A Walk through Johnny von Neumann’s Garden
page 154

Israel Moiseevich Gelfand, Part II
page 162

How to Calculate Proofs: Bridging the Cultural Divide
page 173

On the Measurement of Intangibles. A Principal Eigenvector Approach to Relative Measurement Derived from Paired Comparisons
page 192

Ames Meeting
page 277

Displaced mathematicians (see page 271)
Open Access Journals in Mathematics

Your research wants to be free!
THE FEATURE COLUMN

*monthly essays on mathematical topics*

www.ams.org/featurecolumn

Each month, the Feature Column provides an online in-depth look at a mathematical topic. Complete with graphics, links, and references, the columns cover a wide spectrum of mathematics and its applications, often including historical figures and their contributions. The authors—David Austin, Bill Casselman, Joe Malkevitch, and Tony Phillips—share their excitement about developments in mathematics.

Recent essays include:

- William Thurston
- Mathematical Modeling
- It’s a Small World After All
- What Does a Circle Look Like?
- Old Babylonian Multiplication and Reciprocal Tables
- More Precious than Gold?
- A (Very Short) Detour for the Traveling Salesman
- Archimedes on the Circumference and Area of a Circle
- Weird Rulers
- Arrangements and Duality or How I Learned to Slice a Ham and Cheese Sandwich
- Fragments of Greek Mathematics

**AMS members:** Sign up for the AMS members-only *Headlines & Deadlines* service at [www.ams.org/enews](http://www.ams.org/enews) to receive email notifications when each new column is posted.
Solve the differential equation.

\[ t \ln t \frac{dr}{dt} + r = 7te^t \]

\[ r = \frac{7e^t + C}{\ln t} \]

**WHO HAS THE #1 HOMEWORK SYSTEM FOR CALCULUS?**

**THE ANSWER IS IN THE QUESTIONS.**

When it comes to online calculus, you need a solution that can grade the toughest open-ended questions. And for that there is one answer: WebAssign.

WebAssign’s patent pending grading engine can recognize multiple correct answers to the same complex question. Competitive systems, on the other hand, are forced to use multiple choice answers because, well they have no choice. And speaking of choice, only WebAssign supports every major textbook from every major publisher. With new interactive tutorials and videos offered to every student, it’s not hard to see why WebAssign is the perfect answer to your online homework needs.

It’s all part of the WebAssign commitment to excellence in education. Learn all about it now at [webassign.net/math].
This month’s issue contains a number of special contributions. Freeman Dyson offers a retrospective of the work of John von Neumann. Thomas Saaty discusses how to measure intangibles. Raymond Boute treats the role of proofs in math education. And we have the second installment of our memorial for I. M. Gelfand. There are two articles about giving effective math talks—representing two different points of view. Finally, there is an interview with Abel Prize Laureate Endre Szemerédi and also an interview with outgoing AMS President Eric Friedlander.

—Steven G. Krantz, Editor

**Features**

**154** A Walk through Johnny von Neumann’s Garden

*Freeman Dyson*

**162** Israel Moiseevich Gelfand, Part II

*Vladimir Retakh, Coordinating Editor*

**173** How to Calculate Proofs: Bridging the Cultural Divide

*Raymond T. Boute*

**192** On the Measurement of Intangibles.

A Principal Eigenvector Approach to Relative Measurement Derived from Paired Comparisons

*Thomas L. Saaty*
DEPARTMENTS

About the Cover ......................................................... 271
Mathematics People .................................................... 245

Mathematics Opportunities ......................................... 247
Summer Program for Women Undergraduates, News from the CRM.

For Your Information .................................................. 248
April 2013 Mathematics Awareness Month: Mathematics of Sustainability, EMS Code of Ethical Practice, Correction.

Inside the AMS ............................................................ 249
Math in Moscow Scholarships Awarded, From the AMS Public Awareness Office, AMS Email Support for Frequently Asked Questions, Deaths of AMS Members.

Reference and Book List ............................................... 251
Mathematics Calendar .................................................. 260
New Publications Offered by the AMS ............................... 266
Classified Advertisements .............................................. 272
General Information Regarding Meetings & Conferences of the AMS .................................................. 274
Meetings and Conferences of the AMS ............................... 275
Meetings and Conferences Table of Contents ................. 288

From the AMS Secretary

2012 Election Results .................................................... 256
Nominations by Petition for 2013 AMS Election ................ 257
Call for Nominations for the 2014 Leroy P. Steele Prizes .... 259
Grants to support collaborations between Chinese and U.S./Canadian researchers are made possible through the generosity of Ky and Yu-Fen Fan.

The Fan China Exchange Program is intended to send eminent mathematicians from the U.S. and Canada to make a positive impact on the mathematical research community in China and to bring Chinese scientists in the early stages of their research to the U.S. and Canada to help further their careers. The program encourages host institutions to provide some type of additional support for the travel or living expenses of the visitor and to ensure a suitable length of stay.

Applications received before March 15 will be considered for the following academic year.

For more information on the Fan China Exchange Program and application process see [www.ams.org/employment/chinaexchange.html](http://www.ams.org/employment/chinaexchange.html) or contact the AMS Membership and Programs Department by telephone at 800-321-4267, ext. 4170 (U.S. and Canada), or 401-455-4170 (worldwide), or by email at chinaexchange@ams.org.
AMS Fellows

A hot topic in the air right now is the AMS Fellows program. With the approval of the membership, the AMS has designated about one thousand members of the Society to be Fellows—that is about one mathematician in thirty. A Fellow of the Society is supposed to be a distinguished mathematician who is being recognized for his/her scholarly excellence. (I note that the National Academy elects about one mathematician in three hundred, so it is even more rarified.)

Even though I am now a Fellow, I must confess that I voted against the program just because I could not understand its purpose. As editor of the Notices, I have received several letters protesting the Fellows program (and announcing the writer’s intention to decline the fellowship and resign his/her AMS membership). These letters have provided me with food for thought.

The typical argument of those vilifying the Fellows program is that mathematics is (thankfully) egalitarian and the Fellows program flies in the face of that level playing field. So it is doing no good. I think that this reasoning is naive. Certainly the subject of mathematics is egalitarian—both the loftiest professor at the Institute for Advanced Study and the lowliest lecturer at the University of Northern South Dakota can spend as much time as they like studying, writing about, and publishing about the Axiom of Choice, and nobody will tell them to do otherwise. But the mathematics profession is certainly not egalitarian. There are those in mathematics who have clout and who use it and there are those who do not. There are those with a surfeit of privileges and those without. There are those who are our leaders and who are highly placed. And then there are the rest of us. Such is human nature.

The other scientific societies have had Fellows programs for quite some time, and they find them to be quite useful. Colleagues have told me that at their institutions the statistics department frequently reminds the dean that 20 percent of their faculty are Fellows of the American Statistical Association, thus beating the national average and showing how distinguished they are as a group. When visitors or parents tour the university, they are always excited to see the supercomputing lab and the genetic engineering building, but certainly not the rather drab math building and the seedy-looking math faculty. It is useful, from an administrative point of view, to be able to say that 15 percent of our math faculty are AMS Fellows. The existence of Fellows in any given mathematics department shows that that group consists of serious scholars.

When the National Academy of Sciences was formed, its avowed purpose was to advise the president on matters of science. But eventually the joke became that the main purpose of the National Academy was to debate who the new members should be. There is some danger that the Fellows will serve a similar purpose, with little pretense to advising the AMS president. Since the Fellows program is new, I can only hope that we exercise some wisdom to make it a program with legs and with some intrinsic value.

I note that the number of Fellows at some distinguished schools is surprisingly low. Princeton has just 19, and Yale has just 4. This is apparently because members of these august institutions do not belong to the AMS, hence do not qualify to be Fellows. I think that it is a shame, and reflects poorly on all of us, that our leaders do not belong to our professional society.

I am not going to resign my membership in the AMS. As a Fellow, I am not now able to prove better theorems. I will not do a better job directing my Ph.D. students. I will not get a better parking place. I will probably not be more admired by the secretaries and staff. But I will contribute to the public image of my department and perhaps help it to promote itself. I might also hope to reinforce everyone’s sense of self-worth. We mathematicians have not paid much attention to that sort of thing over the years, and we have suffered for it. It is time for us to learn to help ourselves.

—Steven Krantz
Washington University in St. Louis
Editor, AMS Notices
notices@math.wustl.edu

DOI: http://dx.doi.org/10.1090/noti950
New Prize “EMS Monograph Award” by the EMS Publishing House

On the occasion of our tenth anniversary, we are happy to announce a new prize, open to all mathematicians. The EMS Monograph Award is assigned every two years to the author(s) of a monograph in any area of mathematics that is judged by the selection committee to be an outstanding contribution to its field. The prize is endowed with 10,000 Euro and the winning monograph will be published by the EMS Publishing House in the series “EMS Tracts in Mathematics”.

Submission
The monograph must be original and unpublished, written in English and should not be submitted elsewhere until an editorial decision is rendered on the submission. The first award will be announced in 2014 (probably in the June Newsletter of the EMS); the deadline for submissions is 30 June 2013. Monographs should preferably be typeset in TeX. Authors should send a pdf file of the manuscript by email and a hard copy together with a letter to:

European Mathematical Society Publishing House
ETH-Zentrum SEW A27, Scheuchzerstrasse 70, CH-8092 Zürich, Switzerland
E-mail: info@ems-ph.org

Scientific Committee
John Coates, Pierre Degond, Carlos Kenig, Jaroslav Nesetril, Michael Roeckner, Vladimir Turaev

EMS Tracts in Mathematics

Editorial Board:
Carlos E. Kenig (University of Chicago, USA)
Andrew Ranicki (University of Edinburgh, UK)
Michael Röckner (Universität Bielefeld, Germany, and Purdue University, USA)
Vladimir Turaev (Indiana University, Bloomington, USA)
Alexander Varchenko (University of North Carolina at Chapel Hill, USA)

This series includes advanced texts and monographs covering all fields in pure and applied mathematics. Tracts will give a reliable introduction and reference to special fields of current research. The books in the series will in most cases be authored monographs, although edited volumes may be published if appropriate. They are addressed to graduate students seeking access to research topics as well as to the experts in the field working at the frontier of research.

Most recent titles:
Vol. 17 Anders Björn and Jana Björn: Nonlinear Potential Theory on Metric Spaces
Vol. 16 Marek Jarnicki and Peter Pflug: Separately Analytic Functions
Vol. 15 Ronald Brown, Philip J. Higgins and Rafael Sivera: Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids
Vol. 13 Laurent Bessières, Gérard Besson, Michel Boileau, Sylvain Maillot and Joan Porti: Geometrisation of 3-Manifolds
Vol. 11 Hans Triebel: Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration
Vol. 10 Vladimir Turaev: Homotopy Quantum Field Theory. With Appendices by Michael Müger and Alexis Virelizier
Letters to the Editor

A Proposal Concerning Fellows
Mathematics is egalitarian, open to all with the requisite talent and training, although access to that training may be hampered by accidents of birth and economic circumstance, impediments that society must work to overcome. On the other hand, we mathematicians crown ourselves with a diversity of prizes, however nonegalitarian that may be, or capriciously (as some claim) they may be awarded. About that there is generally no complaint. Now, however, the disquiet caused by the creation of a class of Fellows of the Society is causing some offered this honorific to decline it, and others even to find in it cause for renouncing membership in our Society. The distinction is that omission from an ultra-select class does not diminish our feeling of self-worth but that those long serving the mathematical community may feel their service derogated by exclusion from a much broader group. I suggest that, as with other professional and scientific societies, we honor as Fellows those who have reached some combination of years of membership and age. Those who for years have faithfully toiled in the vineyards should also receive their due reward.

—Murray Gerstenhaber
University of Pennsylvania
mgersten@math.upenn.edu

(Received October 31, 2012)

On Scissor Congruence
The article “WHAT IS... a Scissors Congruence” in the October 2012 Notices reviewed work arising from Hilbert’s problem on the decomposition of two convex polyhedra into pairwise congruent subpolyhedra. I wish to call attention to work on more general decompositions of convex bodies.

The term “scissor congruent” was introduced by Dubins et al. [1] to describe a pair of convex planar bodies A, B having cell decompositions such that there is a bijection between the two-cells of A and those of B with corresponding two-cells congruent through rigid motions.

The main result in [1] is that A and B are scissor congruent iff they have the same area and their boundaries have partitions \( \partial A = \bigcup_i I_i \cup \bigcup_i J_i \) and \( \partial B = \bigcup_i I_i \cup \bigcup_i J_i \) with \( I_i \) and \( J_i \) congruent arcs.

It follows that a square is not scissor congruent to any strictly convex body, and if two ellipses are scissor congruent, then they are congruent.


References

—Morris W. Hirsch
University of Wisconsin
University of California
mwhirsch@chorus.net

(Received November 20, 2012)

Ethics in Mathematics
The leading principle of the professional ethics in mathematics is proving and disproving of claims. Negative claims must be proven as well as positive ones.

Rejection of papers by referees must be also proven. Otherwise, authors can be unprotected against subjective decisions, which can be serious obstacles for the progress of science.

Editors can reject papers if and only if those papers do not preserve the packages of rules for authors and/or they are out of the scope of the respective journals.

There have to be reasonable periods of time for referees’ refereeing, and if the referees violate those periods they have to explain why to editors.

There should not be any a priori advantage of editors and referees over authors; unfortunately, editors as a rule are on the side of referees and reject papers by recommendation of the referees without authors’ responses (and even without a proof that the authors have serious mistakes).

—Viktor Ivanov
Independent Scholar
ivanvvn1@hotmail.com

(Received October 31, 2012)

Refereeing Nirvana
Brian Osserman’s refereeing proposal (Notices, November 2012) reminds me of the running joke about “grunge frosh” during my undergraduate days at Caltech. An older student would be frustrated by a long and tedious calculation, so a typical facetious comment would be to farm out such annoying calculations to an unsuspecting frosh because we older students couldn’t be bothered to do them. Except I don’t think that Brian is joking....

—Mason A. Porter
University of Oxford
porterm@maths.ox.ac.uk

(Received November 25, 2012)
The deadline for nominations for the first annual election process is:

**April 1, 2013**

Learn how to make or support a nomination in the *Requirements and Nominations Guide* at:


Questions:

Contact AMS staff at 800-321-4267, ext. 4113 or by email at amsfellows@ams.org
A Walk through Johnny von Neumann’s Garden

Freeman Dyson

Foundations of Mathematics
Johnny von Neumann left behind him six massive volumes of collected works, assembled and edited by Abraham Taub [1]. The collected works are his garden, containing a large and heterogeneous set of objects that he planted. Each of them grew from a seed, from an idea or a problem that came into his head. He developed the idea or solved the problem and then wrote it down and published it. He wrote fast and published fast, so that the flowers are still fresh. For my talk this morning I decided to take a walk through the garden and see what I could find. Luckily only two of the papers are in Hungarian. He wrote mostly in German until he came to live permanently in the United States at the age of thirty, and after that in English.

Johnny was educated at the famous Lutheran High School in Budapest from age ten to age eighteen. There he had excellent teachers and even more excellent schoolmates. One of the schoolmates was Eugene Wigner, who became an outstanding physicist and a lifelong friend. But Johnny’s father understood that the Lutheran High School was not giving Johnny everything he needed. Johnny had a passion for mathematics going far beyond what the school could teach. So his father hired Michael Fekete, a mathematician from the University of Budapest, to work with Johnny at home. The first flower in Johnny’s garden is a paper, “On the position of zeroes of certain minimum polynomials” [2], published jointly by Fekete and von Neumann when Johnny was eighteen. The style of the paper is dry and professional, following the tradition set by Euclid two thousand years earlier. Almost everything that Johnny wrote as a mathematician is in the Euclidean style, stating and proving theorems one after another with no wasted words.

Although the subject of his first paper was probably suggested by Fekete, the style is already recognizable as Johnny’s. Johnny’s unique gift as a mathematician was to transform problems in all areas of mathematics into problems of logic. He was able to see intuitively the logical essence of problems and then to use the simple rules of logic to solve the problems. His first paper is a fine example of his style of thinking. A theorem which appears to belong to geometry, restricting the possible positions of points where some function of a complex variable is equal to zero, is transformed into a statement of pure logic. All the geometrical complications disappear and the proof of the theorem becomes short and easy. In the whole paper there are no calculations, only verbal definitions and logical deductions.

The next flower in the garden is Johnny’s first solo paper, “On the introduction of transfinite numbers” [3], which he published at age nineteen. This shows where his strongest interests lay at the beginning of his career when he was a young bird ready to leave the nest and stretch out his mathematical wings. His dominating passion then and for the next five years was to understand and reconstruct the logical foundations of mathematics. He was lucky to arrive on the scene at the historical moment when confusion about the foundations of mathematics was at a maximum. In the nineteenth century, Georg Cantor had greatly enlarged the scope of mathematics by creating a marvelous theory of transfinite numbers, giving precise definitions to a vast hierarchy of infinities. Then, at the beginning of the twentieth century,
Bertrand Russell and other critics discovered that Cantor's theory led to logical contradictions. Russell's paradox threw doubt not only onto Cantor's creation of a new world of infinities but also onto the established concepts of classical mathematics. Johnny became aware as soon as he began to talk with Fekete and to read the mathematical literature that mathematics was in a state of crisis. Since Cantor's mathematical reasoning had led to logical absurdities, nobody knew how to draw the line between reliable mathematics and imaginative nonsense. Johnny decided at the age of nineteen that it was his task to resolve the crisis and to put mathematics back onto a firm logical foundation.

The first paragraph of Johnny's first solo paper consists of a single sentence: “The purpose of this work is to make the idea of Cantor's ordinal numbers unambiguous and concrete.” The rest of the paper provides a new definition of ordinal numbers and demonstrates that the new definition leads to the same results as Cantor's old definition. Johnny makes no claim to have resolved the crisis that arose from Cantor's theory. He has only made the crisis more acute by giving Cantor's concepts a sharper definition. To make the crisis more acute means to understand it better, and to understand it better is the first step toward resolving it.

