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of the American Mathematical Society

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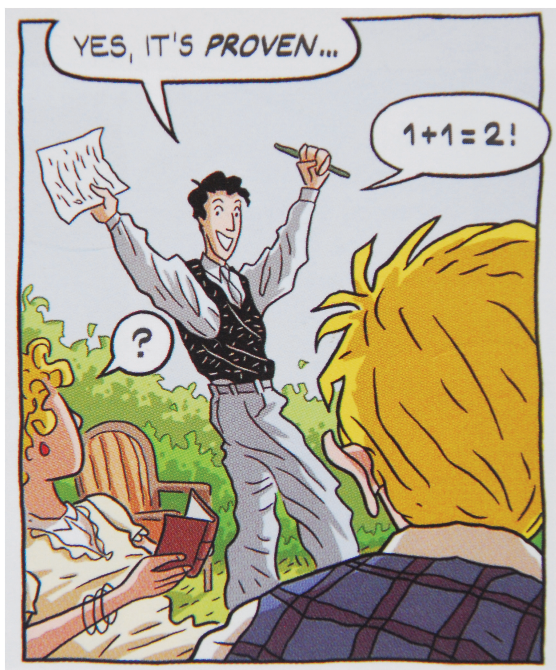

About the Cover: Bertrand Russell: Lover, husband, mathematician, and comic book hero (see page 1477)

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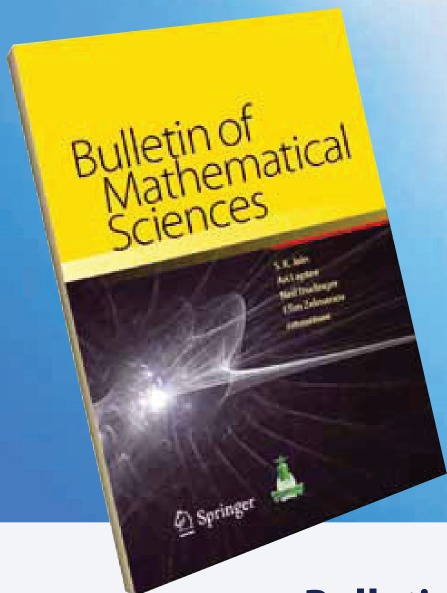

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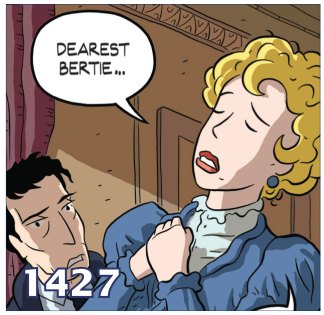
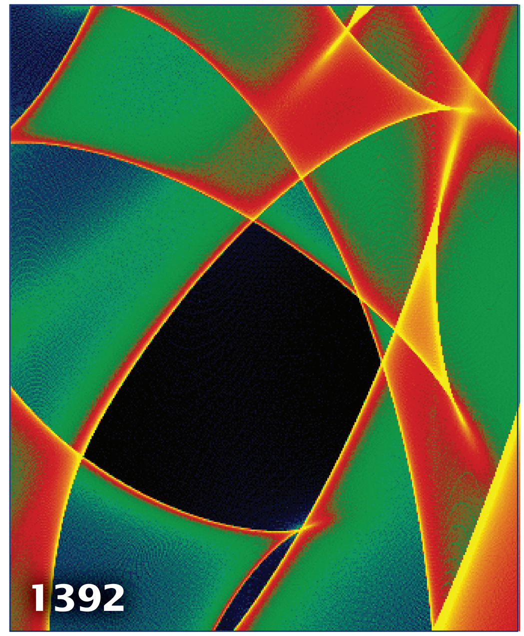
Notices
of the American Mathematical Society

December 2010

Communications

Commentary





Features

Our articles this month have a bent toward algebra and number theory, with a bit of logic and physics thrown in. The piece on topical bias has something to say to all of us. The subjects range from the very classical (Ramanujan and Gödel) to the very modern (cell phone technology and journal editing and gravitational lensing). One of the beauties of mathematics is that its ideas are timeless.

—*Steven G. Krantz*
Editor

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of the American Mathematical Society

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Departments

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I thank Randi D. Ruden for her splendid editorial work, and for helping to assemble this issue. She is essential to everything that I do.

—Steven G. Krantz
Editor

A Request: The Painlevé Project

In recent years the Painlevé equations, particularly the six Painlevé transcendents PI,…,PVI, have emerged as the core of modern special function theory. In the eighteenth and nineteenth centuries, the classical special functions such as the Bessel functions, the Airy function, the Legendre functions, the hypergeometric functions, and so on, were recognized and developed in response to the problems of the day in electromagnetism, acoustics, hydrodynamics, elasticity, and many other areas. In the same way, around the middle of the twentieth century, as science and engineering continued to expand in new directions, a new class of functions, the Painlevé functions, started to appear in applications. The list of problems now known to be described by the Painlevé equations is large, varied, and expanding rapidly. The list includes, at one end, the scattering of neutrons off heavy nuclei, and at the other, the statistics of the zeros of the Riemann-zeta function on the critical line Re $z = 1/2$. And in between, among many others, there is random matrix theory, the asymptotic theory of orthogonal polynomials, self-similar solutions of integrable equations, combinatorial problems such as Ulam's longest increasing subsequence problem, tiling problems, multivariate statistics in the important asymptotic regime where the number of variables and the number of samples are comparable and large, and also random particle systems.

Random matrix theory is a common portal for the Painlevé equations into science and engineering. A striking recent example is the discovery that the statistics of the bus delivery system in Cuernavaca, Mexico, is described with remarkable accuracy by random matrix theory: The bus delivery system in Cuernavaca has particular distinguishing features and is a prototype for the bus transportation system in many cities in Latin America. Equally striking is the recent experimental verification of random matrix behavior in turbulent liquid crystal growth.

Over the years, the properties of the classical special functions—algebraic, analytical, asymptotic, and numerical—have been organized and tabulated in various handbooks such as the Bateman Project or the National Bureau of Standards *Handbook of Mathematical Functions*, edited by Abramowitz and Stegun. What is needed now is a comparable organization and tabulation of the properties—algebraic, analytical, asymptotic, and numerical—of the Painlevé functions. This letter is an appeal to interested parties in the scientific community at large for help in developing such a "Painlevé Project". What we have in mind will be described below.

Although the Painlevé equations are nonlinear, much is already known about their solutions, particularly their algebraic, analytical, and asymptotic properties. This is because the equations are integrable in the sense that they have a Lax-Pair and also a Riemann-Hilbert representation from which the asymptotic behavior of the solutions can be inferred using the nonlinear steepest-descent method. The numerical analysis of the equations is less developed and presents novel challenges; in particular, in contrast to the classical special functions, where the linearity of the equations greatly simplifies the situation, each problem for the nonlinear Painlevé equations arises essentially anew.

As a first step in the Painlevé Project, we have established an e-site, maintained at the National Institute of Standards and Technology (NIST). We ask interested readers to send to the site

(1) pointers to new work on the theory of the Painlevé equations, algebraic, analytical, asymptotic, or numerical;

(2) pointers to new applications of the Painlevé equations;

(3) suggestions for possible new applications of the Painlevé equations;

(4) requests for specific information about the Painlevé equations.

The e-site will work as follows:

(1) You must be a "subscriber" to post messages to the e-site. To become a subscriber, send email to `daniel.lozier@nist.gov`.

(2) To post a message after becoming a subscriber, send email to `PainleveProject@nist.gov`. The message will be forwarded to every subscriber.

(3) See `http://cio.nist.gov/esd/emaildir/lists/painleveproject/threads.html` for the complete archive of posted messages. This archive is visible to anyone, not just subscribers.

(4) See `http://cio.nist.gov/esd/emaildir/lists/painleveproject/subscribers.html` for the complete list of subscribers. This list is visible to anyone, not just subscribers.

Depending on the response to our appeal, we plan to set up a Wiki page for the Painlevé equations, and then ultimately a comprehensive handbook in a style befitting our digital age, along the lines of the hyperlinked version (`http://dlmf.nist.gov`) of the new *NIST Handbook of Mathematical Functions*, edited by Olver, Lozier, Boisvert, and Clark and published by Cambridge University Press. Incidentally, this work contains, for the first time, a chapter on the Painlevé equations.

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Letters to the Editor

Ignore the Statement, Not the Teaching

I found Rainer Schulze-Pillot's strong objections to teaching statements in "Ethics and the plagiarized teaching statement" in the September 2010 *Notices* to be somewhat compelling and perhaps worthy of the strong language that the author uses. Unfortunately, I believe the author's argument, as it appears in the letter, perpetuates an illogic that has far greater negative consequences to our profession than forcing new Ph.D.s to write a brief statement about teaching. The author writes, "We shouldn't try to hire a gifted salesperson but an able mathematician." Really? Doesn't the request for a teaching statement as part of the application process suggest the hiring committee cares about their new hire's potential as a teacher? If so, I suggest that many (most?) of these hiring committees shouldn't try to hire simply a gifted mathematician but an able teacher as well. The problem is much deeper than the teaching statement. It transcends the entire way we prepare new Ph.D.s as well as how we nurture and reward effective teaching in the academe.

—*Julian F. Fleron*
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(Received August 31, 2010)

On the Iceland Map in an AMS Ad

For some months the AMS has been running an ad in the *Notices* (and doubtless elsewhere) which irks me. In the August issue it is on page 839, comparing "A World Without Mathematics" (an older map of Iceland) to "A World With Mathematics" (a modern image of the Earth).

It strikes me that the Iceland map is astonishingly detailed and accurate for one of its age (the sea monsters notwithstanding), and indeed nearly every hit for "old map Iceland" on my usual search engine yields some variant of this map. It (and the many copies that followed) appears to be produced by Abraham Ortelius in 1590, based on a now-lost original by Gudbrandur Thorlaksson, a bishop in Iceland.

I am no expert, but Eli Maor and his historian friends tell me that quite a bit of mathematics is involved in such cartography (especially trigonometry). So I don't see any reason to doubt the claim made on various websites (such as `http://kort.bok.hi.is/`) that "The map is so superior to all earlier maps of Iceland in content and execution…[Thorlaksson] had studied mathematics and astronomy alongside theology. He had calculated the position of Hólar and arrived at an amazingly accurate result." (Apparently his grandson was also quite the astronomer/cartographer and made an even better map several decades later.)

Obviously the Iceland map is made with less sophisticated mathematics than the image of the Earth. However, if geodesy was something Gauss could do without losing his status as a mathematician, perhaps the AMS should be picking something actually done without mathematics to depict "A World Without Mathematics".

—*Karl-Dieter Crisman*
Gordon College

(Received August 31, 2010)

The Gateway Arch

The *Notices* arrives in Australia several months after publication so forgive the lateness of my comment.

It concerns the fact that I am an Australian. However, I have friends in St. Louis and I have visited the marvelous Gateway Arch. I assumed the article was concerned with this arch but the connections were less than explicit. The note at the end confirms the connection with St. Louis. I wonder how many of your readers, especially those outside of the U.S., knew what the title of the article in the February *Notices* referred to?

It seems that here was a missed opportunity to give real interest to the mathematics with a photo of the arch and a few short words about it.

—*Jan Thomas*
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(Received August 31, 2010)

Correction

The article "The public lectures in Hyderabad" in the November 2010 issue of the *Notices* included a reference to the book *Proofs from the Book*. Unfortunately, the name of one of the book's authors, Martin Aigner, was misspelled in the article. The *Notices* apologizes for this error.

Submitting Letters to the Editor

The *Notices* invites readers to submit letters and opinion pieces on topics related to mathematics. Electronic submissions are preferred (`notices-letters@ams.org`); see the masthead for postal mail addresses. Opinion pieces are usually one printed page in length (about 800 words). Letters are normally less than one page long, and shorter letters are preferred.

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Gravity's Action on Light

A. O. Petters

In memory of Vladimir Arnold and David Blackwell

Gravitational lensing is the action of gravity on light. The subject has become a vibrant area in astronomy and mathematical physics with great predictive power. The field cuts across geometry, topology, probability, and singularity theory and interconnects mathematics, physics, and astrophysics.

The first part of this article gives an introduction to the subject with a review of standard results. The rest of the paper brings the reader to the mathematical forefront of the subject with a treatment of some recent research findings and unsolved problems. In addition, the interdisciplinary features of gravitational lensing are highlighted through the following topics: stability and genericity in lens systems, deterministic and stochastic aspects of image counting, local and global geometry of caustics and cosmic shadow patterns, and magnification relations for stable caustics. We add that the article on lensing by Khavinson and Neumann [40] in the June/July 2008 *Notices* complements this one and was focused primarily on the link between the maximum number of zeros of complex rational harmonic functions and the gravitational lensing problem of the maximum number of lensed images.

Our story begins with Einstein.

Arlie Petters (AP) is professor of mathematics, physics, and business administration at Duke University. His email address is `arlie.petters@duke.edu`. *He is thankful to Amir Aazami, Charles Keeton, Alberto Teguia, and Marcus Werner for constructive feedback on the article and acknowledges partial support from NSF grant DMS-0707003 and hospitality at the Petters Research Institute, where part of this work was done.*

Einstein and Gravitational Lensing

Even before completing his general theory of relativity, Einstein [25] had explored by 1911 how strongly gravity would bend light grazing a celestial body and implored astronomers to search the heavens for this effect: "It would be highly desirable that astronomers take up the question raised here, even if the considerations should seem to be insufficiently founded or entirely speculative."

It was not until the completion of general relativity in 1915 that Einstein obtained the correct formula for light's bending angle by a static spherically symmetric compact body of mass $\mathtt{m}$:

$$\hat{\alpha}(r) \approx 4\,\frac{\mathtt{m}_\bullet}{r}, \tag{1}$$

where $\mathtt{m}_\bullet = G\mathtt{m}/c^2$ (gravitational radius of lens) with G the universal gravitational constant, c the speed of light, and r the distance of closest approach of the light ray to the center of the lens.

Today we know that Einstein's approximate bending angle formula (1) is the first term in a series (Keeton and AP 2005 [36]):

(2)

$$\hat{\alpha}(b) = A_1\left(\frac{\mathtt{m}_\bullet}{b}\right) + A_2\left(\frac{\mathtt{m}_\bullet}{b}\right)^2 + A_3\left(\frac{\mathtt{m}_\bullet}{b}\right)^3 + A_4\left(\frac{\mathtt{m}_\bullet}{b}\right)^4 + A_5\left(\frac{\mathtt{m}_\bullet}{b}\right)^5 + \mathcal{O}\left(\frac{\mathtt{m}_\bullet}{b}\right)^6,$$

where b is called the *impact parameter* and $A_1 = 4$, $A_2 = 15\pi/4$, $A_3 = 128/3$, $A_4 = 3465\pi/64$, and $A_5 = 3584/5$. The distance r of closest approach in (1) is coordinate dependent, whereas the quantities b and $\mathtt{m}_\bullet$ are coordinate independent. Consequently, the series (2) is coordinate independent.

The 1919 observational confirmation of the light bending angle equation (1) for the case of

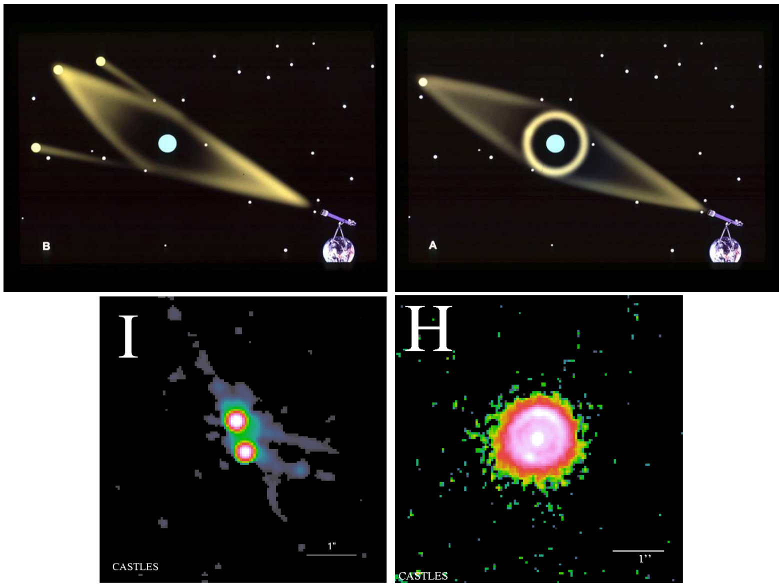

Figure 1. Top row: Einstein predicted that double images (left panel, outer bright spots) will occur if the light source (bright spot between the outer two) is off the line of sight and lensed by an intermediate compact body (large blue spot). He also argued that if the source is on the line of sight, the source will appear as a bright ring, known today as an *Einstein ring* (right panel). Bottom row: An observed example of double-lensed images (left panel) and an Einstein ring image (right panel) of lensed quasars. The "I" and "H" refer to the infrared wavelength bands used for the observations. Credits: Schoendorf (*Duke Magazine*) and CASTLES [18].

lensing by the sun gave observational support to Einstein's gravitational theory over Newtonian gravity. It made Einstein a household name and marked the first observation of gravitational lensing. Einstein also discovered two remarkable lensing properties, which he published in 1936 in a short *Science* article [26]. He showed that *double images* can occur when a background light source is off the line of sight. When the source is on the line of sight, a highly magnified ring-like image, now called an *Einstein ring*, can form; see Figure 1.

Einstein was very skeptical that these effects would be observed and, upon the urging of the Czech engineer Rudi Mandl, he reluctantly published the article [54, p. 7]. Fortunately, the advancement of technology over the next four decades set the foundation for the first observation in 1979 of double images of a lensed quasar. This serendipitous breakthrough discovery by Walsh, Carswell, and Weymann [72] marked the transformation of gravitational lensing from an arena of purely theoretical speculation to a data-driven science!

Today, observations of gravitational lensing signatures in the universe abound. Earth- and space-borne telescopes have found multiple images, rings, arcs, and highly magnified images of lensed sources. For lensing by galaxies, data from lens samples like CASTLES [18], SQLS [70], and SLACS [69] reveal scores of examples of multiply imaged quasars and observed ring/arc systems.

Gravitational Lensing Framework

The Space-Time Geometry

The light rays in gravitational lensing are modeled by null geodesics that ride the geometry of space-time. The Einstein equation is the physical law governing the interplay between the space-time's geometry and its mass-energy content (lenses). Lensing effects arise when multiple light rays have different arrival times at a given spatial location, light rays converge to create caustics, infinitesimal bundles of light rays experience expansion and/or contraction due to the Ricci curvature and shearing due to the Weyl tensor, etc. These effects are far too complicated to address here in a general space-time framework; see Perlick [48]. We restrict ourselves to the space-time setting relevant to astronomical observations.

Most gravitational lenses can be modeled using the *static*, *thin-lens*, *weak-deflection* approximation, because the observer-lens distance[1] d_L and lens-light source distance $d_{L,S}$ are significantly larger than the diameter of the lens and because the bending angles are much less than unity (e.g.,

[1] *The issue of distance in cosmology is a story unto itself. In fact, the distances in gravitational lensing are typically angular diameter distances; see [66, Sec. 4.5] for details.*

less than an arcminute) [54]. Examples of such lenses are planets, stars like our Sun, and galaxies. This approximation fails for lensing near a black hole because bending angles can exceed 360 degrees! Amazingly, however, the static, thin-lens, weak-deflection approximation is unmatched in the power of its predictions that are accessible to current and near-future instrumentation.

Figure 2 illustrates the above approximation, where L is the lens plane and $S = \mathbb{R}^2$ is the light source plane. The (scaled) positions $\mathbf{x}$ and $\mathbf{y}$ in the figure are given by $\mathbf{x} = \mathbf{r}/d_L$ and $\mathbf{y} = \mathbf{s}/d_S$, where $\mathbf{r}$ is the vector of impact of the light ray in the lens plane, $\mathbf{s}$ is the position of the light source on the light source plane S, and d_L and d_S are the respective distances from the observer to L and S. Note that $d_S > d_L \gg |\mathbf{r}|$ in a typical astrophysical setting.

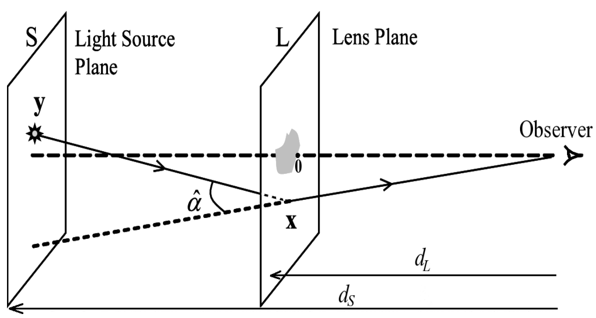

Figure 2. Schematic of thin-lens, weak-deflection single plane gravitational lensing. The dashed line through the origin is the optical axis of the lens system.

The space-time geometry for a static,[2] thin-lens, weak-deflection lens has a metric of the form:

$$\mathbf{g}_{\text{weak}} = -\left(1 + \frac{2\phi}{c^2}\right) dt^2 + a^2(t)\left(1 - \frac{2\phi}{c^2}\right) \mathbf{g}_{\text{Euc}},$$

where $|\phi|/c^2 \ll 1$. Here $t = c\tau$, with c the speed of light and τ cosmological time. The function $a(t)$ accounts for the expansion of the universe. The metric $\mathbf{g}_{\text{Euc}}$ is the standard Euclidean metric on $\mathbb{R}^3$ and ϕ is the time-independent, three-dimensional Newtonian potential of the lens with corresponding mass density concentrated about the lens plane. In astrophysical applications, the metric $\mathbf{g}_{\text{weak}}$ is computed only to first order in ϕ/c^2.

[2] *The static condition means that physically the lens has negligible change over the time interval during which the lensing effect is being observed.*

Deflection Potential

In physical modeling of the lens system in Figure 2, the three-dimensional Newtonian potential ϕ of the lens is "projected" into the lens plane L, creating a two-dimensional Newtonian potential ψ of the lens in L (since the lens is thin and bending angles are small); see [66, Chap. 4] for details. The projected potential ψ is called the *deflection potential* and defined by:

$$\psi(\mathbf{x}) = \frac{2d_{LS}}{c^2 d_L d_S} \int_{\zeta_0}^{\zeta_s} \phi(d_L\mathbf{x}, \zeta)\, \mathrm{d}\zeta,$$

where $(\mathbf{x}, \zeta)$ are rectangular coordinates covering the region of space containing the lens system, the ζ-axis coincides with the optical axis of the lens system, and ζ_0, ζ_s are the respective ζ-locations of the observer and light source plane.

Lensing by ψ produces a weak-deflection bending angle generalizing Einstein's bending angle (1) or, equivalently, the first term in (2), to:

$$\hat{\boldsymbol{\alpha}}(d_L\mathbf{x}) = \frac{d_S}{d_{L,S}} \nabla\psi(\mathbf{x}),$$

where the gradient operator ∇ is relative to rectangular coordinates $\mathbf{x} = (u, v)$ on the lens plane. The physical bending angle is $\hat{\alpha} = |\hat{\boldsymbol{\alpha}}|$. Note that a point mass deflection potential, namely, $\psi(\mathbf{x}) = m \log|\mathbf{x}|$ with $m = 4(d_{L,S}/c^2 d_S)\mathtt{m}$, produces Einstein's bending angle (1).

More formally, the deflection potential is a smooth function $\psi : L \longrightarrow \mathbb{R}$, where $L = \mathbb{R}^2 - A$ models the lens plane, with A a finite set of points representing possible singularities in the lens. The surface mass density κ due to ψ is determined by the two-dimensional Poisson equation:

$$\kappa(\mathbf{x}) = \frac{1}{2}\nabla^2\psi(\mathbf{x}),$$

which is the Einstein equation in the present two-dimensional context. The density κ causes the expansion or contraction of cross-sections of infinitesimal light ray bundles. Similarly, the gravitational tug due to matter produces shearing across such bundles. The *shear* due to ψ is a rank two symmetric trace-free tensor in L, whose independent components $(\Gamma_1(\mathbf{x}), \Gamma_2(\mathbf{x}))$ are defined by:

$$\Gamma_1(\mathbf{x}) = \frac{1}{2}\left[\psi_{\mathrm{uu}}(\mathbf{x}) - \psi_{\mathrm{vv}}(\mathbf{x})\right], \quad \Gamma_2(\mathbf{x}) = \psi_{\mathrm{uv}}(\mathbf{x}).$$

The subscripts indicate partial derivatives relative to (u, v). The *magnitude* of the shear tensor is defined as $\Gamma = \sqrt{\Gamma_1^2 + \Gamma_2^2}$.

Example: Microlensing Potential. This lens models a local region of a galaxy lens using three physical components: (1) a collection of g stars with masses $m_1, \ldots, m_g$ at respective positions $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_g$, (2) a continuous matter component with constant density $\kappa_c \geq 0$ from dark matter, and

(3) a constant shear $\gamma \geq 0$ from the overall gravitational pull of the rest of the galaxy across the local region. The deflection potential and surface mass density of the lens are given respectively by:

$$\psi_g(\mathbf{x}) = \frac{\kappa_c}{2}|\mathbf{x}|^2 - \frac{\gamma}{2}(\mathrm{u}^2 - \mathrm{v}^2) + \sum_{j=1}^{g} m_j \log|\mathbf{x} - \boldsymbol{\xi}_j|$$

and

$$\kappa_g(\mathbf{x}) = \pi\left(m_1\delta_{\boldsymbol{\xi}_1}(\mathbf{x}) + \cdots + m_g\delta_{\boldsymbol{\xi}_g}(\mathbf{x})\right) + \kappa_c,$$

where $\delta_{\boldsymbol{\xi}_j}(\mathbf{x})$ is the Dirac delta centered at $\boldsymbol{\xi}_j$. The set of singularities of ψ_g is $A = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_g\}$. The γ term contributes to the deflection potential ψ_g through a harmonic function and, hence, does not appear in the expression for κ_g. Also, the magnitude Γ_g of the shear tensor converges to the shear from infinity γ as $|\mathbf{x}| \to \infty$. The deflection potential ψ_g typically generates lensed images with angular separations of approximately a micro-arcsecond. For this reason, lensing by ψ_g is called *microlensing*.

Time-Delay Function

The deflection potential ψ induces a family of functions $T : L \times S \to \mathbb{R}$, called a *time-delay family*, defined by:

$$T(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{x} - \mathbf{y}|^2}{2} - \psi(\mathbf{x}),$$

where $S = \mathbb{R}^2$ is the light source plane (Figure 2) and $\mathbf{y} \in S$. Each function $T_{\mathbf{y}} : L \to \mathbb{R}$ $T_{\mathbf{y}}(\mathbf{x}) = T(\mathbf{x}, \mathbf{y})$ in the family T is called a *time-delay function at* $\mathbf{y}$. Physically, the value $T_{\mathbf{y}}(\mathbf{x})$ is proportional to the arrival time difference measured by the observer between a deflected ray traveling from $\mathbf{y}$ to the observer with impact vector $\mathbf{x}$ and a ray traveling from $\mathbf{y}$ to the observer in the absence of lensing. The arrival time of the unlensed ray enters the time-delay function as a constant and is used simply for convenience. Adding a constant to $T_{\mathbf{y}}(\mathbf{x})$ has no impact on lensing observables, since they arise from either differences of the time-delay function at lensed images or partial derivatives of the time-delay function [54, Sec. 3.4].

By Fermat's principle [54, p. 66], the light rays connecting $\mathbf{y}$ to the observer have impact vectors given by the critical points[3] of $T_{\mathbf{y}}$, i.e., solutions $\mathbf{x}$ in L of

(3) $$\nabla T_{\mathbf{y}}(\mathbf{x}) = \mathbf{0},$$

where ∇ is relative to the $\mathbf{x}$ coordinates. The light rays will correspond generically to either minima, maxima, or saddles of the time-delay function.

[3] *Generally, we define a* critical point *of a smooth map f from an n-manifold $\mathcal{N}$ to an m-manifold $\mathcal{M}$ to be a point $x \in \mathcal{N}$, where* $\mathrm{rank}[d_x f] < \min\{n, m\}$.

Lensing Map and Lensed Images

Every time-delay family T induces a transformation $\boldsymbol{\eta} : L \longrightarrow S$, called a *lensing map*, as follows:

$$\boldsymbol{\eta}(\mathbf{x}) = \nabla T_{\mathbf{y}}(\mathbf{x}) + \mathbf{y} = \mathbf{x} - \nabla\psi(\mathbf{x}),$$

where ∇ is the $\mathbf{x}$-gradient. A light ray from $\mathbf{y}$ to the observer is then characterized by a solution $\mathbf{x}$ in L of the *lens equation*

(4) $$\boldsymbol{\eta}(\mathbf{x}) = \mathbf{y}.$$

The action of $\boldsymbol{\eta}$ has a simple intuitive interpretation via its lens equation. Reverse the light ray in Figure 2 and imagine the light ray being shot like a cannon ball from the observer to the lens plane. The ray impacts the lens plane at $\mathbf{x}$ and gets deflected at $\mathbf{x}$ by the gravity of the matter lens there, causing the ray to hit the light source plane at $\mathbf{y}$.

The *lensed images* of a light source at $\mathbf{y}$ are then defined to be elements of the fiber $\boldsymbol{\eta}^{-1}(\mathbf{y})$. By the bijection between solutions of (3) and (4), we can naturally identify each lensed image of $\mathbf{y}$ with its associated critical point of $T_{\mathbf{y}}$. Consequently, when $T_{\mathbf{y}}$ is nondegenerate, we can assign a Morse index $i_{\mathbf{x}}$ to a lensed image $\mathbf{x}$ of $\mathbf{y}$, namely, $i_{\mathbf{x}} = 0$, 1, and 2, respectively, for $\mathbf{x}$ a minimum, saddle, and maximum lensed image. The *parity* of a lensed image $\mathbf{x}$ is defined as the evenness or oddness of $i_{\mathbf{x}}$ for nondegenerate $T_{\mathbf{y}}$. A *positive parity* (respectively, *negative parity*) lensed image is one with an even (respectively, odd) parity. The lensed images in Figure 2 form a positive-negative parity pair; one is a minimum lensed image and the other a saddle.

Magnification of Lensed Images

The *magnification* $M_{\mathbf{y}}(\mathbf{x})$ of a lensed image $\mathbf{x}$ of a light source at $\mathbf{y}$ is given physically as the ratio of the flux of the image to the flux of the light source in the absence of lensing. It can be shown that:

$$M_{\mathbf{y}}(\mathbf{x}) = \frac{1}{|\det[\mathrm{Jac}\,\boldsymbol{\eta}](\mathbf{x})|}, \qquad \boldsymbol{\eta}(\mathbf{x}) = \mathbf{y}.$$

Magnification is a geometric invariant because $M_{\mathbf{y}}(\mathbf{x})$ is the reciprocal of the Gaussian curvature at the critical point $(\mathbf{x}, T_{\mathbf{y}}(\mathbf{x}))$ in the graph of $T_{\mathbf{y}}$. A lensed image $\mathbf{x}$ of $\mathbf{y}$ is *magnified* if $M_{\mathbf{y}}(\mathbf{x}) > 1$ and *demagnified* when $M_{\mathbf{y}}(\mathbf{x}) < 1$. The *signed magnification* of a lensed image $\mathbf{x}$ of $\mathbf{y}$ is $\mu_{\mathbf{y}}(\mathbf{x}) = (-1)^{i_{\mathbf{x}}} M_{\mathbf{y}}(\mathbf{x})$, where $\boldsymbol{\eta}(\mathbf{x}) = \mathbf{y}$ and $i_{\mathbf{x}}$ is the Morse index of $\mathbf{x}$. The *total magnification* of a light source at $\mathbf{y}$ is

$$M_{\mathrm{tot}}(\mathbf{y}) = \sum_{\mathbf{x} \in \boldsymbol{\eta}^{-1}(\mathbf{y})} M_{\mathbf{y}}(\mathbf{x}).$$

The critical point type of a lensed image has physical relevance. *Minimum lensed images are never demagnified* (Schneider 1984 [64]) and, in fact, are typically magnified in real systems [66, 54]. A minimum lensed image $\mathbf{x}_{\min}$ cannot have an arbitrary position relative to the lens but is always located where the lens's surface mass density κ is subcritical, $0 \le \kappa(\mathbf{x}_{\min}) < 1$, and its magnitude of shear is not supercritical, $0 \le \Gamma(\mathbf{x}_{\min}) \le 1$.

A maximum lensed image $\mathbf{x}_{\max}$ is situated where the surface mass density of the lens is supercritical, $\kappa(\mathbf{x}_{\max}) > 1$. For example, a maximum lensed image produced by a galaxy lens would be located near the dense nucleus of the galaxy, causing the image to be highly demagnified, hence difficult to observe.

No restrictions are known for the positions of saddle lensed images due to general lenses. However, computer simulations of microlensing show that saddle lensed images have a tendency to congregate near the positions of point masses; see [54, Sec. 11.6] for a mathematical result on the trajectories of saddle lensed images.

Critical Curves and Caustics

The set of *critical points* of $\boldsymbol{\eta}$ is the locus of all $\mathbf{x}$ in L where $\det[\mathrm{Jac}\,\boldsymbol{\eta}](\mathbf{x}) = 0$. This corresponds to the set of all formally infinitely magnified lensed images of all light source positions in the light source plane S. A curve of critical points is called a *critical curve* of $\boldsymbol{\eta}$. The set of *caustics* of $\boldsymbol{\eta}$ is the set of critical values of $\boldsymbol{\eta}$, which is the set of all light source positions giving rise to at least one infinitely magnified lensed image. The set of caustics of $\boldsymbol{\eta}$ has measure zero. For a point mass lens, which Einstein studied in 1936 [26], the set of critical points forms a circle called an *Einstein ring*, and the set of caustics is a single point; see Figure 1.

Multiplane Lensing Framework

The single-plane lensing extends naturally to k lens planes as depicted in Figure 3.

Let $\psi_i : L_i \longrightarrow \mathbb{R}$ be the deflection potential on the ith lens plane L_i, set $L_{(k)} = L_1 \times \cdots \times L_k$, and let $\mathbf{x}_{k+1} = \mathbf{y}$. A *k-plane time-delay family* induced

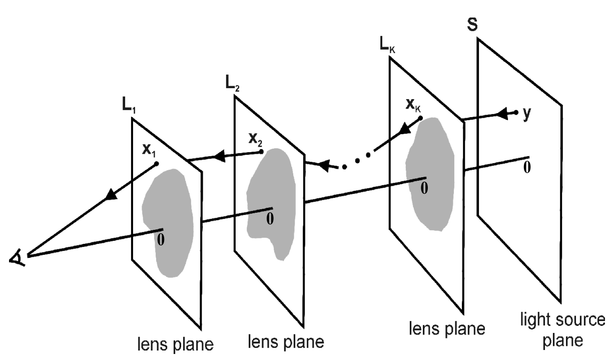

Figure 3. A schematic of multiplane lensing in the static, thin-lens, weak-deflection approximation. Credits: [54, p. 196].

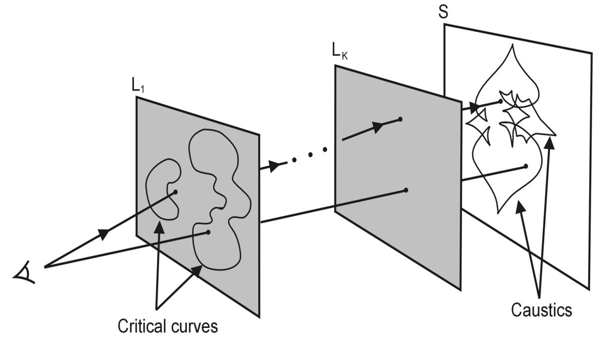

Figure 4. Critical curves and caustics in multiplane lensing. Note the cusp points and fold arcs on the caustics. Credits: [54, p. 203].

by these deflection potentials is the function $T_{(k)} : L_{(k)} \times S \longrightarrow \mathbb{R}$ defined by

$$T_{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}) = \sum_{i=1}^{k} \vartheta_i \left[\frac{|\mathbf{x}_i - \mathbf{x}_{i+1}|^2}{2} - \beta_i \psi_i(\mathbf{x}_i) \right],$$

where ϑ_i and β_i are constants involving physical aspects (e.g., distances) of the lens system. The *k-plane lensing map* generated by $T_{(k)}$ is a map of the form $\boldsymbol{\eta}_{(k)} : P \subseteq L_1 \longrightarrow S$, where $P = \mathbb{R}^2 - B$ with B the set of light ray obstruction points, namely, the set of all points in $\mathbb{R}^2$ through which a backwards traced light ray is obstructed from reaching the light source plane (e.g., if a light ray impacts a star). The notions of lensed image, magnification, critical points, and caustics carry over naturally to the k-plane case.

Remark. The lensing map $\boldsymbol{\eta}_{(k)}$ is generated by the time-delay family $T_{(k)}$ in a manner similar to how Lagrangian maps are formed from their generating family of functions or catastrophe maps are produced from their generating unfoldings. In fact, the Lagrangian map or catastrophe map induced by $T_{(k)}$ is differentiably equivalent to $\boldsymbol{\eta}_{(k)}$.

Gravitational Lensing Software

A publicly available extensive lensing software is GRAVLENS, which was developed by Keeton: `http://redfive.rutgers.edu/~keeton/gravlens/`. The software computes positions, magnifications, and time delays of lensed images for both point and extended light sources and for essentially arbitrary mass distributions in the lens. The forthcoming version 2.0 (expected 2010) will do stochastic lensing and k-plane lensing.

Stability and Genericity in Lensing

Often astronomers develop intuition into lensing from the analysis of overly simplified analytical models. There is then a need for general theoretical lensing results that capture the properties of lens systems that are shared by most such systems and that persist even under the many small perturbations affecting these systems in their cosmic environment. We desire a characterization of the *generic, stable features* of k-lens systems that are independent of the specific fine details of the system.

A *generic property* in a space[4] of maps is a property common to all maps in an open dense subset of the space. Of course, a precise definition of stability is beyond the scope of this article, so we shall employ an informal characterization; see [30] and [54, Chap. 7] for rigorous treatments.

Intuitively, a lensing map $\boldsymbol{\eta}_{(k)}$ is "locally stable" if any sufficiently small perturbation $\widehat{\boldsymbol{\eta}_{(k)}}$ of $\boldsymbol{\eta}_{(k)}$ (not necessarily a linear perturbation) has the same local critical point structure as $\boldsymbol{\eta}_{(k)}$ up to a coordinate change. It can be shown that a lensing map $\boldsymbol{\eta}_{(k)}$ is *locally stable* if and only if its caustics are locally either fold arcs or cusp points. In this case, the set of critical points form disjoint nonself-intersecting curves (critical curves), whose total number we can bound in some cases (e.g., Theorem 8). Note that the lensing map of a point mass lens, which was studied by Einstein in 1936 [26], is not locally stable because the caustic is a point, though the critical curve is a circle; see Figure 1.

A lensing map $\boldsymbol{\eta}_{(k)}$ is *transverse stable* if and only if $\boldsymbol{\eta}_{(k)}$ is locally stable and its caustics curves are "stably distributed", namely, each intersection of fold arcs occurs at a nonzero angle, no more than two fold arcs cross at the same point, no fold arc passes through a cusp point, and no two cusp points coincide.

The theorem below, which was established in the monograph [54] by AP, Levine, and Wambsganss, characterizes the generic properties of k-plane lens systems:

Theorem 1. [54, p. 311] *Let $\mathfrak{T}_{(k)}$ be the space of all k-plane time-delay families $T_{(k)} : L_{(k)} \times S \longrightarrow \mathbb{R}$ and let $\mathfrak{T}^{\star}_{(k)}$ be the subset of $\mathfrak{T}_{(k)}$ of all k-plane time-delay families whose lensing maps are transverse stable. Then $\mathfrak{T}^{\star}_{(k)}$ is open and dense in $\mathfrak{T}_{(k)}$.*

A sketch of the proof of Theorem 1 is beyond the scope of this article. It utilizes the technical machinery of multijet transversality to singularity manifolds. Theorem 1 implies that among the vast array of gravitational lens systems in the cosmos that fall within the static, thin-lens, weak-deflection approximation, the associated lensing map is typically transverse stable. In addition, for a dense subset of light source positions, the corresponding lensed images are either minima, maxima, or generalized saddles, no matter how complex the lens system; see [54, Chap. 8].

[4] *We assume that a space of maps between smooth manifolds has the Whitney C^∞ topology.*

Image Counting

As early as 1912 [61], Einstein had found that a lens consisting of a single star (point mass) will produce *two* lensed images of a background star that is not on a caustic. For two point masses on the same lens plane, Schneider and Weiss 1986 [67] employed lengthy, intricate calculations to show that this lens produces *three* or *five* lensed images of a light source not on a caustic. Naturally, one would like to know how many lensed images are produced by g stars or, generally, by a generic gravitational lens system.

In 1991 AP [49] addressed the image counting problem using *Morse theory under boundary conditions A and B* to obtain counting formulas and a minimum for the total number of images due to a generic k-plane gravitational lens system. An added benefit from the Morse theoretic approach is that it applies to generic k-plane lensing and generalizes naturally to Lorentzian manifolds; see Perlick [48] and references therein.

For simplicity, we present a sample of image counting results due only to lensing by point masses (stars).

Theorem 2. [49] *If a light source, not on a caustic, undergoes single-plane lensing by g point masses, then:*

(1) *There are no maximum lensed images.*
(2) *The total number N of lensed images obeys:*

$$N = 2N_{\min} + g - 1 = 2N_{\text{sad}} - g + 1,$$

where $N_{\min}$ and N_{sad} are the number of minimum and saddle lensed images, respectively.

(3) *The minimum value of N is $g + 1$.*

The counting formula in Theorem 2 is useful for checking whether software that numerically searches the lens equation for microimages has overlooked some—e.g., a system with 10,000 stars has more than that many microimages. The counting formula also shows that the total number of lensed images is even (odd) if and only if the number of point masses is odd (even).

For the maximum number of lensed images, Rhie 2003 [62] constructed an example of lensing by g point masses that produce $5g - 5$ lensed images for $g \geq 2$, which she conjectured is the maximum possible; see [57] for a brief history. Khavinson and Neumann 2006 [39] settled the conjecture by translating the problem into determining the maximum number of zeros of a complex rational harmonic function of the form $r(z) - \bar{z}$, where $r(z) = p(z)/q(z)$ with $p(z)$ and $q(z)$ relatively prime polynomials and $\deg r = \max\{\deg p, \deg q\} = g$. We note that Kuznia and Lundberg 2009 [41] studied the case where $r(z)$ is a Blaschke product and found a maximum of $g + 3$ zeros.

Theorem 3. [62, 39] *A light source, not on a caustic, that is lensed by g point masses on a single lens plane has a maximum number of lensed images of $5g-5$ for $g \geq 2$.*

By Theorems 2 and 3, we have $g+1 \leq N \leq 5g-5$ for $g \geq 2$, which yields that two point masses will produce three, four, or five lensed images. But N has parity (evenness or oddness) opposite to g, so four images cannot be produced. Hence, we immediately recover the result in [67] that two stars produce three or five images of a light source not on a caustic.

The multiplane analogue of Theorem 2 is:

Theorem 4. [49, 51] *If a light source, not on a caustic, undergoes k-plane lensing by point masses with g_i point masses on the ith lens plane, then:*

(1) *There are no maximum lensed images.*

(2) *The total number N of lensed images obeys:*

$$N = 2N_+ - \prod_{i=1}^{k}(1-g_i) = 2N_- + \prod_{i=1}^{k}(1-g_i),$$

where $N_\pm$ is the number of even/odd index lensed images.

(3) *The minimum value of N is $\prod_{i=1}^{k}(g_i+1)$.*

Theorem 4 immediately shows that if any lens plane has only one point mass, then the total number of lensed images is always *even*. This is complemented by the fact that there is an *odd number of lensed images due to k-plane lensing by nonsingular lenses* [49, 51].

No multiplane analogue of Theorem 3 exists as of the writing of this article. However, an upper bound was found by AP [53] in 1997 using a resultant approach:

Theorem 5. [53] *If a light source, not on a caustic, is lensed by g point masses distributed in space with one point mass on each lens plane, then the number of images is bounded as follows:*

$$2^g \leq N \leq 2\left(2^{2(g-1)}-1\right),$$

where the lower bound is sharp (attainable).

The sharp lower bound in Theorem 5 follows from Theorem 4(3) since $g_i = 1$ for $i = 1, \ldots, g$.

Open Problem 1. Determine the maximum number of lensed images due to g point masses distributed in space with one point mass on each lens plane.

There is no global maximum number of lensed images due to lensing by general nonsingular gravitational lenses. This is because one can always add an appropriate smooth mass clump to a nonsingular lens to produce extra images. Note that a global minimum exists for the number of lensed images due to multiplane singular lenses with time-delay functions satisfying Morse boundary conditions A. In fact, the global minimum is also given by Theorem 4(3) [51].

A maximum number of lensed images due to nonsingular lenses can be found in certain special cases. For example, in 2007 Fassnacht, Keeton, and Khavinson proved that an elliptical uniform mass distribution produces a maximum of four images external to the lens [29]. Khavinson and Lundberg 2010 [37] showed that multiple imaging by finitely many disjoint radially symmetric disc lenses is more complex than was originally thought by constructing an example with five such lenses that surprisingly produces twenty-seven lensed images, where twenty-five images would have been expected. Khavinson and Lundberg 2010 [38] also showed that there are at most eight external lensed images due to an elliptic lens with isothermal density, which was followed by recent work of Bergweiler and Eremenko 2010 [14] proving that the maximum number is actually six. This problem involved studying zeros of complex transcendental harmonic functions as opposed to the complex rational harmonic functions found in microlensing.

Open Problem 2. Determine the maximum number of lensed images due to elliptical isothermal lenses distributed over multiple lens planes.

Stochastic Gravitational Lensing

In lensing we often do not know the positions of stars, the locations of dark matter clumps, etc., and have to treat such components of a lens system as random. For these situations, the induced time-delay family and its lensing map are random and connect naturally with the *geometry of random fields.*

The theory of random fields has been developed largely around the tractable case of Gaussian fields. Interested readers may consult, for example, the seminal works of Adler, Berry, Hannay, Longuet-Higgins, Nye, Taylor, Upstill, Worsham, etc.; see [5], [6], [7], [15]. Interestingly, the key random fields in gravitational lensing are, in general, not Gaussian.

Non-gaussianity and Stochastic Microlensing

Consider a random microlensing deflection potential ψ_g where all the point masses have the same mass $m_i = m$ and their positions $\boldsymbol{\xi}_i$ are *independent and uniformly distributed* in the disc $B(\mathbf{0}, R)$ of radius $R = \sqrt{g/\pi}$ centered at the origin of the lens plane. Let $T_{\mathbf{y},g}$, $\boldsymbol{\eta}_g$, and $\mathbf{G}_g = (\Gamma_{1,g}, \Gamma_{2,g})$ be, respectively, the single-plane time-delay function, lensing map, and pair of shear tensor components due to the given ψ_g. Set $T^*_{\mathbf{y},g}(\mathbf{x}) = T_{\mathbf{y},g}(\mathbf{x}) + gm\log R$ and $\boldsymbol{\eta}^*_g(\mathbf{x}) = \boldsymbol{\eta}_g(\mathbf{x})/\sqrt{\log g}$. Denote the probability density functions (p.d.f.'s) of $T^*_{\mathbf{y},g}(\mathbf{x})$, $\boldsymbol{\eta}^*_g(\mathbf{x})$, and $\mathbf{G}_g(\mathbf{x})$ by $f_{T^*_{\mathbf{y},g}(\mathbf{x})}$, $f_{\boldsymbol{\eta}^*_g(\mathbf{x})}$, and $f_{\mathbf{G}_g(\mathbf{x})}$, respectively.

AP, Teguia, and Rider 2009 [55, 56] showed:

Theorem 6. [55, 56] *For the above random microlensing deflection potential ψ_g, the p.d.f.'s $f_{T^*_{\mathbf{y},g}(\mathbf{0})}$, $f_{\boldsymbol{\eta}^*_g(\mathbf{x})}$, and $f_{\mathbf{G}_g(\mathbf{x})}$, where $\mathbf{x} \in B(\mathbf{0}, R)$, are **not** Gaussian for $g = 1, 2, \ldots$. As $g \to \infty$, the asymptotic forms are:*

(1) $f_{T^*_{\mathbf{y},g}(\mathbf{x})}(\mathbf{t}) = f_{\mathrm{Gm},g}(\mathbf{t}) + O(1/g^{3/2})$

(2) $f_{\boldsymbol{\eta}^*_g}(\mathbf{k}) = f_{\mathrm{Gs},g}(\mathbf{k})[1 + E_g(\mathbf{k})] + O\left(1/\log g\right)$

(3) $f_{\mathbf{G}_g(\mathbf{x})}(\mathbf{g}) = f_{\mathrm{Ch},g}(\mathbf{g})[1 + H_g(\mathbf{g})] + O\left(1/g^3\right)$,

where $f_{\mathrm{Gm},g}$, $f_{\mathrm{Gs},g}$, and $f_{\mathrm{Ch},g}$ are gamma, bivariate Gaussian, and stretched bivariate Cauchy densities, respectively. See [55, 56] *for explicit forms of the integrable functions E_g and H_g.*

To illustrate a random $\boldsymbol{\eta}_g$, let $\kappa_c = 0.405$, $\gamma = 0.3$, and $\kappa_* = \pi m = 0.045$. In [55] it was shown that when g is a million, there is a 56% probability that the random lensing map $\boldsymbol{\eta}_g : L \longrightarrow S$ will map a point $\mathbf{x}_0 = (u_0, v_0)$ in L to a point inside a disc of angular radius $r_0 = 0.1$ centered at $\mathbf{a}_0 = \left((1 - \kappa_c + \gamma)u_0, (1 - \kappa_c + \gamma)v_0\right)$. The probability jumps to 97% for mapping $\mathbf{x}_0$ inside a radius $2r_0$ centered at $\mathbf{a}_0$.

Global Expectation and the Kac-Rice Formula

For a random time-delay function $T_{\mathbf{y}}$, let $N_+(D, \mathbf{y})$ be the random number of positive parity lensed images inside a closed disk D in the lens plane. Under appropriate physically reasonable conditions, the theory of random fields [5, 6] yields that the expectation of $N_+(D, \mathbf{y})$ can be obtained using a Kac-Rice type formula [56]:

$$(5)\quad E[N_+(D, \mathbf{y})] = \int_D E\left[\det[\mathrm{Jac}\,\boldsymbol{\eta}](\mathbf{x})\, \mathbf{1}_{\mathcal{G}_A}(\mathbf{x}) \,\middle|\, \boldsymbol{\eta}(\mathbf{x}) = \mathbf{y}\right] f_{\boldsymbol{\eta}(\mathbf{x})}(\mathbf{y}) \mathrm{d}\mathbf{x},$$

where $\mathbf{1}_{\mathcal{G}_A}$ is the indicator function on

$$\mathcal{G}_A = \{\mathbf{x} \in \mathbb{R}^2 : \det[\mathrm{Jac}\,\boldsymbol{\eta}](\mathbf{x}) \in (0, \infty)\},$$

and $f_{\boldsymbol{\eta}(\mathbf{x})}$ is the p.d.f. of the lensing map at $\mathbf{x}$.

For most applications, equation (5) is insufficient because the light source position $\mathbf{y}$ is not known and so is a random vector. It is physically natural then to generalize (5) by averaging out the light source position. This introduces the notion of "global expectation". Let $\{\mathfrak{S}\}$ be a countable compact covering of S where every $\mathfrak{S}$ has the same positive Lebesgue measure $|\mathfrak{S}_0|$. Construct a family $\{\mathbf{Y}_{\mathfrak{S}}\}$ of random light source positions $\mathbf{Y}$, where $\mathbf{Y}_{\mathfrak{S}}$ is uniformly distributed on $\mathfrak{S}$. The *global expectation* of the number of positive parity lensed images will be denoted by $\hat{E}[N_+(D, \mathbf{Y}; \mathfrak{S})]_{\{\mathfrak{S}\}}$ and defined to be the mean of $E[N_+(D, \mathbf{y})]$ over the family $\{\mathbf{Y}_{\mathfrak{S}}\}$.

AP, Rider, and Teguia 2009 [56] used the Kac-Rice technology to obtain:

Theorem 7. [56] *The global expectation of the number of positive parity lensed images in D is:*

$$\hat{E}[N_+(D, Y; \mathfrak{S})]_{\{\mathfrak{S}\}} = \frac{1}{|\mathfrak{S}_0|} \int_D E\left[\det\left[\mathrm{Jac}\,\boldsymbol{\eta}\right](\mathbf{x})\ \mathbf{1}_{\mathcal{G}_A}(\mathbf{x})\right]\, \mathrm{d}\mathbf{x}.$$

Theorem 7 applies to generic lensing scenarios. Furthermore, the theorem is not merely formal because it can be used to calculate the global expected number of minimum lensed images in stochastic microlensing. For example, it was shown in [56] that if $g = 1000$, $\kappa_{\mathrm{tot}} = \kappa_c + \kappa_* = 0.45$, and (γ, κ_*) varies over the physically reasonable values $(0.n, 0.n)$, where $n = 1, 2, 3, 4$, then the global expected number of minimum microimages is between one and three even though there are over 1,000 microimages. Hence, there are few minimum lensed images compared to saddles for these parameters. Bear in mind, however, that the global expected mean number of minimum lensed images is divergent for $(1 - \kappa_{\mathrm{tot}})^2 = \gamma^2$.

Open Problem 3. Develop a general mathematical theory of stochastic gravitational lensing.

Initial steps were taken in [55, 56] for stochastic microlensing and in [33] for the case of lensing by randomly distributed dark matter clumps in galaxies, but a vast array of statistical and probabilistic issues remain unexplored.

Caustics

Counting Caustic Curves and Cusps

For the case of microlensing, some global quantitative results are known about the caustics:

Theorem 8. *Consider a locally stable single-plane lensing map due to the microlensing deflection potential ψ_g. Then:*

(1) [74] *The number of critical curves is at most $2g$.*

(2) [66, 54] *The number of cusps is even.*

(3) [60] *The number of cusps is bounded above as follows:*

$$0 \le N_{\mathrm{cusps}} \le \begin{cases} 12g^2 & \text{if } \gamma > 0 \\ 12g(g-1) & \text{if } \gamma = 0. \end{cases}$$

(4) [52] *The total signed curvature K_f of the fold caustics satisfies:*

$$K_f = -2\pi g.$$

In Theorem 8, the *even number cusp* and *total signed curvature* results extend to generic k-plane lensing (AP 1995 [52]), where for the latter we have

$$K_f = -2\pi|\mathcal{B}|,$$

with $|\mathcal{B}|$ the number of obstruction points in the lens system; also see [54, Sec. 15.4] for more. Figure 5 captures the even-number-cusp result.

Open Problem 4. Determine the maximum number of cusps in microlensing.

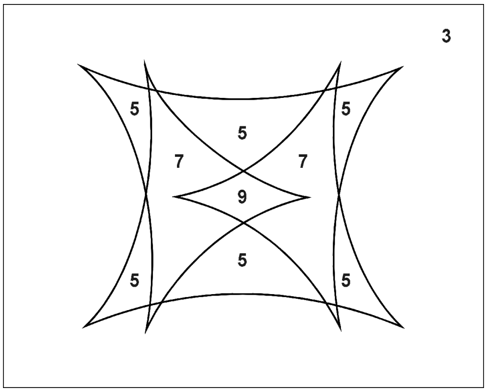

Figure 5. An illustration of the even-number-cusp theorem and local-convexity theorem. For the latter, fold curves are convex (outward curving) locally on the side from which a light source has more images. Each numeral indicates the number of lensed images of a light source in the given region. These caustics are due to two point mass lens with shear $\gamma > 0$. Credits: [74].

We suspect that the techniques involved with this problem are similar to the complex algebraic methods employed by Khavinson and Neumann [39] to address the maximum number of lensed images.

Local Convexity of Caustics

The fold caustic curves due to single-plane lensing by the microlensing deflection potential ψ_g satisfy local convexity, which means that the fold caustic curve is convex (bends outward) when viewed from the side where a light source has more images. This phenomenon was discovered in 1976 by Berry [13] while studying the caustics due to water droplets. The notion was mathematically developed in [54, Sec. 9.3] for the context of gravitational lensing.

Theorem 9. [54, p. 360] *If $\boldsymbol{\eta}_p$ is a locally stable single-plane lensing map induced by a deflection potential ψ with a locally constant surface mass density κ, then the fold arc caustics of $\boldsymbol{\eta}$ are locally convex.*

Theorem 9 applies to the microlensing deflection potential ψ_g since $\kappa_g(\mathbf{x}) = \kappa_c$ for $\mathbf{x} \in L$. Consequently, caustics due to stars cannot bend arbitrarily. Figure 5 gives an example of local convexity in microlensing. Local convexity, however, can be violated for 2-plane lensing by locally constant surface mass densities. For example, 2-plane microlensing can produce teardrop caustics [59].

Caustic Metamorphoses

For the time-delay families considered so far, all physical parameters of the lens system, such as the masses of the lenses, distances between lens planes, redshifts, etc., are assumed fixed. Allowing these parameters to vary will produce higher order caustic singularities whose contours on the light source plane produce fascinating caustic metamorphoses that can be classified into characteristic types. Note that generic optical caustic metamorphoses have distinct global properties enforced for general caustic metamorphoses—e.g., in three-space, saucer shape or pancake caustics cannot occur in optical caustic metamorphoses (Chekanov 1986 [22]).

Generically, there are five local 1-parameter metamorphoses of caustics in the plane: *lips*, *beak-to-beaks*, *swallowtails*, *elliptic umbilics*, and *hyperbolic umbilics* (Zakalyukin 1984 [75], Arnold 1986 [9], Arnold 1991 [10]). All five of these caustic metamorphoses occur in gravitational lensing [66, 54].

Figure 6 depicts an example of four swallowtail metamorphoses due to lensing by a point mass lens with shear $\gamma = 0.2$ and $\kappa_c = 1.21$. For the general microlensing deflection potential ψ_g, only beak-to-beaks, swallowtails, and elliptic umbilics can occur because lips and hyperbolic umbilics violate the local convexity of Theorem 9.

The following results are known about upper bounds on the number of caustic metamorphoses in microlensing:

Theorem 10. *Consider a single-plane lensing map due to the microlensing deflection potential ψ_g. Then:*

(1) [74] *The number of beak-to-beak caustic metamorphoses is at most* $3g - 3$.
(2) [54, p. 536] *The number of elliptic umbilic caustic metamorphoses is at most* $2g - 2$ *for* $\gamma = 0$ *and* $2g$ *for* $\gamma > 0$.

Theorem 10 immediately shows that a single point mass lens with continuous matter and shear cannot produce any beak-to-beak caustic metamorphoses.

Open Problem 5. Determine the maximum number of swallowtail metamorphoses in microlensing.

This problem relates to Open Problem 4 since the maximum number of cusps would be an upper bound for the number of swallowtails.

Elimination of Cusps

Caustic metamorphoses can also eliminate singularities on caustic curves, a result shown for microlensing by AP and Witt in 1996:

Theorem 11. [60] *Let $\boldsymbol{\eta}$ be the single-plane lensing map induced by the microlensing deflection potential ψ_g. For a sufficiently large continuous dark matter density κ_c, all the cusp caustics are eliminated, and the caustics evolve into a disjoint collection of g oval, fold caustic curves.*

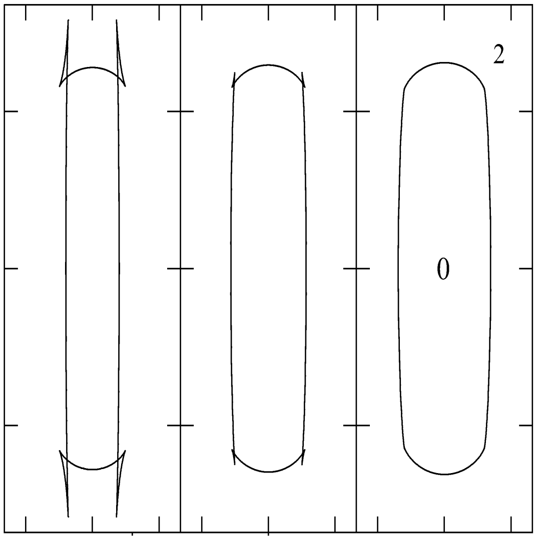

Figure 6. An illustration of swallowtail metamorphoses and the cusp-elimination theorem (Theorem 11). The left panel shows a caustic with four swallowtail caustic metamorphoses, which yields a total of eight cusps. All the cusps get eliminated by the swallowtail caustic metamorphoses. A light source inside the oval caustic in the right panel will have no lensed images, whereas two lensed images are produced for light source positions outside the oval. Hence, the local-convexity theorem is satisfied by the oval caustic. The caustics are produced by a point mass lens with shear $\gamma = 0.2$ and κ_c that increases from left to right through the panels, starting at $\kappa_c = 1.21$ in the left panel. Credits: [54, p. 385].

Theorem 11 shows that the lower bound on the number of cusps in Theorem 8(3) is not trivial but is the minimum number of cusps. Figure 6 gives an example of cusp elimination leading to an oval caustic. The bottom left panel of Figure 8 also shows some oval caustics.

Open Problem 6. Determine whether cusp elimination holds for any locally stable k-plane lensing map, where each stage of the evolution is a k-plane lensing map.

The notion of cusp elimination for general maps was anticipated by Levine [42] as early as 1963 and further explored by Eliashberg [27] in 1970, giving rise to the Levine-Eliashberg cusp-elimination theorem: A locally stable map from a compact, oriented n-manifold into the plane is homotopic to a locally stable map with zero or one cusp if the Euler characteristic of the manifold is even or odd, respectively.

Remark. The Levine-Eliashberg cusp-elimination theorem does not imply that cusp elimination holds in gravitational lensing, since if one starts with a lensing map whose domain is extended to the celestial sphere, then the theorem does not guarantee that each stage of the homotopy is a lensing map. Nevertheless, it was this theorem that inspired Theorem 11.

Arnold's ADE-Family of Caustics

In addition to the light source position $\mathbf{y}$, a lens system has other physical parameters $\mathbf{p} \in \mathbb{R}^{n-2}$, which can represent masses, core radii, shears, redshifts, angular diameter distances, etc. So far, these parameters have been fixed, and only $\mathbf{y}$ was varied.

When we allow the parameters $\mathbf{p}$ to vary as well, we have an n-parameter family $T_{\mathbf{p},\mathbf{y}}$ of time-delay functions that generate higher-order caustics in $\mathbb{R}^{n-2} \times S = \{\mathbf{p}, \mathbf{y}\}$. Slices through these higher-dimensional caustics by the light source plane S produce caustic curves in S with distinctive features; see the sketches by Callahan [19].

The local classification of higher-order caustics for general n-parameter families is well known from singularity theory (e.g., [8, 11, 12, 21, 54, 46]):

- $n = 2$: *folds* A_2 and *cusps* A_3.
- $n = 3$: list for $n = 2$ along with *swallowtails* A_4, *elliptic umbilics* D_4^-, and *hyperbolic umbilics* D_4^+.
- $n = 4$: list for $n = 3$ along with *butterflies* A_5 and *parabolic umbilics* D_5.
- $n = 5$: list for $n = 4$ along with *wigwams* A_6, *second elliptic umbilics* D_6^-, *second hyperbolic umbilics* D_6^+, and *symbolic umbilics* E_6.

The A, D, E notation is due to Arnold, who connected these caustic singularities with Coxeter-Dynkin diagrams of simple Lie algebras with the same designation. Arnold's classification of typical caustics for $n \leq 5$ is:

Theorem 12. [8] *For $n \leq 5$, there is an open dense set in the space of Lagrangian maps of n-dimensional Lagrangian submanifolds such that the caustics of each Lagrangian map in the set are locally from the following list: $A_\ell (1 \leq \ell \leq n+1)$, $D_\ell (4 \leq \ell \leq n+1)$, $E_6 (5 \leq n)$, $E_7 (6 \leq n)$, and $E_8 (7 \leq n)$.*

Figure 7 illustrates the swallowtail A_4, elliptic umbilic D_4^-, and hyperbolic umbilic D_4^+ caustic surfaces from Arnold's ADE-list of caustics in Theorem 12. Generic slices of these caustic surfaces by a plane produce the characteristic shapes of the caustics curves. Note that the figure includes the swallowtail caustic metamorphoses in Figure 6 and three of the five generic local caustic metamorphoses mentioned earlier.

Though Theorem 12 applies to general Lagrangian maps, Guckenheimer [32] showed that it also exhausts the list of typical caustics due to

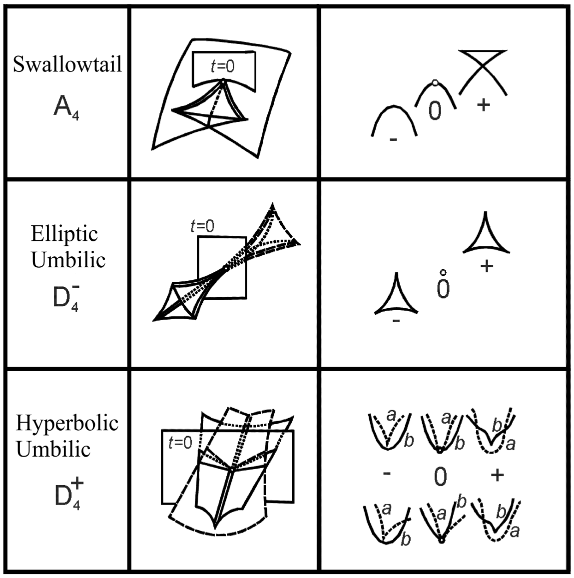

Figure 7. Swallowtail, elliptic umbilic, and hyperbolic umbilic caustic surfaces. The slices through the caustic surfaces produce planar caustic curves with characteristic features that occur in gravitational lensing. Credits: [9].

general optical systems. Since gravitational lens systems form a proper subset of the set of optical systems, we need to address whether Arnold's entire ADE-list of caustics occurs in gravitational lensing.

Gravitational lens models are known to produce at least the seven caustics for the $n = 4$ case in Arnold's Theorem 12 [54]. Shin and Evans 2007 [71] showed that our Milky Way galaxy acting as a lens can produce butterfly caustics. Orban de Xivry and Marshall 2009 [47] also developed an atlas of lensing signatures, predicting that a galaxy with a misaligned disc and nucleus would produce swallowtails and butterflies, binary galaxies would generate elliptic umbilics, and clusters of galaxies would produce hyperbolic umbilics and more.

Gravitational lensing by the Abell 1703 cluster of galaxies already reveals a hyperbolic umbilic signature [47]. Current and planned wide-field optical imaging surveys are expected to find thousands of new lensing signatures, which will likely contain evidence for many ADE-caustics [47].

Remark. Along with Theorem 12, the wider Arnold singularity theory also applies to gravitational lensing. AP 1993 [50] employed the theory to obtain local classifications of the qualitative features of certain key structures in lensing: *lensed image surfaces* (multibranched graphs of all lensed images with respect to light source position), *multibranched graphs of lensed image time delays*, *Maxwell sets* (light source positions for which at least two lensed images have identical time delays), *bicaustics* (paths traced out by cusps during an evolution of caustics), etc.

Caustics and Cosmic Shadows

One of the striking consequences of gravitational lensing is that the gravitational fields due to bodies in the universe such as stars, galaxies, black holes, dark matter, etc., cast *shadow patterns* throughout the cosmos. Some examples are shown in Figure 8.

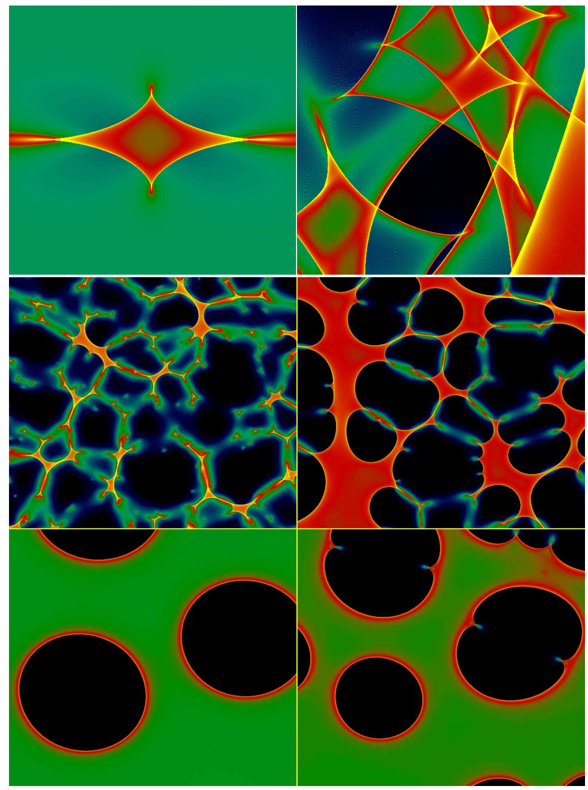

Figure 8. Examples of shadow patterns on the light source plane generated by microlensing. The fold caustic curves are brighter locally on one side, and each cusp has an emanating bright lobe. Elimination of cusps occurs for the middle and bottom rows of panels (clockwise through those panels), resulting in oval caustics bounding demagnified regions. The shadow pattern color scheme is: yellow (brightest) → red → green → blue → black (darkest). Credits: Wambsganss.

Within our framework, the shadow pattern lies on the light source plane and is modeled by assigning at each point $\mathbf{y}$ in the light source plane S the total magnification $M_{\text{tot}}(\mathbf{y})$ of a light source at $\mathbf{y}$. As $\mathbf{y}$ varies, a gradation in magnification is mapped out on the light source plane with caustics as the brightest part of the shadow pattern—Figure 8.

Making use of Theorem 1 we can infer the generic properties of the shadow pattern due

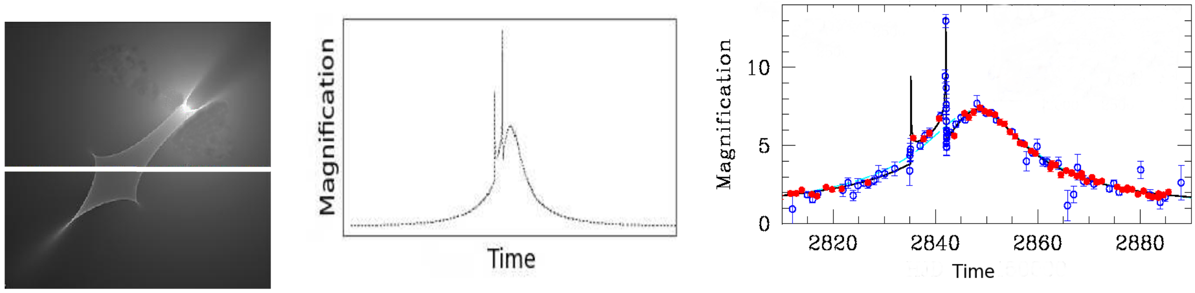

Figure 9. Finding an extrasolar planet using gravitational lensing. *Leftmost panel:* A theoretical shadow pattern due to a foreground star-planet lens. A background star moves from left to right along the white linear path. *Middle panel:* As the background star cuts through the shadow pattern due to lensing by the foreground star-planet lens (trajectory in the left panel), the background star's magnification varies according to the theoretical light curve shown. The spikes are due to the presence of the planet, which creates the caustic curve shown in the leftmost panel. In the absence of a planet, the light curve would have no spikes and be smooth. *Rightmost panel:* The theoretical light curve in the middle fits observational data of a background star being lensed by a star-planet lens located toward the center of our galaxy. The dots are observations made by the MAO (blue) and OGLE (red) groups. Credits: Bond et al., MAO and OGLE collaborations [17].

to multiplane lensing. Two such properties are (cf. Figure 8): (1) a fold caustic curve is brighter locally on one side of the curve and (2) cusp caustics, though they form a set of measure zero, contribute nonzero area to a shadow pattern since a high-magnification lobe emanates from each cusp.

How to Detect an Extrasolar Planet?

The issue of extraterrestrial life has resurged in the media recently with Stephen Hawking's Discovery Channel series. A natural step in the search for life in other parts of our galaxy is to find planets outside our solar system. Gravitational lensing of light sources cutting across shadow patterns provides a powerful tool.

The method is summarized in Figure 9. Suppose that a star and planet lens a background star moving across the line of sight. The star-planet lens creates a shadow pattern on the light source plane (leftmost panel, Figure 9). Note that during the period of observation, the background star typically travels a short distance compared to the scale of the lens system and so its trajectory can be modeled by a linear path.

As the background star cuts across the shadow pattern (leftmost panel, Figure 9), the background star's total magnification will vary according to the brightness gradation in the shadow pattern. The plot of this magnification as a function of position along the linear path is called a *light curve*.

The crossing of caustics by a lensed light source causes significant jumps in the magnification of the source. In the middle panel of Figure 9, the predicted theoretical light curve shows characteristic spikes as the light source crosses the two fold arcs of the caustic. These significant spikes are due to the planet, though the planet is at least a million times less massive than the star. The light curve would be smooth in the absence of the planet [17]. Remarkably, such spikes in light curves have led to the discovery of several extrasolar planets [28].

The alert reader may be disturbed by the shadow pattern in the left panel of Figure 9 due to a star-planet lens. The caustic appears to have five cusps, contradicting the even-number cusp theorem. The issue is resolved in Figure 10.

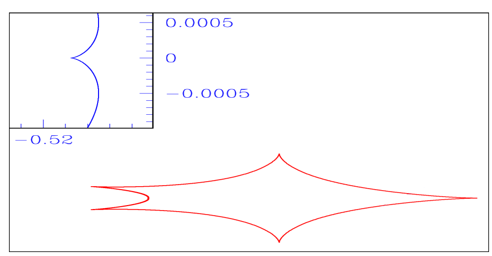

Figure 10. The caustic in the left panel of Figure 9 appears to have five cusps, violating the even-number cusp theorem. There is actually a hidden extra cusp (insert) on the arc joining the two leftmost cusps. Credits: [54, p. 125].

Magnification Relations in Lensing

Qualitative and quantitative results play important roles in gravitational lensing. The former focuses on properties that arise from or are preserved by nonlinear coordinate transformations in the lens and light source planes—e.g., the generic, stable, and topological results of Theorems 1, 4, and 8(4). Quantitative properties are derived from or preserved by linear (ideally, orthogonal) coordinate transformations. Our treatment of magnification relations in this section will be strictly quantitative in the aforementioned sense, while the next section will employ the qualitative local forms of the families of functions in Arnold's ADE-list.

