Remembering Herbert Federer (1920–2010)

Harold Parks, Coordinating Editor

Herbert Federer

Leslie Vaaler

Herbert Federer taught me about life, scholarship, and the world of mathematics; he was my father. When I was a little girl, my father and I would go on walks and he would talk to me. As I remember this communication, he always respected my ability to understand adult topics, so long as they were presented with careful explanation. He spoke deliberately, taking the time to choose words he felt conveyed just what he was trying to say. (Those who knew Herbert Federer will recognize this precision with language.) On our walks, my father was pleased to be asked questions and encouraged further queries by treating them as intelligent responses. In this manner, he gave me the roots of intellectual self-confidence.

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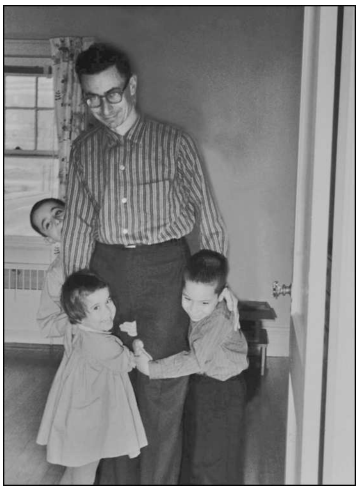
My father wanted me to understand his world, and so he talked to me about teaching, about the Brown University mathematics department, about mathematicians he admired, and about the joys and frustration of being a mathematician. I remember my father telling me about visiting Princeton as a young mathematician and walking with Steenrod on the golf course. He talked to me about working on his thesis the summer before he became a graduate student. He shared with me thoughts about mathematicians being familiar with areas of mathematics other than just their own.

In a 1976 description of his mathematical career to date, he wrote, “I have worked hard to transform this subject from a collection of isolated results into a cohesive body of knowledge. However, my main effort has been directed towards a deeper understanding of concepts significantly related to some classical properties in other parts of mathematics. These interests also led me to write two papers on group theory and homotopy theory.”

Thanks to my father, even before I understood any significant mathematics, I understood that mathematics was an art as well as a science.

My father liked to work at home. I was brought up knowing that it was important not to disturb him, but I also knew that should I knock on his study door, he would always stop and talk with me. One day, as I sat in his study, he explained to me the importance of a mathematician having a “big wastebasket” so that many paths could be tried out, the majority of which would not turn out to be useful. He placed a paper model of a surface in a decorative animal clip I had given him, and when I asked him about it, he talked of minimal surfaces. Prior to the publication of *Geometric Measure Theory* (in 1969 when I was eleven years old), he talked to me about the importance of good mathematical notation, making a bibliography, and proof sheets. Years later, when I wrote a book, these lessons were useful.

My father shared with me his hopes for his book *Geometric Measure Theory*. As stated in the preface, he wished it to serve as a "comprehensive treatise" on the subject for "mature mathematicians" as well as a textbook for very "able students". It was certainly his hope that the book would bring more attention to the subject. My father lamented that certain other areas of mathematics were more fashionable than geometric measure theory and blamed himself for not being a sufficiently good politician.



Herb and his children, Andrew, Wayne, and Leslie.

At some point, after the book was published, most likely when I was an undergraduate student and took a particular interest in algebra, my father took pride in showing me that *Geometric Measure Theory* began with an explanation of exterior algebras. He once wrote that his scientific effort was "directed to the development of geometric measure theory, with its roots and applications in classical geometry and analysis, yet in the functorial spirit of modern topology and algebra."

Professor Federer enjoyed teaching graduate analysis using the second chapter of his book. He was very pleased when he found a hardworking student with talent to understand the material. He always had high standards for himself and for his family, and I am sure he was a demanding teacher.

My father was born on July 23, 1920, in Vienna, Austria. He immigrated to the United States in 1938 and became a naturalized citizen in 1944. He chose never to travel to Europe, and his domestic travel was also quite limited.

Herbert Federer began his undergraduate education at Santa Barbara and then transfered to Berkeley, receiving the degrees B.A. in mathematics and physics in 1942 and Ph.D. in mathematics in 1944. During 1944 and 1945, he served in the U.S. Army at the Ballistic Research Laboratory in Aberdeen. Beginning in 1945, he was a member of the mathematics department at Brown University. He became a full professor in 1951, a Florence Pirce Grant University Professor in 1966, and professor emeritus in 1985. He supervised the Ph.D. theses of ten students.

Herbert Federer and his daughter, Leslie, in 1984.