Johnny's second solo paper, "An axiomatization of set theory" [4], appeared two years later when he was twenty-one years old and a student at the University of Berlin. Set theory means the theory of things and collections of things, considering only their logical relationships and forgetting about their individual qualities. From the point of view of set theory, you and I and stars and planets and words and numbers are all just things and are all treated the same way. Axiomatization means to describe set theory in the same style that Euclid used to describe geometry two thousand years ago, building the theory by logical deduction from a few basic assumptions which he called axioms. Johnny found a new set of axioms for set theory. He hoped that his new axioms could serve as a consistent logical basis for all the useful parts of mathematics while avoiding the paradoxes. But he was well aware that his consistent basis for mathematics was a hope and not a proven reality.

The essential novelty of Johnny's axioms was to introduce two species of objects, which he called "one-things" and "two-things". He used these abstract names in order to avoid possibly misleading impressions that might arise from using more familiar words. To make Johnny's ideas easier to understand, I will use the names "sets" for one-things and "classes" for two-things. So Johnny had a version of set theory with two kinds of objects: the sets, which are in some sense small enough to be handled collectively by the normal rules, and the classes, which are in some sense too big to be handled collectively. The axioms are constructed so that the "class of all sets" exists as a well-defined object. It is a class but not a set. Neither the "set of all sets" nor the "class of all classes" exists in the theory. This simple trick, using different names and different rules for small and large collections, allows Johnny to avoid the logical paradoxes. The paradoxes arose in the older versions of set theory from using the concept "set of all sets" too freely. In Johnny's new version, this concept is forbidden, but the "class of all sets" is allowed, providing the framework for a logical construction of mathematics. The class of all sets is the universe of mathematics, the framework within which all mathematical collectives are defined.

Before writing his paper, Johnny had been talking with David Hilbert in Göttingen. Hilbert was forty years older than Johnny and was the most famous mathematician in the world. Hilbert was passionately promoting a program for resolving the crisis of mathematics by solving the Entscheidungsproblem, the decision problem. To solve the decision problem meant to find a formal method of deciding the truth or falsehood of every mathematical statement. If he could solve the decision problem, that would show that the axioms of mathematics were both consistent and categorical. To be consistent means that they can never prove both a statement and its negation. To be categorical means that for every statement the axioms prove either the statement or its negation. Hilbert proclaimed, with all his authority as spiritual father of mathematicians, that to resolve the crisis of mathematics it was necessary to find a set of axioms that were proven to be both consistent and categorical. Mathematics would only rest on a firm logical foundation if every meaningful mathematical statement could be proved true or false.

At the end of his axiomatization paper, Johnny puts a brief and modest summary of his claims. He does not claim to have resolved the crisis of mathematics. He claims only to have opened the way to a possible resolution by finding a set of axioms that is not known to be self-contradictory. He has not proved that his axioms are consistent, and he has not proved that they are categorical. He ends his paper with two sentences expressing not very diplomatically his skepticism about Hilbert's program: "Even Hilbert's approaches are here powerless, for this objection concerns the categoricity and not the consistency of set-theory. All that we can do now is to recognize that another argument against set-theory has arisen, and that we see no way ahead leading to rehabilitation."

Three years later Johnny published two much longer papers about the foundations of mathematics. One was "On Hilbert's proof theory" [5]. The other was his Ph.D. thesis, with the title "The axiomatization of set theory" [6], an expanded version of the 1925 paper. These two papers show that Johnny was still desperately trying to rescue
mathematics by following Hilbert’s program. Johnny was stuck. He had created a simple and beautiful new set of axioms, which were later shown by Kurt Gödel to be exactly what was needed for understanding the true nature of mathematics, but he did not know what to do with them. At that point, he gave up trying to rescue mathematics and devoted the rest of his life to other things.

Another three years later, in 1931, Kurt Gödel in Vienna proved two theorems that totally devastated the Hilbert program. Gödel proved that no system of axioms for mathematics could be categorical and that no system of axioms could prove itself to be consistent. After Gödel, mathematics could never be the unique compendium of absolute truth that mathematicians from Euclid to Hilbert had imagined. After Gödel, mathematics was a free creation of the human mind, with truth and falsehood depending on human tastes and preferences. For Hilbert and many of his contemporaries, the discoveries of Gödel appeared to be a disaster. Their hopes of building a unique and solid foundation for mathematics had collapsed. But Johnny understood immediately that the new freedom created by Gödel was a gain and not a loss. Johnny said in a public lecture that Gödel was the greatest logician since Aristotle. Johnny regretted that he had not made Gödel’s discoveries himself three years earlier, but he was happy to see that Gödel used his 1925 system of axioms with separate names for sets and classes. Johnny was proud to have made a substantial contribution to the foundations of the new mathematics.

Games and Quanta
The next flower, “Theory of party games” [7], comes from a different corner of the garden. At the age of twenty-four, Johnny had become a professional mathematician with a position as instructor at the University of Berlin. He enjoyed the nightlife of Berlin and was intrigued by the logic of games such as poker and baccarat in which the outcome depends on a mixture of luck and skill. The question, whether a logical strategy exists for a player to have the best chance to win such games, had been raised by the French mathematician Émile Borel. Borel had asked the question but was unable to answer it. Johnny found the answer, which turned out to be a deep mathematical theorem. For a game with only two players, there exists a unique strategy which gives each of them the best outcome on the average. The proof that such a strategy exists is another fine example of Johnny’s style, reducing a problem of calculation to a problem of logic.

The optimum strategy usually requires a large element of randomness so that the moves of the players are truly unpredictable. Player A must throw dice to decide how to move so that Player B cannot win by predicting what Player A will do. In the game of poker the throw of the dice will occasionally require Player A to bet high on a weak hand, a move that is called bluffing. If Player A never bluffs, Player B can win by guessing more accurately the strength of Player A’s cards. At the end of his paper Johnny writes, “The agreement of the mathematical results with the empirically known rules of successful gambling, for example the necessity of bluffing in poker, can be considered as experimental confirmation of our theory.”

For games with three or more players, Johnny found no such elegant solution to the problem. To have the best chance of winning a game with three players, Player A must bribe or threaten Player B to form a coalition against Player C. The players must compete for the roles of the winners, A and B, and try to escape the role of the loser, C. The result of the competition is decided by personal willpower or spite and not by mathematics. At the end of his discussion of the three-person game, Johnny says, “The decisive factor, which is altogether absent from the orderly and equitable two-person game, is combat.”

In another corner of the garden there is a little flower all by itself, a short paper with the title “The division of an interval into a denumerable infinity of identical parts” [8]. This solves a problem raised by the Polish mathematician Hugo Steinhaus. I met Steinhaus after World War II in America. He was one of the few survivors of the group of brilliant mathematicians who emerged in Poland between the wars. Half of them were Jewish and half were Gentiles. The chance of survival was about the same for both, since those who emigrated were mostly Jews and those who survived in Poland were all Gentiles. Johnny solved the Steinhaus problem quickly and never returned to it. The theorem that he proved is counterintuitive, and the proof is astonishing. The theorem is about sets of points on an interval. An interval means a finite piece of a straight line. A denumerable infinity means a collection of objects that can be labeled with whole numbers, 1, 2, 3,..., all the way to infinity. The theorem says that there exists a collection of sets of points S1, S2, S3,..., with the following properties: (1) Every point on the interval belongs to exactly one Sj. (2) The sets Sj are identical in all respects except for position, each Sj being obtained from any other by displacing it bodily through a certain distance along the line.

The theorem is counterintuitive because it is impossible to visualize the sets Sj. If you try to imagine how the points of the set Sj are arranged near the ends, you fail. You fail because the sets are nonmeasurable, and nobody has ever visualized a nonmeasurable set of points. Nonmeasurable sets cannot be constructed using any of the familiar tools of geometry. Johnny’s proof of the theorem is astonishing because it is totally abstract. He never
even mentions the geometry of the sets $S_j$. He gives no clue to their appearance or their construction. He proves their existence by reducing it to a proposition in pure logic and proves the proposition by purely logical arguments. This little paper is the most extreme manifestation of the Johnny style.

During his Berlin years, Johnny made frequent visits to Göttingen, where Heisenberg had recently invented quantum mechanics and Hilbert was the presiding mathematician. Hilbert was intensely interested in quantum mechanics and encouraged collaboration between mathematicians and physicists. From the point of view of Hilbert, quantum mechanics was a mess. Heisenberg had no use for rigorous mathematics and no wish to learn it. Dirac made free use of his famous delta-function, which was defined by a mathematical absurdity: being infinite at a single point and zero everywhere else. When Hilbert remarked to Dirac that the delta-function could lead to mathematical contradictions, Dirac replied, “Did I get into mathematical contradictions?” Dirac knew that his delta-function was a good tool for calculating quantum processes, and that was all he needed. Twenty years later, Laurent Schwartz provided a rigorous basis for the delta-function and proved that Dirac was right. Meanwhile, Johnny worked with Hilbert and published a series of papers cleaning up the mess. For several years, quantum mechanics was Johnny’s main interest. In 1932 he published the book *Mathematical Foundations of Quantum Mechanics* [9], which occupies a substantial piece of his garden.

Johnny’s book was the first exposition of quantum mechanics that made the theory mathematically respectable. The concepts were rigorously defined and the consequences rigorously deduced. Much of the work was original, especially the chapters on quantum statistics and the theory of measurement. I read the book in 1946 when I was still a pure mathematician but already intending to switch my attention to physics. I found it enormously helpful. It gave me what I needed, a mathematically precise statement of the theory, explaining the fine points that the physicists had been too sloppy to mention. From that book I learned most of what I know about quantum mechanics. But then, after I had made the transition to physics and had begun to read the current physics journals, I was surprised to discover that nobody in the physics journals ever referred to Johnny’s book. So far as the physicists were concerned, Johnny did not exist. Of course, their ignorance of Johnny’s work was partly a problem of language. The book was in German, and the first English translation was only published in 1955. But I think even if the book had been available in English, the physicists of the 1940s would not have found it interesting. That was the time when the culture of physics and the culture of mathematics were most widely separated. The culture of physics was dominated by people like Oppenheimer who made friends with poets and art historians but not with pure mathematicians. The culture of mathematics was dominated by the Bourbaki cabal, which tried to expunge from mathematics everything that was not purely abstract. The gap between physics and mathematics was as wide as the gap between science and the humanities described by C. P. Snow in his famous lecture on the two cultures. Johnny was one of the very few people who were at home in all four cultures: in physics and mathematics, and also in science and the humanities.

The central concept in Johnny’s version of quantum mechanics is the abstract Hilbert space. Hilbert space is the infinite-dimensional space in which quantum states are vectors and observable quantities are linear operators. Hilbert had defined and explored Hilbert space long before quantum mechanics made it useful. The unexpected usefulness of Hilbert space arises from the fact that the equations of quantum mechanics are exactly linear. The operators form a linear algebra, and the states can be arranged in multiplets defined by linear representations of the algebra. Johnny liked to formulate physical problems in abstract and general language, so he formulated quantum mechanics as a theory of rings of linear operators in Hilbert space. A ring means a set of operators that can be added or subtracted or multiplied together but not divided. Any physical system obeying the rules of quantum mechanics can be described by a ring of operators. Johnny began studying rings of operators to find out how many different types of quantum systems could exist.

After Johnny had published his quantum mechanics book, he continued for several years to develop the theory of rings of operators. The third volume of his collected works consists entirely of papers on rings of operators. He published seven long papers with a total of more than five hundred pages. I will not discuss these monumental papers this morning. They contain Johnny’s deepest work as a pure mathematician. He proved that every ring of operators is a direct product of irreducible rings that he called factors. He discovered that there are five types of factor, of which only two were previously known. Each of the types has unique and unexpected properties. Exploring the ocean of rings of operators, he found new continents that he had no time to survey in detail. He left the study of the three new types of factor unfinished. He intended one day to publish a grand synthesis of his work on rings of operators. The grand synthesis remains an unwritten masterpiece, like the eighth symphony of Sibelius.

The quantum mechanics book is the last item on my list of flowers that Johnny published in German. It was published in 1932 when he was dividing his time equally between Berlin and Princeton. In
the same year he began writing papers in English. One of his first papers to appear in English was "Proof of the quasi-ergodic hypothesis" [10], which he published in the Proceedings of the National Academy of Sciences to make sure that American mathematicians would read it. This paper solved an important problem in classical mechanics using the same concept of Hilbert space that he had used to solve problems in quantum mechanics. A classical dynamical system is said to be ergodic if after we put it into an initial state and then leave it alone for an infinite time, it comes arbitrarily close to any final state with probability independent of the initial state. Johnny proved that under certain clearly specified conditions, a system is ergodic if and only if there exist no constants of the motion. A constant of the motion means a quantity depending on the state of the system which does not change as the system moves forward in time. Johnny's theorem provides a firm mathematical basis for the assumptions that are customarily made by physicists using classical statistical mechanics. Translated into the sloppy language used by physicists, the theorem says that the time-average of any single trajectory of the system over a long time is equal to the statistical average of all trajectories. Even more sloppily, physicists say that time-averages are equal to ensemble averages, and we use the word ensemble to mean the set of all states of the system.

One of the American mathematicians who read Johnny's paper in the Proceedings of the National Academy was Garrett Birkhoff. Garrett was the son of George Birkhoff, and both father and son were famous mathematicians. Garrett and Johnny became close friends, and Garrett came to Princeton for frequent visits. After Johnny died, Garrett wrote a memoir about the work that Johnny did in the 1930s. Here is a sentence from Garrett's memoir: "Anyone wishing to get an unforgettable impression of the razor edge of von Neumann's mind need merely try to pursue this chain of exact reasoning for himself, realizing that often five pages of it were written down before breakfast, seated at a living room writing-table in a bathrobe."

A minor offshoot of Johnny's thinking about operators in Hilbert space was his invention of continuous geometry, a new kind of geometry in which the dimension of a subspace is a continuous variable. A couple of short papers, "Continuous geometry" [11] and "Examples of continuous geometries" [12], are to be found in his garden. These papers were published in 1936 when Johnny was settled in Princeton. Johnny writes at the beginning, "We will give only the axioms, some comments on them, and then the main definitions and results. A detailed account will appear soon in a mathematical periodical." This is a promise that was never fulfilled. From this time in his life onward he made many such broken promises. He got into the habit of working on a problem, solving it to his own satisfaction, and then not taking the time to publish the results in detail. He gave lectures in Princeton on continuous geometry. His lecture notes were published in a book Continuous Geometry, which appeared in 1960 after his death. The book is boring. It is probably the most boring stuff that ever appeared under Johnny's name. You can tell from the book that Johnny was already bored by continuous geometry while he was giving the lectures. He had good reasons for not publishing the notes while he was alive. He had no need to publish or perish. He was a tenured professor at the Institute for Advanced Study. After 1936 he published only stuff that he considered important and not boring. He became increasingly interested in a wide range of subjects outside pure mathematics. He had, after all, earned a degree in chemical engineering at the ETH in Zurich at the same time as he was studying mathematics in Budapest.

**Bombs and Computers**

The next flower is a report, Theory of Detonation Waves [13], written in 1942, presenting a scholarly and thorough analysis of what happens when chemical high explosives detonate. Johnny had seen his homeland, Hungary, dismembered as a result of military defeat in 1918. He was even more eager than other European Jews to join the fight against Hitler. He was delighted to apply his mathematical skills and his knowledge of engineering to military problems and became a consultant to the United States Army before the United States went to war in 1941. His 1942 report was one of a series providing a theoretical basis for the improvement of military explosives. Military explosives are a delicate compromise between two conflicting requirements: they should detonate with maximum efficiency when fired in anger and should resist detonation with maximum safety when exposed to gunfire or accidental explosions nearby. When you are trying to find the best compromise, it is a great help to have a consultant who understands the chemistry as well as the mathematics.

Johnny's report does not discuss particular weapons but supplies a mathematical theory that designers of weapons can use to optimize designs. When he began work for the military, the applications were to artillery shells and antiship depth charges. In 1943 he was invited by his friend Robert Oppenheimer to visit Los Alamos and apply his ideas to the design of nuclear weapons. His understanding of shock waves made a big contribution to the success of the Los Alamos project. At Los Alamos he saw monstrous numerical calculations carried out laboriously by gangs of human computers. He began to think seriously about the possibility of electronic computers that could do such calculations better and faster than humans. In 1944 he met Herman Goldstine, who was then a
young army officer involved in a project to build a real electronic computer, the ENIAC, at the University of Pennsylvania. Johnny and Herman became close friends. Herman later said of Johnny, “While he was indeed a demi-god, he had made a detailed study of humans and could imitate them perfectly. Actually he had great social presence, a very warm, human personality, and a wonderful sense of humor.” They worked out a plan to do something spectacular with computers as soon as the war was over. In Johnny’s garden there is a paper, “On the principles of large scale computing machines” [14], describing their plans. I won’t say more about this, since Johnny’s work on computers is covered by other speakers.

I got to know Johny personally when I came to the Institute for Advanced Study in 1948. He was then actively engaged in building the institute computer and learning how to use it. He understood from the beginning that two of the most important uses of the machine would be to predict weather and to model climate. He hired engineers to build the machine and meteorologists to use it. The chief engineer was Julian Bigelow, and the chief meteorologist was Jules Charney. Each of them had a gang of young people to do the heavy work, persuading a totally new kind of machine to produce some real science. I enjoyed very much the young people, with their rowdy conversation and irreverent behavior. There was an amusing clash of cultures between these young hooligans and the older members of the institute. As Einstein wrote to his friend the queen of the Belgians when he arrived at the institute in 1933, Princeton was a quaint and ceremonious village populated by demi-gods on stilts. The culture of the older members was based on formal politeness and respect for the academic hierarchy. Johnny and I were on the side of the hooligans.

When Johnny died, the institute quickly got rid of the computer project, and the older culture reasserted itself. No more hooligans were hired, and the breath of fresh air that they had brought to the institute was blown away with them to UCLA and MIT. In 1980 the institute celebrated its fiftieth birthday by publishing a volume with the title A Community of Scholars, 1930–1980, consisting of biographies and bibliographies of the members. Not one of the young hooligans who built the machine and predicted the weather is mentioned in the book. They were not scholarly enough to be officially recognized as belonging to the institute. But there is a flower in Johnny’s garden, a paper, “Numerical integration of the barotropic vorticity equation” [15], by Charney, Fjortoft, and von Neumann, describing their first attempts to predict weather. Since the institute computer was not yet running, they did their calculations with the ENIAC. Using the ENIAC, the numerical simulation moved ahead in time more slowly than the weather that it was supposed to simulate, so there was no real prediction. At the end they express the hope that the institute computer will be fast enough to keep ahead of real time. Four years later, when Johnny’s machine and others like it were running, their hopes were fulfilled. Johnny then announced that a prediction of weather twenty-four hours ahead could be done in less than an hour. That was as far as he was able to go toward his dream of understanding climate. One year later, he was diagnosed with terminal cancer, and three years later he was dead.