Fold and Cusp Magnification Relations

For a locally stable single plane lensing map, the multiple images of a light source near a fold or cusp caustic have distinct configurations locally. Specifically, if the source is near a fold caustic curve, then the multiple images will include a close doublet of images of opposite parity, which then straddle both sides of a critical curve. The image configuration in Figure 11 shows the close doublet for a light source near a fold arc.

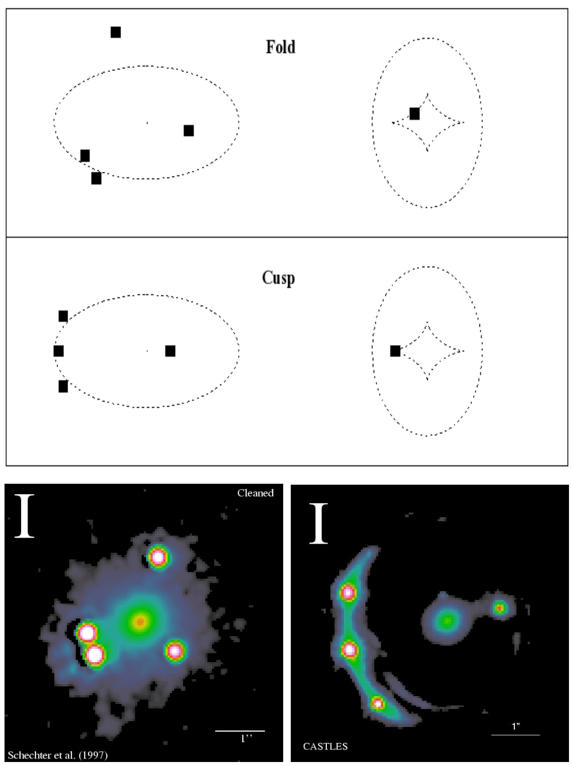

Figure 11. Top two panels: Predicted multiple image configurations (left) when a source is near a fold caustic arc and cusp caustic point (right). The critical curves are on the left, and the caustics are on the right. A close image doublet and triplet occur for the fold and cusp cases, respectively. The lens model used represents an isothermal ellipsoidal galaxy. Bottom panels: Observed four-image configurations (white discs) of the quasar PG1115 with a fold doublet (bottom left panel) and of the quasar RXJ1131 with a cusp triplet. The observations are in the near-infrared I-band. Credits: Keeton et al. [34] (top panels) and the CASTLES [18] (bottom panels).

When a light source is near a cusp caustic point and interior to the arcs abutting the cusp, a close triplet of images occurs locally among the lensed images; see Figure 11. The local doublet and triplet image configurations are also predicted from the local form of a locally stable lensing map about a fold caustic point and cusp caustic point, respectively [54, Sec. 9.1]. Observational confirmation of these predictions is given in the bottom panels of Figure 11.

Blandford and Narayan 1986 [16] showed that the close fold image doublet has a total signed magnification that sums to zero:

$$\mu_1 + \mu_2 = 0, \tag{6}$$

where μ_1 and μ_2 are the signed magnifications of the lensed images in the doublet. Since the lensed images in the doublet have opposite parity, if μ_1 and μ_2 have positive and negative parity, respectively, the fold magnification relation can be written as:

$$\frac{|\mu_1|}{|\mu_2|} = 1.$$

A result similar to (6) holds for the close cusp image triplet (Schneider and Weiss 1992 [68] and Zakharov 1995 [76]):

$$\mu_1 + \mu_2 + \mu_3 = 0,$$

where μ_1, μ_2, and μ_3 are signed magnifications of the lensed images in the triplet. These local magnification relations hold independent of the choice of lens model and have been observed.

Magnification Relations for $D_4^{\pm}$ Caustics

A natural question is whether higher-order caustic magnification relations occur in gravitational lensing. The next theorem due to Aazami and AP 2009 [1] establishes such relations for elliptic and hyperbolic umbilic caustics in lensing.

Let $\boldsymbol{\eta}_{D_4^-}$ and $\boldsymbol{\eta}_{D_4^+}$ denote the respective quantitative local forms of a single-plane lensing map about elliptic umbilic and hyperbolic umbilic caustics; see [66, pp. 200, 201] for their explicit expressions.

Theorem 13. [1] *At any noncaustic point of $\boldsymbol{\eta}_{D_4^{\pm}}$ where a light source has four lensed images (the maximum number), the total signed magnification of the light source satisfies:*

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0.$$

Theorem 13 *cannot* be established using the local qualitative forms for caustics in singularity theory. Those forms arise from local diffeomorphisms that distort the lens and light source planes and can transform a lensing map $\boldsymbol{\eta}$ into a map having nothing to do with gravitational lensing. Instead, the quantitative local forms $\boldsymbol{\eta}_{D_4^-}$ and $\boldsymbol{\eta}_{D_4^+}$ of the lensing map in Theorem 13 arise from linear coordinate transformations that preserve the geometric magnification relations under study. The interested reader may also consult [54, Sec. 9.2] for quantitative local forms of the lensing map about fold and cusp caustics using only orthogonal coordinate transformations.

Lefschetz Theory and Magnification Relations

We outline the idea behind an alternative proof of Theorem 13 given by Werner 2009 [73] using Lefschetz fixed point theory.

Complexify the polynomial map due to an elliptic umbilic or hyperbolic umbilic singularity to obtain holomorphic maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ such that their fixed points are the four real lensed images of a source at a maximal, noncaustic point (noncaustic locations with the maximum number of lensed images). In this domain, it turns out that the fixed point indices become the signed lensed image magnification, $\det[I - D(f)]^{-1} = \mu$, where $D(f)$ is the matrix of first partial derivatives of f in holomorphic coordinates. If f has no fixed points at infinity, then f can be extended to a holomorphic map $F : \mathbb{CP}^2 \to \mathbb{CP}^2$ for which the holomorphic Lefschetz fixed point formula applies, and we recover $F|_{\mathbb{C}^2} = f$ with the usual decomposition $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$. Since the holomorphic Lefschetz number for holomorphic maps on complex projective space is unity, we find:

$$
\begin{aligned}
1 \quad &= \quad L_{\mathrm{hol}}(F) = \sum_{\mathrm{fix}(F)} \frac{1}{\det[I - D(F)]} \\
&= \quad \sum_{\mathrm{fix}(f)} \frac{1}{\det[I - D(f)]} + \sum_{\mathrm{fix}(F|_{\mathbb{CP}^1})} \frac{1}{\det[I - D(F)]} \\
&= \quad \sum_{i=1}^{4} \mu_i + 1,
\end{aligned}
$$

where the second equality in the top line is the holomorphic Lefschetz fixed point formula. The result follows; for details see [73] and the review article [57]. Note that the above argument cannot be applied to all stable caustics since there are caustics with fixed points at infinity (e.g., parabolic umbilics).

Open Problem 7. Generalize the magnification relations to Lorentzian manifolds.

Why Are Magnification Relations Important?

The left panel of Figure 12 shows an observed example of multiple images satisfying the fold magnification-relation theorem (6). However, the right panel of Figure 12 has a close fold image doublet that violates the fold magnification relation. What does the violation of a caustic magnification relation signify physically? In 1998 Mao and Schneider [43] interpreted the violation of the cusp magnification relation as the galaxy lens not being smooth on the scale of the angular separation of the images. In other words, there is *substructure* (not detected by our instruments) on the scale of the image separations that affects the image magnifications, thereby causing the violation.

A rigorous and systematic study of the violation of the fold and cusp magnification relations using data was conducted by Keeton, Gaudi, and AP in 2003 [34] and 2005 [35]. For the data sets used in their studies, they showed that five of the twelve fold doublets and three of the four cusp triplets had to arise from galaxy lenses with substructure.

Today, two candidates have emerged for the substructure in the galaxy lenses producing violations of the caustic magnification relations: *dark matter clumps* (Metcalf and Madau 2001 [44] and Chiba 2002 [23]) and *microlensing* by stars with continuous dark matter and shear from infinity (Schechter and Wambsganss 2002 [65]). Both scenarios employ *stochastic gravitational lensing* because the substructure is assumed to be randomly distributed—for instance, the positions of the dark matter clumps or the stars are random vectors.

The planned deep sky surveys [47] for lensing signatures discussed earlier are expected to find examples of high-order caustic magnification relations as well as their violations, which would lead to evidence of substructure in not only galaxies but also clusters of galaxies.

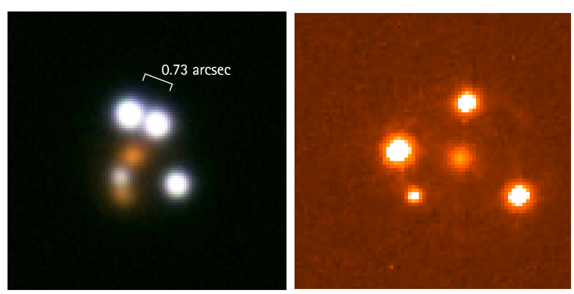

Figure 12. Left panel: Four lensed images (white blobs) of the lensed quasar HE0230-2130. The fold magnification-relation theorem (i.e., $|\mu_1|/|\mu_2| = 1$) is obeyed by the upper close image doublet because the lensed images are a mere 0.73 arcseconds apart with magnification ratio $|\mu_1|/|\mu_2|$ between 0.9841 and 1.0161. The yellow structure is the location of the deflector galaxy. Right panel: Four lensed images (white blobs) of the lensed quasar SDSS0924+0219, where the yellowish central blob identifies the galaxy lens. The leftmost lensed image pair in the right panel is as close as the doublet in the left panel since the right panel pair has an angular separation of 0.74 arcseconds. However, the right panel pair violates the fold magnification-relation theorem since $|\mu_1|/|\mu_2| = 10$! Credits: Burles (left panel) and Schechter (right panel).

The ADE-Magnification Theorem

Aazami and AP 2009 [2] extended their magnification relations in Theorem 13 beyond lensing. They established the result below for the families of functions, not necessarily time-delay families, in Arnold's *ADE* list, which includes caustics with fixed points at infinity.

Let $\mathcal{L}$ be an open subset of $\mathbb{R}^2$ and $S = \mathbb{R}^2$. Consider an n-parameter family F of functions $F_{\mathbf{c},\mathbf{s}} : \mathcal{L} \longrightarrow \mathbb{R}$, where $(\mathbf{c}, \mathbf{s}) \in \mathbb{R}^{n-2} \times S$. For fixed $(\mathbf{c}, \mathbf{s})$, define the *signed magnification* of a critical point $\mathbf{w}_i \in \mathcal{L}$ of $F_{\mathbf{c},\mathbf{s}}$ by:

$$\mathfrak{M}_{\mathbf{c},\mathbf{s}}(\mathbf{w}_i) = \frac{1}{\text{Gauss}\,(\mathbf{w}_i, F_{\mathbf{c},\mathbf{s}}(\mathbf{w}_i))},$$

where the denominator is the Gaussian curvature at $(\mathbf{w}_i, F_{\mathbf{c},\mathbf{s}}(\mathbf{w}_i))$ in the graph of $F_{\mathbf{c},\mathbf{s}}$. Fix $\mathbf{c}_0$ and call $\mathbf{s}' \in S$ a *noncaustic point* if $F_{\mathbf{c}_0,\mathbf{s}'}$ is nondegenerate. Consider the subset $S_0 \subseteq S$ of noncaustic points $\mathbf{s}'$ such that $F_{\mathbf{c}_0,\mathbf{s}'}$ has finitely many critical points, say, $N(\mathbf{s}')$ such points. An element $\mathbf{s}_\times$ of S_0 is called a *maximal, noncaustic point* if $N(\mathbf{s}_\times) = \max_{\mathbf{s}' \in S_0} N(\mathbf{s}')$.

Theorem 14. [2] *Let $F_{\mathbf{c},\mathbf{s}} : \mathcal{L} \longrightarrow \mathbb{R}$ be any n-parameter function in the ADE-family of caustics. Then maximal, noncaustic points exist for each ADE-caustic and for every such point $\mathbf{s}_\times$, the total signed magnification of the critical points $\mathbf{w}_i$ of $F_{\mathbf{c},\mathbf{s}_\times}$ satisfies:*

$$\sum_{i=1}^{N(\mathbf{s}_\times)} \mathfrak{M}_{\mathbf{c},\mathbf{s}_\times}(\mathbf{w}_i) = 0,$$

where $N(\mathbf{s}_\times) = n+1$ with $n \geq 1$ for A_{n+1} caustics, $N(\mathbf{s}_\times) = n+1$ with $n \geq 3$ for D_{n+1} caustics, $N(\mathbf{s}_\times) = 6, 7, 8$ for E_6, E_7, E_8 caustics, respectively.

Observe that since magnification is a reciprocal of Gaussian curvature, the magnification relations in Theorem 14 are geometric invariants. Also, note that the theorem singles out the highest-order caustics of each n-parameter family.

The proof of Theorem 14 in [2] was algebraic, employing the Euler trace formula [11]. Recently, Aazami, AP, and Rabin 2010 [3] gave an alternative proof using a geometric approach with residues on compact orbifolds. Note that similar to the Lefschetz fixed point approach, direct application of the Euler-Jacobi theorem does not directly yield all the ADE-magnification relations due to nonzero "common roots" at infinity. We end the article with a sketch of the proof in [3].

Orbifolds and ADE-Magnification Relations

We begin by reviewing the standard residue approach to magnification relations of Dalal and Rabin 2001 [24], which was generalized by Aazami, AP, and Rabin 2010 [3] to orbifolds. For brevity, the proof in [3] will be illustrated using only one singularity from the *ADE*-caustic family. We select the parabolic umbilic caustic D_5 since it induces a "lens equation with a root at infinity". To make these ideas precise, we use the fact that the parabolic umbilic arises from the four-parameter family

$$F_{\mathbf{c},\mathbf{s}}(\mathrm{u},\mathrm{v}) = \mathrm{u}^2\mathrm{v} \pm \mathrm{v}^4 + c_3\mathrm{v}^3 + c_2\mathrm{v}^2 - s_2\mathrm{v} - s_2\mathrm{u}.$$

This singularity has a maximal, noncaustic point, which we take to be $\mathbf{s}$, and the maximum number of preimages is five [3]. The parabolic umbilic family F induces a map $\mathbf{f}_\mathbf{c}$ between planes via

$$\mathbf{f}_\mathbf{c}(\mathbf{w}) = \nabla F_{\mathbf{c},\mathbf{s}}(\mathbf{w}) + \mathbf{s},$$

where $\mathbf{w} = (\mathrm{u}, \mathrm{v})$. Explicitly: $\nabla F_{\mathbf{c},\mathbf{s}}(\mathbf{w}) = \mathbf{0}$:

$$\begin{aligned} \mathbf{f}_\mathbf{c}(\mathrm{u},\mathrm{v}) &= (f_\mathbf{c}^{(1)}(\mathrm{u},\mathrm{v})\ ,\ f_\mathbf{c}^{(2)}(\mathrm{u},\mathrm{v})) \\ &= (2\mathrm{uv}\ ,\ \mathrm{u}^2 \pm 4\mathrm{v}^3 + 3c_3\mathrm{v}^2 + 2c_2\mathrm{v})\,. \end{aligned}$$

Although the $\mathbf{f}_\mathbf{c}$-preimages $\mathbf{w} = (\mathrm{u}, \mathrm{v})$ have real coordinates, we shall drop that restriction, allowing for $\mathbf{w} \in \mathbb{C}^2$. Call the critical values of $\mathbf{f}_c$ the *caustic points* of $\mathbf{f}_\mathbf{c}$ since the locus of critical values of $\mathbf{f}_\mathbf{c}$ coincides with the set of $\mathbf{s} \in S$ such that $F_{\mathbf{c},\mathbf{s}}$ has at least one degenerate critical point.

Now, for any noncaustic point $\mathbf{s} = (s_1, s_2)$ of $\mathbf{f}_\mathbf{c}$, consider the following meromorphic two-form on $\mathbb{C}^2$:

$$\omega = \frac{d\mathrm{u}\, d\mathrm{v}}{P_1(\mathrm{u},\mathrm{v})P_2(\mathrm{u},\mathrm{v})},$$

where $P_1(\mathrm{u},\mathrm{v}) = f_\mathbf{c}^{(1)}(\mathrm{u},\mathrm{v}) - s_1$ and $P_2(\mathrm{u},\mathrm{v}) = f_\mathbf{c}^{(2)}(\mathrm{u},\mathrm{v}) - s_2$. We are interested only in the poles of ω that are the common roots of P_1 and P_2 as they are $\mathbf{f}_\mathbf{c}$-preimages of $\mathbf{s}$. Call such poles the $\mathbf{f}_\mathbf{c}^{-1}(\mathbf{s})$*-poles*. It can be shown that the residue of ω at $\mathbf{w} \in \mathbf{f}^{-1}(\mathbf{s})$ is precisely the signed magnification $\mathfrak{M}_{\mathbf{c},\mathbf{s}}(\mathbf{w})$, provided $\mathbf{s}$ is not a critical value of $\mathbf{f}_\mathbf{c}$.

Using homogeneous coordinates $[\mathrm{U} : \mathrm{V} : \mathrm{W}]$, where $\mathrm{u} = \mathrm{U}/\mathrm{W}$, $\mathrm{v} = \mathrm{V}/\mathrm{W}$, and $(\mathrm{U}, \mathrm{V}, \mathrm{W})$ is nonzero, extend P_1 and P_2, and hence $\mathbf{f}_\mathbf{c}$, to complex projective space $\mathbb{CP}^2$:

(7)

$$\begin{aligned} P_1(\mathrm{U},\mathrm{V},\mathrm{W})_{\text{hom}} &= 2\mathrm{UV} - s_1\mathrm{W}^2 \\ P_2(\mathrm{U},\mathrm{V},\mathrm{W})_{\text{hom}} &= \mathrm{U}^2\mathrm{W} \pm 4\mathrm{V}^3 + 3c_3\mathrm{V}^2\mathrm{W} \\ &\qquad +2c_2\mathrm{VW}2 - s_2\mathrm{W}^3. \end{aligned}$$

Likewise, extend ω to a 2-form on $\mathbb{CP}^2$ that is homogeneous of degree zero. Its $\mathbf{f}_\mathbf{c}^{-1}(\mathbf{s})$-poles are now the common roots of P_1 and P_2 in $\mathbb{CP}^2$. Note that $\mathbb{C}^2$ corresponds to $\mathrm{W} = 1$ and infinity to $\mathrm{W} = 0$.

The global residue theorem [31] states that the sum of the residues of the $\mathbf{f}_\mathbf{c}^{-1}(\mathbf{s})$-poles of ω on $\mathbb{CP}^2$, which consists of those in $\mathbb{C}^2$ and at infinity, is identically zero. Since the set of poles of ω in $\mathbb{C}^2$ equals $\mathbf{f}_\mathbf{c}^{-1}(\mathbf{s})$, the sum of their residues is the total signed magnification, $\mathfrak{M}_{\text{tot},\mathbf{c}}(\mathbf{s}) = \sum_{\mathbf{w} \in \mathbf{f}_\mathbf{c}^{-1}(\mathbf{s})} \mathfrak{M}_{\mathbf{c},\mathbf{s}}(\mathbf{w})$. Consequently, the total signed magnification $\mathfrak{M}_{\text{tot},\mathbf{c}}(\mathbf{s})$ is equal to minus the sum of the residues of ω at infinity.

Recalling that $\mathbf{s}$ is a maximal, noncaustic point of $\mathbf{f_c}$, let us now examine the behavior of the extended parabolic umbilic map $\mathbf{f_c}$ at infinity in $\mathbb{CP}^2$. Setting $\mathrm{W} = 0$ in equation (7) yields

$$P_1(\mathrm{U},\mathrm{V},0)_{\text{hom}} = 2\mathrm{UV}, \qquad P_2(\mathrm{U},\mathrm{V},0)_{\text{hom}} = \pm 4\mathrm{V}^3.$$

These equations have one nonzero common root, which is the $\mathbf{f_c}^{-1}(\mathbf{s})$-pole of ω at infinity or the $\mathbf{f_c}$-preimage of $\mathbf{s}$ at infinity, namely, the point $[1:0:0]$ in $\mathbb{CP}^2$. As shown in [3], the way around this pole at infinity is to consider an extension to a space other than $\mathbb{CP}^2$, namely, *weighted projective space* $\mathbb{WP}(a_0, a_1, a_2)$, where a_0, a_1, a_2 are positive integers denoting particular "weights" of the space. These are examples of compact *orbifolds*.

Orbifolds generalize manifolds. Whereas a manifold locally looks like an open subset of $\mathbb{R}^n$, an orbifold X locally looks like the quotient of an open subset of $\mathbb{R}^n$ by the action of a finite group G. The analogues of coordinate charts are known as orbifold charts. Like coordinate charts, two overlapping orbifold charts are required to satisfy a compatibility condition. For our purpose, it is best to distinguish an orbifold X by its *singular points*, which are points $p \in X$ whose stabilizer group $G_p \subset G$ is nontrivial. If an orbifold has no singular points, then it is a smooth manifold. Consult [63, 45, 4] for more on orbifolds.

The most common examples of orbifolds are those which arise as quotients of manifolds by compact Lie groups, and weighted projective space is no exception. For example, the orbifold $\mathbb{WP}(3,2,1)$ is defined by the Lie group action $\mathbb{S}^1 \times \mathbb{S}^5 \longrightarrow \mathbb{S}^5 : \big(z, (\mathrm{U},\mathrm{V},\mathrm{W})\big) \longmapsto (z^3\mathrm{U}, z^2\mathrm{V}, z\mathrm{W})$. So $\mathbb{WP}(3,2,1) = \mathbb{S}^5/\mathbb{S}^1$ under this action, where $\mathbb{S}^1 \subset \mathbb{C}$ and $\mathbb{S}^5 \subset \mathbb{C}^3$. Notice that if the action were $\big(z, (\mathrm{U},\mathrm{V},\mathrm{W})\big) \longmapsto (z\mathrm{U}, z\mathrm{V}, z\mathrm{W})$, then the resulting quotient space would be ordinary complex projective space $\mathbb{CP}^2$. In other words, $\mathbb{WP}(1,1,1) = \mathbb{CP}^2$. Similar to $\mathbb{CP}^2$, the space $\mathbb{WP}(3,2,1)$ has $\mathbb{C}^2$ corresponding to $\mathrm{W} = 1$ and infinity to $\mathrm{W} = 0$, and $(\mathrm{U},\mathrm{V},\mathrm{W}) \neq (0,0,0)$. There are no singular points in the $\mathbb{C}^2$ part of $\mathbb{WP}(3,2,1)$ since the stabilizer condition for such points implies $z\mathrm{W} = \mathrm{W}$, which forces $z = 1$ and, hence, the stabilizer group to be trivial.

Covering $\mathbb{WP}(3,2,1)$ with homogeneous coordinates $[\mathrm{U}:\mathrm{V}:\mathrm{W}]$, we see that U and V now have weights 3 and 2, respectively, and relate to the coordinates $(\mathrm{u},\mathrm{v}) \in \mathbb{C}^2$ in a new way:

$$\mathrm{u} = \frac{\mathrm{U}}{\mathrm{W}^3}, \qquad \mathrm{v} = \frac{\mathrm{V}}{\mathrm{W}^2}.$$

Extending the parabolic umbilic map $\mathbf{f_c}$ to $\mathbb{WP}(3,2,1)$ then yields extensions of P_1 and P_2 to the following new homogeneous polynomials, respectively:

$$\begin{cases} 2\mathrm{UV} - s_1\mathrm{W}^5 \\ \mathrm{U}^2 \pm 4\mathrm{V}^3 + 3c_3\mathrm{V}^2\mathrm{W}^2 + 2c_2\mathrm{V}\mathrm{W}^4 - s_2\mathrm{W}^6. \end{cases}$$

The two-form ω then extends to $\mathbb{WP}(3,2,1)$. In $\mathbb{C}^2$, which corresponds to $\mathrm{W} = 1$, we recover the same $\mathbf{f_c}$ as in the $\mathbb{CP}^2$ discussion. At infinity or $\mathrm{W} = 0$, however, the polynomials become:

$$\begin{cases} 2\mathrm{UV} \\ \mathrm{U}^2 \pm 4\mathrm{V}^3. \end{cases}$$

The only common root is $[0:0:0]$, which is not a point in $\mathbb{WP}(3,2,1)$. In other words, there are no preimages at infinity, hence no $\mathbf{f_c}^{-1}(\mathbf{s})$-poles of ω at infinity. All the $\mathbf{f_c}^{-1}(\mathbf{s})$-poles then lie in the $\mathbb{C}^2$ part of $\mathbb{WP}(3,2,1)$, where there are no singular points. By the global residue theorem for compact orbifolds, the total signed magnification of the $\mathbf{f_c}$-preimages of the maximal, noncaustic point $\mathbf{s}$ for the parabolic umbilic then satisfies:

$$\mathfrak{M}_{\text{tot},\mathbf{c}}(\mathbf{s}) = \sum_{i=1}^{5} \mathfrak{M}_i = 0.$$

Employing the above procedure with appropriate choices of weighted projective spaces yields magnification relations for *all* the A, D, E singularities; see [3] for details.

Further Reading

Reference [66] treats the astrophysical aspects of lensing, [54] develops a mathematical theory of lensing for the single and multiplane cases, [57] reviews some mathematical lensing results not covered in this article, and [48] carries out generalizations of lensing to Lorentzian manifolds. The forthcoming book [58] will focus on strong-deflection lensing by black holes and include lensing in Kerr and Fermat geometries.

Figure Credits

Figure 1, top row: Jerry Schoendorf, MAMS; bottom row, CfA-Arizona Space Telescope LEns Survey (CASTLES) website. Figures 2, 7, 9, 11 courtesy of the author. Figures 3, 4, 6, 10 courtesy of the author and with kind permission of Springer Science and Business Media [see 54]. Figure 5, from [74], H. J. Witt and A. O. Petters, authors. Figure 8, Joachim Wambsganss. Figure 12, left panel: Scott Burles; right panel, Paul Schecter.

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The Last Words of a Genius

Ken Ono

This story begins with a cryptic letter written by a dying genius, the clues of which inspired scores of mathematicians to embark on an adventure which resembles an Indiana Jones movie. It is reminiscent of the quest for the Holy Grail, in which skillful knights confront great obstacles. But these knights are mathematicians, and the Grail is replaced by a mathematical "Rosetta Stone" that promises to reveal hidden truths in new worlds.

The Saga

Our drama begins on March 27, 1919, the date of Srinivasa Ramanujan's triumphant, but bittersweet, Indian homecoming. Five years earlier, accepting an invitation from the eminent British mathematician G. H. Hardy, the amateur Ramanujan had left for Cambridge University with the dream of making a name for himself in the world of mathematics. Now, stepping off the ship *Nagoya* in Bombay (now Mumbai), the two-time college dropout, who had intuited unimaginable formulas, returned as a world-renowned number theorist. He had achieved his goal. At the young age of thirty-one, Ramanujan had already made important contributions to a mindboggling array of subjects:[1] the distribution of prime numbers, hypergeometric series, elliptic functions, modular forms, probabilistic number theory, the theory of partitions, and q-series, among others. He had published over thirty papers, including seven with Hardy. In recognition of these accomplishments, Ramanujan was named a Fellow of Trinity College, and he was elected a Fellow of the Royal Society (F.R.S.), an honor shared by Sir Isaac Newton.

Sadly, the occasion of Ramanujan's homecoming was not one of celebration. He was a very sick man; he was much thinner than the rotund Ramanujan his Indian friends remembered. One of the main reasons for his declining health was malnutrition. He had been adhering to a strict vegetarian diet in a time and place with no adequate resources to support it. He also struggled with the severe change in climate. Accustomed to the temperate weather of south India, he did not have or did not wear appropriate clothing to protect him from the cool and damp Cambridge weather. These conditions took their toll, and he became gravely ill. He was diagnosed[2] with tuberculosis, and he returned to India seeking familiar surroundings, a forgiving climate, and a return to good health. Tragically, Ramanujan's health declined over the course of the following year, and he passed away on April 26, 1920, in Madras (now Chennai), with his wife Janaki by his side.

Ramanujan's Last Letter

Amazingly, in spite of his condition, Ramanujan spent his last year working on mathematics in isolation.

> *…through all the pain and fever,…Ramanujan, lying in bed, his head propped up on pillows, kept working. When he required it, Janaki would give him his slate; later she'd gather up the accumulated sheets of mathematics-covered paper …and place them in the big leather box which he brought from England* (see p. 329 of [29]).

Janaki would later remember these last days (see p. 91 of [38]):

> *He was only skin and bones. He often complained of severe pain. In spite of it he was always busy*

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[1] *See* [4, 6, 7, 8, 25, 29, 35] *for more on Ramanujan and his achievements.*

[2] *The diagnosis of tuberculosis is now believed to be incorrect. D. A. B. Young examined the evidence pertaining to Ramanujan's illness, and he concluded that Ramanujan died of hepatic amoebiasis* [42].

doing his mathematics. …Four days before he died he was scribbling.

In a fateful last letter to Hardy, dated January 12, 1920, Ramanujan shared hints (see p. 220 of [7]) of his last theory.

I am extremely sorry for not writing you…I discovered very interesting functions recently which I call "Mock" theta functions.…they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

This mysterious letter set off a great adventure: the quest to realize the meaning behind these last words and to then unearth the implications of this understanding. These words exposed, in an unexplored territory of the world of mathematics, a padlocked wooden gate, beyond which was the promise of unknown mathematical treasures.

The Early Years

The letter, roughly four typewritten pages, consists of formulas for seventeen strange power series and a discussion of their asymptotics and behavior near the boundary of the unit disk. There are no proofs of any kind. Ramanujan also grouped these series based on their "order", a term he did not define. As a typical example, he offered

$$
\begin{aligned}
(1.1)\quad f(q) &= \sum_{n=0}^{\infty} a_f(n)q^n \\
&:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\
&= 1+q-\cdots+17503q^{99}+\ldots,
\end{aligned}
$$

which he called a *third-order mock theta function.* He then miraculously claimed that

$$
(1.2)\qquad a_f(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n-\frac{1}{24}}}\cdot e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}.
$$

Obviously Ramanujan knew much more than he revealed.

G. N. Watson was the first mathematician to take on the challenge. He worked for years, and on November 14, 1935, at a meeting of the London Mathematical Society, he celebrated his retirement as president of the Society with his now famous address [40]:

It is not unnatural that [one's] mode of approach to the preparation of his valedictory address should have taken the form of an investigation into the procedure of his similarly situated predecessors.…I was, however, deterred from this course…[Ramanujan's last] letter is the subject which I have chosen…; I doubt whether a more suitable title could be found for it than the title used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes.

Watson chose the title *The Final Problem: An Account of the Mock Theta Functions.*

He proceeded to describe his findings, a medley of identities and formulas. Using "q-hypergeometric series", he reformulated Ramanujan's examples. For $f(q)$ he proved:

$$
(1.3)\quad f(q) = \frac{2}{\prod_{n=1}^{\infty}(1-q^n)}\cdot\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{(3n^2+n)/2}}{1+q^n}.
$$

He also proved identities relating the mock theta functions to *Mordell integrals*, such as

$$
\begin{aligned}
(1.4)\quad &\int_0^{\infty} e^{-3\pi x^2}\cdot\frac{\sinh \pi x}{\sinh 3\pi x}\,dx = \frac{1}{e^{2\pi/3}\sqrt{3}} \\
&\times\sum_{n=0}^{\infty}\frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2\cdots(1+e^{-(2n+1)\pi})^2}.
\end{aligned}
$$

He concluded by entrusting the quest to the next generation of mathematicians.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. …To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine [Persephone] …

Mathematicians continued the pursuit. In the late 1930s A. Selberg, as a high school student, published his first two mathematical papers on the subject. In the 1950s and 1960s, G. E. Andrews and L. Dragonette, employing Watson's results, finally confirmed Ramanujan's claimed asymptotic (1.2).[3] Many mathematicians, among them B. C. Berndt, Y.-S. Choi, B. Gordon, and R. McIntosh, progressed further along the lines set by Watson. After many technical calculations that run on for pages, these mathematicians mastered the asymptotics of Ramanujan's examples, amassed identities such as (1.3), and obtained analytic transformations relating these examples to integrals such as (1.4).

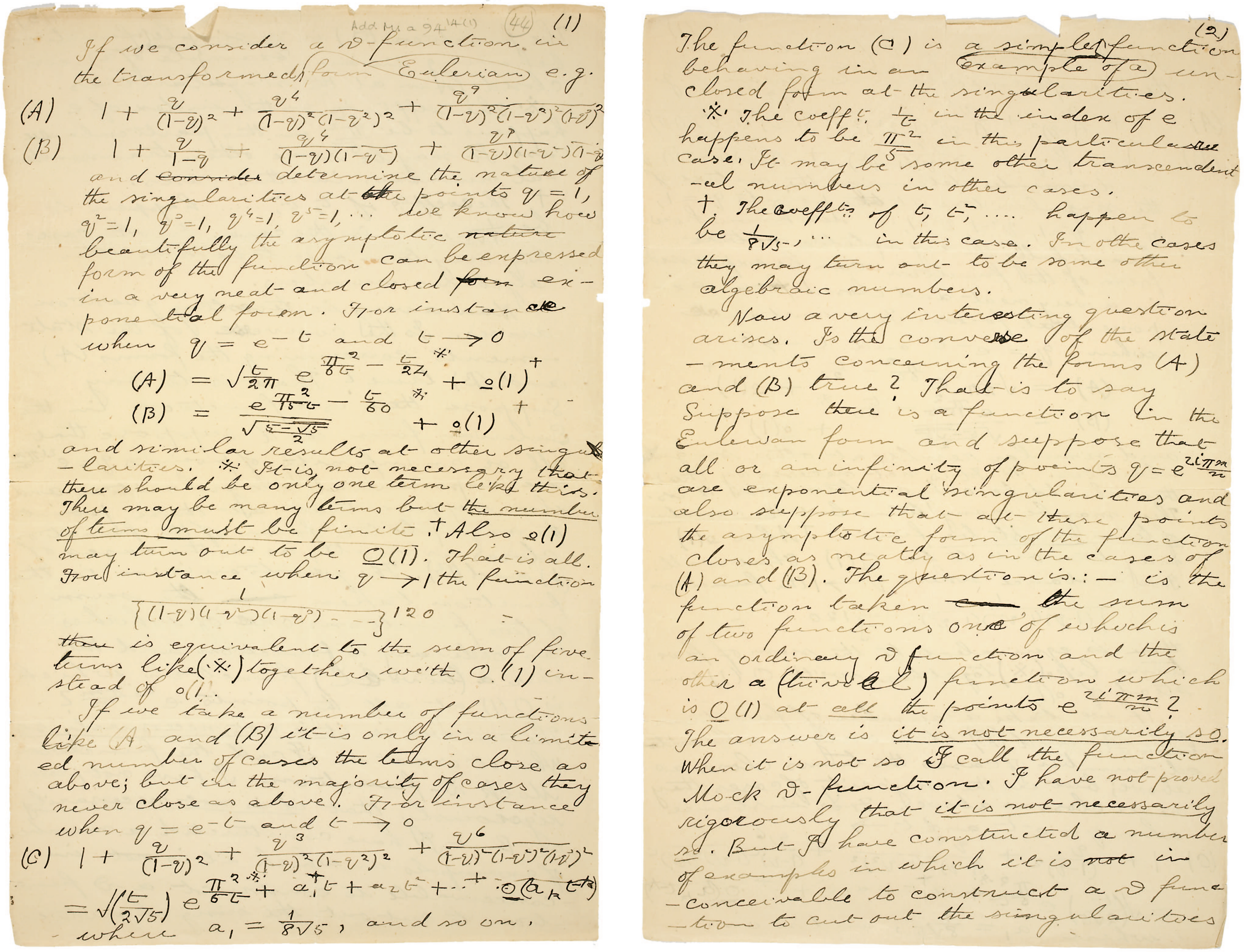
Mathematicians now had a grasp of the padlock which secured the wooden gate. But they still did not know the meaning behind Ramanujan's last words. Mathematicians had gathered a box full of formulas, but there would be little progress for the next ten years.

The "Lost Notebook"

In the spring of 1976, while searching through archived papers from Watson's estate in the Trinity College Library at Cambridge University, Andrews discovered the "Lost Notebook" [2]. The notebook, consisting of over 100 pages of Ramanujan's last works, was archived in a box among assorted papers collected from Watson's estate.

…the notebook and other material was discovered among Watson's papers by Dr. J. M. Whittaker,

[3] *K. Bringmann and the author have obtained an exact formula for $a_f(n)$ [12].*



Add. Ms a 94[14(1)] (44) (1)

If we consider a ϑ-function in the transformed Eulerian form e.g.

(A) $1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

(B) $1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)}$

and ~~consider~~ determine the nature of the singularities at ~~the~~ the points $q=1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1$, ... we know how beautifully the asymptotic ~~nature~~ form of the function can be expressed in a very neat and closed ~~form~~ exponential form. For instance when $q = e^{-t}$ and $t \to 0$

$$(A) = \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}} {}^{*} + o(1)^{\dagger}$$

$$(B) = \frac{e^{\frac{\pi^2}{15t} - \frac{t}{60}}}{\sqrt{\frac{5-\sqrt{5}}{2}}} {}^{*} + o(1)^{\dagger}$$

and similar results at other singularities. * It is not necessary that there should be only one term like this. There may be many terms but the number of terms must be finite. † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q \to 1$ the function

$$\frac{1}{\{(1-q)(1-q^2)(1-q^3)\cdots\}^{120}}$$

~~then~~ is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance when $q = e^{-t}$ and $t \to 0$

(C) $1 + \frac{q}{(1-q)^2} + \frac{q^3}{(1-q)^2(1-q^2)^2} + \frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

$$= \sqrt{\left(\frac{t}{2\sqrt{5}}\right)}\, e^{\frac{\pi^2}{5t}} {}^{*} + a_1 t + a_2 t^2 + \cdots {}^{\dagger} + O(a_k t^k)$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on.

(2)

The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

* The coeff[t] $\frac{1}{t}$ in the index of e happens to be $\frac{\pi^2}{5}$ in this particular case. It may be some other transcendental numbers in other cases.

† The coeff[ts] of $t, t^2, \ldots$ happen to be $\frac{1}{8\sqrt{5}}, \ldots$ in this case. In other cases they may turn out to be some other algebraic numbers.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is:— is the function taken ~~——~~ the sum of two functions one of which is an ordinary ϑ function and the other a (trivial) function which is $O(1)$ at all the points $e^{2i\pi m/n}$? The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is ~~not~~ inconceivable to construct a ϑ function to cut out the singularities

The five pages of Ramanujan's last letter to Hardy... (above and following pages)

who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R. A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts…[2].

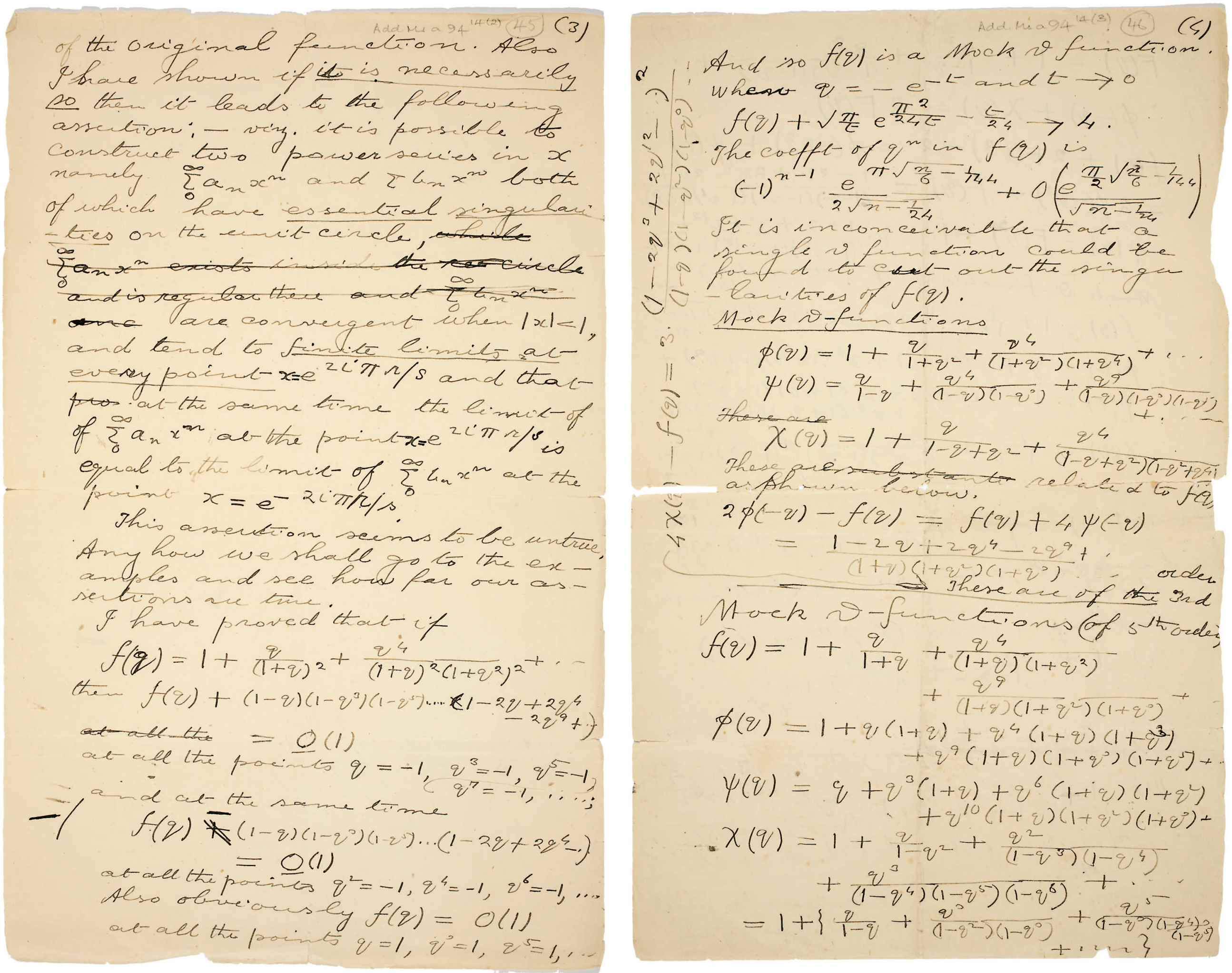
Janaki presumably sent Hardy the large leather box, the one filled with Ramanujan's last papers. Hardy passed it on to Watson in turn.

Although never truly lost, the sheaf of papers had survived the long journey from India only to then lie forgotten in the Trinity College Library. The journey was indeed extraordinary, for the manuscript almost met a catastrophic end. Whittaker, the son of Watson's famous coauthor E. T. Whittaker, recalled, in a letter to G. E. Andrews dated August 15, 1979, the scene of Watson's study at the time of his death in 1965 (see p. 304 of [7]):

…papers covered the floor of a fair sized room to the depth of about a foot, all jumbled together, and were to be incinerated in a few days. One could only make lucky dips [into the rubble] and, as Watson never threw away anything, the result might be a sheet of mathematics, but more probably a receipted bill or a draft of his income tax return for 1923. By an extraordinary stroke of luck one of my dips brought up the Ramanujan material.

The Lost Notebook allowed mathematicians to escape the seemingly eternal morass. In addition to listing some new mock theta functions, the scrawl contained many valuable clues: striking identities and relations, recorded without proofs of any kind. Thanks to these clues, mathematicians found many applications for the mock theta functions: *L*-functions in number theory, hypergeometric functions, partitions, Lie theory, modular forms, physics, and polymer chemistry, to name a few.

The Lost Notebook notably surrendered new sorts of identities that, as we shall see, go on to play a crucial role in the quest. Andrews proved identities [3] relating mock theta functions to



Add. Ms a 94 (45) (3)

of the original function. Also I have shown if it is necessarily so then it leads to the following assertion:— viz. it is possible to construct two power series in x namely $\sum a_n x^n$ and $\sum b_n x^n$ both of which have essential singularities on the unit circle, are convergent when $|x|<1$, and tend to finite limits at every point $x=e^{2i\pi r/s}$ and that at the same time the limit of $\sum_0^\infty a_n x^n$ at the point $x=e^{2i\pi r/s}$ is equal to the limit of $\sum_0^\infty b_n x^n$ at the point $x=e^{-2i\pi r/s}$.

This assertion seems to be untrue. Anyhow we shall go to the examples and see how far our assertions are true.

I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots) = O(1)$ at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, …; and at the same time $f(q) - (1-q)(1-q^2)(1-q^3)\cdots(1-2q+2q^4-\cdots) = O(1)$ at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, … Also obviously $f(q)=O(1)$ at all the points $q=1$, $q^3=1$, $q^5=1$, …

Add. Ms a 94 (46) (4)

And so $f(q)$ is a Mock ϑ function. When $q=-e^{-t}$ and $t\to 0$

$$f(q) + \sqrt{\tfrac{\pi}{t}}\, e^{\frac{\pi^2}{24t}-\frac{t}{24}} \to 4.$$

The coeff of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{e^{\frac{\pi}{2}\sqrt{\frac{n}{6}-\frac{1}{144}}}}{\sqrt{n-\frac{1}{24}}}\right)$$

It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots$$

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \cdots$$

These are related to $f(q)$ as shown below.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \frac{1-2q+2q^4-2q^9+\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$

$$4\chi(q^3) - f(q) = 3\,\frac{(1-2q^3+2q^{12}-\cdots)}{(1-q)(1-q^2)(1-q^3)\cdots}$$

These are of the 3rd order

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \cdots$$

$$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \cdots$$

$$\chi(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$$

$$= 1 + \left\{\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots\right\}$$

indefinite binary quadratic forms. For example, he proved:

(1.5)

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\cdots(1+q^n)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \cdot \left(\sum_{\substack{n+j\ge 0\\ n-j\ge 0}} - \sum_{\substack{n+j<0\\ n-j<0}} \right) (-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}.$$

D. Hickerson confirmed [26] identities in which sums of mock theta functions are infinite products. For example, for $f_0(q)$ and the mock theta function $\Phi(q)$, he showed that

$$f_0(q) + 2\Phi(q^2) = \prod_{n=1}^{\infty} \frac{(1-q^{5n})(1-q^{10n-5})}{(1-q^{5n-4})(1-q^{5n-1})}.$$

As indefinite binary quadratic forms and infinite products appear in modular form theory, these identities finally provided evidence linking mock theta functions to modular forms, the "ordinary theta functions" of Ramanujan's last letter.

A *modular form* is a holomorphic function on the upper half complex plane $\mathbb{H}$ which is tamed by Möbius transformations $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Loosely speaking,[4] a *weight k modular form on a subgroup* $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies

$$f(\gamma\tau) = (c\tau+d)^k f(\tau) \tag{1.6}$$

for all $\gamma \in \Gamma$ and is meromorphic "at the cusps".

Despite this evidence, the essence of Ramanujan's theory continued to elude mathematicians. The problem was that the Lost Notebook is merely a bundle of pages that "…contains over six hundred mathematical formulae listed one after the other without proof[5]…there are only a few words scattered here and there…" (p. 89 and p. 96 of [2]). Instead of furnishing the missing key, the Notebook

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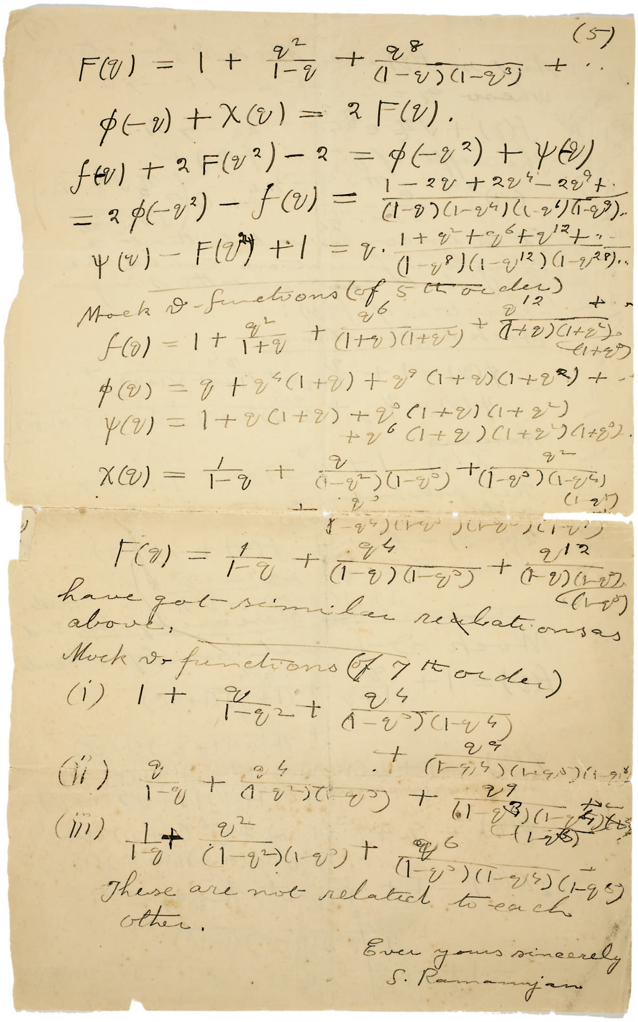
[4] *If k is not an integer, then (1.6) must be suitably modified.*

[5] *Almost all of the results on q-series in the Lost Notebook have now been proved [4].*


(5)

$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots$

$\phi(-q) + \chi(q) = 2F(q)$.

$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(q)$

$= 2\phi(-q^2) - f(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$

$\psi(q) - F(q^4) + 1 = q \cdot \frac{1 + q^2 + q^6 + q^{12} + \cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}$

Mock ϑ-functions (of 5th order)

$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \cdots$

$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^3) + \cdots$

$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3) \cdots$

$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$

$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}$

have got similar relations as above.

Mock ϑ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}$

These are not related to each other.

Even yours sincerely
S. Ramanujan

provided a hammer in the form of countless identities. So armed, mathematicians burst through the wooden gate, only to find a long dusty hallway lined with locked iron doors. When the dust settled, they could read the signs on the doors, and with this knowledge they finally understood the widespread scope of the mock theta functions. At the Ramanujan Centenary Conference in 1987, F. Dyson eloquently summed up the dilemma [21]:

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions...

Zwegers's Thesis

By the late 1990s little progress had been made. Then in 2002, in a brilliant Ph.D. thesis [45] written under D. Zagier, S. Zwegers made sense of the mock theta functions. By understanding the meaning behind identities such as (1.3–1.5) and by notably making use of earlier work of Lerch and Mordell, he found the answer: *real analytic modular forms*. In the solution, one must first "complete" the mock theta functions by adding a nonholomorphic function, a so-called "period integral".

For the mock theta functions $f(q)$ (see (1.1)) and (1.7)

$$\omega(q) = \sum_{n=0}^{\infty} a_{\omega}(n)q^n := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$

Zwegers [44] defined the *vector-valued mock theta function* (here $q := e^{2\pi i\tau}$)

$$F(\tau) = (F_0(\tau),\ F_1(\tau),\ F_2(\tau))^T := (q^{-\frac{1}{24}}f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T.$$

Then using theta functions $g_0(z), g_1(z)$, and $g_2(z)$, where

$$g_0(z) := \sum_{n=-\infty}^{\infty} (-1)^n \left(n + \frac{1}{3}\right) e^{3\pi i\left(n+\frac{1}{3}\right)^2 z}$$

($g_1(z)$ and $g_2(z)$ are similar), he defined the vector-valued nonholomorphic function

(1.8)

$$G(\tau) = (G_0(\tau),\ G_1(\tau),\ G_2(\tau))^T := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z),\ g_0(z),\ -g_2(z))^T}{\sqrt{-i(\tau+z)}}\,dz.$$

He completed $F(\tau)$ to obtain $H(\tau) := F(\tau) - G(\tau)$, and he proved that [44]

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau)$$

and

$$H(-1/\tau) = \sqrt{-i\tau}\cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau),$$

where $\zeta_n := e^{2\pi i/n}$. As $\mathrm{SL}_2(\mathbb{Z}) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\rangle$, this gives a vector version of (1.6), and so the *vector-valued mock theta function* $F(\tau)$ is the *holomorphic part* of the vector-valued real analytic modular form $H(\tau)$.

Generalizing identities such as (1.3), in which mock theta functions are related to "Appell-Lerch" series, Zwegers also produced infinite families of mock theta functions that eclipse Ramanujan's list. For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, he defined the function

$$\mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v;\tau)} \cdot \sum_{n\in\mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n}, \tag{1.9}$$

where $z := e^{2\pi i u}$, $w := e^{2\pi i v}$, $q := e^{2\pi i \tau}$ and $\vartheta(v;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{\pi i\nu} w^{\nu} q^{\nu^2/2}$. Using a function $R(u-v;\tau)$ which resembles the components of (1.8), he then defined

$$\hat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau). \tag{1.10}$$

He proved that $\hat{\mu}(u,v;\tau)$ is a nonholomorphic *Jacobi form*, a function whose specializations at "torsion points" give weight 1/2 real analytic modular forms. This function satisfies transformations that imply (1.6) for these specializations; for example, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then there is an explicit root of unity $\chi(\gamma)$ for which

$$\hat{\mu}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)^{-3}(c\tau+d)^{\frac{1}{2}} e^{-\pi i c(u-v)^2/(c\tau+d)} \cdot \hat{\mu}(u,v;\tau).$$

Exploring New Worlds

Armed with Zwegers's landmark thesis, mathematicians have begun to explore [36, 43] the worlds behind the iron doors. Here we sample some of the discoveries that this author has obtained with his collaborators.

Harmonic Maass Forms

The mock theta functions turn out to be holomorphic parts of distinguished real analytic modular forms, the *harmonic Maass forms*, which were recently introduced by J. H. Bruinier and J. Funke [16].

Loosely speaking, a *weight k harmonic Maass form* is a smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying (1.6) and $\Delta_k(M) = 0$, which also has (at most)[6] linear exponential growth at cusps. Here the *hyperbolic Laplacian* Δ_k, where $\tau = x+iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The Fourier expansions of these forms have been the object of our explorations. In terms of the incomplete gamma function $\Gamma(\alpha;x) := \int_x^\infty e^{-t}t^{\alpha-1}\,dt$, every weight $2-k$ harmonic Maass form $M(\tau)$, where $k > 1$, has an expansion of the form

$$M(\tau) = \sum_{n\gg-\infty} c_M^+(n)q^n + \sum_{n<0} c_M^-(n)\Gamma(k-1, 4\pi|n|y)q^n. \tag{2.11}$$

Obviously, $M(\tau)$ naturally decomposes into two pieces, a *holomorphic part*

$$M^+(\tau) := \sum_{n\gg-\infty} c_M^+(n)q^n$$

and its complement $M^-(\tau)$, the *nonholomorphic part*. The mock theta functions and Zwegers's μ-function give holomorphic parts of weight 1/2 harmonic Maass forms.

Harmonic Maass forms are generalizations of modular forms; a modular form is a harmonic Maass form $M(\tau)$ where $M^-(\tau) = 0$. Because modular forms appear prominently in mathematics, one then expects the mock theta functions and harmonic Maass forms to have far-reaching implications. Our first forays in the long dusty hallway have been profitable, and we have obtained results [12, 13, 14, 15, 17, 32, 36, 37] on a wide array of subjects: partitions and q-series, Moonshine, Donaldson invariants, Borcherds products, and elliptic curves, among others. We now describe some of these results.

Partitions

A *partition* of a nonnegative integer n is any nonincreasing sequence of positive integers that sum to n. If $p(n)$ denotes the number of partitions of n, then Ramanujan famously proved that

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

In an effort to provide a combinatorial explanation of these congruences, Dyson defined the *rank* of a partition to be its largest part minus the number of its parts. For example, the table below includes the ranks of the partitions of 4.

Partition	Rank	Rank mod 5
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

Based on numerics, Dyson [20] made the following conjecture whose truth provides a combinatorial explanation of Ramanujan's congruences modulo 5 and 7.

Conjecture (Dyson). The partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7).

In 1954 A. O. L. Atkin and H. P. F. Swinnerton-Dyer proved [5] Dyson's conjecture.[7]

There is now a robust theory of partition congruences modulo every integer M coprime to 6 [1, 34], and typical congruences look more like

$$p(48037937n + 1122838) \equiv 0 \pmod{17}.$$

One naturally asks: what role, if any, do Dyson's original guesses play within this theory?

[6] *In this paper we use a slightly stronger condition; we assume the existence of "principal parts" at cusps.*

[7] *A short calculation reveals that the obvious generalization of the conjecture cannot hold for 11.*

K. Bringmann and the author [13] investigated this question, and in their work they related $N(r,t;n)$, the number of partitions of n with rank congruent to $r \pmod t$, to harmonic Maass forms. They essentially proved that

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^n$$

is the holomorphic part of a weight 1/2 harmonic Maass form. This result, combined with Shimura's theory of half-integral weight modular forms and the Deligne-Serre theory of Galois representations, implies that ranks "explain" infinite classes of congruences.

Theorem 2.1 (Th. 1.5 of [13]). *If $Q \geq 5$ is prime and $j \geq 1$, then there are positive integers t and arithmetic progressions $An+B$ such that*

$$N(r,t;An+B) \equiv 0 \pmod{Q^j}$$

for every $0 \leq r < t$. In particular, we have that $p(An+B) \equiv 0 \pmod{Q^j}$.

Moonshine

In the late 1970s J. McKay and J. Thompson [39] observed that the first few coefficients of the classical elliptic modular function

$$\begin{aligned} j(z) - 744 &\\ &= q^{-1} + 196884q + 21493760q^2 \\ &\quad + 864299970q^3 + \cdots \end{aligned}$$

are certain linear combinations of the dimensions of the irreducible representations of the *Monster* group. For example, the degrees of the four "smallest" irreducible representations are: 1, 196883, 21296876, and 842609326, and the first few coefficients are:

$$\begin{aligned} 1 &= 1 \\ 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2\cdot 196883 + 2\cdot 1. \end{aligned}$$

J. Conway and S. Norton [18] expanded on these observations, and they formulated a series of deep conjectures, the so-called *Monstrous Moonshine Conjectures*. These conjectures have now been settled, and thanks to the work of many authors, most notably R. E. Borcherds [9], there is a beautiful theory, involving string theory, vertex operator algebras, and generalized Kac-Moody superalgebras, in which connections between objects like the j-function and the Monster are revealed.

In the 1980s, in the spirit of Moonshine, V. G. Kac and D. H. Peterson [27] established the modularity of similar characters that arise in the study of infinite-dimensional affine Lie algebras. As a generalization of this work, ten years ago Kac and M. Wakimoto [28] computed characters of the affine Lie superalgebras $g\ell(m,1)^\wedge$ and $s\ell(m,1)^\wedge$. These characters are not modular, and Kac asked whether they might be related to harmonic Maass forms. Bringmann and the author have confirmed [14] this speculation; these characters are pieces of nonholomorphic modular functions. We consider the character for the $s\ell(m,1)^\wedge$ modules $L(\Lambda_{(s)})$, where $L(\Lambda_{(s)})$ is the irreducible $s\ell(m,1)^\wedge$ module with highest weight $\Lambda_{(s)}$. If $m \geq 2$ and $s \in \mathbb{Z}$, then the work of Kac and Wakimoto implies that

Ramanujan passport photo.

(2.12)

$$tr_{L(\Lambda_{(s)})}q^{L_0} = 2q^{\frac{m-2-12s}{24}} \cdot \frac{\eta(2\tau)^2}{\eta(\tau)^{m+2}} \cdot \sum_{k=(k_1,k_2,\ldots,k_{m-1})\in\mathbb{Z}^{m-1}} \frac{q^{\frac{1}{2}\sum_{i=1}^{m-1}k_i(k_i+1)}}{1+q^{|k|-s}},$$

where $|k| := \sum_{i=1}^{m-1} k_i$ and $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta function. Using the function R in (1.10), Bringmann and the author defined the function

(2.13)

$$\mathcal{T}_{m,s}(\tau) := tr_{L(\Lambda_{(s)})}q^{L_0} - 2^{m-1}q^{\frac{m-2}{24}}\frac{\eta(2\tau)^{2m}}{\eta(\tau)^{2m+1}}R(-s\tau;(m-1)\tau),$$

and they showed that

$$\frac{\eta(\tau)^{2m+1}}{\eta(2\tau)^{2m}} \cdot tr_{L(\Lambda_{(s)})}q^{L_0}$$

is (up to a power of q) a mock theta function. As a consequence, they proved:

Theorem 2.2 (Th. 1.1 of [14]). *If $m \geq 2$ and $s \in \mathbb{Z}$, then $\mathcal{T}_{m,s}(\tau)$ is (up to a power of q) a nonholomorphic modular function.*

Donaldson Invariants

In recent work with A. Malmendier [32], it is shown that the mock theta function

$$M(q) := q^{-\frac{1}{8}} \times \sum_{n=0}^{\infty}\frac{(-1)^{n+1}q^{(n+1)^2}(1-q)(1-q^3)\cdots(1-q^{2n-1})}{(1+q)^2(1+q^3)^2\cdots(1+q^{2n+1})^2}$$

is a "topological invariant" for $\mathbb{C}P^2$. This claim pertains to the differentiable topology of 4-manifolds.

In the early 1980s S. Donaldson proved (for example, see [19]) that the diffeomorphism class of a compact, simply connected, differentiable 4-manifold X is not necessarily determined by its intersection form. In his work, he famously defined the *Donaldson invariants*, diffeomorphism invariants of X obtained as graded homogeneous polynomials on the homology ring with integer coefficients.

There are two families of invariants corresponding to the $SU(2)$ and the $SO(3)$ gauge theories. The author and Malmendier considered the $SO(3)$ case for the simplest manifold, the complex projective plane $\mathbb{C}P^2$ with the Fubini-Study metric. The invariants are difficult to work out even in this case; indeed, they were not computed until the work of L. Göttsche [22] in 1996, assuming the Kotschick-Morgan Conjecture. Göttsche, H. Nakajima, and K. Yoshioka have recently confirmed [23] this provisional description of the Donaldson "zeta-function"

$$Z(p,S) = \sum_{m,n} \phi_{mn} \cdot \frac{p^m S^n}{m!n!}. \tag{2.15}$$

In the mid-1990s G. Moore and E. Witten conjectured [33] "u-plane integral" formulas for this zeta function. Their work relies on the following identifications:

$$\begin{array}{ccc} u\text{-plane integrals} & \longleftrightarrow & \text{Donaldson invariants for } \mathbb{C}P^2 \\ \updownarrow & & \\ \text{elliptic surface} & & \end{array}$$

Here the rational elliptic surface is the universal curve for the modular group $\Gamma_0(4)$, which can be identified with $\mathbb{C}P^1$ minus 3 points with singular fibers. In addition, the rational elliptic surface is endowed with an analytical marking such that the generic fibers correspond to elliptic curves that are parameterized by $\mathbb{C}$-lattices $\langle \omega, \tau\omega \rangle$ in the usual way. Then the u-plane zeta function is given as a "regularized" integral

(2.16)
$$Z_{UP}(p,S) := -\frac{8}{\sqrt{2\pi}} \cdot \int_{UP}^{reg} \frac{du \wedge d\overline{u}}{\sqrt{\operatorname{Im}\tau}} \cdot \frac{d\overline{\tau}}{d\overline{u}} \cdot \frac{\Delta^{\frac{1}{8}}}{\omega^{\frac{1}{2}}} \cdot e^{2up+S^2\hat{T}} \cdot \overline{\eta(\tau)^3}.$$

Here Δ is the discriminant of the corresponding elliptic curves, and $\hat{T}$ is defined by the renormalization flow on the elliptic surface.

The Moore-Witten Conjecture in this case is that $Z(p,S) = Z_{UP}(p,S)$. Using (2.16), the author and Malmendier reformulated this conjecture in terms of harmonic Maass forms arising from $M(q)$, and they then used Zwegers's μ-function to prove the following theorem.

Theorem 2.3 (Th. 1.1 of [32])**.** *The Moore-Witten Conjecture for the $SO(3)$-gauge theory on $\mathbb{C}P^2$ is true. In particular, we have that $Z(p,S) = Z_{UP}(p,S)$.*

Borcherds Products

Recently R. E. Borcherds provided [10, 11] a striking description for the exponents in the infinite product expansion of many modular forms with a *Heegner divisor*. He proved that the exponents in these expansions are coefficients of weight 1/2 modular forms. As an example, the classical Eisenstein series $E_4(\tau)$ factorizes as

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n \\ &= (1-q)^{-240}(1-q^2)^{26760} \cdots = \prod_{n=1}^{\infty} (1-q^n)^{c(n)}, \end{aligned}$$

where the $c(n)$ are the coefficients $b(n^2)$ of a weight 1/2 meromorphic modular form

$$\begin{aligned} G(\tau) &= \sum_{n\geq -3} b(n)q^n \\ &= q^{-3} + 4 - 240q \\ &\quad + 26760q^4 + \cdots - 4096240q^9 + \cdots. \end{aligned}$$

Bruinier and the author [17] have generalized this phenomenon to allow for exponents that are coefficients of weight 1/2 harmonic Maass forms. For brevity, we give examples of *generalized Borcherds products* that arise from the mock theta functions $f(q)$ and $\omega(q)$. To this end, let $1 < D \equiv 23 \pmod{24}$ be square-free, and for $0 \leq j \leq 11$ let

(2.17)
$$\begin{aligned} H_j(\tau) &= \sum_{n \gg -\infty} C(j;n)q^n \\ &:= \begin{cases} 0 & \text{if } j = 0,3,6,9, \\ \left(\frac{j}{3}\right) q^{-1} f(q^{24}) & \text{if } j = 1,5,7,11, \\ \left(\frac{j}{3}\right) 2q^8 \left(\omega(q^{12}) - \omega(-q^{12})\right) & \text{if } j = 2,10, \\ (-1)^{\frac{j}{4}} 2q^8 \left(\omega(q^{12}) + \omega(-q^{12})\right) & \text{if } j = 4,8. \end{cases} \end{aligned}$$

If $e(\alpha) := e^{2\pi i\alpha}$ and $\left(\frac{-D}{b}\right)$ is the Jacobi-Kronecker quadratic residue symbol, then define

$$P_D(X) := \prod_{b \mod D} (1 - e(-b/D)X)^{\left(\frac{-D}{b}\right)}. \tag{2.18}$$

Using this rational function, we then define the *generalized Borcherds product* $\Psi_D(\tau)$ by

$$\Psi_D(\tau) := \prod_{m=1}^{\infty} P_D(q^m)^{C(\overline{m};Dm^2)}. \tag{2.19}$$

The exponents come from (2.17), and $\overline{m}$ is the residue class of m modulo 12.

Theorem 2.4 (§8.2 of [17])**.** *The function $\Psi_D(\tau)$ is a weight 0 meromorphic modular form on $\Gamma_0(6)$ with a discriminant $-D$ Heegner divisor (see §5 of [17] for the explicit divisor).*

This theorem has an interesting consequence for the parity of the partition function. Very little is known about this parity; indeed, it was not even known that $p(n)$ takes infinitely many even and odd values until 1959 [30]. Using Theorem 2.4 and the fact that

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \equiv \sum_{n=0}^{\infty} p(n)q^n \pmod 4,$$

the author proved the following result for the partition numbers evaluated at the values of certain quadratic polynomials.

Theorem 2.5 (Corollary 1.4 of [37]). *If $\ell \equiv 23 \pmod{24}$ is prime, then there are infinitely many m coprime to 6 for which $p\left(\frac{\ell m^2+1}{24}\right)$ is even. Moreover, the first such m is bounded by $12h(-\ell)+2$, where $h(-\ell)$ is the class number of $\mathbb{Q}(\sqrt{-\ell})$.*

Elliptic Curve L-Functions

If $E/\mathbb{Q}$ is an elliptic curve

$$E/\mathbb{Q}:\ y^2 = x^3 + Ax + B,$$

then $E(\mathbb{Q})$, its $\mathbb{Q}$-rational points, forms a finitely generated abelian group. The Birch and Swinnerton-Dyer Conjecture, one of the Clay millennium prize problems, predicts that

$$\mathrm{ord}_{s=1}(L(E,s)) = \text{Rank of } E(\mathbb{Q}),$$

where $L(E,s)$ is the Hasse-Weil L-function for E. There is no known procedure for computing $\mathrm{ord}_{s=1}(L(E,s))$. Determining when $\mathrm{ord}_{s=1}(L(E,s)) \le 1$ for elliptic curves in a family of quadratic twists already requires the deep theorems of Kohnen [31] and Waldspurger [41], and of Gross and Zagier [24]. These results, however, involve very disparate criteria for deducing the analytic behavior at $s=1$.

Using generalized Borcherds products [17], Bruinier and the author have produced a single device that encompasses these criteria. We present a special case of these results. Suppose that E has prime conductor, and suppose further that the sign of the functional equation of $L(E,s)$ is -1. If Δ is a fundamental discriminant of a quadratic field, then let $E(\Delta)$ be the *quadratic twist* elliptic curve

$$E(\Delta):\ \ \Delta y^2 = x^3 + Ax + B.$$

Using harmonic Maass forms and their generalized Borcherds products, the author and Bruinier show that the coefficients of certain harmonic Maass forms encode the vanishing of central derivatives (resp. values) of the L-functions for the elliptic curves $E(\Delta)$.

Theorem 2.6 (Th. 1.1 of [17]). *Assuming the hypotheses above, there is a weight 1/2 harmonic Maass form*

$$M_E(\tau) = \sum_{n\gg-\infty} c_M^+(n)q^n + \sum_{n<0} c_M^-(n)\Gamma(1/2;4\pi|n|y)q^n,$$

and a nonzero constant $\alpha(E)$ that satisfies:

(1) *If $\Delta<0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then*

$$L(E(\Delta),1) = \alpha(E)\cdot\sqrt{|\Delta|}\cdot c_M^-(\Delta)^2.$$

(2) *If $\Delta>0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then $L'(E(\Delta),1)=0$ if and only if $c_M^+(\Delta)$ is algebraic.*

The Path Ahead

As we have seen, Ramanujan's deathbed letter set into motion an implausible adventure, one whose first act is now over. It was about the theory of harmonic Maass forms and its implications for many subjects: partitions and q-series, Moonshine, Donaldson invariants, mathematical physics, Borcherds products, and L-functions of elliptic curves, to name a few. This theory has provided satisfying answers to the first challenges: to understand the meaning behind Ramanujan's last words and to realize the expectation that this understanding would reveal and open new doors in the interconnected world of mathematics.

Every step along the way has evoked wonder—the enigmatic letter, the Lost Notebook, and the work of many minds. If the past is the road map to the future, then the yet unwritten acts promise forays, by intrepid mathematicians of today and tomorrow, into new worlds presently populated with seemingly unattainable mathematical truths.

Acknowledgments

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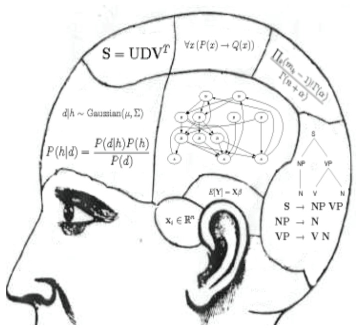

IPAM Graduate Summer School:

Probabilistic Models of Cognition

July 6 – 16, 2011

ORGANIZING COMMITTEE: N. GOODMAN (Stanford), **JOSH TENENBAUM** (MIT), and **ALAN YUILLE** (UCLA)

Introduction

This summer school is motivated by recent advances which offer the promise of building rigorous models for human cognition by applying the mathematical and computational tools developed for designing artificial systems. In turn, the complexity of human cognitive abilities offers challenges which test current theories and drive the development of more advanced tools. The goal is to develop a common mathematical framework for all aspects of cognition, and review how it explains empirical phenomena in the major areas of cognitive science - including vision, memory, reasoning, learning, planning, and language.

The main theoretical theme is to model cognitive abilities as forms of probabilistic inference over structured relational systems such as graphs and generative grammars. We will focus on how the mind learns complex generative models of the world and how it inverts or conditions these models based on observed data to infer world structure. We will pay particular emphasis on vision because this is currently an area of great activity but we will address all aspects of cognition. Other important themes include the combination of logic with probability and the development of probabilistic programming languages.

Participation

This summer school will provide a rare opportunity for the mathematical and cognitive science communities to learn about current research directions in this area. Funding is available to support graduate students and postdoctoral researchers in the early stages of their career, as well as more senior researchers interested in undertaking new research in this area. Encouraging the careers of women and minority mathematicians and scientists is an important component of IPAM's mission and we welcome their applications. The application is available online, and is due **March 1, 2011**.

http://www.ipam.ucla.edu/programs/gss2011

Topical Bias in Generalist Mathematics Journals

Joseph F. Grcar

Generalist mathematics journals exhibit bias toward the branches of mathematics by publishing articles about some subjects in quantities far disproportionate to the production of papers in those areas within all of mathematics. *Bias* is used here because it is the shortest English word with Webster's meaning of "a tendency of a statistical estimate to deviate in one direction from a true value." This paper quantifies the bias, which seems not to be discussed previously, and suggests some consequences of it.

The mathematical topics that are generally agreed to be the major branches of mathematics and the production of papers about them can be determined from the Mathematics Subject Classification and from two databases based on the classification [3]. The classification dates in some form to 1931, when *Zentralblatt für Mathematik und ihre Grenzgebiete* began publishing annual reviews of papers grouped into broad subject areas; *Mathematical Reviews* started publishing similar material in 1940. The American Mathematical Society created hierarchical codes to classify papers for its defunct Mathematical Offprint Service in the late 1960s [8]. This AMS (MOS) Classification quickly became the de facto index for mathematical literature. Both *Zentralblatt* and *Mathematical Reviews* maintain historical databases of indexed papers that can be accessed now through the World Wide Web.

The data presented here for the decade 2000–2009 were gathered from the *Zentralblatt* database in January 2010. The 854,547 items that were available at the time will increase as *Zentralblatt* completely assimilates papers from the recent past. All the records can be retrieved by Mathematics Subject Classification.