Herbert Federer joined the American Mathematical Society in 1943. He served on the invitations committee for the 1958 summer institute, as associate secretary during 1967 and 1968, and as Representative on the National Research Council from 1966 to 1969. He delivered an invited address (New York City, 1951) and was the colloquium lecturer at the August 1977 meeting in Seattle. My father and Wendell Fleming received the 1987 Steele Prize for their 1960 paper "Normal and integral currents".

Professor Federer was an Alfred P. Sloan Research Fellow (1957–1960), a National Science Foundation Senior Postdoctoral Fellow (1964–1965), and a John Guggenheim Memorial Fellow (1975–1976). He became a fellow of the American Academy of Arts and Sciences in 1962 and a member of the National Academy of Sciences in 1975.

My father was a private man. Mathematics and his family were Herbert's two loves. I believe he would not want me to share further personal details of his life, but he would be pleased if this memoir attracted mathematicians to learn more about geometric measure theory, the subject he loved so dearly.

Herbert Federer taught me about life, scholarship, and the world of mathematics. He taught me about love and responsibility. He was a wonderful mathematician and father.

John Wermer

Herb Federer was a remarkable man. He was passionately committed to mathematics and had a very personal approach to all issues, including notation.

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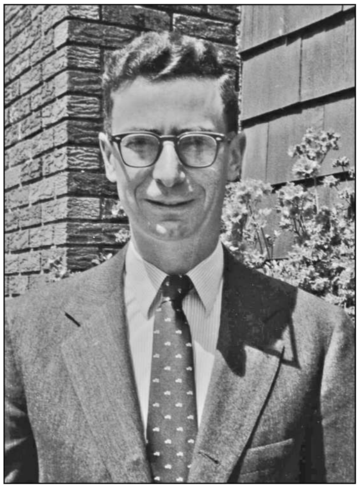

Herbert Federer, 1958.

When I came to Brown University as Herb's junior colleague in 1954, it greatly impressed me that Herb had professional knowledge of and gave graduate courses in and wrote up lecture notes for algebraic topology, differential geometry, algebraic geometry, besides his central fields of interest in real analysis and geometric analysis. When visitors came to speak at Brown, in different areas, they were often eager to consult with Herb on all kinds of mathematical questions.

I remember that when Iz Singer gave a colloquium at Brown on the Atiyah–Singer theorem in its early stages, I understood little of the talk, but Herb understood it very well and realized that something important had happened.

Together with his coworker Wendell Fleming, Herb developed a theory of currents which became a powerful tool in modern geometric analysis, and he wrote his monumental book *Geometric Measure Theory*, which has been very influential.

When I came to Brown in 1954, Herb was very friendly towards me, and I remember him fondly.

William Allard

When I first arrived at Brown University in the fall of 1963, I wasn't sure what to expect. Even though I didn't know much mathematics, I was pretty sure I wanted to study it. I began my graduate career by taking courses in real analysis, taught by Bob Accola; algebra, taught by Than Ward; and complex analysis taught by Herb Federer. These courses, as I thought at the time and as I now realize even more, were taught very well, for which I am now very grateful.

On the first day of class Herb said that only one in five of us would earn a degree; to this day I do not know why he said that—perhaps it was to remind us that graduate study in mathematics was not a cakewalk. He also noticed some ashtrays (yes, ashtrays; this was 1963) in the first or second row of desks and proceeded to deposit them in the wastebasket, making a remark or two as to the low opinion he had of smokers, a group of people which at the time included me.

Herb taught a splendid complex analysis course. He used notes he had developed over the years. Indeed, when Herb decided to learn a subject, he started from the beginning and worked everything out his own way, constantly and laboriously reorganizing as he understood more. He did this, as far as I know, with algebraic topology and algebraic geometry as well as, needless to say, geometric measure theory. It shortly became clear to me how concisely and elegantly he presented the material. More important than this perhaps was his superb organization of the material.

In the class were some talented undergraduates among whom were Blaine Lawson and Joel Pasternack. Herb held office hours every Friday afternoon, which Joel and I nearly always attended. They were wonderful. In spite of the fact that Herb was feared by many students, he was very welcoming to anyone who cared about mathematics and who took the trouble to get to know him. Among other things, I distinctly remember him elaborating on the construction of the universal covering space during one of these office hours; this was something that was fantastic to me at the time. Herb's course lasted for one year. I still have the notes and am contemplating writing them up; I wouldn't be surprised if some of the material, particularly on Riemann surfaces, is not easily accessible in the literature.

Now at this time Herb was fairly well along in the writing of *Geometric Measure Theory*. It turned out that he wanted someone to check it very carefully. He asked me to do this during my second year at Brown. What an opportunity! Of course I agreed and proceeded to read his beautifully handwritten notes for the next three and a half years.