**Summing Up**

In the last decade of his life, Johnny did not find time to write formal mathematical papers. Instead he wrote informal essays, sometimes addressed to his colleagues in the government agencies that supported his work and sometimes to the general public. The last two flowers in my tour of his garden are addressed to the public. They are thoughtful and beautifully written. He took a lot of trouble to think clearly and write simply. The first of the two was titled “The mathematician” [16]. It was published in 1947 as a chapter in a book of essays, *The Works of the Mind*, by a variety of authors. It is a swan song, summarizing in simple words the conclusions that Johnny had reached at the end of his life as a pure mathematician. He had devoted the best years of his life to pure mathematics, when he was, as Newton said of his own early years, “in the prime of my life for invention.” From age nineteen to age twenty-seven he had struggled to build firm logical foundations for pure mathematics, preparing the ground for G"{o}del’s discovery that no set of foundations could be complete. After the G"{o}del revolution, he took advantage of the new freedom to experiment with logical foundations for quantum mechanics and for the discipline that was later given the name of computer science. His essay “The mathematician” describes the development of mathematics as a free creation of the human mind, with foundations either borrowed from empirical science or freely invented.

The main message of Johnny’s essay is stated at the end in words that have become famous among mathematicians: “As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from reality, it is beset with very grave dangers…. At a great distance from its empirical source, or after much abstract inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical. When it shows signs of becoming baroque, then the danger signal is up. It would be easy to give examples, to trace specific evolutions into the baroque and the very high baroque, but this, again, would be too technical. In any event,
whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source, the reinjection of more or less directly empirical ideas. I am convinced that this was a necessary condition to conserve the freshness and the vitality of the subject, and that this will remain equally true in the future." Johnny probably had in mind his elaborate fiddling with continuous geometry as an example of very high baroque, and his plunge into the empirical world of computer science as the rejuvenating return to the source.

After this farewell to pure mathematics, the last seven years of Johnny’s life were divided between running the computer project in Princeton and advising the government in Washington. During this period he became known to the public as a military hardliner. For a few years he publicly advocated a preventive war against the Soviet Union. He became deeply involved with high-level committees considering problems of military strategy. One of the committees, known to historians as the von Neumann Committee, advocated the fateful decision to base United States strategy on a force of intercontinental ballistic missiles, combining the technologies of multistage rocketry and hydrogen bombs. This decision would make it technically possible for the United States to destroy the Soviet Union in forty minutes, with the inevitable consequence that the Soviet Union would be able to destroy the United States with a similar force of missiles a few years later.

The idea of a preventive nuclear war conveys today an impression of militarism gone mad. But to the generation that lived and suffered through the 1930s, the idea had another meaning. It was widely held, especially by liberal intellectuals, that the French and British governments had behaved in a cowardly and immoral fashion when they failed to march into Germany in 1936 to stop Hitler from remilitarizing the Rhineland. A preventive war in 1936, when Germany was still effectively disarmed and incapable of serious resistance against invading forces, might have overturned Hitler’s regime in a few days and saved the fifty million human beings who were to die in World War II. We cannot know whether a preventive war in 1936 would have been either feasible or effective. We know only that the idea of preventive war as a morally acceptable option was widely accepted by the people of Johnny’s generation, who looked back to 1936 as a tragically missed opportunity. To them, the idea of forestalling a terrible catastrophe by a bold preventive action was neither insane nor criminal.

Johnny argued in the 1940s that America was facing the same choice that France and Britain faced in 1936. The Soviet Union was then just beginning to build the industrial base for the mass production of nuclear weapons. Johnny saw the 1940s as the last chance for America to overthrow the Stalin regime, as 1936 had been the last chance to overthrow Hitler, without a war of annihilation. He saw a preventive war in the 1940s as preferable, not only for America but for humanity as a whole, to a war of annihilation later. I am not saying that he was right. I consider it unlikely that preventive war could have achieved its objective either in 1936 or in the 1940s. I am only saying that to talk of Johnny’s advocacy of preventive war without mentioning the events of 1936 which dominated his perception of the moral issues is to miss the main point of his argument.

The last flower on my tour of Johnny’s garden is a paper written for the general public and published in Fortune magazine in June 1955, two months before the onset of his fatal illness. The title is “Can we survive technology?” [17]. Johnny is now no longer concerned with the intellectual problems of mathematicians but with the human problems of war and peace, nuclear weapons and nuclear power, global warming and climate control, computers changing the rules of economics and politics. In the last seven years of his life, as he moved into the centers of power in Washington and made friends with generals and politicians, he understood that the urgent problems of society were human rather than technical. His view of human nature was bleak. “It is just as foolish to complain that people are selfish and treacherous as it is to complain that the magnetic field does not increase unless the electric field has a curl. Both are laws of nature.” His view of the future was equally bleak. “Present awful possibilities of nuclear warfare may give way to others even more awful. After global climate control becomes possible, perhaps all our present involvements will seem simple. We should not deceive ourselves. Once such possibilities become actual, they will be exploited.... The one solid fact is that the difficulties are due to an evolution that, while useful and constructive, is also dangerous. Can we produce the required adjustments with the necessary speed? The most hopeful answer is that the human species has been subjected to similar tests before and seems to have a congenital ability to come through, after varying amounts of trouble. To ask in advance for a complete recipe would be unreasonable. We can specify only the human qualities required: patience, flexibility, intelligence." Johnny possessed these qualities himself. They are still the qualities that we need in order to have the best chance of survival as we move into the world that he created.

References
Israel Moiseevich Gelfand, Part II

Vladimir Retakh, Coordinating Editor

Dusa McDuff

An Encounter with Gelfand

I was Gelfand’s student for six months in the winter of 1969–1970 and am very happy to have this opportunity to honor him. A great inspiration and source of strength for many years, he had a transformative effect on my career.

After undergraduate study at the University of Edinburgh, I had spent two years as a graduate student at Cambridge, solving a well-known problem about von Neumann algebras. My husband was writing a Ph.D. on Innokenty Annensky, a somewhat obscure but well-regarded Russian symbolist poet, and he needed to study in the Moscow Archives. My advisor suggested that I also apply for a British Council scholarship to Moscow, which I did, but he never suggested that I make a plan for what to do while there. So when I got to Moscow and was asked whom I wanted to study with, I was completely unprepared for the question. I said the first name that came to mind, which luckily was Gelfand. (I knew his name because some years earlier I had studied some of his books on distribution theory.) They called him up, and we arranged to meet. “How will I recognize you?” he asked. I explained what I looked like and he told me what he looked like. No doubt we met sometime just before his seminar.

He wanted to know why I was in Moscow (at that time there were not many foreign visitors) and I explained about my husband, David. He asked what I’d done, and I told him of my work in von Neumann algebras. He said, “Well, I am much more interested in the fact that David is studying Annensky than that you have solved this problem about von Neumann algebras.” Then he gave me his recent paper on Gelfand-Fuchs cohomology to read. It was titled “The cohomology of the Lie algebra of vector fields on a manifold”, but I had been so narrowly educated that I didn’t know what cohomology was, what a Lie algebra was, what a vector field was, or what a manifold was.

So he told me what to do. I went to Kirillov’s lectures on Lie groups (I could understand Russian but not speak it); I studied in the library—I remember reading a very well-thumbed copy of Eilenberg and Mac Lane and thinking how strange it was that I was reading this classic math book in English at a table in the Moscow University library; and of course I went to the seminar. That was wonderful. I gave a talk on my work, with Gelfand translating one of my English sentences followed by about ten of his Russian ones. He also talked to me before the seminar (he called it his English lesson) about all kinds of mathematics. He was trying to explain to me how certain ideas emerged, how they were interrelated. Of course I didn’t understand very much at all, but I was amazed at the way he thought about what he was doing. He said, for example, that in some series of papers he’d been groping for an idea that didn’t quite come into focus, so he tried again some years later with a completely different approach. I’d never imagined that mathematics might be thought of as anything other than a collection of definite, though perhaps very beautiful, theorems.
One week he talked to me for over an hour before the seminar. People kept coming in and saying, “Come on, everyone’s waiting.” But he wanted to finish his explanations. I felt rather uncomfortable about this and made some excuse for the next week. That was obviously not the right thing to do, and I didn’t see him for a while, until a few weeks later I gave him a translation that I had made of an article he’d written on biology and nature. “Why did you do this?” he asked. I think he was a bit suspicious of me, at that point. I said he’d asked me to translate it so I could send it to my father (who was a distinguished geneticist interested in the ideas of René Thom). Gelfand, of course, thought that René Thom had a completely wrong approach to biology and wanted his own views to be better known in the West. So by that translation I got back into his good graces. We resumed meeting, but not at the seminar. Instead, he occasionally invited me to his house.

I don’t remember much of the mathematics he taught me then, though certainly we did talk about math. But I remember him reading Pushkin for me, obviously thinking of himself as Mozart. He played some Bach records and made tea (I remember he had a treasure trove of little packets of special teas that people had sent him from all over Russia). He also invited my husband, David, to supper once or twice and told us stories about Mandelstam’s widow (whom he’d known) and various Jewish anecdotes. Once he took me shopping; he bought and gave me all the good classical records he could find. Very cheap, they contained wonderful performances by Russian musicians that he thought I should hear. He said that this too was “teaching me mathematics.” I was a would-be mathematician married to a would-be poet, and it was very important to me that he tried to reach me in this way.

When I got back to Britain, I had quite a difficult time. I’d completely changed my field and for several years didn’t really have anyone to work with. But Gelfand kept in touch. He sent me New Year’s cards, even occasionally with a brief personal note. He told people like Atiyah and Singer about me, and it was clearly on his recommendation that I spent a year at MIT, another very important milestone in my career. So he mentored me in every way he could. I visited Moscow again in 1983 (going for a week or so with the Haefligers and with Jack Milnor). I gave another talk at his seminar, but he thought the topic was uninteresting. Afterwards, he advised me to finish my papers and then move on. So I gradually moved towards symplectic geometry. I gave another talk at his seminar at Rutgers around 1995, and this one he did like.

I also remember that once we were in New York City together (I don’t remember exactly when this was, but sometime before he moved here). He took me to the Frick Collection (which at that time I didn’t know about) and showed me Rembrandt’s self-portrait, how its eyes follow you as you walk around looking at it. I am sure he thought that Rembrandt was seeing him. But that’s okay. I like people who have large ideas.

Vladimir Retakh

Israel Moiseevich Gelfand

I spent many years in close association with Gelfand. Let me try to recall a few moments.

My first “interaction” with him started under circumstances that were unfortunate for me. At seventeen I came to the Moscow State University from the provinces to take the entrance examinations to the Department of Mechanics and Mathematics. I successfully passed my mathematics examinations but failed an oral exam in physics. The same thing happened to a number of Jewish applicants, with the same few examiners. It was 1965, a rather tranquil time in the former Soviet Union. I don’t think that the examiners had any official instructions about whom to admit. They just followed their hearts. Nonetheless, the famous Moscow mathematician Alexander Semenovich Kronrod let everyone know of this circumstance, and somebody told me that the news had reached Gelfand.

Calling a Corresponding Member of the Academy of Science without any prior introduction was a severe violation of the Soviet code of conduct. But my parents were so desperate that my father dared to call Gelfand, who of course did not know him. After hearing the story, Gelfand calmly replied, “Let the boy go to the Pedagogical Institute (a sort of teachers’ college). If he is good enough, he can also attend lectures at the university.” I got the same recommendation from Kronrod, who also promised to supervise my studies in mathematics, and I followed this advice.

Kronrod (the last and perhaps most beloved student of N. N. Luzin) was a “problem solver”. He did not like theories and was not a fan of modern mathematics. Following an old Moscow tradition,
I spent my first semester on problems in real variables, and in my second semester I was lured into seminars at the Moscow State University.

At one point Kronrod caustically asked me, “So what do you understand there?” “Not much,” I confessed. Kronrod shrugged his shoulders: “Then try the Gelfand Seminar. If you go to mathematical seminars just to pray, you should do so in the main synagogue.” So I started to attend the Seminar.

Gelfand’s Seminar has already been described in many places by both participants and nonparticipants. Surprisingly, these descriptions often differ. There is not even agreement about the official starting time for the Seminar. I think that Simon Gindikin is right (he has given, in my opinion, the most accurate description of the Seminar): the official time was 6 to 8 p.m. But in fact, the seminar would start somewhere between 7 or 8 p.m., or even later, and go until 11 p.m. or so. Once or twice per semester Gelfand would start the seminar before 7 p.m. and then tease the latecomers.

At the seminar I was immediately overwhelmed by Gelfand’s erudition and, at the same time, by the way he swooped down on the participants and speakers. Only a few, including Western foreigners and a grandson of A. N. Kosygin (the Soviet prime minister at that time), were shielded from Gelfand’s barbs. One of my advisers claimed that in fact Gelfand interrogated potential speakers in advance for hours and then demonstrated his pseudo-improvisations in public. I think the truth was a little bit of everything. Gelfand could also choose a speaker out of people currently present without any warning.

At the seminar I was immediately overwhelmed by Gelfand’s erudition and, at the same time, by the way he swooped down on the participants and speakers. Only a few, including Western foreigners and a grandson of A. N. Kosygin (the Soviet prime minister at that time), were shielded from Gelfand’s barbs. One of my advisers claimed that in fact Gelfand interrogated potential speakers in advance for hours and then demonstrated his pseudo-improvisations in public. I think the truth was a little bit of everything. Gelfand could also choose a speaker out of people currently present without any warning.

From time to time Gelfand would formulate absolute truths at the seminar. For example, one of his most famous statements was “Everything is representation theory.” Later I understood that such absolute truths were in fact relative: Gelfand’s truths depended heavily on the current situation, and their internal contradictions just created additional drama. Forty years later during a discussion of some noncommutative algebra problems, he changed this famous statement to “Nothing is representation theory”.

Once Gelfand started the seminar with a poll of the distinguished participants Graev, Kirillov, Vishik, and others (they all sat together at the front of the room): “Do you know derived categories? No? How can you do any mathematics without it?” After a few years the pendulum swung in the opposite direction. During a talk on orthogonal polynomials (at which only the senior people paid any attention to the speaker), Gelfand threw out the phrase: “Youngsters are the most conservative people in the world; they always know what is right and what is wrong.”

As I already noted, both speakers and participants were mercilessly ridiculed by Gelfand. However, victims could count on moral support from others. Gelfand’s jokes were not considered a stain on one’s reputation in any way. Rather, for a young person to be a target of Gelfand’s sarcasm was a sign of distinction.

It was clear that foreigners coming to the Seminar considered it a highly exotic adventure. I remember a talk by Lipman Bers. He mentioned a theorem by Maskit and added, “I am proud that Maskit is my former student.” Gelfand reacted immediately, “You cannot say ‘my former student’. This is like saying ‘my former son’.” This was not a linguistic, but rather a cultural difference: Gelfand always saw himself as a father figure to his students and collaborators.

Gelfand’s team always included both permanent and transient members. The most long-term collaborator of Gelfand (for almost sixty years) was M. I. Graev. Some of the team members would get well-posed problems, while others participated in discussions of rather vague ideas. In my own case it was often enough for Gelfand to ask, “Can we do something in this direction?”

While working with the team, Gelfand’s only permanent motto was “I ask only simple questions.” His other reactions were sometimes unpredictable, from “Why are you trying to adjust our project to your own interests? You do not see the big picture,” to “I gave foolish advice and you just blindly followed it. It is enough to have just one fool in our company and it’s me in this case.”

After 1970, with the beginning of emigration from the Soviet Union, Gelfand’s team started changing more radically. Kazhdan, I believe, was the first to emigrate. Others followed his example. In my opinion, the greatest loss for Gelfand himself, as for all of us, was the departure of Joseph Bernstein.

Emigrants at that time disappeared completely behind the iron curtain, and we had a feeling that...
they were lost forever. It was hard. We basically did not have any social life outside of mathematics. Our collaborators were usually our best friends, with whom we discussed everything: mathematics, politics, books, our personal lives, etc. Quiet and soft-spoken Bernstein was open to everyone, especially young mathematicians. He would listen to their vague and sometimes contradictory ideas and then often put them into brilliant and short statements.

The leader of the last Moscow team was Andrei Zelevinsky. He suggested to Gelfand that I be invited to work on their projects. Gelfand started our “negotiations” with a frontal attack. “Well, you are doing some homological algebra but we already have Beilinson for that. If you are going to work with me, you have to start from scratch. In medieval times painter’s pupils worked for years just preparing paints for the master. Do you know what a hypergeometric function is? No? Very well, you can work with me on hypergeometric functions.”

After a few days Gelfand changed tactics. He asked me to open the celebrated handbook of Bateman and Erdelyi and point out the formulas I liked. He reacted to my choices quite positively: “Well, you have some taste. Why were you so defused and Gelfand returned to mathematics. The only hint I got from him was that the theory must be based on Cramer’s rules. Gelfand had been asking about noncommutative determinants every semester since my sophomore year and my friends would always shrug their shoulders: “The old man is losing his grip. Who cares about such things now?”

At the boarding house, Gelfand lived in a tiny room. There was only space for two beds with their nightstands. We would kneel, with our notebooks on the beds, writing our formulas, and Gelfand would laugh like a happy child: “You just look at these formulas; they tell us what to do by themselves. How nice!”

We continued the game with formulas in Gelfand’s kitchen in Moscow. Once a government official phoned Gelfand. He complained that his thirteen-year-old son hated mathematics and asked Gelfand for advice. I expected to hear a cascade of Gelfand’s jokes, but he was dead serious. He asked the boy to pick up the phone and said, “I will give you just three problems: multiply one by one, one by negative one, and negative one by negative one.”

The teenager gave the correct answer to the first two questions and then stopped. “That’s great,” Gelfand said, “you already know two thirds of all mathematics; you just need to try a little bit to get the rest of it”. It was the best lesson in pedagogy I ever had.

My extended family emigrated to the U.S. in 1993. I spent a year as a visiting scholar at Harvard and Rutgers and then got my first teaching job at Oklahoma State University. I was almost scared to death at that time: my knowledge of the American Midwest was limited to O. Henry’s stories, and among Russians there circulated a lot of terrible tales about American students.