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Figure 1 displays the percentage of all mathematics papers that address each of the major branches of mathematics enumerated by Subject Classification. The present 5-digit codes begin with the 2-digit numbers that reflect the coarsest level of differentiation, namely, the major branches of mathematics. Currently, sixty-three of the 2-digit numbers are assigned. Each paper so catalogued receives one primary code and optionally any number of secondary codes. For example, when the data were gathered, 22,443 papers from 2000–2009 in the *Zentralblatt* database had either a primary or a secondary code in the "group theory" classification, 20. Thus approximately 22443/854547, or 2.63 percent, of all mathematics papers discussed group theory during 2000–2009. This value is recorded in Figure 1.

The gray shading in Figure 1 indicates papers for which the associated classification is not the primary subject. Such papers are prevalent across mathematics, so they are included in the count of papers for each subject. Because a paper may contribute to several subjects, the percentages for all classes sum to 156.5. The red, white, and blue shades in the figure distinguish subjects in a manner to be explained.

The American Mathematical Society publishes three print journals "devoted to research articles in all areas of mathematics". The extent to which these journals fulfill the Society's promise of inclusiveness can be examined by calculating percentages similar to those in Figure 1 but specific to the journals in question. The data presented here are for *Proceedings of the American Mathematical Society*; similar observations can be made for the *Journal* and the *Transactions*. The *Zentralblatt* database holds 4,758 papers from the *Proceedings* during the past decade, of which 309 had a primary or secondary code beginning with 20. Therefore, of all papers in the *Proceedings*, roughly 309/4758, or 6.49 percent, discuss group theory. This percentage for the generalist journal, 6.49, contrasts with the 2.63 percent among all mathematics papers. The fraction of papers about group theory in the journal is over twice the fraction in mathematics as a whole.

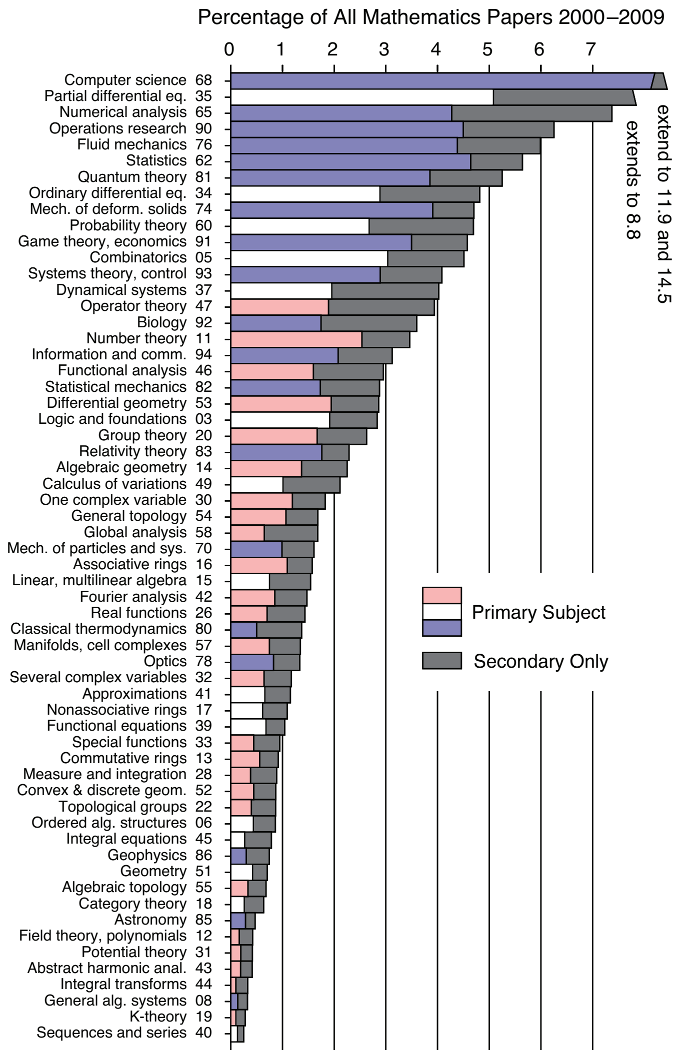

Figure 1. The distribution of effort in mathematical research is indicated by the percentage of all mathematics papers addressing a particular Mathematics Subject Classification. Three subject classes are omitted: general 00, history 01, and education 97. Papers for which a subject is primary are indicated by red, white, or blue shading (the shading is explained elsewhere). Papers for which a subject is only secondary are indicated by grey shading.

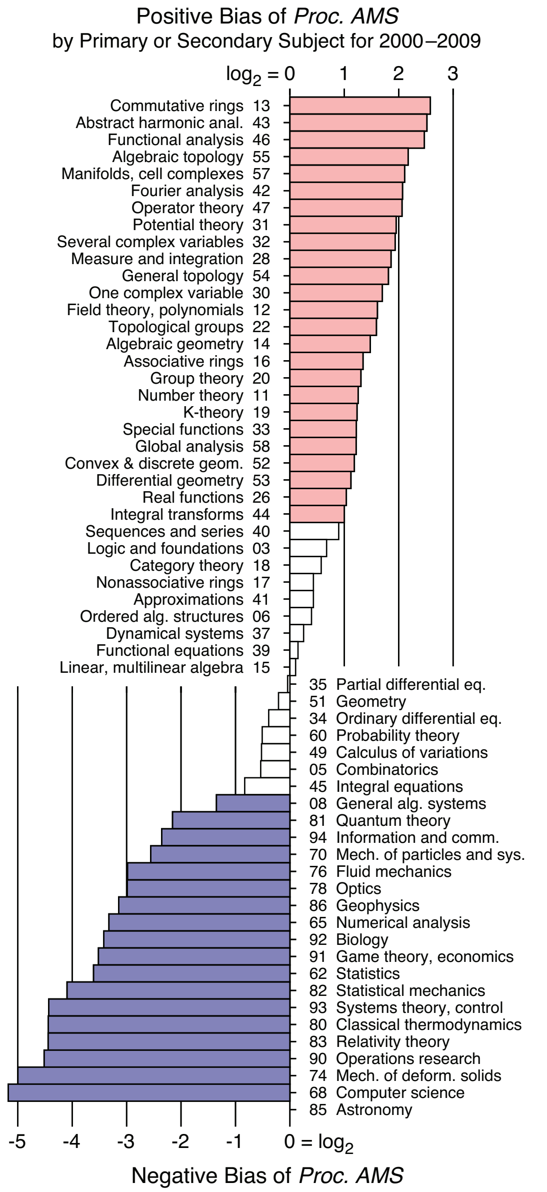

Figure 2. Topical bias in *Proceedings of the American Mathematical Society*. Bias is the ratio of the fraction of publications in the journal for a given subject to the fraction of publications in all of mathematics for that subject. Note that the scale is logarithmic to the base 2. The journal had no papers about the one subject at the bottom. Subjects with strong positive or negative bias have red or blue shading, respectively, to ease their identification in other figures.

The bias of a journal for or against a given branch of mathematics may be defined as the ratio of the fraction of papers about the subject in the journal to the fraction of papers about the subject in all of mathematics. It is convenient to represent the ratio as a base 2 logarithm, so a bias in favor has a positive value and a bias against has a negative value. The *Proceedings of the American Mathematical Society* thus has a positive bias for group theory of 6.49/2.63 or $2.47 = 2^{+1.31}$.

Figure 2 displays the biases of the *Proceedings* toward all the branches of mathematics. The wide range of values indicates that the journal is unrepresentative of mathematics research. It has a strong bias (2^{+1} to over 2^{+2}) in favor of twenty-five subjects that are colored red in Figures 1 and 2. Comparing the figures reveals a strong positive bias for three subjects that are of interest to relatively few mathematicians, in that each subject accounts for less than 1 percent of all mathematics papers: commutative rings 13 (bias $2^{+2.58}$), abstract harmonic analysis 43 (bias $2^{+2.52}$), and algebraic topology 55 (bias $2^{+2.17}$). In contrast, the *Proceedings* has a neutral to slightly negative bias for three subjects that are of interest to many more mathematicians, in that each of them constitutes over 4 percent of all mathematics papers: ordinary differential equations 34 (bias $2^{-0.39}$), probability theory 60 (bias $2^{-0.51}$), and combinatorics 05 (bias $2^{-0.54}$).

The ratio of biases for two subjects is easily seen to equal the ratio of the conditional probabilities that papers about the respective subjects would appear in the journal. Viewed in this way, the biases of the *Proceedings* are remarkable. For example, the journal is roughly $6.5 \approx 2^{+2.17}/2^{-0.54}$ times more likely to be the publisher of a paper about algebraic topology than about combinatorics.

The journal has a strong negative bias (2^{-1} to much below) against eighteen subjects that are colored blue in Figure 2. The same shading in Figure 1 reveals that the strong negative bias occurs for many branches of mathematics about which most papers are written. Indeed, the *Proceedings* is strongly biased against seven of the ten most heavily published subjects. Consequently, the generalist journal neglects subjects to which many mathematicians contribute or whose very creation is associated with some mathematicians. To cite a few examples: Tukey [1] for statistics 62, Lax [11] for numerical analysis 65, Moser [9] for mechanics 70, Ladyzhenskaya [4] for fluid mechanics 76, Dobrushin [10] for statistical mechanics 82, Dantzig [2] for operations research 90, Shannon [5] for information theory 94, and, of course, Wiener [6] for systems theory 93 and von Neumann [7] for game theory 91.

The *Transactions* has roughly the same biases as the *Proceedings* (Figure 3). Among heavily published subjects that appear in quantity in either journal, the *Proceedings* has stronger positive biases for functional analysis 46 and for operator theory 47.

Explanations for the biases are in a sense circular, in that authors submit where history suggests acceptance is likely, or that editorial

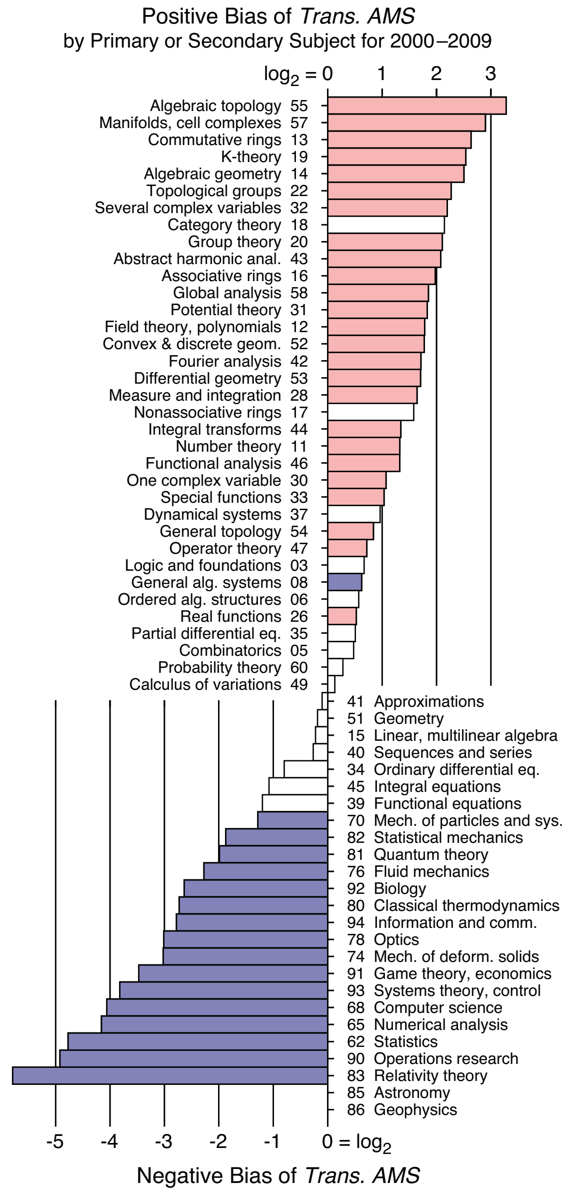

Figure 3. Topical bias in *Transactions of the American Mathematical Society*. The scale is logarithmic to the base 2. Subjects are shaded as in Figure 2 to ease comparison with the *Proceedings*. The journal had no papers about the two subjects at the bottom.

boards more accurately evaluate the familiar. For example, not all branches of mathematics employ just the telegraphic style that is prevalent in these journals. Thus biases are self-perpetuating, although surely procedures could be found so that submissions that are exceptions to what is usually published do not prove the rule.

The significant question raised by the data is not how biases occur or how to manage them; rather, the question is whether the present topical distribution in generalist journals best serves mathematics. Biases in professional journals impart an illusory picture of a field that can be dangerous if it becomes so pervasive as to affect the evolution of the underlying subject matter. In the present case, because the sponsoring society does not explain the topical selectivity of its generalist journals, and because the literature does not examine the issue, in the absence of clarifying public discussion, readers may assume that the contents of the flagship journals confirm prejudices about the branches of the field. If unchecked over many years, these opinions may influence decisions about curricula, publications, and staffing that can fragment the research community.

Such dissolution of mathematics may be occurring already, as evidenced by the proliferation and growth of fields, particularly since the middle of the last century, that have considerable mathematical content but whose faculties and professional societies have little overlapping membership. When different fields sponsor different branches of mathematics, then the subfields may adopt the cultures of their sponsors, so the branches of mathematics eventually may come to disagree over acceptable idiom, notation, rigor, and terminology. Barriers among the branches of mathematics entail a large opportunity cost, so to speak, because possible collaborations and synergies may not be realized. Moreover, as governments increasingly view scientific research as a component of national wealth, disparities may be expected to grow in the allocation of resources to the fields that encompass different branches of mathematics. In this way the fragmentation of mathematical research cedes to nonmathematicians a greater degree of responsibility to choose which branches of mathematics to encourage and how they should develop.

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Book Review

Logicomix: An Epic Search for Truth

Reviewed by Judith Roitman

Logicomix: An Epic Search for Truth
Apostolos Doxiadis and Christos H. Papadimitriou
with art by Alecos Papadatos and Annie di Donna
Bloomsbury USA, 2009
US$22.95, 352 pages, Paperback
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Four-Sentence Description

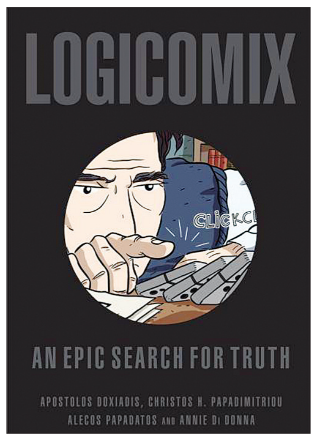
Logicomix is a graphic novel about Bertrand Russell, focusing on his and other's work on the foundations of mathematics. Its structure is a storyline within a frame within a frame. Being a graphic novel, art is a major component. And there is a terrific appendix, called *Notebook*, combining minibiographies of many mathematicians and philosophers (including Aristotle, Euclid, and Leibniz) with a detailed, clearly written glossary.

Art

The art is superb, in the lineage of Hergé and his great creation Tintin (I'm not the first person to notice this), sometimes called the European style: no overmuscled superheroes, no depressed losers, and no extreme caricatures. People look reasonably natural, color is important but not overwhelming, there's a sense of life and movement and a visual sense of excitement when excitement is called for. Maybe too much excitement at times; see below.

Storyline

Little Bertie grows up with enough family secrets, family insanity, and emotional deprivation to fuel several Brontë novels (by several Brontës). Like many other smart kids in similar circumstances, he turns to mathematics for stability. Unlike most other smart kids in similar circumstances, when he goes to college (Cambridge) he realizes mathematics has no real foundations, so he decides to provide them. Only to discover the worm in the apple, a.k.a. Russell's paradox (the set of all sets which do not contain themselves). With Whitehead, he writes the *Principia Mathematica* to provide the foundation and cast the worm from the apple (via the theory of types). He encounters many famous mathematicians and philosophers. Despite his attachment to reason, he falls in and out of love, marries several times, and has a huge crush on Whitehead's wife. He meets Wittgenstein. He questions his own work. He starts his long involvement in politics and has an unsuccessful flirtation with experimental education. This is not a chronological description; several of these things happen at once.



The Inner Frame

The story is told by Russell himself during a talk he gave at an unnamed American university on September 4, 1939, the day Britain declared war on Germany.[1] Accosted by a group of students wanting him to tell America not to go to war, he invites them into the talk, tells them that he disagrees with them, and tells the story of his life and work as a way of explaining his views.

The Outer Frame

But the book does not open with Russell talking in 1939. It opens with Apostolos Doxiadis,

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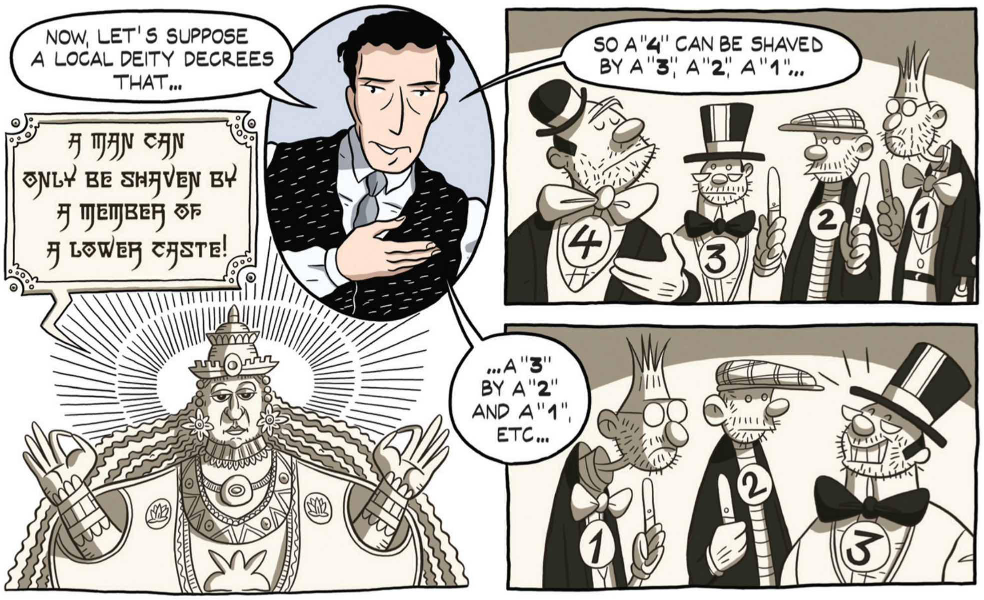
[1]*I'm not sure if there really was such a talk, and if there was, whether it took the form it took in the book. As we are reminded throughout, this is a graphic novel.*



the lead author, introducing us to his colleagues (including the dog Manga)[2] and describing what they are trying to do. The team will continue to appear periodically, sometimes explaining a little mathematics (Christos Papadimitriou does most of this), sometimes trying to figure out what tack to take, sometimes commenting on the greater human meaning of what they are writing about. The outer frame also ends the book, as the team watches a performance of the last scene of the *Oresteia*, when Athena invites the Furies to remain in Athens as a benevolent force, a.k.a. the marriage of reason (Athena) and passion (the Furies).[3]

The Team

This is not the first time Apostolos Doxiadis has written a novel embedded in mathematics; he is the author of *Uncle Petros and Goldbach's Conjecture*. Christos Papadimitriou is an eminent computer scientist at Berkeley and the author of *Turing (A Novel About Computation)* whose eponymous hero is an interactive tutoring program. Alecos Papadatos (who did the drawings, and who became attracted to animation in high school as a way of explaining geometry to his classmates—think locus of points) and his wife Annie Di Donna (who did the color) both have extensive backgrounds in animation. Di Donna is French; the others are Greek.[4] The team (minus Papadimitriou) has branched out with a graphic piece in the *Financial Times*[5] on Claude Levi-Strauss, staying within the outer frame of the team itself.[6] Minor members of the team are the visual researcher and letterer Anne Bardy (who, through her involvement with a performance of the *Oresteia*, has an important role), Dimitris Karatzaferis, and Thodoris Paraskeva, the inkers.

logicomix.com

Yes, there is a webpage, an excellent one. It is clear from the webpage that *Logicomix* has a lot of enthusiastic fans, some of them quite famous, and has won some well-deserved awards. There is a nice *Behind the scenes* tab which gives you a glimpse of how a graphic novel is actually created. But most revealing are the links to talks that Doxiadis (with assistance from Papadatos) gave at Cambridge and Birmingham. The Cambridge talk, in particular, opens with Doxiadis talking about his own life: his extreme distaste for mathematics until, in early adolescence, he suddenly fell in love with it. Within a year he was at Columbia University doing mathematics, which he continued to do for about eight years both at Columbia and in Paris. Then, suddenly, faced with a difficult family situation back in Greece, he lost his love for the subject, left it not quite completely (aside from *Uncle Petros* and *Logicomix*, there's a play about Gödel), and returned to his original loves of writing, cinema, and theater. He has also been working with Barry Mazur for a number of years on a project about mathematics and narrative.

Intellectual Content

If you are writing about people who are seriously invested in intellectual work, it's helpful to talk about their ideas, and, early on, Papadimitriou is introduced as the guy who will make sure that what the team says about logic makes sense. The way logic and mathematics is explicated is terrific and worth discussing. In one word: patience. In seven words: patience embedded in story embedded in art. You have to keep it interesting; your reader isn't worried about passing a test. But you can't pander, either; an insulted reader will stop reading. So, for example, type theory is explained during a croquet game by a caste society in which you can only be shaved by a lower caste member

[2]*Not a reference to Japanese graphic novels, but Greek slang for somebody who is really cool.*

[3]*Full disclosure: my classicist husband courted me by casting me in the role of Athena, with the great classicist Gareth Morgan as all of the Furies put together.*

[4]*So why, in the English translation of the Greek original, is Di Donna saddled with a French accent while her colleagues speak perfect English? Eet ees not fair!*

[5]*February 27, 2010.*

[6]*And using* Oresteia *to illustrate Levi-Strauss's structuralism.*

Illustrations courtesy of LOGICOMIX, © Bloomsbury USA.

(since N is the index set, no one in the lowest caste gets shaved). It takes eleven panels to complete this intellectual arc, including panels with somewhat comically drawn men labeled 1, 2, 3, 4. . . carrying razors, and a so-called local deity who looks like a cross between Buddha and a maharajah. Plenty of time for the reader/viewer to come to grips with what is going on. Brilliant.

What Art Brings to the Table and What It Takes Away

If *Logicomix* were just a nicely drawn graphic novel about Bertrand Russell and other mathematicians/philosophers with a few fairly clear explanations of basic set theory thrown in (such as the Hilbert Hotel and Russell's paradox), it would not have gotten nor have deserved the attention and praise it has received. What *Logicomix* does that few works in any medium do is to make intellectual passion palpable. That is its greatest strength. And it's here that its form becomes its substance. For example, flipping at random, there is a two-page spread of Wittgenstein with a thought bubble saying, "The **meaning** of the **world** does not reside **in** the **world**"[7] while he stands in the blasted landscape of a World War I battlefield with strange clouds and vapors, a visually muffled explosion in the near distance, and a haze around the dark moon.[8] Stunning. You just can't do this with words alone.

And that is where the problems come in. The team keeps reminding us that this is a graphic novel, but when it presents (1) powerful visual images of Russell visiting an insane Cantor and (2) powerful visual images of how this visit awakens Russell's personal demons resulting in (3) a powerfully drawn three-page nightmare, the admission in an afterword that Russell and Cantor actually never met doesn't overcome the image of Cantor that has been implanted in the reader's/viewer's mind. There are many scenes like this, powerfully drawn scenes that never happened, most of them involving some kind of madness. For me, the most problematic—the reality must have been painful enough—is depicting Hilbert lecturing on mathematics outdoors, pushing away his adolescent son who clings to him desperately to avoid being hauled away to an insane asylum. The son is pried away from the father, who, when someone expresses sympathy, responds "I have no son."[9]

Crazy

Which brings us to a major theme of *Logicomix*: the connection between logic and insanity. I would say the purported connection, but it's clear that Doxiadis thinks there's something to this—the phrase used on p. 217 is "Logic from Madness"—and he certainly marshals his evidence: if his logicians aren't crazy (Gödel, Frege, Cantor)

[7] *Boldface as in the text, where it is done more subtly.*

[8] *Wittgenstein actually was a soldier in the front lines and wrote his revolutionary* Tractatus Logico-Philosophicus*—which he later disavowed—while a prisoner of war.*

[9] *Hilbert did have a son who, at fifteen, was taken to an insane asylum where he eventually died; Hilbert never visited him there. Whatever judgment we may make of this, the situation depicted in* Logicomix *is far more cruel.*

they have craziness in the family (Russell, Hilbert, Wittgenstein);[10] this linkage is a major subject of the team's discussions, and in some sense, the book is structured by it.

On the first page we learn that the story we are about to hear is "so sad". On p. 24 we have a reference to Gian-Carlo Rota's article "on the curiously high rate of psychosis in the lives of the founders of logic," and on this page the theme is declared: not madness from logic but: "They became logicians from madness." On p. 78 we learn that as a young man Russell wrote that he would have killed himself if not for mathematics. On p. 83 we have Russell saying that as a young man he decided that madness was a disease pulling weak spirits away from the natural harmony of reason,[11] and hence he should devote himself to reason as his way out of his "dark legacy".[12] On p. 201 Papadimitriou summarizes Doxiadis's thesis as that the foundational quest is a spiritual tragedy, which Doxiadis does not deny; two pages later we learn that "less tortured characters would not have found this price [e.g., 362 pages in the *Principia* to prove that 1 + 1 = 2] worth paying." On p. 217, after getting lost in the back streets of Athens relying on childhood memories to find his way, a trip in which he was physically attacked and, in a separate incident, robbed, Papadimitriou decides that Doxiadis is correct and that the madness came from confusing reality with intellectual maps,[13] at which Doxiadis has the thought bubble "what a perfect definition of insanity!" On p. 230 we learn that it was Russell's "character, his insecurities, his neuroses which drove him to logic." On p. 282 Doxiadis conjectures that "maybe what brings them to logic is fear of ambiguity and emotion."

Sigh.

Your local neighborhood bartender is a bartender because of her character, her insecurities, her neuroses (and her socioeconomic situation at key times and places, not to mention her personal circumstances)—why should Bertrand Russell be any different? What's the big deal? People do not read carefully, and the casual reader could easily be left with an impression that logicians are crazy. My own first impression was that the authors believed that logicians are crazy—the detail of the previous paragraph is meant to qualify this impression. The authors undercut their own case: not every logician who appears is crazy, not every logician is motivated by madness. Most important, the main character is a major counterexample. Madness did run rampant in Russell's family, in both previous and later generations.[14] He may have feared it, and various aspects of his personal life (sexual relationships, attitudes toward parenting, involvement with progressive education, and perhaps—depending on point of view—his politics) may have a bit of a meshugeneh quality, but he seemed nevertheless to have been eminently sane.

Tragedy and Triumph

Was Russell's great effort in the *Principia*, as he himself seems to have felt, a failure? As an adolescent I delighted in Boole's locution "dog dog = dog" and repeated it to my friends, no doubt to their great annoyance, but "$x \cap x = x$" is mathematically powerful in a way that "dog dog = dog" is not. Things explode when the right notation comes along; *Principia* was an important part of this.

[10] *The judgment on Wittgenstein's sanity is not clear.*

[11] *My paraphrase. I am not a Russell scholar and don't know if he really said anything like this, but if he did it's especially poignant since no one is as rational as a paranoid schizophrenic.*

[12] *It is certain that Russell was devoted to reason.*

[13] *Relating to the map theme, on p. 241 Wittgenstein comes to a breakthrough understanding of the relation between language and reality while observing German army brass manipulating toy soldiers and toy cannons while planning a battle.*

[14] *Note that none of these mad relatives were logicians.*

Yet, except for his paradox, Russell's name doesn't come up in logic texts. No mathematical logic text cites *Principia*'s 362-page proof that 1 + 1 = 2. Except perhaps as a historical curiosity, mathematical logic texts generally don't cite *Principia* at all. Mathematicians don't think about or work within the theory of types.[15] Mathematics is usually done without much regard for a foundational basis, and when you want a foundational basis you turn to the (quite different) Cantor-Zermelo-Fraenkel-von Neumann approach to set theory; or, if you prefer, you turn to category theory. The mathematically oriented pioneers of mathematical logic and set theory at the beginning of the twentieth century generally did not turn to Russell and Whitehead for inspiration, and neither do their descendants.

With at least one exception, and that's a big one: Gödel not only read the *Principia*[16] but entitled his pioneering monograph on incompleteness theorems *On Formally Undecidable Propositions of* Principia Mathematica *and Related Systems*. Of course, what the incompleteness theorems say is that Russell's quest for mathematical certainty is doomed. You can summarize this by: the *Principia* was, in some sense, crucial to that part of Gödel's work which, as a by-product, destroyed the purpose of *Principia*.

But while Doxiadis sees this as a tragedy, Papadimitriou sees the narrative arc as a triumph, leading to Turing and computer science: "No, it's a total triumph! and it abounds in happy endings, the happiest being that the tools of reason are at everybody's fingertips!" Cue the *Oresteia*.

Gender

A final word on gender. The characters in *Logicomix* tend to be, if you'll excuse the expression, incomplete. The men tend to be bumbling fools caught up in their intellectual enthusiasms, occasionally commenting on the stupidity of women. The women tend to be practical, aware of the physical world that the men scarcely notice. There are, of course, counterexamples to this polarity (e.g., Evelyn Whitehead thinks that she is dying when it's only an anxiety attack), but the overall pattern is clear, even in the team's contemporary interactions. Anne rescues Papadimitriou from his disastrous adventure. Annie is full of clever commentaries on the intellectual action, but they are commentaries only; the actual ideas spring from Doxiadis and Papadimitriou. Only Athena seems to be completely human. Maybe you have to be a god to be fully human, but I would hope not.

[15] *Except, of course, for mathematical logicians who specialize in it as one would specialize in, say, commutative groups.*

[16] *Perhaps the only person who ever did in its entirety, as* Logicomix *points out.*

Division Algebras and Wireless Communication

B. A. Sethuraman

The aim of this note is to bring to the attention of a wide mathematical audience the recent application of division algebras to wireless communication. The application occurs in the context of communication involving multiple transmit and receive antennas, a context known in engineering as MIMO—short for multiple input, multiple output. While the use of multiple receive antennas goes back to the time of Marconi, the basic theoretical framework for communication using multiple transmit antennas was only published about ten years ago. The progress in the field has been quite rapid, however, and MIMO communication is widely credited with being one of the key emerging areas in telecommunication. Our focus here will be on one aspect of this subject: the formatting of transmit information for optimum reliability.

Recall that a division algebra is an (associative) algebra with a multiplicative identity in which every nonzero element is invertible. The center of a division algebra is the set of elements in the algebra that commute with every other element in the algebra; the center is itself just a commutative field, and the division algebra is naturally a vector space over its center. We consider only division algebras that are finite-dimensional as such vector spaces. Commutative fields are trivial examples of these division algebras, but they are by no means the only ones: for instance, class-field theory tells us that over any algebraic number field K, there is a rich supply of noncommutative division algebras whose center is K and are finite-dimensional over K.

Interest in MIMO communication began with the papers [21, 10, 24, 11], in which it was established that MIMO wireless transmission could be used both to decrease the probability of error and to increase the amount of information that can be transmitted. This result caught the attention of telecommunication operators, particularly since MIMO communication does not require additional resources in the form of either a larger slice of the radio spectrum or increased transmitted power.

The basic setup is as follows: Complex numbers $Re^{\imath\phi}$, encoded as the amplitude (R) and phase (ϕ) of a radio wave, are sent from t transmit antennas (one number from each antenna), and the encoded signals are then received by r receive antennas. The presence of obstacles in the environment, such as buildings, causes attenuation of the signals; in addition, the signals are reflected several times and interfere with one another. The combined degradation of the signals is commonly referred to as fading, and achieving reliable communication in the presence of fading has been the most challenging aspect of wireless communication. The received and transmitted signals are modeled by the relation

$$Y_{r\times 1} = \theta H_{r\times t} X_{t\times 1} + W_{r\times 1}$$

where X is a $t\times 1$ vector of information signals, Y is an $r\times 1$ vector of received signals, W is an $r\times 1$ vector of additive noise, H is an $r\times t$ matrix

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that models the fading, and θ is a real number chosen to multiply the information signals so as to fit the power available for transmission. Under the most commonly adopted model, the entries of the noise vector W and the channel matrix H are assumed to be Gaussian complex random variables that are independent and identically distributed with zero mean. (A Gaussian complex random variable is one of the form $w = x + \imath y$, where x and y are real Gaussian random variables that are independent and have the same mean and variance. The modulus of such a random variable, and in particular the magnitude of each fading coefficient h_{ij}, is then Rayleigh distributed. This model is hence also known as the Rayleigh fading channel model.) It is the presence of fading in the channel that distinguishes this model from more classical channels, where the primary source of disturbance is the additive Gaussian noise W.

A common engineering model is to assume that the channel characteristics (i.e., the fading coefficients h_{ij}) stay constant in some fixed but small time interval and that these characteristics are known to the receiver but not the transmitter. (This is known as *coherent* transmission.) If each antenna can transmit n times during such an interval, then the transmission process is compartmentalized into blocks of length n: each antenna transmits n times, and each receiver waits to receive all n transmissions before processing them. A common simplifying assumption is to take $r = t = n$, and the equation above is accordingly modified to read

$$Y_{n\times n} = \theta H_{n\times n} X_{n\times n} + W_{n\times n}. \tag{1}$$

Thus the ith column of Y, θX, and W represent (respectively) the received vectors, the transmitted information, and the additive noise from the ith transmission. A measure of the power available during a single transmission from all n antennas, i.e., a single use of the telecommunication channel, is the *signal-to-noise ratio* (SNR) ρ. Recall that the Frobenius norm $||X||_F$ of $X = (x_{i,j})$ equals $\sqrt{\sum_{i,j} |x_{i,j}|^2}$. Since the power required to send a complex number varies as the square of its modulus, the normalization constant θ must satisfy $\theta^2 ||X||_F^2 \le n\rho$.

A subset S of the nonzero complex numbers known as the *signal set* is selected as the alphabet (a common situation is that S is a finite subset of size q of the nonzero Gaussian integers $\mathbb{Z}[\imath] - \{0\}$), and a k-tuple $(s_1, s_2, \dots, s_k)$, $s_i \in S$, constitutes the message that the transmitter wishes to convey to the receiver. Thus there are q^k messages in all, and it is assumed that each message is equally likely to be transmitted. A *space-time code* is then a one-to-one map $X\colon S^k \to M_n(\mathbb{C})$; we write $\mathcal{X}$ for $X(S^k)$. The transmitted matrix $\theta X_{n\times n}$ in Equation (1) is thus drawn from the set $\theta\mathcal{X}$ as $(s_1, s_2, \dots, s_k)$ vary in S^k. Often $\mathcal{X}$ itself is referred to as the space-time code. It is typically assumed that the map X is "linear in S^k", that is, it is the restriction to S^k of a group homomorphism $\langle S\rangle^k \to M_n(\mathbb{C})$, where $\langle S\rangle$ is the additive subgroup of $\mathbb{C}$ generated by S. (The term "space-time" refers to the fact that information $(s_1, s_2, \dots, s_k)$ is packaged in the spatial direction by sending it out through several physically separated transmit antennas and in the time direction by sending it out in n consecutive transmissions.)

Under the information-theoretic framework developed by Shannon in 1948 ([18]) and adopted ever since within the telecommunication community, the amount of information conveyed by a message in this setting is equal to $\log_2(q^k)$ "bits". Since this amount of information is conveyed in n transmissions over the MIMO channel, the rate of information transmission is then given by $\frac{k}{n}\log_2(q)$ bits per channel use. When q and n are fixed a priori, the quantity k serves as a measure of information rate.

Reliability of communication is commonly measured by the probability P_e of incorrectly decoding the transmitted message at the receiver. The pairwise error probability $P_e(i,j)$ (for $i \neq j$) is the probability that message i is transmitted and message j is decoded. Performance analysis of MIMO communication systems typically focuses on the pairwise error probability, as it is easier to estimate and also because the error probability P_e can be upper and lower bounded in terms of the pairwise error probability.

It was shown in [21, 11] that for a fixed SNR (i.e., power) ρ, in order to keep the pairwise error probability low, the space-time code $\mathcal{X}$ must meet the two criteria below, of which the first is primary:

(1) *Rank Criterion:* For $\underline{s} := (s_1, s_2, \dots, s_k)$ and $\underline{s'} := (s'_1, s'_2, \dots, s'_k)$ with $\underline{s} \neq \underline{s'}$, the difference matrix
$$X(\underline{s}) - X(\underline{s'})$$
must have full rank n, i.e., it must be invertible.

(2) *Coding Gain Criterion:* For $\underline{s}$ and $\underline{s'}$ as above, $\underline{s} \neq \underline{s'}$, the modulus of the determinant of difference
$$|\det(\theta X(\underline{s}) - \theta X(\underline{s'}))|$$
must be as large as possible.

Clearly, the second criterion comes into play only when the first criterion has been met but then subsumes it. Each criterion impacts a different communication parameter, and the two are hence stated independently. Note that one cannot arbitrarily scale the matrices X to increase the coding gain because the assumption of fixed ρ, along with the relation $\theta^2||X||_F^2 \le n\rho$, would simply cause a corresponding decrease in θ. Note too that one cannot increase the quantity k (a proxy for the rate of information) arbitrarily, as this would create a larger set of matrices θX all circumscribed to lie within a sphere of radius $\sqrt{n\rho}$, which would then cause the determinant of their differences to get smaller, thereby going against the second criterion.

Satisfying the Rank Criterion

The earliest space-time code, for two antennas, was given by an engineer, Alamouti ([1]): given an arbitrary signal set S, he chose $X\colon S^2 \to M_2(\mathbb{C})$ to be

$$X(s_1, s_2) = \begin{pmatrix} s_1 & -\overline{s_2} \\ s_2 & \overline{s_1} \end{pmatrix} \tag{2}$$

(where $\overline{s_i}$ stands for complex conjugation). It is easy to see that the rank criterion is immediately met. Writing $s_1 = u_1 + \imath u_2$, $s_2 = u_3 + \imath u_4$, each such matrix can be expressed in the form $X(s_1, s_2) = \sum_{i=1}^{4} u_i A_i$. The 2×2 complex matrices A_j are such that for any complex 2×2 channel matrix H, the collection of 2×2 matrices $\{HA_i\}$ is pairwise orthogonal when regarded as vectors in $\mathbb{R}^8$ by writing out sequentially the real and imaginary parts of each entry of the $\{HA_i\}$. The expansion above makes it possible to do a least squares estimation of the u_j from the received matrix Y, also considered as a vector in $\mathbb{R}^8$ as above, by projecting onto the respective matrices HA_j (we will consider this in more detail later). It is this property that makes the Alamouti code so easy to decode, and, not surprisingly, the code has since been adopted into the IEEE 802.11n "Wireless LAN" standard. In applications, the $\{s_1, s_2\}$ are typically drawn from a subset of $\mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$.

Alamouti's code led to a furious search among engineers and coding theorists for generalizations for higher numbers of antennas. Much of the early work (see [22], for example) focused on combinatorial methods. The matrix X in Equation (2) is almost unitary: it satisfies $XX^\dagger = (s_1\overline{s_1} + s_2\overline{s_2})I_2$, where the superscript $\dagger$ stands for transpose conjugate, and I_2 stands for the 2×2 identity matrix. Not surprisingly, early workers (see [22], for example) sought $n \times n$ matrices $X(s_1, \ldots, s_k)$ whose entries come from the set $\{\pm s_j, \pm\overline{s_j}, \pm\imath s_j, \pm\imath\overline{s_j},\ j = 1, \ldots, k\}$ and satisfy

$$XX^\dagger = (s_1\overline{s_1} + \cdots + s_k\overline{s_k})I_n. \tag{3}$$

This quickly leads to a necessary condition: the existence of $2k-1$ complex $n \times n$ matrices A_i satisfying $A_i^\dagger A_i = I_n$, $A_i^\dagger = -A_i$, and $A_iA_j = -A_jA_i$ for $1 \le i < j \le 2k-1$. These are, of course, the Hurwitz-Radon-Eckmann matrices, and classical results of Hurwitz-Radon-Eckmann (see [6], for instance) severely limits the values of k for which such matrices can exist. If $n = 2^a(2b+1)$, then the Hurwitz-Radon-Eckmann result says that the maximum possible value of k equals $(a+1)$. Thus $k = n$ if and only if $n = 2$, $k \le \frac{3n}{4}$ for $n > 2$, and $k \le \frac{n}{2}$ for $n > 4$. It follows that these generalizations of the Alamouti code transmit too few information symbols for more than two transmit antennas. (A similar analysis of the matrices A_i using representations of Clifford algebras was made by Tirkkonen and Hottinen in [20].)

In 2001 Sundar Rajan, a professor of communication engineering at the Indian Institute of Science, introduced the problem of designing matrices $X(s_1, \ldots, s_k)$ satisfying the rank criterion to this author. Given his algebraic background, this author could recognize easily that matrices arising from embeddings of fields and division algebras can be utilized to solve this problem. Let $f\colon D \to M_n(\mathbb{C})$ be an embedding, i.e., an (injective) ring homomorphism of a division algebra D into the $n \times n$ matrices over $\mathbb{C}$. Then for $X_1 = f(d_1)$ and $X_2 = f(d_2)$ $(X_1 \ne X_2)$, $X_1 - X_2$ must necessarily be invertible. This is because $d_1 - d_2$, being a nonzero element of the division algebra D, is automatically invertible, and since f is a homomorphism, the same must also be true of $X_1 - X_2$. Thus the matrices in $f(D)$ automatically satisfy the rank criterion. Using this observation, Sundar Rajan, his Ph.D. student Shashidhar, and this author ([19]) proposed several schemes for constructing space-time codes from various signal sets. For each signal set S and for each n, they constructed suitable division algebras D, suitable embeddings $f\colon D \to M_n(\mathbb{C})$, and suitable injective maps $X\colon S^k \to f(D)$, for suitable k.

For simplicity of construction in the noncommutative case, the authors of [19] used *cyclic division algebras* for their codes. A cyclic division algebra is constructed from two data: a field extension K/F of degree n that is Galois with cyclic Galois group $\langle\sigma\rangle$, and a nonzero element $\gamma \in F$ that satisfies the property that for any $i = 1, \ldots, n-1$, γ^i is not a norm[1] from K to F. As a K-vector space, the algebra is expressible as

$$D = \bigoplus_{i=0}^{n-1} Ku^i$$

where u is a symbol. The multiplication in this algebra is given by the relations $uk = \sigma(k)u$ for all $k \in K$, and $u^n = \gamma$. The bilinearity of multiplication, along with these relations, then allows us to determine the product of any two elements of D. One can prove that this construction indeed yields a division algebra with center F. (Such a division algebra is said to be of *index* n.)

There is a well-known embedding of such a D into $M_n(K)$ that sends $k_0 + k_1u + \cdots k_{n-1}u^{n-1}$ to

[1] *This is a sufficient condition to obtain a cyclic division algebra.*

$$
(4)\quad \begin{bmatrix} k_0 & \gamma\sigma(k_{n-1}) & \gamma\sigma^2(k_{n-2}) & \dots & \gamma\sigma^{n-1}(k_1) \\ k_1 & \sigma(k_0) & \gamma\sigma^2(k_{n-1}) & \dots & \gamma\sigma^{n-1}(k_2) \\ k_2 & \sigma(k_1) & \sigma^2(k_0) & \dots & \gamma\sigma^{n-1}(k_3) \\ k_3 & \sigma(k_2) & \sigma^2(k_1) & \dots & \gamma\sigma^{n-1}(k_4) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n-2} & \sigma(k_{n-3}) & \sigma^2(k_{n-4}) & \dots & \gamma\sigma^{n-1}(k_{n-1}) \\ k_{n-1} & \sigma(k_{n-2}) & \sigma^2(k_{n-3}) & \dots & \sigma^{n-1}(k_0) \end{bmatrix}
$$

By taking F to be various subfields of $\mathbb{C}$ containing $\mathbb{Q}(S)$ (the field generated by the elements of S over $\mathbb{Q}$) in this formulation, and for each such F taking various K and γ, a wide variety of space-time codes can be constructed for a wide range of signal sets. For further simplicity of construction, particularly in the selection of the element γ above, the authors of [19] chose all their base fields F to contain transcendental elements; in most cases, their cyclic extensions K/F were of the form $K_0(x)/F_0(x)$, where K_0/F_0 is a cyclic extension of number fields, and x is a transcendental. In these cases, the authors' construction yielded codes $X\colon S^{n^2} \to M_n(\mathbb{C})$, i.e., with $k = n^2$.

Alamouti's original code above arises as a special case of this formulation: the matrices of Equation (2) are just the matrices of Equation (4) above specialized to the cyclic algebra $(\mathbb{C}/\mathbb{R}, \sigma, -1)$, where σ stands for complex conjugation. This is nothing other than Hamilton's quaternions: the four-dimensional $\mathbb{R}$ algebra $\mathbb{R} \oplus \mathbb{R}\imath \oplus \mathbb{R}j \oplus \mathbb{R}k$ subject to the relations $\imath^2 = j^2 = -1$, $\imath j = -j\imath = k$. (The signal set in Alamouti's construction is contained in K instead of F, unless of course S is real.)

Satisfying the Coding Gain Criterion

The coding community immediately recognized the potential of cyclic division algebras as a fundamental tool for constructing space-time codes and began to work with the coding paradigm introduced in [19]. However, there was still a drawback. Although the specific codes of [19] certainly satisfied the rank criterion, their performance was not satisfactory. The reason for this became clear: the specific division algebras of [19] were proposed only for mathematical simplicity—merely as easy examples of the larger paradigm of division algebras—and were not optimized for the coding gain performance criterion above. The use of transcendental numbers in the codes in [19] caused the determinants of the difference matrices to come arbitrarily close to zero and limited their performance.

This situation was quickly remedied in [2] by a very clever technique. To provide a lower bound on the moduli of the determinants of the difference of code matrices, the authors Belfiore, Rekaya, and Viterbo first constructed division algebras from cyclic extensions $K/\mathbb{Q}(\imath)$ and $\gamma \in \mathbb{Z}[\imath]$, but then restricted the various k_i in the matrix (4) above to entries in $\mathcal{O}_K$, the ring of integers of K. The net result, as can easily be seen, is that the determinant of the difference of any two such matrices will live in $\mathbb{Z}[\imath]$ and therefore will have modulus bounded below by 1. Moreover, this will be true no matter how large a subset of $\mathbb{Z}[\imath]$ is used as the signal set. They called this last property the "nonvanishing determinant property", and they called the specific code they proposed the Golden Code. It was so named for the Golden Ratio that appears naturally: it is derived from the division algebra $(\mathbb{Q}(\imath, \sqrt{5})/\mathbb{Q}(\imath), \sigma, \imath)$. Here, σ is the automorphism of $K = \mathbb{Q}(\imath, \sqrt{5})$ that sends $\sqrt{5}$ to $-\sqrt{5}$ and acts as the identity on $\mathbb{Q}(\imath)$. A $\mathbb{Z}[\imath]$-basis for $\mathcal{O}_K$ is given by 1 and $\phi = \frac{1+\sqrt{5}}{2}$. Write ψ for $\sigma(\phi) = \frac{1-\sqrt{5}}{2}$. For a signal set $S \subset \mathbb{Z}[\imath] \subset \mathbb{Q}(\imath)$ (the most common kind of signal set), this code sends S^4 to $M_n(\mathbb{C})$ via the matrix

$$
(5)\quad \frac{1}{\sqrt{5}} \begin{pmatrix} s_{0,1}\alpha + s_{0,2}\alpha\phi & \imath(s_{1,1}\theta + s_{1,2}\theta\psi) \\ s_{1,1}\alpha + s_{1,2}\alpha\phi & s_{0,1}\theta + s_{0,2}\theta\psi \end{pmatrix}.
$$

Here, the $\frac{1}{\sqrt{5}}$ scale factor, $\alpha = 1 + \imath(1 - \phi)$, and $\theta = \sigma(\alpha) = 1 + \imath(1 - \psi)$ are used to shape the code (more on this later). Comparing with the matrix (4) above and ignoring the scale factor, we see that $k_0 = s_{0,1}\alpha + s_{0,2}\alpha\phi$ and $k_1 = s_{1,1}\alpha + s_{1,2}\alpha\phi$. Note that this code encodes four information symbols in each matrix. (A variant of this code, also based on the division algebra $(\mathbb{Q}(\imath, \sqrt{5})/\mathbb{Q}(\imath), \sigma, \imath)$, also incorporating the shaping criterion described later, is currently part of the IEEE 802.16e "WiMAX" standard. The Alamouti code based on the quaternions is also part of this standard.)

With the introduction of cyclic division algebras as a fundamental construction paradigm and with the use of codes constructed with entries from $\mathcal{O}_K$ for suitable extensions of $\mathbb{Q}(\imath)$, the subject of space-time coding took off. It is harmless and very often actually useful to assume that the signal set S is infinite: typically, S is assumed to be one of the standard lattices $\mathbb{Z}$, $\mathbb{Z}[\imath]$ or the Eisenstein lattice $\mathbb{Z}[\omega]$, where ω stands for the primitive third root of unity $\frac{-1+\sqrt{-3}}{2}$. (Under these assumptions the code forms an additive group, so one only needs to consider the rank of $X(s_1, \dots, s_k)$ and the modulus of the determinant $|\det \theta X(s_1, \dots, s_k)|$ in the rank criterion and the coding gain criterion.) Coding theorists immediately looked for specific constructions of division algebras of the form $(K/F, \sigma, \gamma)$ for the cases where $F = \mathbb{Q}$, $F = \mathbb{Q}(\imath)$, and $F = \mathbb{Q}(\sqrt{-3})$, corresponding to signal sets equaling one of the three lattices above. While such constructions have been known in principle to mathematicians working with division algebras, the coding theorists absorbed the necessary number-theoretic background in very short order and explicitly constructed division algebras over such fields for all indices n ([15] and [7]). (The hard task here is to select $\gamma \in \mathcal{O}_F$ so that it has the property that γ^i is not a norm from K to F for $i = 1, \dots, n-1$.) In all such cases, an $\mathcal{O}_F$-basis β_j of $\mathcal{O}_K$ is chosen, and each k_i is written as $\sum_{j=1}^n s_{i,j}\beta_j$ for $s_{i,j}$ in the signal set. Thus n^2 elements from the signal set are coded in each matrix, and by construction, the determinant of each matrix is nonzero and lies in one of the discrete lattices above. The modulus of the determinant will therefore be bounded below by the length of the shortest vector in the lattice so the code will have the nonvanishing determinant property.

Other Performance Measures

In parallel, as the subject became better understood, several additional performance criteria started to be imposed on codes. In a fundamental paper ([25]), Zheng and Tse provided a precise quantification of the trade-off (known as the diversity-multiplexing gain or "DMG" trade-off) between information rate and reliability. They defined numerical measures for each of the benefits and showed that the pair of benefits lie in a region of the first quadrant whose upper boundary is a piecewise linear concave up curve. In the paper [7], Vijay Kumar and his students showed that all codes constructed from cyclic division algebras with the additional nonvanishing determinant property will automatically perform at the upper boundary of this region and will hence be "DMG optimal". This of course further cemented the use of cyclic division algebras for code construction.

Another set of criteria was proposed by Oggier and coworkers in the paper [17]. One first rewrites the matrix (4) as a single $n^2 \times 1$ vector. When $k_i = \sum_{j=1}^n s_{i,j}\beta_j$ for $s_{i,j}$ in the signal set and β_j an $\mathcal{O}_F$ basis for $\mathcal{O}_K$, this $n^2 \times 1$ vector can be expressed as $M.\overline{v}$, where M is an $n^2 \times n^2$ matrix and $\overline{v}$ is the column vector $(s_{0,1}, s_{0,2}, \dots, s_{i,j}, \dots, s_{n-1,n})^T$. One now requires that the matrix M be *unitary* and that $|\gamma| = 1$. The first condition is called good shaping, and the idea behind it is that this forces the average energy needed to send the vector $\overline{v}$ without coding to be the same as that needed to send it in the coded matrix form (4). The condition $|\gamma| = 1$ causes the average energy transmitted per antenna to be equal for all transmissions. Oggier and coworkers called such codes "perfect" and constructed perfect codes for $n = 2, 3, 4$, and 6. This was followed by work of Elia and coworkers ([8]), who constructed perfect codes for all values of n and additionally showed that perfect codes satisfy other information-theoretic properties such as information-losslessness (a concept introduced by Damen and coworkers in [4]) and approximate universality (a concept introduced by Tavildar and Viswanath in [23]).

The mathematics needed for the work on perfect codes is quite interesting. Analyzing the condition that M be unitary, we find that it is sufficient to make the following matrix unitary:

$$U(\{\beta_1, \dots, \beta_n\}) = \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \sigma(\beta_1) & \cdots & \sigma(\beta_n) \\ & \vdots & \\ \sigma^{n-1}(\beta_1) & \cdots & \sigma^{n-1}(\beta_n) \end{bmatrix}.$$

Here, it is not necessary that the β_j be an $\mathcal{O}_F$ basis of $\mathcal{O}_K$; it is sufficient that they be an $\mathcal{O}_F$ linearly independent subset of $\mathcal{O}_K$. (So, for example, in the Golden Code (5) above, α is chosen so that with $\beta_1 = \alpha$ and $\beta_2 = \alpha\phi$, the matrix

$$\begin{pmatrix} \alpha & \alpha\phi \\ \theta & \theta\psi \end{pmatrix}$$

is unitary after being multiplied by the $\frac{1}{\sqrt{5}}$ scale factor.) So the question is: how to find $\mathcal{O}_F$ submodules of $\mathcal{O}_K$ that satisfy this unitary condition? For $n = 2^b$, it is easy to see that for the field $K = \mathbb{Q}(\zeta)$ and $F = \mathbb{Q}(\imath)$, where ζ is a primitive 2^{b+2}th root of unity, the various powers of ζ are $\mathbb{Z}[\imath]$-linearly independent and satisfy the unitary condition above. For odd n, Elia and coworkers use a construction due to B. Erez [9] that was needed in a different context: Erez was showing that for certain cyclic extensions $K/\mathbb{Q}$ with Galois group G, the square root of the inverse different is a free $\mathbb{Z}[G]$ module that has an orthogonal basis with respect to the usual trace form on K that sends x, y to $Tr_{K/\mathbb{Q}}(xy)$.

The most recent performance criteria for space-time codes, and in some sense the most mathematically exciting, have come from Lahtonen and coworkers ([13]). For the usual cases in which S is one of $\mathbb{Z}$, $\mathbb{Z}[\imath]$, $\mathbb{Z}[\omega]$, it is easy to see from the linearity of the code matrices X that on writing each X as an $n^2 \times 1$ vector as above and separating the real and imaginary parts, one gets a full lattice in $\mathbb{R}^{2n^2}$, i.e., the additive group generated by $2n^2$ linearly independent vectors in $\mathbb{R}^{2n^2}$. We refer to this lattice as the *code lattice*. After normalizing all code matrices so that $\inf_{X \in \mathcal{X}} |\det(X)| = 1$, they postulate that codes whose lattice points are the most dense in $\mathbb{R}^{2n^2}$ will have the best performance, and, indeed, they find this is borne out in several circumstances by simulations. To obtain a suitable numerical measure for the relative density, they invert the situation: they normalize the code lattice to have fundamental volume 1 instead. Thus they define the normalized minimum determinant of a code lattice Λ of rank $2n^2$ in a $\mathbb{Q}(\imath)$ division algebra of index n (embedded in $M_n(\mathbb{C})$) as the minimum of the moduli of the determinants $|\det(X(s_1, \dots, s_{n^2}))|$ as $X(s_1, \dots, s_{n^2})$ runs through the lattice, divided by the fundamental volume of Λ. Since a smaller fundamental volume represents a higher density, the goal is to construct codes whose code lattice Λ would maximize this ratio among all full lattices in the division algebra.

Recall that if D is a division algebra with center F and if R is a subring of F whose quotient field is F, then an R-order in D is a subring T of D containing R that is finitely generated as an R-module and satisfies $TF = D$. A maximal R-order is one that is maximal with respect to inclusion. In the typical situation where S is one of $\mathbb{Z}$, $\mathbb{Z}[\imath]$, or $\mathbb{Z}[\omega]$, so F is one of $\mathbb{Q}$, $\mathbb{Q}(\imath)$, or $\mathbb{Q}(\sqrt{-3})$, and where the k_i of the matrices in (4) are constrained to lie in $\mathcal{O}_K$ and $\gamma \in \mathcal{O}_F$, the code matrices of (4) naturally form an S-order. Thus the code matrices have a dual structure of an S-order and a full lattice in $\mathbb{R}^{2n^2}$. Lahtonen and coworkers investigate the interplay between these two structures. They ask: How will the code's performance as measured by its normalized minimum determinant vary if, in addition to carrying its natural structure of a full $\mathbb{Z}$-lattice in $\mathbb{R}^{2n^2}$, we choose our code to form an arbitrary S-order inside an F-division algebra? In these cases, the minimum modulus of the determinants of the code matrices is 1, so it follows from the definition of the normalized minimum determinant that the smaller the fundamental volume of the lattice the better the code. If T_1 and T_2 are S-orders and Λ_{T_1} and Λ_{T_2} the corresponding lattices with fundamental volumes V_{T_1} and V_{T_2}, then $T_1 \subseteq T_2$ implies $\Lambda_{T_1} \subseteq \Lambda_{T_2}$, which in turn means that $V_{T_2} \leq V_{T_1}$. It follows therefore that the best normalized minimum determinant will arise when a maximal order is used for the code. The authors then relate the fundamental volume of the code lattice to the $\mathbb{Z}$-discriminant of the maximal order and then invoke known formulas for discriminants of maximal orders to compute the best normalized minimum determinant of codes arising from $\mathcal{O}_F$ orders inside a given division algebra. In particular, they show (for the fields $\mathbb{Q}(\imath)$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}$) that the best division algebras to use will be ones that are ramified at precisely two of the "smallest" primes of the field (where the size of a prime $P =< \pi >$ is defined to be the modulus $|\pi|$). Thus, for $\mathbb{Q}(\imath)$ for example, one needs to transmit on a code arising from a maximal order inside a division algebra ramified only at $(1+\imath)$ and $(2+\imath)$ (or $(2-\imath)$). (Much of this was part of Vehkalahti's Ph.D. thesis.)

One of the drawbacks of using maximal orders is that the corresponding code lattice may not have good shape. Thus optimizing a code for minimum normalized determinant may destroy any optimization for shape. The recent work of Raj Kumar and Caire ([3]) proposes a very clever technique of mapping lattice points to certain *cosets* of a suitably chosen sublattice of a standard cubic lattice; this smooths out an irregular lattice and gives it better shape. In particular, their technique applies to codes from lattices from maximal orders and provides a further performance boost in such cases.

Key Challenge: Decoding

What are some of the key problems that need to be solved in space-time codes? Perhaps the biggest engineering challenge in the subject is the issue of decoding. The problem quite simply is the following: given the received vectors in Y (see Equation (1)), determine the entries of the matrix X that represent the original information. Assume that k symbols are coded in the matrix X and that the entries of X are linear in the signal entries $s_1, \ldots, s_k$ (typically arising from $\mathbb{Z}$, $\mathbb{Z}[\imath]$, or $\mathbb{Z}[\omega]$). By writing out sequentially the real and imaginary parts of each entry of Y, W, and $s_1, \ldots, s_k$, we may rewrite Equation (1) as $\widetilde{Y} = Z\overline{v} + \widetilde{W}$. Here Z is a $2n^2 \times 2k$ real matrix that depends on H, θ, and the parameters of the code matrix X, $\overline{v}$ is the signal vector $(x_1, y_1, \ldots, x_i, y_i, \ldots, x_k, y_k)^T$ with x_i and y_i being the real and imaginary parts of s_i, and similarly for $\widetilde{Y}$ and $\widetilde{W}$. If the columns of Z were orthonormal, decoding would be quite simple: we would have $Z^T\widetilde{Y} = \overline{v} + Z^T\widetilde{W}$ with $Z^T\widetilde{W}$ also having independent, identically distributed Gaussian entries. Hence, under maximum likelihood estimation, $\overline{v}$ can be taken to be the closest vector in S^k (viewed inside the Euclidean space $\mathbb{R}^{2k}$) to $Z^T\widetilde{Y}$. This is a very simple and computationally fast scheme: we march through $Z^T\widetilde{Y}$ component pair by component pair, and we find the element of the signal lattice S closest to that component pair.

The process above is called *single symbol decoding.* (For $k < n^2$ this is the same as orthogonal projection onto the subspace of $\mathbb{R}^{2n^2}$ determined by the columns of Z.) There are some nice situations in which the matrix Z is (essentially) orthogonal; this happens in the case of the Alamouti code, and more generally, in the codes satisfying Equation (3). The matrix Z for such codes satisfies $ZZ^T = \theta^2 Tr(HH^\dagger)I_{2k}$. We may divide the relation $\widetilde{Y} = Z\overline{v} + \widetilde{W}$ by $\theta\sqrt{Tr(HH^\dagger)}$. The entries of the new noise vector $1/(\theta\sqrt{Tr(HH^\dagger)})\widetilde{W}$ are still independent identically distributed Gaussian, while the columns of the matrix $1/(\theta\sqrt{Tr(HH^\dagger)})Z$ are now orthonormal. Thus single symbol decoding can be employed in all these cases.

But for other codes Z is rarely orthogonal! In general, given that the entries of W are independent identically distributed Gaussian, for maximum likelihood estimation one needs to search in S^{n^2} (viewed inside the Euclidean space $\mathbb{R}^{2n^2}$) for that vector $\overline{v} = (x_1, y_1, \ldots, x_{n^2}, y_{n^2})^T$ such that $Z\overline{v}$ is closest to $\widetilde{Y}$. (Here we will assume that $k = n^2$, as is usually the case for codes from cyclic division algebras.) This can no longer be accomplished symbol by symbol, and one needs to search in the full space S^{n^2} instead of just in S. There is an algorithm called the *sphere decoding algorithm* (see [5], for instance) that accomplishes this search in an intelligent manner, but as is to be expected of any search in S^{n^2}, even this algorithm gets very cumbersome once n exceeds 2. (One must keep in mind that the search for $\overline{v}$ such that $Z\overline{v}$ is closest to $\widetilde{Y}$ is essentially a closest lattice point search, and this is known to be NP-hard. What saves the day is that the received vectors $\widetilde{Y}$ are not random but have a Gaussian distribution about the lattice vectors $Z\overline{v}$. In [12], Hassibi and Vikalo show that under certain technical assumptions, the expected complexity of the sphere decoding algorithm is polynomial, although the worst-case complexity is exponential.)

Since Z is rarely orthogonal, we may ask whether we can take advantage of the obvious algebraic structure of the code and simplify the closest vector problem for our particular application. A very clever set of ideas of Luzzi et al. [16] does just that and gives an approximate solution to the decoding problem for the Golden Code (Equation (5)) by reducing the situation to the action of $SL_2(\mathbb{C})$ on three-dimensional hyperbolic space $\mathbb{H}^3$. Their work is a veritable tour de force of the application of abstract mathematics to engineering problems. Their goal is to approximate the channel matrix H (normalized to have determinant 1) by an element U of determinant 1 in the $\mathbb{Z}[\imath]$-order $R = (\mathcal{O}_K/\mathbb{Z}[\imath], \sigma, \imath)$. Writing $H = EU$ with E simply being the error HU^{-1}, they argue that choosing U so that the Frobenius norm of $E^{-1} = UH^{-1}$ is minimized approximates the original problem by the following: given a vector Y in $\mathbb{C}^{n^2}$ and an unknown vector S in $\mathbb{Z}[\imath]^{n^2}$ determine a "best" estimate of S if the difference vector $W = Y - S$ is known to be *approximately* independent identically distributed Gaussian (in a suitable sense). Given this assumption about the noise vector W, a reasonable way to proceed is to assume that W is actually independent identically distributed Gaussian. In this situation, the maximum-likelihood estimate of S is obtained by taking the ith entry of S to be the lattice point in $\mathbb{Z}[\imath]$ closest to the ith entry of Y. The authors find that their scheme gives a fast and acceptably accurate decoding.

What is fascinating is the mathematics behind their choice of U. First, they need to determine generators and relations for the group of norm 1 units $\mathcal{U}_1(R)$ of R (i.e., the set of multiplicatively invertible elements of R whose determinant as a code matrix is 1). In general, it is very difficult to find these for orders in division algebras, but in the case of certain special quaternion algebras over number fields, generators and relations for $\mathcal{U}_1(R)$ are known. Much of the idea behind this goes back to Poincaré. The norm 1 units in the order R above (modulo the subgroup $\{\pm 1\}$) turns out to be a Kleinian group, i.e., a discrete subgroup of the projective special linear group $PSL_2(\mathbb{C})$. As a subgroup of $PSL_2(\mathbb{C})$, $U_1(R)$ (modulo $\{\pm 1\}$) acts on the upper half-space model of hyperbolic 3-space $\mathbb{H}^3$ as a group of orientation-preserving isometries, and Poincaré's fundamental polyhedron theorem gives a set of generators and relations for such a group in terms of certain automorphisms of a fundamental domain for the group. Given a point P in $\mathbb{H}^3$, the Dirichlet polyhedron centered on P is the closure of the set of points x such that $d_H(x, P) < d_H(g(x), P)$ for all $g \in U_1(R)$ (modulo $\{\pm 1\}$), $g \neq 1$, where d_H is the hyperbolic metric on $\mathbb{H}^3$. The authors construct a Dirichlet polyhedron centered on $J = (0, 0, 1)$; this is a fundamental domain for $U_1(R)$. From this polyhedron, using Poincaré's theorem and a computer search, they determine a set of generators of $\mathcal{U}_1(R)$. They do this ahead of time and store the results. Next, in real time, given a fading matrix H (normalized to have determinant 1), they need to find an element U of $\mathcal{U}_1(R)$ such that the Frobenius norm of UH^{-1} is minimized. They observe that viewing UH^{-1} as an element of $PSL_2(\mathbb{C})$ acting on $\mathbb{H}^3$, the Frobenius norm of UH^{-1} is just 2 $cosh\ d_H(J, UH^{-1}(J))$, where J and d_H are as above. Since U is an isometry, they must find $U \in \mathcal{U}_1(R)$ that minimizes $cosh\ d_H(U^{-1}(J), H^{-1}(J))$. From the definition of Dirichlet polyhedra, this means that they need to find a Dirichlet polyhedron centered on some $U^{-1}(J)$ which contains $H^{-1}(J)$. They use the geometry of $\mathbb{H}^3$ relative to the action of $U_1(R)$ to find such a U: they just need to repeatedly consider the various Dirichlet polyhedra centered on J and the various $g_i(J)$, where the g_i run through the generators of $\mathcal{U}_1(R)$ that they have computed ahead of time, along with their inverses.

Role of Mathematicians

What is the role of mathematicians in this field? The subject is clearly very mathematical; yet, unlike classical coding theory, which now has a mathematical life of its own and can, for instance, be thought of as a theory of subspaces of vector spaces over finite fields, the center of gravity of space-time codes currently lies very solidly in engineering. There is as yet no deep independent "mathematics of space-time codes": the driving force behind the subject consists of fundamental engineering problems that need to be solved before MIMO wireless communication reaches its full practical potential, particularly for three or more antennas. This author therefore believes that, as things stand now, isolated mathematical investigations of space-time codes that are not grounded in concrete engineering questions would very likely lead to sterile results. At least for now, mathematicians can best contribute to the subject by working in collaboration with engineers who are motivated by fundamental engineering questions. This author has found that the leading engineers in the field already have a practical and intuitive understanding of much abstract mathematics but welcome help from trained mathematicians. (This author has also found that they are a genuine pleasure to collaborate with.) There is clearly a lot of work for mathematicians to do: particularly in decoding systems with large numbers of receive and transmit antennas, but also in other areas of MIMO communication that we have not touched upon in this article, such as cooperative communication in networks, or noncoherent communication, where the matrix H is not known to either the receiver or the transmitter.

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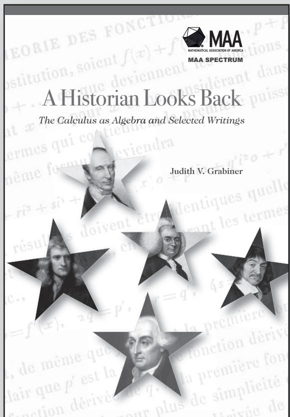

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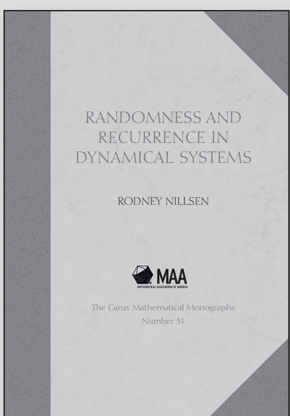

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WHAT IS...

a Mock Modular Form?

Amanda Folsom

The tale has been told and retold over time. The year: 1913. An unlikely correspondence begins between prominent number theorist G. H. Hardy and (then) poor Indian clerk S. Ramanujan. A mathematical collaboration between the two persists for the remainder of Ramanujan's lifetime: a mere seven years, until his untimely death at the age of thirty-two. New to the tale yet rooted in the Hardy-Ramanujan era is the modern notion of "mock modular form", not defined in the literature until nearly a century later, in 2007, by D. Zagier. Today, the story seems to hail from the distant past, but its mathematical harvest, its intrigue, has not let up. For example, the Hardy-Ramanujan story has been described recently as "one of the most romantic stories in the history of mathematics" by Zagier (2007), and decades prior to this, G. N. Watson describes Ramanujan's mathematics as inspiring in him a great "thrill" (1936). Far too many mathematicians to list, including G. Andrews, B. Berndt, K. Bringmann, K. Ono, H. Rademacher, and S. Zwegers, have perpetuated the legacy of Ramanujan's mathematics. Ramanujan's life story has even been dramatically reinterpreted in the 2007 work of fiction *The Indian Clerk* by David Leavitt.

An aspect of the allure to the Hardy-Ramanujan story is the spawn of a mathematical mystery surrounding the content of Ramanujan's deathbed letter to Hardy, his last. Watson, in his presidential address to the London Mathematical Society, prophetically declared the subject of the last letter, Ramanujan's "mock theta functions", to be "the final problem". The mathematical visions of Ramanujan were telling, particularly the seventeen peculiar functions of the last letter, which he dubbed "mock theta functions" and which appeared along with various properties and relations between them, yet with little to no explanation. (A handful of other such functions appear in Ramanujan's so called "lost notebook", unearthed by G. Andrews in 1976, and in work of G. N. Watson.) Curiously, the "mock theta functions" were reminiscent of modular forms, which, loosely speaking, are holomorphic functions on the upper half complex plane equipped with certain symmetries. In fact, Ramanujan used the term "theta function" to refer to what we call a modular form, so his choice of terminology "mock theta function" implies he thought of his functions as "fake"' or "pseudo" modular forms.

For example, one of Ramanujan's mock theta functions is given by $f(q) := \sum_{n\geq 0} q^{n^2}/(-q;q)_n^2$, where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. Modular forms by nature are also equipped with q-series expansions, where $q := e^{2\pi i\tau}$, τ is the variable in the upper half complex plane, and Dedekind's η-function, a well-known modular form, satisfies $q^{1/24}\eta^{-1}(\tau) = \sum_{n\geq 0} q^{n^2}/(q;q)_n^2$. These series expansions for $f(q)$ and $q^{1/24}\eta^{-1}(\tau)$ barely differ. In fact, Ramanujan observed other analytic properties shared by his mock theta functions and modular forms, providing a *description* of what he calls a "mock theta function"—and a notoriously vague *definition*.

Historically, one reason why the mock theta functions became an object of fascination for so many lies in the theory of integer partitions. For any natural number n, a partition of n is defined to be any nonincreasing sequence of positive integers whose sum is n. So, for example, there are three

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partitions of $3 : 1+1+1, 2+1$, and 3, and if $p(n)$ denotes the number of partitions of n, $p(3) = 3$. A useful tool for studying $p(n)$, as with many functions on $\mathbb{N}$, is its "generating function", which is of the form $P(x) := 1 + \sum_{n\geq 1} p(n)x^n$, where x is a variable. It is not difficult to see by a counting argument that $P(x) = \prod_{n\geq 1}(1-x^n)^{-1}$. Returning to modular forms, it is also well known that $\eta(\tau)$ has an infinite q-product expansion, and upon replacing x with $q := e^{2\pi i\tau}$ in $P(x)$, one finds the following relationship between the modular form $\eta(\tau)$ and the partition generating function $P(x)$: $P(q) = q^{1/24}/\eta(\tau)$.

Many combinatorial generating functions like $P(x)$ are related to modular forms with q-infinite product expansions in similar ways, and having such relationships often allows one to use the theory of modular forms to further explain various aspects of the combinatorial functions. A famous example of this is the work of Hardy, Ramanujan, and H. Rademacher, who first used the theory of modular forms to describe the asymptotic behavior of $p(n)$ as $n \to \infty$. The mock theta functions stood out in that they too seemed to be related to combinatorial functions. We now know that the mock theta function $f(q)$, for example, is related to F. Dyson's rank-generating function, where the rank of a partition is equal to its largest part minus the number of its parts.

By the time of the Ramanujan Centenary Conference in 1987, it had become clear that "the mock theta functions give us tantalizing hints of a grand synthesis still to be discovered," as Dyson said. "Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of the modular forms…This," he said, "remains a challenge for the future." Despite the volumes of literature produced by many famous mathematicians on the subject since the Hardy-Ramanujan era, the "what *is*" remained unanswered for eighty-two years.

Enter S. Zwegers, a 2002 doctoral student under D. Zagier, whose thesis finally provided long awaited explanations, one being: while the mock theta functions were indeed not modular, they could be "completed" (after multiplying by a suitable power of q) by adding a certain nonholomorphic component, and then packaged together to produce real analytic vector valued functions that exhibit appropriate modular behavior. This one-sentence description does not do justice to the scope of Zwegers's results, which are much broader and also realize the mock theta functions within other contexts.

Zwegers's breakthrough didn't simply put an end to the mystery surrounding the mock theta functions; it (fortunately) opened the door to many more unanswered questions! Notably, K. Bringmann, K. Ono, and collaborators developed an overarching theory of "weak Maass forms" (see the work of J. Bruinier and J. Funke for a debut appearance in the literature), a space of functions to which we now understand the mock theta functions belong. Weak Maass forms are nonholomorphic modular forms that are also eigenfunctions of a certain differential (Laplacian) operator. (For the reader familiar with usual Maass forms or the more modern "Langlands program", while there is a small intersection with the theory of weak Maass forms, the theories are distinct in that the required growth condition on weak Maass forms is relaxed, for example, hence the descriptor "weak".)