Over the years a number of people, many of whom are excellent mathematicians, have complained about the book being too difficult or not having enough motivation or not being friendly to the reader. This continues to puzzle me. I guess I believe that anyone who wants to learn geometric measure theory will have to suffer in doing so because of the inherent technicality of the subject. But I believe Herb's book affords the diligent and patient reader a path to the goal of learning large parts of the subject which minimizes the pain. I must admit, however, that there were several times when I would suggest that he ought to say a bit more than he did in various proofs. He never accepted my suggestions, saying words like, "But don't you see; I have said all that needs to be said." Oh, well. Then there is the famous Theorem 4.5.9 which has thirty-one parts! I have come to believe that there are many different ways

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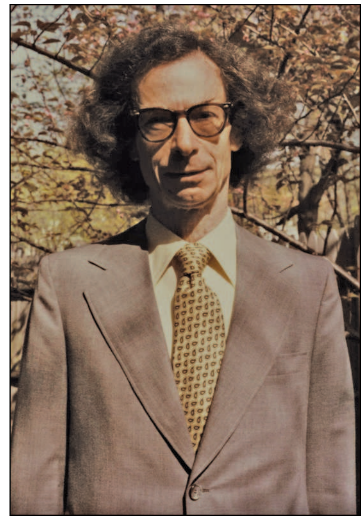
to approach learning and inventing mathematics and that, for some, Herb's way won't work. But I remain convinced that his book is a remarkable and extremely valuable part of the literature. It represents the culmination of many years of work by a talented craftsman absolutely dedicated to his work.

It turns out that, during the last five years or so, after having left the field around twenty years ago to do other things, I have again been working on geometric measure theory, so I have had many occasions to revisit *Geometric Measure Theory* and have been amazed by how clear and efficient the presentation is.

I would now like to elaborate on some of the material in *Geometric Measure Theory*. At this point in time, the most important parts of the book are Chapters Two, Three, and Four, entitled "General measure theory", "Rectifiability", and "Homological integration theory", respectively. In Chapters Two and Three we find a beautiful and efficient development of, among other things, Hausdorff measure and everything one might want to know about the images and level sets of Lipschitz functions on Euclidean space. Chapter Two ends with the statement and proof of the famous Besicovitch–Federer theorem on rectifiability and nonrectifiability. One also finds a treatment of Haar measure as well as the fine structure of real analytic and semianalytic sets. I must also mention the marvelous Morse–Sard–Federer theorem on the regularity of the level sets of highly differentiable functions.

In Chapter Four we find the theory of currents. (I must admit I have always found the title "Homological integration theory" to be a bit pretentious.) Of course currents were introduced by de Rham many years earlier. But de Rham did not treat the *rectifiable* currents; these form natural spaces in which one finds the solution of many variational problems like, most notably, the Plateau problem of minimizing area with a prescribed boundary. Rectifiable and integral currents first appeared in the landmark 1960 paper "Normal and integral currents" by Federer and Fleming. I have reread large parts of this chapter recently and have been delighted by the clarity and efficiency of the presentation. In my opinion, this chapter remains the best reference for this material today.

Chapter One is entitled "Grassmann algebra". Here we find a beautiful treatment of *metric* multilinear algebra including exterior algebra. Again, I don't believe there is a better treatment of this subject.


Herbert Federer, 1979.

Finally, we come to Chapter Five, "Applications to the calculus of variations". Here we find a treatment of Almgren's regularity theory for elliptic variational problems which had been published right before Federer wrote this chapter. We also find a treatment of Simons's work on minimizing cones which appeared in 1968 as well as the De Giorgi–Federer dimension reduction trick for applying regularity theory for the area integrand in dimension n to obtain regularity results in dimension $n + 1$. As Federer himself predicted, the results and techniques in this chapter have been superseded by later work. Thus, perhaps a bit sadly, I have to say that Chapter Five is not where one goes to study regularity theory. This is not the place to give the many relevant references for the state of the art in this area.

The aforementioned Besicovitch–Federer rectifiability theory was used in "Normal and integral currents" as well as in Chapter Four to obtain the fundamental compactness results for integral currents which in turn give existence results in the calculus of variations. Owing to the work of many people, most notably Almgren, this rectifiability theory is no longer necessary to obtain compactness theorems. Indeed, I find this later work more appealing geometrically.

In closing, let me point out that in the last twenty years or so there has been a flowering of work in geometric measure theory not just in the United States but also in Europe. Herb Federer, perhaps as much as anyone, laid the foundations.