As usual, Gelfand defused the situation: “Are you a professional or not? As a professional you must be able to teach at elementary school, to give talks interesting to Harvard faculty, and everything in between. Just one piece of advice: make your
students comfortable in your classes—one cannot teach without this."

After spending a year at Oklahoma State University and another at Penn State University, I returned to Boston. My wife and I rented a small apartment near Harvard Square. A memorial conference for Garrett Birkhoff was scheduled on April 1st of that year and Gelfand was invited as a keynote speaker.

Gelfand came to Boston a few days early, and we spent this time preparing his talk on lattice theory. Unfortunately, a huge snowfall was expected in Boston right before the conference and the organizers at Harvard suggested cancelling the event. Many interested people would not be able to be there, the audience might be sparse, and so on. Gelfand surprised them by insisting that the show must go on under any circumstances. After all, no one worthy of being in the audience would miss it because of a little snow.

Gelfand’s interests were at once intense and wide-ranging. Perhaps the most remarkable illustration of this is what happened on that snowy day he spoke at Harvard. To make the morning commute easier, Gelfand had spent the night in our apartment on a small futon in the living room. During our breakfast on April 1 my wife offered him some eggplant. “It is good”, Gelfand said, “but the way you cook eggplant is totally wrong. I will teach you how to do it.”

At that point I knew what might happen and had to act to prevent a cooking lesson that might last several hours. I pleaded, “It’s late.” In fact we started on our way through the snowdrifts.

Gelfand’s talk, scheduled for one hour, lasted about two hours. Actually, he wanted to continue, but the listeners began to be restless. After the talk Gelfand approached me: “Do you understand why I agreed to come? I did not know Birkhoff. It is all about lattices. Maybe they can replace category theory, which is too rigid.”

We never returned to this subject, and in a few months I left Boston for Arkansas. Gelfand did not call me very often, just two or three times a day. The first call was usually at 8 a.m. After two years I started my tenure at the same university as Gelfand and the number of his everyday phone calls doubled.

Acknowledgment

Many thanks to Mark Saul and to my Rutgers colleagues whose remarks greatly improved the initial version of this story.

Serge Tabachnikov

Gelfand’s School by Correspondence

This note concerns a major contribution of I. Gelfand to mathematical education: the School by Correspondence for high school students at Moscow State University, founded by Gelfand in 1964 and commonly known as Gelfand’s School.

The first half of the 1960s was a relatively liberal time in the history of the former Soviet Union. It was also the beginning of space exploration, and the general excitement generated by Sputnik extended to the exact sciences, including mathematics. This was the time when special high schools for physics and mathematics appeared in many large cities of the country. Let me mention two notable examples. The first is the Boarding High School No. 18 at Moscow State University, founded by A. Kolmogorov in 1963 for talented students from across the country. The second is the famous Moscow High School No. 2. Many well-known mathematicians, now working in various countries, are alumni of these and other similar elite schools.

Gelfand’s School reached out to a much broader population of students, and its approach was very democratic: the goal was to provide a quality mathematical education to motivated and talented students, mostly from smaller towns and rural areas, who did not have access to specialized mathematical education available at major scientific centers. For example, as a matter of principle, Gelfand’s School would not admit students from Moscow. The education in Gelfand’s School was free.

Structure of the School

Gelfand’s School was complementary to the school curriculum (which was the same in all schools in the USSR) and covered the last three years of high school. Unlike the common practice in this country, talented and motivated students were not accelerated through the curriculum but rather were offered a substantial enrichment and an in-depth study of the familiar “school mathematics”.

Let me describe the school’s mode of operation. Every student received, roughly once a month, a brochure covering a particular topic. These booklets contained some theoretical material, numerous examples of problem solving, and a test, usually consisting of two parts: mandatory and optional (each having a dozen problems or so). The student worked on this material for about a month, culminating in writing detailed solutions.

Serge Tabachnikov is professor of mathematics at Pennsylvania State University. His email address is tabachn@math.psu.edu.
to the test and mailing it to Gelfand’s School at Moscow.

There the tests were graded, mostly by teaching assistants recruited from undergraduate and graduate students of the Faculty of Mathematics of Moscow State University (it had a large population of students, with an incoming class of about 500, and offered five years of undergraduate studies). The ratio of TAs to students at Gelfand’s School was about 1:10; ideally, a group of students from the same town or the same school was assigned to the same TA. To ensure quality, grading was multilayered: TAs were organized in groups, each group had a leader who cosigned each graded test and was personally responsible for the quality.

In fact, the word “grading” is not quite appropriate: the main goal was to teach the students, rather than just to assign a grade for their test. A typical work, sent back to the student, contained numerous remarks on the weak and strong points of the work, written in the margins, and oftentimes a detailed overall review of the progress of the student. Ideally, the same TA was responsible for the same student during his or her three years of education at Gelfand’s School.

In its heyday, Gelfand’s School had dozens of branch schools based at regional universities and pedagogical institutes. These branch schools had a similar structure; they supplemented the educational materials from the central school with those of their own.

The Program

The curriculum of Gelfand’s School changed with time, but there was a hard core of topics and textbooks from the early days on. First of all, there were the classic books [1] and [2] (I refer to the English translations, not the Russian originals), written in 1964, that is, for the very first class of the school. They were followed by [5] and later by [3] and [4]. It is refreshing to compare these slim—yet very substantial—books with numerous bloated and inflated textbooks that are, unfortunately, so common! For example, The Method of Coordinates [1] has less than ninety pages, including numerous figures. Its last section concerns the geometry of the 4-dimensional cube; the section culminates with the problem of describing the family of 3-dimensional sections of a 4-dimensional cube, orthogonal to its main diagonal; this problem was discussed in detail.

Let me mention some other topics that are traditionally included in the curriculum: combinatorics, arithmetic of integers, polynomials, word problems, areas of polygons, equations and inequalities, complex numbers, logarithmic and exponential equations, space geometry, series and limits. Some topics were designed for more advanced students and were optional, for example, introduction to game theory (the title of the brochure was “Instructive games”).

It is worth mentioning that some of the assignments, especially in the last year of study, were designed to prepare for the entrance exams to universities (which were highly specialized; for example, at a faculty of mathematics, a student had to take four entrance exams: written mathematics, oral mathematics, oral physics, and an essay).

Authors, Personnel, and Teaching Assistants

A brief look at the references to this memoir shows that I. Gelfand was a coauthor of most of the books that made up the backbone of the educational material of the school. Gelfand attracted many a talented author to this work: E. Glagoleva, V. Gutenmacher, A. Kirillov, A. Shen, E. Shnol, A. Toom, N. Vasilyev, and others. Most of the authors were professional mathematicians, in some cases very prominent ones (A. Kirillov). Many brochures for the school were also developed by members of the school staff.

Let me say a few words about the first—and so far the only—director of the Gelfand School, V. F. Ovchinnikov. I. Gelfand had a rare gift of choosing unique people for his “teams”; V. Ovchinnikov was a perfect choice for the director. He was the founding principal of the Moscow High School No. 2 for physics and mathematics that was mentioned above. As a curiosity, let me mention that there was a law in the former Soviet Union that prohibited the same person to be a director of more than one organization simultaneously; as an exception, the appointment of Ovchinnikov as the director of Gelfand’s School was signed by the Soviet Prime Minister A. N. Kosygin.

The staff of the Gelfand School consisted of six to ten teachers and program coordinators and about the same number of administrative assistants. Some of the former were previously

1 The excellence of this school was not limited to physics and mathematics: literature, history, and other “ideological” subjects were taught exceptionally well by a team of brilliant teachers. For its students, the school was indeed an oasis of freedom. This eventually attracted the attention of the authorities, leading to a crackdown on the school and the firing of the principal and many teachers.
Explaining geometry to 8th graders, 1968.

Explaining solutions to Olympiad problems, 1968.

teachers of mathematics at School No. 2, whom Ovchinnikov invited to work at Gelfand’s School. As to the latter, most of them were work-study students of Moscow State University. The author of this note worked at Gelfand’s School in 1979–1988 (in particular, combining this full-time job with graduate studies). One of the responsibilities of program coordinators was to run a special training session for teaching assistants where these students were introduced to the next topic in the curriculum of the school and were prepared for the next grading cycle.

Many hundreds of undergraduate and graduate students of mathematics of Moscow State University served as Gelfand School TAs. Every Soviet student was supposed to do some “service”. The nature of these “services” varied; for example, one of them was helping the police to patrol the streets (this was not a very popular job). In contrast, teaching assistantship at Gelfand’s School was a desirable and prestigious service, so the school had no problem recruiting the necessary number of TAs. Many very well-known mathematicians served as teaching assistants in their undergraduate and graduate years. To represent three different generations, let me mention just three names: Mikhail Shubin (Northeastern), Victor Vassiliev (Moscow), and Maxim Kontsevich (IHES). Many teaching assistants were themselves alumni of Gelfand’s School (there were years when close to a quarter of the incoming class at the Faculty of Mathematics of the Moscow State University consisted of graduates of Gelfand’s School).

Students

To be admitted to Gelfand’s School, a student needed to pass a written entry exam (the first class of 1964 consisted of 1,472 students selected from more than 6,000 applicants). The total number of students of the school over its forty-six years of work well exceeds 200,000. The majority of these students continued their education at good universities leading to careers in mathematics, science, and technology. I am not aware of any comprehensive data on the graduates, in particular, on those who became mathematicians. Let me just mention one notable example: Edward Frenkel (UC Berkeley) was a student at Gelfand’s School in the early 1980s.

Along with individual students, Gelfand’s School had “collective students”. A collective student was one of a group of students from the same class that worked under supervision of their teacher (usually this was an after-class activity, akin to a math club). It was up to the teacher how to organize the work of this group; what it shared with an individual student was the same assignment and the same test. One collective student produced a single test which was graded the same way as those of individual students. An additional bonus was that the teachers who participated in this program substantially expanded their mathematical and pedagogical horizons; in effect, this was a form of continued education for high school teachers.

Gelfand’s School Today

Today Gelfand’s School is alive and well. It is now called an “Open Lycée”, having eight departments that, in addition to mathematics, offer courses in biology, physics, history, economics, chemistry, philology, and law. The full course in mathematics now spans five years. Many things have changed in Russia (including the name of the country), but Gelfand’s School has survived all the disturbances and cataclysms, and it continues to provide high quality mathematical education to a large number of students across the country.

Another Gelfand Story

Let me finish with a Gelfand story that concerned me personally. In the mid 1980s, I submitted a proposal to my supervisors at Gelfand’s School on how to improve the geometry courses that we offered at the time. This was a rather bold and not very well-balanced proposal; anyway, it ended up on I. Gelfand’s desk. One day, my telephone rang. It was Gelfand speaking. “Do you want to hear a joke?” he asked. The joke goes like this: A crow sits on a branch of a tree. A rabbit runs by. “Hello, crow, what are you doing?” “I am showing off.” “May I join you?” “You are welcome.” So the rabbit climbs the tree and sits next to the crow. Then a fox runs by, asks the same question, and also joins them. So does a wolf. Finally, a bear walks by. “May I join you?” the bear asks. “Please do.” When
the bear sits next to the rest of the animals, the branch breaks and all the animals fall down except the crow, who flies and cries, “Before showing off, learn to fly!”

Gelfand continued to explain to me that, so to speak, I wanted to feed the students with pastries and candies whereas what they really needed was whole grain bread. I remember this lesson to this day.²

References


Mark Saul

Gelfand at 92

In October 2005 I got an email message from a woman with a Russian name whom I didn't know. I immediately knew that it was actually from I. M. Gelfand and that the person writing was one of his many minions, on whom he relied for communication.

“Professor Gelfand would like to speak to you,” it read, coldly. But I knew there was more to it than that. Professor Gelfand and I had spoken every six months or so, and I had last seen him two years ago. But it was I who always called him, not the other way around. Something’s up. Something’s wrong.

A phone call verified this. Gelfand was in the hospital and wanted to speak with me. It must be serious. Maybe he’s thinking that it’s his time, and he wants to say goodbye to his friends and colleagues. So I called around to others who knew and worked with him. This took awhile, and by the time I reached someone, it was to find out that Gelfand was back home. He had stumbled and became concerned about his health, so went to the hospital, from which he was now released.

I called Gelfand at home. His voice was a bit weaker than before, but still clear. His hearing, too, had deteriorated a bit. After repeating everything twice, I was handed over to yet another minion, who spoke only Russian, and I had to repeat my responses in that language. Gelfand asked when I could come see him—again not typical of him.

This was all in mid October. My schedule and his did not allow us to meet until some weeks later. It had been raining for a week, and traffic was snarled. So I was two hours late. No matter. I came to the door of Gelfand’s nondescript suburban home and was greeted by Irina, the telephone minion, who looked after him when no one else was home. I was ushered in and sat down at a small table just outside the kitchen. On the table was a pile of folders, obviously work in progress. Some separate sheets of paper bore equations, symbols, and notes. Three softcover Russian books lay nearby.

“One minute,” Irina said, and left the room.

But it was almost five minutes that I waited. After looking at every object in the room several times, objects that were not new to me, I heard a quiet, rhythmic bump-bump, bump-bump, bump-bump. Before I could see what it was, I knew what it was. It was an old man with a walker, making slow but purposeful progress around the corner of the corridor and towards me. It was Gelfand.

He sat down in a chair with a well-worn cushion and fastened his still bright blue eyes on me. He was very glad to see me, very glad to talk. He talked slowly at that point. His hands shook like birds fluttering against the bars of their cage. His thoughts quickly outgrew the prison of his mind—an old struggle, but now they were caught in the prison of his body—a new one.

He was still thinking, his mind roving widely over an intellectual landscape while his body remained in that old chair with its old cushions. He pointed to the three books recently published in Moscow. Two bore his name (as well as those of coauthors). A third was by a former student, with a handwritten message to Gelfand on the title page, thanking him for all he had given the author. The books were about applications of mathematics to medical diagnosis. Gelfand slowly explained to me some of the issues. His eyes blazed with energy, just as I had remembered them. But now I had to look closely to see his face reflecting the joy of discovery, as it used to. For him, it was more than simply a sense of accomplishment. It was a joy in the knowledge itself, as the pride of a parent can be for the child itself and not just for the parent’s own role.

I remember a conversation of about six years ago, one September. I had gone on vacation that summer with my family, while Gelfand was in France, working with some European mathematicians.

“Did you have a good summer?” I asked. “Did you get to relax?”

“I didn’t have time to relax. I learned many new things,” said the 86-year-old winner of the MacArthur, Kyoto, and Wolf prizes (not to mention

²Eventually I wrote a little book on polynomials for Gelfand’s School. This book is still in use there. Mark Saul is the director for the Center for Mathematical Talent at the Courant Institute of Mathematical Sciences. His email address is mes37@nyu.edu.
the Order of Lenin—which he almost never does).

Now Gelfand perused the pages of the new books with me, commenting on how he wrote a paper which started this chapter, or had an idea which begat that one. His name was sprinkled about the books. I realized that he must have had many categories of minion. Some cooked his meals. Others tended to his email. Still others took his ideas, worked with them, and put them into print or into practice. It dawned on me that this was exactly my role when we worked on educational issues together. It was a collaboration, but the freshest ideas, and the most basic ones, were usually his. He was already an old man, even then. His mind, however, was a young one, even now.

Yes, he took pride in the accomplishments of his students. But the relationship was not the usual one between teacher and student. He taught them how to think, how to investigate things mathematically. But he remained with them, it seems, even on the next lowest level—the level not just of processing ideas, but of the particular ideas themselves. His mind overflowed with them, and he seemed to have had the ability to surround himself with people who were worthy receptacles, who ensured that the abundance of ideas did not fall on parched earth.

We spoke about what he might do to support education and how I might contribute to his continuing work. We talked about our colleagues’ work in America. He would not talk about the past, although I was curious. He mentioned that one Russian colleague did not know how to deal with the American system as well as he did. This colleague was only seven years old, after all, when Stalin died. I tried to ask him how his experiences under Stalin helped him adjust to America, but this topic did not interest him.

I admired some new paintings which had appeared on the wall. They were the work of his daughter, who was studying art. He beamed and rose. He showed me some more work. Haltingly, using the walker, he led me into his bedroom, where the work was displayed. Irina, hearing him move, immediately came downstairs, following behind him lest he fall. Various bits of hospital apparatus lay around the bed. Life was more complicated for him than he let on.

When we got back to the living room, Gelfand turned the conversation to our trigonometry book. “We must revise it,” he commented. I asked what needed revision. He replied immediately, “The Pythagorean Theorem. The proof. It’s wrong.”

I asked for more details. His hands fluttered over to a pencil and began drawing. His lines were wiggles, like the outlines of the aged elephants in the Babar books. But I knew what he was drawing.

He drew a large square. He proceeded around its perimeter, putting a point on each side, a short distance from each vertex. I knew, because I knew this proof, that the short distances were meant to be equal. He connected these points, forming four right triangles and a square inside. He wrote down \((a + b)^2\). I finished the diagram, marking the short segment “b” and the remainder of each side “a”. Then I marked the hypotenuse of each triangle, which is a side of the inner square, with “c”.

“Yes,” he said, and wrote down “\(= c^2 + 4(ab/2)\)”. The area of the big square was equal to the area of the inner square plus four times the area of one right triangle. Expanding, this proved the theorem. I smiled. I knew this proof and so did he, but somehow we used a more complicated one in the trigonometry book.

“So this is how we must revise the book,” I said.

“Yes,” he said and sat silently for a minute. He often fell silent during a conversation, sometimes for many minutes, to think about something. One learned patience with this seemingly antisocial behavior. Sometimes, after the silence, he would walk over to the phone and make a call or read some email or take a book from the shelf and show me a poem of Pasternak. His mind often behaved like a magnificent wild animal, which he was trying to use as a beast of burden. Eventually the flow of the conversation would return, but one never knew how or when.

My patience eventually paid off. He broke the silence: “It is wrong.”

I confessed that I didn’t see the error. He smiled a bit, and his eyes lit up. He enjoyed this intellectual cat-and-mouse game, a game which betrayed the art of a master teacher. I knew what was coming. He would tell me as little as possible until I got the idea myself.