How do the mock theta functions fit into this framework of weak Maass forms and lead to the answer to the question "what *is* a mock modular form"? As alluded to above, to associate modular behavior to a given mock theta function $m(q)$, one first needs to 1) define a suitable multiple $h(q) := q^M m(q)$ for some $M \in \mathbb{Q}$, and 2) add to $h(q)$ an appropriate nonholomorphic function $g^*(\tau)$, constructed from a modular theta series $g(\tau)$, dubbed *the shadow* of h (after Zagier). The final object $\widehat{h}(\tau) := h(q) + g^*(\tau)$ is then a nonholomorphic modular form. (Implicit here is that one must also replace q by $e^{2\pi i\tau}, \tau \in \mathbb{H}$, in the function $h(q)$.) Thus, one gains modularity at the expense of holomorphicity: $h(q)$ is holomorphic but not modular, while $\widehat{h}(\tau)$ is modular but not holomorphic. In particular, the function $\widehat{h}(\tau)$ is a weak Maass form, whose holomorphic part is essentially the mock theta function $m(q)$.

Loosely speaking then, a "mock modular form" is a holomorphic part of a weak Maass form. One particularly beautiful example of a mock modular form due to Zagier is the generating function for Hurwitz class numbers of algebraic number theory, whose completion (associated weak Maass form) is the so-called Zagier-Eisenstein series, and whose shadow is given by the classical modular theta function $\sum_{n\in\mathbb{Z}} q^{n^2}$.

The ability to realize the mock theta functions within the theory of weak Maass forms has led to many important discoveries. One notable example towards Dyson's "challenge for the future" within partition theory is due to Bringmann and Ono, who show that generalized rank-generating functions (which include the mock theta function $f(q)$ as a special case) are mock modular forms. Their work not only led to deeper results in the theory of partitions by making use of the theory of weak Maass forms but also exhibited a new perspective on the roles played by modular forms in both theories.

To grasp a more precise formulation of mock modular forms, note that functions exhibiting modular behavior (said to satisfy modular "transformations") come equipped with an integer or half-integer "weight" k. One has two clear formulations of the space $\mathbb{M}_k$ of weight k mock modular forms due to Zagier. First, consider the space $\mathfrak{M}_k$ of real analytic functions that exhibit a modular transformation of weight k and satisfy suitable growth conditions. Understanding the roles played by the shadows g, and also the differential Laplacian operator, leads to the fact that the space $\widehat{\mathbb{M}}_k := \{F \in \mathfrak{M}_k \mid \partial_\tau (y^k \partial_{\overline{\tau}} F) = 0\}$ not only consists of weight k weak Maass forms (with special eigenvalue $\frac{k}{2}(1 - \frac{k}{2})$) but is also isomorphic to the space of mock modular forms $\mathbb{M}_k$ of weight k by mapping $h \in \mathbb{M}_k$ with shadow g to its completion $\widehat{h}! = h + g^* \in \widehat{\mathbb{M}}_k$. Second, one may also realize $\mathbb{M}_k$ via the exact sequence $0 \to M_k^! \to \mathbb{M}_k \xrightarrow{S} M_{2-k} \to 0$, where M_{2-k} is the space of holomorphic modular forms of weight $2-k$, $M_k^!$ is the space of weakly holomorphic modular forms of weight k (allowing additional poles at points called cusps) and $S(h) := g$ for the mock modular form h with shadow g. While this definition as stated is arguably the most natural, it may be of interest to consider other generalizations, some of which are currently being explored.

The tale of mock theta functions, mock modular forms, and weak Maass forms, while rooted in analytic number theory, has bled into many other areas of mathematics: sometimes mock modular forms are combinatorial generating functions, sometimes they answer questions about the nonvanishing of L-functions, sometimes they are related to class numbers, sometimes they are characters for affine Lie superalgebras, and sometimes they tell us about topological invariants—to name just a few of their many roles. Watson was right: "Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder…."

What *is* next?

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Every Waking Moment Ky Fan (1914–2010)

Bor-Luh Lin

Photographs courtesy of the author.

Ky Fan

Ky Fan passed away on March 22, 2010, at the age of ninety-five in Santa Barbara, California. He was born in Hangzhou, China, on September 19, 1914. He enrolled in National Peking University in 1932. Despite an interest in engineering, he pursued studies in mathematics due in part to the influence of his uncle, Zuxun Feng, who was chair of the Department of Mathematics at Peking University. As a junior in college, Fan was inspired by a visit of E. Sperner and translated into Chinese the book by O. Schreier and E. Sperner, *Einführung in die Analytische Geometrie und Algebra*. This translation, published in 1935 with the title *Analytical Geometry and Algebra I, II*, became a standard textbook in China. In fact, in 1953, almost twenty years after its initial publication, I used the same text as an undergraduate at National Taiwan University for a course on advanced geometry. This book so inspired me that I sought to join Fan at the University of Notre Dame to pursue a Ph.D. in mathematics. By the time Fan graduated from Peking University in 1936, he had also translated a book of Landau on number theory and ideal theory and had coauthored with a colleague a book on number theory. He was an instructor at Peking University in 1936–1939. In 1939 he was selected by the China-France Education Foundation to receive a Boxer Scholarship. This national competition provided one student with a chance to pursue a degree in mathematics in Europe. Working with Maurice Fréchet, he received his D.Sci. from the University of Paris in just two years, with a thesis with the title *Sur quelques notions fondamentales de l'analyse générale*. He was a French National Science Fellow at Centre National de la Recherche Scientifique in 1941–1942 and a member of the Institut Henri Poincaré in 1942–1945. By 1945 he had already published twenty-five papers on abstract analysis and topology, including the monograph *Introduction à la topologie combinatorire, I. Initiation* (Vubert, Paris), written with M. Fréchet.

Fan was at the Institute for Advanced Study at Princeton in 1945–1947. As an assistant of John von Neumann, and inspired by H. Weyl, he developed an interest in operator theory, matrix theory, minimax theory, and game theory. Later he extended his interests to systems of inequalities and fixed point theory. At that time many in these fields were studying finite games and optimizations in finite-dimensional spaces. Fan, however, pioneered the fields of infinite games and inequalities in infinite-dimensional linear spaces. He made fundamental and groundbreaking contributions and continued to be a leader in these areas throughout his career. He also made major contributions to the geometry of Banach spaces, convex analysis, combinatorial topology, topological groups, and analytical function theory.

Fan taught at the University of Notre Dame from 1947 to 1960. For many summers during this period, he was at the National Bureau of Standards, Oak Ridge National Laboratory, and Argonne National Laboratory. He taught at Wayne State University (1960–1961), Northwestern University (1961–1965), and the University of

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California at Santa Barbara starting in 1965. Upon his retirement from UC Santa Barbara in 1985, an international symposium was held to celebrate his many contributions to mathematics. The symposium resulted in the publication of a monograph with the title *Nonlinear and Convex Analysis: Proceedings In Honor of Ky Fan* (Lecture Notes in Pure and Applied Mathematics, Dekker, **107**(1987)). He was chair of the Department of Mathematics at UC Santa Barbara in 1968–1969. He held visiting positions at the University of Texas-Austin, Universität Hamburg, the Université Paris IX, and Università degli Studi di Perugia. He was elected a member of Academia Sinica in 1964 and served as the director of the Institute of Mathematics at the Academia Sinica from 1978–1984. Two issues of the *Bulletin of the Institute of Mathematics, Academia Sinica*—V. 2, No. 2, 1974, and V. 3, No. 1, 1975—were dedicated to him for his sixtieth birthday. In 1989 he was a Distinguished Visiting Scholar at the Chinese University in Hong Kong and received an Honorary Professorship from Peking University. After the visit, he donated his whole collection of mathematics books and treatises to Peking University. In 1990 he was awarded the degree Docteur Honoris Causa from the Université de Paris-Dauphine and was the featured speaker at the Conference on Directions in Matrix Theory (Fourth Auburn Linear Algebra Conference). His lecture at the Auburn conference, his publication lists up to 1992, a brief bibliography, and a list of his Ph.D. students appeared in *Linear Algebra and its Applications* 162–164:1-2(1992), 1–22. In 1993 the T.I.Tec/K.E.S. Conference on Nonlinear and Convex Analysis in Tokyo was dedicated to Fan in recognition of his fundamental contributions to the field. In 2011 the seventh International Conference on Nonlinear and Convex Analysis in Hirosaki will be dedicated to his memory. In 1994 an entire issue of *Topological Methods in Nonlinear Analysis* was dedicated to him for his eightieth birthday. He served on many editorial boards. He was a distinguished editor of *Linear Algebra and its Applications*, and he was one of the founding editors of the *Journal of Mathematical Analysis and Applications* and the *Journal of Nonlinear and Convex Analysis*. The latter will publish a special issue in memory of Fan.

In 1999 Ky Fan and his wife, Yu-Fen Fan, made a gift of approximately US$1 million to the American Mathematical Society. The fund was used to establish the Ky and Yu-Fen Fan Endowment to support and to foster collaborations between Chinese mathematicians and mathematicians in other parts of the world, especially North America, and to support mathematically talented high school students in the United States. The AMS uses the fund from the Fan Endowment to fund the China Exchange Program, which provides grants to Chinese mathematics departments to bring visitors from the rest of the world, as well as grants to North American departments to bring in visitors from China. The program also supports occasional conferences in China and improvement of library holdings in Chinese institutions. The Fan Endowment has also provided grants to assist programs in the United States that nurture mathematically talented high school students and has supported the Ky and Yu-Fen Fan Scholarships within the AMS Epsilon Scholarships Program for high school students. As noted by AMS past president Felix Browder, "The impact of Ky and Yu-Fen's generosity will be felt for years to come."

Fan published about 130 papers, many of which made fundamental contributions to several fields in pure and applied mathematics. From the famed Ky Fan inequalities (1951) to the Fan condition (1956), to the Ky Fan minimax inequality (1972) his papers shaped the fields of linear and nonlinear functional analysis, linear algebra, convex analysis, and optimization. Fan had a knack for identifying the central problems in a field of study and presenting them in the most general concise statements, with new approaches and elegant proofs. He brought seeming unrelated areas together to create new mathematics. As a result, his papers were often cited and pointed to new research directions. His contributions concerning fixed points and minimax inequalities have had a major impact in the development of nonlinear functional analysis. He made significant contributions to locations and identified maximum eigenvalues of matrices. His works found wide applications to mathematical economics, differential equations, potential theory, and numerical analysis. His research demonstrated the beauty of combining pure and applied mathematics. There are a large number of theorems, lemmas, inequalities, equalities, conditions, norms, etc., that bear the name of Ky Fan. The bibliography lists some major reference books that contain many of his contributions. We now briefly discuss the Ky Fan inequality, the Fan condition, and the Ky Fan minimax inequality.

Ky Fan, China, 1939.

Ky Fan Minimax Inequality (K. Fan, A minimax inequality and applications, *Inequalities III, Proc. of the Third Symposium on Inequalities*, Acad. Press (1972), 103–113).

> Let X be a nonempty compact convex set in a Hausdorff topological vector space E.

Suppose f is a real-valued function defined on $X \times X$ such that:

(a) For each fixed x in X, $f(x,y)$ is a lower semicontinuous function of y on X,

(b) For each fixed y in X, $f(x,y)$ is a quasi-concave function of x on X.

Then the minimax inequality

$$\min_{y\in X}\max_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x)$$

holds.

Applications of this inequality to fixed points, differential equations, and potential theory were given in the paper. It was later found to be equivalent to the Brouwer fixed point theorem but much more powerful and easier to use. Many important equilibrium theorems in mathematical economics follow quickly from the Ky Fan inequality.

Ky Fan Inequality (K. FAN, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Natl. Acad. Sci. USA*, 37 (1951), 760–766).

Let $A_1, A_2, \ldots, A_m$ be m ($m \ge 1$) completely continuous operators in Hilbert space H. For each $j = 1, 2, \ldots, m$, let $\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{ji}, \ldots$ be the eigenvalues of $A_j^* A_j$. Then for every positive integer n:

$$\max \left| \sum_{i=1}^{n} (U_1 A_1 \cdots U_m A_m x_i, x_i) \right|$$

$$= \sum_{i=1}^{n} (\lambda_{1i}\lambda_{2i} \cdots \lambda_{mi}),$$

$$\max \left| \det_{1\le i,k\le n} (U_1 A_1 \cdots U_m A_m x_i, x_k) \right|$$

$$= \prod_{i=1}^{n} (\lambda_{1i}\lambda_{2i} \cdots \lambda_{mi}),$$

where, for both maxima, $U_1, U_2, \ldots, U_m$ independently run over all unitary operators and $\{x_1, x_2, \ldots, x_n\}$ runs over all n orthonormal elements in H.

Let A, B be completely continuous operators in Hilbert space H. If $\{\lambda_i\}$, $\{\kappa_i\}$, and $\{\sigma_i\}$ are the eigenvalues of A^*A, B^*B, and $(A+B)^*(A+B)$ respectively, then

$$\Phi(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \ldots, \sqrt{\sigma_n})$$

$$\le \Phi\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right)$$

$$+\Phi(\sqrt{\kappa_1}, \sqrt{\kappa_2}, \ldots, \sqrt{\kappa_n})$$

holds for every symmetric gauge function Φ of any number n variables.

These inequalities generalized inequalities of von Neumann and Weyl and were listed by J. Dieudonné in *A Panorama of Pure Mathematics, as seen by N. Bourbaki*, Acad. Press (1982) as a major contribution to operator theory.

Fan Condition (K. FAN, *On systems of linear inequalities, linear inequalities and related systems*, Annals of Math. Studies, Princeton Univ. Press, vol. 38 (1956), 99–156).

Let X be a real linear space. For any linear functionals $f_i, i = 1, 2, \ldots, n$, and real numbers $c_i, i = 1, 2, \ldots, n$, there exists $x \in X$ such that $f_i(x) \ge c_i, i = 1, 2, \ldots, n$, if and only if for any nonnegative numbers $a_i, i = 1, 2, \ldots, n$, the relation $\sum_{i=1}^{n} a_i f_i = 0$ implies $\sum_{i=1}^{n} a_i c_i \le 0$.

The Fan condition has many applications and is a fundamental result in linear programming. The following is a nonlinear version of the Fan condition and is one of the major tools in nonlinear programming and convex analysis. It appeared in the paper "Systems of inequalities involving convex functions", by K. Fan, I. Glicksberg, and A. J. Hoffman, *Proc. Amer. Math. Soc.* **8** (1957), 617–622.

Let K be a nonempty convex set in a real vector space X. Let $f_i \colon K \to \mathbb{R}, i = 1, 2, \ldots, n$, be convex functions. Then $f_i(x) < 0, i = 1, 2, \ldots, n$, has a solution $x \in K$ if and only if there exist real numbers $\lambda_i \ge 0$, $i = 1, 2, \ldots, n$, not all zero, such that $\sum_{i=1}^{n} \lambda_i f_i(x) \ge 0$ for all $x \in K$.

Although Fan's contributions to mathematics are undisputed, I can think of no better way to honor his memory than to describe his role as a mentor and adviser. Fan was widely known to be a very rigorous teacher, one who expected the best of his students. Over his career, Fan had twenty-two Ph.D. students. When he left Notre Dame in 1960, John C. Cantwell, Ronald J. Knill, Robert E. Mullins, John O. Riedl Jr., and I followed him to Northwestern to complete our Ph.D. theses. The five of us were given thesis topics in five completely different areas. These areas were so diverse that we found no common ground among us to discuss the individual problems, and they were chosen so that none of the topics were closely related to any of Fan's published papers. He had decided the best way to train us was by pushing us to develop our own research directions rather than simply follow in his footsteps.

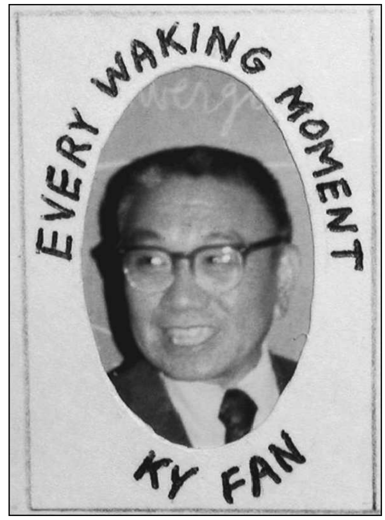
His style of teaching was unique, if occasionally intimidating. In his courses, we were given long lists of references to study. These references were in English, French, German, or Russian. The book of exercises he compiled to accompany these references was equally daunting. The content of his courses was usually about double that of a typical course. But, as with everything Fan did, the courses were beautifully organized. I still remember his lectures. He would start at the far corner of the blackboard and gradually cover the entire blackboard as the class went on. His lecture would always finish in the far right corner of the blackboard as class ended.

He demanded nothing less than complete dedication from his students. He considered his lectures to be performances and demanded the students' total attention. Those who neglected to take notes would be the first to be called upon in class, and he was not shy about letting the entire class know if any answer was stupid or irrelevant. It was not long before all of the students were assiduously taking notes and preparing answers to the many possible questions that could come at any moment.

He demanded that his graduate students think about mathematics all the time. In 1975 graduate students at UC Santa Barbara surprised him on his sixtieth birthday with a T-shirt imprinted with his photo, surrounding which, in large black letters, were the words, "EVERY WAKING MOMENT". Fan, of course, was smiling in the middle.

Dennis Wildfogel, a Ph.D. student of Fan at UC Santa Barbara in 1974, provides a story about Fan: "As many people know, Dr. Fan's doctoral adviser was the noted mathematician Maurice Fréchet, and Fréchet's adviser was the even more famous Jacques Hadamard. Fréchet died while I was working on my thesis. Shortly thereafter, someone pointed out to Fan that Hadamard had lived until age ninety-seven, Fréchet until ninety-four, and concluded that Fan would live to be ninety-one. Now, for most people in their late fifties (as Dr. Fan was at that time), being told you will live until age ninety-one would be good news. But Dr. Fan resented any limitation on his opportunity to do mathematics, so he replied testily, 'How do you know it's a straight line?!? Maybe it is a parabola."' For additional stories on Fan's teaching and mentoring, see the blog `drfantales.blogspot.com`, created by Wildfogel.

As a mathematician and teacher, he expected perfection and total devotion. As a mentor and as a person, he was a traditional Chinese scholar; unfailingly kind, courteous, generous, and humble. He went out of his way to help his students or anyone who was interested in mathematics. Even when he was in a wheelchair in the later part of his life, he would still respond to my letters with advice and encouragement. Any news I or other students had on progress in our studies in mathematics would make him happy. In 1994 he sent me a preprint of a paper to be published in *Proceedings of AMS*, with a note that this was his way of celebrating his eightieth birthday. He lived the way he taught his students: EVERY WAKING MOMENT was spent thinking about and working on mathematics. His love and dedication to mathematics continue to be a model for all of us, and his impact will be felt on the field of mathematics for generations to come.



T-shirt from graduate students, UC Santa Barbara, 1975.

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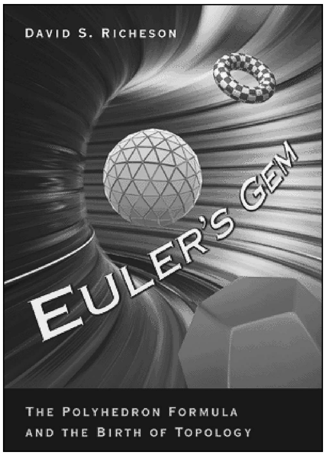
Book Review



Euler's Gem

Reviewed by Jeremy L. Martin

Euler's Gem
David S. Richeson
Princeton University Press, 2008
US$27.95, 332 pages
ISBN-13: 978-0691126777

When you admit to a stranger on an airplane that you are a professional mathematician, what happens next? "I never liked mathematics." "I'm no good with numbers." "I used to be good at math until I got to calculus." "Isn't math boring?" "What do you guys actually *do* all day?" Why not take such a response as an invitation to pull out a scratchpad and do a little teaching? One of my favorite topics is Euler's polyhedral formula: it is simple and elegant, it is not just about arithmetic or calculus, and it requires hardly any technical background to understand. Even strangers on airplanes are capable of looking at five or six examples, conjecturing that $V - E + F = 2$ for all polyhedra, and asking good questions: "But how can you prove that it's always true?" "Does it work if the polyhedron has a hole in it?" Admittedly, the scene doesn't always end this happily, but we mathematicians need to be able to communicate our discipline to strangers on airplanes, not to mention prospective students, deans, members of Congress, and small children.

David Richeson's *Euler's Gem* does an outstanding job of explaining serious mathematics to a general audience, and I plan to recommend it to the next stranger I meet on an airplane. The book is structured as a "tour guide" to the history of geometry and topology, revolving around Euler's formula and organized roughly chronologically: from the study of polyhedra in ancient Greek geometry to the discovery, proofs, and generalizations of Euler's formula in the seventeenth, eighteenth, and nineteenth centuries, to such diverse modern topics as knot theory, fixed-point theorems, curvature, the classification of surfaces, homology theory, and the Poincaré conjecture. The book is primarily intended for a lay audience, but there is also much of interest to professional mathematics students, teachers, and researchers. While a few of the book's generalizations about mathematical history and aesthetics are a bit simplistic or even one-sided, the wealth of clear and engaging exposition outweighs these occasional flaws.

The historical development of geometry, from its Greek roots to its modern form, is a recurring theme. An example is the discussion of the attempts of Lhuilier and others, in the early nineteenth century, to generalize Euler's formula to nonconvex polyhedra. Lhuilier's approach, incorporating contributions for the number of "tunnels", "cavities", and "inner polygons" (Richeson's terms), as well as vertices, edges, and faces, might seem a little misguided with the benefit of hindsight; wouldn't it be easier to phrase everything in terms of cell complexes and homology? Yes, it would, but in 1810 no one knew what a cell complex (or for that matter a manifold) was. The reader can see the origins of these modern ideas by comparing Lhuilier's work with that of Listing (whose "spatial complexes" Richeson describes briefly on pp. 249–250) and finally modern topology as developed by Poincaré. Even an expert should be able to benefit from seeing the evolution of "standard" mathematical definitions from simple and intuitive to complex and precise; this evolution is something not found in most graduate courses or textbooks.

Some of the generalizations about mathematics history may be oversimplifications: for instance,

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"Euler's predecessors were so focused on metric properties that they missed this fundamental interdependence. Not only did it not occur to them that they should count the features on a polyhedron, they did not even know which features to count" (p. 85). Kepler and Descartes did in fact count pieces of polyhedra, and their work is described elsewhere in the book. Kepler had observed the phenomenon of polar duality for regular solids and the fact that dualizing reverses the ordered triple (V, E, F), what a modern combinatorialist would call the f-vector. Meanwhile, a century before Euler, Descartes had observed[1] a formula closely related to Euler's: $P = 2F + 2V - 4$, where P is the number of plane angles. Each angle contains two edges, and each edge belongs to four angles (two on each end), so $P = 2E$, implying Euler's formula. Whether or not you think that Descartes deserves equal naming rights with Euler (a topic that has been debated), it seems clear that Kepler and Descartes were at least counting *something.* No doubt Euler's work was a major turning point and had far more direct impact than that of Descartes, but it is an overstatement to claim that Euler was the first to realize the applicability of counting in geometry. (On the other hand, Euler's original proof was indeed, as Richeson says, "a precursor to modern combinatorial proofs" (p. 67): calculate $V - E + F$ for a given polyhedron by slicing off a tetrahedron at a time and total the contributions from the individual tetrahedra. For those, like me, who think of Euler's formula as completely combinatorial, it is interesting to learn that the first rigorous proof, due to Legendre, is fundamentally geometric: project the polyhedron onto a sphere and then apply the Harriot-Girard theorem, which says that the area of a geodesic triangle on the unit sphere equals its angle sum minus π.)

The big historical picture may be slightly fuzzy, but the exposition of substantial mathematics is uniformly clear and concrete, with lots of pictures and examples, and sensibly organized. Several of the chapters stand on their own and would work well as self-contained reading assignments in a geometry course for mathematics majors or future secondary-school teachers. For example, the description of the classification of surfaces by their Euler characteristics and orientability (Chapters 16–17) is an absorbing, self-contained mathematical story, told at an appropriate level of technicality, in which terms such as "isomorphism invariant" and "orientable" receive clear, simple definitions that avoid unnecessary technicalities, without sacrificing accuracy. The theme of intrinsic versus extrinsic geometry (what properties of a curve or surface depend on how it is embedded in space?) is given the attention it deserves, leading into a chapter on the lovely subject of knot theory, with lots of pictures and explanations that make substantial mathematics (like the Seifert surface, an orientable surface in R^3 having a given knot as boundary) appealing and fun. If your seatmate complains that geometry is boring, here is an antidote.

Further geometry topics that receive excellent treatment include Descartes' theorem on solid angles of polyhedra and its continuous analogue, the Gauss-Bonnet theorem, which measures the total curvature of a surface in terms of its Euler characteristic. The presentation is clear and self-contained, requiring little more background than the fact that the sum of angles in an n-sided polygon is $(n - 2)\pi$—hardly too much to ask. The explanation of curvature is concrete; wisely, the calculus details are banished to footnotes. (My one small complaint: the biographical sketch of Gauss somewhat breaks up the flow of the mathematical story.) Richeson's explanation of homology (Chapter 23) was one of my favorite parts of the book, and I wish I had read it before taking algebraic topology as a graduate student—all those long exact sequences would have made a lot more sense if I had known what they were trying to measure.

Unfortunately, this outstanding section is followed by a mistaken explanation: "Kepler's observation [polar duality] is Poincaré duality in disguise. We are free to exchange the roles of i-dimensional and $(n - i)$-dimensional simplices." The intent is good, but the details are inaccurate: the f-vectors of two polar dual polytopes are the reverses of each other, whereas Poincaré duality says (among other things) that the Betti numbers of a *single* manifold form a palindrome. Although both statements are superficially concerned with symmetry, they are hardly the same thing in disguise. I cannot resist inserting a plug for combinatorics here: I would have liked to see a section about another relevant (and quite beautiful) duality for polytopes, namely the Dehn-Sommerville equations. Briefly, the f-vector of a simplicial polytope can be transformed into another invariant called the h-vector (for example, an octahedron has f-vector $(6, 12, 8)$ and h-vector $(1, 3, 3, 1)$), which carries the same information; the Dehn-Sommerville equations say that the h-vector is a palindrome. After describing homology and Poincaré duality, it would have been a natural next step to define the h-vector, to state the Dehn-Sommerville equations with an example or two, and perhaps to sketch the beautiful geometric proof by Bruggesser and Mani [1]. Perhaps this is something to look forward to in the second edition.

[1] *In a set of papers lost in a shipwreck after Descartes' death and not brought to public light until 1860—Richeson provides the juicy details in Chapter 9.*

I was disappointed by the book's discussion of the use of computers in mathematics, particularly Appel and Haken's 1976 proof of the four-color theorem (the first solution of a major open problem that relied on a computer to check a large finite number of cases):

> Although most people came to believe that [Appel and Haken's] proof was correct, most pure mathematicians found the proof inelegant, unsatisfying, and unsporting. It was as if Evel Knievel boasted that he could cross the Grand Canyon on his motorcycle, only to build a bridge and use it to make the crossing. Perhaps it is how mountain climbing purists feel about the use of bottled oxygen in high-altitude climbing. (p. 143)

This is unnecessarily dismissive and it neglects to present the other point of view: that Appel and Haken's work did us all a big favor by introducing a powerful new tool in doing mathematics and that whether it is "sporting" is a moot point, because mathematics is the richer for having any proof at all of the four-color theorem. The passage also pays scant attention to the fact that computer-aided proofs are much more widely accepted in the mathematical community today than they were in 1976. Later on the same page, we read,

> Perhaps some day someone will create a black box that proves theorems.... Some would say that this would take the fun out of mathematics and make it less beautiful.

Yet some would say the reverse: computers can help us discover and create beauty. Consider the development of automated summation techniques; they may take some of the fun out of proving hypergeometric identities, but being able to delegate such tasks to a computer frees up lots of mathematician-hours to do other things that a machine can't do. In addition, the mathematics underlying hypergeometric summation is itself quite beautiful, and it's hard to imagine any hypothetical black box being built without much more complex and beautiful mathematics (as a starting point, see the articles on formal proof in the October 2008 issue of the *Notices*). Describing the artistic and aesthetic sides of mathematics is a noble goal, but I am concerned that the quoted passages are counterproductive. We should portray ourselves not as purists who disdain the use of nontraditional tools but as scientists who are willing to be open to new methods.

It is easier to criticize a problematic sentence than to praise an entire well-written chapter. Overall, I found much more to like than to criticize in *Euler's Gem*. At its best, the book succeeds at showing the reader a lot of attractive mathematics with a well-chosen level of technical detail. I recommend it both to professional mathematicians and to their seatmates.

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What Does the Free Will Theorem Actually Prove?

Sheldon Goldstein, Daniel V. Tausk, Roderich Tumulka, and Nino Zanghì

Conway and Kochen have presented a "free will theorem" [4, 6] which they claim shows that "if indeed we humans have free will, then [so do] elementary particles." In a more precise fashion, they claim it shows that for certain quantum experiments in which the experimenters can choose between several options, no deterministic or stochastic model can account for the observed outcomes without violating a condition "MIN" motivated by relativistic symmetry. We point out that for stochastic models this conclusion is not correct, while for deterministic models it is not new.

In the way the free will theorem is formulated and proved, it only concerns deterministic models. But Conway and Kochen have argued [4, 5, 6, 7] that "randomness can't help," meaning that stochastic models are excluded as well if we insist on the conditions "SPIN", "TWIN", and "MIN". We point out a mistake in their argument. Namely, the theorem is of the form

(1) deterministic model with SPIN & TWIN & MIN $\Rightarrow$ contradiction,

and in order to derive the further claim, which is of the form

(2) stochastic model with SPIN & TWIN & MIN $\Rightarrow$ contradiction,

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Conway and Kochen propose a method for converting any stochastic model into a deterministic one [4]:

> let the stochastic element…be a sequence of random numbers (not all of which need be used by both particles). Although these might only be generated as needed, **it will plainly make no difference to let them be given in advance**. [emphasis added]

In this way, (2) would be a corollary of (1) if the conversion preserved the properties SPIN, TWIN, and MIN. However, Conway and Kochen have neglected to check whether they are preserved, and indeed, as we will show, the conversion preserves only SPIN and TWIN but not MIN. We do so by exhibiting a simple example of a stochastic model satisfying SPIN, TWIN, and MIN. As a consequence, no method of conversion of stochastic models into deterministic ones can preserve SPIN, TWIN, and MIN. More directly, our example shows that (2) is false. Contrary to the emphasized part of the above quotation, letting the randomness be given in advance makes a big difference for the purpose at hand.

The relevant details are as follows. The reasoning concerns a certain experiment in which, after a preparation procedure, two experimenters (A and B), located in space-time regions that are spacelike separated, can each choose between several options for running the experiment. We denote by $\mathfrak{a}$ (resp., by $\mathfrak{b}$) the choice of A (resp., of B) and by O_A (resp., O_B) the outcome of A (resp., of B). The data collected from this experiment can be represented by a joint probability distribution $\mathbb{P}_{\mathfrak{a}\mathfrak{b}}(O_A, O_B)$ for the outcomes (O_A, O_B) that depends on the choices $(\mathfrak{a}, \mathfrak{b})$. Experimenter A chooses $\mathfrak{a} = (x, y, z)$ from a certain set of forty orthonormal bases of $\mathbb{R}^3$, B chooses $\mathfrak{b} = w$ from a certain set of thirty-three unit vectors in $\mathbb{R}^3$. SPIN asserts that the outcome O_A obtained by A is always one of the triples 110, 101, or 011, and the outcome O_B obtained by B is either 0 or 1. TWIN asserts that whenever $w = x$ (resp., $w = y/w = z$), O_B coincides with the first (resp., second/third) digit of O_A. Quantum mechanics predicts data $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B)$ that satisfy SPIN and TWIN, given explicitly in Table 1.

$\mathbb{P}_{\mathfrak{ab}}$	$O_B = 0$	$O_B = 1$
$O_A = 011$	$\frac{1}{3}(w \cdot x)^2$	$\frac{1}{3}[1 - (w \cdot x)^2]$
$O_A = 101$	$\frac{1}{3}(w \cdot y)^2$	$\frac{1}{3}[1 - (w \cdot y)^2]$
$O_A = 110$	$\frac{1}{3}(w \cdot z)^2$	$\frac{1}{3}[1 - (w \cdot z)^2]$

Table 1. Joint probability distribution of outcomes as predicted by quantum mechanics, with $\cdot$ denoting the scalar product of vectors in $\mathbb{R}^3$.

A *stochastic model* for the data $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B)$ means, for the purpose at hand, a probability measure $\mathbb{P}^\Lambda$ (that does not depend on $\mathfrak{a}$ and $\mathfrak{b}$) on some measurable space Λ and, for each $\lambda \in \Lambda$ and $\mathfrak{a}$ and $\mathfrak{b}$, a probability measure $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda)$ on the set $\{110, 101, 011\} \times \{0, 1\}$ of possible outcomes such that, when λ is averaged over with $\mathbb{P}^\Lambda$, the data $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B)$ are obtained:

$$(3) \quad \mathbb{P}_{\mathfrak{ab}}(O_A, O_B) = \int_\Lambda \mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda)\, d\mathbb{P}^\Lambda(\lambda).$$

A *deterministic model* for the data $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B)$ is a stochastic model such that each $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda)$ is supported by a single outcome, i.e., one for which there are functions θ_A and θ_B such that:

$$(4) \quad \mathbb{P}_{\mathfrak{ab}}(O_A = \theta_A(\mathfrak{a}, \mathfrak{b}, \lambda),\ O_B = \theta_B(\mathfrak{a}, \mathfrak{b}, \lambda)|\lambda) = 1,$$

for all $\mathfrak{a}$, $\mathfrak{b}$, and λ.

The MIN condition is formulated in a somewhat vague way [6]:

> The MIN Axiom: Assume that the experiments performed by A and B are spacelike separated. Then experimenter B can freely choose any one of the thirty-three particular directions w, and $[O_A]$ is independent of this choice. Similarly and independently, A can freely choose any one of the forty triples x, y, z, and $[O_B]$ is independent of that choice.[1]

What does MIN mean for a deterministic model? According to Conway and Kochen [6]:

> It is possible to give a more precise form of MIN by replacing the phrase "$[O_B]$ is independent of A's choice" by "if $[O_A]$ is determined by B's choice, then its value does not vary with that choice."

[1] *Here and in the following quotation, we have adapted the notation by putting $[O_A]$ for "a's response" and $[O_B]$ for "b's response".*

That is, MIN asserts that the function θ_A does not depend on $\mathfrak{b}$ and the function θ_B does not depend on $\mathfrak{a}$:

$$(5) \quad \theta_A(\mathfrak{a}, \mathfrak{b}, \lambda) = \theta_A(\mathfrak{a}, \lambda), \quad \theta_B(\mathfrak{a}, \mathfrak{b}, \lambda) = \theta_B(\mathfrak{b}, \lambda).$$

What does MIN mean for a stochastic model? Conway and Kochen do not say precisely, as the above quotation deals only with the case of a deterministic model ("if $[O_A]$ is determined by B's choice"), but the most reasonable interpretation is a condition known as *parameter independence* [11, 12]: *for any given λ, the distribution of O_A does not depend on $\mathfrak{b}$, and the distribution of O_B does not depend on $\mathfrak{a}$*:

$$(6) \quad \mathbb{P}_{\mathfrak{ab}}(O_A|\lambda) = \mathbb{P}_{\mathfrak{a}}(O_A|\lambda), \quad \mathbb{P}_{\mathfrak{ab}}(O_B|\lambda) = \mathbb{P}_{\mathfrak{b}}(O_B|\lambda).$$

Note that for deterministic models (6) is the same as (5).

An example of a stochastic model satisfying SPIN, TWIN, and MIN (understood as (6)) is obtained from "rGRWf", the relativistic Ghirardi–Rimini–Weber theory with flash ontology [14, 15], but much simpler examples are possible. As a second example, one may simply take $(\Lambda, \mathbb{P}^\Lambda)$ to be the trivial probability space containing just one element (so that λ is a constant and can be ignored). Then, according to the definition of stochastic models, the data themselves form a stochastic model. That is, take $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda) = \mathbb{P}_{\mathfrak{ab}}(O_A, O_B)$ as given by Table 1. We know that this stochastic model satisfies SPIN and TWIN, and it also satisfies (6), since, for all $\mathfrak{a}$ and $\mathfrak{b}$, the marginal distribution of O_A is uniform and the marginal distribution of O_B gives probability 1/3 to 0 and 2/3 to 1. As a third (and even simpler) example, let us drop the requirement (3) that the stochastic model agrees with the data predicted by quantum mechanics and focus just on satisfying SPIN, TWIN, and MIN. Take $(\Lambda, \mathbb{P}^\lambda)$ to be trivial as before. If $\mathfrak{b} = w$ coincides with coordinate x (resp., y/z) of $\mathfrak{a}$, then let $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda)$ give probability 1/3 to each of $(110, 1)$, $(101, 1)$, $(011, 0)$ (resp., to each of $(110, 1)$, $(101, 0)$, $(011, 1)$/to each of $(110, 0)$, $(101, 1)$, $(011, 1)$) and probability zero to the other three possible values of (O_A, O_B). If w coincides with none of x, y, z, then let $\mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda)$ give probability 1/9 to each of $(110, 0)$, $(101, 0)$, $(011, 0)$ and probability 2/9 to each of $(110, 1)$, $(101, 1)$, $(011, 1)$. Then SPIN and TWIN are obviously true, and (6) is true because the marginal distributions of O_A and O_B are the same as in the previous example.

To illustrate explicitly why (6) breaks down when putting all randomness in the past, let us consider a specific conversion method of stochastic models into deterministic ones that Conway and Kochen have proposed [6] in response to earlier criticisms of their claims concerning the viability of rGRWf [15]:

we can easily deal with the dependence of the distribution of flashes on the external fields F_A [= $\mathfrak{a}$] and F_B [= $\mathfrak{b}$], which arise from the two experimenters' choices of directions x, y, z, and w. There are $40 \times 33 = 1320$ possible fields in question. For each such choice, we have a distribution $X(F_A, F_B)$ of flashes, i.e., we have different distributions $X_1, X_2, \ldots, X_{1320}$. Let us be given "in advance" all such random sequences, with their different weightings as determined by the different fields. Note that for this to be given, nature does not have to know in advance the actual free choices F_A (i.e., x, y, z) and F_B (i.e., w) of the experimenters. Once the choices are made, **nature need only refer to the relevant random sequence X_k in order to emit the flashes in accord with rGRWf**. [emphasis added]

The problem here is that the deterministic model obtained from this method of conversion manifestly violates MIN, because if nature were to follow the recipe suggested in the emphasized part of the quotation above, then she would have to use the value of $k = k(x, y, z, w)$ depending on both experimenters' choices, $\mathfrak{a} = (x, y, z)$, and $\mathfrak{b} = w$, in order to produce any of the outcomes O_A, O_B.

The conclusion that there are some predictions of quantum theory that cannot be obtained by a deterministic model satisfying parameter independence is not new. As noted by Jarrett in 1984 [11], for a stochastic model *Bell's locality condition* [1, 2]

$$\mathbb{P}_{\mathfrak{ab}}(O_A, O_B|\lambda) = \mathbb{P}_{\mathfrak{a}}(O_A|\lambda)\, \mathbb{P}_{\mathfrak{b}}(O_B|\lambda) \tag{7}$$

is (straightforwardly) equivalent to the conjunction of parameter independence (6) and another condition known as *outcome independence*,

$$\begin{aligned} \mathbb{P}_{\mathfrak{ab}}(O_A|O_B, \lambda) &= \mathbb{P}_{\mathfrak{ab}}(O_A|\lambda)\,, \\ \mathbb{P}_{\mathfrak{ab}}(O_B|O_A, \lambda) &= \mathbb{P}_{\mathfrak{ab}}(O_B|\lambda)\,. \end{aligned} \tag{8}$$

For a deterministic model, (8) is always trivially satisfied, as the distributions $\mathbb{P}_{\mathfrak{ab}}(O_A|\lambda)$ and $\mathbb{P}_{\mathfrak{ab}}(O_B|\lambda)$ each assign probability 1 to a single outcome, so any further information (such as the other outcome) is redundant. Thus, for a deterministic model, parameter independence is equivalent to locality, which Bell showed in 1964 [1] to be incompatible with some predictions of quantum mechanics. Therefore, deterministic models in agreement with quantum predictions must violate parameter independence. Even from the very same experiment as considered by Conway and Kochen, this conclusion was derived before in [13, 10, 3, 8] and [9, Section 4.2.1], in [9] using only SPIN and TWIN.

It has been suggested to one of us (R. T.) by Simon Kochen that our understanding of MIN is too weak, that MIN should be regarded as requiring that the actual outcome itself of A be independent of B's choice, and not just its probability distribution. We are unable to see why this is a reasonable requirement for a stochastic theory—or even what exactly it should mean. Be that as it may, the existence of the examples described here demonstrates that any such variant of MIN for a stochastic model would either be unreasonable (or worse) or would fail to be preserved under conversion of the model to a deterministic one.

Acknowledgments

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The Surprise Examination Paradox and the Second Incompleteness Theorem

Shira Kritchman and Ran Raz

Few theorems in the history of mathematics have inspired mathematicians and philosophers as much as Gödel's incompleteness theorems. The first incompleteness theorem states that, for any rich enough[1] consistent mathematical theory,[2] there exists a statement that cannot be proved or disproved within the theory. The second incompleteness theorem states that for any rich enough consistent mathematical theory, the consistency of the theory itself cannot be proved (or disproved) within the theory.

In this paper we give a new proof for Gödel's second incompleteness theorem, based on Kolmogorov complexity, Chaitin's incompleteness theorem, and an argument that resembles the surprise examination paradox.

We then go the other way around and suggest that the second incompleteness theorem gives a possible resolution of the surprise examination paradox. Roughly speaking, we argue that the flaw in the derivation of the paradox is that it contains a hidden assumption that one can prove the consistency of the mathematical theory in which the derivation is done; which is impossible by the second incompleteness theorem.

The First Incompleteness Theorem

Gödel's original proof for the first incompleteness theorem [Gödel31] is based on the *liar paradox*.

> **The liar paradox:** consider the statement "this statement is false." The statement can be neither true nor false.

Gödel considered the related statement "this statement has no proof." He showed that this statement can be expressed in any theory that is capable of expressing elementary arithmetic. If the statement has a proof, then it is false; but since in a consistent theory any statement that has a proof must be true, we conclude that if the theory is consistent, the statement has no proof. Since the statement has no proof, it is true (over $\mathbb{N}$). Thus, if the theory is consistent, we have an example for a true statement (over $\mathbb{N}$) that has no proof.

The main conceptual difficulty in Gödel's original proof is the self-reference of the statement "this statement has no proof." A conceptually simpler proof of the first incompleteness theorem, based on *Berry's paradox*, was given by Chaitin [Chaitin71].

> **Berry's paradox:** consider the expression "the smallest positive integer not definable in under eleven words." This expression defines that integer in under eleven words.

To formalize Berry's paradox, Chaitin uses the notion of *Kolmogorov complexity*. The Kolmogorov complexity $K(x)$ of an integer x is defined to be the length (in bits) of the shortest computer program that outputs x (and stops). Formally, to define $K(x)$ one has to fix a programming language, such as

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[1]*We require that the theory can express and prove basic arithmetical truths. In particular, ZFC and Peano arithmetic (PA) are rich enough.*

[2]*Here and below, we consider only first-order theories with recursively enumerable sets of axioms. For simplicity, let us assume that the set of axioms is computable.*

LISP, Pascal or C++. Alternatively, one can define $K(x)$ by considering any *universal Turing machine*.

Chaitin's incompleteness theorem states that for any rich enough consistent mathematical theory, there exists a (large enough) integer L (depending on the theory and on the programming language that is used to define Kolmogorov complexity) such that, for any integer x, the statement "$K(x) > L$" cannot be proved within the theory.

The proof given by Chaitin is as follows. Let L be a large enough integer. Assume for a contradiction that, for some integer x, there is a proof for the statement "$K(x) > L$". Let w be the first proof (say, according to the lexicographic order) for a statement of the form "$K(x) > L$". Let z be the integer x such that w proves "$K(x) > L$". It is easy to give a computer program that outputs z: the program enumerates all possible proofs w, one by one, and for the first w that proves a statement of the form "$K(x) > L$", the program outputs x and stops. The length of this program is a constant $+ \log L$. Thus, if L is large enough, the Kolmogorov complexity of z is less than L. Since w is a proof for "$K(z) > L$" (which is a false statement), we conclude that the theory is inconsistent.

Note that the number of computer programs of length L bits is at most 2^{L+1}. Hence, for any integer L, there exists an integer $0 \le x \le 2^{L+1}$, such that $K(x) > L$. Thus, for some integer x, the statement "$K(x) > L$" is a true statement (over $\mathbb{N}$) that has no proof.

A different proof for Gödel's first incompleteness theorem, also based on Berry's paradox, was given by Boolos [Boolos89] (see also [Vopenka66, Kikuchi94]). Other proofs for the first incompleteness theorem are also known (for a recent survey, see [Kotlarski04]).

The Second Incompleteness Theorem

The second incompleteness theorem follows directly from Gödel's original proof for the first incompleteness theorem. As described above, Gödel expressed the statement "this statement has no proof" and showed that, if the theory is consistent, this is a true statement (over $\mathbb{N}$) that has no proof. Informally, because the proof that this is a true statement can be obtained within any rich enough theory, such as Peano arithmetic (PA) or ZFC, if the consistency of the theory itself can also be proved within the theory, then the statement can be proved within the theory, which is a contradiction. Hence, if the theory is rich enough, the consistency of the theory cannot be proved within the theory.

Thus the second incompleteness theorem follows directly from Gödel's original proof for the first incompleteness theorem. However, the second incompleteness theorem doesn't follow from Chaitin's and Boolos's simpler proofs for the first incompleteness theorem. The problem is that these proofs only show the existence of a true statement (over $\mathbb{N}$) that has no proof, without giving an explicit example of such a statement.

A different proof for the second incompleteness theorem, based on Berry's paradox, was given by Kikuchi [Kikuchi97]. This proof is model-theoretic and seems to us somewhat less intuitive for people who are less familiar with model theory. For previous model-theoretic proofs for the second incompleteness theorem see [Kreisel50] (see also [Smoryński77]).

Our Approach

We give a new proof for the second incompleteness theorem, based on Chaitin's incompleteness theorem and an argument that resembles the *surprise examination paradox* (also known as the *unexpected hanging paradox*).

> **The surprise examination paradox:** the teacher announces in class: "next week you are going to have an exam, but you will not be able to know on which day of the week the exam is held until that day." The exam cannot be held on Friday, because otherwise, the night before the students will know that the exam is going to be held the next day. Hence, in the same way, the exam cannot be held on Thursday. In the same way, the exam cannot be held on any of the days of the week.

Let T be a (rich enough) mathematical theory, such as PA or ZFC. For simplicity, the reader can assume that T is ZFC, the theory of all mathematics; thus any mathematical proof, and in particular any proof in this paper, is obtained within T.

Let L be the integer guaranteed by Chaitin's incompleteness theorem. Thus, for any integer x, the statement "$K(x) > L$" cannot be proved (in the theory T) unless the theory is inconsistent. Note, however, that for any integer x such that $K(x) \le L$, there is a proof (in T) for the statement "$K(x) \le L$", simply by giving the computer program of length at most L that outputs x and stops and by describing the running of that computer program until it stops.

Let m be the number of integers $0 \le x \le 2^{L+1}$ such that $K(x) > L$. (The number m is analogous to the day of the week on which the exam is held in the surprise examination paradox.) Recall that because the number of computer programs of length L bits is at most 2^{L+1}, there exists at least one integer $0 \le x \le 2^{L+1}$ such that $K(x) > L$. Hence, $m \ge 1$.

Assume that $m = 1$. Thus there exists a single integer $x \in \{0, \ldots, 2^{L+1}\}$ such that $K(x) > L$, and every other integer $y \in \{0, \ldots, 2^{L+1}\}$ satisfies

$K(y) \leq L$. In this case, one can prove that x satisfies $K(x) > L$ by proving that every other integer $y \in \{0, \ldots, 2^{L+1}\}$ satisfies $K(y) \leq L$ (and recall that there is a proof for every such statement). Because we proved that $m \geq 1$, the only x for which we didn't prove $K(x) \leq L$ must satisfy $K(x) > L$.

Thus if $m = 1$, then for some integer x the statement "$K(x) > L$" can be proved (in T). But we know that for any integer x the statement "$K(x) > L$" cannot be proved (in T) unless the theory is inconsistent. Hence, if the theory is consistent, $m \geq 2$. As we assume that T is a rich enough theory, we can prove the last conclusion in T. That is, we can prove in T that if T is consistent, then $m \geq 2$.

Assume for a contradiction that the consistency of T can be proved within T. Thus we can prove in T the statement "$m \geq 2$". In the same way, we can work our way up and prove that $m \geq i + 1$, for every $i \leq 2^{L+1} + 1$. In particular, $m > 2^{L+1} + 1$, which is a contradiction, as $m \leq 2^{L+1} + 1$ (by the definition of m).

The Formal Proof

To present the proof formally, one needs to be able to express provability within T, in the language of T. The standard way of doing that is by assuming that the language of T contains the language of arithmetics and by encoding every formula and every proof in T by an integer, usually referred to as the Gödel number of that formula or proof. For a formula A, let $\ulcorner A \urcorner$ be its Gödel number. Let $\mathrm{Pr}_T(\ulcorner A \urcorner)$ be the following formula: *there exists w that is the Gödel number of a T-proof for the formula A*. Intuitively, $\mathrm{Pr}_T(\ulcorner A \urcorner)$ expresses the provability of the formula A. Formally, the formulas $\mathrm{Pr}_T(\ulcorner A \urcorner)$ satisfy the so-called Hilbert-Bernays derivability conditions (see, for example, [Mendelson97]):

1. If T proves A, then T proves $\mathrm{Pr}_T(\ulcorner A \urcorner)$.
2. T proves: $\mathrm{Pr}_T(\ulcorner A \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner A \urcorner) \urcorner)$.
3. T proves: $\mathrm{Pr}_T(\ulcorner A \to B \urcorner) \to (\mathrm{Pr}_T(\ulcorner A \urcorner) \to \mathrm{Pr}_T(\ulcorner B \urcorner))$.

The consistency of T is usually expressed as the formula $\mathrm{Con}(T) \equiv \neg\mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$. In all that comes below, $T \vdash A$ denotes "T proves A". We will prove that $T \nvdash \mathrm{Con}(T)$, unless T is inconsistent.

For our proof, we will need two facts about provability of claims concerning Kolmogorov complexity. First, we need to know that $\mathrm{Con}(T) \to \neg\mathrm{Pr}_T(\ulcorner K(x) > L \urcorner)$. We will use the following form of Chaitin's incompleteness theorem (see, for example, [Kikuchi97], Theorem 3.3).

$$(1) \qquad T \vdash \mathrm{Con}(T) \to \forall x \in \{0, \ldots, 2^{L+1}\} \; \neg\mathrm{Pr}_T(\ulcorner K(x) > L \urcorner).$$

Second, we need to know that $(K(y) \leq L) \to \mathrm{Pr}_T(\ulcorner K(y) \leq L \urcorner)$. We will use the following form (formally, this follows, as $K(y) \leq L$ is a Σ_1 formula; see, for example, [Kikuchi97], Theorem 1.2 and Section 2).

$$(2) \qquad T \vdash \forall y \in \{0, \ldots, 2^{L+1}\} \; ((K(y) \leq L) \to \mathrm{Pr}_T(\ulcorner K(y) \leq L \urcorner)).$$

Assume for a contradiction that T is consistent and $T \vdash \mathrm{Con}(T)$. Then, by Equation 1,

$$(3) \qquad T \vdash \forall x \in \{0, \ldots, 2^{L+1}\} \; \neg\mathrm{Pr}_T(\ulcorner K(x) > L \urcorner).$$

We will derive a contradiction by proving by induction that, for every $i \leq 2^{L+1} + 1$, $T \vdash (m \geq i + 1)$, where m is defined as in the previous section. Because, obviously, $T \vdash (m \leq 2^{L+1} + 1)$, this is a contradiction to the assumption that T is consistent and $T \vdash \mathrm{Con}(T)$. Because we already know that $T \vdash (m \geq 1)$, we already have the base case of the induction. Assume (the induction hypothesis) that for some $1 \leq i \leq 2^{L+1} + 1$,

$$T \vdash (m \geq i).$$

We will show that $T \vdash (m \geq i + 1)$ as follows. Let $r = 2^{L+1} + 1 - i$.

1. By the definition of m, $T \vdash (m = i) \to \exists$ different $y_1, \ldots, y_r \in \{0, \ldots, 2^{L+1}\} \bigwedge_{j=1}^{r} (K(y_j) \leq L)$.
2. Hence, by Equation 2, $T \vdash (m = i) \to \exists$ different $y_1, \ldots, y_r \in \{0, \ldots, 2^{L+1}\} \bigwedge_{j=1}^{r} \mathrm{Pr}_T(\ulcorner K(y_j) \leq L \urcorner)$.
3. For every different $y_1, \ldots, y_r \in \{0, \ldots, 2^{L+1}\}$, and every $x \in \{0, \ldots, 2^{L+1}\} \setminus \{y_1, \ldots, y_r\}$, $T \vdash (m \geq i) \to \left(\bigwedge_{j=1}^{r}(K(y_j) \leq L) \to (K(x) > L)\right)$, (by the definition of m), and hence by Hilbert-Bernays derivability conditions, $T \vdash \mathrm{Pr}_T(\ulcorner m \geq i \urcorner) \to \left(\bigwedge_{j=1}^{r} \mathrm{Pr}_T(\ulcorner K(y_j) \leq L \urcorner) \to \mathrm{Pr}_T(\ulcorner K(x) > L \urcorner)\right)$.
4. By the previous two items, $T \vdash ((m = i) \wedge \mathrm{Pr}_T(\ulcorner m \geq i \urcorner)) \to \exists x \in \{0, \ldots, 2^{L+1}\} \; \mathrm{Pr}_T(\ulcorner K(x) > L \urcorner)$.
5. As $T \vdash (m \geq i)$ (by the induction hypothesis), $T \vdash \mathrm{Pr}_T(\ulcorner m \geq i \urcorner)$. Hence, $T \vdash (m = i) \to \exists x \in \{0, \ldots, 2^{L+1}\} \; \mathrm{Pr}_T(\ulcorner K(x) > L \urcorner)$.
6. Hence, by Equation 3, $T \vdash \neg(m = i)$.
7. Hence, as $T \vdash (m \geq i)$, $T \vdash (m \geq i + 1)$.

□

A Possible Resolution of the Surprise Examination Paradox

In the previous sections we gave a proof for Gödel's second incompleteness theorem by an argument that resembles the surprise examination paradox. In this section we go the other way around and suggest that the second incompleteness theorem gives a possible resolution of the surprise examination paradox. Roughly speaking, we argue that the flaw in the derivation of the paradox is that it contains a hidden assumption that one can prove the consistency of the mathematical theory in which the derivation is done, which is impossible by the second incompleteness theorem.

The important step in analyzing the paradox is the translation of the teacher's announcement into a mathematical language. The key point lies in the formalization of the notions of surprise and knowledge.

As before, let T be a rich enough mathematical theory (say, ZFC). Let $\{1,\ldots,5\}$ be the days of the week and let m denote the day of the week on which the exam is held. Recall the teacher's announcement: "next week you are going to have an exam, but you will not be able to know on which day of the week the exam is held until that day." The first part of the announcement is formalized as $m \in \{1,\ldots,5\}$. A standard way that appears in the literature to formalize the second part is by replacing the notion of **knowledge** by the notion of **provability** [Shaw58, Fitch64] (for a recent survey see [Chow98]). The second part is rephrased as "on the night before the exam you will not be able to prove, using this statement, that the exam is tomorrow," or, equivalently, "for every $1 \le i \le 5$, if you are able to prove, using this statement, that $(m \ge i) \to (m = i)$, then $m \ne i$." This can be formalized as the following statement that we denote by S (the statement S contains both parts of the teacher's announcement):

$$S \equiv [m \in \{1,\ldots,5\}] \bigwedge_{1\le i\le 5} \left[\Pr_{T,S}(\ulcorner m \ge i \to m = i \urcorner) \to (m \ne i)\right],$$

where $\Pr_{T,S}(\ulcorner A \urcorner)$ expresses the provability of a formula A from the formula S in the theory T (formally, $\Pr_{T,S}(\ulcorner A \urcorner)$ is the formula: *there exists w that is the Gödel number of a T-proof for the formula A from the formula S*). Note that the formula S is self-referential. Nevertheless, it is well known that this is not a real problem and that such a formula S can be formulated (see [Shaw58, Chow98]; for more about this issue, see below).

Let us try to analyze the paradox when the teacher's announcement is formalized as the above statement S. We will start from the last day. The statement $m \ge 5$ together with $m \in \{1,\ldots,5\}$ imply $m = 5$. Hence, $\Pr_{T,S}(\ulcorner m \ge 5 \to m = 5 \urcorner)$, and by S we can conclude $m \ne 5$. Thus S implies $m \in \{1,\ldots,4\}$. In the same way, working our way down, we can prove $\Pr_{T,S}(\ulcorner m \ge 4 \to m = 4 \urcorner)$, and by S we can conclude $m \ne 4$. In the same way, $m \ne 3$, $m \ne 2$, and $m \ne 1$. In other words, S implies $m \notin \{1,\ldots,5\}$. Thus S contradicts itself.

The fact that S contradicts itself gives a certain explanation for the paradox; the teacher's announcement is just a contradiction. On the other hand, we feel that this formulation doesn't fully explain the paradox: Note that, because S is a contradiction, it can be used to prove any statement. So, for example, on Tuesday night the students can use S to prove that the exam will be held on Wednesday. Is it fair to say that this means that they **know** that the exam will be held on Wednesday? No, because they can also use S to prove that the exam will be held on Thursday. Thus we conclude that, since S is a contradiction, **provability** from S doesn't imply **knowledge**. Recall, however, that the very intuition behind the formalization of the teacher's announcement as S was that the notion of *knowledge* can be replaced by the notion of *provability*. But if provability from S doesn't imply knowledge, the statement S doesn't seem to be an accurate translation of the teacher's announcement into a mathematical language.

Is there a better way to formalize the teacher's announcement? To answer this question, let us analyze the situation from the students' point of view on Tuesday night. There are three possibilities:

1. On Tuesday night, the students are not able to prove that the exam will be held on Wednesday.
2. On Tuesday night, the students are able to prove that the exam will be held on Wednesday, but they are also able to prove for some other day that the exam will be held on that day.
   (Note that this possibility can only occur if the system is inconsistent and is in fact equivalent to the inconsistency of the system).
3. On Tuesday night, the students are able to prove that the exam will be held on Wednesday, and they are not able to prove for any other day that the exam will be held on that day.

We feel that only in the third case is it fair to say that the students know that the exam will be held on Wednesday. They know that the exam will be held on Wednesday only if they are able to prove that the exam will be held on Wednesday, and they are not able to prove for any other day that the exam will be held on that day.

We hence rephrase the second part of the teacher's announcement as "for every $1 \le i \le 5$, if one can prove (using this statement) that $(m \ge i) \to (m = i)$, and there is no $j \ne i$ for which one can prove (using this statement) $(m \ge i) \to (m = j)$, then $m \ne i$". Thus the teacher's announcement is the following statement:[3]

$$S \equiv [m \in \{1,\ldots,5\}] \bigwedge_{1\le i\le 5} \left[\left(\Pr_{T,S}(\ulcorner m \ge i \to m = i \urcorner) \bigwedge_{1\le j\le 5, j\ne i} \neg\Pr_{T,S}(\ulcorner m \ge i \to m = j \urcorner)\right) \to (m \ne i)\right].$$

Let us try to analyze the paradox when the teacher's announcement is formalized as

[3] *This statement is equivalent to one of the suggestions (the statement I_5) made by Halpern and Moses* [HM86]. *However, the analysis of the paradox there is different from the one shown here and makes no use of Gödel's second incompleteness theorem.*

the new statement S. As before, $m \geq 5$ together with $m \in \{1, \ldots, 5\}$ imply $m = 5$. Hence, $\mathrm{Pr}_{T,S}(\ulcorner m \geq 5 \to m = 5 \urcorner)$. However, this time one cannot use S to conclude $m \neq 5$, as it is possible that for some $j \neq 5$ we also have $\mathrm{Pr}_{T,S}(\ulcorner m \geq 5 \to m = j \urcorner)$. This happens iff the system $T + S$ is inconsistent. Formally, this time one cannot use S to deduce $m \neq 5$, but rather the formula

$$\mathrm{Con}(T, S) \to (m \neq 5),$$

where $\mathrm{Con}(T, S) \equiv \neg\mathrm{Pr}_{T,S}(\ulcorner 0 = 1 \urcorner)$ expresses the consistency of $T + S$. Because by the second incompleteness theorem one cannot prove $\mathrm{Con}(T, S)$ within $T + S$, we cannot conclude that S implies $m \neq 5$ and hence cannot continue the argument.

More precisely, because S doesn't imply $m \in \{1, \ldots, 4\}$, but rather $\mathrm{Con}(T, S) \to m \in \{1, \ldots, 4\}$, when we try to work our way down we do not get the desired formula $\mathrm{Pr}_{T,S}(\ulcorner m \geq 4 \to m = 4 \urcorner)$ but rather the formula

$$\mathrm{Pr}_{T,S}(\ulcorner \mathrm{Con}(T, S) \wedge (m \geq 4) \to m = 4 \urcorner),$$

which is not enough to continue the argument.

Thus our conclusion is that if the students believe in the consistency of $T + S$, the exam cannot be held on Friday, because on Thursday night the students will know that if $T + S$ is consistent the exam will be held on Friday. However, the exam can be held on any other day of the week because the students cannot prove the consistency of $T + S$.

Finally, for completeness, let us address the issue of the self-reference of the statement S. The issue of self-referentiality of a statement goes back to Gödel's original proof for the first incompleteness theorem. The self-reference is what makes Gödel's original proof conceptually difficult and what makes the teacher's announcement in the surprise examination paradox paradoxical.

To solve this issue, Gödel introduced the technique of diagonalization. The same technique can be used here. To formalize S, we will use the notation $a \Rightarrow b$ to indicate implication between Gödel numbers a and b. That is, $a \Rightarrow b$ is a statement indicating that a is a Gödel number of a statement A and b is a Gödel number of a statement B, such that $A \to B$. We will also need the function $\mathrm{Sub}(a, b)$ that represents substitution of b in the formula with Gödel number a. That is, if a is a Gödel number of a formula $A(x)$ with free variable x and b is a number, then $\mathrm{Sub}(a, b)$ is the Gödel number of the statement $A(b)$.

Let $\nu_{ij} \equiv \ulcorner m \geq i \to m = j \urcorner$. Denote by $Q(x)$ the formula

$$Q(x) \equiv [m \in \{1, \ldots, 5\}] \bigwedge_{1 \leq i \leq 5} \left[\left(\mathrm{Pr}_T(\mathrm{Sub}(x, x) \Rightarrow \nu_{ii}) \bigwedge_{1 \leq j \leq 5, j \neq i} \neg\mathrm{Pr}_T(\mathrm{Sub}(x, x) \Rightarrow \nu_{ij}) \right) \to (m \neq i) \right].$$

Let q be the Gödel number of the formula $Q(x)$. The statement S is formalized as $S \equiv Q(q)$. To see that this statement is the one that we are interested in, denote by s the Gödel number of S and note that $s = \mathrm{Sub}(q, q)$. Thus

$$S \equiv [m \in \{1, \ldots, 5\}] \bigwedge_{1 \leq i \leq 5} \left[\left(\mathrm{Pr}_T(s \Rightarrow \nu_{ii}) \bigwedge_{1 \leq j \leq 5, j \neq i} \neg\mathrm{Pr}_T(s \Rightarrow \nu_{ij}) \right) \to (m \neq i) \right].$$

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Fields Medals Awarded

On August 19, 2010, four Fields Medals were awarded at the opening ceremonies of the International Congress of Mathematicians (ICM) in Hyderabad, India. The medalists are ELON LINDENSTRAUSS, NGÔ BAO CHÂU, STANISLAV SMIRNOV, and CÉDRIC VILLANI.

The Fields Medals are given every four years by the International Mathematical Union (IMU). Although there is no formal age limit for recipients, the medals have traditionally been presented to mathematicians under forty years of age as an encouragement to future achievement. The medal is named after the Canadian mathematician John Charles Fields (1863–1932), who organized the 1924 ICM in Toronto. At a 1931 meeting of the Committee of the International Congress, chaired by Fields, it was decided that funds left over from the Toronto ICM "should be set apart for two medals to be awarded in connection with successive International Mathematical Congresses." In outlining the rules for awarding the medals, Fields specified that the medals "should be of a character as purely international and impersonal as possible." During the 1960s, in light of the great expansion of mathematics research, the possible number of medals to be awarded was increased from two to four. Today the Fields Medal is recognized as the world's highest honor in mathematics.

Previous recipients of the Fields Medal are: Lars V. Ahlfors and Jesse Douglas (1936); Laurent Schwartz and Atle Selberg (1950); Kunihiko Kodaira and Jean-Pierre Serre (1954); Klaus Roth and René Thom (1958); Lars Hörmander and John W. Milnor (1962); Michael F. Atiyah, Paul J. Cohen, Alexander Grothendieck, and Stephen Smale (1966); Alan Baker, Heisuke Hironaka, Sergei Novikov, and John G. Thompson (1970); Enrico Bombieri and David Mumford (1974); Pierre R. Deligne, Charles Fefferman, Grigorii A. Margulis, and Daniel G. Quillen (1978); Alain Connes, William P. Thurston, and Shing-Tung Yau (1982); Simon K. Donaldson, Gerd Faltings, and Michael H. Freedman (1986); Vladimir Drinfel'd, Vaughan F. R. Jones, Shigefumi Mori, and Edward Witten (1990); Jean Bourgain, Pierre-Louis Lions, Jean-Christophe Yoccoz, and Efim Zelmanov (1994); Richard Borcherds, William Timothy Gowers, Maxim Kontsevich, and Curtis T. McMullen (1998); Laurent Lafforgue and Vladimir Voevodsky (2002); and Andrei Okounkov, Grigori Perelman, (medal declined), Terence Tao, and Wendelin Werner (2006).

Elon Lindenstrauss

Citation: "For his results on measure rigidity in ergodic theory and their applications to number theory."

Elon Lindenstrauss

Photographs courtesy of ICM/IMU.

Elon Lindenstrauss has developed extraordinarily powerful theoretical tools in ergodic theory, a field of mathematics initially developed to understand celestial mechanics. He then used them, together with his deep understanding of ergodic theory, to solve a series of striking problems in areas of mathematics that are seemingly far afield. His methods are expected to continue to yield rich insights throughout mathematics for decades to come.

Ergodic theory studies dynamical systems, which are simply mathematical rules that describe how a system changes over time. So, for example, a dynamical system might describe a billiard ball ricocheting around a frictionless, pocketless billiard table. The ball will travel in a straight line until it hits the side of the table, which it will bounce off of as if from a mirror. If the table is rectangular, this dynamical system is pretty simple and predictable, because a ball sent in any direction will end up bouncing off each of the four walls at a consistent angle. But suppose, on the other hand, that the billiard table has rounded ends like a stadium. In that case, a ball from almost any starting position headed in almost any direction will shoot all over the entire stadium at endlessly varying angles. Systems with this kind of complicated behavior are called "ergodic".

The way that mathematicians pin down this notion that the trajectories spread out all over the space is through the notion of "measure invariance". A measure can be thought of as a more flexible way to compute area, and having an invariant measure essentially ensures that if two regions of the space in some sense have equal areas, points will travel into them the same percentage of the time. By contrast, in the rectangular table (which of course is not ergodic), the center will get very little traffic in most directions.

In many dynamical systems, there is more than one invariant measure, that is, more than one way of computing area for which almost all the trajectories will go into equal areas equally often. In fact, there are often infinitely many invariant measures. What Lindenstrauss showed, however, is that in certain circumstances, there can be only a very few invariant measures. This turns out to be an extremely powerful tool, a kind of hammer that can break hard problems open.

Lindenstrauss then adroitly wielded his hammer to crack some hard problems indeed. One example of this is in an area called "Diophantine approximations", which is about finding rational numbers that are usefully close to irrational ones. Pi, for example, can be approximated pretty well as 22/7. The rational number 179/57 is a bit closer, but because its denominator is so much larger, it's not as convenient an approximation. In the early nineteenth century, the German mathematician Johann Dirichlet proposed one possible standard for judging the quality of an approximation: The imprecision of a rational approximation p/q should be less than $1/(q^2)$. He then went on to show, in a not very difficult proof, that there are infinitely many approximations to any irrational number that meet this standard. (To put this in formula form, he showed that for any real number α, there are infinitely many integers p and q such that $|\alpha - p/q| < 1/q^2$.)

Eighty years ago the British mathematician John Edensor Littlewood proposed an analogue to Dirichlet's statement to approximate two irrational numbers at once: It should be possible, he figured, to find approximations p/q to α and r/q to β so that the product of the imprecision of the two approximations would be as small as you please. (In formula form, the claim is that for any real numbers α and β and any tiny positive quantity ε you like, there will be approximations p/q to α and r/q to β so that $|\alpha - p/q| \times |\beta - r/q| < \varepsilon/q^3$.) He gave the problem to his graduate students, thinking it should not be that much harder than Dirichlet's proof. But the Littlewood conjecture turned out to be extraordinarily difficult, and until recently, no substantial progress had been made on it.

Then Lindenstrauss brought his ergodic theory tools to the problem, in joint work with Manfred Einsiedler and Anatole Katok. Ergodic theory might seem an odd choice for a problem that does not involve dynamical systems or time, but such unlikely pairings are sometimes the most powerful. Here's one way of reformulating Littlewood's problem to see a connection: First imagine a unit square, and glue the top edge to the bottom edge to make a cylinder. Now glue the right edge to the left edge and you'll get a shape called a torus that looks like a donut. You can roll up the entire coordinate plane to this same shape by gluing any point (x, y) to the point whose x-coordinate is the fractional part of x and whose y-coordinate is the fractional part of y. This torus is the space of our dynamical system. We can then define a transformation by taking any point (x, y) to another point $(x+\alpha, y+\beta)$. If α and β are irrational (or more precisely, not rationally related), this dynamical system will be ergodic. The Littlewood conjecture then becomes the claim that you can make these trajectories suitably close to the origin by applying the transformation enough times. The number of times you apply the transformation becomes the denominator of the fractions approximating α and β.

Using a reformulation of the Littlewood conjecture in terms of a more complex dynamical system, the team made a huge step of progress on the conjecture: They showed that if there are any pairs of numbers for which the conjecture is false, there are only a very few of them, a negligible portion of them all.

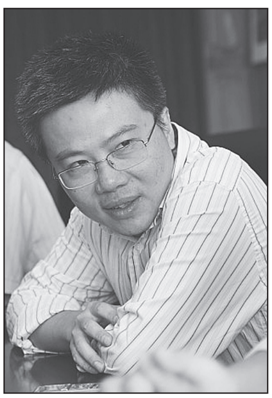
Another example of the power of Lindenstrauss's work is his proof of the first nontrivial case of the arithmetic quantum unique ergodicity conjecture. Ergodic systems come up frequently in physics, because as soon as you have three bodies interacting, for example, the system starts to behave in a somewhat ergodic fashion. But if those interactions happen at the quantum scale, you cannot describe them with the ordinary tools of ergodic theory, because quantum theory does not allow for well-defined paths of points at well-defined positions; instead, you can only consider the probability that a point will exist in a particular position at a particular time. Analyzing such systems mathematically has proven extraordinarily difficult, and physicists have had to rely on numerical simulations alone, without a firm mathematical underpinning.

The quantum unique ergodicity conjecture says, roughly, that if you calculate area using the measure that is natural in classical dynamics, then as the energy of the system goes up, this probability distribution becomes more evenly distributed over the space. Furthermore, this measure is the only one for which that is true. Lindenstrauss was able to prove this in an arithmetic context for particular kinds of dynamical systems, creating one of the first major, rigorous advances in the theory of quantum chaos.

Elon Lindenstrauss was born in 1970 in Jerusalem. He received his Ph.D. in mathematics from the Hebrew University of Jerusalem in 1999. He is professor of mathematics at Hebrew University and at Princeton University. He has been a member of the Institute for Advanced Study, Princeton; a Clay Mathematics Institute Long-Term Prize Fellow; a visiting member of the Courant Institute of Mathematical Sciences at New York University; and an assistant professor at Stanford University. His distinctions include the Leonard M. and Eleanor B. Blumenthal Award for the Advancement of Research in Pure Mathematics (2001), the 2003 Salem Prize, the European Mathematical Society Prize (2004), and the Anna and Lajos Erdős Prize in Mathematics (2009).

Ngô Bao Châu

Citation: "For his proof of the fundamental lemma in the theory of automorphic forms through the introduction of new algebro-geometric methods."



Ngô Bao Châu

Ngô Bao Châu removed one of the great impediments to a grand, decades-long program to uncover hidden connections between seemingly disparate areas of mathematics. In doing so, he provided a solid foundation for a large body of theory and developed techniques that are likely to unleash a flood of new results.

The path to Ngô's achievement began in 1967, when the mathematician Robert Langlands had a mind-boggling bold vision of a sort of mathematical wormhole connecting fields that seemed to be light years apart. His proposal was so ambitious and unlikely that when he first wrote of it to the great number theorist André Weil, he began with this sheepish note: "If you are willing to read [my letter] as pure speculation I would appreciate that; if not—I am sure you have a wastebasket handy." Langlands then laid out a series of dazzling conjectures that have proven to be a road map for a large area of research ever since.

The great majority of those conjectures remain unproven and are expected to occupy mathematicians for generations to come. Even so, the progress on the program so far has been a powerful engine for new mathematical results, including Andrew Wiles's proof of Fermat's Last Theorem and Richard Taylor's proof of the Sato-Tate conjecture. The full realization of Langlands's program would unify many of the fields of modern mathematics, including number theory, group theory, representation theory, and algebraic geometry.