Robert Hardt

Since 1967 Herbert Federer was an inspiring scholar and excellent mentor to me. His outstanding works have had a crucial influence on the development of geometric calculus of variations and the study of rectifiable sets and geometric measures. It was my great fortune to have had him as a teacher and Ph.D. advisor at Brown University from 1967 to 1971. These were turbulent years globally and locally with the Cold War, the Vietnam War, the student protests, and the reforms at the universities. Nevertheless, it was also a period of

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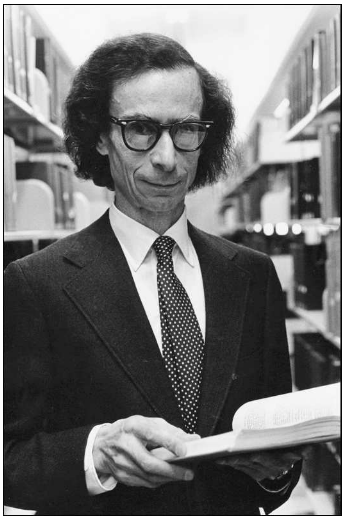


Brown University Archives, John Foraste, photographer.

Herbert Federer, mid-1970s.

great mathematics, and I felt the excitement of the coming-of-age of geometric calculus of variations in the beautiful works of Ennio De Giorgi, E. R. "Peter" Reifenberg, Herbert Federer, and Wendell Fleming. Their papers introduced various natural higher-dimensional generalizations of the classical two-dimensional Plateau problem of finding a surface (e.g., soap film) of least area spanning a given boundary curve. The objects discovered in these papers include various geometric weak limits of manifolds and polyhedra and have proven to have wide applications. The work by Federer and Fleming on normal and integral currents is still wonderful reading, whether in the Steele Prize-winning paper [FF60] or embedded in Federer's fundamental book [Fed69].

As another sample of Federer's insights, I want to call attention to the delightful 1965 paper [Fed65] which I believe has created linkages between Riemannian, complex, and algebraic geometry. His proof of the mass minimality of arbitrary complex subvarieties of Kähler manifolds greatly facilitated the birth of the now widely studied subject of *calibration theory*, in which many different special closed, possibly singular, forms provide variational information on associated geometric objects. In [Fed59], Federer introduced the important co-area formula, which involves fiber integration for changing variables with a Lipschitz map from one manifold to another one of a smaller dimension. In [Fed65], Federer generalized this to rectifiable and normal currents, where densities and orientations are involved. He recognized that the consequent theory of *slicing* could describe and be useful for numerous intersection theory phenomena in algebraic topology and in differential and algebraic geometry.

As is indicated in the paper [Fed65] and was a part of the spirit of all of his publications, one of the characteristics of Federer's work was his love of and his dedication to many kinds of mathematics. Certainly he was expert in all types of analysis, old and new, but few analysts know about his 1946 and 1956 papers that treat free groups and spectral sequences or about his string of outstanding papers that solved numerous open problems in the then-popular theory of Lebesgue area. Whenever Federer became interested in a new subject (e.g., algebraic topology or algebraic geometry), he would go to the library and load up on large stacks of classic and modern books and journals. He'd then spend many weeks reading them, teach a graduate course, and ultimately produce a large collection of (unfortunately unpublished) notes. The various lecture notes that I have seen were extensive, likely to the chagrin of some students. Yet to most students he was an inspiration through his hard work and the reach of his research. He had one principle point of advice for students, and this he indicated by the only sign on his door—a long, vertically stacked series of small stickers that said "Read, Read, Read, ..." In contrast to the narrow reading habits of most mathematicians, Federer once said, "I never read *any* mathematics that I didn't eventually use." I believe that Federer was well influenced by Hassler Whitney in having this breadth of mathematical interests as well as in the direction of his research. See Federer's enthusiastic review [Fed58] of Whitney's book and Federer's later paper [Fed75].

Herbert Federer had a strong sense of scholarship, as is evident in all of his writings. He was extremely careful and not too quick to publish. He once advised that after completing a paper, one should put it in the desk drawer for one month and then bring it out and reread it to find mistakes "as if you were the author's worst enemy." I remember when he pulled the paper [Fed70] out of the drawer. This now well-known paper involved estimating the dimension of the singular set of solutions of the codimension-one oriented Plateau problem. He had written it some time earlier but waited to submit it until he was sure that the singular set could in fact be nonempty. This was established by Bombieri, De Giorgi, and Giusti. In retrospect, this delay in publication was probably not a good idea because the important technique of this paper, now referred to as *Federer induction*, has proven to have wide applicability, not only to other area-minimizing problems but also to energy-minimizing harmonic maps and other systems of elliptic PDE's.