I knew, but at the same time I was concerned. How could this classic proof be wrong? Was his mind getting old after all? I suppressed the thought. If it was right, I would know soon enough. If it was wrong, I would learn something. The wager was an easy one to make.

I was wrong, of course. His hands again took up the pencil and drew a diagram which I eventually resolved into two right triangles symmetric about a line, with the same hypotenuse. “When are two triangles congruent?” he asked me.
Uncharacteristically, he answered himself, “When they can be made to coincide.” He didn’t know the English word “coincide”, so he used the Russian. With a gesture whose irony would unfold with the dialogue, I agreed, placing my right hand, fingers extended, exactly over my left.

“These two triangles cannot be made to coincide. You must reflect one to get the other. There are two definitions.” I knew what he was saying, but not why he was saying it. We had talked years ago about orientation in the plane and how areas of polygons can be given a sign according to the orientation of their perimeters. The subject had arisen with the discussion of Hero’s formula, and we had decided not to include the issue of signed areas in the trigonometry text. But what does this matter have to do with the proof at hand?

I didn’t understand, and told him so. The four triangles in the diagram all had the same orientation. Even if you didn’t allow for reflections, they could be made to coincide. You didn’t need to “flip”. Didn’t he see this? I knew better than to ask Gelfand this last question. But I harbored a mental reservation. Happily, I found that I still had some patience.

Gelfand had more. He just looked at me. I thought he didn’t understand. But of course the shoe was on the other foot. “Look,” I said. “You trace this triangle clockwise. Then you trace the next one clockwise. A rotation brings one onto the other. There’s no need to flip.”

Gelfand was not impressed. He merely asked, “Rotation about what point?”

The question was too easy. There was some ulterior motive for it. “Around this one.” I indicated the center of the square.

Then suddenly I saw what he had seen, and I gasped. The triangles indeed had the same orientation. But if you trace the triangles in a clockwise orientation, the four hypotenuses form a square with a counterclockwise orientation.

!!!

“So the areas are equal only in modulus. We have to account for this in our book.” He said a few more things, but the words were inconsequential in comparison with the lessons, both mathematical and pedagogical, that I had learned. The teacher in him knew that I would eventually figure things out and that this process would do me more good than if he had told me what he was thinking. This was his gift to me: a gift of knowledge, a gift of patience, a gift of trust. He could have shown me his idea directly. I would have been duly impressed with the insight and would have let others know. Instead, he let me discover the insight—and thus demonstrated his gift both as a mathematician and as a teacher.

I began thinking about other dissection proofs of the Pythagorean theorem. Is there always this anomaly? Or is the problem with the particular argument we are using? I was thinking again about Hero’s formula and the role orientation might play there. But it was 8:15, it had taken me three hours to get there through snarled traffic, and Gelfand was getting tired.

So we adjourned. I agreed to call and to do a few more errands for him.

The trip back was uneventful. The rain had stopped and the traffic cleared. That’s all I remember about it because I was occupied by the picture of Gelfand at 92, by the diagrams he drew, by the flaw in the proof, and by the art of teaching so beautifully laid out to me.

I am occupied still. What led me to that incredible insight? Was it Gelfand’s question about the center of rotation? Or simply his silence, a communication in itself that there was something more to see than I had seen? In what did the act of teaching consist? These are not simple questions, and I’m not sure what methodologies will allow us to answer them.

There were more thoughts as I drove: Was it an act of courage for him to stay active like this? I wasn’t sure it was. His experience might simply have been that of living, and thinking, and creating, as most of ours is of living, and breathing, and eating. But I can certainly count it a lesson in courage among the lessons I learned from Israel Moiseevich Gelfand that night.
Beginning February 1, the AMS is accepting applications for the AMS-Simons Travel Grants program. Each grant provides an early career mathematician with $2,000 per year for two years to reimburse travel expenses related to research. Sixty new awards will be made in 2013. Individuals who are not more than four years past the completion of the PhD are eligible. The department of the awardee will also receive a small amount of funding to help enhance its research atmosphere.

The deadline for 2013 applications is March 31, 2013.

Applicants must be located in the United States or be U.S. citizens. For complete details of eligibility and application instructions, visit:

How to Calculate Proofs: Bridging the Cultural Divide

Raymond T. Boute

This article argues that one of the most neglected opportunities in many branches of mathematics is symbolic reasoning for the “logical” parts rather than just for the algebraic, analytic, etc., parts, where symbolic calculation has been a well-established routine since Viète and Descartes [4], [5].

This is far from suggesting that symbolic reasoning should be promoted from neglect to supremacy or that it is always the best choice: we will discuss its place among other styles. One of several things we do want to emphasize is the advantage of being able to calculate with predicates and quantifiers [10], [23], [25] as fluently as is common for derivatives, integrals, and sums (\(\forall\) and \(\exists\) being much simpler than \(\Sigma\)). Even if, for a specific problem, this ability is not exploited by choice, at least it will have provided a broader basis for making choices.

The treatment is introductory, addressing anyone interested in augmenting their palette of reasoning strategies, including mathematics educators and, directly or indirectly, students.

As a matter of terminology, one of the styles we will discuss, and also endorse for logic, is the calculational style [22], [24] typical in algebra and analysis. The leftmost example in Figure 1 is taken from an engineering textbook [6, p. 91], intentionally leaving symbols unexplained to focus only on the shape of the argument as expressions chained by relational operators (=, \(\leq\)). Normally at each step one also explicitly mentions the rule used (definition, theorem, \ldots).

\[
\begin{align*}
\frac{1}{n} \sum_{x} p^n(x) l_n(x) \\
= \frac{1}{n} \sum_{x} p^n(x) [1 - \log q^n(x)] \\
= \frac{1}{n} + \frac{1}{n} L(p^n; q^n) + H_n(\theta) \\
= \frac{1}{n} \log(p^n, q^n) + H_n(\theta) \\
\leq \frac{1}{n} + H_n(\theta)
\end{align*}
\]

Figure 1. Calculational (left) and equational (right) derivations in common mathematics.

This style of argument is called calculational, and if all relational operators are pure equality, as in the rightmost example [12], it is equational. Such arguments are commonly appreciated for their sense of “smoothness”, especially if there are no interruptions by prose in the chain.

Examples appear in every nonspecialist mathematics journal, a random sample being the CMJ issue at the time of this writing [14, p. 381]. Of course, more sizeable calculations are structured hierarchically, often form part of a larger theory or project, and are interspersed with expository text. Such issues are orthogonal to the one at hand and are discussed later.

Equations form such a powerful tool in problem solving because they delegate substantial parts of reasoning (especially the tricky ones) to symbolic calculation. Practicing fosters a parallel intuition for symbol manipulation, enhancing the intuition in the domain of discourse (or even preceding it when exploring new domains). For instance, in deriving \((a + b) \cdot (a - b) = a^2 - b^2\), the essence is the cancellation in the intermediate form \(a^2 + b \cdot a - a \cdot b - b^2\), and the underlying intuition is
of a symbolic nature, not explained as simply in another way.

**Setting the Scene: High School Algebra**

In a problem-solving view of theorem proving, the demonstrandum is not always known in advance but can be a solution to a problem (to be **discovered**). Educationally, this teaches students to think beyond just proving what is given.

For reference, we show a typical high school algebra calculation in great detail to highlight how, in later examples, logical calculations use exactly the same style, just with other rules.

One of the first topics one learns in high school algebra is “simplifying” expressions by algebraic rules such as **commutativity**, **associativity**, and **distributivity**. An example is “*Simplify (a + b) · (a − b)*” as a variant of “*Prove (a + b) · (a − b) = a^2 − b^2*”.

Of course, “simplify” is context-dependent; here “*Convert to sum of products form*” would be more specific. The reverse problem (starting from \(a^2 − b^2\)) requires a little “inventive” step. In “*Prove*” variants, a heuristic [23] is starting from the side that minimizes such sights.

---

**Figure 2.** The calculational nature of common high school algebra: an example.

Figure 2 shows the calculation to various degrees of detail, exposing the hierarchy [2], [33], [35]. We briefly comment. Even in high school one soon learns to do symbolic head calculation, and in larger contexts one would write the result as Figure 2(a). For exposing the crucial step, namely the cancellation, Figure 2(b) is a possibility. However, Figure 2(b) involves tacit conventions such as omitting parentheses and using “−” for (additive) inversion as well as subtraction (overloading).

Figure 2(c) is more explicit. We reserve (Rearrange) for any mix of associativity, commutativity, and distributivity. Figure 2(d) shows the most detail we give here. In a hypertext environment [2], [33], [35], (d) can be seen as obtained by “clicking” on the (justification)s in (c). The property \(a · b = −(a · b)\) is proven for rings (abstract) in [28, p. 89], but in high school it is proven in literally the same way for numbers (concrete). Here are some crucial observations.

- The calculations are essentially formal: using rules without interpreting the expressions.
- Notation and rules are sufficiently robust to make formal use safe (mature formalism).
- The theorem \((a + b) · (a − b) = a^2 − b^2\) is **discovered**, not set in advance as a goal.

Another advantage is clarity: in the detailed calculation, the steps use the rules so explicitly that one can infer these rules by “reverse engineering”, comparing consecutive lines.

All of this is common practice in high school. Who would derive \((a + b) · (a − b) = a^2 − b^2\) using prose? The same question can be asked for logical reasoning.

---

**Rationale: The Roles of Diversity, Logic as Algebra, and Formality**

Sangwin [46] emphasizes the role of equations in problem solving from secondary school onwards. He quotes Pólya as saying that […] the most important task of mathematical instruction in the secondary school is to teach the setting up of equations to solve word problems.

Yet, there is a caveat. Students may become so exclusively focused on equations as “the” easy shortcut in problem solving that some teachers advise them to explore different methods as well, since diversity of methods and perspectives enhances insight and understanding. My favorite example is deriving simple geometric solutions to typical calculus problems—also because calculus deserves appreciation for its intrinsic beauty rather than just for its applications.

Diversity notwithstanding, often a problem is stated in symbolic form either as given or by translation from prose to facilitate the solution. In that case, it would be a setback not to exploit the formal properties of the symbolism insofar as it is mature for practical use.

Indeed, Boyer [11, p. 180] distinguishes three stages in the development of algebra. He presents it as an oversimplified first approximation, but it is relevant to any branch of mathematics and can be made more accurate by the qualifications added between [ ].

1. **rhetorical**: writing everything out in words [and reasoning correspondingly in words];
2. *syncopation*: adopting abbreviations [but still without systematic formal rules];

3. *symbolic*: final stage [with systematic syntax plus systematic formal calculation rules].

Algebra for polynomials reached the syncopation stage with Diophantus, while the symbolic stage was achieved by Viète and Descartes [4], [5]. It was Leibniz’s dream for logical reasoning to become symbolic in a similar way (“calculemus”). Progress in this direction started with Boole, and the developments in formal logic over the subsequent one hundred fifty years are well documented.

Yet most logic textbooks present symbolic logic at first but regress to rhetoric logic in later chapters, illustrating what we meant earlier by “mature for practical use”. Not surprisingly, symbolic logic is even less used in nonspecialist mathematics. For instance, Taylor [49] deplores that many mathematicians still use \( \forall \) and \( \exists \) as syncopation, often obscuring the logical structure of arguments. With the typical \( \delta \)-\( \epsilon \) arguments from analysis as an example, he concludes that examiners **fully deserve the garbage that they get in return**.

The explanation may well be that mathematicians miss the smoothness of algebra in the formal logic of most logic textbooks and might better appreciate Halmos and Givant’s *Logic as Algebra* [25]. (Halmos also stated years earlier that \( \forall \) and \( \exists \) are better used not at all than as mere shorthand.) How to bring the idea of logic as algebra into practice is shown in mathematics journals by Gries and others [24], and additional introductory material can be found in the computer science literature [2], [10], [22], [23], some of it even usable in high school [3].

Many will remember the disasters caused in mathematics education around 1960 by the injudicious and coarse manner in which formality and abstraction were introduced [1]. Mathematics has recovered, but, unfortunately for mathematics, formality itself was blamed and has not recovered, as generations of mathematicians still seem to be smarting from the trauma.

Indeed, the backlash was severe, including a long period of deemphasizing proofs. Steven Krantz [32] noted that classic texts in the style of Rudin [43] are often no longer suitable, or appear to be inaccessible, to the present crop of students and clarified this in an email saying, *Many of the students who show up in a real analysis course do not know what a proof is.*

Many texts on proofs in recent years indicate a revived interest, yet some still reflect the “formalism trauma” in urging students to write proofs in prose [51]. This perpetuates the double standard of symbolic algebra/calculus versus rhetoric logic, causing a stylistic rift within mathematics. Lamport [33] notes that *the structure of mathematical proofs has not changed in 300 years. … Proofs are still written like essays, in a stilted form of ordinary prose.*

The predilection of students towards equations in problem solving may be turned into an advantage by teaching logic with the familiar calculational flavor of algebra and thus considerably lowering the threshold for formality, in particular symbolic reasoning.

An important technical observation [2] is that the calculational style typical in algebra and calculus (Figures 1 and 2) and illustrated for logic in later examples can be seen as a streamlined form of so-called natural deduction [42], a formalization of the informal reasoning style used in mathematics since antiquity. Whenever it facilitates reasoning, forms can be intermixed [23].

Whatever style is used, it is advantageous to embrace rather than eschew formality. Lamport shows in [33] and with different examples in [35] how proofs in the classical deductive style can be vastly improved by formalizing definitions and theorems and structuring proofs hierarchically.

There can be no doubt that proof assistants and other computer support for logic reasoning [26], [27] are potentially as useful in everyday mathematics as Mathematica, Maple, etc.

However, currently the threshold is rather high. Mathematicians routinely use the formal notation from calculus, linear algebra, and so on, which are quite faithfully supported (including deficiencies!) by the software tools. By contrast, few are used to formalize the logical parts of their arguments, and using a proof assistant requires diligent formalization at every stage, as for using Maple and Mathematica, but there the formality looks “normal”.

The threshold can be eliminated by routinely formalizing logic arguments. The motivation is not that “the software requires it” but because it leads to better arguments also “on paper”.

### A Mini-Gallery Illustrating Logic Reasoning by Calculation

The examples cover a variety of topics and different aspects of the calculational style.

### A Detailed Example

Example 3.3.5 from [51, p. 119] is chosen because its author notes that it is the most complex proof thus far, and some readers on the Amazon website mention that it was the first place in [51] where they experienced difficulties.

The operators used are defined in Figure 3. The reference definitions from [51] are transliterated into a more uniform notation in view of later examples from other sources. Of course, among the
wide variety of quantifier notations in the literature, the differences between the form $\forall v \in S(p)$ in [51] and the form $\forall v : S.p$ used here may appear negligible, yet they are a matter of language orthogonality and robustness. The equivalent variants for $\emptyset$ and $\cup$ are obtained by eliminating $\{ | \}$, using $y \in \{x | P(x)\} \equiv P(y)$. The actual example concerns the following:

**Task.** Suppose $B$ is a set and $\mathcal{F}$ is a family of sets. Prove that if $\cup \mathcal{F} \subseteq B$, then $\mathcal{F} \subseteq \emptyset B$.

How this is done in [51] and the resulting rhetorical proof are shown in Figure 4. An equational derivation, simultaneously discovering the theorem and its converse, is shown in Figure 5.

**Clarification and Comparison of the Strategies**

The scratch work in Figure 4(a) is a condensed version of [51, p. 119] and lists successive **Givens-Goal** tables and explanations. The various strategies used are developed in [51] but are beyond the scope of this article. Some involve introducing arbitrary objects at various points into the proof [51, p. 119]. The discussion by Lamport [35, p. 3] of the questions this may raise for beginners is enlightening.

For calculational derivations, the simple strategy from high school algebra suffices: for each step, look at the shape of the expressions to determine what yields progress: using a definition or a rule from logic. All definitions used in Figure 5 are easily identified in Figure 3. In fact, by the reverse engineering as in Figure 2(d), one can reconstruct the rightmost column of Figure 3 from Figure 5. This even holds for the logic rules used, for instance, by comparing successive lines:

<table>
<thead>
<tr>
<th>Name Symbol</th>
<th>Location in [50]</th>
<th>Reference definition</th>
<th>Equivalent variant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subset</td>
<td>p. 58, item 5</td>
<td>$A \subseteq B \equiv \forall x. x \in A \Rightarrow x \in B$</td>
<td>$A \subseteq B \equiv \forall x. x \in B$</td>
</tr>
<tr>
<td>Power set</td>
<td>p. 75, Def. 2.3.2</td>
<td>$\emptyset A = {x</td>
<td>x \in A}$</td>
</tr>
<tr>
<td>Union</td>
<td>p. 77, Def. 2.3.5</td>
<td>$\mathcal{U} \mathcal{F} = {x</td>
<td>\exists A : \mathcal{F}. x \in A}$</td>
</tr>
</tbody>
</table>

Legend: $A$ and $B$ are sets and $\mathcal{F}$ is a family (as a set) of sets

---

**Figure 3.** Definition of subset, power set, and union (of a family of sets), from [50].

<table>
<thead>
<tr>
<th>Explanation</th>
<th>Givens</th>
<th>Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0. Proving $\mathcal{U} \mathcal{F} \subseteq B \Rightarrow \mathcal{F} \subseteq \emptyset B$</td>
<td>$\mathcal{U} \mathcal{F} \subseteq B$</td>
<td>$\mathcal{F} \subseteq \emptyset B$</td>
</tr>
<tr>
<td>1. Goal 0 is $\forall x. x \in \mathcal{F} \Rightarrow x \in \emptyset B$</td>
<td>$\mathcal{U} \mathcal{F} \subseteq B$</td>
<td>$x \in \emptyset B$</td>
</tr>
<tr>
<td>2. Goal 1 is $\forall y. y \in x \Rightarrow y \in B$</td>
<td>$\mathcal{U} \mathcal{F} \subseteq B$</td>
<td>$x \in \emptyset B$</td>
</tr>
<tr>
<td>3. Try stronger goal $y \in \mathcal{U} \mathcal{F}$</td>
<td>$\mathcal{U} \mathcal{F} \subseteq B$</td>
<td>$y \in \mathcal{U} \mathcal{F}$</td>
</tr>
<tr>
<td>4. Expand goal 3</td>
<td>$\mathcal{U} \mathcal{F} \subseteq B$</td>
<td>$\exists A \in \mathcal{F}. y \in A$</td>
</tr>
<tr>
<td>5. Observe that $x$ is a witness for $A$ in goal 4, so the proof is done.</td>
<td>$\mathcal{U} \mathcal{F} \subseteq B$</td>
<td>$y \in x$</td>
</tr>
</tbody>
</table>

(a) Scratch work

**Theorem.** Suppose $B$ is a set and $\mathcal{F}$ is a family of sets. If $\mathcal{U} \mathcal{F} \subseteq B$ then, $\mathcal{F} \subseteq \emptyset B$.