Langlands's vision was of a bridge across a division in mathematics dating all the way back to Euclid's time, that between magnitude and multitude. Magnitudes are the mathematical form of butter, a continuous smear of stuff that can be divided up into pieces as small as you please. Lines and curves, planes, the space we live in, and even higher-dimensional spaces are all magnitudes, and they are commonly studied with the tools of geometry and analysis. Multitudes, on the other hand, are like beans, discrete objects that can be put in piles but cannot be split without losing their essence. The whole numbers are the canonical example of multitudes, and they are studied with the tools of number theory. Langlands predicted that certain numbers that arise in analysis—specifically, the eigenvalues of certain operators on differential forms on particular Riemannian manifolds, called automorphic forms—were actually a code that, if unraveled, would classify fundamental objects in the arithmetic world.

One of the tools developed from the Langlands program is the "Arthur-Selberg trace formula", an equation that shows precisely how geometric information can calculate arithmetic information. That is valuable in itself and, furthermore, is a building block in proving Langlands's principle of functoriality, one of the great pillars of his program. But Langlands ran across an annoying stumbling block in trying to use the trace formula. He kept encountering complicated finite sums that clearly seemed to be equal, but he couldn't quite figure out how to show it. It seemed like a straightforward problem, one that could be solved with a bit of combinatorial fiddling, so he called it a "lemma"—the term for a minor but useful result—and assigned it to a graduate student.

When the graduate student could not prove it, he tried another. Then he worked on it himself. Then he consulted with other mathematicians. At the same time as everyone continued to fail to prove it, the critical need for the result became increasingly clear. So the problem came to have a slightly grander title: the "fundamental lemma".

After three decades of work, only a few special cases had yielded to proof. The lack of a proof was such a roadblock to progress that many mathematicians had begun simply assuming it was true and developing results that depended upon it, creating a huge body of theory that would come crashing down if it turned out to be false.

Ngô Bao Châu was the one to finally break the problem open. The complicated identities in the fundamental lemma, he realized, could be seen as arising naturally out of sophisticated mathematical objects known as Hitchen fibrations. His approach was entirely novel and unexpected: Hitchen fibrations are purely geometric objects that are close to mathematical physics, nearly the last thing anyone expected to be relevant to this problem in the purest of pure math.

But it was instantly clear that he had made a profound connection. His approach turned the annoying, fiddly complexity of the fundamental lemma into a simple, natural statement about Hitchen fibrations. Even before he had managed to complete the proof, he had achieved something even more impressive: he had created genuine understanding.

Furthermore, by putting the problem in this much bigger framework, Ngô gave himself powerful new tools to assault it with. In 2004 he proved some important and difficult special cases working with his former thesis advisor, Gérard Laumon, and, in 2008, using his new methods, he cracked the problem in its full generality.

Ngô's methods are so novel that mathematicians expect them to break open a number of other problems as well. A prime target is another piece of Langlands's program, his "theory of endoscopy".

His techniques might even point the way toward a proof of the full principle of functoriality, which would be close to a full realization of Langlands's original vision. Langlands himself, who is now more than seventy years old and still hard at work, has developed a highly speculative but enticing approach to the problem. It is still far from clear that these ideas will lead to a proof, but if they do, they will have to rely on the kinds of geometric ideas that Ngô has introduced.

Ngô Bao Châu was born on June 28, 1972, in Hanoi, Vietnam. After secondary school in Vietnam, he moved to France and studied at the Université Paris 6, École Normale Supérieure de Paris. He completed his Ph.D. degree in Orsay under the supervision of Gérard Laumon. He is currently professor in the Faculté des Sciences at Orsay and a member of the Institute for Advanced Study in Princeton. In September 2010 he began his new appointment at the University of Chicago. Jointly with Laumon, Ngô was awarded the Clay Research Award in 2004. In 2007 he was awarded the Sophie Germain Prize and the Oberwolfach Prize.

Stanislav Smirnov

Citation: "For the proof of conformal invariance of percolation and the planar Ising model in statistical physics."

Stanislav Smirnov has put a firm mathematical foundation under a burgeoning area of mathematical physics. He gave elegant proofs of two longstanding, fundamental conjectures in statistical physics, finding surprising symmetries in mathematical models of physical phenomena.

Though Smirnov's work is highly theoretical, it relates to some surprisingly practical questions.

Stanislav Smirnov

For instance, when can water flow through soil and when is it blocked? For it to flow, small-scale pores in the soil must link up to provide a continuous channel from one place to another. This is a classic question in statistical physics, because the large-scale behavior of this system (whether the water can flow through a continuous channel of pores) is determined by its small-scale, probabilistic behavior (the chance that at any given spot in the soil, there will be a pore).

It is also a natural question to model mathematically. Imagine each spot in the soil as lying on a grid or lattice, and color the spot blue if water can flow and yellow if it can't. Determine the color of each spot by the toss of a coin (heads for yellow, tails for blue), using a coin that might be weighted rather than fair. If a path of blue spots crosses from one side of a rectangle to the other, the water can pass from one side to the other.

Such "percolation models" behave in a remarkable way. For extreme values, the behavior is as you might expect: If the coin is heavily weighted against blue, the water almost certainly will not flow, and if it is heavily weighted toward blue, the water almost certainly will. But the probability of flow does not change evenly as the percentage of blue spots increases. Instead, the water is almost certainly going to be blocked until the percentage of blue spots reaches some threshold value, and once it does, the probability that the water will flow starts surging upward. This threshold is called the "critical point". Abrupt change of behavior like this is a bit like what happens to water as it heats: suddenly, at a critical temperature, the water boils. For that reason, this phenomenon is commonly called a phase transition.

But of course, real soil does not come with neat, evenly spaced horizontal or vertical pores. So to apply this model to the real world, a couple of troublesome questions arise. First, how fine should the lattice grid be? Physicists are most interested in understanding processes at the molecular scale, in which case the grid should be very small indeed. Mathematicians then ask about the relationship between models with ever-smaller grids. Their hope is that as the grids get finer, the models will get closer and closer to one single model that effectively has an infinitely fine grid, called a "scaling limit".

To see why it is not obvious that the scaling limit will exist, imagine choosing a particular percentage of blue spots for a lattice and calculating the probability that the pores will line up to form a crossing. Then make the grid size smaller and calculate it again. As the grids get finer, the crossing probabilities may get closer and closer to some number, the way the numbers 1.9, 1.99, 1.999, 1.9999…get closer and closer to 2. In that case, this number will be the crossing probability for the scaling limit. But it is imaginable that the crossing probabilities will jump around and never converge toward a limit, like the sequence of numbers 2, 4, 2, 4, 2, 4… In that case, should the crossing probability for the scaling limit be 2 or 4? There is no good answer, so we have to say that the scaling limit does not exist.

Another potentially problematic question is what shape lattice to use. Even if we restrict ourselves to two dimensions, there are many choices: square lattices, triangular lattices, rhombic lattices, and so forth. Ideally, the model would be "universal", so that the choice of lattice shape does not matter, but that is not obviously true.

Physicists are pretty sure that neither of these potential problems is so bad. Using physical intuition, they have argued convincingly that the model will indeed approach a well-defined scaling limit as the grid gets finer. Furthermore, though the choice of lattice shape does affect the critical point, physicists have persuaded themselves that it will not affect many of the other properties they are interested in.

Physicists have figured out even more about two-dimensional lattices, including finding evidence for a surprising and beautiful symmetry. Imagine taking a lattice of any shape and stretching it or squinching it but leaving the angles all the same. The Mercator projection of the globe is an example of this: Greenland is huge, since distances are changed, but latitude and longitude lines nevertheless stay at right angles. Physicists have convinced themselves that if you transform two-dimensional percolation models in this way, it will not change their scaling limits (as long as you are near the critical points). Or, to use the technical term, they are persuaded that scaling limits are "conformally invariant".

In 1992 John Cardy, a physicist at the University of Oxford, used this insight to achieve one of percolation theory's big goals: a precise formula that calculates the crossing probabilities of the scaling limits of two-dimensional lattices near the critical point. The only problem was that, although his physical arguments were persuasive, neither he nor anyone else could turn that physical intuition into a mathematical proof.

In 2001 Smirnov put all this physical theory on a firm mathematical foundation. He proved that scaling limits are conformally invariant, though only for the triangular lattice (the shape that pennies, for example, fall into naturally when laid flat on a table and packed tightly together). In the process, he also proved the correctness of Cardy's formula for triangular lattices. His proof used an approach independent of ones used earlier by physicists that provided fundamental new insights. It also provided a critical missing step in the theory of Schramm-Loewner evolution, an important, recently developed method in statistical physics.

In another major achievement, Smirnov used similar methods to understand the Ising model, which describes such phenomena as magnetism, gas movement, image processing, and ecology. Just as with percolation, the large-scale behaviors of these phenomena are determined by their probabilistic, small-scale behavior. Consider, for example, magnetism. The atoms in a piece of iron behave like miniature magnets, with the electrons moving around the nucleus, creating a miniature magnetic field. The atoms try to pull their neighbors into the same alignment as their own. When enough atoms have their north poles pointing the same direction, the iron as a whole becomes magnetic. Mathematicians model this by visualizing the atoms as lying on the nodes of a lattice, with statistical rules that determine whether they are aligned with their north poles pointing up or down.

Like the percolation model, the Ising model undergoes a phase transition: as the iron is heated, the atoms vibrate more quickly, and if it is heated above a certain point, the vibrations are so strong that neighboring atoms suddenly no longer hold one another in alignment and the piece as a whole begins to lose its magnetism.

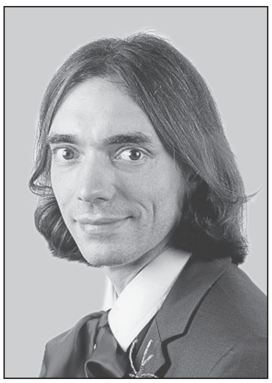
The same questions that mathematicians and physicists worry about in percolation also apply to the Ising model. The grid should be extremely small, since it is operating on the atomic level. So as the grid mesh gets finer and finer, does the model converge toward some infinitely fine version, a scaling limit? Furthermore, how does the lattice shape affect the critical point and other properties? And what happens if one stretches or squishes the lattice without changing the angles; does the scaling limit change?

For this model, too, Smirnov was able to show that the models do indeed converge toward a scaling limit as the grid mesh gets finer and that they are unaffected by stretches and squishes—that is, that they are conformally invariant. Later, with Dmitry Chelkak, he established universality, extending the results to a wide range of different lattices. He has also done significant work in analysis and dynamical systems. His work will continue to enrich both mathematics and physics in the future.

Stanislav Smirnov was born in 1970 in St. Petersburg, Russia. He received his Ph.D. from the California Institute of Technology in 1996 under the direction of Nikolai Makarov. He held a Gibbs Instructorship at Yale University and short-term positions at the Institute for Advanced Study, Princeton, and the Max Planck Institute for Mathematics (MPIM), Bonn. He moved to Sweden in 1998 and became professor at the Royal Institute of Technology and researcher at the Swedish Royal Academy of Sciences in 2001. Since 2003 he has been a professor at the University of Geneva, Switzerland. His distinctions include the St. Petersburg Mathematical Society Prize (1997), the Clay Research Award (2001), the Salem Prize (2001), the Gran Gustafsson Research Prize (2001), the Rollo Davidson Prize (2002), and the European Mathematical Society Prize (2004).

Cédric Villani

Citation: "For his proofs of nonlinear Landau damping and convergence to equilibrium for the Boltzmann equation."



Cédric Villani

Cédric Villani has provided a deep mathematical understanding of a variety of physical phenomena. At the center of much of his work is his profound mathematical interpretation of the physical concept of entropy, which he has applied to solving major problems inspired by physics. Furthermore, his results have fed back into mathematics, enriching both fields through the connection.

Villani began his mathematical career by reexamining one of the most shocking and controversial theories of nineteenth-century physics. In 1872 Ludwig Boltzmann studied what happens when the stopper is removed on a gas-filled beaker and the gas spreads around the room. Boltzmann explained the process by calculating the probability that a molecule of gas would be in a particular spot with a particular velocity at any particular moment—before the atomic theory of matter was widely accepted. Even more shockingly, though, his equation created an arrow of time.

The issue was this: when molecules bounce off each other, their interactions are regulated by Newton's laws, which are all perfectly time-reversible; that is, in principle, we could stop time, send all the molecules back in the direction they had come from, and they would zip right back into the beaker. But Boltzmann's equation is not time-reversible. The molecules almost always go from a state of greater order (e.g., enclosed in the beaker) to less order (e.g., spread around the room). Or, more technically, entropy increases.

Over the next decades, physicists reconciled themselves to entropy's emergence from time-reversible laws, and, indeed, entropy became a key tool in physics, probability theory, and information theory. A key question remained unanswered, though: How quickly does entropy increase? Experiments and numerical simulations could provide rough estimates, but no deep understanding of the process existed.

Villani, together with his collaborators Giuseppe Toscani and Laurent Desvillettes, developed the mathematical underpinnings needed to get a rigorous answer, even when the gas starts from a highly ordered state that has a long way to go to reach its disordered, equilibrium state. His discovery had a completely unexpected implication: though entropy always increases, sometimes it does so faster and sometimes more slowly. Furthermore, his work revealed connections between entropy and apparently unrelated areas of mathematics, such as Korn's inequality from elasticity theory.

After this accomplishment, Villani brought his deep understanding of entropy to another formerly controversial theory. In 1946 the Soviet physicist Lev Davidovich Landau made a mind-bending claim: that, in certain circumstances, a phenomenon can approach equilibrium without increasing entropy.

In a gas, the two phenomena always go together. Gas approaches equilibrium by spreading around a room, losing any order it initially had and increasing entropy as much as possible. But Landau argued that plasma, a gas-like form of matter that contains so much energy that the electrons get ripped away from the atoms, was a different story. In plasma, the free-floating charged particles create an electrical field that in turn drives their motion. This means that unlike particles in a gas, which affect the motion only of other particles they happen to smash against, plasma particles influence the motion of faraway particles that they never touch as well. That means that Boltzmann's equation for gases does not apply—and the Vlasov-Poisson equation that does is time-reversible, and hence does not involve an increase in entropy. Nevertheless, plasma, like gas, spreads out and approaches an equilibrium state. It was believed that this happened only because of the collisions between atoms. But Landau argued that even if there were no collisions, the plasma would move toward equilibrium because of a decay in the electric field. He proved it—but only for a simplified linear approximation of the Vlasov-Poisson equation.

Despite a huge amount of study over the next six decades, little progress was made in understanding how this equilibrium state comes about or in proving Landau's claim for the full Vlasov-Poisson equation. Last year, Villani, in collaboration with Clément Mouhot, finally came to a deep understanding of the process and proved Landau right.

A third major area of Villani's work initially seemed to have nothing to do with entropy, until Villani found deep connections and transformed the field. He became involved in optimal transport theory, which grew out of one of the most practical of questions: Suppose you have a bunch of mines and a bunch of factories, in different locations, with varying costs for moving the ore from each particular mine to each particular factory. What is the cheapest way to transport the ore?

This problem was first studied by the French mathematician Gaspard Monge in 1781 and rediscovered by the Russian mathematician Leonid Kantorovich in 1938. Kantorovich's work on this problem blossomed into an entire field (linear programming), won him the Nobel Prize in economics in 1975, and spread into a remarkable array of areas, including meteorology, dynamical systems, fluid mechanics, irrigation networks, image reconstruction, cosmology, the placement of reflector antennas—and, in the last couple of decades, mathematics.

Villani and Felix Otto made one of the critical connections when they realized that gas diffusion could be understood in the framework of optimal transport. An initial configuration of gas particles can be seen as the mines, and a later configuration can be seen as the factories. (More precisely, it is the probability distribution of the particles in each case.) The farther the gas particles have to move to go from one configuration to the other, the higher the cost.

One can then imagine each of these possible configurations as corresponding to a point in an abstract mountainous landscape. The distance between two points is defined as the optimal transport cost, and the height of each point is defined by the entropy (with low points having high entropy). This gives a beautiful way of understanding what happens as gas spreads out in a room: it is as though the gas rolls down the slopes of this abstract terrain, its configurations changing as specified by the points on the downward path.

Now suppose that a fan is blowing when you open the beaker of gas, so that the gas does not spread uniformly as it diffuses. Mathematically, this can be modeled by considering the space in which the gas is spreading to be distorted or curved. Villani and Otto realized that the curvature of the space in which the gas spreads would translate into the topography of the abstract landscape. This connection allowed them to apply the rich mathematical understanding of curvature (in particular, Ricci curvature, which was critical in the recent solution of the Poincaré conjecture) to answer questions about optimal transport.

Furthermore, Villani and John Lott were able to take advantage of these links with optimal transport to further develop the theory of curvature. For example, mathematicians had not had a way of defining Ricci curvature at all in some situations, such as at a sharp corner. Villani and Lott (and simultaneously, using complementary tools, Karl-Theodor Sturm) were able to use the connection with optimal transport to offer a definition and push the mathematical understanding of curvature to new, deeper levels. This depth of understanding and development of novel connections between different areas is typical of Villani's work.

Cédric Villani was born in 1973 in France. After studying mathematics at École Normale Supérieure in Paris from 1992 to 1996, he was appointed assistant professor there. He received his Ph.D. in 1998. Since 2000 he has been a full professor at École Normale Supérieure de Lyon. He has held semester-long visiting positions in Atlanta, Berkeley, and Princeton. His distinctions include the Jacques Herbrand Prize of the French Academy of Science (2007), the Prize of the European Mathematical Society (2008), the Henri Poincaré Prize of the International Association for Mathematical Physics, and the Fermat Prize (2009). In 2009 he was appointed director of the Institut Henri Poincaré in Paris and part-time visitor at the Institut des Hautes Études Scientifiques.

—From IMU news releases

2010 Gauss Prize Awarded

Photograph courtesy of ICM/IMU—B. Eymann–Académie des Sciences.

Yves Meyer

On August 19, 2010, the 2010 Carl Friedrich Gauss Prize for Applications of Mathematics was awarded at the opening ceremonies of the International Congress of Mathematicians (ICM) in Hyderabad, India. The prizewinner is Yves Meyer of the École Normale Supérieure de Cachan, France. He was honored "for fundamental contributions to number theory, operator theory and harmonic analysis, and his pivotal role in the development of wavelets and multiresolution analysis."

The Carl Friedrich Gauss Prize for Applications of Mathematics was instituted in 2006 to recognize mathematical results that have opened new areas of practical applications. It is granted jointly by the International Mathematical Union and the German Mathematical Society and is awarded every four years at the ICM.

"Whenever you feel competent about a theory, just abandon it." This has been Meyer's principle in his more than four decades of outstanding mathematical research work. He believes that only researchers who are newborn to a theme can show imagination and make big contributions. Accordingly, Meyer has passed through four distinct phases of research activity corresponding to his explorations in four disparate areas—quasicrystals, the Calderón-Zygmund program, wavelets, and the Navier-Stokes equation. The varied subjects that he has worked on are indicative of his broad interests. In each one of them Meyer has made fundamental contributions. His extensive work in each would suggest that he does not leave a field of research that he has entered until he is convinced that the subject has been brought to its logical end.

The seeds for Meyer's highly original approach in every branch of mathematics that he has ventured into were perhaps sown early in his career. He started his research career after having been a high school teacher for three years following his university education. He completed his Ph.D. in 1966 in just three years. "I was my own supervisor when I wrote my Ph.D.," Meyer has said. This individualistic perspective to a problem has been his hallmark till this day.

In 1970 Meyer introduced some totally new ideas in harmonic analysis (a branch of mathematics that studies the representation of functions or signals as a superposition of some basic waves) that turned out to be useful not only in number theory but also in the theory of the so-called quasicrystals. There are certain algebraic numbers called the Pisot-Vijayaraghavan numbers and certain numbers known as Salem numbers. These have some remarkable properties that show up in harmonic analysis and Diophantine approximation (approximation of real numbers by rational numbers). For instance, the Golden Ratio is such a number. Yves Meyer studied these numbers and proved a remarkable result. Meyer's work in this area led to notions of Meyer and model sets which played an important role in the mathematical theory of quasicrystals.

Quasicrystals are space-filling structures that are ordered but lack translational symmetry and are aperiodic in general. Classical theory of crystals allows only two-, three-, four-, and sixfold rotational symmetries, but quasicrystals display fivefold symmetry and symmetry of other orders. Just like crystals, quasicrystals produce modified Bragg diffraction, but where crystals have a simple repeating structure, quasicrystals exhibit more complex structures, like aperiodic tilings. Penrose tilings are an example of such an aperiodic structure that displays fivefold symmetry. Meyer studied certain sets in the n-dimensional Euclidean space (now known as a Meyer set) that are characterized by a certain finiteness property of its set of distances. Meyer's idea was that the study of such sets includes the study of possible structures of quasicrystals. This formal basis has now become an important tool in the study of aperiodic structures in general.

In 1975 Meyer collaborated with Ronald Coifman on what are called Calderón-Zygmund operators. The important results that they obtained gave rise to several other works by others, which have led to applications in areas such as complex analysis, partial differential equations, ergodic theory, number theory, and geometric measure theory. This approach of Meyer and Coifman can be looked on as the interplay between two opposing paradigms: the classical complex-analytic approach and the more modern Calderón-Zygmund approach, which relies primarily on real-variable techniques. Nowadays, it is the latter approach that dominates, even for problems that actually belong to the area of complex analysis.

The Calderón-Zygmund approach was the result of the search for new techniques because the complex-analytic methods broke down in higher dimensions. This was done by S. Mihlin, A. Calderón, and A. Zygmund, who investigated and resolved the problem for a wide class of operators, which we now refer to as singular integral operators, or Calderón-Zygmund operators. These singular integral operators are much more flexible than the standard representation of an operator, according to Meyer. His collaborative work with Coifman on certain multilinear integral operators has proved to be of great importance to the subject. With Coifman and Alan MacIntosh he proved the boundedness and continuity of the Cauchy integral operator, which is the most famous example of a singular integral operator, on all Lipschitz curves. This had been a long-standing problem in analysis.

Meyer credits the research phase on wavelets, which have had a tremendous impact on signal and image processing, with having given him a second scientific life. A wavelet is a brief wave-like oscillation with amplitude that starts out at zero, increases, and decreases back to zero, like what may be recorded by a seismograph or heart monitor. But in mathematics these are specially constructed to satisfy certain mathematical requirements and are used in representing data or other functions. As mathematical tools they are used to extract information from many kinds of data, including audio signals and images. Sets of wavelets are generally required to analyze the data. Wavelets can be combined with portions of an unknown signal by the technique of convolution to extract information from the unknown signal.

Representation of functions as a superposition of waves is not new. It has existed since the early 1800s, when Joseph Fourier discovered that he could represent other functions by superposing sines and cosines. Sine and cosine functions have well-defined frequencies but extend to infinity; that is, although they are localized in frequency, they are not localized in time. This means that, although we might be able to determine all the frequencies in a given signal, we do not know when they are present. For this reason a Fourier expansion cannot represent properly transient signals or signals with abrupt changes. For decades scientists have looked for more appropriate functions than these simple sine and cosine functions to approximate choppy signals.

To overcome this problem, several solutions have been developed in the past several decades to represent a signal in the time and the frequency domain at the same time. The effort in this direction began in the 1930s with the Wigner transform, a construction by Eugene Wigner, the famous mathematician-physicist. Basically wavelets are building blocks of function spaces that are more localized than Fourier series and integrals. The idea behind the joint time-frequency representations is to cut the signal of interest into several parts and analyze each part separately with a resolution matched to its scale. In wavelet analysis, appropriate approximating functions that are contained in finite domains thus become very suitable for analyzing data with sharp discontinuities.

The fundamental question that the wavelet approach tries to answer is how to cut the signal. The time-frequency domain representation itself has a limitation imposed by the Heisenberg uncertainty principle that both the time and frequency domains cannot be localized to arbitrary accuracy simultaneously. Therefore, unfolding a signal in the time-frequency plane is a difficult problem, which can be compared to writing the score and listening to the music simultaneously. So groups in diverse fields of research developed techniques to cut up signals localized in time according to resolution scales of their interest. These techniques were the precursors of the wavelet approach.

The wavelet analysis technique begins with choosing a wavelet prototype function, called the mother wavelet. Time-resolution analysis can be performed with a contracted, high-frequency version of the mother wavelet. Frequency-resolution analysis can be performed with a dilated, low-frequency version of the same wavelet. The wavelet transform or wavelet analysis is the most recent solution to overcome the limitations of the Fourier transform. In wavelet analysis the use of a fully scalable modulated window solves the signal-cutting problem mentioned earlier. The window is shifted along the signal, and for every position the spectrum (the transform) is calculated. Then this process is repeated several times with a slightly shorter (or longer) window for every new cycle. The result of this repetitive signal analysis is a collection of time-scale representations of the signal, each with different resolution; in short, multiscale resolution or multiresolution analysis. Simply put, the large scale is the big picture, whereas the small scale shows the details. It is like zooming in without loss of detail. That is, wavelet analysis sees both the forest and trees.

In geophysics and seismic exploration, one could find models to analyze waveforms propagating underground. Multiscale decompositions of images were used in computer vision because the scale depended on the depth of a scene. In audio processing, filter banks of constant octave bandwidth (dilated filters) applied to the analysis of sounds and speech and to handle the problem of Doppler shift multiscale analysis of radar signals were evolved. In physics, multiscale decompositions were used in quantum physics by Kenneth G. Wilson for the representation of coherent states and also to analyze the fractal properties of turbulence. In neurophysiology, dilation models had been introduced by the physicist G. Zweig to model the responses of simple cells in the visual cortex and in the auditory cochlea. Wavelet analysis would bring these disparate approaches together into a unifying framework. Meyer is widely acknowledged as one of the founders of wavelet theory.

In 1981 Jean Morlet, a geologist working on seismic signals, had developed what are known as Morlet wavelets, which performed much better than the Fourier transforms. Actually, Morlet and Alex Grossman, a physicist whom Morlet had approached to understand the mathematical basis of what he was doing, were the first to coin the term "wavelet" in 1984. Meyer heard about the work and was the first to realize the connection between Morlet's wavelets and earlier mathematical constructs, such as the work of Littlewood and Paley, used for the construction of functional spaces and for the analysis of singular operators in the Calderón-Zygmund program.

Meyer studied whether it was possible to construct an orthonormal basis with wavelets. (An orthonormal basis is like a coordinate system in the space of functions and, like the familiar coordinate axes, each base function is orthogonal to the other. With an orthonormal basis you can represent every function in the space in terms of the basis functions.) This led to his first fundamental result in the subject of wavelets in a Bourbaki seminar article that constructs a lot of orthonormal bases with Schwarz class functions (functions that have values only over a small region and decay rapidly outside). This article was a major breakthrough that enabled subsequent analysis by Meyer. "In this article," says Stéphane Mallat, "the construction of Meyer had isolated the key structures in which I could recognize similarities with the tools used in computer vision for multiscale image analysis and in signal processing for filter banks."

A Mallat-Meyer collaboration resulted in the construction of mathematical multiresolution analysis and a characterization of wavelet orthonormal bases with conjugate mirror filters that implement a first wavelet transform algorithm that performed faster than the fast Fourier transform (FFT) algorithm. Thanks to the Meyer-Mallat result, wavelets became much easier to use. One could now do a wavelet analysis without knowing the formula for the mother wavelet. The process was reduced to simple operations of averaging groups of pixels together and taking differences, over and over. The language of wavelets also became more comfortable to electrical engineers.

In the 1980s the digital revolution was all around, and efficient algorithms were critically needed in signal and image processing. The JPEG standard for image compression was developed at that time. In 1987 Ingrid Daubechies, a student of Grossman, while visiting the Courant Institute at New York University and later during her appointment at AT&T Bell Labs, discovered a particular class of compactly supported conjugate mirror filters, which were not only orthogonal (like Meyer's) but were also stable and which could be implemented using simple digital filtering ideas. The new wavelets were simple to program, and they were smooth functions, unlike some of the earlier jumpy functions. Signal processors now had a dream tool: a way to break up digital data into contributions of various scales.

Combining Daubechies's and Mallat's ideas, one could do a simple orthogonal transform that could be rapidly computed in modern digital computers. Daubechies wavelets turn the theory into a practical tool that can be easily programmed and used by a scientist with a minimum of mathematical training. Meyer's first Bourbaki paper actually laid the foundations for a proper mathematical framework for wavelets. That marked the beginning of modern wavelet theory. In recent years wavelets have begun to provide an interesting alternative to Fourier transform methods.

Interestingly, Meyer's first reaction to the work of Grossman and Morlet was "So what! We harmonic analysts knew all this a long time ago!" But he looked at the work again and realized that Grossman and Morlet had done something different and interesting. He built on the difference to eventually formulate his basis construction. "Meyer's construction of the orthonormal bases and his subsequent results in the area were the key discovery that opened the door to all further mathematical developments and applications. Meyer was at the core of the catalysis that brought together mathematicians, scientists, and engineers that built up the theory and resulting algorithms," says Mallat.

Since the work of Daubechies and Mallat, applications that have been explored include multiresolution signal processing, image and data compression, telecommunications, fingerprint analysis, statistics, numerical analysis, and speech processing. The fast and stable algorithm of Daubechies was improved subsequently in a joint work between Daubechies and Albert Cohen, Meyer's student, which is now being used in the new standard JPEG2000 for image compression and is now part of the standard toolkit for signal and image processing. Techniques for restoring satellite images have also been developed based on wavelet analysis.

More recently, he has found a surprising connection between his early work on the model sets used to construct quasicrystals—the "Meyer sets"—and "compressed sensing", a technique used for acquiring and reconstructing a signal utilizing the prior knowledge that it is sparse or compressible. Based on this, he has developed a new algorithm for image processing. A version of such an algorithm has been installed in the space mission Herschel of the European Space Agency, which is aimed at providing images of the oldest and coldest stars in the universe.

"To my knowledge," says Wolfgang Dahmen, "Meyer has never worked directly on a concrete application problem." Thus Meyer's mathematics provide good examples of how the investigations of fundamental mathematical questions often yield surprising results that benefit humanity.

Yves Meyer was born in 1939. After graduating from Ecole Normale Supérieure, Paris, in 1960, he taught high school for three years, then took a teaching assistantship at Université de Strasbourg. He received his Ph.D. from Strasbourg in 1966. He has been a professor at École Polytechnique and Université Paris-Dauphine and has also held a full research position at Centre National de la Recherche Scientifique. He was professor at École Normale Supérieure de Cachan, France, from 1999 to 2009 and is currently professor emeritus. He is a foreign honorary member of the American Academy of Arts and Sciences. He has also been awarded a doctorate (Honoris causa) by Universidad Autonoma de Madrid.

—from an IMU news release

2010 Nevanlinna Prize Awarded

On August 19, 2010, the 2010 Rolf Nevanlinna Prize was awarded at the opening ceremonies of the International Congress of Mathematicians (ICM) in Hyderabad, India. The prizewinner is Daniel Spielman of Yale University.

In 1982 the University of Helsinki granted funds to award the Nevanlinna Prize, which honors the work of a young mathematician (less than forty years of age) in the mathematical aspects of information science. The prize is presented every four years by the International Mathematical Union (IMU). Previous recipients of the Nevanlinna Prize are: Robert Tarjan (1982), Leslie Valiant (1986), Alexander Razborov (1990), Avi Wigderson (1994), Peter Shor (1998), Madhu Sudan (2002), and Jon Kleinberg (2006).

Spielman was honored "for smoothed analysis of linear programming, algorithms for graph-based codes and applications of graph theory for numerical computing." Linear programming (LP) is one of the most useful tools in applied mathematics. It is basically a technique for the optimization of an objective function subject to linear equality and inequality constraints. Perhaps the oldest algorithm for LP (an algorithm is a finite sequence of instructions for solving a computational problem; it is like a computer program) is what is known as the simplex method. The simplex algorithm was devised by George Dantzig way back in 1947 and is extremely popular for numerically solving linear programming problems even today. In geometric terms, the constraints define a convex polyhedron in a high-dimensional space, and the simplex algorithm reaches the optimum by moving from one vertex to a neighboring vertex of the polyhedron. The method works very well in practice, even though in the worst case (in artificially constructed instances) the algorithm takes exponential time. Thus understanding the complexity of LP and that of the simplex algorithm have been major problems in computer science. Mathematicians have been puzzled by the general efficacy of the simplex method in practice and have long tried to establish this as a mathematical theorem. Although worst-case analysis is an effective tool to study the difficulty of a problem, it is not an effective tool to study the practical performance of an algorithm. An alternative is the average case analysis, in which the performance is averaged over all instances of a given size, but real instances of the problem arising in practice may not be the same as average case instances. So average case analysis is not always appropriate.

Photograph courtesy ICM/IMU.

Daniel Spielman

In 1979 L. G. Kachian proved that LP is in P; that is, there is a polynomial time algorithm for linear programming, and this led to the discovery of polynomial algorithms for many optimization problems. The general belief then was that there was a genuine tradeoff between being good in theory and being good in practice; that these two may not coexist for LP. However, Narendra Karmarkar's interior point method in 1984 and subsequent variants thereof, whose complexity is polynomial, shattered this belief. Interior point methods construct a sequence of feasible points, which lie in the interior of the polyhedron but never on its boundary (as against the vertex-to-vertex path of the simplex algorithm), that converge to the solution. Karmarkar's algorithm and the later theoretical and practical discoveries are competitive with, and occasionally superior to, the simplex method.

But in spite of these developments, the simplex method remains the most popular method for solving LP problems, and there was no satisfactory explanation for its excellent performance till the beautiful concept of smoothed analysis introduced by Spielman and Shang-Hua Teng enabled them to prove a mathematical theorem.

Smoothed analysis gives a more realistic analysis of the practical performance of the algorithm than using worst-case or average-case scenarios. It provides "a means", according to Spielman, "of explaining the practical success of algorithms and heuristics that have poor worst-case behavior". Smoothed analysis is a hybrid of worst-case and average-case analysis that inherits the advantages of both. "Spielman and Teng's proof is really a tour de force," says Gil Kalai, a professor of mathematics at the Hebrew University in Jerusalem and an adjunct professor of mathematics and computer science at Yale.

The "smoothed complexity" of an algorithm is given by the performance of the algorithm under slight random perturbations of the worst-case inputs. If the smoothed complexity is low, then the simplex algorithm, for example, should perform well in practical cases in which input data are subject to noise and perturbations. That is, although there may be pathological examples in which the method fails, slight modifications of any pathological example yield a "smooth" problem on which the simplex method works very well. "Through smoothed analysis, theorists may find ways to appreciate heuristics they may have previously rejected. Moreover, we hope that smoothed analysis may inspire the design of successful algorithms that might have been rejected for having poor worst-case complexity," Spielman has said.

The Association for Computing Machinery and the European Association for Theoretical Computer Science awarded the 2008 Gödel Prize to Spielman and Teng for developing the tool. Spielman and Teng's paper "Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time", in the *Journal of the ACM*, was also one of the three winners of the 2009 Fulkerson Prize awarded jointly by the American Mathematical Society (AMS) and the Mathematical Programming Society (MPS).

Since its introduction in 2001, smoothed analysis has been used as a basis for considerable research on problems ranging over mathematical programming, numerical analysis, machine learning, and data mining. "However," writes Spielman, "we still do not know if smoothed analysis, the most ambitious and theoretical of my analyses, will lead to improvements in practice."

A second major contribution by Spielman is in the area of coding. Much of the present-day communication uses coding, either for preserving secrecy or for ensuring error-free communication. Error-correcting codes (ECCs) are the means by which interference in communication is compensated. ECCs play a significant role in modern technology, from satellite communications to computer memories. In ECC, one can recover the correct message at the receiving end, even if a number of bits of the message were corrupted, provided the number is below a threshold. Spielman's work has aimed at developing codes that are quickly encodable and decodable and that allow communication at rates approaching the capacity of the communication channel.

In information theory, low-density parity-check code (LDPC) is a linear ECC, which corrects on a block-by-block basis and enables message transmission over a noisy channel. LDPC codes are constructed using a certain class of sparse graphs (in which the number of edges is much less than the possible number of edges). They are capacity-approaching codes, which means that practical constructions exist that allow the noise threshold to be set very close to the limit of what is theoretically possible. Using certain iterative techniques, LDPC codes can be decoded in time linear to their block length.

An important technique to make both coding and decoding efficient is based on extremely well connected but sparse graphs called expanders. It is this counterintuitive and apparently contradictory feature of expanders—that they are sparse and yet highly connected—that makes expanders very useful, points out Peter Sarnak, a professor of mathematics at Princeton University. Spielman and his coauthors have done fundamental work using such graph theoretic methods and have designed very efficient methods for coding and decoding. The most famous of Spielman's papers in this field was "Improved low-density parity check codes using irregular graphs", which shared the 2002 Information Theory Society Paper Award. This paper demonstrated that irregular LDPCs could perform much better than the more common ones that use regular graphs on the additive white Gaussian noise (AWGN) channel.

This was an extension of the earlier work on efficient erasure-correcting codes by Spielman and others that introduced irregular LDPC codes and proved that they could approach the capacity of the so-called binary erasure channel. So these codes provide an efficient solution to problems such as packet-loss over the Internet and are particularly useful in multicast communications. They also provide one of the best-known coding techniques for minimizing power consumption required to achieve reliable communication in the presence of white Gaussian noise. Irregular LDPC codes have had many other applications, including the recent DVB-S2 (Digital Video Broadcasting-Satellite v2) standard.

Spielman has applied for five patents for ECCs that he has invented, and four of them have already been granted by the U.S. Patent Office.

Combinatorial scientific computing is the name given to the interdisciplinary field in which one applies graph theory and combinatorial algorithms to problems in computational science and engineering. Spielman has recently turned his attention to one of the most fundamental problems in computing: the problem of solving a system of linear equations, which is central to scientific and engineering applications, mathematical programming, and machine learning. He has found remarkable linear time algorithms based on graph partitioning for several important classes of linear systems. These have led to considerable theoretical advances, as well as to practically good algorithms.

"The beautiful interface between theory and practice, be it in mathematical programming, error-correcting codes, the search for sparsifiers meeting the so-called Ramanujan bound, analysis of algorithms, computational complexity theory or numerical analysis, is characteristic of Dan Spielman's work," says Kalai.

Daniel Spielman was born in 1970 in Philadelphia, Pennsylvania. He received his B.A. in mathematics and computer science from Yale University in 1992 and his Ph.D. in applied mathematics from the Massachusetts Institute of Technology in 1995. He spent a year as a National Science Foundation (NSF) postdoctoral fellow in the Computer Science Department at University of California, Berkeley, and then taught at the Applied Mathematics Department of MIT until 2005. Since 2006 he has been Professor of Applied Mathematics and Computer Science at Yale University.

—from an IMU news release

2010 Chern Medal Awarded

Photograph courtesy of ICM/IMU.

Louis Nirenberg

On August 19, 2010, the first Chern Medal Award was presented at the opening ceremonies of the International Congress of Mathematicians (ICM) in Hyderabad, India. The awardee is Louis Nirenberg of the Courant Institute of Mathematical Sciences, New York University. He was honored "for his role in the formulation of the modern theory of nonlinear elliptic partial differential equations and for mentoring numerous students and postdocs in this area."

The Chern Medal Award was instituted for 2010 in memory of the outstanding mathematician Shiing-Shen Chern. It will be awarded jointly with the Chern Medal Foundation (CMF) to an individual whose lifelong outstanding achievements in the field of mathematics warrant the highest level of recognition.

Shiing-Shen Chern devoted his life to mathematics, both in active research and education, and to nurturing the field whenever the opportunity arose. He obtained fundamental results in all the major aspects of modern geometry and founded the area of global differential geometry. Chern exhibited keen aesthetic tastes in his selection of problems, and the breadth of his work deepened the connections of modern geometry with different areas of mathematics.

The award consists of a medal and a monetary prize of US$500,000. It is required that half of the award be donated to organizations of the recipient's choice to support research, education, outreach, or other activities to promote mathematics. Chern was generous during his lifetime in his personal support of the field, and it is hoped that this philanthropy requirement for the promotion of mathematics will set the stage and the standard for mathematicians to continue this generosity on a personal level.

Louis Nirenberg is undoubtedly one of the outstanding analysts of the twentieth century. His work has had major influence in the development of different areas of mathematics and their applications. In particular, he has been a leader in most of the developments in the theory of linear and nonlinear partial differential equations (PDEs) and related areas of analysis, the basic mathematical tools of modern science. PDEs arise in physics and geometry when systems depend on several variables simultaneously, and the most interesting ones are nonlinear. The importance of PDEs is clear in the fact that, of the seven million-dollar Millennium Problems posed by the Clay Mathematics Institute, three are in or are related to PDEs. Nirenberg's work in PDEs is deep and fundamental. He developed intricate connections between analysis and differential geometry and applied them to the theory of fluid phenomena and other physical processes.

Nirenberg's thesis concerned a fundamental issue in geometry: the solution to an embedding problem in differential geometry that Hermann Weyl had posed around 1916. *Given a Riemannian metric on a unit sphere with positive Gauss curvature, can you embed this two-sphere isometrically in three-space as a convex surface?* The thesis itself was a window to Nirenberg's main interests—PDEs, and especially elliptic PDEs. In giving a positive answer to Weyl's question, Nirenberg used ideas due to Charles Morrey. He solved the problem by reducing it to a problem in nonlinear PDE. This equation is also an elliptic PDEs. (In elliptic PDEs, the coefficients satisfy a positivity condition, and these have applications in almost all areas of mathematics, as well as numerous applications in physics. As with a general PDE, an elliptic PDE may have nonconstant coefficients and be nonlinear. The basic example of an elliptic PDE is Laplace's equation.)

As mathematician J. Mawhin has pointed out in his recent tribute to Nirenberg, ellipticity is a key word in Nirenberg's mathematical work. More than one-third of Nirenberg's articles contain the word "elliptic" in their titles, and a much larger fraction of them deal with elliptic equations or systems. "There is hardly any aspect of those equations he has not considered," Mawhin said. Despite their variety, elliptic PDEs have a well-developed theory to which Nirenberg contributed in large measure. In 1953, the first year in which he published, he published three further papers, two of which involved nonlinear elliptic PDEs, to which he returned in later years.

Through the next twenty years, with several collaborators, he developed the theory of elliptic equations satisfying certain regularity criteria that he had formulated. Although proofs for the existence and uniqueness of weak solutions to elliptic problems were well known, Nirenberg addressed the much more difficult question of how regular (or well behaved) the weak solution is. (A weak solution to a PDE is one for which the derivatives appearing in the equation may not all exist but which nevertheless is deemed to satisfy the equation in a certain dual sense.) Today, Nirenberg's method of differences for proving the interior and boundary regularity is part of a graduate student's education in PDEs.

A high point in his research in this area is the extension of this work, which was done in collaboration with Shmuel Agmon and Avron Douglis on a priori estimates for general linear elliptic systems. This is one of the most widely quoted results in analysis. As Nirenberg has said, the objective was to obtain "general estimates for general systems under general boundary conditions". The citation for the Steele Prize for Lifetime Achievement, which he received from the American Mathematical Society (1994), said: "Nirenberg is a master of the art and science of obtaining and applying a priori estimates in all fields of analysis."

Essential questions about regularity for Navier-Stokes equations, whose solutions determine fluid motion, are still open, and it is one of the Clay Institute's Millennium Problems. Among the best results today is the Caffarelli-Kohn-Nirenberg estimate of the measure of the set of singularities. Nirenberg feels that the problem will be solved soon but that it will require more input from harmonic analysis.

Building on earlier estimates of Alberto Calderón and Antoni Zygmund, he and Joseph Kohn introduced the concept of a pseudodifferential operator, a generalization of the idea of parametrix for a partial differential operator, which is useful in addressing the question of regularity of solutions to elliptic boundary value problems. They were addressing a problem that involved singular integral operators (integral operators that are not mathematically defined at a point), and they needed facts about certain properties of singular integral operators, but these were not available in the literature. So they developed what they needed, and what resulted was the extremely useful notion of a pseudodifferential operator. This has helped generate copious developments. Another important work was the one with François Treves on the solvability of general linear PDEs. Some other highlights are his research on the regularity of free boundary problems with David Kinderlehrer and Joel Spruck. Such problems have found a wide range of applications, including flame propagation.

As mentioned earlier, Nirenberg, a pioneer in nonlinear PDEs, has turned his attention again at various stages to fully nonlinear equations and made striking breakthroughs. An example is the series of papers on the existence of smooth solutions of equations of Monge-Ampère type, which are nonlinear second-order PDEs with special symmetries, with Luis Caffarelli and Spruck. Nirenberg's study of symmetric solutions of nonlinear elliptic equations using moving plane methods, with Basilis Gidas and Wei Ming Ni, and later with Henri Berestycki, is an ingenious application of the maximum principle. Mawhin has called Nirenberg a "Paganini of the maximum principle". Thanks to Nirenberg's ideas and methods, this has developed into a beautiful theory and has led to applications in combustion theory.

The following famous quotations by Nirenberg are indicative of his approach to handling nonlinearity; in Nirenberg's view, the problem determines the method. In his invited lecture at the Stockholm International Congress of Mathematicians (ICM 1962), he said: "Most results for nonlinear problems are still obtained via linear ones, despite the fact that the problems are nonlinear and not because of it." In the same lecture he also said, commenting on someone's result, "The nonlinear character of the equation is used in an essential way; indeed, he obtains results because of the nonlinearity and not despite it."

Nirenberg's wide range of interests also includes differential geometry and topology, in which he has made significant contributions. In harmonic analysis a function of bounded mean oscillation (BMO) is a real-valued function whose mean oscillation is bounded (finite). Motivated by Fritz John's earlier work in elasticity, Nirenberg, in association with John, investigated for the first time the topology of the space of such functions and gave a precise definition for it. The space is also sometimes called the John-Nirenberg space. The results were crucial for later work of Charles Fefferman on Hardy spaces. This function space has subsequently been used in many parts of analysis and in martingale theory. More recently, motivated by some nonlinear problems in physics, Nirenberg returned to this field and, in collaboration with Haim Brezis, investigated the space of functions with vanishing mean oscillation (VMO). The VMO functions form a predual for the first Hardy space. This function space is actually a subspace of BMO. Nirenberg and Brezis extended the "degree theory" in topology to mappings belonging to VMO, a result that took topologists by surprise. A fundamental question in the study of complex manifolds is: *When is an almost complex structure given by a complex structure?* A manifold is a topological space that can be locally described in terms of simpler, well-understood spaces such as Euclidean spaces. For a manifold to be a complex manifold (in which operations with imaginary numbers can be defined), the existence of an almost complex structure is necessary but not sufficient. (An almost complex manifold is a smooth manifold equipped with a structure that, roughly speaking, defines operations with imaginary numbers on each tangent space, the differentiable manifold attached to every point.) That is, every complex manifold is an almost complex manifold, but not vice versa.

In 1957 Nirenberg, with his student August Newlander, proved a fundamental result that answered this question, which had been open for many years. According to Nirenberg, André Weil and Shiing-Shen Chern had drawn his attention to the problem of proving the integrability condition for almost complex structures. Its resolution paved the way for its use in the study of many aspects of complex manifolds, particularly deformation theory. Although the problem is linear, Newlander and Nirenberg's proof reduces the problem to a system of nonlinear PDEs such that each equation involves derivatives with respect to only one complex variable. Nirenberg's tremendous insight into the properties of PDEs and his unique ability to connect PDEs, analysis, and geometry runs through all of his work.

Inequalities have had a special attraction for Nirenberg, and there are several important inequalities associated with his name. A very important result is the set of Gagliardo-Nirenberg inequalities. The AMS citation called it a "minor gem". Nirenberg's love for inequalities comes from his long association with Kurt Friedrichs at the Courant Institute of Mathematical Sciences at New York University, which Nirenberg joined after receiving his bachelor's degree in physics and mathematics from McGill University in 1945. Though he wanted to go into physics, thanks to an excellent physics teacher at McGill, he turned to mathematics on the advice of Richard Courant. Nirenberg has said that Friedrichs was a major influence on him and that Friedrichs's views of mathematics very much formed his view.

"Friedrichs was a great lover of inequalities, and that affected me very much. The point of view was that inequalities are more interesting than equalities, the identities," Nirenberg said in an interview with AMS. Elsewhere he has said, "I love inequalities. So if somebody shows me a new inequality, I say: Oh! that is beautiful, let me think about it." He had wanted to study for his Ph.D. under Friedrichs but instead studied under Jim Stoker, finishing in 1949. But Friedrichs's influence is clearly visible in Nirenberg's choice of problems, which are often drawn from physics. He does not distinguish between "pure" and "applied" mathematics, an attitude that has resulted from spending his entire career at Courant, "where there is just mathematics and people are interested in pure and applied problems," a heritage of Courant and Friedrichs. Though Nirenberg has said that he is more of a problem solver than one who develops theories, his approach to problems has resulted in the formulation of theories in different areas of mathematics.

Nirenberg is recognized for his excellent lectures and lucid expository writing. He has published over 185 papers and has had 46 students and 245 descendants, according to the Mathematics Genealogy Project. In each of the past ten years, the top fifteen cited papers in mathematics have included at least two by Nirenberg, according to MathSciNet. The fact that nearly 90 percent of Nirenberg's papers are written collaboratively shows how Nirenberg, for over six decades, has shared his knowledge and mathematical insight with mathematicians from all over the world. He is also known to be very generous and enthusiastically presents the results of other mathematicians in his lectures and survey papers.

Kohn has said this of him: "Nirenberg's career has been an inspiration; his numerous students, collaborators, and colleagues have learned a great deal from him. Aside from mathematics, Nirenberg has taught all of us the enjoyment of travel, movies, and gastronomy. An appreciation of Nirenberg also must include his ever-present sense of humor. His humor is irrepressible, so that on occasion it makes its way to the printed page."

In his AMS interview, Nirenberg said: "I wrote one paper with Philip Hartman that was elementary but enormous fun to do. That's the thing I try to get across to people who don't know anything about mathematics: what fun it is! One of the wonders of mathematics is you go somewhere in the world and you meet other mathematicians, and it's like one big family. This large family is a wonderful joy."

Louis Nirenberg was born in 1925 in Hamilton, Ontario, Canada. After receiving his bachelor's degree from McGill University in 1945, he obtained his M.S. (1947) and Ph.D. (1949) degrees from New York University. Nirenberg then joined the faculty of NYU. He was one of the original members of the Courant Institute of Mathematical Sciences. After spending his entire academic career at Courant, he retired in 1999 and is now emeritus professor. His distinctions include the 1959 Bôcher Prize of the AMS, the Jeffrey-Williams Prize of the Canadian Mathematical Society in 1987, and the AMS Steele Prize for lifetime achievement in 1994. In 1982 he (along with the late V. I. Arnold) was the first recipient in mathematics of the Crafoord Prize, established by the Royal Swedish Academy of Sciences in areas not covered by the Nobel Prizes. In 1995 he received the National Medal of Science, the highest honor in the United States for contributions to science.

—from an IMU news release

2009 Fulkerson Prizes

The 2009 Delbert Ray Fulkerson Prizes were presented at the 20th International Symposium on Mathematical Programming, held August 23 to August 28, 2009, in Chicago. Listed below are the names of the authors who received the Fulkerson Prizes, the titles of their prizewinning papers, and the prize citations.

MARIA CHUDNOVSKY, NEIL ROBERTSON, PAUL SEYMOUR, and ROBIN THOMAS, "The strong perfect graph theorem", *Annals of Mathematics*, volume 164, issue 1, 2006, pages 51–229.

Claude Berge introduced the class of perfect graphs in 1960, together with a possible characterization in terms of forbidden subgraphs. The resolution of Berge's strong perfect graph conjecture quickly became one of the most sought-after goals in graph theory. The pursuit of the conjecture brought together four important elements: vertex colorings, stable sets, cliques, and clique covers. Moreover, D. R. Fulkerson established connections between perfect graphs and integer programming through his theory of antiblocking polyhedra. A graph is called perfect if for every induced subgraph H the clique-covering number of H is equal to the cardinality of its largest stable set. The strong perfect graph conjecture states that a graph is perfect if and only if neither it nor its complement contains, as an induced subgraph, an odd circuit having at least five edges. The elegance and simplicity of this possible characterization led to a great body of work in the literature, culminating in the proof by Chudnovsky, Robertson, Seymour, and Thomas, which was announced in May 2002, just one month before Berge passed away. The long, difficult, and creative proof by Chudnovsky and her colleagues is one of the great achievements in discrete mathematics.

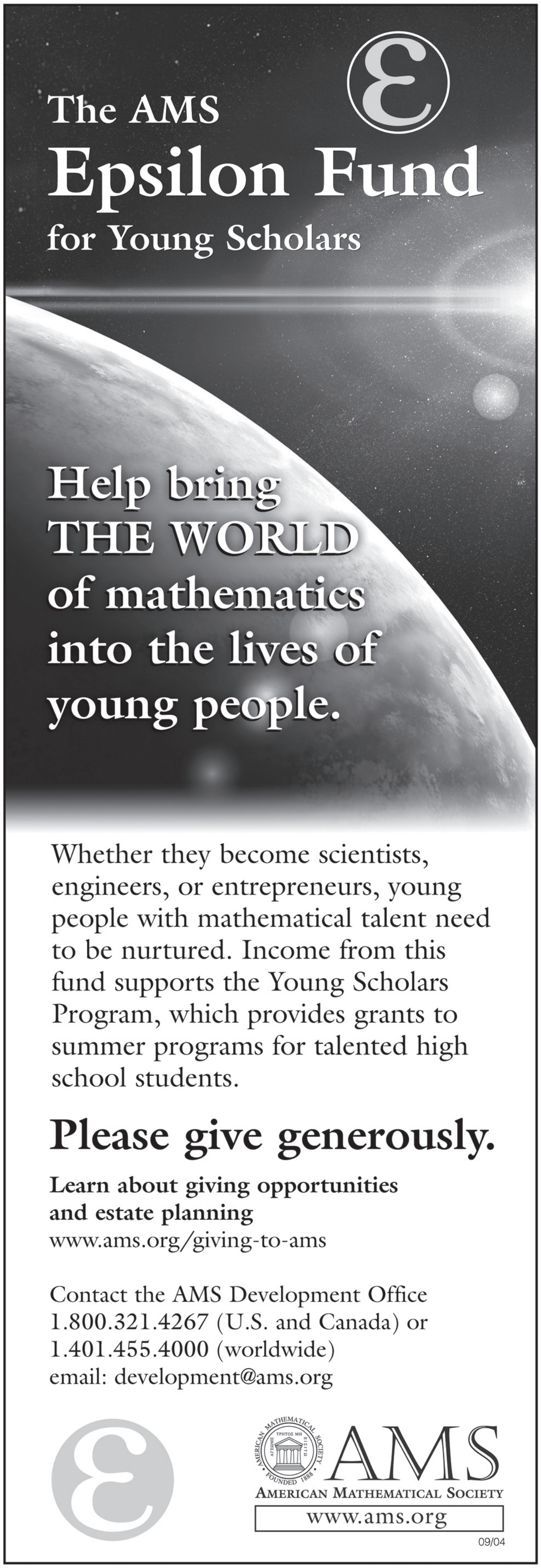
Maria Chudnovsky is in the Department of Mathematics at Princeton University. Neil Robertson is in the Department of Mathematics at the Ohio State University. Paul Seymour is in the Department of Mathematics and the Program in Applied and Computational Mathematics at Princeton University. Robin Thomas is in the School of Mathematics at the Georgia Institute of Technology.

DANIEL A. SPIELMAN and SHANG-HUA TENG, "Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time", *Journal of the ACM*, volume 51, issue 3, 2004, pages 385–463.

George Dantzig's simplex algorithm for linear programming is a fundamental tool in applied mathematics. The work of Spielman and Teng is an important step toward providing a theoretical understanding of the algorithm's great success in practice, despite its known exponential worst-case behavior. The smoothed analysis introduced by the authors fits nicely between overly pessimistic worst-case results and the average-case theory developed in the 1980s. In smoothed analysis, the performance of an algorithm is measured under small perturbations of arbitrary real inputs. The Spielman-Teng proof that the simplex algorithm runs in polynomial time under this measure combines beautiful technical results that intersect multiple areas of discrete mathematics. Moreover, the general smoothed-analysis framework is one that can be applied in many algorithmic settings, and it is now established as an important technique in theoretical computer science. The LP-specific techniques used by Spielman and Teng have interesting interpretations regarding the Hirsch conjecture, and they provide new insights into the good behavior of the simplex algorithm.

Daniel Spielman is in the Department of Computer Science at Yale University. Shang-Hua Teng is in the Department of Computer Science at Boston University.

THOMAS C. HALES, "A proof of the Kepler conjecture", *Annals of Mathematics*, volume 162,



issue 3, 2005, pages 1063–1183, and Samuel P. Ferguson, "Sphere Packings, V. Pentahedral Prisms", *Discrete and Computational Geometry*, volume 36, issue 1, 2006, 167–204.

In 1611 Johannes Kepler asserted that the densest packing of equal-radius spheres is obtained by the familiar cannonball arrangement. This statement is known as the Kepler conjecture, and it is a component of Hilbert's eighteenth problem. After four centuries, Ferguson and Hales have now proven Kepler's assertion. The Ferguson-Hales proof develops deep connections between sphere packings and mathematical programming, making heavy use of linear programming duality and branch and bound to establish results on the density of candidate configurations of spheres. The beautiful geometric arguments and innovative use of computational tools make this a landmark result in both geometry and discrete mathematics.

Hales and Ferguson received the AMS David P. Robbins Prize in 2007 for this work. Hales presented a special lecture on the proof at ISMP 2000 in Atlanta. Ferguson's paper provides a difficult step in the Kepler proof. This award is given to both researchers to appropriately acknowledge their individual contributions to this remarkable result and also to have the full proof of the Kepler theorem.

Thomas Hales is in the Department of Mathematics at the University of Pittsburgh. Samuel P. Ferguson is at the National Security Agency.

About the Prize

The Delbert Ray Fulkerson Prize recognizes outstanding papers in the area of discrete mathematics. Established in 1979, the prize is sponsored jointly by the Mathematical Programming Society (MPS) and the AMS. Up to three awards of US$1,500 each are presented at each (triennial) international symposium of the MPS. Originally, the prizes were paid out of a memorial fund administered by the AMS that was established by friends of the late Delbert Ray Fulkerson to encourage mathematical excellence in the fields of research exemplified by his work. The prizes are now funded by an endowment administered by MPS.

The prize is presented for papers published during the six calendar years preceding the year in which the prize is given. The prize is given for single papers, not series of papers or books, and in the event of joint authorship, the prize is divided. The topics of papers considered for the prize include graph theory, networks, mathematical programming, applied combinatorics, and related subjects.

—Announcement of the Fulkerson Prize Committee

Bertrand Russell: Lover, Husband, Mathematician, and Comic Book Hero

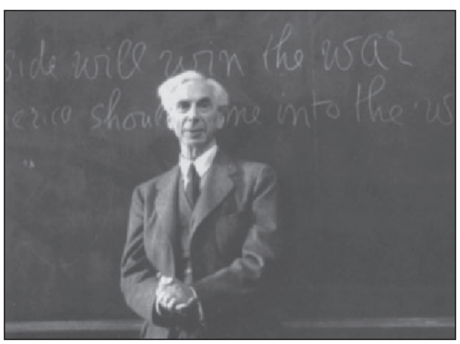
This month's cover images are panels from the book *Logicomix*, reviewed by Judith Roitman in this issue. We wondered what it takes to turn mathematical history into an interesting graphical novel. Some of our questions are answered on the book's website

`http://www.logicomix.com`

but several are not, so we submitted questions to the principal members of the group that produced it. The matters that most intrigued us were, to what extent was the book based on fact? What prompted the group to take on such an unusual and difficult project?

The main narrative of your book is tied to an autobiographical lecture by Bertrand Russell given around the beginning of World War II. What is the main source for this story line?

Apostolos Doxiadis: The event of the lecture itself was inspired by a photo of Russell in front of a blackboard in a lecture hall of an American university (not known to us) sometime in 1939, as well as what we could make of the text on the board. We knew that Russell was lecturing in the U.S. at the time and that by the end of September 1939 he had already abandoned his fanatical pacifist position in regard to the new war. We found in one of his letters the phrase that said roughly, "I am still at heart a pacifist, but the prospect of a Europe divided between Hitler and Stalin is too hard to bear." Otherwise, the occasion, the circumstances, the subject of the lecture, the actual lecture, the reaction to the lecture (picketing by isolationists), the lecture's content, and all the dialogue with the audience are invented. As for the narrative of Russell's life, we used mostly Ray Monk's biography, as well as Russell's autobiography, correspondence, and various autobiographical texts (*My Philosophical Development* and others). On the whole, we stayed very close to the events of Russell's life. We cut out a lot, for example the fact that he had an older brother, who was in fact the one who introduced him to Euclid, and his many love affairs up to 1939. But we did not consciously deviate from reality, except in the cases mentioned in our Notes inside the book, especially in the case of inventing meetings with people (such as Frege) where there was only evidence for meetings of ideas through reading, writing, and correspondence.



Bertrand Russell, around 1939.

Another major point of deviation from recorded history was in accentuating the interest of Russell in the developments in mathematical logic after WW I and his meeting with Wittgenstein in the Hague. Though he did visit the Vienna Circle in Vienna in the late 1920s and/or early 1930s and was one of their idols, there was no evidence that he followed developments in mathematical logic closely at this time. He certainly was not present (neither was Hilbert) at Gödel's announcement of the incompleteness theorems.

Christos Papadimitriou: We based much on Monk's biography and some (like the moment of epiphany about life and the world by Evelyn Whitehead's sick bed) on BR's autobiography (not a totally reliable source, because the man writes like the devil and can convince you of anything he chooses). But we favored plot over strict historical truth time and again. When my colleague Paolo Mancosu chastised these liberties, I was bold enough to respond, "But, Paolo, Marc Anthony never spoke at Caesar's funeral."

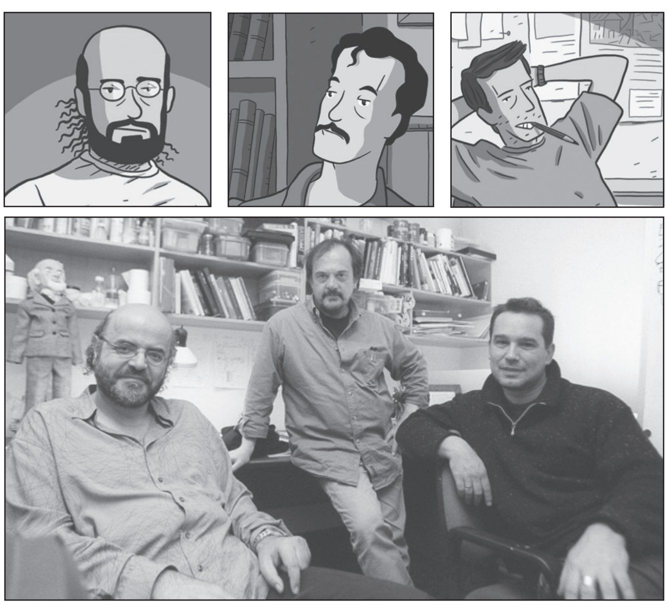
Somewhere the origin of this project is attributed to the remark of Gian-Carlo Rota (in his book Indiscrete Thoughts)*, "It cannot be a complete coincidence that several outstanding logicians of the twentieth century found shelter in asylums at some time in their lives." But many people have remarked on the madness of some of your heroes. So if this origin story is true, maybe there was some preconditioning? Why were you reading Rota, anyway?*

Doxiadis: I studied mathematics back in the 1970s and was very much into it, as well as the Hardyesque, extremist version of math-for-art's-sake purist ideology. I was at Columbia but had friends at Princeton and often visited and went to the institute, where I absorbed, among other things, the Gödel mystique. I heard stories about Gödel (even saw him once in the lounge, wrapped up in layers of sweaters and coats) and read, in awe, Nagel and Newman's *Gödel's Proof*. Many years later, after the writing of my novel *Uncle Petros and Goldbach's Conjecture* (which to me was a novel about a hero in search of a great goal, not—as it was to many mathematical readers—a novel "about" mathematics), I began to think a lot, once again, about the world of mathematics and was in a sense drawn back into it, though never into active research. I had not stayed in any sense in touch with actual research or progress, but started to read again, fascinated, about the history and philosophy of mathematics. I also read many of the new (by now) mathematical biographies—I was always fascinated by mathematicians as people and relished the new wealth of material that appeared after the year 2000. I also have read most of the published books by well-known mathematicians that are autobiographical, or philosophical-epistemological (apart from the older famous ones, like Poincaré, Hadamard, Wiener, Weil, Hardy; also later ones like Halmos, Hersh, etc.). Among these was Rota's *Discrete Thoughts* and *Indiscrete Thoughts*, which are particularly well-written and stimulating.

So the origin of the story, for me, is my interest in the story of the search for the foundations. But I never wanted to write the history of an idea. I am a writer and am interested in the stories of people—of specific, concrete human beings. I had read the lives of several mathematicians and a lot about the story of mathematics in the latter half of the nineteenth and early twentieth centuries. But it was Rota's comment that made me think again about the core of the story, the motivation and the pathology of the protagonists (not in the sense of "too much logic drove them mad," which I think is silly, but in the sense that a certain kind of personality was drawn to the task of foundational work).

Papadimitriou: When Apostolos and I met for the first time over lunch in the summer of 2001, he mentioned to me Rota's remark (I knew Gian-Carlo from my MIT days), and we confessed to each other our fascination with this story. Martin Davis' book *Engines of Logic* had just come out, and Apostolos was working on his wonderful play *Seventeenth Night* about Gödel. The idea to write something about the foundations quest came soon after, and the graphic novel idea soon after that. I reacted negatively to this one (I knew next to nothing about the genre), but I did become a believer soon enough. My special interest in this story is summarized in the last pages, namely, the amazing reversal that turned the spectacular failure of the foundations quest into the advent of the computer.

Doxiadis: I think it's important to say that we puzzled a lot about what would give this story unity, i.e., make it one story instead of many stories—a novel, as it were, rather than a collection of short stories. Russell and his life provided a natural answer to this, though we did not land on it until after some months of thinking and reading



Christos Papadimitriou, Apostolos Doxiadis, and Alecos Papadatos—as cartoons and in real life.

about the material. To give him substance as hero/storyteller, we had to (partly artificially) keep his interest in the quest alive after 1918.

The fourth principal member of the team, Annie Di Donna.

Finishing this project must have taken a fair amount of concentration. How much work was it?

Doxiadis: Logicomix took five years of full-time work to complete, and, yes, it involved a lot of concentration. For some it was really full full-time (mostly for Alecos and Annie). For me it was more or less half-time, in the sense that after a long synopsis we worked on with Christos, I wrote the script in ten-page batches, often with as much as two or three months in between—Alecos and Annie were taking much longer to draw and color than I took to write the script.

Alecos Papadatos: When Apostolos came in the studio to discuss his idea, Annie and I understood immediately that this tale was larger than mathematics as we knew it. Given that Apostolos was largely known as a gifted storyteller, we knew that the making of such a graphic narrative was going to be quite an adventure. And it was! Russell was a great character to draw. Full of passion and yet reserved, with a great sense of humor. There was a lot of work in representing his facial expressions and body positions and yet communicating what (probably) went into his mind, especially during his lecture scenes. Frege, Cantor, and Hilbert were great characters in the script and also fun to draw, but only Russell had this characteristic of being there and elsewhere at the same moment.

Annie Di Donna: When I think back to the trajectory of the project, I remember each of us carefully and methodically using precise narrative encoding, whether for the idea or the scriptwriting, character design, or the use of color. I often smile thinking that this encoding is mainly based on emotion, but they have a mathematical logic of their own.

Is there anything special you'd like to comment on for an audience of mathematicians?

Papadimitrious: Logicomix is about an era of innocence when the absolute truth was something one could have hoped for, with Mathematics the brightest promise: "We shall know." In contrast, Computer Science was born (through Turing's 1936 paper) painfully aware of her own limitations. As a computer scientist who is functionally a mathematician (my day job is to prove theorems about computation), I have looked at the quest as the rich and tragic story that lies at the root of the duality of what I am and do.

Doxiadis: To me mathematical research is one of the most romantic endeavors in the world, and I wish that more mathematicians would remember this when they speak about mathematics to outsiders. The unmathematical hoi polloi don't know this, you see—in fact the great majority don't even suspect it. So someone must tell them!

—Bill Casselman
Graphics Editor
(`notices-covers@ams.org`)

The *Notices* could not function without the generous contributions of many scholars and other professionals. Through editing, refereeing, and advising, they all contribute much wisdom. We take this opportunity to name and to offer thanks to those who participated in the issues for 2010.

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Mathematics People

Hakobyan Awarded Artin Junior Prize

HRANT HAKOBYAN of Kansas State University has been awarded the 2010 Emil Artin Junior Prize in Mathematics. Hakobyan was chosen for his joint paper with David A. Herron, "Euclidean quasiconvexity", published in the *Annales Academiæ Scientiarum Fennicæ Mathematica* **33** (2008), 205–230. Established in 2001, the Emil Artin Junior Prize in Mathematics carries a cash award of US$1,000 and is presented usually every year to a student or former student of an Armenian university under the age of thirty-five for outstanding contributions to algebra, geometry, topology, and number theory—the fields in which Emil Artin made major contributions. The prize committee consisted of A. Basmajian, Y. Movsisyan, and V. Pambuccian.

—Victor Pambuccian

Tirole Awarded CME/MSRI Prize

JEAN TIROLE of the Industrial Economics Institute and the Toulouse School of Economics has been awarded the 2010 CME Group-MSRI Prize in Innovative Quantitative Applications. The prize recognizes individuals "who contribute original concepts and innovation in the use of mathematical, statistical or computational methods for the study of the behavior of markets and, more broadly, of economics." According to the prize selection committee, Tirole's "use of game theory and information theory in his economic analysis of markets, institution regulation, and financial crises represents the forefront of mathematics applied to real-world contexts." The award carries a cash prize of US$25,000.

—From a CME/MSRI announcement

NDSEG Fellowships

Fourteen young mathematicians have been awarded National Defense Science and Engineering Graduate (NDSEG) Fellowships by the Department of Defense (DoD) for 2010. The fellowships are sponsored by the United States Army, Navy, and Air Force. As a means of increasing the number of U.S. citizens trained in disciplines of military importance in science and engineering, DoD awards fellowships to individuals who have demonstrated ability and special aptitude for advanced training in science and engineering.

The following are the names of the fellows in mathematics, their institutions, and the offices that awarded the fellowships: ADAM BACKER, Office of Naval Research (ONR); SARAH CONSTANTIN, Yale University, Air Force Office of Scientific Research (AFOSR); MAX ENGELSTEIN, Yale University, Army Research Office (ARO); MICHAEL FLEDER, Courant Institute of Mathematical Sciences, New York University, AFOSR; JASON LEE, ONR; SHAUN MAGUIRE, AFOSR; ANDREW MANION, Princeton University, ARO; ALISON MILLER, Princeton University, ARO; RAMI MOHIEDDINE, University of California, Los Angeles, AFOSR; SHRENIK SHAH, Princeton University, ARO; NIKE SUN, Stanford University, AFOSR; NEAL WADHWA, Massachusetts Institute of Technology, ONR; TEGAN WEBSTER, Rensselaer Polytechnic Institute, AFOSR; JOSHUA ZAHL, University of California, Los Angeles, ONR.

—From an NDSEG announcement

NSF Postdoctoral Fellowships Awarded

The Mathematical Sciences Postdoctoral Research Fellowship program of the Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) awards fellowships each year for postdoctoral research in pure mathematics, applied mathematics and operations research, and statistics. Following are the names of the fellowship recipients for 2010, together with their Ph.D. institutions (in parentheses) and the institutions at which they will use their fellowships.

SCOTT ARMSTRONG (University of California, Berkeley), University of Chicago; JAMES BIRD (Harvard University), Massachusetts Institute of Technology; JEFFREY CASE (University of California, Santa Barbara), Princeton University; JONATHAN CHAIKA (Rice University), University of Chicago; STEPHEN CURRAN (University of California, Berkeley), University of California, Los Angeles; KIRIL DATCHEV

(University of California, Berkeley), Massachusetts Institute of Technology; Mark Davenport (Rice University), Stanford University; Gabriel Drummond-Cole (City University of New York), Northwestern University; Daniel Erman (University of California, Berkeley), Stanford University; Pascal Getreuer (University of California, Los Angeles), École Normale Superieure de Cachan; James Gill (Washington University, St. Louis), University of Washington; Nathan Glatt-Holtz (University of Southern California), Indiana University; Thomas Goldstein (University of California, Los Angeles), Stanford University; Matt Holzer (Boston University), University of Minnesota; Kimberly Hopkins (University of Texas, Austin), University of California, Los Angeles; Ryan Hynd (University of California), Courant Institute, New York University; Zachary Kilpatrick (University of Utah), University of Pittsburgh; Stephen Kleene (Johns Hopkins University), Massachusetts Institute of Technology; Robin Koytcheff (Stanford University), Brown University; Brian Lehmann (Massachusetts Institute of Technology), University of Michigan; Guoying Lei (University of California), California Institute of Technology; Adam Levine (Columbia University), Brandeis University; Katrina Ligett (Carnegie Mellon University), Cornell University; Ricky Liu (Massachusetts Institute of Technology), University of Minnesota; Carl Mautner (University of Texas, Austin), Harvard University; Wesley Pegden (Rutgers University), New York University; Emily Peters (University of California, Berkeley), Massachusetts Institute of Technology; Jonathan Pottharst (Harvard University), Boston University; Amanda Redlich (Massachusetts Institute of Technology), Rutgers University; Lisa Rogers (Rensselaer Polytechnic Institute), New York University; Benjamin Rossman (Massachusetts Institute of Technology), Tokyo Institute of Technology; Anne Shiu (University of California, Berkeley), Duke University; Charles Smart (University of California, Berkeley), Courant Institute, New York University; Kelli Talaska (University of Michigan), University of California, Berkeley; Kevin Tucker (University of Michigan), University of Utah; Bianca Viray (University of California, Berkeley), Brown University; Catherine Williams (University of Washington, Seattle), Columbia University; and Paulette Willis (University of Iowa), University of Houston.

—NSF announcement

B. H. Neumann Awards Given

John Dowsey of the mathematics education department, University of Melbourne, has received a B. H. Neumann Award from the Australian Mathematics Trust. He has been a member of the Problems Committee of the Mathematics Challenge for Young Australians, for which he served as deputy chair, helping to develop materials. He has been a member of the committee that composes the problems for the Australian Intermediate Mathematical Olympiad (AIMO), essentially the national Olympiad for students up to year 10. Cheryl Praeger of the University of Western Australia was also honored with a Neumann Award; she is best known for her work in group theory, algebraic graph theory, and combinatorial designs.