Federer was a real stickler for precision, organization, and referencing. His notation was logical, even if it wasn't always common. All these characteristics are evident in his seminal book *Geometric Measure Theory*. Its appearance in 1969 was timely, as it brought together earlier studies of geometric Hausdorff-type measures, work on rectifiability of sets and measures of general dimension, and the fast developing theory of geometric higher-dimensional calculus of variations. All of the arguments in his text exhibited exceptional completeness. That said, this book is not for the casual reader because his writing tends to be particularly concise. Forty years after the book's publication, the richness of its ideas continue to make it both a profound and indispensable work. Federer once told me that, despite more than a decade of his work, the book was destined to become obsolete in the next twenty years. He was wrong. This book was just like his car, a Plymouth Fury wagon, purchased in the early 1970s that he somehow managed to keep going for almost the rest of his life. Today, the book *Geometric Measure Theory* is still running fine and continues to provide thrilling rides for the youngest generation of geometric measure theorists.

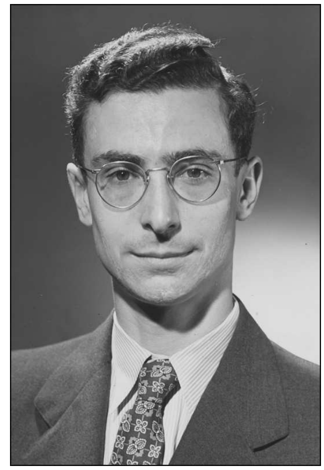
William P. Ziemer

I was both shocked and deeply saddened to learn of the death of Herbert Federer. I was shocked because to me Federer was a giant and giants are supposed to go on forever. I was deeply saddened because one of my primary sources of inspiration was to be no more.

In fact, Federer was considered a giant by many mathematicians because of his profound influence in geometric analysis. Federer, one of the creators of geometric measure theory (GMT), is perhaps best known for his fundamental development of the subject, which culminated in his publication of a treatise in 1969, with the same name, [Fed69]. The book, nearly 700 pages, is written in a manner which commands both admiration and respect because of its virtually flawless presentation of a wide range of mathematical subjects and is written in a style that is unique to Federer. The book, as well as all of his work, was carefully prepared in handwritten notes and includes an extensive bibliography of approximately 230 items. The manuscript is about ten inches thick and is characterized by the degree to which it attains perfection. This is an attribute that is shared with all of his writings. I know of only one small errata sheet.

I first met Federer in 1958 when I entered Brown University as a graduate student. (Coincidentally, this was the same time that Wendell Fleming, my Ph.D. mentor, joined the faculty at Brown.) I was impressed by how friendly and warm he was to my wife, Suzanne, and me. In fact, shortly after our first meeting, he insisted that I call him "Herb", something that I had difficulty in doing for a long time.

Despite the fact that Herb is best known for his work in geometric measure theory, this occupied only the second half of his career. The first half, from 1943–1960, was also a highly productive period with several of his papers laying the groundwork for his subsequent work in GMT. In fact, most of his papers in this period were devoted to area theory, a subject which has been lost to most researchers today. Because of its intrinsic beauty and because several of the fundamental advances in GMT can be traced to his ideas in area theory, I will focus on his achievements in this field. His first published paper, in this period and in his career, was the result of his asking A. P. Morse, who later turned out to be his Ph.D. mentor, for a problem to test whether he was capable of being a research mathematician. The answer became abundantly clear in the joint paper with Morse that Federer had the right stuff [FM43].



Herbert Federer, circa 1948–1949.

Brown University Library.

The problem of what should constitute the area of a surface confounded researchers for many years. In 1914 Carathéodory defined a k-dimensional measure in $\mathbb{R}^n$ in which he proved that the length of a rectifiable curve coincides with its one-dimensional measure. In 1919 Hausdorff, developing Carathéodory's ideas, constructed a continuous scale of measures. After this, it became obvious that area should be regarded as a two-dimensional measure and should establish the well-known integral formulas associated with area. Later Lebesgue's definition, somewhat modified by Frechet, of the area as being the lower limit of areas of approximating polyhedra became the dominant one. It became dominant partly because of its successful application in the solution of the classical Plateau problem. It had the notable feature of lower semicontinuity, which is crucial in the calculus of variations.

Federer's next two papers, [Fed44a], [Fed44b], mark the beginning of his research on Lebesgue area, a field that was dominated by two influential mathematicians, Lamberto Cesari and Tibor Radó.