**Proof.** Suppose $\mathcal{U} \mathcal{F} \subseteq B$. Let $x$ be an arbitrary element of $\mathcal{F}$. Let $y$ be an arbitrary element of $x$. Since $y \in x$ and $x \in \mathcal{F}$, clearly $y \in \mathcal{U} \mathcal{F}$. But then since $\mathcal{U} \mathcal{F} \subseteq B$, $y \in B$. Since $y$ was an arbitrary element of $x$, we can conclude that $x \in B$, so $x \in \emptyset B$. But $x$ was an arbitrary element of $\mathcal{F}$, so this shows that $\mathcal{F} \subseteq \emptyset B$, as required.

(b) Final writeup

---

**Figure 4.** Construction of a rhetorical proof and the final writeup, from [50, pp. 119–120].
Theorem 0.20 If there is a one-to-one function on a set $A$ to a subset of a set $B$ and there is also a one-to-one function on $B$ to a subset of $A$, then $A$ and $B$ are equipollent.

Proof Suppose that $f$ is a one-to-one map of $A$ into $B$ and $g$ is one-to-one on $B$ to $A$. It may be supposed that $A$ and $B$ are disjoint. The proof of the theorem is accomplished by decomposing $A$ and $B$ into classes which are most easily described in terms of parthenogenesis. A point $x$ (of either $A$ or $B$) is an ancestor of a point $y$ iff $y$ can be obtained from $x$ by successive application of $f$ and $g$ (or $g$ and $f$). Now decompose $A$ into three sets: let $A_E$ consist of all points of $A$ which have an even number of ancestors, let $A_O$ consist of points which have an odd number of ancestors, and let $A_I$ consist of points which have infinitely many ancestors. Decompose $B$ similarly and observe: $f$ maps $A_E$ onto $B_O$ and $A_I$ onto $B_I$, and $g^{-1}$ maps $A_O$ onto $B_E$. Hence the function which agrees on $f$ on $A_E \cup A_I$ and agrees with $g^{-1}$ on $A_O$ is a one-to-one map from $A$ onto $B$. 

The intuitively elegant form of the proof of theorem 0.20 is due to G. Birkhoff and S. MacLane. John L. Kelley, General Topology, pp. 28–29. Van Nostrand (1955)
\[(\exists f : A \to B . \text{inj } f) \land (\exists g : B \to A . \text{inj } g) \Rightarrow \exists h : A \to B . \text{bij } B \ h\]

(Enter scope)
\[\forall f : (A \to B) . \text{inj } f . \forall g : (B \to A) . \text{inj } g . \exists h : A \to B . \text{bij } B \ h\]

⇒ (Introd. H)
\[\exists h : A \to B . \ h \in \mathcal{R} \ H \wedge \text{bij } B \ h\]

where H := \{F : \mathcal{P} A . \ fF \cup gA\ F\}

(Change var. \(\exists\))
\[\exists F : \mathcal{D} \ H . \ H \in A \to B \wedge \text{bij } B \ F \wedge \mathcal{R} \ F\]

(Elimin. H)
\[\exists F : \mathcal{P} A . \ \mathcal{F} A \ F = A \ \mathcal{F} A \wedge gA \ F = B \ \mathcal{F} F\]

(Elimin. \(\tilde{g}\))
\[\mathcal{F} A \ F = A \ \mathcal{F} \ (B \ \mathcal{F} F)\]

(Calc. fixpt.)
\[\exists F : \mathcal{P} A . \ F = \mathcal{U} \{C : \mathcal{P} A | C \subseteq A \ \mathcal{F} (B \ \mathcal{F} C)\}\]

(One-pt. rule \(\exists\))
\[\mathcal{U} \{X : \mathcal{P} S \in \mathcal{P} S\} \in \mathcal{P} S\]

\[\mathcal{U} \{X : \mathcal{P} S | p \in \mathcal{P} S\} \in \mathcal{P} S\]

Figure 7. Top level of a calculational proof for the Schröder-Bernstein theorem.

- Right distributivity ⇒∃: (∃v:S . p) ⇒ q = ∃v:S . p = q (v not free in q)

This aspect is mentioned only to illustrate how informative this style is to a reader.

Discussion
In the rhetorical proof of Figure 4, “clearly” and “we can conclude that” stand for omitted steps, justifications, and references to the definitions for the reader to figure out. The structure, visible in the scratch work, is absent in the final writeup. Most of the prose amounts to fragmenting the logical and circumventing quantified expressions instead of using their power. The given/goal approach covers only one direction at a time.

The calculation in Figure 5 is only slightly longer but much more detailed; nothing is left unexplained. Of course, shortcuts are possible, similar to Figure 2(c).

Finally, guided by the shape of the expressions (“letting the symbols do the work”), the end result was discovered by calculation, as in Figure 2, and the resulting theorem is stronger because it is a logical equality, not just implication, as in the task description. We also found a general purpose intermediate result for \(\cup\text{-conversion}\): \(\cup F \subseteq B \equiv \forall A : F . A \subseteq B\) (used in a later example). As a general purpose result, it merits factorizing out as a separate theorem.

A More Advanced Example: The (Cantor-) Schröder-Bernstein Theorem
This example is illustrative for many reasons. Books on set theory mention it as the first theorem whose proof is “difficult” [47, p. 94]. Lamport [33, p. 604] reports a proof by Kelley [30], noting that it is terse because it is written for a “more sophisticated audience”, but also that the refinement needed to explain it in an introductory class reveals that it is wrong! This proof should be readable via Google Books, but pages are often hidden, so we quote the text in Figure 6.

Some may call such text a proof, but at best it is only a proof outline. Discarding details as “evident” can cause errors. An informal proof has the effect that one may tend to read it only as a series of hints, mentally patching errors along the way by letting some intuitive interpretation prevail over what is actually written. Educationally, of course, deliberate sloppiness sets very poor examples, unlike honest mistakes, since nothing can guarantee total absence of errors.

Turning an informal outline into a genuine proof requires formalizing and refining the proof obligations, which increases the length dramatically, in this case several pages [33, p. 604]. Using auxiliary concepts such as even/odd numbers or chains [13] causes extra proof obligations. Therefore we try using only concepts from the theorem’s domain of discourse: set theory.

The top level of a calculational proof is shown in Figure 7. Little triangles stand for parts (underlined for easy visual retrieval) not copied from an earlier line to avoid repetition. This top-level proof is shown only to convey the flavor and should be seen in the spirit of Figure 1, since giving a list of all formal definitions and further expansion is far beyond the scope of this paper. Readers familiar with the theorem will infer most notation. The central ones are:

- \(f\ F\) for restriction of function \(f\) to set \(F\) (definition via abstraction: \(f\ F : x : F \cap F . f x\))
- \(\mathcal{F} F\) for the image of set \(F\) under function \(f\); thus \(\mathcal{F} F = \{f x | x : D \cap F . f x\}\)
- \(A \ \mathcal{F} F\) for set difference: \(x \in A \ \mathcal{F} F \equiv x \in A \wedge x \notin F\)
- \(F \cup g\) to merge \(f\) and \(g\) (if functions are seen as sets of pairs, \(D \cap D g \equiv \emptyset \Rightarrow f \cup g = f \cup g\))
- A twelve-page note with all formal definitions, the derivation of the associated toolkit, the expansion to lower-level detail of Figure 7, the derivation of several variants for the (Elimin. \(\tilde{g}\))-step, and
**Theorem** Let $A$ be a set and $s : \mathcal{P} A \rightarrow \mathcal{P} A$ isotonic ($\forall C, C' : (\mathcal{P} A)^2. C \subseteq C' \Rightarrow sC \subseteq sC'$). If we define $D := \{ C : \mathcal{P} A | C \subseteq sC \}$, then $F := \bigcup D$ satisfies $F = sF$.

**Proof** With the given assumptions of the theorem, $F \in \mathcal{P} A$ (lemma as exercise) and $F = \bigcup D \Rightarrow \text{Weakening to } \subseteq \bigcup D \subseteq F$

- $\equiv \langle \text{U-elimination} \rangle \forall C : D. C \subseteq F$
- $\Rightarrow \langle \text{Isotony } s \rangle \forall C : D. sC \subseteq sF$
- $\equiv \langle \text{Definition } D \rangle \forall C : D. C \subseteq sC \wedge sC \subseteq sF$
- $\Rightarrow \langle \text{Transitivity } \subseteq \rangle \forall C : D. C \subseteq sF$
- $\equiv \langle \text{U-introduction} \rangle \bigcup D \subseteq sF$
- $\equiv \langle F = \bigcup D \rangle F \subseteq sF \quad (\text{a})$

$\Rightarrow \langle \text{Isotony } s \rangle sF \subseteq s(sF) \quad \text{— Alternative format: } \Delta \wedge sF \subseteq s(sF)$

- $\equiv \langle \text{Definition } D \rangle sF \in D$
- $\Rightarrow \langle \text{U-subset} \rangle sF \subseteq \bigcup D$
- $\equiv \langle F = \bigcup D \rangle sF \subseteq F$
- $\Rightarrow \langle (\text{a}), \text{antisym. } \subseteq \rangle F = sF$

Figure 8. A simple fixpoint property.

Matching fixpoint calculations can be obtained by sending an email to the author. Still, as promised, we illustrate the use of $\bigcup$-conversion in proving the fixpoint property for (Calc. fixpt.) in Figure 8. Although $\bigcup$-conversion is a logical equality, we rename it $\bigcup$-elimination or $\bigcup$-introduction, depending on the direction of the calculation, as extra reading aids.

A few simple conventions, not elaborated here, reduce writing, unless one prefers writing expressions in full as is customary in calculus.

**Intermezzo: When Is a “Proof by Contradiction” a Proof by Contradiction?**

The strategies of proving $p \Rightarrow q$ as $(p \wedge \neg q) \Rightarrow q$ or, by the deduction theorem, assuming $p$ and $\neg q$ and proving $q$ are potentially very powerful due to the “free” extra hypothesis $\neg q$. This free bonus explains the near-lyrical acclamations of famous mathematicians like Godfrey H. Hardy. Hence the following is likely to raise some hackles but hopefully will lead to fruitful discussion.

By reflecting the logical structure explicitly at all levels, the calculational style teaches us to consider more critically how far the potential of the “$\neg q$ hypothesis” is effectively exploited, in particular to obtain simple(r) proofs. It turns out that in many theorems typically presented as examples, the true essence does not depend on the $\neg q$ hypothesis and even provides “free” generalizations by simply discarding that hypothesis. Often the extraction of the central concept is made more informative by recasting it in problem form, making the original theorem an evident corollary. Here are some examples, the first being inspired by [50], yet adapted here.

- Euclid’s theorem saying that the number of primes is infinite. More informative is: "For any prime $p$, each prime factor of $1 + \prod \{ q : \text{Primes} | q \leq p \}$ is larger than $p$."

  The proof of this variant is the essence of Euclid’s classic proof yet does not depend on the $\neg q$ hypothesis. The original theorem is a direct corollary.

- The irrationality of $\sqrt{2}$. Two variants stated as problems (discussed later) are:
  - When is the square root of an integer rational? (Answer: iff it is an integer)
  - Given a prime $p$, for which positive natural $k$ is $\sqrt[k]{p}$ rational? (Answer: iff $k = 1$)

- Thousands of years more recent are typical proofs in abstract algebra about uniqueness of unit elements or inverses. Many texts typically assume, for instance, that $u \neq u'$, then prove $u = u'$ without using $u \neq u'$ and emphatically conclude “Contradiction!”.

A hypothesis is spurious in a proof if it can be eliminated while maintaining the original simplicity.

Returning to the irrationality of $\sqrt{2}$, comparing the simplicity for different strategies requires that all details are included (within the proof or as lemmas) down to basic principles.

Most proofs proceed via the existence of a reduced fraction of natural numbers whose square is 2 and derive a contradiction with the fraction
Theorem 6.4.5. \( \sqrt{2} \) is irrational.

Proof. Suppose that \( \sqrt{2} \) is rational. This means that \( \exists q \in \mathbb{Z}^+ \exists p \in \mathbb{Z}^+ (p/q = \sqrt{2}) \), so the set \( S = \{ q \in \mathbb{Z}^+ | \exists p \in \mathbb{Z}^+ (p/q = \sqrt{2}) \} \) is nonempty. By the well-ordering principle, we can let \( q \) be the smallest element of \( S \). Since \( q \in S \), we can choose some \( p \in \mathbb{Z}^+ \) such that \( p/q = \sqrt{2} \). Therefore \( p^2/q^2 = 2 \), so \( p^2 = 2q^2 \) and therefore \( p^2 \) is even. We now apply the theorem from Example 3.4.2, which says that for any integer \( x \), \( x \) is even iff \( x^2 \) is even. Since \( p^2 \) is even, \( p \) must be even, so we can choose some \( p \in \mathbb{Z}^+ \) such that \( p = 2\bar{p} \). Therefore \( p^2 = 4\bar{p}^2 \), and substituting this into the equation \( p^2 = 2q^2 \) we get \( 4\bar{p}^2 = 2q^2 \), so \( 2\bar{p}^2 = q^2 \) and therefore \( q^2 \) is even. Appealing to Example 3.4.2 again, this means that \( q \) must be even, so we can choose some \( q \in \mathbb{Z}^+ \) such that \( q = 2\tilde{q} \). But then \( \sqrt{2} = p/q = (2\tilde{p})/(2\tilde{q}) = \tilde{p}/\tilde{q} \), so \( \tilde{q} \in S \). Clearly \( q < q \), so this contradicts the fact that \( q \) was chosen to be the smallest element of \( S \). Therefore \( \sqrt{2} \) is irrational. \( \square \)

Figure 9. A typical traditional proof of the irrationality of \( \sqrt{2} \).

being reduced. Such a proof [31, p. 295] is quoted in Figure 9. The proof of Example 3.4.2 (x is even iff \( x^2 \) is even) is an essential part, and its scratch work takes nearly two pages. It is based on even/odd distinction and hence is not generalizable to the theorem \( p \) divides \( a^2 \) implies \( p \) divides \( a \), which Sagher [45] identifies as a crucial result about primes not available in the classical age.

Lamport [33] gives a slightly different proof, with more detail, using properties of the gcd.

The example in Figure 10(b) of free generalization and avoidance of spurious contradictory hypotheses is inspired by the essence of Sagher’s remarkably simple argument, still by contradiction [45], shown in Figure 10(a). Sagher tacitly uses a basic theorem for integers [28]: for \( d > 0 \), the integer quotient \( q := D/d \) and remainder \( r := D – dq \) satisfy \( 0 \leq r < d \). For easy reference, we call this Euclid’s Integer Division Theorem (IDT),\(^1\)

In Figure 10(b) this is made very explicit by extensive detail and by introducing redundant \( D \), \( q \), \( r \) rather than writing \( d\sqrt{n} – d(d/\sqrt{n} + d) \) in full, all making the ideas clearer to the reader.

For tidiness and WLOG, rationality issues are better handled via natural numbers (\( \mathbb{N} = \mathbb{Z}_{\geq 0} \)), but here we decided to remain close to the formulations quoted from the literature sources.

To solve the problem Given a prime \( p \), for which positive natural \( k \) is \( \sqrt[p]{p} \) rational?, the simplest approach is to introduce a multiplicity function \( \mu \) such that \( \mu_p \) \( n \) is the multiplicity of a prime factor \( p \) in a nonzero natural number \( n \). Existence and uniqueness follow from the unique factorization theorem [28, p. 19]. The properties of interest here are \( \mu_p \) \( p = 1 \) and the logarithmic-style rules \( \mu_p (n \cdot m) = \mu_p n + \mu_p m \) and \( \mu_p n^k = k \cdot \mu_p n \). This makes the solution a simple exercise (Figure 11).

\(^1\)Generalizable to reals: for any real \( D \) and \( d \), \( d \neq 0 \), there exist unique integer \( q \) and real \( r \) satisfying \( D = dq + r \) and \( 0 \leq r < |d| \), which justifies jointly defining div (\( \div \)) by \( D \div d = q \) and mod (\( \mod \)) by \( D \mod d = r \) [9].

\(^2\)Here standing for “quod est demonstrandum.”
Problem Given a prime $p$, for which positive natural $k$ is $\sqrt{k}$ rational?

Solution For any prime number $p$ and any nonzero natural number $k$,

$$\sqrt{k} \equiv \text{(Def. Ratpos)} \quad \exists n, m : \mathbb{N}_0 \cdot \exists D : \mathbb{Z} \cdot D = d\sqrt{n}$$

$$\equiv \text{(One-pt. rule 3)} \quad \exists d : \mathbb{N}_0 \cdot d\sqrt{n} \in \mathbb{Z}$$

$$\equiv \text{(Prop. calc., def. <)} \quad \forall d' : \mathbb{N} \cdot d' < d \Rightarrow d'\sqrt{n} \in \mathbb{Z} \Rightarrow d' \leq d'$$

$$\Rightarrow \text{(Inst. } d' := r, 0 \leq r \text{)} \quad \exists \text{let } q := D + d ; r := D - dq \text{ in } r < d \wedge r\sqrt{n} \in \mathbb{Z} \Rightarrow r = 0$$

$$\equiv \text{(Simplify)}$$

$$\equiv \text{(By IDT: } r < d \text{)} \quad \forall r\sqrt{n} \in \mathbb{Z} \Rightarrow r = 0$$

$$\equiv \text{(trading sub } 0 \text{)} \quad \forall d' : \mathbb{N} \cdot \exists d^2, p \in \mathbb{Z} \cdot d' = d \Rightarrow d = d'$$

$$\Rightarrow \text{(Inst. } d' := D + d \text{)} \quad \forall d' \in \mathbb{Z} \cdot d' \neq 0$$

$$\equiv \text{(Arithmetic)} \quad \exists n, m : \mathbb{Z} \cdot 1 = k \cdot \mu_p n$$

$$\equiv \text{(Dummy change)} \quad \exists j : \mathbb{Z}_0 \cdot 1 = k \cdot j$$

$$\equiv \text{(k > 0)} \quad k = 1$$

The converse, $\sqrt{n} \in \mathbb{Z} \Rightarrow \text{Ratpos } \sqrt{n}$, is trivial (technically). So the answer is $\text{Ratpos } \sqrt{n} \equiv \sqrt{n} \in \mathbb{Z}$.