—From an Australian Mathematics Trust announcement

Pi Mu Epsilon Student Paper Presentation Awards

Pi Mu Epsilon (PME), the U.S. honorary mathematics society, makes annual awards to recognize the best papers by undergraduate students presented at a PME student paper session. This year PME held a session in conjunction with the Mathematical Association of America MathFest in Pittsburgh, Pennsylvania, August 4–7, 2010. The AMS and the American Statistical Association sponsor awards to student speakers for excellence in exposition and research. Each awardee received a check for US$150. The names, chapters, institutions, and paper titles of the award-winning students follow.

Matt Alexander, Ohio Xi Chapter, Youngstown State University, "Discrete consideration of Aleksandrov's projection theorem"; Erica Evans, Ohio Iota Chapter, Denison University, "Knot mosaics: Results and open questions"; Richard Freedman, North Carolina Lambda Chapter, Wake Forest University, "Understanding hailstone sequences using a new coding process"; Jennifer Garbett, Ohio Pi Chapter, Kenyon College, "Modeling the Manduca sexta midgut"; Michael Joseph, Ohio Lambda Chapter, John Carroll University, "Patterns in primitive Pythagorean triples"; Sepideh Khavara, Ohio Xi Chapter, Youngstown State University, "Modeling of regulation of gene expressions in the presence of toxic selenite in modeling gastric emptying"; Kelley Moran, Maryland Theta Chapter, Goucher College, "Modeling gastric emptying"; Joseph Patt, Ohio Iota Chapter, Denison University, "Putting numbers on the board: Enumeration of knot mosaics"; Scott Powers, North Carolina Beta Chapter, University of North Carolina, Chapel Hill, "Evaluating statistical methodology for gene set analyses"; and Lindsay Van Leir, Virginia Delta Chapter, Roanoke College, "Mapping the liberal arts: The graph theory behind your degree".

—From a Pi Mu Epsilon announcement

My Recollections of Dirk Struik

Editor's Note: *These recollections of Dirk Struik (1894–2000), written by his last doctoral student, appear here to mark the tenth anniversary of Struik's death. A full obituary appeared in the June/July 2001 issue of the* Notices.

It was in 1959, as a graduate student in the mathematics department of the Massachusetts Institute of Technology, that I first met Dirk Struik. Having made some progress as an undergraduate at New York University in solving the century-old unsolved problem of creating a general, yet elementary, geometric definition of the concept of surface area in any number of dimensions, I asked Dr. Struik to be my mentor for a doctoral thesis on the origins and development of the infinitesimal calculus.

Dirk's unexpected retirement from MIT in 1960 put a hold on my plans, but a decade later our paths crossed again, and with the opportunity made available through a special doctoral program developed by Fairleigh Dickinson University in New Jersey, I was able to pursue my doctoral plans under MIT Professor Emeritus Dirk Struik.

At my first doctoral advisement meeting, held in Dirk's study in Belmont, MA, he suggested that my thesis should be used to enrich the education of advanced undergraduate mathematics majors by designing and developing a sequence of course units based upon selected key episodes in the history of the calculus. Dirk's role as a great teacher could not have been revealed in any better way than this!

On the other hand, Dirk the researcher strongly encouraged me to complete my work in solving the century-old problem on surface area and then publish a paper that could be incorporated into one of the episodes in my thesis. With Dirk's urging, in 1972 I published (with L. V. Toralballa, my undergraduate mentor) an eight-page paper in the *Pacific Journal of Mathematics* that essentially solved this problem. (The problem had its roots in 1868 with the publication of J. A. Serret's "incorrect" definition of surface area, which was discredited by H. A. Schwarz's famous counterexample published in 1882.)

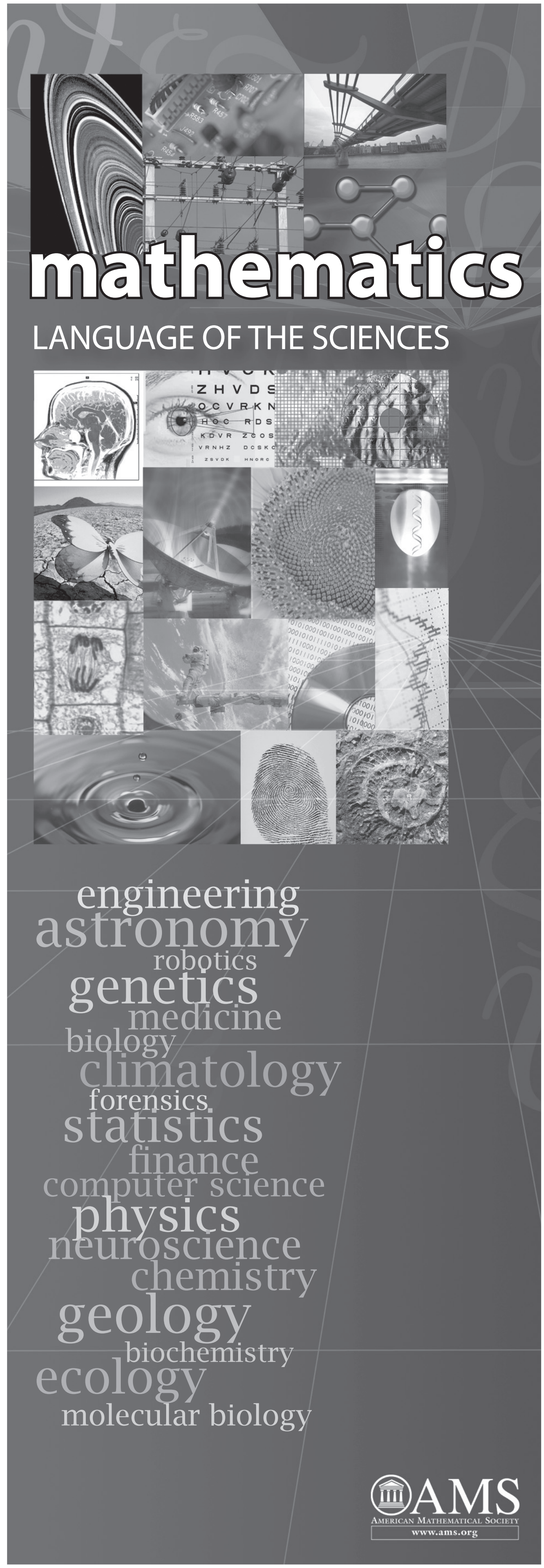
Though outstanding in so many areas, the three things that particularly impressed me about Dirk and that I will always remember were his acute insights into the history of mathematics, his truly remarkable memory, and, most importantly, the consideration and humanity he showed in our working relationship over the years.

At one of our many meetings, Dirk loaned me a copy of Lacroix's *Differential and Integral Calculus* and suggested I read it from cover to cover. Dated December 12, 1816, it was an early edition translated from the French that Dirk had acquired in 1940 from a bookstore in Harvard Square.

At our final meeting in the late 1970s, Dirk approved my thesis while we shared a bottle of his favorite sherry, and then, as I was leaving, he presented me with his coveted early edition of Lacroix's famous book—a gift I shall always cherish from a man I will always remember!

For a number of years after our final meeting, I would receive and respond to occasional handwritten notes from Dirk in which he would inquire about my work and publications. When I first learned of the plans for his 100th birthday celebration to be held at Brown University on September 30, 1994, I decided to surprise him with a special birthday gift of my latest publication on early Greek mathematics. This article was dedicated to Dirk and appeared in the fall 1994 issue of the *New York State Mathematics Teachers' Journal* just prior to his birthday celebration. Indeed, this was a very small repayment for Dirk's years of friendship and support!

—Louis Alpert
Bronx Community College of the City University of New York



Mathematics Opportunities

NSF Computing Equipment and Instrumentation Programs

The Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) plans a limited number of awards for the support of computing environments for research in the mathematical sciences. SCREMS (Scientific Computing Research Environments for the Mathematical Sciences) supports computing environments dedicated to research in the mathematical sciences. Proposals may request support for the purchase of computing equipment and limited support for professional systems administrators or programmer personnel for research computing needs. These grants are intended to support research projects of high quality that require access to advanced computing resources. Requests for routine upgrades of standard desk-environment workstations or laptop computers are not appropriate for this program. Awards are made to provide support for specific research projects rather than to provide general computing capacity. Proposers are encouraged to include projects involving symbolic and algebraic computations, numerical computations and simulations, and graphical representations (visualization) in aid of the research.

For more information see the website `http://www.nsf.gov/pubs/2007/nsf07502/nsf07502.htm`. The deadline for proposals is **January 27, 2011**.

—From an NSF announcement

National Academies Christine Mirzayan Graduate Fellowship Program

The Christine Mirzayan Science and Technology Policy Graduate Fellowship Program of the National Academies is designed to engage graduate science, engineering, medical, veterinary, business, and law students in the analysis and creation of science and technology policy and to familiarize them with the interactions of science, technology, and government. As a result, students develop essential skills different from those attained in academia and make the transition from graduate student to professional.

Applications for the fellowships are invited from scholars from graduate through postdoctoral levels in any physical, biological, or social science field or any field of engineering, medicine and health, or veterinary medicine, as well as business, law, education, and other graduate and professional programs. Postdoctoral scholars should have received their Ph.D.s within the past five years.

The fall session for 2011 will run August 29–November 18, 2011; there will be no summer session in 2011. The stipend for the 12-week session is US$8,240. The deadline for receipt of materials for the fall program is **May 1, 2011**. More information and application forms and instructions can be found on the website `http://sites.nationalacademies.org/PGA/policyfellows/index.htm` or by contacting The National Academies Christine Mirzayan Science and Technology Policy Graduate Fellowship Program, 500 Fifth Street, NW, Room 508, Washington, DC 20001; telephone: 202-334-2455; fax: 202-334-1667; email: `policyfellows@nas.edu`.

—From a National Academies announcement

NDSEG Fellowships

As a means of increasing the number of U.S. citizens trained in disciplines of military importance in science and engineering, the Department of Defense (DoD) awards National Defense Science and Engineering Graduate (NDSEG) Fellowships each year to individuals who have demonstrated ability and special aptitude for advanced training in science and engineering. The fellowships are awarded for a period of three years for study and research leading to doctoral degrees in any of fifteen scientific disciplines.

The NDSEG Fellowship Program is open only to applicants who are citizens or nationals of the United States. NDSEG Fellowships are intended for students at or near the beginning of their graduate studies in science or engineering. Applicants must have received or be on track to receive their bachelor's degrees by fall of 2011. Fellows selected in spring 2011 must begin their fellowship tenure in fall 2011. Fellowships are tenable only at U.S. institutions of higher education offering doctoral degrees in the scientific and engineering disciplines specified. Fellows will receive full tuition and stipends for 12-month tenures: US$30,500 for the first year, US$31,000 for the second year, and US$31,500 for the third year. Applications are encouraged from women; persons with disabilities; and minorities, including members of ethnic minority groups such as African American, American Indian and Alaska Native, Asian, Native Hawaiian and other Pacific Islander, Hispanic, or Latino.

Complete applications must be submitted electronically by **December 17, 2010**. Application forms are available online at `http://ndseg.asee.org/apply_online`. For further information, see `http://ndseg.asee.org/`.

—From an NDSEG announcement

MfA Fellowship Program

The Math for America Foundation (MfA) sponsors the MfA Fellowship Program, which trains mathematically talented individuals to become high school mathematics teachers in New York City; Boston; Los Angeles; Washington, DC; San Diego; or Utah. The fellowship provides an aggregate stipend of up to US$100,000 over five years, a full-tuition scholarship for a master's-level teaching or teacher credentialing program at one of MfA's partner universities, and ongoing support mechanisms, including mentoring and professional development.

Candidates should hold a bachelor's degree with substantial coursework in mathematics and should be new to teaching and able to demonstrate a strong interest in teaching. Candidates must be U.S. citizens or permanent residents of the United States. The deadline for applications is **January 21, 2011.** For more detailed information, see the website at `http://www.mathforamerica.org/`.

—From an MfA announcement

AWM Essay Contest

To increase awareness of women's ongoing contributions to the mathematical sciences, the Association for Women in Mathematics (AWM) is holding an essay contest for biographies of contemporary women mathematicians and statisticians in academic, industrial, and government careers.

The essays will be based primarily on interviews with women who are currently working in mathematical sciences careers. The contest is open to students in the following categories: 6th–8th grades, 9th–12th grades, and college undergraduates. At least one winning submission will be chosen from each category. Winners will receive a prize, and their essays will be published online at the AWM website. A grand prize winner will have his or her submission published in the *AWM Newsletter* as well. The deadline for entries is **February 27, 2011.**

In addition to student entries, organizers are currently seeking women mathematicians to volunteer as the subjects of these essays. For more information, see `http://www.awm-math.org/biographies/contest.html`.

—From an AWM announcement

News from the Mathematical Biosciences Institute

The Mathematical Biosciences Institute (MBI) at the Ohio State University is accepting applications for Early Career Awards during the 2011–12 year on Stochastics in Biological Systems. In addition, the MBI is accepting applications for Postdoctoral Fellows to start September 2011.

Early Career Awards enable recipients to be in residence for stays of at least three months during the program year. Awardees will engage in an integrated program of tutorials and workshops tied to the scientific theme and are expected to interact with local and visiting researchers. More information about the 2011–12 program can be found at `http://mbi.osu.edu/2010/scientific2011.html`. Early Career Awards are aimed at nontenured scientists who currently have continuing employment and who hold a doctorate in any of the mathematical, statistical, and computational sciences or in any of the biological, medical, and related sciences.

Postdoctoral Fellows are immersed in the topics of the MBI emphasis year programs (see `http://mbi.osu.edu`). The MBI Postdoctoral Fellows engage in a three-year integrated program of tutorials, working seminars or journal clubs, and workshops and in interactions with their mathematical and bioscience mentors. These activities are geared toward providing the tools to pursue an independent research program with an emphasis on collaborative research in the mathematical biosciences. MBI-facilitated activities for Postdoctoral Fellows are tailored to the needs of each young scientist.

Applications for the Early Career Awards and the Postdoctoral Fellow positions should be submitted online at `http:www.mathjobs.org/jobs/mbi`. Applicants should provide a curriculum vitae, a research statement, and three letters of recommendation. For additional information contact Rebecca Martin (`rebecca@mbi.osu.edu` or 614-292-3648). The deadline for applications is **December 16, 2010.**

—From an MBI announcement

For Your Information

AMS Endorses Postdoc Date Agreement

For the past eleven years the American Mathematical Society has led the effort to gain broad endorsement for the following proposal:

That mathematics departments and institutes agree not to require a response prior to a certain date (usually early in February of a given year) to an offer of a postdoctoral position that begins in the fall of that year.

This proposal is linked to an agreement made by the National Science Foundation (NSF) that the recipients of the NSF Mathematical Sciences Postdoctoral Fellowships would be notified of their awards at the latest by the end of January.

This agreement ensures that our young colleagues entering the postdoctoral job market have as much information as possible about their options before making a decision. It also allows departmental hiring committees adequate time to review application files and make informed decisions. From our perspective, this agreement has worked well and has made the process more orderly. There have been very few negative comments. Last year, one hundred forty-two mathematics and applied mathematics departments and three mathematics institutes endorsed the agreement.

Therefore, we propose that mathematics departments again collectively enter into the same agreement for the upcoming cycle of recruiting, with the deadline set for Friday, February 4, 2011. The NSF has already agreed that it will complete its review of applications by January 26, 2011, at the latest and that all applicants will be notified electronically at that time. Recipients of the postdoctoral fellowships will also receive notification of their awards by U.S. mail.

The American Mathematical Society is facilitating the process by sending this email message to all doctoral-granting mathematics and applied mathematics departments and mathematics institutes. The list of departments and institutes endorsing this agreement will be widely announced on the AMS website beginning November 1, 2010, and will be updated weekly. The *Notices of the AMS* also plans a brief news article in the January 2011 issue about this agreement.

We ask that you view a proposed updated version of last year's formal agreement at `http://www.ams.org/employment/postdoc-offers.html`, along with last year's list of adhering departments.

IMPORTANT: To streamline this year's process for all involved, we ask that you notify Ellen Maycock at the AMS (`ejm@ams.org`) IF AND ONLY IF:

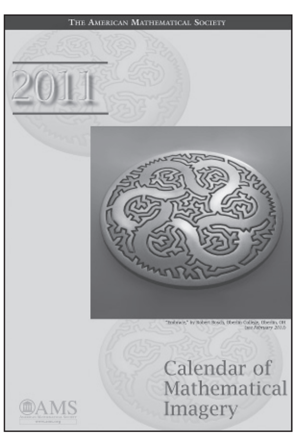
(1) your department is not listed and you would like to be listed as part of the agreement or

(2) your department is listed and you would like to withdraw from the agreement and be removed from the list.

Please feel free to email us with questions and concerns. Thank you for consideration of the proposal.

—Donald McClure, Executive Director,
and Ellen Maycock, Associate Executive Director,
American Mathematical Society

Inside the AMS



From the AMS Public Awareness Office

2011 Calendar of Mathematical Imagery

To request a complimentary copy of the 2011 calendar featuring selected images from Mathematical Imagery albums (see `www.ams.org/mathimagery`), please email `paoffice@ams.org` with subject line "2011calendar-notices". Please limit your order to three copies so that others may also have the opportunity to receive a copy.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers
`paoffice@ams.org`

Reference and Book List

*The **Reference** section of the* Notices *is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.*

Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are `notices@math.wustl.edu` in the case of the editor and `notices@ams.org` in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Upcoming Deadlines

November 18, 2010: Full proposals for NSF Graduate Research Fellowships. See `http://www.nsf.gov/pubs/2010/nsf10604/nsf10604.htm`.

November 19, 2010: Proposals for research programs at CRM. See `http://www.crm.cat/RPapplication`.

December 1, 2010: Applications for AMS Centennial Fellowships. See `http://www.ams.org/ams-fellowships/` or write to the Membership and Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; `prof-serv@ams.org`; 401-455-4105.

December 1, 2010: Letters of intent for proposals for thematic programs at the Bernoulli Center. See the website `http://bernoulli.epfl.ch/new/index.php`.

December 3, 2010: Entries for Ferran Sunyer i Balaguer Prize. See `http://ffsb.iec.cat`.

December 15, 2010: Applications for Fields Institute Postdoctoral Fellowships. See `www.fields.utoronto.ca/proposals/postdoc.html`.

December 15, 2010: Abstracts for contributed papers for ICIAM. See the website `http://www.iciam2011.com/`.

December 15, 2010: Applications for PIMS Postdoctoral Fellowships. See `http://www.pims.math.ca/scientific/postdoctoral`.

December 16, 2010: Applications for Mathematical Biosciences Institute (MBI) Early Career Awards and Postdoctoral Fellowships. See "Mathematics Opportunities" in this issue.

December 17, 2010: Applications for Department of Defense (DoD) National Defense Science and Engineering Graduate (NDSEG) Fellowships. See "Mathematics Opportunities" in this issue.

January 10, 2011: Applications for American Association of University Women (AAUW) Selected Professions

Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

Fellowships. See http://www.aauw.org/fga/fellowships_grants/selected.cfm or contact the AAUW Educational Foundation, Selected Professions Fellowships, Dept. 60, 301 ACT Drive, Iowa City, IA 52243-4030; telephone: 319/337-1716, ext. 60; e-mail: aauw@act.org.

January 15, 2011: Applications for Jefferson Science Fellows (JSF) program. See http://sites.nationalacademies.org/PGA/Jefferson/PGA_046612, email jsf@nas.edu, or telephone 202-334-2643.

January 15, 2011: Applications for AMS-AAAS Mass Media Summer Fellowships. See http://www.aaas.org/programs/education/MassMedia; or contact Stacey Pasco, Manager, Mass Media Program, AAAS Mass Media Science and Engineering Fellows Program, 1200 New York Avenue, NW, Washington, DC 20005; telephone 202-326-6441; fax 202-371-9849; email spasco@aaas.org. Further information is also available at http://www.ams.org/policy/government/fellow-awards/fellow-awards and through the AMS Washington Office, 1527 Eighteenth Street, NW, Washington, DC 20036; telephone 202-588- 1100; fax 202-588-1853; email amsdc@ams.

January 21, 2011: Applications for Math for America Foundation (MfA) Fellowship Program. See "Mathematics Opportunities" in this issue.

January 27, 2011: Proposals for NSF Computing Equipment and Instrumentation Programs (SCREMS). See "Mathematics Opportunities" in this issue.

February 1, 2011: Applications for AWM Mentoring Travel Grants. See http://www.awm-math.org/travelgrants.html#standard.

February 1, 2011: Applications for AWM Travel Grants. See http://www.awm-math.org/travelgrants.html#standard.

February 4, 2011: Full proposals for NSF Mathematical Sciences Research Institutes. See http://www.nsf.gov/pubs/2010/nsf10592/nsf10592.htm?WT.mc_id=USNSF_25&WT.mc_ev=click.

February 15, 2011: Applications for AMS Congressional Fellowship. See http://www.ams.org/programs/ams-fellowships/ams-aaas/ams-aaas-congressional-fellowship or contact the AMS Washington Office at 202-588-1100, email: amsdc@ams.org.

February 21, 2011: Applications for EDGE for Women Summer Program. See http://www.edgeforwomen.org/?page_id=5.

February 27, 2011: Entries for Association for Women in Mathematics (AWM) Essay Contest. See "Mathematics Opportunities" in this issue.

May 1, 2011: Applications for National Academies Christine Mirzayan Graduate Fellowship Program for fall 2011. See "Mathematics Opportunities" in this issue.

May 1, 2011: Applications for AWM Travel Grants. See http://www.awm-math.org/travelgrants.html#standard.

October 1, 2011: Applications for AWM Travel Grants. See http://www.awm-math.org/travelgrants.html#standard.

October 1, 2011: Nominations for the 2012 Emanuel and Carol Parzen Prize. Contact Thomas Wehrly, Department of Statistics, 3143 TAMU, Texas A&M University, College Station, Texas 77843-3143.

NSF Mathematics Education Staff

The Directorate for Education and Human Resources (EHR) of the National Science Foundation (NSF) sponsors a range of programs that support educational projects in mathematics, science, and engineering. Listed below is contact information for those EHR program officers whose fields are in the mathematical sciences or mathematics education. These individuals can provide information about the programs they oversee, as well as information about other EHR programs of interest to mathematicians. The postal address is: Directorate for Education and Human Resources, National Science Foundation, 4201 Wilson Boulevard, Arlington, VA 22230. The EHR webpage is http://www.nsf.gov/dir/index.jsp?org=EHR.

Office of the Assistant Director

Joan Ferrini Mundy
Acting Assistant Director
703-292-4682
jferrini@nsf.gov

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Lee Zia
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IMU Executive Committee

The Executive Committee of the International Mathematical Union (IMU) consists of ten voting members elected for four-year terms: the four officers (president, two vice presidents, and secretary) and six other members. The retiring president is an ex-officio member of the Executive Committee without vote for a period of four years. The current members (terms January 1, 2011, to December 31, 2014) of the IMU Executive Committee are:

President:
Ingrid Daubechies (United States)

Secretary:
Martin Grötschel (Germany)

Vice Presidents:
Christiane Rousseau (Canada)
Marcelo Viana (Brazil)

Members at Large:
Manuel de León (Spain)
Yiming Long (China)
Cheryl E. Praeger (Australia)
Vasudevan Srinivas (India)
John Francis Toland (United Kingdom)
Wendelin Werner (France)

Ex Officio:
László Lovász, Past President (Hungary)

Book List

The Book List highlights books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. When a book has been reviewed in the Notices*, a reference is given to the review. Generally the list will contain only books published within the last two years, though exceptions may be made in cases where current events (e.g., the death of a prominent mathematician, coverage of a certain piece of mathematics in the news) warrant drawing readers' attention to older books. Suggestions for books to include on the list may be sent to* `notices-booklist@ams.org`.

*Added to "Book List" since the list's last appearance.

Apocalypse When?: Calculating How Long the Human Race Will Survive, by Willard Wells. Springer Praxis, June 2009. ISBN-13:978-03870-983-64.

The Archimedes Codex: How a Medieval Prayer Book Is Revealing the True Genius of Antiquity's Greatest Scientist, by Reviel Netz and William Noel. Da Capo Press, October 2007. ISBN-13: 978-03068-1580-5. (Reviewed September 2008.)

Bright Boys: The Making of Information Technology, by Tom Green. A K Peters, April 2010. ISBN-13: 978-1-56881-476-6.

The Calculus of Friendship: What a Teacher and Student Learned about Life While Corresponding about Math, by Steven Strogatz. Princeton University Press, August 2009. ISBN-13: 978-0-691-13493-2. (Reviewed June/July 2010.)

The Cult of Statistical Significance: How the Standard Error Costs Us Jobs, Justice, and Lives, by Stephen T. Ziliak and Deirdre N. McCloskey, University of Michigan Press, February 2008. ISBN-13: 978-04720-500-79. (Reviewed October 2010.)

Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics, by Amir Alexander. Harvard University Press, April 2010. ISBN-13: 978-06740-466-10. (Reviewed November 2010.)

Euler's Gem: The Polyhedron Formula and the Birth of Topology, by David S. Richeson. Princeton University Press, September 2008. ISBN-13: 97-80691-1267-77. (Reviewed in this issue.)

Here's Looking at Euclid: A Surprising Excursion through the Astonishing World of Math, by Alex Bellos. Free Press, June 2010. ISBN-13: 978-14165-882-52.

The Housekeeper and the Professor, by Yoko Ogawa. Picador, February 2009. ISBN-13: 978-03124-278-01. (Reviewed May 2010.)

How to Read Historical Mathematics, by Benjamin Wardhaugh. Princeton University Press, March 2010. ISBN-13: 978-06911-401-48.

Isaac Newton on Mathematical Certainty and Method, by Niccolò Guicciardini. MIT Press, October 2009. ISBN-13: 978-02620-131-78.

Logicomix: An Epic Search for Truth, by Apostolos Doxiadis and Christos Papadimitriou. Bloomsbury USA, September 2009. ISBN-13: 978-15969-145-20. (Reviewed in this issue.)

Logic's Lost Genius: The Life of Gerhard Gentzen, by Eckart Menzler-Trott, Craig Smorynski (translator), Edward R. Griffor (translator). AMS-LMS, November 2007. ISBN-13: 978-0-8218-3550-0.

The Mathematical Mechanic: Using Physical Reason to Solve Problems, by Mark Levi. Princeton University Press, 2009. ISBN-13: 978-0691140209.

Mathematicians: An Outer View of the Inner World, by Mariana Cook. Princeton University Press, June 2009. ISBN-13: 978-0-691-13951-7. (Reviewed August 2010.)

Mathematicians Fleeing from Nazi Germany: Individual Fates and Global Impact, by Reinhard Siegmund-Schultze. Princeton University Press, July 2009. ISBN-13: 978-0-691-14041-4. (Reviewed November 2010.)

Mathematics in Ancient Iraq: A Social History, by Eleanor Robson. Princeton University Press, August 2008. ISBN-13: 978-06910-918-22. (Reviewed March 2010.)

Mathematics in India, by Kim Plofker. Princeton University Press, January 2009. ISBN-13: 978-06911-206-76. (Reviewed March 2010.)

The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook, by Victor J. Katz et al. Princeton University Press, July 2007. ISBN-13: 978-0-6911-2745-3.

**A Motif of Mathematics: History and Application of the Mediant and the Farey Sequence*, by Scott B. Guthery. Docent Press, September 2010. ISBN-13 978-4538-105-76.

More Mathematical Astronomy Morsels, by Jean Meeus. Willmann-Bell, 2002. ISBN 0-943396743.

Mrs. Perkins's Electric Quilt: And Other Intriguing Stories of Mathematical Physics, Paul J. Nahin, Princeton University Press, August 2009. ISBN-13: 978-06911-354-03.

Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity, by Loren Graham and Jean-Michel Kantor. Belknap Press of Harvard University Press, March 2009. ISBN-13: 978-06740-329-34.

Nonsense on Stilts: How to Tell Science from Bunk, by Massimo Pigliucci. University of Chicago Press, May 2010. ISBN-13: 978-02266-678-67.

Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present, by George G. Szpiro. Princeton University Press, April 2010. ISBN-13: 978-06911-399-44.

The Numerati, by Stephen Baker. Houghton Mifflin, August 2008. ISBN-13: 978-06187-846-08. (Reviewed October 2009.)

Our Days Are Numbered: How Mathematics Orders Our Lives, by Jason Brown. Emblem Editions, April 2010. ISBN-13: 978-07710-169-74.

A Passion for Discovery, by Peter Freund. World Scientific, August 2007. ISBN-13: 978-9-8127-7214-5

Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century, by Masha Gessen. Houghton Mifflin Harcourt, November 2009. ISBN-13: 978-01510-140-64.

Pioneering Women in American Mathematics: The Pre-1940 Ph.D.'s, by Judy Green and Jeanne LaDuke. AMS, December 2008. ISBN-13:978-08218-4376-5.

Plato's Ghost: The Modernist Transformation of Mathematics, by Jeremy Gray. Princeton University Press, September 2008. ISBN-13: 978-06911-361-03. (Reviewed February 2010.)

Probabilities: The Little Numbers That Rule Our Lives, by Peter Olofsson. Wiley, March 2010. ISBN-13: 978-04706-244-56.

Proofs from THE BOOK, by Martin Aigner and Günter Ziegler. Expanded fourth edition, Springer, October 2009. ISBN-13: 978-3-642-00855-9

Pythagoras' Revenge: A Mathematical Mystery, by Arturo Sangalli. Princeton University Press, May 2009. ISBN-13: 978-06910-495-57. (Reviewed May 2010.)

Recountings: Conversations with MIT Mathematicians, edited by Joel Segel. A K Peters, January 2009. ISBN-13: 978-15688-144-90.

Roger Boscovich, by Radoslav Dimitric (Serbian). Helios Publishing Company, September 2006. ISBN-13: 978-09788-256-21.

Sacred Mathematics: Japanese Temple Geometry, by Fukagawa Hidetoshi and Tony Rothman. Princeton University Press, July 2008. ISBN-13: 978-0-6911-2745-3.

The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions, by Shing-Tung Yau (with Steve Nadis). Basic Books, September 2010. ISBN-13: 978-04650-202-32.

The Solitude of Prime Numbers, by Paolo Giordano. Pamela Dorman Books, March 2010. ISBN-13: 978-06700-214-82. (Reviewed September 2010.)

Solving Mathematical Problems: A Personal Perspective, by Terence Tao. Oxford University Press, September 2006. ISBN-13: 978-0-199-20560-8. (Reviewed February 2010.)

Sphere Packing, Lewis Carroll, and Reversi, by Martin Gardner. Cambridge University Press, July 2009. ISBN-13: 978-0521756075.

The Strangest Man, by Graham Farmelo. Basic Books, August 2009. ISBN-13: 978-04650-182-77.

Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving, by Sanjoy Mahajan. MIT Press, March 2010. ISBN-13: 978-0-262-51429-3.

Survival Guide for Outsiders: How to Protect Yourself from Politicians, Experts, and Other Insiders, by Sherman Stein. BookSurge Publishing, February 2010. ISBN-13: 978-14392-532-74.

Symmetry in Chaos: A Search for Pattern in Mathematics, Art, and Nature, by Michael Field and Martin Golubitsky. Society for Industrial and Applied Mathematics, second revised edition, May 2009. ISBN-13: 978-08987-167-26.

Teaching Statistics Using Baseball, by James Albert. Mathematical Association of America, July 2003. ISBN-13: 978-08838-572-74. (Reviewed April 2010.)

Tools of American Math Teaching, 1800–2000, by Peggy Aldrich Kidwell, Amy Ackerberg-Hastings, and David Lindsay Roberts. Johns Hopkins University Press, July 2008. ISBN-13: 978-0801888144. (Reviewed January 2010.)

What's Luck Got to Do with It? The History, Mathematics and Psychology of The Gambler's Illusion, by Joseph Mazur. Princeton University Press, July 2010. ISBN: 978-0-691-13890-9.

The Unfinished Game: Pascal, Fermat, and the Seventeenth-Century Letter That Made the World Modern, by Keith Devlin. Basic Books, September 2008. ISBN-13: 978-0-4650-0910-7.

Zeno's Paradox: Unraveling the Ancient Mystery behind the Science of Space and Time, by Joseph Mazur. Plume, March 2008 (reprint edition). ISBN-13: 978-0-4522-8917-8.

Mathematics Calendar

December 2010

2–4 **4th Global Conference on Power Control and Optimization (PCO'2010)**, Damai Puri Resort, Kucing, Sarawak, Malaysia. (May 2010, p. 675)

4–5 **Palmetto Number Theory Series XIV**, University of South Carolina, Columbia, South Carolina. (Oct. 2010, p. 1165)

4–6 **2010 CMS Winter Meeting**, Coast Hotel and Suites, Vancouver (BC), Canada. (Jun./Jul. 2010, p. 784)

5–10 **66th Annual Deming Conference on Applied Statistics**, Atlantic City, New Jersey. (Jun./Jul. 2010, p. 784)

* 6–7 **International Conference on Education and Management Technology, ICEdMaT '10**, Open Learning Society, Pokhera, Nepal.
Description: This conference provides opportunities for the delegates to exchange new ideas and application experiences face to face, to establish business or research relations and to find global partners for future collaboration. Moreover Emerge of education & Management Technologies, popularly known as Technology, has brought the use of computing devices to a new level.
Information: `http://www.icedmat.com`.

6–10 **AIM Workshop: Waves and multiscale processes in the tropics**, American Institute of Mathematics, Palo Alto, California. (May 2010, p. 675)

6–10 **34ACCMCC: The 34th Australasian Conference on Combinatorial Mathematics & Combinatorial Computing**, The Australian National University, Canberra, Australia. (Oct. 2010, p. 1165)

6–10 **MSRI—Random Matrix Theory and its Applications II**, Mathematical Sciences Research Institute, Berkeley, California. (Dec. 2009, p. 1482)

9–11 **International Conference on Recent Development in Mathematical Sciences and its Applications (ICRDMSA-2010)**, Calcutta Mathematical Society AE-374, Sector-I, Salt Lake Kolkata-700064, West-Bengal, India. (Aug. 2010, p. 904)

10–14 **School on Information and Randomness 2010**, Universidad de la Frontera, Pucón, Chile. (Oct. 2010, p. 1165)

13–17 **AIM Workshop: Random matrices**, American Institute of Mathematics, Palo Alto, California. (May 2010, p. 676)

13–17 **Conference in Geometry and Global Analysis Celebrating P. Gilkey's 65th Birthday**, Universidade de Santiago de Compostela/ Universidade da Coruña Santiago de Compostela, A Coruña, Spain. (Jun./Jul. 2010, p. 784)

13–17 **Conference "Quantization of Singular Spaces"**, University of Aarhus, Denmark. (Jun./Jul. 2010, p. 784)

15–16 **The International Conference on Mathemtics Education Research**, Malacca, Malaysia. (May 2010, p. 676)

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.
An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.
In general, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to `notices@ams.org` or `mathcal@ams.org`.
In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.
The complete listing of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.
The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: `http://www.ams.org/`.

17–21 **The 15th Asian Technology Conference in Mathematics (ATCM 2010)**, University of Malaya, Kuala Lumpur, Malaysia. (Apr. 2010, p. 551)

17–22 **International Congress of Chinese Mathematicians (ICCM)**, Opening ceremony on December 17 in the Great Hall of the People, Beijing, China. (Nov. 2010, p. 1347)

19–21 **"Mathematical Sciences for Advancement of Science and Technology" (MSAST 2010)**, IMBIC Hall, Salt Lake, Kolkata (Calcutta), India. (Apr. 2010, p. 551)

* 22–24 **CONIAPS XII:12th Conference of the International Academy of Physical Sciences**, University of Rajasthan, Jaipur, India. (Nov. 2010, p. 1347)
Focal Theme: Emerging Interfaces of Physical Sciences.
Information: `http://www.iaps.in`.

22–24 **XIIth International Conference of International Academy of Physical Sciences (CONIAPS XII)**, University of Rajasthan, Jaipur, India. (Nov. 2010, p. 1347)

25–27 **International Conference on Current trends in Mathematics**, Allahabad, Uttar Pradesh, India. (May 2009, p. 659)

27–30 **The 76th Annual Conference of the Indian Mathematical Society**, S. V. National Institute of Technology, SURAT -395007, Gujarat, India. (Jun./Jul. 2010, p. 784)

29–31 **ICCAM 2010: "International Conference on Computational and Applied Mathematics" Symposium Partial Differential Equations:Modeling, Analysis and Numerical Methods**, First Hotel Bangkong 2 Soi Somprasong 1, Petchaburi Road, Tanonphayathai, Rajthavee, Bangkok 10400 Thailand. (Nov. 2009, p. 1361)

January 2011

3–5 **ICMS 2011: International Conference on Mathematical Sciences in Honour of Professor A. M. Mathai**, St. Thomas College Pala, Kottayam 686574, Kerala, India. (May 2010, p. 676)

* 3–5 **IMA Special Event: First Abel Conference: A Mathematical Celebration of John Tate**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota.
Description: The Abel Conference is an annual conference series that focuses on the mathematics of the Abel Prize winners. The conference topics will be centered around the important mathematical areas impacted by the work of the Abel Prize recipient, and when appropriate, on the impact of the work on applications. The first conference will be in honor of John Tate, recipient of the 2010 Abel Prize.
Information: `http://www.ima.umn.edu/2010-2011/SW1.3-5.11/`.

* 5–7 **First International Conference on Algebra, Topology and Topological Algebras**, Facultad de Ingenierĭa of the Universidad Veracruzana, Calzada Adolfo Ruiz Cortines s/n, Fraccionamiento Costa Verde, Boca del Río, Veracruz, México.
Description: The essential character of Topological Algebras is the simultaneous consideration of two structures on the same set. An algebraic structure, that of an algebra, and a topological structure ensuring the consistency in between algebra and topology, leading as vital components, to an important object of Functional Analysis that we call Topological Algebras. Among the main applications of topological algebras can be mentioned: differential geometry of smooth manifolds, mathematical physics, quantum relativity, quantum cosmology, topological homological algebra, topological algebraic geometry, sheaf theory and K-theory. This event intends to gather specialists in the interaction of these fields.
Invited speakers: Mati Abel, Hugo Arizmendi, Marina Haralampidou, Mohamed Oudadess, Manuel Sanchis, Thomas Tonev, and Wiesław Zelazko.
Information: `http://icatta.izt.uam.mx`.

6–9 **Joint Mathematics Meetings**, New Orleans, Louisiana. (Oct. 2010 p. 1192)

7–8 **First Lisbon Research Workshop on Economics and Econometrics of Education**, ISEG: Institute for Economics and Management, Lisbon Technical University, Lisbon, Portugal. (Nov. 2010 p. 1348)

10–14 **AIM Workshop: Sustainability Problems**, American Institute of Mathematics, Palo Alto, California. (Nov. 2010 p. 1348)

10–14 **Algorithmic Game Theory**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California. (Jun./Jul. 2010, p. 784)

10–14 **IMA Workshop: High Performance Computing and Emerging Architectures**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

10–May 20 **MSRI Future Scientific Programs: Arithmetic Statistics**, Mathematical Sciences Research Institute, Berkeley, California. (Aug. 2009, p. 864)

10–May 20 **MSRI Future Scientific Programs: Free Boundary Problems, Theory and Applications**, Mathematical Sciences Research Institute, Berkeley, California. (Aug. 2009, p. 864)

12–14 **Statistical Methods for Meteorology and Climate Change**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 904)

13–14 **Connections for Women: Free Boundary Problems, Theory and Applications**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 784)

14–16 **2011 International Conference on Intelligent Structure and Vibration Control (ISVC 2011)**, Chongqing, China. (Oct. 2010 p. 11)

17–21 **AIM Workshop: Deformation theory, patching, quadratic forms, and the Brauer group**, American Institute of Mathematics, Palo Alto, California. (May 2010, p. 676)

18–21 **Efficiency of the Simplex Method; Quo Vadis Hirsch Conjecture?**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California. (Jun./Jul. 2010, p. 785)

18–21 **Introductory Workshop: Free Boundary Problems, Theory and Applications**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)

* 21–22 **National Conference on Recent Frontiers in Applied Dynamical Systems (NCRFADS 2011)**, Department of Mathematics, School of Science and Humanities, Karunya University, Karunya Nagar, Coimbatore- 641 114, Tamil Nadu, India.
Description: The objective of this conference is to share the recent advancement and progress in the theory and applications of dynamical systems. Post graduate, research scholars, scientists and academicians in all fields of differential equations, dynamical systems and applications including modeling and computations are invited to attend and share their experiences in the field of their expertise. This conference will provide unique national forum where exciting interactions and communications would take place among the participants and also brings fruitful directions and collaborations to the national community in the field of dynamical systems. The participants will certainly benefit from this conference as it provides a great opportunity to foster their knowledge in dynamical systems and also their friendship among them with other participants.
Information: `http://www.karunya.edu/sh/maths/ncrfads2011`.

* 22 **Workshop on Analytic Algorithmics and Combinatorics**, Holiday Inn San Francisco Golden Gateway, San Francisco, California.
Description: The aim of the ANALCO workshop is to provide a forum for the presentation of original research in the analysis of algorithms and associated combinatorial structures. We invite both papers that

study properties of fundamental combinatorial structures that arise in practical computational applications (such as permutations, trees, strings, tries, and graphs) and papers that address the precise analysis of algorithms for processing such structures, including: average-case analysis; analysis of moments, extrema, and distributions; probabilistic analysis of randomized algorithms, and so on. Submissions that present significant new information about classic algorithms are welcome, as are new analyses of new algorithms that present unique analytic challenges. We also invite submissions that address tools and techniques for the analysis of algorithms and combinatorial structures, both mathematical and computational.
Information: `http://www.siam.org/meetings/analco11/`.

* 22 **Workshop on Algorithm Engineering and Experiments**, Holiday Inn San Francisco Golden Gateway, San Francisco, California.
Description: The aim of the ALENEX workshop is to provide a forum for presentation of original research in the implementation and experimental evaluation of algorithms and data structures. We invite submissions that present significant case studies in experimental analysis (such studies may tighten, extend, or otherwise improve current theoretical results) or in the implementation, testing, and evaluation of algorithms for realistic environments and scenarios, including specific applied areas (for example, databases, networks, operations research, computational biology and physics, computational geometry, and the World Wide Web) that present unique challenges in their underlying algorithmic problems. We also invite submissions that address methodological issues and standards in the context of empirical research on algorithms and data structures. The scientific program will include time for discussion and debate of topics in this rapidly evolving research area.
Information: `http://www.siam.org/meetings/alenex11/`.

23–25 **ACM-SIAM Symposium on Discrete Algorithms (SODA11)**, Holiday Inn, San Francisco Golden Gateway, San Francisco, California. (Mar. 2010, p. 434)

* 24–28 **DANCE Winter School RTNS2011 (Recent Trends in Nonlinear Science 2011)**, Vilanova i la Geltru, Spain.
Description: This is the eighth Winter School in Dynamical Systems of the DANCE Spanish network.
Courses: Yakov Pesin (PSU), Smooth ergodic theory; Gabor Stepan (Budapest Univ. Tech. Economics), Delay equations with applications to engineering; Yingfei Yo (Georgia Tech), Multi-frequency oscillations in dynamical systems.
Registration: The registration fee is 330 euros. It includes attendance and materials. People registering early are entitled to have a reduced fee of 300 euros. There will be a number of registration and registration plus accommodation grants for young participants.
Deadlines: Registration Period: from September 15th to November 14th, 2010. Registration Period with reduced fee or application for financial support: Before October 31st, 2010. Communication, by e-mail, of awarded grants: November 12th, 2010. Public Publication of the list of accepted participants: November 19th, 2010. Payment of Registration Fee: from November 22nd to December 23rd, 2010.
Information: `http://www.dance-net.org/rtns2011`.

26–28 **International Conference on Operations Research and Optimization 2011 (ORO2011)**, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran. (Sept. 2010 p. 1034)

27–28 **Connections for Women: Arithmetic Statistics**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)

31–February 4 **Brazilian Operator Algebras Symposium**, Pousada Villas del Sol y Mar, Jurere Beach, Florianopolis, Brazil. (Nov. 2010 p. 1165)

31–February 4 **Introductory Workshop: Arithmetic Statistics**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)

31–March 4 **Complex and Riemannian Geometry**, CIRM, Marseille, France. (Oct. 2010 p. 1165)

February 2011

* 1–12 **XI Winter Russian Diffiety School**, "Academic Gymnasium", St. Petersburg State University, St. Petersburg, Russia.
Description: The strategic goal of the Diffiety School is to produce experts in a new area of Mathematics called "Secondary Calculus". "Secondary Calculus" is the result of a natural evolution of the classical geometrical theory of partial differential equations (PDE) originated by Sophus Lie. In particular, it allows the construction of a general theory of PDE, in the same manner as algebraic geometry does with respect to algebraic equations. There are strong indications that secondary calculus may become a natural language for quantum field theory, just in the same way as standard calculus is for classical physics. From the mathematical point of view, Secondary Calculus is a complex mathematical construction putting into a natural interrelation many parts of modern mathematics such as commutative and homological algebra, algebraic and differential topology, differential geometry, etc.
Information: `http://www.levi-civita.org/Activities/DiffietySchools/xiwds`.

6–8 **3rd International Conference on Wireless Information Networks & Business Information System (WINBIS'11)**, Open Learning Society, Kathmandu, Nepal. (Jun./Jul. 2010, p. 785)

7–12 **Complex Geometry—Extremal metrics: Evolution equations and stability**, CIRM, Marseille, France. (Aug. 2010, p. 904)

12–17 **Combinatorics and Analysis in Spatial Probability - ESF-EMS-ERCOM Conference**, Eindhoven, The Netherlands. (Nov. 2010 p. 1348)

14–19 **Fourth School and Workshop on Mathematical Methods in Quantum Mechanics**, Casa della Gioventù, University of Padova, Bressanone, Italy. (Aug. 2010, p. 904)

16–19 **International Conference on Operator Theory**, Monastir, Tunisia. (Jun./Jul. 2010, p. 785)

18–20 **Second International Conference on Emerging Applications of Information Technology (EAIT 2011)**, Kolkata, India. (Oct. 2010 p. 1165)

19–20 **Palmetto Number Theory Series XV**, Clemson University, Clemson, South Carolina. (Oct. 2010 p. 1165)

* 21–25 **NCTS(Taiwan)-CPT(France) Joint Workshop on Symplectic Geometry and Quantum Symmetries in Mathematical Physics**, National Center for Theoretical Sciences, National Tsing-Hua University, Hsin-Chu, Taiwan.
Description: An NCTS-CPT Joint Workshop on "Symplectic Geometry and Quantum Symmetries in Mathematical Physics" will be held at the National Center for Theoretical Sciences in Taiwan on February 21-25, 2011. The aim of this workshop is to bring together mathematicians and mathematical physicists with various backgrounds to discuss prospective advances in the areas covered by the title. Another purpose is to foster interactions between several groups of French and Taiwanese scientists working in those fields. Interested researchers from all countries are invited to attend. We hope to attract Ph.D. students, post doctoral students and young scholars.
Information: `http://math.cts.nthu.edu.tw/Mathematics/2011Taiwan-FranceWorkshop.htm`.

25–26 **International Conference on Logic, Information, Control and Computation -ICLICC 2011**, Department of Mathematics, Gandhigram Rural Institute, Deemed University, Gandhigram, Dindigul, Tamil Nadu, India. (Nov. 2010 p. 1348)

25—March 5 **The Homotopy Interpretation of Constructive Type Theory**, Oberwolfach Mathematical Research Institute, Oberwolfach, Germany. (Nov. 2010 p. 1348)

* 28–March 4 **Geometric flows in finite or infinite dimension**, CIRM, Marseille, France.
Description: This workshop will gather pure and applied mathematicians interested in geometric evolution equations in finite or infinite dimension. As far as the chosen topics are concerned, we will try to focus on geometric flows which are less known than the traditional Ricci-flow and to present also significant applications outside geometry. These flows, perhaps less known or less "popular", appear naturally in various geometric contexts (Riemannian, pseudo-Riemannian or sub-Riemannian geometry), and also on infinite dimensional objects (e.g. diffeomorphism groups). Our objective is to present applications of geometric flows to other fields than geometry, like mathematical physics, mechanics, image processing, etc. The topics covered by the workshop will include: mean curvature flow, Gaussian curvature flow, geodesic flows on diffeomorphism groups and spaces of embeddings, applications of geodesic flows in fluid mechanics, applications of geometric flows in image processing.
Information: `http://www.latp.univ-provence.fr/geom2011/index.php/welcome/week5`.

28–March 4 **SIAM Conference on Computational Science and Engineering (CSE11)**, Grand Sierra Resort and Casino, Reno, Nevada. (Jun./Jul. 2010, p. 785)

March 2011

* 2–5 **Integration, Vector Measures and Related Topics IV. Dedicated to Joe Diestel**, University of Murcia, Murcia, Spain.
Aim: Of this four day conference is to bring together experienced and novice researchers interested in Integration, Vector Measures and their Applications. The conference will feature a series of plenary and short lectures as well as a mini-course and contributed posters on recent advances in the subject. The previous meetings of this series of conferences were held in Valencia in 2004, Sevilla in 2006 and Eichstätt in 2008.
Support: Partial support for a small number of participants is expected to be available. Recent recipients of doctoral degrees and predoc students are encouraged to apply.
Organizer: The Functional Analysis Group of the University of Murcia.
Sponsors: UMU, MCIN, iMath Consolider, Fundacion Seneca CARM.
Information: `http://www.um.es/beca/Murcia2011/`.

7–11 **Free Boundary Problems, Theory and Applications Workshop**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)

7–11 (NEW DATE) **IMA Workshop: Computing in Image Processing, Computer Graphics, Virtual Surgery, and Sports**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

12–13 **AMS Southeastern Section Meeting**, Georgia Southern University, Statesboro, Georgia. (Sept. 2010, p. 1034)

14–June 10 **Probability and Discrete Mathematics in Mathematical Biology**, Institute for Mathematical Sciences, National University of Singapore, Singapore. (Nov. 2010, p. 1348)

14–June 17 **Navigating Chemical Compound Space for Materials and BioDesign**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California. (Jan. 2010, p. 76)

16–18 **IAENG International Conference on Operations Research 2011**, Royal Garden Hotel, Kowloon, Hong Kong. (Aug. 2010, p. 904)

17–19 **The 45th Annual Spring Topology and Dynamical Systems Conference**, University of Texas at Tyler, Tyler, Texas. (Oct. 2010, p. 1165)

18–20 **AMS Central Section Meeting**, University of Iowa, Iowa City, Iowa. (Sept. 2010, p. 1034)

* 21–24 **SIAM Conference on Mathematical & Computational Issues in the Geosciences**, Hilton Long Beach & Executive Meeting Center, Long Beach, California.
Description: From points of view ranging from science to public policy, there is a growing interest in modeling and simulation of geosystems and their applications. Some examples include petroleum exploration and recovery, underground waste disposal and cleanup of hazardous waste, earthquake prediction, weather prediction, and global climate change. Such modeling is fundamentally interdisciplinary; physical and mathematical modeling at appropriate scales, physical experiments, mathematical theory, probability and statistics, numerical approximations, and large-scale computational algorithms all have important roles to play.
Information: `http://www.siam.org/meetings/gs11/`.

21–25 **AIM Workshop: Hypergraph Turán Problem**, American Institute of Mathematics, Palo Alto, California. (Aug. 2010, p. 905)

28–April 1 **International Conference on Homotopy and Non-Commutative Geometry**, Batumi State University, Batumi, Republic of Georgia. (Jun./Jul. 2010, p. 785)

28–April 1 **International workshop: Unlikely intersections in algebraic groups and Shimura varieties**, Scuola Normale Superiore, Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy. (Oct. 2010, p. 1165)

April 2011

2–3 **Midwest Graduate Student Topology and Geometry Conference**, Michigan State University, East Lansing, Michigan. (Oct. 2010, p. 1165)

7–9 **ICMCS'11 - IEEE co-sponsored Conference 2nd International Conference on Multimedia Computing and Systems**, Ouarzazate, Morocco. (Aug. 2010, p. 905)

9–10 **AMS Eastern Section Meeting**, College of the Holy Cross, Worcester, Massachusetts. (Sept. 2010, p. 1034)

11–15 **Arithmetic Statistics**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)

11–15 **IMA Workshop: Societally Relevant Computing**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

14–16 **The First International Conference on Multidimensional Finance, Insurance and Investment: ICMFII 2011**, Hammamet, Tunisia. (Nov. 2010, p. 1348)

17–19 **7th IMA Modelling in Industrial Maintenance and Reliability**, Sidney Sussex College, University of Cambridge, United Kingdom. (Sept. 2010, p. 1035)

18–22 **Computational Statistical Methods for Genomics and Systems Biology**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

27–28 **Third Conference on Mathematical Sciences (CMS'2011)**, Department of Mathematics, Faculty of Science and Information Technology, Zarqa Private University, Zarqa, Jordan. (Aug. 2010, p. 905)

* 28–30 **SIAM International Conference on Data Mining**, Hilton Phoenix East/Mesa, Mesa, Arizona.
Description: Data mining is an important tool in science, engineering, industrial processes, healthcare, business, and medicine. The datasets in these fields are large, complex, and often noisy. Extracting knowledge requires the use of sophisticated, high performance and principled analysis techniques and algorithms, based on sound theoretical and statistical foundations. These techniques in turn require implementations that are carefully tuned for performance; powerful

visualization technologies; interface systems that are usable by scientists, engineers, and physicians as well as researchers; and infrastructures that support them.
Information: http://www.siam.org/meetings/sdm11/.

30–May 1 **AMS Western Section Meeting**, University of Nevada, Las Vegas, Nevada. (Sept. 2010, p. 1035)

May 2011

1–August 31 **MITACS International Focus Period on Advances in Network Analysis and its Applications**, Locations throughout Canada. (Apr. 2010, p. 552)

2–4 **Statistical Issues in Forest Management**, Université Laval, Québec City, Canada. (Aug. 2010, p. 905)

9–13 **Causal Inference in Health Research**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

16–19 **Analysis of Survival and Event History Data**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

* 16–19 **SIAM Conference on Optimization**, Darmstadtium Conference Center, Darmstadt, Germany.
Description: The SIAM Conference on Optimization will feature the latest research in theory, algorithms, and applications in optimization problems. In particular, it will emphasize large-scale problems and will feature important applications in networks, manufacturing, medicine, biology, finance, aeronautics, control, operations research, and other areas of science and engineering. The conference brings together mathematicians, operations researchers, computer scientists, engineers, and software developers; thus it provides an excellent opportunity for sharing ideas and problems among specialists and users of optimization in academia, government, and industry.
Information: http://www.siam.org/meetings/op11/.

22–26 **SIAM Conference on Applications of Dynamical Systems (DS11)**, Snowbird Ski and Summer Resort, Snowbird, Utah. (Mar. 2010, p. 434)

* 22–27 **Progress on Difference Equations 2011**, Dublin City University, Dublin, Ireland.
Description: The conference is about difference equations, discrete dynamical systems, and their applications in all fields. A special theme will be their applications to Financial Mathematics.
Information: http://www.dcu.ie/maths/pode2011/index.shtml.

* 25–28 **Sixth International Conference on Dynamic Systems and Applications**, Morehouse College, Atlanta, Georgia.
Topics: Dynamical systems, computational mathematics, simulation, stochastic/deterministic: Differential equations, partial differential equations, integral equations, integro-differential equations, difference equations, and related topics.
Information: http://www.dynamicpublishers.com/icdsa6.htm.

June 2011

5–7 **National Conference On Nonlinear Analysis and Applications**, Department of Mathematics, H.N.B. Garhwal University, Campus Pauri, Pauri Garhwal, Uttarakhand. (Jun./Jul. 2010, p. 786)

6–9 **Copula Models and Dependence**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

6–10 **IMA Workshop: Large-scale Inverse Problems and Quantification of Uncertainty**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

* 6–10 **Low-dimensional manifolds and high-dimensional categories**, UC Berkeley, Berkeley, California.
Description: The conference will be in honor of Michael Freedman's 60th birthday. One of the main goals of this conference is to promote cross-fertilization between experts on 4-manifolds and experts on higher categories and quantum field theories.
Organizers: Ian Agol, UC Berkeley; Rob Kirby, UC Berkeley; Slava Krushkal, University of Virginia; Peter Teichner, UC Berkeley; Abigail Thompson, UC Davis; Kevin Walker, Microsoft Research.
Information: email: ianagol@math.berkeley.edu.

13–16 **2011 International Conference on Applied Mathematics and Interdisciplinary Research**, Nankai University, Tianjin, China. (Sept. 2010, p. 1166)

13–17 **AIM Workshop: The Cohen-Lenstra heuristics for Class Groups**, American Institute of Mathematics, Palo Alto, California. (Nov. 2010, p. 1349)

13–19 **Strobl2011 - From Abstract to Computational Harmonic Analysis**, Bifeb, Strobl, Salzburg, Austria. (Nov. 2010, p. 1349)

14–17 **2011 World Conference on Natural Resource Modeling**, Ottowa, Canada. (Sept. 2010, p. 1035)

* 16–18 **Lorentz Geometry in Mathematics and in Physics**, Institut de Recherche Mathématique Avancée, University of Strasbourg, France.
Description: This is the 87th conference in the series "Encounters between Mathematicians and Theoretical Physicists". The theme is "Lorentz Geometry in Mathematics and in Physics". The Encounter is dedicated to Norbert A'Campo for his 70th birthday.
Organizers: The organisers are Charles Boubel and Athanase Papadopoulos.
Invited speakers: Thierry Barbot (Avignon), Mauro Carfora (Pavia), Mihalis Dafermos (Cambridge), Charles Frances (Orsay), Jacques Franchi (Strasbourg), Helmut Friedrich (MPI, Golm), Kirill Krasnov (Nottingham), Catherine Meusburger (Hamburg), Karim Noui (Tours), Vladimir Matveev (Jena), Jean-Marc Schlenker (Toulouse), Abdelghani Zeghib (ENS Lyon).
Language: All talks are in English. Some of the talks will be survey talks intended for a general audience.
Registration: Graduate students and young mathematicians are welcome. Registration is required (and free of charge), at the URL link. Hotel booking can be asked for through the registration link.
Information: http://www-irma.u-strasbg.fr/article1044.html.

* 19–25 **49th International Symposium on Functional Equations**, Graz, Austria.
Topics: Functional equations and inequalities, mean values, functional equations on algebraic structures, Hyers-Ulam stability, regularity properties of solutions, conditional functional equations, functional-differential equations, iteration theory; applications of the above, in particular to the natural, social, and behavioral sciences.
Local Organizer: Jens Schwaiger, Institut für Mathematik, Karl-Franzens-Universität Graz, Heinrichstr. 36, A-8010 Graz, Austria; email: jens.schwaiger@uni-graz.at.
Scientific Committee: J. Aczél (Honorary Chair; Waterloo, ON, Canada), W. Benz (Honorary Member, Hamburg, Germany), Z. Daróczy (Honorary Member, Debrecen, Hungary), R. Ger (Chair; Katowice, Poland), Zs. Pales (Debrecen, Hungary), J. Rätz (Bern, Switzerland), L. Reich (Graz, Austria), and A. Sklar (Chicago, IL, USA).
Information: Participation at these annual meetings is by invitation only. Those wishing to be invited to this or one of the following meetings send details of their interest and, preferably, publications (paper copies) and/or manuscripts with their postal and e-mail address to: Roman Ger, Institute of Mathematics, Silesian University, Bankowa 14, PL-40-007 Katowice, Poland (romanger@us.edu.pl) before December 15, 2010.

22–24 **3rd IMA International Conference Mathematics in Sport**, The Lowry, Salford Quays, United Kingdom. (Sept. 2010, p. 1035)

July 2011

* 3–9 **22nd International Workshop on Operator Theory and Applications (IWOTA 2011)**, Universidad de Sevilla, Spain.
Description: IWOTA workshops bring together mathematicians and engineers working in operator theory and its applications to related fields, ranging from classical analysis, differential and integral equations, complex and harmonic analysis to mathematical physics, mathematical system and control theory, signal processing and numerical analysis. IWOTA gathers leading experts from all over the world for an intensive exchange of information and opinion, and for tracing the future developments in the field.
Information: `http://congreso.us.es/iwota2011`.

* 4–8 **Conference on Several Complex Variables on the Occasion of Professor Jozef Siciak' 80th birthday**, Jagiellonian University, Krakow, Poland.
Description: The main aim of the conference is to promote, encourage, cooperate, and bring together researchers in the fields of differential & difference equations. All areas of differential & difference equations will be represented with special emphasis on applications. It will be a mathematically enriching and socially exciting event.
Information: `http://gamma.im.uj.edu.pl/~complex2011/`.

4–10 **International Conference on Topology and its Applications (ICTA), 2011**, Department of Mathematics, COMSATS Institute of Information Technology (CIIT), Islamabad, Pakistan. (Oct. 2010, p. 1166)

* 5–8 **International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA-2011)**, Babeş-Bolyai University, Cluj-Napoca, Romania.
Description: This conference follows the previous successful editions of ICNODEA held in Cluj-Napoca in 2001, 2004 and 2007. The purpose of ICNODEA-2011 is to explore new developments in the theory of nonlinear operators and its applications to integral, differential and partial differential equations and inclusions.
Main topics: Fixed point theory and its applications; critical point theory and its applications; nonlinear ODES, PDEs and Integral equations; dynamical systems; applications in biology, medicine, economics, physics and engineering. The Program will include: 50-min. plenary lectures; 30- min. invited lectures; 20-min. contributed talks and a poster session.
Information: `http://www.cs.ubbcluj.ro/~icnodeacj/index.htm`.

6–8 **The 2011 International Conference of Applied and Engineering Mathematics**, Imperial College London, London, United Kingdom. (Nov. 2010, p. 1349)

6–8 **IMA Conference on Nonlinearity and Coherent Structures**, University of Reading, United Kingdom. (Sept. 2010, p. 1035)

10–16 **International Conference on Rings and Algebras in Honor of Professor Pjek-Hwee Lee**, National Taiwan University, Taipei, Taiwan. (Sept. 2010, p. 1035)

11–15 **The 10th International Conference on Finite Fields and their Applications**, Ghent, Belgium. (Jun./Jul. 2010, p. 786)

12–15 **The 6th SEAMS-GMU 2011 International Conference on Mathematics and Its Applications; Workshop on Financial Mathematics and Workshop on Dynamical System in Biology**, Gadjah Mada University, Yogyakarta, Indonesia. (Oct. 2010, p. 1166)

* 15–30 **XIV Summer Diffiety School**, Levi-Civita Institute, Santo Stefano del Sole (AV), Italy.
Description: The strategic goal of the Diffiety Schools is to continuously produce experts in a new area of Mathematics called "Secondary Calculus". "Secondary Calculus" is the result of a natural evolution of the classical geometrical theory of partial differential equations (PDE) originated by Sophus Lie. In particular, it allows the construction of a general theory of PDE, in the same manner as algebraic geometry does with respect to algebraic equations. There are strong indications that secondary calculus may become a natural language for quantum field theory, just in the same way as standard calculus is for classical physics. From the mathematical point of view, Secondary Calculus is a complex mathematical construction putting into a natural interrelation many parts of modern mathematics such as commutative and homological algebra, algebraic and differential topology, differential geometry, etc.
Information: `http://www.levi-civita.org/Activities/DiffietySchools/xivds`

18–22 **7th International Congress on Industrial and Applied Mathematics - ICIAM 2011**, Vancouver, BC, Canada. (Jun./Jul. 2010, p. 786)

18–22 **Geometry & Topology Down Under – A Conference in Honour of Hyam Rubinstein**, The University of Melbourne, Australia. (Sept. 2010, p. 1035)

* 21–27 **Loops' 11**, Trest, Czech Republic.
Description: Loops '11 aspires to be a forum for all aspects of loops and quasigroups. We intend not only to highlight new relevant results in algebra and geometry, but also to foster connections to combinatorics, group theory, cryptography and physics. All manifestations of the quasigroup concept are welcome at the conference. A series of lectures and seminars, called Workshops Loops '11, will be held immediately before the conference and is aimed at both active researchers and graduate students. 21.-23.07. Workshops Loops'11 25.-27.07. Conference Loops'11
Information: `http://www.karlin.mff.cuni.cz/~loops11`.

25–27 **SIAM Conference on Control and Its Applications (CT11)**, Hyatt Regency Baltimore, Baltimore, Maryland. (Sept. 2010, p. 1035)

25–29 **AIM Workshop: Branching Problems for Unitary Representations**, Max Planck Institute for Mathematics, Bonn, Germany. (Nov. 2010, p. 1349)

26–29 **Conference in Harmonic Analysis and Partial Differential Equations in honour of Eric Sawyer**, Fields Institute, Toronto, Canada. (Jun./Jul. 2010, p. 786)

August 2011

1–5 **Categories, Geometry and Physics**, Santa Marta, Colombia. (Jun./Jul. 2010, p. 786)

1–5 **Conference in Honour of Søren Asmussen—New Frontiers in Applied Probability**, Sandbjerg Estate, Sønderborg, Denmark. (Jun./Jul. 2010, p. 786)

8–13 **Formal and analytic solutions of differential and difference equations**, Mathematical Research and Conference Center in Bedlewo, Poland. (Sept. 2010, p. 1036)

15-19 **AIM Workshop: Graph and Hypergraph Limits**, American Institute of Mathematics, Palo Alto, California. (Sept. 2010, p. 1036)

29–September 2 **11th International Workshop on Orthogonal Polynomials, Special Functions, and Applications**, Universidad Carlos III de Madrid, c/Universidad, 30 28911 Leganes, Madrid, Spain. (Nov. 2010, p. 1349)

September 2011

7–9 **IMA Hot Topics Workshop: Instantaneous Frequencies and Trends for Nonstationary Nonlinear Data**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Oct. 2010, p. 1166)

10–11 **AMS Eastern Section Meeting**, Cornell University, Ithaca, New York. (Sept. 2010, p. 1036)

10–16 **Turning Dreams into Reality: Transformations and Paradigm Shifts in Mathematics Education**, Rhodes University, Grahamstown, South Africa. (Feb. 2010, p. 307)

* 12–16 **8th International Conference on Combinatorics on Words, WORDS 2011**, Czech Technical University in Prague, Prague, Czech Republic.
Description: The central topic of WORDS conferences is the mathematical theory of words (i.e., finite or infinite sequences of symbols taken from a finite alphabet) from all points of view: combinatorial, algebraic, algorithmic, as well as its applications to physics, biology, linguistics and others.
Information: `http://words2011.fjfi.cvut.cz/`.

12–16 **25th IFIP TC 7 Conference on System Modeling and Optimization**, University of Technology, Berlin, Germany. (Nov. 2010, p. 1349)

12–16 **AIM Workshop: L^2 invariants and their relatives for finitely generated groups**, American Institute of Mathematics, Palo Alto, California. (Aug. 2010, p. 905)

12–16 **Mathematical and Computational Approaches in High-Throughput Genomics**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California. (Sept. 2010, p. 1036)

19–23 **IMA Workshop: High Dimensional Phenomena**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Oct. 2010, p. 1166)

* 19–23 **The Sixteenth Asian Technology Conference in Mathematics (ATCM 2011)**, Abant izzet Baysal University, Bolu, Turkey.
Description: The ATCM 2011 is an international conference to be held in Turkey that will continue addressing technology-based issues in all Mathematical Sciences. Thanks to advanced technological tools such as computer algebra systems (CAS), interactive and dynamic geometry, and hand-held devices, the effectiveness of our teaching and learning, and the horizon of our research in mathematics and its applications continue to grow rapidly. The aim of this conference is to provide a forum for educators, researchers, teachers and experts in exchanging information regarding enhancing technology to enrich mathematics learning,teaching and research at all levels. English is the official language of the conference.
Information: `http://atcm2011.org`.

24–25 **AMS Western Section Meeting**, Wake Forest University, Winston Salem, North Carolina. (Sept. 2010, p. 1036)

October 2011

14–16 **AMS Central Section Meeting**, University of Nebraska-Lincoln, Lincoln, Nebraska. (Sept. 2010, p. 1036)

22–23 **AMS Western Section Meeting**, University of Utah, Salt Lake City, Utah. (Sept. 2010, p. 1036)

24–28 **IMA Workshop: Large Graphs: Modeling, Algorithms and Applications**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Oct. 2010, p. 1166)

November 2011

14–18 **IMA Workshop: Large Data Sets in Medical Informatics**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota.

* 19–21 **International Conference on Analysis and its Applications**, Department of Mathematics, Aligarh Muslim University, Aligarh, India.
Description: The main aim of the conference is to promote, encourage, cooperate, and bring together theoreticians and multi–disciplinary and inter–disciplinary researchers in the fields of nonlinear analysis; operator theory; fixed point theory; set-valued analysis; variational analysis including variational inequalities; convex analysis; smooth and nonsmooth analysis; wavelet analysis; Fourier analysis; modern methods in summability and approximation theory; sequence spaces and matrix transformations.
Information: `http://www.amu.ac.in/conference/icaa2011`.

The following new announcements will not be repeated until the criteria in the next to the last paragraph at the bottom of the first page of this section are met.

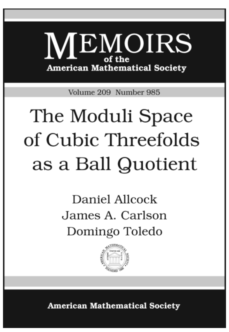
March 2012

* 5–9 **5th International Conference on High Performance Scientific Computing**, Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, Hanoi, Vietnam.
Description: Mathematical modeling, numerical simulation, methods for optimization and control, parallel computing: architectures, algorithms, tools, and environments, software development, applications of scientific computing in physics, mechanics, hydrology, chemistry, biology, medicine, transport, logistics, site location, communication, scheduling, industry, business, finance, etc.
Plenary Speakers: Frank Allgoewer (Stuttgart), Ralf Borndoerfer (Berlin), Ingrid Daubechies (Princeton), Mats Gyllenberg (Helsinki), Karl Kunisch (Graz), Bob Russell (Burnaby), Volker Schulz (Trier), Christoph Schwab (Zurich), Tamas Terlaky (Bethlehem, PA).
Deadlines: Deadline for registration and submission of abstracts: September 29, 2011. Notification of acceptance for presentation: December 21, 2011. Deadline for submission of data to apply for a business visa: January 6, 2012. Deadline for hotel reservation: January 6, 2012. Deadline for submission of full papers for the conference proceedings: May 12, 2012.
Information: `http://hpsc.iwr.uni-heidelberg.de/HPSCHanoi2012`.December 2010

New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to http://www.ams.org/bookstore-email.

Algebra and Algebraic Geometry



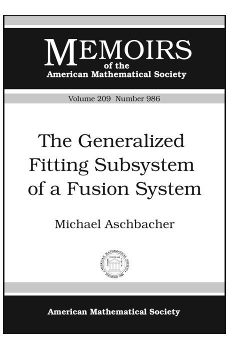
The Moduli Space of Cubic Threefolds as a Ball Quotient

Daniel Allcock, *University of Texas at Austin, TX*, and **James A. Carlson** and **Domingo Toledo**, *University of Utah, Salt Lake City, UT*

Contents: Moduli of smooth cubic threefolds; The discriminant near a chordal cubic; Extension of the period map; Degeneration to a chordal cubic; Degeneration to a nodal cubic; The Main theorem; The monodromy group and hyperplane arrangement; Bibliography; Index.

Memoirs of the American Mathematical Society, Volume 209, Number 985

February 2011, 70 pages, Softcover, ISBN: 978-0-8218-4751-0, LC 2010037801, 2000 *Mathematics Subject Classification:* 32G20; 14J30, **Individual member US$39.60**, List US$66, Institutional member US$52.80, Order code MEMO/209/985



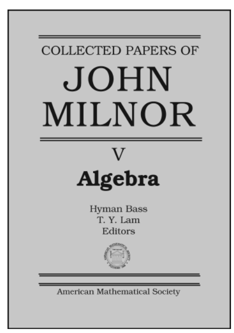
The Generalized Fitting Subsystem of a Fusion System

Michael Aschbacher, *California Institute of Technology, Pasadena, CA*

Contents: Introduction; Background; Direct products; $\mathcal{E}_1 \wedge \mathcal{E}_2$; The product of strongly closed subgroups; Pairs of commuting strongly closed subgroups; Centralizers; Characteristic and subnormal subsystems; $T\mathcal{F}_0$; Components; Balance; The fundamental group of $\mathcal{F}^c$; Factorizing morphisms; Composition series; Constrained systems; Solvable fusion systems; Fusion systems in simple groups; An example; Bibliography.

Memoirs of the American Mathematical Society, Volume 209, Number 986

February 2011, 110 pages, Softcover, ISBN: 978-0-8218-5303-0, LC 2010038097, 2000 *Mathematics Subject Classification:* 20D20, 55R35, **Individual member US$42**, List US$70, Institutional member US$56, Order code MEMO/209/986



Collected Papers of John Milnor

V. Algebra

Hyman Bass, *University of Michigan, Ann Arbor, MI*, and **T. Y. Lam**, *University of California, Berkeley, CA*, Editors

In addition to his seminal work in topology, John Milnor is also an accomplished algebraist, producing a spectacular agenda-setting body of work related to algebraic K-theory and quadratic forms during the five-year period 1965–1970. These papers, together with other (some of them previously unpublished) works in algebra are assembled here in this fifth volume of Milnor's Collected Papers. They constitute not only an important historical archive, but also, thanks to the clarity and elegance of Milnor's mathematical exposition, a valuable resource for work in the fields treated. In addition, Milnor's papers are complemented by detailed surveys on the current state of the field in two areas. One is on the congruence subgroup problem, by Gopal Prasad and Andrei Rapinchuk. The other is on algebraic K-theory and quadratic forms, by Alexander Merkurjev.

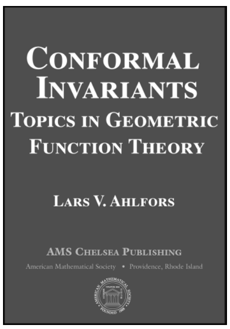
This item will also be of interest to those working in geometry and topology.