In [Fed44b], Federer considers the problem that is perhaps the central question in area theory, the answer to which had been sought by many researchers.

(1) It asks for the type of multiplicity function that, when integrated over the range of f with respect to Hausdorff measure, will yield the Lebesgue area of f.

In this paper his results imply that if all the partial derivatives of f exist everywhere in a region T, then the Lebesgue area can be represented as the integral of the crude multiplicity function

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$N(f, T, y)$, which denotes the number of times in T that f takes the value y.

The paper [Fed47] really lays the foundation for the development of GMT. Up to the time of this paper, A. S. Besicovitch had studied the geometric properties of plane sets of finite Carathéodory linear measure and these studies were extended by A. P. Morse and J. F. Randolph. The corresponding problems for two-dimensional measures over three-dimensional space are connected with the theory of surface area. This paper contains a discussion of these properties for a large class of k-dimensional (outer) measures over n-dimensional space and also develops some of the fundamental tools of GMT. For example, he shows that any set $E \subset \mathbb{R}^n$ with finite k-dimensional Hausdorff measure can be decomposed into rectifiable and nonrectifiable parts. Then Federer applies the preceding theory to show that the Hausdorff measure of a two-dimensional nonparametric surface in $\mathbb{R}^3$ equals the Lebesgue area of the map defining the surface.

The problem of finding a suitable multiplicity function such that its integral over the range of f will yield the Lebesgue area of f remained intractable until Federer brought some notions of algebraic topology to bear. In [Fed46], an area is defined for all continuous k-dimensional surfaces in terms of the stable values of their projections into k-dimensional subspaces; the area thus defined is lower semicontinuous. Its relation to Lebesgue area is only partially settled in this paper.

Then, in [Fed48], results were announced which represent generalizations to n dimensions of previous material known only in the two-dimensional case. The topological index, which had been used as a principal tool in the two-dimensional case, is replaced by the topological degree, expressed in terms of Čech cohomology groups, and the use of the Hopf Extension Theorem, which allows the stable multiplicity function to be determined by merely counting the number of essential domains of $f^{-1}(U)$, where U is a domain in $\mathbb{R}^n$. The techniques of algebraic topology are fully applied.

The key to extending the theory of Lebesgue area from two-dimensional surfaces in $\mathbb{R}^3$ to surfaces in $\mathbb{R}^n$ was the generalization of Cesari's inequality from $\mathbb{R}^3$ to $\mathbb{R}^n$ [Ces42]. That inequality states that the Lebesgue area of a mapping $f : X \to \mathbb{R}^3$ is dominated by the sum of the areas of its projection onto the three coordinate planes. Here X denotes a finitely triangulable subset of the plane. In [Fed55], Federer proved the extension of this inequality to $\mathbb{R}^n$, which was a monumental achievement as it necessitated the complete development of the length of light mappings defined on an arbitrary metric space, thus foretelling the directions of modern day GMT. Here, the length of a light mapping $f : X \to Y$, where X is assumed to be a locally compact, separable metric space and Y an arbitrary metric space, is defined as the supremum of $\sum \operatorname{diam}[f(C)]$ where the supremum is taken over all countable disjoint families of nondegenerate continua in X. So, with this result, the theory of Lebesgue area for surfaces in $\mathbb{R}^3$ can be essentially generalized to surfaces in $\mathbb{R}^n$.

The paper [Fed55] is one of Federer's best efforts in area theory. In particular, it contains the basic idea that led to the fundamental result, the Deformation Theorem of GMT [FF60, §5]. It appears in the *Annals of Mathematics* because it was rejected for publication by the *Transactions* despite the fact that it was of the highest quality. This bothered Federer considerably and he contemplated leaving the field. Fortunately, he did not, and thus his best work was yet to come. For example, he and Demers went on to improve the results in [Fed55] by showing that in the case of a flat mapping, a mapping in which both the domain space and range space are of the same dimension, the k-dimensional Lebesgue measure equals the integral of a new multiplicity function which is defined in terms of norms of cohomology classes [DF59].

The paper [Fed59] establishes a very useful result in GMT, known as the co-area formula. In its most elementary form, it states that if f is real-valued, then the total variation of f can be expressed in terms of integration of f over the fibers of f with respect to $(n-1)$-dimensional Hausdorff measure. In its more general form, the formula is valid for any Lipschitz mapping from X to Y where X and Y are separable Riemannian manifolds of class 1 with respective dimensions n and k, $n \geq k$.