Figure 10. Illustrating the avoidance of spurious contradictory hypotheses.

Problem Given a prime $p$, for which positive natural $k$ is $\sqrt{p}$ rational?

Solution For any prime number $p$ and any nonzero natural number $k$,

$$\sqrt{p} \equiv \text{(Def. Ratpos)} \quad \exists n, m : \mathbb{N}_0 \cdot \exists D : \mathbb{Z} \cdot D = d\sqrt{n}$$

$$\equiv \text{(Arithmetic)} \quad \exists n, m : \mathbb{N}_0 \cdot p \cdot m^k = n^k$$

$$\Rightarrow \text{(Leibniz)} \quad \exists n, m : \mathbb{N}_0 \cdot p \cdot (p \cdot m^k) = \mu_p n^k$$

$$\equiv \text{(Properties } \mu \text{)} \quad \exists n, m : \mathbb{N}_0 \cdot 1 + k \cdot \mu_p m = k \cdot \mu_p n$$

$$\equiv \text{(Arithmetic)} \quad \exists n, m : \mathbb{N}_0 \cdot 1 = k \cdot (\mu_p n - \mu_p m)$$

$$\Rightarrow \text{(Dummy change)} \quad \exists j : \mathbb{Z}_0 \cdot 1 = k \cdot j$$

$$\equiv \text{(k > 0)} \quad k = 1$$

Figure 11. Characterizing the rational roots of a prime number.

as a rather slick and fashionable way in rhetoric (politics, law) to discredit an opponent’s viewpoints and was similarly considered a clever ploy in mathematics. Archimedes [44] is reputed to have occasionally “poisoned” his reports with deliberate mistakes to challenge his reader(s) [41]. Perhaps even more subtle is hiding the intuitive thinking leading to a discovery, and what better way to cover one’s tracks than a proof by contradiction? In a fascinating account of rediscovered writings by Archimedes, The Archimedes Codex [41, p. 143] contains an example of a spurious contradiction, although it is unclear whether that reflects Archimedes’ argument.

To conclude, proofs by contradiction should not be rejected but handled with care.
Problem Given $P := [L, U]$, $N := [-U, -L]$, $s : \mathbb{R}_{>0}$. Simplify as far as possible:
\[ \exists f : P \cup N \cdot \exists f' : P \cup N \cdot k, k' : \mathbb{Z}_2 \cdot f - ks = f' - k's. \] (A)

Solution The inner quantification is simplified by writing $f - ks = f' - k's$ as $f - f' = (k - k')s$. The change of variables $\ell := k - k'$ results in $\exists \ell : \mathbb{Z}_{s0} \cdot f - f' = \ell s$.

In expanding the outer quantifications, we exploit symmetry. Also introducing the convenient short-hand $Q_x \equiv \exists \ell : \mathbb{Z}_{s0} \cdot x = \ell s$, domain split for the first $\exists$ yields
\[ (\exists f : P \cdot \exists f' : P \cup N \cdot Q_{f-f'}) \lor (\exists f : N \cdot \exists f' : P \cup N \cdot Q_{f-f'}). \]

Letting $f, f' := f - f', f'$, the second term becomes $\exists f : P \cdot \exists f' : N \cup P \cdot Q_{f-f'}$. Now $Q_{x} \equiv Q_x$ and $N \cup P = P \cup N$, so both terms are equal and collapse since $p \lor p \equiv p$.

By another domain split for $\exists$, we first simplify the second (more interesting) term. Here $f \in P \land f' \in N$ implies $f - f' \in [2L, 2U]$, which we add as a conjunct to help remove $f$ and $f'$ as follows:
\[ \exists f : P \cdot \exists f' : N \cdot \exists \ell : \mathbb{Z}_{s0} \cdot x = \ell s \land f - f' \in [2L, 2U] \] (Leibniz)
\[ \exists f : P \cdot \exists f' : N \cdot \exists \ell : \mathbb{Z}_{s0} \cdot f - f' = \ell s \land \ell s \in [2L, 2U] \] (Swap $\exists$)
\[ \exists \ell : \mathbb{Z}_{s0} \cdot \exists f : P \cdot \exists f' : N \cdot f - f' = \ell s \land \ell s \in [2L, 2U] \] (Distrib. $\lor/\exists$)
\[ \exists \ell : \mathbb{Z}_{s0} \cdot \ell s \in [2L, 2U] \land \exists f : P \cdot \exists f' : N \cdot f - f' = \ell s \] (Arithmetic)
\[ \exists f : P \cdot \exists f' : N \cdot f = f' \land \ell s \] (One-pt. rule $\exists$)
\[ \exists f' : P \cdot f - \ell s \in N \] (Eliminate $f$)
\[ \exists \ell : \mathbb{Z}_{s0} \cdot \ell s \in [2L, 2U] \] (B)

By $\forall v : [2L, 2U] \cdot \exists f : P \cdot f - v \in N$ (proof: witness $f := v/2$)

Similarly using $f \in P \land f' \in N \Rightarrow f - f' \in [-B, B]$, the first term of (B) becomes
\[ (\exists \ell : \mathbb{Z}_{s0} \cdot \ell s \in [-B, B]) \lor (\exists \ell : \mathbb{Z}_{s0} \cdot \ell s \in [2L, 2U]) \] (B).

By symmetry (as before), the first term equals $\exists \ell : \mathbb{Z}_{s0} \cdot \ell s \in [0, B]$. From $s \geq 0$ we get $\ell s \in [0, B] \Rightarrow \ell \geq 0$ and $\ell s \in [2L, 2U] \Rightarrow \ell \geq 0$. Hence, by (Trading sub $\exists$),
\[ (\exists \ell : \mathbb{N}_{s0} \cdot \ell s \in [0, B]) \lor (\exists \ell : \mathbb{N}_{s0} \cdot \ell s \in [2L, 2U]) \] (B).

A last simplification requires checking a hunch that the first term is absorbed.
\[ \exists \ell : \mathbb{N}_{s0} \cdot \ell s \in [0, B] \] (B).

\[ \Rightarrow (\text{Eliminate } \ell) \quad s \leq B \quad \text{— Converse simple but not needed} \] (B).

\[ \let \ell \ell \Rightarrow (\exists \ell : \mathbb{N}_{s0} \cdot \ell s \leq B) \Rightarrow s \leq B \] (proof: $\forall \ell : \mathbb{N}_{s0} \cdot \ell s \leq B \Rightarrow s \leq B \land \ell \leq B$)

\[ \Rightarrow (\text{Weaken: } B \leq 2B) \quad s \leq 2B \] (B).

\[ \equiv (\text{Euclid IDT}) \quad 0 \leq 2U - (2U \div s)s < s \land s \leq 2B \] (B).

\[ \equiv (\text{Arithm. } \leq) \quad 2L < (2U \div s)s \leq 2U \land s \leq 2B \] (B).

\[ \Rightarrow (2U \div s \in \mathbb{N}_{s0}) \quad \exists \ell : \mathbb{N}_{s0} \cdot \ell s \in [2L, 2U] \] (B).

Seeing no further simplification, we present the final result as
\[ \exists \ell : \mathbb{N}_{s0} \cdot 2L/\ell \leq s \leq 2U/\ell. \] (C)
Solving a Small But Puzzling Problem in Bandpass Sampling

This example shows that “back-of-an-envelope” calculations (quick calculations to resolve an unclear practical issue) are helpful not only with algebra or calculus but also for problems with a logical flavor.

As noted in a recent historical survey [17], sampling has interested scientists since Cauchy. In principle, a bandlimited signal can be faithfully reconstructed from samples taken at a rate that is at least twice the bandwidth. However, guaranteeing the absence of aliasing requires intricate choices. Indeed, in a renowned textbook, Lyons [39] points out many incorrect results in the literature, showing that the problem is puzzling and not evident to get right. He also presents a correct answer but based on exactly the same kind of informal arguments as the faulty ones. We show here how a formal solution resolves the issue in a straightforward way.

We first recall some well-known results. Let a signal \( x \) with Fourier transform \( \mathcal{F}x \) be sampled with sampling period \( T \) (or rate \( s := 1/T \)). The usual model is pointwise multiplication with delta functions distance \( T \) apart. The sampled version \( \hat{x}_T \) has Fourier transform \( \mathcal{F}\hat{x}_T \)

\[
(\mathcal{F}\hat{x}_T)(\omega) = \int_{-\infty}^{\infty} x(t) \sum_{k=-\infty}^{\infty} e^{jkt}\omega e^{-jot} dt
\]

\[= \sum_{k=-\infty}^{\infty} (\mathcal{F}x)(\omega - k\Omega) \text{ where } \Omega := 2\pi/T. \tag{1}\]

This shows that \( \mathcal{F}\hat{x}_T \) consists of an infinite number of replicas of \( \mathcal{F}x \), spaced \( \Omega \) apart. The usual reconstruction with sinc functions is faithful if and only if the replicas do not overlap.

The resulting requirements for bandpass sampling are formalized as follows. For convenience, we switch from \( \omega - k\Omega \) to \( f - ks \) (\( \omega = 2\pi f, \Omega = 2\pi s, s > 0 \)). A signal is bandlimited if its spectrum is zero outside an interval \( P := [L, U] \), called the support (for the positive frequencies), and the bandwidth is \( B := U - L \).

Including the negative frequencies \( N := [-U, -L] \), the support is \( S := P \cup N \), so the support \( R_k(s) \) for the \( k \)-th replica is given by \( R_k(s) = \{f - ks \mid f \in S\} \). Sampling rates \( s \) causing (undesirable) overlap are given by

\[
(2) \text{ Overlap}(s) \equiv \exists k, k': \mathbb{Z}^2 \ni \exists f : \mathbb{R}, f \in R_k(s) \cap R_{k'}(s),
\]

which is our basic form. The problem is deriving a practical engineering formula.

The calculation is done in two phases. The first is shown in Figure 12. Every step reflects head calculation, with details in the appendix, which also shows the second phase in the hierarchical layout illustrated thus far. For a change, Figure 13 shows how one might present it for readers less familiar with the rules for quantifiers than with those for sums or integrals (without prose, as in Figure 1). Although linked by prose, the steps are purely calculational: transforming expressions, guided by their shape. Proper quantifier rules preserve algebraic style, whereas the common reflex on seeing \( \exists x : S \cdot p \) is “Well, let’s pick such an \( x \).” The chain is even equational, and the reader may wish to compare this with handling rhs(2) \( \Rightarrow \) rhs(3) and rhs(3) \( \Rightarrow \) rhs(2) separately. Anyway, we have proven

\[
(3) \text{ Overlap}(s) \equiv \exists \ell : \mathbb{N}_{\geq 0}, 2L/\ell \leq s \leq 2U/\ell.
\]

Here we give some useful interpretations for special cases. For baseband sampling (\( L = 0 \) Hz, \( U = B \)), the formula yields the usual \( s > 2B \) requirement. The steps based on Euclid’s IDT show that this is also a necessary condition for bandpass sampling (extractable as a theorem). In view of (3), it is sufficient only if \( L \) is a multiple of \( B \) and provided the spectrum has no significant content at the edges \( L \) and \( U \) of the interval \( [L, U] \).

Two afterthoughts. The original “back-of-an-envelope” calculation grew significantly by adding detail. Also, given the many errors in the literature, how dependable are the calculations shown, not checked by another human or by computer? Yet, adhering to the diversity principle, we devised a 2D \( f, s \) diagram where every formal step has a geometrical interpretation, and (3) appears as a diagram via a simple coordinate transformation (horizontal shift, vertical scaling) without redrawing, as described in [8].

Note on Strategy and on Representativeness of Examples

Many examples show that maintaining logical equality (\( \equiv \)) saves the double effort of a ping-pong argument (exploring \( \Rightarrow \) and \( \Leftarrow \) separately). This is a bonus that is not available when the two directions inherently differ, as in the last line of Figure 10(b). Still, in exploration (outcome not known in advance), it is a good strategy to maintain \( \equiv \) as long as possible. Once a weakening or strengthening step is taken, the reverse will have to be handled separately, but that is no reason to abandon ship: internal subchains may still be equational and perhaps interesting as separate theorems.

Another issue is representativeness of illustrations. Readers of an earlier version wondered whether the examples might be too elementary. These were chosen to illustrate style only, and examples must not be so large as to require several articles by themselves (e.g., the Odd Order Theorem). Still, new examples are now added, some matching the least elementary theorems in a representative sample of nonspecialist texts on
theorem proving. In selecting from such texts and in other branches as well (real/complex analysis, from δ-ε arguments [10] to Hilbert spaces and applications in communications engineering), we found that the calculational style consistently made proofs and problem solving noticeably easier for both readers and writers.

A set of examples representative for all branches of mathematics would fill textbooks, but there is no doubt that every reader will find experimenting with logic as algebra rewarding.

Views on Formality, Calculation, Problem Solving and Proofs: The Cultural Divide

While it is true that the practical use of formal logic developed most strongly in computer science by necessity [16], it is arguably more accurate and constructive to recognize that the actual cultural divide lies within mathematics. Indeed, recall the main points of this article:

1. Precise notation deserves to be embraced rather than avoided. We learn symbols like + and = in first grade. Words are used only for reading equalities like 2 + 3 = 5 aloud but not for writing them. Why write “and” rather than ∧ or “for all” rather than ∀?

2. Formal rules permit us to reason with symbols, overcoming the drawbacks of informal arguments. They make the difference between mere syncopation and mature symbolism. Who would prove \((a + b) \cdot (a - b) = a^2 - b^2\) in words?

3. Propositional and quantifier calculus are as useful for logic-type arguments as algebra and integral calculus are for the kind of problems that interested Descartes and Newton.

Formal rules as perfected by Descartes have replaced word arguments in algebra to a large extent without fully eliminating prose, which also has a useful role. Such balance is equally relevant for logic reasoning, but today it is still heavily weighted on the syncopation side.

With the risk of stating the obvious, we emphasize that the class of problems considered are those whose formulation is already given in a symbolic setting or when translation is rewarding. By contrast, there are many informally stated problems whose formalization would entail a huge loss of abstraction and generality (except with a disproportionate amount of effort) yet which can be solved rigorously without symbolism. Every reader will have some favorite examples.

\(^3\)There are also many problems where a very direct ad hoc formal axiomatization enables applying formal logic without overhead.

The cultural divide of interest here is between symbolic reasoning in algebra/calculus and rhetoric logic. In this context, the distinction between formality in general and its embodiment in the calculational style hardly matters, and otherwise is sufficiently clear from the context.

Initial Reactions

Colleagues practicing formal reasoning (calculational or not [33],[35]) in nonspecialist mathematics report mutually similar initial reactions from other engineers and mathematicians who have always done the logical parts in prose. This issue appears even more sensitive than others with a formative element, which, as Mason observed for word problems [40], “sometimes provoke strong emotional reactions from otherwise calm adults.”

Interestingly, every single argument contra would make a no less valid case against the no less formal style exemplified by Figure 2, well-established for hundreds of years. One could easily leave it at that, letting the reader make the balance. However, a concrete list is instructive.

- **Formal reasoning uses symbols, making it less accessible to readers unfamiliar with them.** (i) Of course, every discourse must take into account the background of the intended audience. (ii) There are sound arguments for introducing basic logic symbols much earlier than is customary: truth tables are even simpler than addition tables.

- **Formal reasoning is not as natural as reasoning with words.** (i) The term “natural” is loaded; it usually refers to custom: no mathematician nowadays would consider Figure 2 less natural than a corresponding argument in words. (ii) Logic itself is not “natural”: Aristotle’s logic came not from nature but from hard thinking [44]. (iii) Everything “natural” can be improved; we may thank Diophantus for improving on “natural algebra”.

- **This is not how mathematicians work.** (i) Not all people work in the same way: many combine intuition from the domain of discourse with symbolic intuition, one enriching the other. (ii) Mathematics benefits as much as engineering from diversity in methods.

- **It is unclear how calculational arguments “scale up”.** Does algebra or logic scale up? Presumably the remark means How does one present a huge proof as one big calculation? It is doubtful whether anyone would want to do this: proofs with logic content, just like algebraic-style ones (Figure 2), form part of a larger whole. As in engineering, complexity and size are mastered by hierarchical design, modularization, and proper interfaces.
• **Formal reasoning requires adding much detail that may be uninteresting or distracting.** This is discussed in detail in [2], [33]: with modularization and hierarchical structuring, the writer can present his work such that the reader may focus on preferred parts.

• **The traditional mathematical style always worked fine for me.** (equally personal): For me too, but doing logic as algebra increases my pleasure in doing mathematics.

These initial reactions are the most representative ones, being also reported by others. As with any topic, much depends on individual temperament, background, and work area.

**Values, Beliefs, and Attitudes**

Values one would think to be generally shared by all people interested in mathematics often turn out to be a source of serious discord. Some are directly relevant to the issues in this article and hence deserve some discussion.

Morris Kline aptly noted that *More than anything else mathematics is a method.* [31, p. 454]. The purpose is not stated, but the completion for effective reasoning seems most appropriate and sufficiently general, and will be our main (not unique) basis for appreciation.

A seemingly obvious value is correctness. In computing science it is the ultimate criterion. In practice one often has to settle for less. Mathematicians may ask: when can one be certain that a solution or proof is correct? Except for very simple cases (who is to judge?), the only fitting answer is “never”. Still, checking by peers or by computer can increase confidence.

Utilitarian values only shift the issue. In the end, “the proof of the pudding is in the eating”, which does not mean just survival, but enjoyment. Mathematics is not easy (everyone hits walls), so a truly valid reason for doing it is pleasing the mind, which includes sharing. Arnol’d considers some of these efforts as wasting energy on (rock-climbing-type) exercises [1], but his viewpoint is selective and generally not as harsh as this quote suggests.

What one expects from a proof or a solution to a problem greatly influences how one approaches it or presents it. The idea of “persuading” or “convincing” someone else by a proof (or by an article!) is illusory: more pertinent is gaining (by the writer) or offering (to the reader) further insight and understanding, often helpful to draw conclusions with greater confidence.

