and there is no Room 0, at least at the outset). This inconsistency is easily fixed by treating Room 1 as an exception, but what are we to make of the fact that Hotel Infinity has infinitely many floors? If Room n lies between Room $n - 1$ and Room $n + 1$ for all n (as is strongly implied by the text), then which numbers are on the hotel's higher floors? Indeed, you can lead any young reader to see that Rooms 1 through 1,234,566 must all be on the ground floor, so that room 1,234,566 cannot play the pivotal role required by the plot. And, leaving that aside, if there were infinitely many floors, why couldn't the whole numbers and fractions be accommodated by putting the whole numbers on the ground floor, the fractions with denominator 2 on the second floor, the fractions with denominator 3 on the third floor, etc.?

Since this is a work of fiction designed to awaken the imagination, I view these imperfections of the book as a plus, not a minus; if you know a child who likes this book, you might try to lead him or her to discover these inconsistencies with a little bit of Socratic prodding (and perhaps challenge the child to redesign the hotel in various ways). At some point or other, the question may arise whether there could be a hotel with more than one floor such that Room n lies between Room $n - 1$ and Room $n + 1$ for all $n > 1$. At this point the child might embark on a project equivalent to proving the axiom of induction, and experience both confusion and frustration. This would be an excellent occasion for explaining that when we learn or create mathematics, confusion is often a good thing: it means we have understood a tension between two opposed ideas that must somehow be reconciled. Indeed, if you are a mathematical researcher, you might explain to the child that the way you make a living is by finding good things to be confused about and then trying to un-confuse yourself.

The topic of confusion leads us to the title character of the book, the unnamed cat, who is the reader's surrogate, and who can serve as a stand-in for both the future mathematician and the future nonmathematician. The cat's role is to express puzzlement at what is really going on, when everyone else seems content that a solution has been found. The cat can see that the move-everyone-to-the-next-room trick has worked, but is mystified as to how the trick works. Since all the rooms were full before, and all the rooms are full now, and one new guest has been accommodated, there must be a new room in the hotel somewhere—but where is it? Ekeland wisely does not introduce a character to resolve the cat's confusion. Some confusions need to be left unresolved, and revisited from year to year as we gain new ways of thinking.

Most mathematicians, as young students, played the role of the cat at one time or another, feeling (and perhaps voicing) confusion in a classroom situation in which the other students, who were satisfied with a more superficial level of understanding, didn't see anything to be confused about. Our schools need teachers who understand that confusion can sometimes be evidence of a deeper approach to the subject matter. Indeed, who can say how many potential mathematicians were driven away from mathematics at an early age by classmates and teachers who made them feel stupid for feeling rightly confused about deep matters?

In the end the cat opts to leave Numberland for a place that is easier to understand, namely, our own world (more specifically, Corsica—which may be an arbitrary or personal choice of Ekeland's, or may hold some meaning that eludes me). The cat still dreams of Numberland, but she enjoys living in a place where puzzlement is not a fact of daily life. Like Alice, or the Dorothy of the MGM version of *The Wizard of Oz*, the cat's sojourn in a land governed by strange rules has given her a heightened appreciation of the mundane (though unlike Alice or Dorothy, she ends up on Earth as a refugee, not a returning native).

In this final stage of her journey the cat strikes me as a stand-in for the student who retreats from the counterintuitive constructs of abstract mathematics in favor of the concrete and the graspable. Whether these students become engineers or accountants or artists, what we mathematicians hope for them is not that they become good at solving fanciful puzzles like the ones the Hilberts face, but that they accord some respect to the challenge of these puzzles, and that, in some corner of their minds, they have an esthetic response to such puzzles and their solutions. Such "dreams of Numberland" should be part of the residue that students are left with after their mathematical education is completed.

We should not expect all of our students to want to live in Numberland, or even to visit very often, but we should hope they will acquire the view of mathematics that is tacitly advertised by Ekeland and O'Brien: a view of mathematics as not just a mountain of facts but also a fountain of paradox.