Contents: *Algebras and groups:* Introduction; On the structure of Hopf algebras, preprint; On the structure of Hopf algebras; Remarks on infinite-dimensional Lie groups; The representation rings of some classical groups; Growth of finitely generated solvable groups; *The congruence subgroup problem:* Introduction; On unimodular groups over number fields (preprint, 1965); Solution of the congruence subgroup problem for $SL_n (n \geq 3)$ and $Sp_{2n} (n \geq 2)$; On a functorial property of power residue symbols; On polylogarithms, Hurwitz zeta functions, and the Kubert identities; Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre; *Algebraic K-theory and quadratic forms:* Introduction; On isometries of inner product spaces; Algebraic K-theory and quadratic forms; Symmetric inner product spaces over a Dedekind domain (preprint, 1970); Symmetric inner products in characteristic 2; Developments in algebraic K-theory and quadratic forms after the work of Milnor; Index.

Collected Works, Volume 19

January 2011, approximately 408 pages, Hardcover, ISBN: 978-0-8218-4876-0, LC 2010035148, 2000 *Mathematics Subject Classification:* 00B60, 11E70, 11E81, 19D45, 20H05; 20D10, 22E65, 57T05, **AMS members US$71.20**, List US$89, Order code CWORKS/19.5

Analysis



Conformal Invariants

Topics in Geometric Function Theory

Lars V. Ahlfors

Most conformal invariants can be described in terms of extremal properties. Conformal invariants and extremal problems are therefore intimately linked and form together the central theme of this classic book which is primarily intended for students with approximately a year's background in complex variable theory. The book emphasizes the geometric approach as well as classical and semi-classical results which Lars Ahlfors felt every student of complex analysis should know before embarking on independent research.

At the time of the book's original appearance, much of this material had never appeared in book form, particularly the discussion of the theory of extremal length. Schiffer's variational method also receives special attention, and a proof of $|a_4| \leq 4$ is included which was new at the time of publication. The last two chapters give an introduction to Riemann surfaces, with topological and analytical background supplied to support a proof of the uniformization theorem.

Included in this new reprint is a Foreword by Peter Duren, F. W. Gehring, and Brad Osgood, as well as an extensive errata.

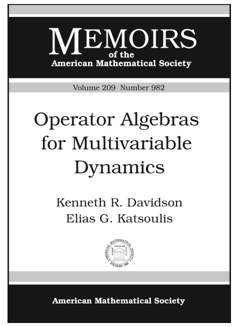
…encompasses a wealth of material in a mere one hundred and fifty-one pages. Its purpose is to present an exposition of selected topics in the geometric theory of functions of one complex variable, which in the author's opinion should be known by all prospective workers in complex analysis. From a methodological point of view the approach of the book is dominated by the notion of conformal invariant and concomitantly by extremal considerations. …It is a splendid offering.

— ***Reviewed for Math Reviews by M. H. Heins in 1975***

Contents: Applications of Schwarz's lemma; Capacity; Harmonic measure; Extremal length; Elementary theory of univalent functions; Löewner's method; The Schiffer variation; Properties of the extremal functions; Riemann surfaces; The uniformization theorem; Bibliography; Index; Errata.

AMS Chelsea Publishing, Volume 371

December 2010, 160 pages, Hardcover, ISBN: 978-0-8218-5270-5, LC 2010035576, 2000 *Mathematics Subject Classification:* 30-02, **AMS members US$31.50**, List US$35, Order code CHEL/371.H



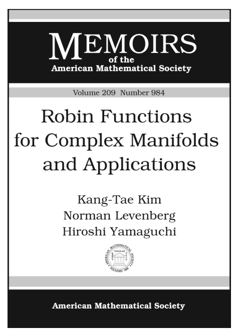
Operator Algebras for Multivariable Dynamics

Kenneth R. Davidson, *University of Waterloo, ON, Canada*, and **Elias G. Katsoulis**, *East Carolina University, Greenville, NC*

Contents: Introduction; Dilation theory; Recovering the dynamics; Semisimplicity; Open problems and future directions; Bibliography.

Memoirs of the American Mathematical Society, Volume 209, Number 982

February 2011, 53 pages, Softcover, ISBN: 978-0-8218-5302-3, LC 2010037690, 2000 *Mathematics Subject Classification:* 47L55; 47L40, 46L05, 37B20, 37B99, **Individual member US$36**, List US$60, Institutional member US$48, Order code MEMO/209/982



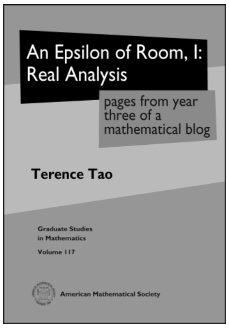
Robin Functions for Complex Manifolds and Applications

Kang-Tae Kim, *Pohang University of Science and Technology, South Korea*, **Norman Levenberg**, *Indiana University, Bloomington, IN*, and **Hiroshi Yamaguchi**, *Shiga University, Japan*

Contents: Introduction; The variation formula; Subharmonicity of $-\lambda$; Rigidity; Complex Lie groups; Complex homogeneous spaces; Flag space; Appendix A; Appendix B; Appendix C; Bibliography.

Memoirs of the American Mathematical Society, Volume 209, Number 984

February 2011, 111 pages, Softcover, ISBN: 978-0-8218-4965-1, LC 2010038099, 2000 *Mathematics Subject Classification:* 32U10; 32E10, 32M05, **Individual member US$42**, List US$70, Institutional member US$56, Order code MEMO/209/984



An Epsilon of Room, I: Real Analysis

pages from year three of a mathematical blog

Terence Tao, *University of California, Los Angeles, CA*

In 2007 Terry Tao began a mathematical blog to cover a variety of topics, ranging from his own research and other recent developments in mathematics, to lecture notes for his classes, to nontechnical puzzles and expository articles. The first two years of the blog have already been published by the American Mathematical Society. The posts from the third year are being published in two volumes. The present volume consists of a second course in real analysis, together with related material from the blog.

The real analysis course assumes some familiarity with general measure theory, as well as fundamental notions from undergraduate analysis. The text then covers more advanced topics in measure theory, notably the Lebesgue-Radon-Nikodym theorem and the Riesz representation theorem, topics in functional analysis, such as Hilbert spaces and Banach spaces, and the study of spaces of distributions and key function spaces, including Lebesgue's L^p spaces and Sobolev spaces. There is also a discussion of the general theory of the Fourier transform.

The second part of the book addresses a number of auxiliary topics, such as Zorn's lemma, the Carathéodory extension theorem, and the Banach-Tarski paradox. Tao also discusses the epsilon regularisation argument—a fundamental trick from soft analysis, from which the book gets its title. Taken together, the book presents more than enough material for a second graduate course in real analysis.

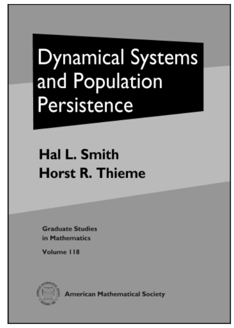
The second volume consists of technical and expository articles on a variety of topics and can be read independently.

Contents: Real analysis; Related articles; Bibliography; Index.

Graduate Studies in Mathematics, Volume 117

December 2010, approximately 333 pages, Hardcover, ISBN: 978-0-8218-5278-1, LC 2010036469, 2000 *Mathematics Subject Classification:* 42-01, 46-01, **AMS members US$49.60,** List US$62, Order code GSM/117

Applications



Dynamical Systems and Population Persistence

Hal L. Smith and **Horst R. Thieme**, *Arizona State University, Tempe, AZ*

The mathematical theory of persistence answers questions such as which species, in a mathematical model of interacting species, will survive over the long term. It applies to infinite-dimensional as well as to finite-dimensional dynamical systems, and to discrete-time as well as to continuous-time semiflows.

This monograph provides a self-contained treatment of persistence theory that is accessible to graduate students. The key results for deterministic autonomous systems are proved in full detail such as the acyclicity theorem and the tripartition of a global compact attractor. Suitable conditions are given for persistence to imply strong persistence even for nonautonomous semiflows, and time-heterogeneous persistence results are developed using so-called "average Lyapunov functions".

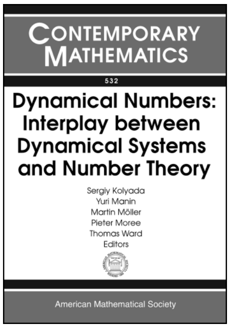
Applications play a large role in the monograph from the beginning. These include ODE models such as an SEIRS infectious disease in a meta-population and discrete-time nonlinear matrix models of demographic dynamics. Entire chapters are devoted to infinite-dimensional examples including an SI epidemic model with variable infectivity, microbial growth in a tubular bio-reactor, and an age-structured model of cells growing in a chemostat.

Contents: Introduction; Semiflows on metric spaces; Compact attractors; Uniform weak persistence; Uniform persistence; The interplay of attractors, repellers, and persistence; Existence of nontrivial fixed points via persistence; Nonlinear matrix models: Main act; Topological approaches to persistence; An SI endemic model with variable infectivity; Semiflows induced by semilinear Cauchy problems; Microbial growth in a tubular bio-reactor; Dividing cells in a chemostat; Persistence for nonautonomous dynamical systems; Forced persistence in linear Cauchy problems; Persistence via average Lyapunov functions; Tools from analysis and differential equations; Tools from functional analysis and integral equations; Bibliography; Index.

Graduate Studies in Mathematics, Volume 118

January 2011, approximately 411 pages, Hardcover, ISBN: 978-0-8218-4945-3, LC 2010033476, 2000 *Mathematics Subject Classification:* 37N25, 92D25, 92D30; 37B25, 37Lxx, **AMS members US$60**, List US$75, Order code GSM/118

Differential Equations



Dynamical Numbers: Interplay between Dynamical Systems and Number Theory

Sergiy Kolyada, *National Academy of Science of Ukraine, Kiev, Ukraine*, **Yuri Manin**, *Max Planck Institute for Mathematics, Bonn, Germany*, **Martin Möller**, *Goethe-Universität, Frankfurt, Frankfurt am Main, Germany*, **Pieter Moree**, *Max Planck Institute for Mathematics, Bonn, Germany*, and **Thomas Ward**, *University of East Anglia, Norwich, United Kingdom*, Editors

This volume contains papers from the special program and international conference on Dynamical Numbers which were held at the Max-Planck Institute in Bonn, Germany in 2009.

These papers reflect the extraordinary range and depth of the interactions between ergodic theory and dynamical systems and number theory. Topics covered in the book include stationary measures, systems of enumeration, geometrical methods, spectral methods, and algebraic dynamical systems.

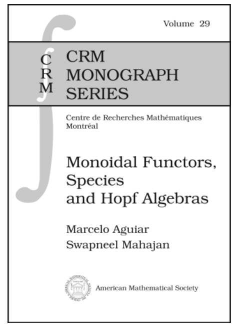
This item will also be of interest to those working in number theory.

Contents: H. Furstenberg and **E. Glasner**, Stationary dynamical systems; **J. Smillie** and **C. Ulcigrai**, Geodesic flow on the Teichmüller disk of the regular octagon, cutting sequences and octagon continued fractions maps; **P. Kůrka**, Expansion of rational numbers in Möbius number systems; **P. Moussa**, Localisation of algebraic integers and polynomial iteration; **J. Marklof**, Horospheres and Farey fractions; **T.-C. Dinh** and **N. Sibony**, Exponential mixing for automorphisms on compact Kähler manifolds; **A. M. Vershik**, Orbit theory, locally finite permutations and Morse arithmetic; **A. I. Danilenko** and **A. V. Solomko**, Ergodic abelian actions with homogeneous spectrum; **I. Kapovich** and **T. Nagnibeda**, Geometric entropy of geodesic currents on free groups; **F. Pakovich**, **C. Pech**, and **A. K. Zvonkin**, Laurent polynomial moment problem: A case study; **D. Lind**, **K. Schmidt**, and **E. Verbitskiy**, Entropy and growth rate of periodic points of algebraic $\mathbb{Z}^d$-actions; **M. Pollicott** and **R. Sharp**, Statistics of matrix products in hyperbolic geometry; **E. Lanneau**, Infinite sequence of fixed point free pseudo-Anosov homeomorphisms on a family of genus two surfaces.

Contemporary Mathematics, Volume 532

December 2010, 242 pages, Softcover, ISBN: 978-0-8218-4958-3, LC 2010027232, 2000 *Mathematics Subject Classification:* 11J70, 20F65, 22D40, 30E05, 37A15, 37A20, 37A30, 37A35, 54H20, 60B15, **AMS members US$63.20**, List US$79, Order code CONM/532

Discrete Mathematics and Combinatorics



Monoidal Functors, Species and Hopf Algebras

Marcelo Aguiar, *Texas A&M University, College Station, TX*, and **Swapneel Mahajan**, *Indian Institute of Technology, Mumbai, India*

This research monograph integrates ideas from category theory, algebra and combinatorics. It is organized in three parts.

Part I belongs to the realm of category theory. It reviews some of the foundational work of Bénabou, Eilenberg, Kelly and Mac Lane on monoidal categories and of Joyal and Street on braided monoidal categories, and proceeds to study higher monoidal categories and higher monoidal functors. Special attention is devoted to the notion of a bilax monoidal functor which plays a central role in this work.

Combinatorics and geometry are the theme of Part II. Joyal's species constitute a good framework for the study of algebraic structures associated to combinatorial objects. This part discusses the category of species focusing particularly on the Hopf monoids therein. The notion of a Hopf monoid in species parallels that of a Hopf algebra and reflects the manner in which combinatorial structures compose and decompose. Numerous examples of Hopf monoids are given in the text. These are constructed from combinatorial and geometric data and inspired by ideas of Rota and Tits' theory of Coxeter complexes.

Part III is of an algebraic nature and shows how ideas in Parts I and II lead to a unified approach to Hopf algebras. The main step is the construction of Fock functors from species to graded vector spaces. These functors are bilax monoidal and thus translate Hopf monoids in species to graded Hopf algebras. This functorial construction of Hopf algebras encompasses both quantum groups and the Hopf algebras of recent prominence in the combinatorics literature.

The monograph opens a vast new area of research. It is written with clarity and sufficient detail to make it accessible to advanced graduate students.

This item will also be of interest to those working in algebra and algebraic geometry and mathematical physics.

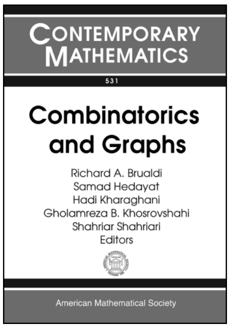
Titles in this series are co-published with the Centre de Recherches Mathématiques.

Contents: *Monoidal categories:* Monoidal categories; Graded vector spaces; Monoidal functors; Operad Lax monoidal functors; Bilax monoidal functors in homological algebra; 2-monoidal categories; Higher monoidal categories; *Hopf monoids in species:* Monoidal structures on species; Deformations of Hopf monoids; The Coxeter complex of type A; Universal constructions of Hopf monoids; Hopf monoids from geometry; Hopf monoids from combinatorics; Hopf monoids in colored species; *Fock functors:* From species to graded vector spaces; Deformations of Fock functors; From Hopf monoids to Hopf algebras: Examples; Adjoints of the Fock functors; Decorated Fock functors and creation-annihilation; Colored Fock functors; *Appendices:* Categorical preliminaries;

Operads; Pseudomonoids and the looping principle; Monoids and the simplicial category; *References:* Bibliography; Notation index; Author index; Subject index.

CRM Monograph Series, Volume 29

November 2010, 784 pages, Hardcover, ISBN: 978-0-8218-4776-3, LC 2010025240, 2000 *Mathematics Subject Classification:* 05A30, 16T30, 18D10, 18D35, 20B30, 81R50; 05A18, 05B35, 05C25, 05E05, 05E45, 06A11, 06A15, 16T25, 18D05, 18D20, 18D25, 18D50, 18G30, 18G35, 20F55, 51E24, 81S05, **AMS members US$135.20**, List US$169, Order code CRMM/29



Combinatorics and Graphs

Richard A. Brualdi, *University of Wisconsin, Madison, WI*, **Samad Hedayat**, *University of Illinois at Chicago, IL*, **Hadi Kharaghani**, *University of Lethbridge, AB, Canada*, **Gholamreza B. Khosrovshahi**, *IPM, Tehran, Iran*, and **Shahriar Shahriari**, *Pomona College, Claremont, CA*, Editors

This volume contains a collection of papers presented at the international conference IPM 20—Combinatorics 2009, which was held at the Institute for Research in Fundamental Sciences in Tehran, Iran, May 15–21, 2009.

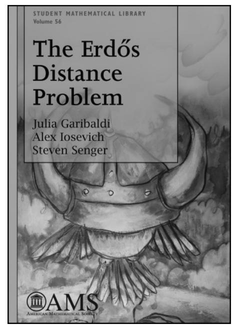
The conference celebrated IPM's 20th anniversary and was dedicated to Reza Khosrovshahi, one of the founders of IPM and the director of its School of Mathematics from 1996 to 2007, on the occasion of his 70th birthday.

The conference attracted an international group of distinguished researchers from many different parts of combinatorics and graph theory, including permutations, designs, graph minors, graph coloring, graph eigenvalues, distance regular graphs and association schemes, hypergraphs, and arrangements.

Contents: **A. E. Brouwer**, The eigenvalues of oppositeness graphs in buildings of spherical type; **S. Akbari**, **M. Ghanbari**, and **S. Jahanbekam**, On the dynamic chromatic number of graphs; **A. H. Berliner**, **R. A. Brualdi**, **L. Deaett**, **K. P. Kiernan**, **S. A. Meyer**, and **M. W. Schroeder**, Signed domination of graphs and (0,1)-matrices; **A. S. Hedayat** and **W. Zheng**, Totally balanced test-control incomplete crossover designs and their statistical applications; **E. Bannai** and **E. Bannai**, Euclidean designs and coherent configurations; **J. Širáň** and **Y. Wang**, Maps with highest level of symmetry that are even more symmetric than other such maps: Regular maps with largest exponent groups; **A. R. Barghi**, A note on finite groups determined by a combinatorial property; **S. Akbari**, **S. Alikhani**, **M. R. Oboudi**, and **Y. H. Peng**, On the zeros of domination polynomial of a graph; **S. Akbari**, **A. Doni**, **M. Ghanbari**, **S. Jahanbekam**, and **A. Saito**, List coloring of graphs with cycles of length divisible by a given integer; **Q. Wang**, On generalized Lucas sequences; **A. Sakzad** and **M.-R. Sadeghi**, On cycle-free lattices; **S. Akbari**, **M. R. Oboudi**, and **S. Qajar**, On the rational independence roots; **W. H. Haemers** and **F. Ramezani**, Graphs cospectral with Kneser graphs; **R. P. Stanley**, A survey of alternating permutations; **A. Mohammadian** and **B. Tayfeh-Rezaie**, The spectrum of the McKay-Miller-Širáň graphs; **D. Dellamonica, Jr.**, **P. Frankl**, and **V. Rödl**, A theorem on incidence matrices and quasirandom hypergraphs; **M. Hasheminezhad** and **B. D. McKay**, Combinatorial estimates by the switching method; **J. H. Koolen**, **W. S. Lee**, and **W. J. Martin**, Characterizing completely regular codes from an algebraic viewpoint; **W. H. Holzmann**, **H. Kharaghani**, and **W. Orrick**, On the real unbiased Hadamard matrices; **R. M. Wilson**, The proportion of various graphs in graph-designs; **F. Didehvar**, **A. D. Mehrabi**, and **F. Raee B.**, On unique independence weighted graphs.

Contemporary Mathematics, Volume 531

December 2010, 264 pages, Softcover, ISBN: 978-0-8218-4865-4, LC 2010026895, 2000 *Mathematics Subject Classification:* 05A05, 05B05, 05B20, 05B25, 05C15, 05C22, 05C35, 05C50, 05D05, 05E30, **AMS members US$71.20**, List US$89, Order code CONM/531



The Erdős Distance Problem

Julia Garibaldi, **Alex Iosevich**, *University of Rochester, NY*, and **Steven Senger**, *University of Missouri-Columbia, MO*

The Erdős problem asks, What is the smallest possible number of distinct distances between points of a large finite subset of the Euclidean space in dimensions two and higher. The main goal of this book is to introduce the reader to the techniques, ideas, and consequences related to the Erdős problem. The authors introduce these concepts in a concrete and elementary way that allows a wide audience—from motivated high school students interested in mathematics to graduate students specializing in combinatorics and geometry—to absorb the content and appreciate its far reaching implications. In the process, the reader is familiarized with a wide range of techniques from several areas of mathematics and can appreciate the power of the resulting symbiosis.

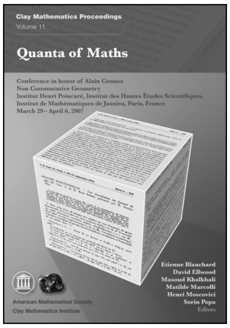
The book is heavily problem oriented, following the authors' firm belief that most of the learning in mathematics is done by working through the exercises. Many of these problems are recently published results by mathematicians working in the area. The order of the exercises is designed both to reinforce the material presented in the text and, equally importantly, to entice the reader to leave all worldly concerns behind and launch head first into the multifaceted and rewarding world of Erdős combinatorics.

Contents: Introduction; The $\sqrt{n}$ theory; The $n^{2/3}$ theory; The Cauchy-Schwarz inequality; Graph theory and incidences; The $n^{4/5}$ theory; The $n^{6/7}$ theory; Beyond $n^{6/7}$; Information theory; Dot products; Vector spaces over finite fields; Distances in vector spaces over finite fields; Applications of the Erdős distance problem; Hyperbolas in the plane; Basic probability theory; Jensen's inequality; Bibliography; Biographical information; Index of terminology.

Student Mathematical Library, Volume 56

January 2011, approximately 161 pages, Softcover, ISBN: 978-0-8218-5281-1, LC 2010033266, 2000 *Mathematics Subject Classification:* 05-XX, 11-XX, 42-XX, 51-XX, **AMS members US$23.20**, List US$29, Order code STML/56

General Interest



Quanta of Maths

Etienne Blanchard, *University of Paris 7, France*, **David Ellwood**, *Clay Mathematics Institute, Cambridge, MA*, **Masoud Khalkhali**, *University of Western Ontario, London, ON, Canada*, **Matilde Marcolli**, *California Institute of Technology, Pasadena, CA*, **Henri Moscovici**, *Ohio State University, Columbus, OH*, and **Sorin Popa**, *University of California, Los Angeles, CA*, Editors

The work of Alain Connes has cut a wide swath across several areas of mathematics and physics. Reflecting its broad spectrum and profound impact on the contemporary mathematical landscape, this collection of articles covers a wealth of topics at the forefront of research in operator algebras, analysis, noncommutative geometry, topology, number theory and physics.

Specific themes covered by the articles are as follows:

- entropy in operator algebras, regular C^*-algebras of integral domains, properly infinite C^*-algebras, representations of free groups and 1-cohomology, Leibniz seminorms and quantum metric spaces;
- von Neumann algebras, fundamental Group of II_1 factors, subfactors and planar algebras;
- Baum-Connes conjecture and property T, equivariant K-homology, Hermitian K-theory;
- cyclic cohomology, local index formula and twisted spectral triples, tangent groupoid and the index theorem;
- noncommutative geometry and space-time, spectral action principle, quantum gravity, noncommutative ADHM and instantons, non-compact spectral triples of finite volume, noncommutative coordinate algebras;
- Hopf algebras, Vinberg algebras, renormalization and combinatorics, motivic renormalization and singularities;
- cyclotomy and analytic geometry over F_1, quantum modular forms;
- differential K-theory, cyclic theory and S-cohomology.

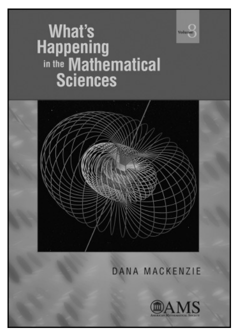
This item will also be of interest to those working in algebra and algebraic geometry and geometry and topology.

Titles in this series are co-published with the Clay Mathematics Institute (Cambridge, MA).

Contents: P. Baum, **N. Higson**, and **T. Schick**, A geometric description of equivariant K-homology for proper actions; **D. Bisch**, **P. Das**, and **S. K. Ghosh**, The planar algebra of diagonal subfactors; **E. Blanchard**, K_1-injectivity for properly infinite C^*-algebras; **S. Brain** and **G. Landi**, Families of monads and instantons from a noncommutative ADHM construction; **D. Burghelea**, Cyclic theory for commutative differential graded algebras and S-cohomology; **P. Cartier**, Vinberg algebras, Lie groups and combinatorics; **A. H. Chamseddine**, Noncommutative geometry as the key to unlock the secrets of space-time; **J. Cuntz** and **X. Li**, The regular C^*-algebra of an integral domain; **M. Dubois-Violette**, Noncommutative coordinate algebras; **A. Guionnet**, **V. F. R. Jones**, and **D. Shlyakhtenko**, Random matrices, free probability, planar algebras and subfactors; **N. Higson**, The tangent groupoid and the index theorem; **M. Karoubi**, Le théorème de périodicité en K-théorie hermitienne; **M. Khalkhali**, A short survey of cyclic cohomology; **D. Kreimer**, The core Hopf algebra; **V. Lafforgue**, Propriété (T) renforcée et conjecture de Baum-Connes; **J.-L. Loday** and **M. Ronco**, Combinatorial Hopf algebras; **Yu. I. Manin**, Cyclotomy and analytic geometry over $\mathbb{F}_1$; **M. Marcolli**, Motivic renormalization and singularities; **F. Martin** and **A. Valette**, Free groups and reduced 1-cohomology of unitary representations; **H. Moscovici**, Local index formula and twisted spectral triples; **A. Perez** and **C. Rovelli**, Observables in quantum gravity; **S. Popa** and **S. Vaes**, On the fundamental group of II_1 factors and equivalence relations arising from group actions; **M. A. Rieffel**, Leibniz seminorms for "matrix algebras converge to the sphere"; **J. Simons** and **D. Sullivan**, Structured vector bundles define differential K-theory; **E. Størmer**, Entropy in operator algebras; **R. Wulkenhaar**, Non-compact spectral triples with finite volume; **G. Yu**, A characterization of the image of the Baum-Connes map; **D. Zagier**, Quantum modular forms.

Clay Mathematics Proceedings, Volume 11

December 2010, 675 pages, Softcover, ISBN: 978-0-8218-5203-3, LC 2010034787, 2000 *Mathematics Subject Classification:* 58B34, **AMS members US$103.20**, List US$129, Order code CMIP/11



What's Happening in the Mathematical Sciences, Volume 8

Dana Mackenzie

The goal of the series is to shed light on topics on the leading edge of mathematical research in a way that is accessible to the mathematical layperson. The articles frequently combine mathematics with physics, and are written in a lively style that should be accessible to anyone with genuine interest and some college-level experience in mathematics and science.

—***Choice***

What's Happening in the Mathematical Sciences showcases the remarkable recent progress in pure and applied mathematics. Once again, there are some surprises, where we discover new properties of familiar things, in this case tightly-packed tetrahedra or curious turtle-like shapes that right themselves. Mathematics also has played significant roles in current events, most notably the financial crisis, but also in screening for breast cancer. The Netflix competition to find a better algorithm for recommending videos to subscribers demonstrated how deeply mathematics is used behind the scenes in our everyday lives.

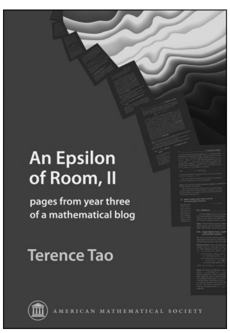
Mathematicians have settled several important conjectures in the past few years. In topology, the recently solved Kervaire invariant conjecture tells us about exotic spheres in high dimension. The Weinstein conjecture, proved by Cliff Taubes, guarantees periodicity in certain important dynamical systems. A very old dynamical system—the game of billiards—received two innovative makeovers. First, mathematicians proved the existence of "wandering" trajectories in an inside-out version of the game, called "outer billiards," which some researchers consider a toy model for planetary motion. Second, mathematicians proved two different versions of the Quantum Unique Ergodicity conjecture, which says that a quantum-mechanical billiard ball behaves, in the long term (and at high energies) similarly to a classical billiard ball. The proof uses ideas from pure number theory dating back to Ramanujan. Finally, in another area of statistical physics, mathematicians showed that the transition from an unmixed to a mixed system often happens, relatively speaking, in the blink of an eye.

Dana Mackenzie, a science and mathematics writer, makes the mathematics and the applications easily comprehensible, by calling on common sense or on similar but familiar phenomena. The stories invite you into the exciting world of modern mathematics, with its thrill of discovery and the anticipation of what is still to come. Anyone with an interest in mathematics, from high school teachers and college students to engineers and computer scientists, will find something of interest here. The stories are well told and the mathematics is gripping.

Contents: As one heroic age ends, another begins; A brave new symplectic world; The ultimate billiard shot; SimaPatient; Instant randomness; Accounting for taste; In search of quantum chaos; 3-D surprises; Mathematics and the financial crisis.

What's Happening in the Mathematical Sciences, Volume 8

January 2011, approximately 136 pages, Softcover, ISBN: 978-0-8218-4999-6, 2000 *Mathematics Subject Classification:* 00A06, **AMS members US$18.40**, List US$23, Order code HAPPENING/8



An Epsilon of Room, II

pages from year three of a mathematical blog

Terence Tao, *University of California, Los Angeles, CA*

There are many bits and pieces of folklore in mathematics that are passed down from advisor to student, or from collaborator to collaborator, but which are too fuzzy and nonrigorous to be discussed in the formal literature. Traditionally, it was a matter of luck and location as to who learned such "folklore mathematics". But today, such bits and pieces can be communicated effectively and efficiently via the semiformal medium of research blogging. This book grew from such a blog.

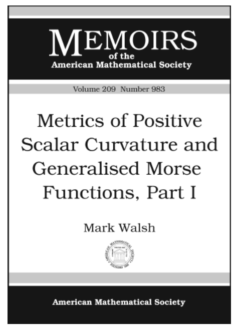
In 2007 Terry Tao began a mathematical blog to cover a variety of topics, ranging from his own research and other recent developments in mathematics, to lecture notes for his classes, to nontechnical puzzles and expository articles. The first two years of the blog have already been published by the American Mathematical Society. The posts from the third year are being published in two volumes. This second volume contains a broad selection of mathematical expositions and self-contained technical notes in many areas of mathematics, such as logic, mathematical physics, combinatorics, number theory, statistics, theoretical computer science, and group theory. Tao has an extraordinary ability to explain deep results to his audience, which has made his blog quite popular. Some examples of this facility in the present book are the tale of two students and a multiple-choice exam being used to explain the $P = NP$ conjecture and a discussion of "no self-defeating object" arguments that starts from a schoolyard number game and ends with results in logic, game theory, and theoretical physics.

The first volume consists of a second course in real analysis, together with related material from the blog, and it can be read independently.

Contents: Expository articles; Technical articles; Bibliography; Index.

January 2011, approximately 252 pages, Softcover, ISBN: 978-0-8218-5280-4, 2000 *Mathematics Subject Classification:* 00A99, **AMS members US$33.60**, List US$42, Order code MBK/77

Geometry and Topology



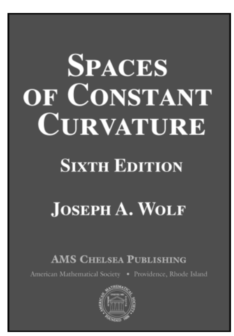
Metrics of Positive Scalar Curvature and Generalised Morse Functions, Part I

Mark Walsh, *Westfälische Wilhelms-Universität Münster, Germany*

Contents: Definitions and preliminary results; Revisiting the surgery theorem; Constructing Gromov-Lawson cobordisms; Constructing Gromov-Lawson concordances; Gromov-Lawson concordance implies isotopy for cancelling surgeries; Gromov-Lawson concordance implies isotopy in the general case; Appendix: Curvature calculations from the surgery theorem; Bibliography.

Memoirs of the American Mathematical Society, Volume 209, Number 983

February 2011, 80 pages, Softcover, ISBN: 978-0-8218-5304-7, LC 2010037798, 2000 *Mathematics Subject Classification:* 53-02, 55-02, **Individual member US$41.40**, List US$69, Institutional member US$55.20, Order code MEMO/209/983



Spaces of Constant Curvature

Sixth Edition

Joseph A. Wolf, *University of California, Berkeley, CA*

This book is the sixth edition of the classic *Spaces of Constant Curvature*, first published in 1967, with the previous (fifth) edition published in 1984. It illustrates the high degree of interplay between group theory and geometry. The reader will benefit from the very concise treatments of riemannian and pseudo-riemannian manifolds and their curvatures, of the representation theory of finite groups, and of indications of recent progress in discrete subgroups of Lie groups.

Part I is a brief introduction to differentiable manifolds, covering spaces, and riemannian and pseudo-riemannian geometry. It also contains a certain amount of introductory material on symmetry groups and space forms, indicating the direction of the later chapters. Part II is an updated treatment of euclidean space form.

Part III is Wolf's classic solution to the Clifford–Klein Spherical Space Form Problem. It starts with an exposition of the representation theory of finite groups. Part IV introduces riemannian symmetric spaces and extends considerations of spherical space forms to space forms of riemannian symmetric spaces. Finally, Part V examines space form problems on pseudo-riemannian symmetric spaces. At the end of Chapter 12 there is a new appendix describing some of the recent work on discrete subgroups of Lie groups with application to space forms of pseudo-riemannian symmetric spaces. Additional references have been added to this sixth edition as well.

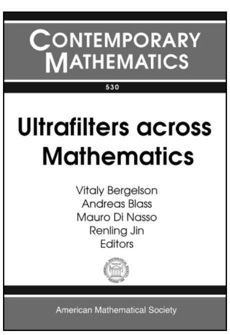
This item will also be of interest to those working in algebra and algebraic geometry.

Contents: *Riemannian geometry:* Affine differential geometry; Riemannian curvature; *The Euclidean space form problem:* Flat Riemannian manifolds; *The spherical space form problem:* Representations of finite groups; Vincent's work on the spherical space form problem; The classification of fixed point free groups; The solution to the spherical space form problem; *Space form problems on symmetric spaces:* Riemannian symmetric spaces; Space forms of irreducible symmetric spaces; Locally symmetric spaces of non-negative curvature; *Space form problems on indefinite metric manifolds:* Spaces of constant curvature; Locally isotropic manifolds; Appendix to Chapter 12; References; Additional references; Index.

AMS Chelsea Publishing, Volume 372

January 2011, 420 pages, Hardcover, ISBN: 978-0-8218-5282-8, LC 2010035675, 2000 *Mathematics Subject Classification:* 53-02, 53C21, 53C30, 53C35, 53C50, 20C05, 22C05; 14L35, 17B45, 20D99, **AMS members US$54**, List US$60, Order code CHEL/372.H

Logic and Foundations



Ultrafilters across Mathematics

Vitaly Bergelson, *Ohio State University, Columbus, OH*, **Andreas Blass**, *University of Michigan, Ann Arbor, MI*, **Mauro Di Nasso**, *Università di Pisa, Italy*, and **Renling Jin**, *College of Charleston, SC*, Editors

This volume originated from the International Congress "ULTRAMATH: Applications of Ultrafilters and Ultraproducts in Mathematics", which was held in Pisa, Italy, from June 1–7, 2008.

The volume aims to present the state-of-the-art of applications in the whole spectrum of mathematics which are grounded on the use of ultrafilters and ultraproducts. It contains two general surveys on ultrafilters in set theory and on the ultraproduct construction, as well as papers that cover additive and combinatorial number theory, nonstandard methods and stochastic differential equations, measure theory, dynamics, Ramsey theory, algebra in the space of ultrafilters, and large cardinals.

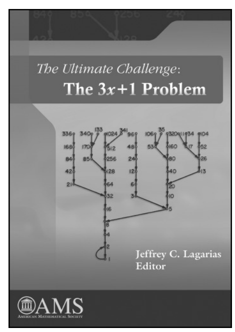
The papers are intended to be accessible and interesting for mathematicians who are not experts on ultrafilters and ultraproducts. Greater prominence has been given to results that can be formulated and presented in non-special terms and be, in principle, understandable by any mathematician, and to those results that connect different areas of mathematics, revealing new facets of known important topics.

Contents: **V. Benci**, **S. Galatolo**, and **M. Ghimenti**, An elementary approach to stochastic differential equations using the infinitesimals; **V. Bergelson**, Ultrafilters, IP sets, dynamics, and combinatorial number theory; **A. Blass**, Ultrafilters and set theory; **D. H. Fremlin**, Measure-centering ultrafilters; **N. Hindman** and **D. Strauss**, Algebra in the space of ultrafilters and Ramsey theory; **R. Jin**, Ultrapower of ℕ and density problems; **H. J. Keisler**, The ultraproduct construction; **I. Neeman**, Ultrafilters and large cardinals.

Contemporary Mathematics, Volume 530

December 2010, 200 pages, Softcover, ISBN: 978-0-8218-4833-3, LC 2010025146, 2000 *Mathematics Subject Classification:* 03C20, 03E05, 03H05, 05C55, 28E15, **AMS members US$55.20**, List US$69, Order code CONM/530

Number Theory



The Ultimate Challenge

The $3x + 1$ Problem

Jeffrey C. Lagarias, *University of Michigan, Ann Arbor, MI*, Editor

The $3x + 1$ problem, or Collatz problem, concerns the following seemingly innocent arithmetic procedure applied to integers: If an integer x is odd then "multiply by three and add one", while if it is even then "divide by two". The $3x + 1$ problem asks whether, starting from any positive integer, repeating this procedure over and over will eventually reach the number 1. Despite its simple appearance, this problem is unsolved. Generalizations of the problem are known to be undecidable, and the problem itself is believed to be extraordinarily difficult.

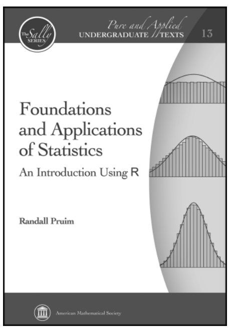
This book reports on what is known on this problem. It consists of a collection of papers, which can be read independently of each other. The book begins with two introductory papers, one giving an overview and current status, and the second giving history and basic results on the problem. These are followed by three survey papers on the problem, relating it to number theory and dynamical systems, to Markov chains and ergodic theory, and to logic and the theory of computation. The next paper presents results on probabilistic models for behavior of the iteration. This is followed by a paper giving the latest computational results on the problem, which verify its truth for $x < 5.4 \cdot 10^{18}$. The book also reprints six early papers on the problem and related questions, by L. Collatz, J. H. Conway, H. S. M. Coxeter, C. J. Everett, and R. K. Guy, each with editorial commentary. The book concludes with an annotated bibliography of work on the problem up to the year 2000.

Contents: *Survey papers:* **J. Lagarias**, The $3x + 1$ problem: An overview; **J. Lagarias**, The $3x + 1$ problem and its generalizations; *Survey papers:* **M. Chamberland**, A $3x + 1$ Survey: Number theory and dynamical systems; **K. R. Matthews**, Generalized $3x + 1$ mappings: Markov chains and ergodic theory; **P. Michel** and

M. Margenstern, Generalized $3x + 1$ functions and the theory of computation; *Stochastic modelling and computation papers:* **A. V. Kontorovich** and **J. Lagarias**, Stochastic models for the $3x + 1$ and $5x + 1$ problems and related problems; **T. O. e Silva**, Empirical verification of the $3x + 1$ and related conjectures; *Reprinted early papers:* **H. S. M. Coxeter**, Cyclic sequences and Frieze patterns; **J. H. Conway**, Unpredictable iterations; **C. J. Everett**, Iteration of the number-theoretic function $f(2n) = n, f(2n + 1) = 3n + 2$; **R. K. Guy**, Don't try to solve these problems!; **L. Collatz**, On the motivation and origin of the $(3n + 1)$-problem; **J. H. Conway**, FRACTRAN: A simple universal programming language for arithmetic; *Annotated bibliography:* **J. Lagarias**, The $3x + 1$ problem: An annotated bibliography (1963–1999).

December 2010, approximately 348 pages, Hardcover, ISBN: 978-0-8218-4940-8, 2000 *Mathematics Subject Classification:* 11B83, 37A45; 11B37, 68Q99, **AMS members US$47.20**, List US$59, Order code MBK/78

Probability and Statistics



Foundations and Applications of Statistics

An Introduction Using R

Randall Pruim, *Calvin College, Grand Rapids, MI*

Foundations and Applications of Statistics simultaneously emphasizes both the foundational and the computational aspects of modern statistics. Engaging and accessible, this book is useful to undergraduate students with a wide range of backgrounds and career goals.

The exposition immediately begins with statistics, presenting concepts and results from probability along the way. Hypothesis testing is introduced very early, and the motivation for several probability distributions comes from p-value computations. Pruim develops the students' practical statistical reasoning through explicit examples and through numerical and graphical summaries of data that allow intuitive inferences before introducing the formal machinery. The topics have been selected to reflect the current practice in statistics, where computation is an indispensible tool. In this vein, the statistical computing environment R is used throughout the text and is integral to the exposition. Attention is paid to developing students' mathematical and computational skills as well as their statistical reasoning. Linear models, such as regression and ANOVA, are treated with explicit reference to the underlying linear algebra, which is motivated geometrically.

Foundations and Applications of Statistics discusses both the mathematical theory underlying statistics and practical applications that make it a powerful tool across disciplines. The book contains ample material for a two-semester course in undergraduate probability and statistics. A one-semester course based on the book will cover hypothesis testing and confidence intervals for the most common situations.

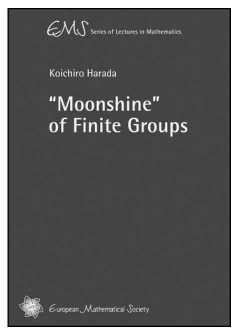
Contents: Summarizing data; Probability and random variables; Continuous distributions; Parameter estimation and testing; Likelihood-based statistics; Introduction to linear models; More linear models; A brief introduction to R; Some mathematical preliminaries; Geometry and linear algebra review; Review of Chapters 1–4; Hints, answers, and solutions to selected exercises; Bibliography; Index to R functions, packages, and data sets; Index.

Pure and Applied Undergraduate Texts, Volume 13

March 2011, approximately 615 pages, Hardcover, ISBN: 978-0-8218-5233-0, 2000 *Mathematics Subject Classification:* 62-01; 60-01, **AMS members US$68**, List US$85, Order code AMSTEXT/13

New AMS-Distributed Publications

Algebra and Algebraic Geometry



"Moonshine" of Finite Groups

Koichiro Harada, *Ohio State University, Columbus, OH*

This is an almost verbatim reproduction of the author's lecture notes written in 1983–84 at Ohio State University, Columbus. A substantial update is given in the bibliography.

Over the last 20 plus years there has been energetic activity in the field of finite simple group theory related to the monster simple group. Most notably, influential works have been produced in the theory of vertex operator algebras from research that was stimulated by the moonshine of the finite groups. Still, we can ask the same questions now that we did 30–40 years ago: What is the monster simple group? Is it really related to the theory of the universe as it was vaguely so envisioned? What lies behind the moonshine phenomena of the monster group? It may appear that we have only scratched the surface. These notes are primarily reproduced for the benefit of readers who wish to start learning about modular functions used in moonshine.

This item will also be of interest to those working in number theory.

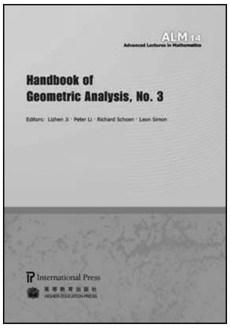
A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: Modular functions and modular forms; Dedekind eta function; "Moonshine" of finite groups; Multiplicative product of η functions; Appendix. Genus zero discrete groups; Bibliography.

EMS Series of Lectures in Mathematics, Volume 12

September 2010, 83 pages, Softcover, ISBN: 978-3-03719-090-6, 2000 *Mathematics Subject Classification:* 20B05, 11F03, **AMS members US$25.60**, List US$32, Order code EMSSERLEC/12

Analysis



Handbook of Geometric Analysis

Number 3

Lizhen Ji, *University of Michigan, Ann Arbor, MI*, **Peter Li**, *University of California, Irvine, CA*, and **Richard Schoen** and **Leon Simon**, *Stanford University, CA*, Editors

Geometric analysis combines differential equations and differential geometry, an important aspect of which is to solve geometric problems by studying differential equations. Besides some known linear differential operators such as the Laplace operator, many differential equations arising from differential geometry are nonlinear. A particularly important example is the Monge-Ampère equation. Applications to geometric problems have also motivated new methods and techniques in differential equations. The field of geometric analysis is broad and has had many striking applications.

This handbook of geometric analysis, the third to be published in the ALM series, provides introductions to and surveys of important topics in geometric analysis and their applications to related fields. It can be used as a reference by graduate students and researchers.

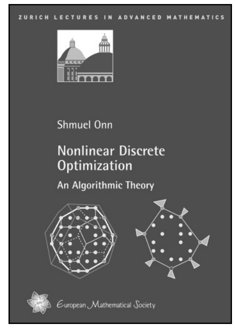
A publication of International Press. Distributed worldwide by the American Mathematical Society.

Contents: M. T. Anderson, A survey of Einstein metrics on 4-manifolds; **S. Brendle** and **R. Schoen**, Sphere theorems in geometry; **C. Gerhardt**, Curvature flows and CMC hypersurfaces; **N. C. Leung**, Geometric structures on Riemannian manifolds; **T.-J. Li**, Symplectic Calabi-Yau surfaces; **D. H. Phong** and **J. Sturm**, Lectures on stability and constant scalar curvature; **X.-P. Zhu**, Analytic aspect of Hamilton's Ricci flow.

International Press

August 2010, 472 pages, Softcover, ISBN: 978-1-57146-205-3, 2000 *Mathematics Subject Classification:* 01-02, 53-06, 58-06, **AMS members US$52**, List US$65, Order code INPR/92

Discrete Mathematics and Combinatorics



Nonlinear Discrete Optimization

An Algorithmic Theory

Shmuel Onn, *Israel Institute of Technology, Haifa, Israel*

This monograph develops an algorithmic theory of nonlinear discrete optimization. It introduces a simple and useful setup, which enables the polynomial time solution of broad fundamental classes of nonlinear combinatorial optimization and integer programming problems in variable dimension. An important part of this theory is enhanced by recent developments in the algebra of Graver bases. The power of the theory is demonstrated by deriving the first polynomial time algorithms in a variety of application areas within operations research and statistics, including vector partitioning, matroid optimization, experimental design, multicommodity flows, multi-index transportation and privacy in statistical databases.

This monograph is intended for graduate students and researchers. It is accessible to anyone with standard undergraduate knowledge and mathematical maturity.

This item will also be of interest to those working in number theory.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: Introduction; Convex discrete maximization; Nonlinear integer programming; *n*-Fold integer programming; Multiway tables and universality; Nonlinear combinatorial optimization; Bibliography; Index.

Zurich Lectures in Advanced Mathematics, Volume 13

September 2010, 147 pages, Softcover, ISBN: 978-3-03719-093-7, 2000 *Mathematics Subject Classification:* 05Axx, 05Cxx, 05Dxx, 05Exx, 11Dxx, 11Hxx, 11Pxx, 13Pxx, 14Qxx, 15Axx, 15Bxx, 51Mxx, 52Axx, 52Bxx, 52Cxx, 62Hxx, 62Kxx, 62Qxx, 65Cxx, 68Qxx, 68Rxx, 68Wxx, 90Bxx, 90Cxx, **AMS members US$33.60**, List US$42, Order code EMSZLEC/13

General Interest

European Congress of Mathematics

Amsterdam, July 14–18, 2008

André Ran, *Vrije University, Amsterdam, The Netherlands*, **Herman te Riele**, *CWI, Amsterdam, The Netherlands*, and **Jan Wiegerinck**, *University of Amsterdam, The Netherlands*, Editors

The European Congress of Mathematics, held every four years, has established itself as a major international mathematical event. Following those in Paris (1992), Budapest (1996), Barcelona (2000), and Stockholm (2004), the Fifth European Congress of Mathematics (5ECM) took place in Amsterdam, The Netherlands, July 14–18, 2008, with about 1000 participants from 68 different countries.

Ten plenary and thirty-three invited lectures were delivered. Three science lectures outlined applications of mathematics in other sciences: climate change, quantum information theory, and population dynamics. As in the four preceding EMS congresses, ten EMS prizes were granted to very promising young mathematicians. In addition, the Felix Klein Prize was awarded, for the second time, for an application of mathematics to a concrete and difficult industrial problem. There were twenty-two minisymposia, spread over the whole mathematical area. Two round table meetings were organized: one on industrial mathematics and one on mathematics and developing countries.

As part of the 44th Nederlands Mathematisch Congres, which was embedded in 5ECM, the so-called Brouwer lecture was presented. It is the Netherlands' most prestigious award in mathematics, organized every three years by the Royal Dutch Mathematical Society. Information about Brouwer was given in an invited historical lecture during the congress.

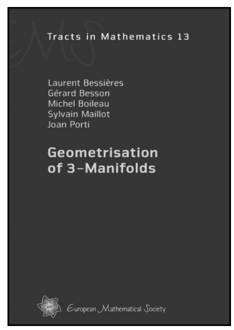
These proceedings contain a selection of the contributions to the congress, providing a permanent record of the best of what mathematics offers today.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: *Invited Lectures:* **J. A. Carrillo** and **J. Rosado**, Uniqueness of bounded solutions to aggregation equations by optimal transport methods; **B. Edixhoven**, On the computation of the coefficients of modular forms; **M. Einsiedler**, Effective equidistribution and spectral gap; **W. Lück**, Survey on aspherical manifolds; **S. A. Merkulov**, Wheeled props in algebra, geometry and quantization; **O. R. Musin**, Positive definite functions in distance geometry; **J. Nešetřil** and **P. Ossona de Mendez**, From sparse graphs to nowhere dense structures: decompositions, independence, dualities and limits; **J. Fuchs**, **T. Nikolaus**, **C. Schweigert**, and **K. Waldorf**, Bundle gerbes and surface holonomy; **C. Teleman**, Topological field theories in 2 dimensions; *Lecture on Invitation by the KWG:* **D. van Dalen**, The Revolution of 1907—Brouwer's dissertation; *Plenary Lectures:* **J. Bourgain**, New developments in combinatorial number theory and applications; **J.-F. Le Gall**, Large random planar maps and their scaling limits; **F. Loeser**, Geometry and non-archimedean integrals; **M. Marcolli**, Feynman integrals and motives; **N. Reshetikhin**, Topological quantum field theory: 20 years later; *Prize Lectures:* **O. Holtz** and **N. Shomron**, Computational complexity and numerical stability of linear problems; **B. Klartag**, High-dimensional distributions with convexity properties; **L. Saint-Raymond**, Some recent results about the sixth problem of Hilbert: hydrodynamic limits of the Boltzmann equation; **A. Smoktunowicz**, Graded algebras associated to algebraic algebras need not be algebraic; Author index.

April 2010, 488 pages, Hardcover, ISBN: 978-3-03719-077-7, 2000 *Mathematics Subject Classification:* 00Bxx, **AMS members US$78.40**, List US$98, Order code EMSEMC/2008

Geometry and Topology



Geometrisation of 3-Manifolds

Laurent Bessières and **Gérard Besson**, *Université Joseph Fourier, Grenoble, France*, **Michel Boileau**, *Université Paul Sabatier, Toulouse, France*, **Sylvain Maillot**, *Université Montpellier II, France*, and **Joan Porti**, *Universitat Autonoma de Barcelona, Spain*

The geometrisation conjecture was proposed by William Thurston in the mid 1970s in order to classify compact 3-manifolds by means of a canonical decomposition along essential, embedded surfaces into pieces that possess geometric structures. It contains the famous Poincaré Conjecture as a special case.

In 2002 Grigory Perelman announced a proof of the geometrisation conjecture based on Richard Hamilton's Ricci flow approach and presented it in a series of three celebrated arXiv preprints. Since then there has been an ongoing effort to understand Perelman's work by giving more detailed and accessible presentations of his ideas or alternative arguments for various parts of the proof.

This book is a contribution to this endeavor. Its two main innovations are first a simplified version of Perelman's Ricci flow with surgery, which is called Ricci flow with bubbling-off, and secondly a completely different and original approach to the last step of the proof. In addition, special effort has been made to simplify and streamline the overall structure of the argument and make the various parts independent of one another.

A complete proof of the geometrisation conjecture is given, modulo pre-Perelman results on Ricci flow, Perelman's results on the $\mathcal{L}$-functional and κ-solutions, as well as the Colding–Minicozzi extinction paper. The book can be read by anyone already familiar with these results or willing to accept them as black boxes. The structure of the proof is presented in a lengthy introduction which does not require knowledge of geometric analysis. The bulk of the proof is the existence theorem for Ricci flow with bubbling-off, which is treated in parts I and II. Part III deals with the long-time behaviors of Ricci flow with bubbling-off. Part IV finishes the proof of the geometrisation conjecture.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: The Geometrisation conjecture; *Part I. Ricci flow with bubbling-off: definitions and statements:* Basic definitions; Piecing together necks and caps; κ-noncollapsing, canonical geometry and pinching; Ricci flow with (r, δ, κ)-bubbling-off; *Part II. Ricci flow with bubbling-off: existence:* Choosing cutoff parameters; Metric surgery and the proof of Proposition A; Persistence; Canonical neighbourhoods and the proof of Proposition B; κ-noncollapsing and the proof of Proposition C; *Part III. Long-time behaviour of Ricci flow with bubbling-off:* The thin-thick decomposition theorem; Refined estimates for long-time behaviour; *Part IV. Weak collapsing and hyperbolisation:* Collapsing, simplicial volume and strategy of proof; Proof of the weak collapsing theorem; A rough classification of 3-manifolds; Appendix A. 3-manifold topology; Appendix B. Comparison geometry; Appendix C. Ricci flow; Appendix D. Alexandrov spaces; Appendix E. A sufficient condition for hyperbolicity; Bibliography; Index.

EMS Tracts in Mathematics, Volume 13

September 2010, 247 pages, Hardcover, ISBN: 978-3-03719-082-1, 2000 *Mathematics Subject Classification:* 57-02, 57M50, 53C44, **AMS members US$51.20**, List US$64, Order code EMSTM/13

Classified Advertisements

Positions available, items for sale, services available, and more

ALABAMA

UNIVERSITY OF ALABAMA AT BIRMINGHAM
Department of Mathematics

The Department of Mathematics at the University of Alabama at Birmingham (UAB) is soliciting applications for a tenure-track assistant professor position beginning August 15, 2011. Applicants whose research is compatible with the department's strengths in differential equations, differential geometry, dynamical systems, mathematical physics, and topology, including computational aspects of these areas, are encouraged to apply. Those with expertise in geometric or harmonic analysis, inverse problems, or probability are of particular interest in this search. For additional information about the department please visit: `http://www.math.uab.edu`.

Applicants should have demonstrated the potential to excel in one of the research areas mentioned and in teaching at all levels of instruction. They should also be committed to professional service including departmental service. Postdoc experience is preferred.

Applications should include a curriculum vita with a publication list, a statement of future research plans, a statement on teaching experience and philosophy, and minimally three letters of reference with at least one letter addressing teaching experience and ability. We prefer applications and all other materials be submitted electronically at: `http://www.mathjobs.org`, although applicants may submit an application including an AMS cover sheet to:

Math Faculty Search
Department of Mathematics
The University of Alabama at Birmingham
Birmingham, AL 35294-1170

The department and university are committed to building a culturally diverse workforce and strongly encourage applications from women and individuals from underrepresented groups. UAB has an active NSF-supported ADVANCE program and a Spouse Relocation Program to assist in the needs of dual career couples. UAB is an Affirmative Action/Equal Employment Opportunity employer.

000101

UNIVERSITY OF ALABAMA, TUSCALOOSA
Department of Mathematics

The Department of Mathematics invites applications for a tenure-track position at the assistant professor level in the general area of modern analysis, with the appointment to begin on August 16, 2011. We are particularly interested in multivariate operator theory and analysis on function spaces; however, we are also interested in other major areas of modern analysis. Candidates must possess a doctoral degree in mathematics or a closely related field by August 16, 2011. Experience in teaching and research is expected. Applicants should apply online at: `http://facultyjobs.ua.edu`; attach a curriculum vita along with a letter of application and arrange for three letters of recommendation to be sent to: Chair of the Analysis Search Committee, Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487-0350. Applications will be reviewed on an ongoing basis and will continue to be accepted until the position is filled. The University of Alabama is an Equal Opportunity/Affirmative Action employer and actively seeks diversity among its employees. Women and minority candidates are strongly encouraged to apply. For more information about the department and the university visit our website at `math.ua.edu`.

000121

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

The 2010 rate is $3.25 per word. No discounts for multiple ads or the same ad in consecutive issues. For an additional $10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: January 2011 issue–October 28, 2010; February 2011 issue–November 29, 2010; March 2011 issue–December 28, 2010; April 2011 issue–January 30, 2011; May 2011 issue–February 28 2011; June/July 2011 issue–April 28, 2011.

U.S. laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the U.S. cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to U.S. laws. Details and specific wording may be found on page 1373 (vol. 44).

Situations wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the U.S. and Canada or 401-455-4084 worldwide for further information.

Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or send email to `classads@ams.org`. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 20904. Advertisers will be billed upon publication.

CALIFORNIA

UNIVERSITY OF CALIFORNIA LOS ANGELES
Department of Mathematics
Faculty Positions
Academic Year 2011-2012

The Department of Mathematics solicits applications at the level of tenure-track/tenure faculty with particular emphasis in applied mathematics. Salary is commensurate with the level of experience.

We also plan to make temporary and visiting appointments in the categories 1-5 below. Depending on the level, candidates must give evidence of potential or demonstrated distinction in scholarship and teaching.

(1) E. R. Hedrick Assistant Professorships. Salary is $61,200 and appointments are for three years. The teaching load is four quarter courses per year.

(2) Computational and Applied Mathematics (CAM) Assistant Professorships. Salary is $61,200 and appointments are for three years. The teaching load is normally reduced by research funding to two quarter courses per year.

(3) Program in Computing (PIC) Assistant Adjunct Professorships. Salary is $65,500. Applicants for these positions must show very strong promise in teaching and research in an area related to computing. The teaching load is four one-quarter programming courses each year and one seminar every two years. Initial appointments are for one year and possibly longer, up to a maximum service of four years.

(4) Assistant Adjunct Professorships and Research Postdocs. Normally appointments are for one year, with the possibility of renewal. Strong research and teaching background required. The salary range is $53,200-$59,500. The teaching load for adjuncts is six quarter courses per year.

(5) Simons Postdoctoral Fellows. Salary is $70,000 increased by 3% each year. Appointments are for three years. The teaching load is four courses over the three-year period. Candidates must show exceptional research promise. Simons Postdoctoral Fellowships will be restricted to candidates who receive the Ph.D. in the academic year immediately preceding that in which they would become Simons Postdoctoral Fellows. This position is funded by the Simons Foundation.

If you wish to be considered for any of these positions you must submit an application and supporting documentation electronically via: `http://www.mathjobs.org`.

For fullest consideration, all application materials should be submitted on or before December 8, 2010. Ph.D. is required for all positions.

UCLA and the Department of Mathematics have a strong commitment to the achievement of excellence in teaching and research and diversity among its faculty and staff. The University of California is an Equal Opportunity/Affirmative Action Employer. The University of California asks that applicants complete the Equal Opportunity Employer survey for Letters and Science at the following URL: `http://cis.ucla.edu/facultysurvey`. Under Federal law, the University of California may employ only individuals who are legally authorized to work in the United States as established by providing documents specified in the Immigration Reform and Control Act of 1986.

000042

UNIVERSITY OF CALIFORNIA LOS ANGELES
Institute for Pure and Applied Mathematics

The Institute for Pure and Applied Mathematics (IPAM) at UCLA is seeking an Associate Director (AD), to begin a two-year appointment on July 1, 2011. The AD is expected to be an active and established research mathematician or scientist in a related field, with experience in conference organization. The primary responsibility of the AD will be running individual programs in coordination with the organizing committees. More information on IPAM's programs can be found at: `http://www.ipam.ucla.edu`. The selected candidate will be encouraged to continue his or her personal research program within the context of the responsibilities to the institute. For a detailed job description and application instructions, go to: `http://www.ipam.ucla.edu/jobopenings/assocdirector.aspx`. Applications will receive fullest consideration if received by February 1, 2011, but we will accept applications as long as the position remains open. UCLA is an Equal Opportunity/Affirmative Action Employer.

000122

UNIVERSITY OF SOUTHERN CALIFORNIA
Department of Mathematics

The Department of Mathematics in the College of Letters, Arts, and Sciences of the University of Southern California seeks to fill the following positions. The start date for all positions is August 2011.

Tenure-Track Assistant Professorship. Subject area: open, with a preference for candidates in geometry, topology, and related fields. Candidates should have demonstrated excellence in research and a strong commitment to graduate and undergraduate education. A Ph.D. is required.

Assistant Professor Non-Tenure-Track. Subject area: any field of mathematics of interest to senior members of the department. Candidates should demonstrate great promise in research and evidence of strong teaching. This is a three-year non-tenure-track appointment with a teaching load of three semester courses per year. A Ph.D. is required.

To apply, please submit the following materials: letter of application and curriculum vitae, including your email address, telephone and fax numbers, preferably with the standardized AMS Cover Sheet. Candidates should also arrange for at least three letters of recommendation to be sent, one of which addresses teaching skills. Applications through `MathJobs` at: `http://www.mathjobs.org` are preferred. Otherwise, all materials should be mailed to:

Search Committee
Department of Mathematics
College of Letters Arts and Sciences
University of Southern California
3620 Vermont Avenue, KAP 108
Los Angeles, CA 90089-2532.

Review of applications will begin November 15, 2010, and will continue until the positions are filled. Additional information about the USC College Department of Mathematics can be found at our website: `http://college.usc.edu/mathematics/home/`.

USC strongly values diversity and is committed to equal opportunity in employment. Women and men, and members of all racial and ethnic groups are encouraged to apply.

000100

COLORADO

UNIVERSITY OF COLORADO AT COLORADO SPRINGS
Assistant Professor of Mathematics

The Department of Mathematics, University of Colorado at Colorado Springs (UCCS), invites applications for a tenure-track assistant professor in mathematics, effective August 2011. The candidate must have a Ph.D. in mathematics or related field at the time of appointment. The position requires teaching, research, and service. Information on how to apply for the position may be found at: `http://mathjobs.org`.

000113

CONNECTICUT

YALE UNIVERSITY
Department of Mathematics
Analysis and Combinatorics

The Department of Mathematics of Yale University invites applications for a position as a tenured associate or full professor in the area of analysis and combinatorics. We seek scholars with a record of outstanding achievement in research who are accomplished teachers at both the undergraduate and graduate level. We are interested in candidates with a breadth of expertise in the above mentioned area.

Yale is an Affirmative Action/Equal Opportunity Employer. Qualified Women and members of minority groups are encouraged to apply. Submit applications and supporting material through `http://MathJobs.org`; email: `gibbs.committee@math.yale.edu` by January 15, 2011.

000108

ILLINOIS

UNIVERSITY OF ILLINOIS AT CHICAGO
Department of Mathematics, Statistics, and Computer Science

The department has active research programs in a broad spectrum of centrally important areas of pure mathematics, computational and applied mathematics, combinatorics, mathematical computer science and scientific computing, probability and statistics, and mathematics education. See `http://www.math.uic.edu` for more information.

Applications are invited for the following position, effective August 16, 2011. Final authorization of the position is subject to the availability of state funding.

Research Assistant Professorship. This is a non-tenure-track position, normally renewable annually to a maximum of three years. This position carries a teaching responsibility of three courses per year, and the expectation that the incumbent play a significant role in the research life of the department. The salary for AY 2010-2011 for this position is $55,000. Applicants must show evidence of outstanding research potential in mathematics, computer science, statistics, mathematics education or related field, and should expect to have a Ph.D. or equivalent degree by the start date.

Applicants should include a vita, research and teaching statements, and at least three (3) letters of recommendation. Applications should be submitted through `http://mathjobs.org`. No applications will be accepted by surface mail or email. To ensure full consideration, application materials must be received by December 31, 2010, but applications will be accepted through January 31, 2011. Minorities, persons with disabilities, and women are particularly encouraged to apply. UIC is an AA/EOE.

000102

UNIVERSITY OF ILLINOIS AT CHICAGO
Department of Mathematics, Statistics, and Computer Science
Tenure-Track
Assistant Professor in Statistics

The Department of Mathematics, Statistics, and Computer Science invites applications for a tenure-track assistant professor position in statistics or probability. The position is effective August 16, 2011, and the salary is negotiable. Applicants must have expertise in statistics, probability, or related areas, a demonstrated commitment to research and teaching, and should expect to have a Ph.D. or equivalent degree by the start date. Final authorization of the position is subject to the availability of state funding.

The department has active research programs in a broad spectrum of centrally important areas of pure mathematics, computational and applied mathematics, mathematical computer science, probability and statistics, and mathematics education. See `http://www.math.uic.edu` for more information.

Applicants should include a vita, research and teaching statements, and at least three (3) letters of recommendation. Applications should be submitted through: `http://mathjobs.org`. No applications will be accepted by surface mail or email. To ensure full consideration, application materials must be received by December 1, 2010, but applications will be accepted through January 15, 2011. However, we will continue considering candidates until the position is filled. Minorities, persons with disabilities, and women are particularly encouraged to apply. UIC is an AA/EOE.

000103

INDIANA

INDIANA UNIVERSITY BLOOMINGTON
Department of Mathematics
Zorn Research Postdoctoral Fellowships

The Department of Mathematics seeks applications for two Zorn Research Postdoctoral Fellowships beginning in the fall of 2011. These are three-year, non-tenure-track positions with reduced teaching loads. Outstanding candidates with a recent Ph.D. in any area of pure or applied mathematics are encouraged to apply. Zorn fellows are paired with mentors with whom they have compatible research interests. The department maintains strong research groups in all of the principal fields of mathematics. Bloomington is located in the forested hills of southern Indiana and offers a rich variety of musical and cultural attractions.

Applicants should submit an AMS cover sheet, a curriculum vitae, a research statement, and a teaching statement using the online service provided by the AMS at: `http://www.mathjobs.org`. If unable to do so, send application materials to the address below. Applicants should arrange for four letters of recommendation, including one evaluating teaching experience. Please ask reference writers to submit their letters electronically through `http://www.mathjobs.org`. If they are unable to do so, they may also send their letters to the following address: Zorn Postdoctoral Fellowships Search Committee, Department of Mathematics, Indiana University, 831 East 3rd Street, Rawles Hall, Bloomington, IN 47405-7106. Applications should be received by December 15, 2010. Indiana University is an Equal Opportunity/Affirmative Action Employer

000094

VALPARAISO UNIVERSITY
Department of Mathematics

The Department of Mathematics and Computer Science announces a tenure-track position available at the rank of assistant professor, beginning August 2011. Area of specialization in the mathematical sciences is open, although expertise in algebra, topology, or statistics is welcome. Applications from women and minorities are encouraged and teaching experience is a plus. Duties include: teach 5 courses (21 credit hours) per year, continued scholarly activity, and service to the department and university. Candidates should be willing to work in a scholarly community committed to Christian higher education in the Lutheran tradition. Send letter of application, vita, statement of teaching philosophy, summary of research plans, and three letters of recommendation to Department of Mathematics and Computer Science Search Committee, Valparaiso University, Valparaiso, IN 46383, or electronically to: `joan.steffen@valpo.edu`. More information about the department can be found at: `http://www.valpo.edu/mcs`. Review of applications will begin on December 1. Valparaiso University does not unlawfully discriminate and aims to employ persons of various backgrounds and experiences to help constitute a diverse community. Its entire EOE policy can be found at: `http://www.valpo.edu/equalopportunity/index.php`.

000117

LOUISIANA

LOUISIANA STATE UNIVERSITY
Department of Mathematics
ASSISTANT/ASSOCIATE PROFESSOR
(Three positions)

Applications are invited for three anticipated assistant or associate professor positions in the Department of Mathematics at Louisiana State University. The department will continue to expand its professorial faculty over the next several years. Applications are invited for positions in the areas of geometric analysis, geometry and topology, algebra, analysis, applied mathematics, and combinatorics. Required Qualifications: Ph.D. or equivalent degree in mathematics or related field. Additional Qualifications Desired: Research excellence as well as commitment to graduate and undergraduate education. Salary and rank will be commensurate with qualifications and experience. Minorities and women are strongly encouraged to apply. An offer of employment is contingent on a satisfactory pre-employment background check. We will begin reviewing applications on January 18, 2011, and will continue to review applications through February 28, 2011, or until the candidates are selected. Apply online and view a more detailed ad at: `http://www.lsusystemcareers.lsu.edu`. Position #031823/031556/005892. LSU system is an Equal Opportunity/Equal Access Employer.

000095

LOUISIANA STATE UNIVERSITY
Department of Mathematics
POSTDOCTORAL RESEARCHER
(Anticipated/NSF VIGRE/Two positions)

The Department of Mathematics at Louisiana State University invites applications for two anticipated National Science Foundation Vertical InteGration of Research and Education (NSF VIGRE) Postdoctoral Researcher positions. Applications are invited for positions in the areas of geometric analysis, geometry and topology, algebra, analysis, applied mathematics, and combinatorics. Required Qualifications: Ph.D. or an equivalent degree in mathematics or a related area. A.B.D. candidates will be considered, but must have Ph.D. by August 2011. Additional Qualifications Desired: Research excellence as well as a commitment to graduate and undergraduate education. Special Requirements: As required by the VIGRE grant, postdoctoral researchers must be U.S. citizens, U.S. nationals, or U.S. permanent residents. An offer of employment is contingent on a satisfactory pre-employment background check. We will begin reviewing applications on January 18, 2011, and will continue to review applications until the positions are filled. Minorities and women are strongly encouraged to apply. Apply online and view a more detailed ad at: `http://www.lsusystemcareers.lsu.edu`. Position #000813 and #015716. LSU system is an Equal Opportunity/Equal Access Employer.

000088

MARYLAND

JOHNS HOPKINS UNIVERSITY
Department of Mathematics
Non-Tenure-Track
J. J. Sylvester Assistant Professor

Subject to availability of resources and administrative approval, the Department of Mathematics solicits applications for non-tenure-track assistant professor positions beginning fall 2011.

The J. J. Sylvester Assistant Professorship is a three-year position offered to recent Ph.D.s with outstanding research potential. Candidates in all areas of pure mathematics, including analysis, mathematical physics, geometric analysis, complex and algebraic geometry, number theory, and topology are encouraged to apply. The teaching load is three courses per academic year.

To submit your applications go to: `http://www.mathjobs.org/jobs/jhu`. Applicants are strongly advised to submit their other materials electronically at this site.

If you do not have computer access, you may mail your application to: Appointments Committee, Department of Mathematics, Johns Hopkins University, 404 Krieger Hall, Baltimore, MD 21218. Application should include a vita, at least four letters of recommendation of which one specifically comments on teaching, and a description of current and planned research. Write to: `cpoole@jhu.edu` for questions concerning these positions. Applications received by November 15, 2010, will be given priority. The Johns Hopkins University is an Affirmative Action/Equal Opportunity Employer. Minorities and women candidates are encouraged to apply.

000089

MASSACHUSETTS

BOSTON COLLEGE
Department of Mathematics
Postdoctoral Position

The Department of Mathematics at Boston College invites applications for a postdoctoral position beginning September 2011. This position is intended for a new or recent Ph.D. with outstanding potential in research and excellent teaching. This is a 3-year visiting assistant professor position, and carries a 2-1 annual teaching load. Research interests should lie within number theory or representation theory or related areas. Candidates should expect to receive their Ph.D. prior to the start of the position and have received the Ph.D. no earlier than spring 2010.

Applications must include a cover letter, description of research plans, curriculum vitae, and four letters of recommendation, with one addressing the candidate's teaching qualifications. Applications received no later than January 1, 2011, will be assured our fullest consideration. Please submit all application materials through: `http://MathJobs.org`.

Applicants may learn more about the department, its faculty, and its programs and about Boston College at `http://www.bc.edu/math`. Email inquiries concerning this position may be directed to: `postdoc-search@bc.edu`. Boston College is an Affirmative Action/Equal Opportunity Employer. Applications from women, minorities, and individuals with disabilities are encouraged.

000056

BOSTON UNIVERSITY
Department of Mathematics and Statistics
Tenure-Track Position

The Department of Mathematics and Statistics at Boston University invites applications at the tenure-track assistant professor level in number theory, especially in the field of automorphic representation theory. Ph.D. required. The position will begin in September 2011, subject to administrative approval. Strong commitment to research and teaching is essential. Please submit the AMS Application Cover Sheet, CV, research statement, teaching statement, and at least three letters of recommendation, one of which addresses teaching, to `http://mathjobs.org`. Alternatively, send all material to NT Search, Department of Mathematics and Statistics, Boston University, 111 Cummington St., Boston, MA 02215. Application deadline: January 3, 2011. Boston University is an Affirmative Action/Equal Opportunity Employer.

000118

NORTHEASTERN UNIVERSITY
Department of Mathematics
Assistant/Associate Professor
Tenure-Track Position

The Department of Mathematics at Northeastern University invites applicants for a tenure-track position at the assistant professor level to start as early as September of 2011. We seek a junior faculty member at the assistant professor level. Exceptionally qualified candidates could be considered at a higher level. Appointments are based on exceptional research contributions in mathematics combined with strong commitment and demonstrated success in teaching. Outstanding candidates with research in any area of mathematics and with an interest and ability to collaborate across disciplines and units in the university are urged to apply. Minimum Qualifications: Candidates must have a Ph.D., research experience, and demonstrated evidence of excellent teaching ability. Review of applications will begin immediately. Complete applications received by November 1, 2010, will be guaranteed full consideration. Additional applications will be considered until the position is filled. To apply, all applicants must complete the following two steps: 1. Visit "Careers at Northeastern" at: `http://psoft.neu.edu/psc/neuhrprdpub/EMPLOYEE/HRMS/c/NEU_HR.NEU_JOBS.GBL`. Click on "Faculty Positions" and search for the current position under the College of Science. Applications can also be submitted by visiting the College of Science website at: http://www.northeastern.edu/cos/ and clicking on the Faculty Positions button. 2. Submit at least three letters of recommendation at: http://www.mathjobs.org. These letters should address the applicant's research accomplishments and supply evidence that the applicant has the ability to communicate and teach effectively. Northeastern University is an Equal Opportunity, Affirmative Action Educational Institution and Employer, Title IX University. Northeastern University particularly welcomes applications from minorities, women, and persons with disabilities. Northeastern University is an E-Verify Employer.