This result has generated great interest and has led to many applications and generalizations. For example, [FR60] established a co-area formula for $f \in BV(\mathbb{R}^n)$, while [MSZ03] proved it for a suitable class of Sobolev mappings.

The paper [Fed60] establishes that the Lebesgue area of a nonparametric surface in $\mathbb{R}^n$ is equal to the $(n-1)$-dimensional Hausdorff measure of its graph. This was proved previously in [Fed47] when $n = 3$, and thus this answers the question that Federer pursued in his first publication in area theory [Fed44a].

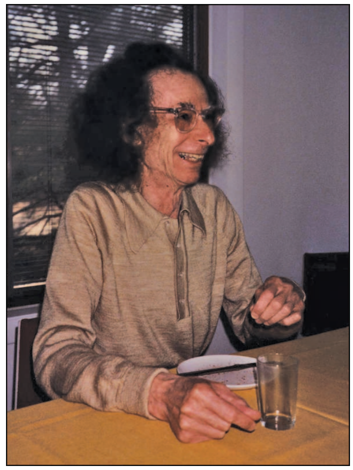
As for the question that was posed in (1), the answer was provided in his last publication on the subject [Fed61]. Let $f : X \to \mathbb{R}^n$ be a continuous mapping where X is a compact manifold of dimension $k \leq n$. Assuming that f has finite Lebesgue area and that either $k = 2$ or that the range of f has $(k+1)$-dimensional Hausdorff measure 0, Federer proves that there exists a unique current-valued measure μ defined over M_f, the middle space associated with f, such that the total variation of μ is equal to the Lebesgue area of f. Moreover, the density of μ, with respect to k-dimensional Hausdorff measure, yields a multiplicity function that provides the answer to the question posed in (1). While Herb was writing this paper, he said that he intended to write it very concisely because he knew that area theory was a dying field and that the paper would not generate much interest. By that time, he was already consumed with the development of GMT.

Even for the casual reader of Herbert Federer's work, it becomes clear that he brings an incredible arsenal of tools to bear on the problem at hand. It is also clear that his determination to learn essentially everything about a problem is highly unusual, for example, taking a period of seventeen years to answer the question raised in (1). He once told me that he uses everything he has learned in his work. This becomes apparent to virtually anyone who has studied his papers. Consider the following quote from G. Bailey Price in his review of Federer's paper [Fed46]: "The paper as a whole is characterized by the treatment of problems and the employment of methods of great generality. The author uses many results from two of his previous papers [Fed44a], [Fed44b]. In addition, he employs a wide variety of powerful tools selected freely from the theory of topological groups, measure theory, integration theory, the theory of functions of real variables, topology and other fields of modern mathematics." As an indication of how he has inspired others to carry on his work, one may note that the number of citations to his book in *Mathematical Reviews* is nearly 1,500, and one should look at the recent work of those who have extended Federer's work to metric spaces; cf. [AK00a], [AK00b], [Mal03], and the references therein.

Herb told me that while he was writing his book, he "was inscribing his epitaph on his tombstone." Indeed, he has and it is our good fortune that he has done it so indelibly because his legacy will be the source of inspiration into the far distant future.

Wendell Fleming

Herbert Federer is remembered for his many deep and original contributions to geometric measure theory (GMT) beginning with his 1945 paper on the Gauss–Green theorem [Fed45]. His work has had a profound influence. It is difficult to imagine that the rapid growth of GMT beginning in the 1960s, as well as its subsequent influence on other areas of mathematics and applications, could have happened without Federer's groundbreaking efforts. His book [Fed69] is a classic reference. He gave the Colloquium Lectures at the 1977 Summer AMS Meeting in Seattle. The manuscript for those lectures appears as [Fed78] and provides a summary of results in GMT through the late 1970s, including historical background. Nonspecialists may find [Fed78] a useful complement to the more detailed development in [Fed69].



Herbert Federer, 1990.

I first met Herb Federer at the 1957 Summer AMS Meeting at Penn State. Afterwards, he suggested to the mathematics department at Brown that I might be offered an assistant professorship. An offer was made, which I accepted. Upon our arrival in Providence in the fall of 1958, my wife and I were warmly welcomed by Herb and Leila Federer. The academic year 1958–1959 was the most satisfying time of my mathematical life. Our joint work on normal and integral currents was done then. This involved many blackboard sessions at Brown, as well as evening phone calls at home. Both Herb and I had heavy teaching loads (by present day standards) and families with young children. Herb undertook the task of organizing our results into a systematic, coherent form, which appeared as [FF60].