000119

TUFTS UNIVERSITY
Department of Mathematics
Non-Tenure-Track Assistant Professorship
Algebraic Geometry

Applications are invited for a non-tenure-track assistant professorship to begin September 1, 2011, with an initial appointment for one year renewable for two more. Applicants must show promise of outstanding research in the area of algebraic geometry and related fields. Preference will be given to candidates whose interests bridge those of Tufts faculty including classical algebraic geometry, algebraic groups and equivariant cohomology. Applicants must also show evidence of excellent teaching. The teaching load will be two courses per semester. Applications should include a cover letter, curriculum vitae, a research statement, and a teaching statement. All of these documents should be submitted electronically through http://www.mathjobs.org. In addition, applicants should arrange for three letters of recommendation to be submitted electronically on their behalf through http://www.mathjobs.org. If a recommender cannot submit online, we will also accept signed PDF attachments sent to: montserrat.teixidoribigas@tufts.edu or paper letters mailed to AG Search Committee Chair, Department of Mathematics, 503 Boston Avenue, Tufts University, Medford, MA 02155. Review of applications will begin on January 2, 2011, and will continue until the position is filled. Tufts University is an Affirmative Action/Equal Opportunity Employer. We are committed to increasing the diversity of our faculty. Members of underrepresented groups are strongly encouraged to apply.

000106

TUFTS UNIVERSITY
Department of Mathematics
Tenure-Track Assistant Professorship
Dynamical Systems and Geometry

Applications are invited for a tenure-track assistant professorship to begin September 1, 2011. Applicants must show promise of outstanding research in the area of dynamical systems and geometry, that is, the study of dynamical aspects of geometric problems or applications of dynamical systems to geometry. Possible specialties include, but are not limited to, actions of the mapping-class group, Teichmüller flows, geodesic flows in non-positive curvature, dynamics or rigidity of group actions, dynamics on the boundary at infinity. Preference will be given to candidates whose interests bridge those of Tufts faculty in geometric group theory, topology, and dynamical systems. Ph.D. required. Applicants must also show evidence of excellent teaching. The teaching load will be two courses per semester. Applications should include a cover letter, curriculum vitae, a research statement, and a teaching statement. All of these documents should be submitted electronically through http://www.mathjobs.org. In addition, applicants should arrange for three letters of recommendation to be submitted electronically on their behalf through http://www.mathjobs.org. If a recommender cannot submit online, we will also accept signed PDF attachments sent to: DGHiring@elist.tufts.edu or paper letters mailed to DG Search Committee Chair, Department of Mathematics, 503 Boston Avenue, Tufts University, Medford, MA 02155. Review of applications will begin on December 1, 2010, and will continue until the position is filled. Tufts University is an Affirmative Action/Equal Opportunity Employer. We are committed to increasing the diversity of our faculty. Members of underrepresented groups are strongly encouraged to apply.

000105

TUFTS UNIVERSITY
Department of Mathematics
Norbert Wiener Assistant Professorship
Integral Geometry and Harmonic Analysis

Applications are invited for a Norbert Wiener Assistant Professorship. This is a non-tenure-track appointment starting on September 1, 2011, for one year and renewable to a maximum of three years. Applicants must show promise of outstanding research in the area of integral geometry and harmonic analysis. Possible specialties include, but are not limited to, Radon transforms, harmonic analysis on Lie groups and homogeneous spaces, representation theory, microlocal analysis, and aspects of geometric analysis, convexity theory, algebraic analysis, and Lie theory related to integral geometry and harmonic analysis. Ph.D. required by the appointment date. Preference will be given to candidates whose interests bridge those of Tufts faculty. Applicants must also show evidence of excellent teaching. The teaching load will be two courses per semester. Applications should include a cover letter, curriculum vitae, a research statement, and a teaching statement. All of these documents should be submitted electronically through http://www.mathjobs.org. In addition, applicants should arrange for three letters of recommendation to be submitted electronically on their behalf through http://www.mathjobs.org. If a recommender cannot submit online, we will also accept signed PDF attachments sent to: IGHAHiring@elist.tufts.edu or paper letters mailed to IGHA Search Committee Chair, Department of Mathematics, 503 Boston Avenue, Tufts University, Medford, MA 02155. Review of applications will begin on December 1, 2010, and will continue until the position is filled. Tufts University is an Affirmative Action/Equal Opportunity Employer. We are committed to increasing the diversity of our faculty. Members of underrepresented groups are strongly encouraged to apply.

000107

TUFTS UNIVERSITY
Tenure-Track Assistant Professorship
Scientific Computing

Applications are invited for a tenure-track assistant professorship to begin September 1, 2011. Applicants must show promise of outstanding research in the area of scientific computing and evidence of strength in teaching a broad range of courses in mathematics, including upper-level undergraduate and graduate courses in applied mathematics. The teaching load will be two courses per semester. Preference will be given to candidates who show potential for interaction with existing applied mathematics research efforts in the department, including computational partial differential equations, computational neuroscience, numerical linear algebra, and inverse problems.

Applications should include a cover letter, curriculum vitae, a research statement, and a teaching statement. All of these documents should be submitted electronically through: http://www.mathjobs.org. In addition, applicants should arrange for three letters of recommendation to be submitted electronically on their behalf through http://www.mathjobs.org. If a recommender cannot submit online, we will accept signed PDF attachments sent to: misha.kilmer@tufts.edu or paper letters mailed to SC Search Committee Chair, Department of Mathematics, Bromfield-Pearson Hall, Tufts University, Medford, MA 02155. Review of applications will begin on Nov. 22, 2010, and will continue until the position is filled. Tufts University is an

Affirmative Action/Equal Opportunity Employer. We are committed to increasing the diversity of our faculty. Members of underrepresented groups are strongly encouraged to apply.

000097

NEBRASKA

UNIVERSITY OF NEBRASKA-LINCOLN
Department of Mathematics

Applications are invited for one tenure-track, associate/full professor of mathematics with emphasis in teacher education, starting August 2011.

Successful candidates must have a Ph.D. in mathematics or a related field. The ideal candidate will have a record of scholarly achievement, outstanding teaching, and demonstrated potential for leading an externally funded, interdisciplinary research team with expertise in the mathematical education of teachers. Teaching duties will emphasize mathematics courses for teachers. The successful candidate will work with faculty in the department and the Center for Science, Mathematics and Computer Education to provide leadership in the mathematical education of teachers; to contribute to the success of two current NSF grants, NebraskaMATH and NebraskaNOYCE; to obtain and lead externally funded projects; to develop partnerships with Nebraska schools and districts; and to contribute to the intellectual, research, and instructional life of UNL.

Applicants should send a letter of application, a CV, separate statements addressing research and teaching, and three or four letters of reference, at least one of which should address teaching, to: Mathematics Teacher Education Search Committee, Department of Mathematics, University of Nebraska-Lincoln, 203 Avery Hall, Lincoln, NE 68588-0130. Use of the AMS application cover sheet is encouraged. To be considered for the position, applicants must complete the Faculty/Administrative Information Form at: http://employment.unl.edu, requisition #100621. Review of applications will begin December 10, 2010, and continue until a suitable candidate is found. For more information see the department's website at: http://www.math.unl.edu.

The University of Nebraska has an active National Science Foundation ADVANCE gender equity program, and is committed to a pluralistic campus community through affirmative action, equal opportunity, work-life balance, and dual careers.

000115

NEVADA

UNIVERSITY OF NEVADA, LAS VEGAS
Department of Mathematical Sciences
Assistant Professor–Applied Analysis

The Department of Mathematical Sciences invites applications for a full-time, 9-month, tenure-track assistant professor position in applied analysis, commencing fall 2011. The successful candidate is expected to teach effectively at the undergraduate and graduate levels, publish in refereed research journals, pursue external funding, and supervise graduate students. The successful candidate will also be expected to provide service to the department, college, university, and the profession. Applicants must have a Ph.D. in mathematical sciences, a demonstrated expertise in applied analysis, and a minimum of two years of postdoctoral experience. Preference will be given to candidates who demonstrate commitment to excellence in both teaching and scholarly activity. For full details, please visit our website at: http://jobs.unlv.edu or call (702) 895-2894 for assistance. UNLV is an Equal Opportunity/Affirmative Action Educator and Employer Committed to Achieving Excellence Through Diversity.

000114

NEW HAMPSHIRE

DARTMOUTH COLLEGE
John Wesley Young Research Instructorships

The John Wesley Young Instructorship is a postdoctoral, two- to three-year appointment intended for promising Ph.D. graduates with strong interests in both research and teaching and whose research interests overlap a department member's. Current research areas include numerical analysis, complex systems, combinatorics, geometry, logic, non-commutative geometry, number theory, operator algebras, probability, set theory and topology. Instructors teach four ten-week courses distributed over three terms, though one of these terms in residence may be free of teaching. The assignments normally include introductory, advanced undergraduate, and graduate courses. Instructors usually teach at least one course in their own specialty. This appointment is for 26 months with a monthly salary of $4,833 and a possible 12 month renewal. Salary includes two-month research stipend for instructors in residence during two of the three summer months. To be eligible for a 2011-2014 Instructorship, candidate must be able to complete all requirements for the Ph.D. degree before September 2011. Applications may be obtained at: http://www.math.dartmouth.edu/activities/recruiting/ or http://www.mathjobs.org. Position ID:JWY # 2240. General inquiries can be directed to Annette Luce, Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, New Hampshire 03755-3551. At least one referee should comment on applicant's teaching ability; at least two referees should write about applicant's research ability. Applications received by January 5, 2011, receive first consideration; applications will be accepted until position is filled. Dartmouth College is committed to diversity and strongly encourages applications from women and minorities. There are 2 positions available to fill.

000073

NEW JERSEY

RUTGERS UNIVERSITY-CAMDEN
Department of Mathematical Sciences

The Department of Mathematical Sciences at Rutgers University-Camden invites applications for a tenure-track position at the assistant professor level. Candidates with strong background in analysis and an established research record are encouraged to apply. The department is particularly interested in applicants whose research activities are related to the current faculty research areas such as differential equations, dynamical systems, stochastic processes, and geometric control theory. The appointee should have a good potential to interact with our Center for Computational and Integrative Biology and its newly established Ph.D. program. Appointment is effective July 1, 2011. Application material, which must include a current vita, a statement of research plan, and at least three letters of recommendation should be sent to: Search Committee, Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102. Rutgers University is an Equal Opportunity Employer. Qualified women and minority group members are urged to apply.

000116

RUTGERS UNIVERSITY-NEW BRUNSWICK
Mathematics Department

The Mathematics Department of Rutgers University-New Brunswick invites applications for the following positions which may be available September 2011.

TENURE-TRACK ASSISTANT PROFESSORSHIP: Subject to availability of funding, the department expects at least two openings at the level of tenure-track assistant professor. Candidates must have the Ph.D. and show a strong record of research accomplishments in pure or applied mathematics, and concern for teaching. The department itself has hiring priorities in algebra and applied probability, but outstanding candidates in any field of pure or applied mathematics will be considered. In regard to one of the positions the department is particularly interested in applicants with fields of research that would make a joint appointment with another department/institute a possibility. The normal annual teaching load for research-active faculty is 2-1. Review of applications begins immediately.

HILL ASSISTANT PROFESSORSHIPS and NON-TENURE-TRACK ASSISTANT PROFESSORSHIPS: These are both three-year nonrenewable positions. Subject to availability of funding, the department may have several open positions of these types. The Hill Assistant Professorship carries a reduced teaching load of 2-1 for research; candidates for it should have received the Ph.D., show outstanding promise of research ability in pure or applied mathematics, and have concern for teaching. The non-tenure-track assistant professorship carries a teaching load of 2-2; candidates for it should have received the Ph.D., show evidence of superior teaching accomplishments and promise of research ability. Review of applications begins January 1, 2011.

Applicants for the above position(s) should submit a curriculum vitae (including a publication list) and arrange for four letters of reference to be submitted, one of which evaluates teaching.

Applicants should first go to the website `http://www.mathjobs.org/jobs` and fill out the AMS Cover Sheet electronically. It is essential you fill out the cover sheet completely, including naming the positions being applied for (TTAP, HILL, NTTAP, respectively) giving the AMS Subject Classification number(s) of area(s) of specialization, and answering the question about how materials are being submitted. The strongly preferred way to submit the CV, references, and any other application materials is online at: `http://www.mathjobs.org/jobs`. If necessary, however, application materials may instead be mailed to: Search Committee, Dept. of Math-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019.

Review of applications will begin on the dates indicated above, and will continue until openings are filled. Updates on these positions will appear on the Rutgers Mathematics Department webpage at: `http://www.math.rutgers.edu`.

Rutgers is an Affirmative Action/Equal Opportunity Employer and encourages applications from women and minority-group members.

000093

NEW YORK

BINGHAMTON UNIVERSITY
Department of Mathematical Sciences

The Department of Mathematical Sciences at Binghamton University (State University of New York at Binghamton) invites applications for at least one position at the assistant professor level beginning fall 2011. First consideration will be given to candidates in geometry, topology and related areas of mathematics. Otherwise, consideration will be given to candidates in probability/statistics, analysis, and all areas of mathematics in which the department already has some presence. Preferred method of application via `mathjobs.org`. Paper applications also accepted: CV, three reference letters and evidence of research and teaching abilities to: Fernando Guzman, Chair, Department of Mathematical Sciences, Binghamton University, Binghamton NY 13902-6000; email: `hiring@math.binghamton.edu`. Binghamton University is an Equal Opportunity/Affirmative Action Employer.

000112

NORTH CAROLINA

WAKE FOREST UNIVERSITY
Department of Mathematics

Applications are invited for one or more Teacher Scholar Postdoctoral Fellow positions in mathematics beginning July 2011. We seek highly qualified candidates who have a commitment to excellence in both teaching and research. Primary consideration will be given to candidates whose research interests overlap with existing faculty. A Ph.D. in mathematics or a related area is required. The department has 20 members and offers both a B.A. and a B.S. in mathematics, with an optional concentration in statistics, a B.S. in interdisciplinary mathematics, and a B.S. in each of mathematical business and mathematical economics. The department has a graduate program offering an M.A. in mathematics. A complete application will include a letter of application, curriculum vitae, teaching statement, research statement, graduate transcripts, and three letters of recommendation. The letter of application should explain why the teaching and research environment at Wake Forest is a good fit for the candidate. Review of applications will begin on January 1, 2011, and will continue until the positions are filled. Applicants are encouraged to post materials electronically at: `http://www.mathjobs.org`. Hard copy can be sent to Stephen Robinson, Wake Forest University, Department of Mathematics, P.O. Box 7388, Winston-Salem, NC 27109. (`sbr@wfu.edu`; `http://www.math.wfu.edu`). AA/EO employer.

000071

WAKE FOREST UNIVERSITY
Department of Mathematics

Applications are invited for one tenure-track position in statistics at the assistant professor level beginning July 2011. In exceptional cases applications at a level higher than assistant professor may be considered. We seek highly qualified candidates who have a commitment to excellence in both teaching and research. A Ph.D. in statistics or a closely related area is required. The department has 20 members and offers both a B.A. and a B.S. in mathematics, with an optional concentration in statistics, a B.S. in interdisciplinary mathematics, and a B.S. in each of mathematical business and mathematical economics. The department has a graduate program offering an M.A. in mathematics. A complete application will include a letter of application, curriculum vitae, teaching statement, research statement, graduate transcripts, and three letters of recommendation. The review of applications will begin on December 1, 2010, and will continue until the position is filled. Applicants are encouraged to post materials electronically at: `http://www.mathjobs.org`. Hard copy can be sent to Stephen Robinson, Wake Forest University, Department of Mathematics, P.O. Box 7388, Winston-Salem, NC 27109. (`sbr@wfu.edu`; `http://www.math.wfu.edu`). AA/EO Employer.

000072

PENNSYLVANIA

BRYN MAWR COLLEGE
Department of Mathematics
Assistant Professor

The Department of Mathematics at Bryn Mawr College invites applications for a full-time, tenure-track assistant professor position to begin July 1, 2011.

We are seeking an analyst for a tenure-track position. Candidates must have completed a doctorate in mathematics by the starting date and must show promise in research and a serious commitment to undergraduate and graduate teaching. It is expected that the new faculty member be able to teach core courses in the mathematics major, to supervise senior theses, and to contribute to the graduate analysis component in our graduate program. We welcome candidates with an applied focus.

Applications must be submitted online through `MathJobs.Org` at: `http://www.mathjobs.org/jobs` and should include a cover letter, a curriculum vitae, a description of research, a statement of teaching philosophy, and three or more letters of reference, at least one of which discusses the applicant's teaching. The review of applications will begin on December 1, 2010, and continue until the position is filled.

Located in suburban Philadelphia, Bryn Mawr College is a highly selective liberal arts college for women who share an intense commitment to intellectual inquiry, an independent and purposeful vision of their lives, and a desire to make meaningful contributions to the world. Bryn Mawr comprises an undergraduate college with 1,300 students, as well as coeducational graduate programs in social work and in some humanities, sciences, and mathematics. The college promotes faculty excellence in both research and teaching, and has strong consortial relationships with Haverford College, Swarthmore College, and the University of Pennsylvania. Bryn Mawr College is an Equal Opportunity Employer; minority candidates and women are especially encouraged to apply.

000104

SOUTH CAROLINA

UNIVERSITY OF SOUTH CAROLINA
Department of Mathematics
Tenure-track Assistant Professor

Applications are invited for a tenure-track assistant professor position in applied and computational mathematics for research related to modeling and computation of biomaterials and function of engineered tissues. Candidates should have a Ph.D. in mathematics or a related field, and have sufficient background in mathematical modeling, mathematical/numerical analysis, simulation, and/or visualization of biomaterials to direct a vigorous interdisciplinary research program. They should also have a firm commitment to excellence in teaching at both the graduate and undergraduate levels. The successful candidate is expected to play an active role in the Interdisciplinary Mathematics Institute at USC and to interact effectively with researchers at the College of Engineering and Computing and the College of Medicine at the University of South Carolina, as well as at the Advanced Tissue Biofabrication Center at the Medical University of South Carolina (in Charleston, SC). The position is associated with and is expected to support a NSF EPSCOR RII award (see `http://www.scepscoridea.org/EPSCoR/2009NSFRII.html`) that will build strength in biofabrication technology in the state of South Carolina. The beginning date for the position will be August 16, 2011. The initial screening of applicants will begin on Dec. 10, 2010. The position will remain open until it is filled. For full consideration, all supporting materials should be submitted electronically through `http://www.mathjobs.org` as soon as possible, which should include a cover letter and a complete AMS STANDARD COVER sheet addressed to: Hiring Committee, Department of Mathematics, University of South Carolina, Columbia, SC 29208; a detailed curriculum vitae with a summary of research accomplishments and goals; and four letters of recommendation. One letter should appraise the applicant's teaching ability. The email address email: `hiring@math.sc.edu` can be used for further inquiries.

The University of South Carolina is an affirmative action, equal opportunity employer. Women and minorities are encouraged to apply. The University of South Carolina does not discriminate in educational or employment opportunities or decisions for qualified persons on the basis of race, color, religion, sex, national origin, age, disability, sexual orientation or veteran status.

000110

TEXAS

TEXAS TECH UNIVERSITY
Department of Mathematics and Statistics

The Department of Mathematics and Statistics at Texas Tech University anticipates filling two open-rank positions, one in probability and/or statistics and the other with a preference given to candidates in computational mathematics or mathematics education. Texas Tech is in the midst of a period of growth toward becoming a tier-one university, and our department has active research groups across many areas of mathematics, statistics, and mathematics education. It is expected that new faculty will be significantly engaged in nationally visible scholarship supported by extramural funding, teaching in the undergraduate and graduate programs, and service to the department, college, and university. Candidates with very strong records who will bring externally sponsored research to Texas Tech will be considered for associate or full professor ranks. To be considered, candidates should apply for position number 2011TLF026 (math) or 2011TLF037 (stats) at: `http://jobs.texastech.edu` (include a vita, AMS cover sheet and research and teaching statements) and have at least three letters of reference sent to Robert Kirby, Hiring Committee Chair, Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042. Review of applications will begin November 1, 2010, and continue until the position is filled. Lubbock, which is located in West Texas, is a commercial, medical, and cultural hub with a population of more than 200,000. Texas Tech University is an Affirmative Action/Equal Opportunity Employer. We strongly encourage applications from women, minorities, and veterans, and we consider the needs of dual career couples.

000099

VIRGINIA

THE UNIVERSITY OF VIRGINIA
Department of Mathematics

The University of Virginia is seeking a distinguished mathematician to chair its Department of Mathematics and hold the title of Marvin Rosenblum Professor of Mathematics (as a tenured full professor). The individual is to lead a concerted effort to build on the department's strengths and to fulfill its potential as a top center of mathematical research and teaching. The chair has the responsibility to oversee all departmental activities related to the faculty, graduate and undergraduate students, and staff. He or she should have both a strong record of research in a central area of mathematics as well as a proven ability to provide academic leadership.

The University of Virginia, founded and designed by Thomas Jefferson, is one of the nation's foremost public universities. The Department of Mathematics, situated in the College of Arts and Sciences, encompasses major research programs in core areas of mathematics, including algebra, analysis, geometry/topology, and applied mathematics. It has 27 faculty members, as well as 3 Whyburn post-doctoral instructors, 54 graduate students, and approximately 345 undergraduate majors. The department has an active visitors program. Additional information about the department is available at: `http://www.math.virginia.edu`.

To apply, please submit the following required documents electronically at: `http://www.MathJobs.org`. A cover letter, an AMS Standard Cover Sheet, a curriculum vitae which includes a publication list and a description of administrative experience, and four letters of recommendation are required.

In addition, all candidates are required to complete a Candidate Profile through the University of Virginia's employment system, which is `http://Jobs@UVA` and is located at: `http://jobs.virginia.edu`.

To submit your candidate profile, please search for posting number 0606066 and follow the directions for applying to this posting. Your application process will not be complete until all required documents are available on `MathJobs` and you receive a confirmation number for your Candidate Profile from `Jobs@UVA`.

For additional information, contact:

Search Committee
Department of Mathematics
University of Virginia
PO Box 400137
Charlottesville, VA 22904-4137;
or
email: `Math-Hiring@Virginia.EDU`

Priority consideration will be given to applications received by February 1, 2011; however, this position remains open to applications until filled.

Women and members of underrepresented groups are encouraged to apply. The University of Virginia is an Affirmative Action/Equal Opportunity Employer and is strongly committed to building diversity within its community.

000096

BRAZIL

ICMC-UNIVERSIDADE DE SÃO PAULO
FAPESP Postdoctoral Research Fellowship

The Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo at São Carlos, S.P., Brazil, welcomes applications for a postdoctoral position in singularity theory. The position is supported by the FAPESP grant 08/54222-6. The candidates will be selected based on the following documents: curriculum vitae, short research project, letter indicating the motivation for the position and short description of their expertise. The deadline for proposals is December 30, 2010. Applications should be send to Professor Maria Aparecida Soares Ruas (email: `maasruas@icmc.usp.br`).

000098

CHINA

EAST CHINA NORMAL UNIVERSITY
Center for Partial Differential Equations
Applicants for Postdoctoral Positions

The successful candidates are expected to be young researchers with Ph.D. degrees in mathematics or related areas, with a strong research record in at least one of the following areas: analysis/ computation/modeling. More information about the positions as well as the introduction of the Center are available at: `http://postdoctor.ecnu.edu.cn/details.aspx?id=31`.

000111

TAIWAN

NATIONAL TSING HUA UNIVERSITY
Department of Mathematics

The Department of Mathematics at National Tsing Hua University of Taiwan invites applications for all levels of faculty positions in pure and applied mathematics. Applications received by January 1, 2011 are given full consideration, but all applications are considered until positions are filled. Send signed cover letter, CV, research statement, three letters of recommendation, copies of publications to: Chairman, Department of Mathematics, National Tsing Hua University, 101 Kuang Fu Road, Hsinchu 300, Taiwan. For more information, visit our webpage: `http://www.math.nthu.edu.tw`.

000120

Co-Sponsored Conferences

Energy, Growth, Form, and Collective Behavior: The 2011 AAAS Meeting in Washington, DC

The 2011 Annual Meeting of the American Association for the Advancement of Science will be February 17–21, in Washington, DC. The theme of this year's meeting is "Science Without Borders". The mathematics program at this year's annual meeting features mathematics applied to topics of international interest. This program features symposia sponsored by Section A (Mathematics) of the AAAS.

The annual meeting is organized into symposia which have three or more speakers, and often a discussant who reflects on the talks that are given. Section A is sponsoring four symposia this year, featuring outstanding expository talks by prominent mathematicians and scientists. The four symposia sponsored by Section A this year are:

- Mathematics and Collective Behavior, organized by Warren Page, City University of New York. (Scheduled speakers: Iain Couzin, Pierre Degond, and Andrea L. Bertozzi)
- Mathematics and Our Energy Future, organized by Russel Caflisch, Institute for Pure and Applied Mathematics, and Mary Lou Zeeman, Bowdoin College. (Scheduled speakers: Martin Z. Bazant, Keith Promislow, and Ian Dobson)
- Growth and Form in Mathematics, Physics, and Biology, organized by L. Mahadevan, Harvard University, and Edward Aboufadel, Grand Valley State University. (Scheduled speakers: L. Mahadevan, Yves Couder, and Alan Newell)
- Explaining Phase Transitions, organized by David Lightfoot, Georgetown University. (Scheduled speakers: Jeffrey Lidz, David Lightfoot, Martina Morris, Douglas H. Erwin, and James Yorke)

Other symposia that will be of interest to the mathematical community include:

- Using Quantitative Content Analysis to Assess the Likelihood of Terrorist Violence
- Estimating Earth's Human Carrying Capacity
- Teaching and Learning in the Digital Age: Reliable Resources Across the Disciplines
- Transcending Gender and Ethnic Barriers to Full STEM Participation
- Science Without Borders: Learning from TIMSS Advanced 2008
- The Crowd and the Cloud—The Future of Online Collaboration
- Surprise…It's Science! Reaching new audiences in unconventional ways with festivals

The above symposia are only a few of the over 150 AAAS program offerings in the physical, life, social, and biological sciences. For further information, including the schedule of talks, go to `www.aaas.org/meetings`.

AAAS annual meetings are the showcases of American science, and they encourage participation by mathematicians and mathematics educators. Section A acknowledges the generous contributions of the American Mathematical Society for travel support. The AAAS Program Committee is genuinely interested in offering symposia on pure and applied mathematical topics of current interest, and in previous years there have been symposia on subjects such as fairness in politics, origami, mathematics and the brain, quantum information theory, and the changing nature of mathematical proof.

The 2012 meeting will be February 16–20, 2012, in Vancouver, BC, Canada. The Steering Committee for Section A seeks organizers and speakers who can present substantial new material in an accessible manner to a large scientific audience. All are invited to attend the Section A Committee business meeting in Washington on Friday, February 18, 2010, at 7:00 PM, where we will brainstorm ideas for symposia. In addition, I invite you to send me, and encourage your colleagues to send me, proposals for future AAAS annual meetings.

The following are the members of the Steering Committee for Section A from February 2010 to February 2011:

Chair: Kenneth Millett (University of California, Santa Barbara)

Chair-Elect: John H. Ewing (Math for America)

Retiring Chair: Keith Devlin (Stanford University)

Secretary: Edward Aboufadel (Grand Valley State University)

Members at Large:

Claudia Neuhauser (University of Minnesota)

Warren Page (City University of New York)

Tony Chan (Hong Kong University of Science and Technology)

Mary Ellen Bock (Purdue University)

—Edward Aboufadel, Secretary of Section A of the AAAS
`aboufade@gvsu.edu`

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `http://www.ams.org/meetings/`. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

Pucón, Chile

December 15–18, 2010

Wednesday – Saturday

Meeting #1066

First Joint International Meeting between the AMS and the Sociedad de Matematica de Chile.
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: August 2010
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/internmtgs.html`.

AMS Invited Addresses

Ricardo Baeza, Universidad de Talca, Chile, *p-cohomological dimension of fields of characteristic p.*

Igor Dolgachev, University of Michigan, *Cremona groups and their subgroups.*

Andres Navas, Universidad de Santiago de Chile, *Probabilistic, dynamical and topological aspects of orderable groups.*

Rodolfo Rodriguez, Universidad de Concepcion, *Numerical approximation of the spectrum of the curl operator.*

Gunther Uhlmann, University of Washington, *Inside-out: Inverse problems.*

S. R. Srinivasa Varadhan, New York University, *Large deviations.*

AMS Special Sessions

Algebra and Model Theory, **Thomas Scanlon**, University of California, Berkeley, **Xavier Vidaux**, Universidad de Concepcion, **Charles Steinhorn**, Vassar College, and **Alf Onshuus**, Universidad de los Andes, Columbia.

Algebraic Modeling of Knotted Objects, **Vaughan F. R. Jones**, University of California, Berkeley, **Jesús Juyumaya**, Universidad de Valparaìso, **Louis H. Kauffman**, University of Illinois at Chicago, and **Sofia Lambropoulou**, National Technical University of Athens.

Applications of Differential and Difference Equations in Biology and Ecology, **J. Robert Buchanan**, Millersville University, **Fernando Córdova**, Universidad Católica de Maule, and **Jorge Velasco Hernandez**, Institute Nacional de Petroleo.

Arithmetic of Quadratic Forms and Integral Lattices, **Maria Ines Icaza**, Universidad de Talca, Chile, **Wai Kiu Chan**, Wesleyan University, and **Ricardo Baeza**, Universidad de Talca, Chile.

Automorphic Forms and Dirichlet Series, **Yves Martin**, Universidad de Chile, and **Solomon Friedberg**, Boston College.

Complex Algebraic Geometry, **Giancarlo Urzua** and **Eduardo Cattani**, University of Massachusetts.

Foliations and Dynamics, **Andrés Navas**, Universidad de Santiago de Chile, and **Steve Hurder**, University of Illinois at Chicago.

Group Actions: Probability and Dynamics, **Andrés Navas**, Universidad de Santiago de Chile, and **Rostislav Grigorchuk**, University of Texas.

Inverse Problems and PDE Control, **Matias Courdurier**, Pontificia Universidad Católica de Chile, **Axel Osses**, Universidad de Chile, and **Gunther Uhlmann**, University of Washington.

Non-Associative Algebras, **Alicia Labra**, Universidad de Chile, and **Kevin McCrimmon**, University of Virginia.

Probability and Mathematical Physics, **Hui-Hsiung Kuo**, Louisiana State University, and **Rolando Rebolledo**, Pontificia Universidad Católica de Chile.

Representation Theory, **Jorge Soto Andrade**, Universidad de Chile, and **Philip Kutzko**, University of Iowa.

Spectral Theory and Mathematical Physics, **Bruno Nachtergaele**, University of California, Davis, and **Rafael Tiedra**, Pontificia Universidad Católica de Chile.

New Orleans, Louisiana

New Orleans Marriott and Sheraton New Orleans Hotel

January 6–9, 2011

Thursday – Sunday

Meeting #1067

Joint Mathematics Meetings, including the 117th Annual Meeting of the AMS, 94th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2010
Program first available on AMS website: November 1, 2010
Program issue of electronic *Notices*: January 2011
Issue of *Abstracts*: Volume 32, Issue 1

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: Expired

AMS-MAA Program Updates

Between the Folds, Sunday, 5:00 p.m.–6:00 p.m. All participants are invited to this documentary film presentation that explores the science, art, creativity, and ingenuity of many of the world's best paper folders. This award-winning film profiles ten artists and theoretical scientists who fuse mathematics and sculpture in the medium of origami. The works—ranging from abstract sculptures to detailed animals—are truly stunning.

AMS Program Updates

The title of the panel discussion at 2:30 p.m. on Saturday sponsored by the Committee on Science Policy is **A Conversation with Sastry Pantula, the New Director of the Division of Mathematical Sciences and the National Science Foundation**, and features an introduction by **Rebecca Goldin**, George Mason University.

The title of the panel discussion at 8:30 a.m. on Sunday sponsored by the Committee on Education is **Teaching Elementary Math Is Not Elementary: How Mathematicians Can Help, and Why**. The panelists are **Johnette Roberts**, Singapore Math Project Coordinator, City of Baker School System; **Kristin Umland**, University of New Mexico; **Hyman Bass**, University of Michigan; **Kenneth I. Gross**, University of Vermont and Director of the Vermont Mathematics Initiative. The panel will be moderated by **Lawrence F. Gray**, University of Minnesota.

As the mathematics learned in the elementary grades forms the foundation for the mathematics taught at the middle and secondary levels and in college, the role of the elementary teacher is of crucial importance in laying the foundation for students' success in later mathematics courses and ultimately for pursuing scientific and technological careers. For this reason, teaching mathematics in the early grades calls for a substantial and robust understanding of mathematics that goes well beyond what the children are expected to learn, and it is becoming widely recognized that mathematicians have a significant part in helping elementary school teachers develop such understanding. The panel represents both mathematicians and elementary school teachers who are actively engaged in this process. During the second half of the session, the audience will be invited to participate in the discussion.

MAA Program Updates

Reunion for Those Interested in Refocusing College Algebra, organized by **Donald B. Small**, U.S. Military Academy, Friday, 5:30 p.m.–7:30 p.m. A national movement to refocus college algebra has developed over the past ten years. Refocus courses emphasize modeling in the problem-solving sense, elementary data analysis, and interpreting graphs. A familiar exercise is to plot given data, fit a curve to the scatterplot, and use the resulting function for predictive purposes. Pedagogically, refocused courses are student-centered and emphasize group work. Developing communication skills and self-confidence are important characteristics of these courses. The reunion will consist of a two-hour discussion/presentation of refocusing college algebra. Several people who are active in the movement will describe their activities. In order to ensure adequate space, persons planning on attending should contact Don Small at `don-small@usma.edu`. Sponsored by MAA CRAFTY.

The **SIGMAA on Mathematicians in Business, Industry and Government** will hold a business meeting on Thursday, 5:30 p.m.–6:30 p.m.

The **SIGMAA on Mathematical and Computational Biology Guest Lecture** at 7:00 p.m. on Friday will be given by **Lisa Fauci**, Tulane University, on *The Biofluiddynamics of Swimming and Pumping: Recent Thoughts*.

The **SIGMAA on the History of Mathematics Guest Lecture** will be given at 6:30 p.m. on Thursday by **Joe Albree**, Auburn University Montgomery, on *Bridges of Trigonometry in the Anglo-American Colonies and the University States*.

Activities of Other Organizations

Women in the Mathematical Sciences: Looking Back, Looking Forward, organized by **Terrell Hodge**, Western Michigan University, and **Maura Mast**, University of Massachusetts Boston; Friday, 1:00 p.m.–2:30 p.m. 2011 marks the 40th anniversary of two pioneering entities created to identify and promote opportunities for women in the mathematical community: the Association for Women in Mathematics and the Joint Committee for Women in the Mathematical Sciences. This panel, including **Mary Gray**, American University; **Jim Lewis**, University of Nebraska-Lincoln; **Jill Pipher**, Brown University; **Jean Taylor**, Rutgers University; and **Marie Vitulli**, University of Oregon; will mark this anniversary by bringing together individuals who are leaders in the mathematics community and who have worked with AWM, JCW, and other organizations to promote equal access and equal opportunity for women and girls in the mathematical sciences. The panelists will offer perspectives on issues faced by women in the mathematical sciences over each of the last four decades; reflect on changes or constants in representation, accomplishments, attitudes, and other trends; and scan the horizons for opportunities and challenges in the upcoming decades. The panel will be moderated by Terrell Hodge. Sponsored by the AMS-ASA-AWM-IMS-MAA-NCTM-SIAM Joint Committee on Women in the Mathematical Sciences.

National Association of Mathematicians

The title of **NAM's Claytor Address** at 1:00 p.m. on Saturday by **Edray Herber Goins** is *Galois Representations and L-Series: A Tour Through Mathematics*.

The title of **NAM's Cox-Talbot Address** given after the banquet on Saturday night by **Robert Bozeman** is *Increasing the Pool of Underrepresented Mathematicians*.

The title of **NAM's Panel Discussion** at 9:00 a.m. on Sunday is *NAM honors the Life of Dr. David Harold Blackwell*.

Social Events

University of Chicago Alumni Reception, Friday, 6:00 p.m.–7:00 p.m.

University of Kansas Math Alumni and Friends Reception, Friday, 5:45 p.m.–7:15 p.m.

Reception for Alumni and Friends of the Department of Mathematics at North Carolina State University, Friday, 6:00 p.m.–8:00 p.m. All alumni, friends, and participants in the Department of Mathematics programs (e.g., REU, REU+, REG, IMSM, RTG) at North Carolina State University are invited to attend and meet old friends and to hear about recent events in the department. H'ors d'oeuvres and drinks will be provided (see `http://www.math.ncsu.edu/` or contact Hien Tran, `tran@math.ncsu.edu`, for more information).

Statesboro, Georgia

Georgia Southern University

March 12–13, 2011

Saturday – Sunday

Meeting #1068

Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: January 2011
Program first available on AMS website: January 27, 2011
Program issue of electronic *Notices*: March 2011
Issue of *Abstracts*: Volume 32, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: November 23, 2010
For abstracts: January 20, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jason A. Behrstock, Lehman College (CUNY), *The quasi-isometric classification of 3-manifold groups*.

Gordana Matic, University of Georgia, *Title to be announced*.

Jeremy T. Tyson, University of Illinois at Urbana-Champaign, *Title to be announced*.

Brett D. Wick, Georgia Institute of Technology, *Title to be announced*.

Special Sessions

Advances in Biomedical Mathematics (Code: SS 4A), **Yangbo Ye**, University of Iowa, and **Jiehua Zhu**, Georgia Southern University.

Advances in Optimization (Code: SS 20A), **Goran Lesaja**, Georgia Southern University.

Algebraic Geometry (Code: SS 18A), **Jing Zhang**, State University of New York at Albany, **Roya Beheshti Zavareh**, Washington University in St. Louis, and **Qi Zhang**, University of Missouri at Columbia.

Algebraic and Geometric Combinatorics (Code: SS 13A), **Drew Armstrong**, University of Miami, and **Benjamin Braun**, University of Kentucky.

Applied Combinatorics (Code: SS 2A), **Hua Wang**, Georgia Southern University, **Miklos Bona**, University of Florida, and **Laszlo Szekely**, University of South Carolina.

Categorical Topology (Code: SS 9A), **Frederic Mynard**, Georgia Southern University, and **Gavin Seal**, EPFL, Lausanne.

Control Systems and Signal Processing (Code: SS 14A), **Zhiqiang Gao**, Cleveland State University, **Frank Goforth**, Georgia Southern University, **Thomas Yang**, Embry-Riddle Aeronautical University, and **Yan Wu**, Georgia Southern University.

Dynamic Equations on Time Scales with Applications (Code: SS 17A), **Billur Kaymakcalan**, Georgia Southern University, and **Bonita Lawrence**, Marshall University.

Fractals and Tilings (Code: SS 3A), **Ka-Sing Lau**, The Chinese University of Hong Kong, **Sze-Man Ngai**, Georgia Southern University, and **Yang Wang**, Michigan State University.

Geometric Group Theory (Code: SS 7A), **Xiangdong Xie**, Georgia Southern University, **Jason A. Behrstock**, Lehman College, CUNY, and **Denis Osin**, Vanderbilt University.

Geometric Mapping Theory in Euclidean and Non-Euclidean Spaces (Code: SS 11A), **Jeremy Tyson**, University of Illinois at Urbana-Champaign, **David A. Herron**, University of Cincinnati, and **Xiagdong Xie**, Georgia Southern University.

Harmonic Analysis and Applications (Code: SS 5A), **Dmitriy Bilyk**, University of South Carolina, **Laura De Carli**, Florida International University, **Alex Stokolos**, Georgia Southern University, and **Brett Wick**, Georgia Institute of Technology.

Harmonic Analysis and Partial Differential Equations (Code: SS 1A), **Paul A. Hagelstein**, Baylor University, **Alexander Stokolos**, Georgia Southern University, **Xiaoyi Zhang**, IAS Princeton and University of Iowa, and **Shijun Zheng**, Georgia Southern University.

Homological Methods in Commutative Algebra (Code: SS 6A), **Alina C. Iacob**, Georgia Southern University, and **Adela N. Vraciu**, University of South Carolina.

Low Dimensional Topology and Contact and Symplectic Geometry (Code: SS 19A), **Gordana Matic**, University of Georgia, and **John Etnyre**, Georgia Institute of Technology.

Matrix Theory and Numerical Linear Algebra (Code: SS 8A), **Richard S. Varga**, Kent State University, and **Xiezhang Li**, Georgia Southern University.

Nonlinear Analysis of PDEs (Code: SS 15A), **Ronghua Pan**, Georgia Institute of Technology, **Tristan Roy**, Institute for Advanced Study, and **Shijun Zheng**, Georgia Southern University.

Set-theoretic Topology (Code: SS 16A), **Frederic Mynard**, Georgia Southern University, and **Peter Nyikos**, University of South Carolina.

Sparse Data Representations and Applications (Code: SS 10A), **Alexander Petukhov** and **Alex Stokolos**, Georgia Southern University, **Ahmed Zayed**, DePaul University, and **Inna Kozlov**, Holon Institute of Technology, Department of Computer Science.

Symplectic and Poisson Geometry (Code: SS 12A), **Yi Lin**, Georgia Southern University, **Alvaro Pelayo**, Washington University, St. Louis, and **Francois Ziegler**, Georgia Southern University.

Iowa City, Iowa

University of Iowa

March 18–20, 2011

Friday – Sunday

Meeting #1069

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: January 2011
Program first available on AMS website: February 5, 2011
Program issue of electronic *Notices*: March 2011
Issue of *Abstracts*: Volume 32, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: November 30, 2010
For abstracts: January 25, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Mihai Ciucu, Indiana University, *Title to be announced.*

David Damanik, Rice University, *Title to be announced.*

Kevin Ford, University of Illinois Urbana-Champaign, *Title to be announced.*

Chiu-Chu Liu, Columbia University, *Title to be announced.*

Special Sessions

Algebraic Combinatorics (Code: SS 19A), **Mihai Ciucu**, Indiana University.

Algebraic K-Theory and Homotopy Theory (Code: SS 8A), **Teena Gerhardt**, Michigan State University, and **Daniel Ramras**, New Mexico State University.

Analytic and Algebraic Number Theory (Code: SS 5A), **Ling Long**, Iowa State University, and **Yangbo Ye**, University of Iowa.

Commutative Ring Theory (Code: SS 6A), **Daniel D. Anderson**, University of Iowa, and **David F. Anderson**, University of Tennessee Knoxville.

Computational Medical Imaging (Code: SS 21A), **Jun Ni** and **Lihe Wang**, University of Iowa.

Geometric Commutative Algebra and Applications (Code: SS 7A), **David Anderson**, University of Washington, and **Julianna Tymoczko**, University of Iowa.

Global and P-adic Representation Theory (Code: SS 3A), **Muthukrishnan Krishnamurthy**, **Philip Kutzco**, and **Yangbo Ye**, University of Iowa.

Graph Theory (Code: SS 17A), **Maria Axenovich**, **Lale Ozkahya**, and **Michael Young**, Iowa State University.

History of Mathematics (Code: SS 13A), **Colin McKinney**, Bradley University.

Modelling, Analysis and Simulation in Contact Mechanics (Code: SS 1A), **Weimin Han**, University of Iowa, and **Mircea Sofonea**, University of Perpignan.

Nonlinear Partial Differential Equations (Code: SS 20A), **Hongjie Dong**, Brown University, and **Dong Li**, **Lihe Wang**, and **Xiaoyi Zhang**, University of Iowa.

Numerical Analysis and Scientific Computing (Code: SS 14A), **Kendall E. Atkinson**, **Bruce P. Ayati**, **Weimin Han**, **Laurent O. Jay**, **Suely Oliveira**, and **David Stewart**, University of Iowa.

Recent Advances in Hyperbolic and Kinetic Problems (Code: SS 15A), **Tong Li**, University of Iowa, and **Hailing Liu**, Iowa State University.

Recent Developments in Nonlinear Evolution Equations (Code: SS 4A), **Yinbin Deng**, Central China Normal University, **Yong Yu** and **Yi Li**, University of Iowa, and **Shuangjie Peng**, Central China Normal University.

Recent Developments in Schubert Calculus (Code: SS 9A), **Leonardo Mihalcea**, Baylor University.

Representations of Algebras (Code: SS 2A), **Frauke Bleher**, University of Iowa, and **Calin Chindris**, University of Missouri.

Spectral Theory (Code: SS 12A), **David Damanik**, Rice University, and **Christian Remling**, University of Oklahoma.

Stochastic Processes with Applications to Mathematical Finance (Code: SS 18A), **Igor Cialenco**, Illinois Institute of Technology, and **José E. Figueroa-López**, Purdue University.

Thin Position (Code: SS 11A), **Jesse Johnson**, Oklahoma State University, and **Maggie Tomova**, University of Iowa.

Topological Problems in Molecular Biology (Code: SS 16A), **Isabel K. Darcy**, University of Iowa, **Stephen D. Levene**, University of Texas at Dallas, and **Jonathan Simon**, University of Iowa.

Universal Algebra and Order (Code: SS 10A), **John Snow**, Concordia University, **Jeremy Alm**, Illinois College, **Clifford Bergman**, Iowa State University, and **Kristi Meyer**, Wisconsin Lutheran College.

Worcester, Massachusetts

College of the Holy Cross

April 9–10, 2011

Saturday – Sunday

Meeting #1070

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: February 2011
Program first available on AMS website: March 10, 2011
Program issue of electronic *Notices*: April 2011
Issue of *Abstracts*: Volume 32, Issue 3

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: December 21, 2010
For abstracts: February 15, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Vitaly Bergelson, Ohio State University, *Title to be announced.*

Kenneth M. Golden, University of Utah, *Title to be announced.*

Walter D. Neumann, Columbia University, *What does a complex surface really look like?*

Natasa Sesum, University of Pennsylvania, *Title to be announced.*

Special Sessions

Celestial Mechanics (Code: SS 16A), **Glen R. Hall**, Boston University, and **Gareth E. Roberts**, College of the Holy Cross.

Combinatorial Representation Theory (Code: SS 14A), **Cristina Ballantine**, College of the Holy Cross, and **Rosa Orellana**, Dartmouth College.

Combinatorics of Coxeter Groups (Code: SS 19A), **Dana C. Ernst**, Plymouth State University, and **Matthew Macauley**, Clemson University.

Complex Analysis and Banach Algebras (Code: SS 1A), **John T. Anderson**, College of the Holy Cross, and **Alexander J. Izzo**, Bowling Green State University.

Computability Theory and Applications (Code: SS 18A), **Brooke Andersen**, Assumption College.

Difference Equations, Dynamics, and Applications (Code: SS 3A), **Michael Radin**, Rochester Polytechnic Institute, and **M. R. S. Kulenovic** and **O. Merino**, University of Rhode Island.

Geometric and Topological Problems in Curvature (Code: SS 17A), **Megan Kerr** and **Stanley Chang**, Wellesley College.

Geometry and Applications of 3-Manifolds (Code: SS 13A), **Abhijit Champanerkar** and **Ilya Kofman**, College of Staten Island, CUNY, and **Walter Neumann**, Barnard College, Columbia University.

Geometry of Nilpotent Lie Groups (Code: SS 11A), **Rachelle DeCoste**, Wheaton College, **Lisa DeMeyer**, Central Michigan University, and **Maura Mast**, University of Massachusetts-Boston.

History and Philosophy of Mathematics (Code: SS 5A), **James J. Tattersall**, Providence College, and **V. Frederick Rickey**, United States Military Academy.

Interactions between Dynamical Systems, Number Theory, and Combinatorics (Code: SS 9A), **Vitaly Bergelson**, The Ohio State University, and **Dmitry Kleinbock**, Brandeis University.

Mathematical and Computational Advances in Interfacial Fluid Dynamics (Code: SS 15A), **Burt S. Tilley**, Worcester Polytechnic Institute, and **Lou Kondic**, New Jersey Institute of Technology.

Mathematics and Climate (Code: SS 8A), **Kenneth M. Golden**, University of Utah, **Catherine Roberts**, College of the Holy Cross, and **MaryLou Zeeman**, Bowdoin College.

Modular Forms, Elliptic Curves, L-functions, and Number Theory (Code: SS 20A), **Sharon Frechette** and **Keith Ouellette**, College of the Holy Cross.

New Trends in College and University Faculty Engagement in K–12 Education (Code: SS 21A), **Jennifer Beineke**, Western New England College, and **Corri Taylor**, Wellesley College.

Number Theory, Arithmetic Topology, and Arithmetic Dynamics (Code: SS 10A), **Michael Bush**, Smith College, and **Farshid Hajir**, University of Massachusetts, Amherst.

Physically Inspired Higher Homotopy Algebra (Code: SS 4A), **Thomas J. Lada**, North Carolina State University, and **Jim Stasheff**, University of North Carolina, Chapel Hill.

Random Processes (Code: SS 7A), **Andrew Ledoan**, Boston College, and **Steven J. Miller** and **Mihai Stoiciu**, Williams College.

The Algebraic Geometry and Topology of Hyperplane Arrangements (Code: SS 6A), **Graham Denham**, University of Western Ontario, and **Alexander I. Suciu**, Northeastern University.

Topics in Partial Differential Equations and Geometric Analysis (Code: SS 12A), **Maria-Cristina Caputo**, University of Arkansas, and **Natasa Sesum**, Rutgers University.

Topological, Geometric, and Quantum Invariants of 3-manifolds (Code: SS 2A), **David Damiano**, College of the Holy Cross, **Scott Taylor**, Colby College, and **Helen Wong**, Carleton College.

Undergraduate Research (Code: SS 22A), **David Damiano**, College of the Holy Cross, **Giuliana Davidoff**, Mount Holyoke College, **Steve Levandosky**, College of the Holy Cross, and **Steven J. Miller**, Williams College.

Las Vegas, Nevada

University of Nevada, Las Vegas

April 30 – May 1, 2011

Saturday – Sunday

Meeting #1071

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: February 2011
Program first available on AMS website: March 17, 2011
Program issue of electronic *Notices*: April 2011
Issue of *Abstracts*: Volume 32, Issue 3

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: January 1, 2011
For abstracts: March 8, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Elizabeth Allman, University of Alaska, *Title to be announced*.

Danny Calegari, California Institute of Technology, *Title to be announced*.

Hector Ceniceros, Stanford University, *Title to be announced*.

Tai-Ping Liu, Stanford University, *Title to be announced*.

Special Sessions

Advances in Modeling, Numerical Analysis and Computations of Fluid Flow Problems (Code: SS 2A), **Monika Neda**, University of Nevada, Las Vegas.

Computational Algebra, Groups and Applications (Code: SS 7A), **Benjamin Fine**, Fairfield University, **Gerhard Rosenberger**, University of Hamburg, Germany, and **Delaram Kahrobaei**, City University of New York.

Extremal Combinatorics (Code: SS 6A), **Jozsef Balogh**, University of California San Diego, and **Ryan Martin**, Iowa State University.

Flow-Structure Interaction (Code: SS 9A), **Paul Atzberger**, University of California Santa Barbara.

Geometric Group Theory and Dynamics (Code: SS 12A), **Matthew Day**, **Danny Calegari**, and **Joel Louwsma**, California Institute of Technology, and **Andy Putnam**, Rice University.

Geometric PDEs (Code: SS 1A), **Matthew Gursky**, Notre Dame University, and **Emmanuel Hebey**, Université de Cergy-Pontoise.

Multilevel Mesh Adaptation and Beyond: Computational Methods for Solving Complex Systems (Code: SS 4A), **Pengtao Sun**, University of Nevada, Las Vegas, and **Long Chen**, University of California Irvine.

Nonlinear PDEs and Variational Methods (Code: SS 11A), **David Costa** and **Hossein Tehrani**, University of Nevada, Las Vegas, and **Zhi-Qiang Wang**, Utah State University.

Partial Differential Equations Modeling Fluids (Code: SS 5A), **Quansen Jiu**, Capital Normal University, Beijing, China, and **Jiahong Wu**, Oklahoma State University.

Recent Advances in Finite Element Methods (Code: SS 3A), **Jichun Li**, University of Nevada, Las Vegas.

Recent Developments in Stochastic Partial Differential Equations (Code: SS 8A), **Igor Cialenco**, Illinois Institute of Technology, and **Nathan Glatt-Holtz**, Indiana University, Bloomington.

Special Session on Computational and Mathematical Finance (Code: SS 13A), **Hongtao Yang**, University of Nevada, Las Vegas.

Topics in Modern Complex Analysis (Code: SS 10A), **Zair Ibragimov**, California State University, Fullerton, **Zafar Ibragimov**, Urgench State University, and **Hrant Hakobyan**, Kansas State University.

Ithaca, New York

Cornell University

September 10–11, 2011

Saturday – Sunday

Meeting #1072

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: June 2011
Program first available on AMS website: July 28, 2011
Program issue of electronic *Notices*: September 2011
Issue of *Abstracts*: Volume 32, Issue 4

Deadlines

For organizers: February 10, 2011
For consideration of contributed papers in Special Sessions: May 24, 2011
For abstracts: July 19, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Mladen Bestvina, University of Utah, *Title to be announced*.

Nigel Higson, Pennsylvania State University, *Title to be announced*.

Gang Tian, Princeton University, *Title to be announced*.

Katrin Wehrheim, Massachusetts Institute of Technology, *Title to be announced*.

Special Sessions

Difference Equations and Applications (Code: SS 1A), **Michael Radin**, Rochester Institute of Technology.

Set Theory (Code: SS 2A), **Paul Larson**, Miami University, Ohio, **Justin Moore**, Cornell University, and **Ernest Schimmerling**, Carnegie Mellon University.

Winston-Salem, North Carolina

Wake Forest University

September 24–25, 2011

Saturday – Sunday

Meeting #1073

Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: June 2011
Program first available on AMS website: August 11, 2011
Program issue of electronic *Notices*: September 2011
Issue of *Abstracts*: Volume 32, Issue 4

Deadlines

For organizers: February 24, 2011
For consideration of contributed papers in Special Sessions: June 7, 2011
For abstracts: August 2, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Benjamin B. Brubaker, Massachusetts Institute of Technology, *Title to be announced*.

Shelly Harvey, Rice University, *Title to be announced*.

Allen Knutson, Cornell University, *Title to be announced*.

Seth M. Sullivant, North Carolina State University, *Title to be announced*.

Lincoln, Nebraska

University of Nebraska-Lincoln

October 14–16, 2011

Friday – Sunday

Meeting #1074

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: August 2011
Program first available on AMS website: September 1, 2011
Program issue of electronic *Notices*: October 2011
Issue of *Abstracts*: Volume 32, Issue 4

Deadlines

For organizers: March 14, 2011
For consideration of contributed papers in Special Sessions: June 28, 2011
For abstracts: August 23, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Lewis Bowen, Texas A&M University, *Title to be announced*.

Emmanuel Candes, Stanford University, *Title to be announced* (Erdős Memorial Lecture).

Alina Cojocaru, University of Illinois at Chicago, *Title to be announced*.

Michael Zieve, University of Michigan, *Title to be announced*.

Special Sessions

Association Schemes and Related Topics (Code: SS 1A), **Sung Y. Song**, Iowa State University, and **Paul Terwilliger**, University of Wisconsin, Madison.

Salt Lake City, Utah

University of Utah

October 22–23, 2011

Saturday – Sunday

Meeting #1075

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2011
Program first available on AMS website: September 8, 2011
Program issue of electronic *Notices*: October 2011
Issue of *Abstracts*: Volume 32, Issue 4

Deadlines

For organizers: March 22, 2011
For consideration of contributed papers in Special Sessions: July 5, 2011
For abstracts: August 30, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Graeme Milton, University of Utah, *Title to be announced*.

Lei Ni, University of California San Diego, *Title to be announced*.

Igor Pak, University of California Los Angeles, *Title to be announced*.

Monica Visan, University of California Los Angeles, *Title to be announced*.

Special Sessions

Special Session on Geometric, Combinatorial, and Computational Group Theory (Code: SS 1A), **Eric Freden**, Southern Utah University, and **Eric Swenson**, Brigham Young University.

Port Elizabeth, Republic of South Africa

Nelson Mandela Metropolitan University

November 29 – December 3, 2011

Tuesday – Saturday

Meeting #1076

First Joint International Meeting between the AMS and the South African Mathematical Society.
Associate secretary: Matthew Miller
Announcement issue of *Notices*: June 2011
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

Invited Addresses

Mark J. Ablowitz, University of Colorado, *Title to be announced*.

Peter Sarnak, Princeton University, *Title to be announced*.

Robin Thomas, Georgia Institute of Technology, *Title to be announced*.

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2012

Wednesday – Saturday
Joint Mathematics Meetings, including the 118th Annual Meeting of the AMS, 95th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2011
Program first available on AMS website: November 1, 2011
Program issue of electronic *Notices*: January 2012
Issue of *Abstracts*: Volume 33, Issue 1

Deadlines

For organizers: April 1, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Honolulu, Hawaii

University of Hawaii

March 3–4, 2012
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: March 2012
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: August 3, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Tampa, Florida

University of South Florida

March 10–11, 2012
Saturday – Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: March 2012
Issue of *Abstracts*: To be announced

Deadlines
For organizers: August 10, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Washington, District of Columbia

George Washington University

March 17–18, 2012
Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: March 2012
Issue of *Abstracts*: To be announced

Deadlines
For organizers: August 17, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Lawrence, Kansas

University of Kansas

March 30 – April 1, 2012
Friday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2012
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

New Orleans, Louisiana

Tulane University

October 13–14, 2012
Saturday – Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2012
Issue of *Abstracts*: To be announced

Deadlines
For organizers: January 13, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 9–12, 2013
Wednesday – Saturday
Joint Mathematics Meetings, including the 119th Annual Meeting of the AMS, 96th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: October 2012
Program first available on AMS website: November 1, 2012
Program issue of electronic *Notices*: January 2012
Issue of *Abstracts*: Volume 34, Issue 1

Deadlines

For organizers: April 1, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Ames, Iowa

Iowa State University

April 27–28, 2013

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2013
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 27, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Special Sessions

Operator Algebras and Topological Dynamics (Code: SS 1A), **Benton L. Duncan**, North Dakota State University, and **Justin R. Peters**, Iowa State University.

Alba Iulia, Romania

June 27–30, 2013

Thursday – Sunday
Associate secretary: Robert J. Daverman
Announcement issue of *Notices*: To be announced
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center, Baltimore Hilton, and Marriott Inner Harbor

January 15–18, 2014

Wednesday – Saturday
Joint Mathematics Meetings, including the 120th Annual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Matthew Miller
Announcement issue of *Notices*: October 2013
Program first available on AMS website: November 1, 2013
Program issue of electronic *Notices*: January 2013
Issue of *Abstracts*: Volume 35, Issue 1

Deadlines

For organizers: April 1, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Antonio, Texas

Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio

January 10–13, 2015

Saturday – Tuesday
Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2015
Issue of *Abstracts*: Volume 36, Issue 1

Deadlines

For organizers: April 1, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention & Trade Center and the Sheraton Seattle Hotel

January 6–9, 2016

Wednesday – Saturday

Joint Mathematics Meetings, including the 122nd Annual Meeting of the AMS, 99th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2015
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2016
Issue of *Abstracts*: Volume 37, Issue 1

Deadlines

For organizers: April 1, 2015
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017

Wednesday – Saturday

Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2017
Issue of *Abstracts*: Volume 38, Issue 1

Deadlines

For organizers: April 1, 2016
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center

January 10–13, 2018

Wednesday – Saturday

Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Matthew Miller
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2017
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

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April issue: 457–592	**September issue: 937–1072**	

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(*Page numbers given are to the main meeting announcements. See `http://www.ams.org/meetings` for full details and updates.)

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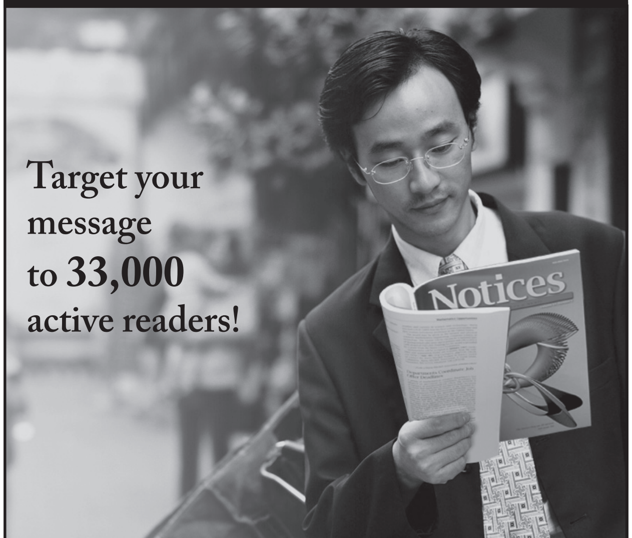

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WHAT IS...

2011 Joint Mathematics Meetings Advance Registration/Housing Form

Name ______________________________
(please write name as you would like it to appear on your badge)

Mailing Address ______________________________

____

Telephone ____________________ Fax: ____________________

In case of emergency (for you) at the meeting, call: Day # ____________ Evening #: ____________

E-mail Address ______________________________

Acknowledgment of this registration and any hotel reservations will be sent to the e-mail address given here, unless you check this box: *Send by U.S. Mail* ☐

Affiliation for badge ____________________ (company/university) Nonmathematician guest badge name: ____________________ (accompanying registered mathematician; please note charge below)

☐ **I DO NOT want my program and badge to be mailed to me on 12/10/10. (Materials will be mailed to the address listed above unless you check this box.)**

Registration Fees

Membership ✓ all that apply. First row is eligible for JMM member registration fee.

❑ AMS ❑ MAA ❑ ASL ❑ CMS ❑ SIAM
❑ ASA ❑ AWM ❑ NAM ❑ YMN

Joint Meetings	**by Dec 15**	**at mtg**	**Subtotal**
❑ Member AMS, MAA, ASL, CMS, SIAM	US$ 224	US$ 294	
❑ Nonmember	US$ 349	US$ 453	
❑ Graduate Student (Member of AMS or MAA)	US$ 49	US$ 59	
❑ Graduate Student (Nonmember)	US$ 76	US$ 86	
❑ Undergraduate Student	US$ 42	US$ 52	
❑ High School Student	US$ 5	US$ 10	
❑ Unemployed	US$ 49	US$ 59	
❑ Temporarily Employed	US$ 181	US$ 210	
❑ Developing Countries Special Rate	US$ 49	US$ 59	
❑ Emeritus Member of AMS or MAA	US$ 49	US$ 59	
❑ High School Teacher	US$ 49	US$ 59	
❑ Librarian	US$ 49	US$ 59	
❑ Press	US$ 0	US$ 0	
❑ Nonmathematician Guest	US$ 15	US$ 15	
			$________

AMS Short Course 1: *Computational Topology (1/4–1/5)*			
❑ Member of AMS or MAA	US$ 100	US$ 134	
❑ Nonmember	US$ 140	US$ 170	
❑ Student, Unemployed, Emeritus	US$ 48	US$ 69	
			$________

AMS Short Course 2: *Evolutionary Game Dynamics (1/4–1/5)*			
❑ Member of AMS or MAA	US$ 100	US$ 134	
❑ Nonmember	US$ 140	US$ 170	
❑ Student, Unemployed, Emeritus	US$ 48	US$ 69	
			$________

MAA Short Course: *What is a Matroid? Theory and Applications from the ground up. (1/4–1/5)*			
❑ Member of MAA or AMS	US$ 150	US$ 160	
❑ Nonmember	US$ 200	US$ 210	
❑ Student, Unemployed, Emeritus	US$ 75	US$ 85	
			$________

MAA Minicourses (see listing in text)

I would like to attend: ❑ One Minicourse ❑ Two Minicourses
Please enroll me in MAA Minicourse(s) #_____ and/or #_____
In order of preference, my alternatives are: #_____ and/or #_____
Price: US$75 for each minicourse.
(For more than 2 minicourses call or email the MMSB.) $________

Graduate Program Fair

❑ Graduate Program Table US$ 60 N/A $________

Employment Center *Please go to http://www.ams.org/emp-reg to register. For further information contact Steve Ferrucci at emp-info@ams.org.*

Events with Tickets

❑ Graduate Student/First Time Attendee Reception (1/6) (no charge)
MER Banquet (1/7) US$ 55 #___Chicken #___Veg #___Kosher
AWM Banquet (1/7) US$ 60 #___Chicken #___Salmon #___Veg #___Kosher
NAM Banquet (1/8) US$ 53 #___Chicken #___Veg #___Kosher
AMS Banquet (1/9) US$ 53 #___Chicken #___Fish #___Veg #___Kosher
(Additional fees may apply for Kosher meals.)
$________

Total for Registrations and Events $________

Payment

Registration & Event Total (total from column on left) $ __________
Hotel Deposit (only if paying by check) $ __________

Total Amount To Be Paid **$** __________
(Note: A US$5 processing fee will be charged for each returned check or invalid credit card. Debit cards are not accepted.)

Method of Payment

❑ **Check**. Make checks payable to the AMS. Checks drawn on foreign banks must be in equivalent foreign currency at current exchange rates. For all check payments, please keep a copy of this form for your records.

❑ **Credit Card**. All major credit cards accepted. For your security, we do not accept credit card numbers by postal mail, e-mail or fax. If the MMSB receives your registration form by fax or postal mail, we will contact you at the phone number provided on this form. For questions, contact the MMSB at mmsb@ams.org.

Signature: ______________________________

❑ **Purchase Order** #____________________ (please enclose copy)

Other Information

Mathematical Reviews field of interest # __________
How did you hear about this meeting? Check one: ❑ Colleague(s) ❑ *Notices* ❑ *Focus* ❑ Internet
❑ This is my first Joint Mathematics Meetings.
❑ I am a mathematics department chair.
❑ For planning purposes for the MAA Two-year College Reception, please check if you are a faculty member at a two-year college.
❑ I would like to receive promotions for future JMM meetings.
❑ Please do not include my name on any promotional mailing lists.
❑ Please ✓ this box if you have a disability requiring special services.

Mailing Address/Contact:

Mathematics Meetings Service Bureau (MMSB)
P. O. Box 6887
Providence, RI 02940-6887 **Fax**: 401-455-4004
Telephone: 401-455-4143 or 1-800-321-4267 x4143; **E-mail:** mmsb@ams.org

Deadlines *Please register by the following dates for:*

To be eligible for the complimentary room drawing:	**Nov. 5, 2010**
For housing reservations, badges/programs mailed:	**Nov. 19, 2010**
For housing changes/cancellations through MMSB:	**Dec. 6, 2010**
For advance registration for the Joint Meetings, short courses, minicourses,and tickets:	**Dec. 15, 2010**
For 50% refund on banquets, cancel by:	**Dec. 27, 2010***
For 50% refund on advance registration, minicourses & short courses, cancel by:	**Dec. 31, 2010***

***no refunds after this date**

Registration for the Joint Meetings is not required for the short courses but it is required for the minicourses & Employment Center.

2011 Joint Mathematics Meetings Hotel Reservations – New Orleans, LA

Please see the hotel page in the announcement or web for detailed information on each hotel. To ensure accurate assignments, please rank hotels in order of preference by writing 1, 2, 3, etc., in the column on the left and by circling the requested room type and rate. If the rate or the hotel requested is no longer available, you will be assigned a room at a ranked or unranked hotel at a comparable rate. Please call the MMSB for details on suite configurations, sizes, availability, etc. Suite reservations can only be made through the MMSB to receive the convention rate. Reservations at the following hotels must be made through the MMSB to receive the convention rates listed. Reservations made directly with the hotels before December 17 will be changed to a higher rate. All rates are subject to a 13% sales/TID tax, plus an additional occupany tax of US$3. **Guarantee requirements: First night deposit by check (add to payment on reverse of form) or a credit card guarantee.**

❏ Deposit enclosed (see front of form)

❏ Hold with my credit card. **For your security, we do not accept credit card numbers by postal mail, e-mail or fax.** If the MMSB receives your registration form by postal mail or fax, we will contact you at the phone number provided on the reverse of this form.

Date and Time of Arrival ____________ **Date and Time of Departure** ____________

Name of Other Room Occupant ____________ **Arrival Date** ________ **Departure Date** ________ **Child (give age(s)** ____

Name of Other Room Occupant ____________ **Arrival Date** ________ **Departure Date** ________ **Child (give age(s)** ____

Order of choice	Hotel	Single	Double 1 bed	Double 2 beds	Triple 2 beds	Triple 2 beds w/cot	Triple - king or queen w/cot	Quad 2 beds	Quad 2 beds w/cot	Suites Starting rates
	New Orleans Marriott (Co-headquarters)									
	Regular Rate	US$ 158	US$ 168	US$ 168	US$ 188	N/A	US$ 188	US$ 208	N/A	US$ 325
	Student Rate	US$ 120	US$ 120	US$ 120	US$ 140	N/A	US$ 140	US$ 160	N/A	N/A
	Sheraton New Orleans (Co-headquarters)									
	Regular Rate	US$ 158	US$ 178	US$ 178	US$ 203	N/A	US$ 228	US$ 228	N/A	US$ 299
	Club Level	US$ 189	US$ 199	US$ 199	US$ 219	N/A	US$ 244	US$ 239	N/A	N/A
	Student Rate	US$ 120	US$ 120	US$ 120	US$ 140	N/A	US$ 165	US$ 160	N/A	N/A
	JW Marriott New Orleans									
	Regular Rate	US$ 148	US$ 158	US$ 158	US$ 178	N/A	US$ 178	US$ 195	N/A	US$ 850
	Astor Crowne Plaza New Orleans									
	Regular Rate	US$ 119	US$ 119	US$ 119	US$ 139	N/A	US$ 159	US$ 159	N/A	US$ 269
	Student Rate	US$ 109	US$ 109	US$ 109	US$ 129	N/A	US$ 149	US$ 149	N/A	N/A

Special Housing Requests:.

❏ I have disabilities as defined by the ADA that require a sleeping room that is accessible to the physically challenged. My needs are: ____________

❏ Other requests: ____________

❏ I am a member of a hotel frequent-travel club and would like to receive appropriate credit. The hotel chain and card number are: ____________

E-mail confirmations (no paper) will be sent by all hotels if an e-mail address is provided.

If you are not making a reservation, please check off one of the following:

❏ I plan to make a reservation at a later date.

❏ I will be making my own reservations at a hotel not listed. Name of hotel: ____________

❏ I live in the area or will be staying privately with family or friends.

❏ I plan to share a room with ____________ , who is making the reservations.

Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Matthew Miller, Department of Mathematics, University of South Carolina, Columbia, SC 29208-0001, e-mail: miller@math.sc.edu; telephone: 803-777-3690.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at** www.ams.org/meetings/.

Meetings:

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 92 in the January 2010 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit http://www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences: (see http://www.ams.org/meetings/ for the most up-to-date information on these conferences.)

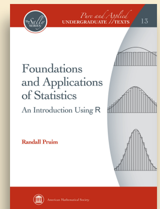
February 17–21, 2011: AAAS Meeting in Washington, DC (Please see www.aaas.org/meetings for more information.)

June 12–July 2, 2011: MRC Research Communities, Snowbird, Utah. (Please see http://www.ams.org/amsmtgs/mrc.html for more information.)

July 4–7, 2011: von Neumann Symposium on Multimodel and Multialgorithm Coupling for Multiscale Problems, Snowbird, Utah. (Please see http://www.ams.org/meetings/amsconf/symposia/symposia-2011 for more information.)

AMERICAN MATHEMATICAL SOCIETY

NEW RELEASES *from the* AMS

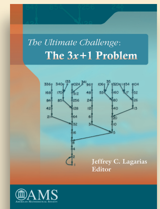


Foundations and Applications of Statistics

An Introduction Using R

Randall Pruim, *Calvin College, Grand Rapids, MI*

Pure and Applied Undergraduate Texts, Volume 13; 2011; approximately 615 pages; Hardcover; ISBN: 978-0-8218-5233-0; List US$85; AMS members US$68; Order code AMSTEXT/13

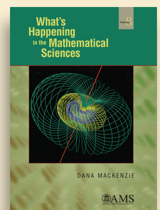


The Ultimate Challenge

The $3x + 1$ Problem

Jeffrey C. Lagarias, *University of Michigan, Ann Arbor, MI*, Editor

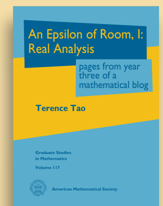
2010; approximately 342 pages; Hardcover; ISBN: 978-0-8218-4940-8; List US$59; AMS members US$47.20; Order code MBK/78



What's Happening in the Mathematical Sciences, Volume 8

Dana Mackenzie

What's Happening in the Mathematical Sciences, Volume 8; 2010; approximately 136 pages; Softcover; ISBN: 978-0-8218-4999-6; List US$23; AMS members US$18.40; Order code HAPPENING/8

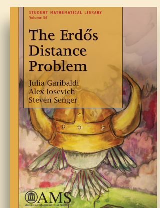


An Epsilon of Room, I: Real Analysis

pages from year three of a mathematical blog

Terence Tao, *University of California, Los Angeles, CA*

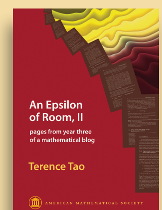
Graduate Studies in Mathematics, Volume 117; 2010; approximately 333 pages; Hardcover; ISBN: 978-0-8218-5278-1; List US$62; AMS members US$49.60; Order code GSM/117



The Erdős Distance Problem

Julia Garibaldi, Alex Iosevich, *University of Rochester, NY*, and Steven Senger, *University of Missouri-Columbia, MO*

Student Mathematical Library, Volume 56; 2010; approximately 161 pages; Softcover; ISBN: 978-0-8218-5281-1; List US$29; AMS members US$23.20; Order code STML/56



An Epsilon of Room, II

pages from year three of a mathematical blog

Terence Tao, *University of California, Los Angeles, CA*

2011; approximately 252 pages; Softcover; ISBN: 978-0-8218-5280-4; List US$42; AMS members US$33.60; Order code MBK/77

AMS BOOKSTORE

For many more publications of interest, visit the AMS Bookstore

www.ams.org/bookstore

TEXTBOOKS FROM THE AMS

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Notices of the American Mathematical Society

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0002-9920(201012)57:11;1-3