During the 1960s there was a lot of activity in GMT at Brown. We had strong Ph.D. students and several visitors. Among the visitors were Peter Reifenberg, who was at Brown in the summer of 1963, and Ennio De Giorgi who visited during the spring semester of 1964. Reifenberg had found another highly original approach to the higher-dimensional Plateau problem [Rei60]. Unfortunately, his promising career ended when he died in a mountaineering accident in 1964.

Herbert Federer set very high standards for his mathematical work and expected high quality work from his students. He was fair-minded and very careful to give proper credit to the work of other people. He was generous with his time when serious mathematical issues were at stake. Federer was the referee for John Nash's 1956 *Annals of*

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Herb and his wife, Leila, in 1979.

Mathematics paper "The imbedding problem for Riemannian manifolds". This involved a collaborative effort between the author and referee over a period of several months. In the final accepted version, Nash stated, "I am profoundly indebted to H. Federer, to whom may be traced most of the improvements over the first chaotic formulation of this work."

By the 1970s I had left GMT to work on stochastic control. When Herb and I met in later years, we didn't discuss mathematics very much, but we always exchanged updates about our children.

In the 1950s Lebesgue area theory had reached a mature state. It had succeeded in providing existence theorems for two-dimensional geometric problems of the calculus of variations, including the Plateau problem. In the area theory formulation, the minimum is achieved among surfaces of a prescribed topological type which have as boundary a given curve. A very different formulation would be needed to study calculus of variations problems with $k \geq 2$, in which only the $(k-1)$-boundary but not the topological type of the k-dimensional comparison surfaces is given. One such formulation is in terms of L. C. Young's generalized surfaces [You51]. Young was my Ph.D. advisor. I came to Brown expecting to continue working in a generalized surface setting. However, Federer soon convinced me of the advantages of developing instead a theory expressed in terms of de Rham's theory of currents. His wisdom and foresight in this regard have been amply justified by developments in GMT which followed our joint paper [FF60].

The k-dimensional Plateau problem in n-dimensional Euclidean $\mathbb{R}^n$ is to minimize k-dimensional area in a suitably defined class of objects with given $(k-1)$-dimensional boundary. The objects which Federer and I considered are called integral currents. Our paper provided a theorem about the existence of k-area minimizing integral currents. There remained the notoriously difficult "regularity question", which is to prove smoothness of the support of an integral current which minimizes k-area, except at points of a singular set of lower Hausdorff dimension. Examples show that in dimensions $1 < k < n-1$, the singular set can have Hausdorff dimension $k-2$. The earliest partial regularity results were due to De Giorgi [DG61] and Reifenberg [Rei64]. Federer's Ph.D. student Fred Almgren and coauthors later made remarkable further progress on the regularity problem for a larger class of elliptic variational integrands [Whi98], [Tay99]. This required persistent, courageous efforts. References [Fed69, Chapter 5] and [Fed78, Section 10] also give systematic accounts of results for the regularity problem up to 1977.

In codimension one ($k = n-1$) it seemed at first that area minimizing currents might have no singular points. This turned out to be correct for $n \leq 7$ by results of De Giorgi, Almgren, and Simons. However, Bombieri, De Giorgi, and Giusti [BDGG69] gave an example of a cone in $\mathbb{R}^8$ which provides a seven-dimensional area-minimizing integral current with a singularity at the vertex. In [Fed70], Federer showed that, for codimension one, this example is generic in the sense that the singular set can have Hausdorff dimension at most $n-8$.

In the integral current formulation, orientations are assigned to tangent k-spaces. These orientations vary continuously on the regular part of the support of any k-area minimizing integral current. Another formulation of the Plateau problem is in terms of Whitney-type flat chains with coefficients in the group Z_2 of integers mod 2. This formulation, in effect, ignores orientations. Federer showed in [Fed70] that for this "nonoriented" version of the Plateau problem, the singular set has Hausdorff dimension at most $k-2$ for arbitrary k. This is essentially the best possible result for the nonoriented version.

Federer also made notable contributions to the theory of weakly differentiable functions on $\mathbb{R}^n$ with applications to Fourier analysis [Fed68], [Fed69, Section 4.5], [Fed78, Section 5]. These include sharp results which extend to $n > 1$ the fact that a function of one variable of bounded variation has everywhere finite left and right limits which differ only on a countable set. Federer's results are included in the lengthy Theorem 4.5.9 in [Fed69]. The statement of that theorem provides a comprehensive list of properties of functions on $\mathbb{R}^n$ with first-order partial derivatives which are measures (in the Schwartz distribution sense). Any such function with compact support corresponds to a normal current of dimension $k = n$.

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