The conclusions are central for typical proofs in computer science (e.g., program correctness), which are characterized by the need for attention to detail rather than conceptual depth. Newly written programs that, at first, seem “obviously correct” to their programmer (in this scenario assumed to be unfamiliar with formal methods) nearly always contain fatal flaws that appear equally obvious when pointed out.

Bentley illustrates this with binary search [7], finding an item in a sorted list using a very simple principle people instinctively use in everyday life: look at any place in the list (most efficiently halfway, but that is another issue), compare to the item sought, and, depending on the outcome, repeat the search in the part before or after the initially chosen place. In a test for experienced programmers, given ample time, only about 10 percent got it right. This example concerns a program of half a dozen lines, so what to think about millions of lines?

Depending on the problem, these elements benefit from obviating formalization via a direct insight conveyed in words or precisely reside in the formalization itself. For instance, the *point-free style*, i.e., expressions without domain variables, shown for set theory by Tarski and Givant [48], is appreciated for its elegant algebraic flavor, and it also works for quantifiers [10]. Lamport notes [35]: *A proof should be beautiful mathematics. Its beauty lies in its logical structure, not in its prose.*

Quoting some sources was felt useful since many people find the above criteria irrelevant.

One would expect less discord about diversity of perspectives, which (arguably more than anything else) enhances understanding and insight. Yet a colleague once criticized it as “confusing to students, since they would not know which method to use on an exam.” No comment.

Discord also exists about the value of “monster theorems”, the holy grail for some mathematicians. Arnol’d considers some of these efforts as *wasting energy on (rock-climbing-type) exercises* [1], but his viewpoint is selective and generally not as harsh as this quote suggests.

Much has been written about beauty and elegance in mathematics, which are ultimately subjective criteria, but common elements seem to be economy of argument (obviously not through omission, but by conceptual clarity), providing insight, and effort-free generalizability.

Bertrand Russell,

*The Study of Mathematics*

It is very desirable in instruction, not merely to persuade the student of the accuracy of important theorems, put to persuade him in a way which itself has, of all possible ways, the most beauty.

March 31, 1931
always appreciated by mathematicians (Kelley’s proof of the Schröder-Bernstein theorem comes to mind), yet sometimes it is driven home: all early ideas for proving Fermat’s last theorem failed in a “small” detail, and the story goes that a famous German professor replied to such proposals with polite letters that differed only in mentioning the line number identifying the error.

Since proofs for program correctness contain few internal results relevant for extraction, in the presentation the details are often provided “exactly where needed”, as emphasized in [2], [33]. Presenting them as a theorem or lemma before the main proof makes the less curious reader wonder about their purpose; presenting them afterwards forces the impatient reader to search.

In mathematics, physics, and engineering, insight and understanding are at least as important as the conclusions, and some internal results may be interesting by themselves. As is customary in papers and textbooks, such results are integrated as theorems in a little ad hoc theory or large general theory relevant to the domain of discourse. If sufficiently general, these theorems are placed before the “main” theorem (which may even have lost this status!); if routine, they can be placed as lemmas after the theorem (even for the impatient).

Students should learn to write in a way that is helpful to readers. Velleman’s How To Prove It: A Structured Approach [51] describes many ways to write better proofs. In the preface (page xii) he observes that indenting proofs helps to make the underlying structure of the proof clear. On page 92 he observes that Using the notation and rules of logic can be very helpful when you are figuring out the strategy for a proof. He emphasizes in both cases that mathematicians do not present their proofs that way and also that they just write the steps with no explanation of how they thought of them.

Perpetuating abuse of notation is a major factor in the cultural divide. A great disappointment to beginning students (and to anyone with a critical mind) is the discrepancy between the reputation of mathematics as being precise and the many abuses of notation that they are forced to accept during their formative years. The intellectual effort is usually minor, but faith in the “honesty” of mathematics as a scientific discipline is damaged by this double standard.

Perhaps more importantly, abuse of notation is an impediment to precise rules and formal reasoning beyond syncopation. This issue will be taken up later.

On Bridging the Gap
Linguistic Symbiosis

A very interesting remark by one of the reviewers is a reference to the work of Ganesalingam [19] and the conclusion that the problem really is the immaturity of computer processing of natural languages. As discussed next, the premises are wrong, but they do not prevent the work from being very valuable and a continuation of current “good practices” of letting formalisms and formalization attain symbiosis with natural language.

Quite misleading is [20, p. 3], contrasting a very unappetizing formula from Russell’s Principia with an example from a textbook called “modern” because it is more recent but otherwise uses a style that would have looked familiar two hundred years ago. This is just an anecdotal detail.

According to [19, p. 26], The primary function of symbolic mathematics is to abbreviate material that would be cumbersome to state with text alone. However, that precisely describes syncopation, the stage of algebra reached with Diophantus. What really counts is formal manipulation for algebra achieved by Viète and Descartes [4], [5]. This is not an expressiveness issue: \( a^2 - b^2 = (a + b) \cdot (a - b) \) can be written in words as The difference between the squares of two numbers equals the product of their sum and their difference (in the same order), which does not sound overly verbose. Not expressiveness, but convenience for reasoning led to symbolism.

Ganesalingam recognizes the superior expressiveness of formal mathematics and gives an example [19, p. 22] to show that formal expressions typical in integral calculus have no analogue in natural, computer, or other languages. Indeed, how would one read formula (1) in words? In integral calculus, expressiveness requires symbolism. Incidentally, in calculations, mathematicians are quite patient in repeating integral signs literally in successive lines, even where only the integrand changes.

By contrast, expressiveness is not a primary issue in quantified expressions: given a proper formalism, they can be conveniently read aloud in words (if there are not too many). At the same time, this may explain why they remained stuck in the syncopation stage [49] and in traditional reasoning are disassembled by the “let’s pick an \( x \)” reflex. When using quantifier calculus instead, successive lines look much like calculations with integrals, except that we have shown a little less patience with repeated quantifications that do not change.

The above considerations explain the following symbiosis, which we found very helpful. In teaching formalization of problems, we emphasize expressing them in precise natural language.
without contorted syntax, yet in such a way that (near-) literal formalization is possible. Conversely, our formalism is designed such that reading logical formulas aloud literally yields grammatically correct sentences. Interestingly, this significantly helped notational engineering.

A Small Example: Set Membership and Binding
This example illustrates the symbiosis between correct grammar in natural language and in formalism design.

Consider the following equations in equally familiar-looking notations:

(4) \[ S \subseteq T \equiv \forall x : S . x \in T, \]
(5) \[ S \subseteq T \equiv \forall x \in S . x \in T. \]

The right-hand side in (4) is read “For all \( x \) in \( S \), \( x \) is in \( T \)”, which is grammatically correct. Consistently, the right-hand side in (5) is read “For all \( x \) is in \( S \), \( x \) is in \( T \)”. Bad marks for grammar in grade school!

The problem in (5) is overloading \( \in \) for two purposes: as the usual set membership operator in \( x \in T \) and for introducing (declaring) a variable \( x \) by \( x \in S \). Overloading is avoided in (4) in an obvious manner. Of course, correct grammar is not the goal but is a good “sanity check”.

Technically, \( \in \) is the standard set membership operator and is better reserved for that purpose. Also, \( x \in T \), read “\( x \) is in \( T \)” or “\( x \) is an element (or member) of \( T \)”, is a proposition (a boolean expression for computer scientists) that may be true or false depending on \( x \) and \( T \).

By contrast, \( x : S \) is a binding (declaration). It has no truth value, but in \( \forall x : S . x \in T \) it has a scope, namely the part after the period. The binding specifies that \( x \in S \) within the scope.

Before revealing the bigger picture, we consider a more illustrative example.

The expressions \( \{ x : \mathbb{R} \mid x > 7 \} \) and \( \{ x^3 \mid x : \mathbb{N} \} \) directly look familiar; most readers will not even notice the colon, and all will read the expressions correctly. However, consider the general pattern. Let \( v \) be a variable, \( p \) a propositional expression, \( S \) and \( T \) sets, and \( e \) any expression. The general patterns for sets are \( \{ v : S \mid p \} \) and \( \{ e \mid v : T \} \). Many authors would write \( \{ v \in S \mid p \} \) and \( \{ e \mid v \in T \} \), but \( \{ x \in S \mid x \in T \} \) fits both patterns, an ambiguity! With proper distinction between binding and set membership, \( \{ x : S \mid x \in T \} = S \cap T \) and \( \{ x \in S \mid x : T \} \subseteq \mathbb{B} \) (booleans). Thinking that \( \{ x \in S \mid x : T \} \) is not useful is incorrect and not relevant: the issue is that the syntax in \( \{ x \in S \mid x \in T \} \) is ambiguous and that correction is very simple.

Intermezzo: The Bigger Picture
Here is an outline of the language design aspects of the formalism used in [10]. Expecting completeness would cause misleading oversimplification.

All abstraction is unified by function abstraction: \( \nu : S \mapsto p.e \) (\( \nu \) optional) denotes a function whose domain is the set of all \( \nu \) in \( S \) satisfying \( p \) and that maps \( \nu \) to \( e \). So \( n : \mathbb{Z} \land n > 0.2 \cdot n \) and \( n : \mathbb{N} \land 0.2 \cdot n \) both denote the function that doubles natural numbers, and \( f = x : \mathcal{D} f . f(x) \).

Predicates are boolean-valued functions. Quantifiers are predicates on predicates: if \( P \) is a predicate, \( \forall p \equiv P = x : \mathcal{D} P . 1 \) and \( \exists p \equiv P \neq x : \mathcal{D} P . 0 \). This supports pointfree expression (as for \( \sum f \), \( \lim_d f \), etc.) and the common pointwise style via function abstraction, as in (4).

The function range operator \( \mathcal{R} \) is defined by \( y \in \mathcal{R} f \equiv \exists x : \mathcal{D} f . y = f(x) \). A synonym is \( \{ \} \). Together with the “abbreviations” \( \nu : S \mapsto p \) for \( v : S \mapsto p . v \) and \( e \mapsto v : S \) for \( v : S . e \), this yields familiar-looking forms such as \( \{ v : S \mid p \} \) and \( \{ e \mid v : T \} \).

Warning: This outline does not cover foundational issues. The notation is intended to be “portable” across various set-theoretic foundations, from ZF [47] to NFU [29]. For instance, a simple interface to Suppes’s formulation comprises two quantifiers \( \forall \) and \( \exists \) from [47, pp. 3–4], used for that purpose only and not needed in “working-type” calculations, as in the examples. Further explanation would lead us astray from the subject matter of this article.

Still, this notational digression demonstrates the simplicity of designing unambiguous notation while maintaining close resemblance to the “average” of the various disparate notations throughout the literature. It also ensures the feasibility of the discipline proposed below.

The Bane of Notational Negligence
Notational negligence is a major impediment against eliminating the cultural divide. At the same time, it is the easiest to remove and hence a logical first step, were it not for various human factors discussed next.

In the literature and in many professions, a text full of spelling mistakes and grammatical errors is considered disgraceful, and reviewers as well as editors would do their best to ensure the necessary corrections. Yet, in mathematical texts, sloppy conventions are tolerated uncritically. Often one notices a supercilious attitude (others have called it machismo) whereby notational conventions are considered too unimportant to deserve some care, or poor notation is defended by (false) claims of “common practice”. Interestingly, such a casual attitude is typical in areas where discourse has
not yet progressed beyond the syncopation stage. Unfortunately, it creates an obvious vicious circle impeding the development of symbolic reasoning, where notation should be sufficiently trustworthy for uninterpreted use.

A pervasive form of sloppiness concerns some of the most useful forms of expression in mathematics: variables and functions. In particular, the distinction between free and bound variables and the rules governing them seem to be insufficiently appreciated.\(^4\) For instance, in [51] the definition “Truth set of \(P(x) = \{x \mid P(x)\}\)” has \(x\) occurring free on the left and bound on the right of the equality sign. The incorrect appearance of \((x)\) in the “truth set of \(P(x)\)” is analogous to talking about “the function \(f(x)\)” when actually meaning “the function \(f\)\,” which invoked the justified wrath of calculus professors fifty years ago.

Unfortunately, the good advice of these professors is still necessary today: it suffices to Google "function \(f(x)\)” or, even worse, "\(x = x(t)\)” to know how things stand.

In a widely disseminated paper about education in signal processing [36], Lee and Varaiya criticize evidently erroneous conventions in classical analysis:

Most texts call the expression \(x(t)\) a function. A better interpretation is that \(x(t)\) is an element in the range of the function \(x\). The difficulty with the former interpretation becomes obvious when talking about systems. Many texts pay lip service to the notion that a system is a function by introducing a notation like \(y(t) = T(x(t))\). This makes no distinction between the value of the function at \(t\) and the function \(y\) itself.

Why does this matter? Consider our favorite type of system, an LTI system. We\(^5\) write \(y(t) = x(t) * h(t)\) to indicate convolution. Under any reasonable interpretation of mathematics, this would seem to imply that \(y(t - \tau) = x(t - \tau) * h(t - \tau)\). But it is not so! How is a student supposed to conclude that \(y(t - 2\tau) = x(t - \tau) * h(t - \tau)\)? This sort of sloppy notation could easily undermine the students’ confidence in mathematics.

Still, some later texts in the same field stick to defective conventions, often pointing out the errors but perpetuating them with the (fortunately false) “common practice” argument.

Gold [21] reports that many students do not know the difference between a function and an expression until some computer algebra system forces them to recognize it.

One of the consequences observed in my own students is the difficulty in understanding higher-order functions, for instance, the fact that \(f(x)\) can be a function, e.g., if \(f\) is a higher-order function defined by \((f(x))(y) = x + y\) (all parentheses optional, but written for emphasis).

Similar misuse of notation is writing \(x[n]\) instead of just \(x\) for a sequence. Even worse, as noted in [36], is writing \(x[n]\) for the sampled version of a continuous signal written \(x(t)\) in the same context. The mathematics literature does not do any better: many authors write a sequence \(x\) as \(x_j\) or even as \(\{x_j\}\); one wonders how they would write the singleton set with \(x_j\).

Reviewers and editors of mathematical texts let all kinds of junk notation pass routinely, although this attitude is fully equivalent to accepting pidgin in literary texts.

A few authors even suggest that sloppy formalism is in the interest of the students, preparing them for worse to come. This is a dangerous misconception. The best basis for students to learn to cope with the wide diversity of often defective formalisms is a flawless formalism. Not only does this provide a reliable reference frame but it also instills a sense of discrimination [22], [36]. Defects create a disparity between the image of mathematics as being very exact and the actual practice of condoning sloppy and misleading use of notation. No wonder many students feel that symbolism is primarily obfuscation or even that mathematics is yet another hoax!

**Conclusion**

As for many issues, education is a crucial element in bridging a methodological gap. Ample evidence has been given that learning to practice faithful formalization and true symbolism (not syncopation) should be an integral part of mathematics education. Most effective for dealing with logical arguments is the ability to calculate with predicates and quantifiers as fluently as is customary for derivatives and integrals.

Suitable study material can be found on the Web, and there is also the excellent textbook by Gries and Schneider [23]. One reason why [23] should be complemented by material from the Web is that it does not cover pointfree expression (in particular for quantifiers) and a systematic approach to bindings (although it contains no errors in this respect). Yet there are high expectations for a new edition that has been in preparation for some years.

A further complement is provided by the many texts taking the more traditional approach, such as [51], because they are a treasure trove of

\(^4\)It is significant that lambda calculus is little known in mathematics yet is basic knowledge in computer science.

\(^5\)The pronoun “we” might be misleading; apparently the intended meaning is “some authors”.

---

188 Notices of the AMS Volume 60, Number 2
Figure 14. Bandpass sampling: (a') details of concretizing the support overlap formula.

interesting example problems that can improve any introductory course on constructing proofs. For the time being, such a combination of various sources is the best basis for helping students to develop a sense of discrimination and quality, together with proficiency in formulating problems and theories, in problem solving and in mathematical reasoning and discourse.

From a traditional perspective, these concerns might be considered university level, but thorough studies and contributions by others [3] have shown the desirability and feasibility of preparing symbolic reasoning already in high school.

Acknowledgement
The author wishes to thank Leslie Lamport for many useful suggestions.

Appendix: Organizing and Presenting Sizeable Calculations
For various reasons, large symbolic calculations are preferably neither elaborated nor presented in one stretch but are split into parts more convenient for both writer and reader. Obviously, this is common practice throughout the mathematics, physics, and engineering literature.

A few more remarks may be useful, using Figures 14 and 15 as examples. These figures directly illustrate how a larger calculation can be cut into parts at convenient places.

Even then, the calculations would not normally be presented with all details “in plain view”. In a hypertext environment, only top level justification(s) and resulting formulas would appear, to be expanded only when the reader wishes. In a regular text environment, the familiar forms of presentation have shown their value: splitting lower-level calculations as well into “chunks”, interspersing textual comments, and highlighting the main intermediate results as numbered formulas. For the example of Figure 15, the textbook variant would be Figure 13.

References
Given \( P := [L, U], N := [-U, -L] \), \( s : \mathbb{R}_{>0} \), we simplify (A) as follows.

\[
\exists f : P \cup N . \exists f' : P \cup N . \exists k, k' : \mathbb{Z}^2 . f - ks = f' - k's' \tag{A}
\]

\( \equiv \) (Simplify inner \( \exists \))

\[
\equiv (\text{Arithmetic}) \land \exists k, k' : \mathbb{Z}^2 . f - f' = (k - k')s
\]

\( \equiv (\text{Change var}) \land \exists \ell : \mathbb{Z}^2 . f - f' = \ell s
\]

\( \equiv (\text{Shorthand}) \quad \text{let} \quad Q_x \equiv \exists \ell : \mathbb{Z}^2 . x = \ell s \quad \text{in} \quad \exists f : P \cup N . \exists f' : P \cup N . Q_{f-f'}
\]

\( \equiv (\text{Exploit symmetry})
\]

\[
\equiv (\text{Dom. split} \ \exists) \land (\exists f : P . \exists f' : P \cup N . Q_{f-f'}) \lor (\exists f : N . \exists f' : P \cup N . Q_{f-f'})
\]

\( \equiv (\text{Change var}) \land (\exists f : P . \exists f' : P \cup N . Q_{f-f'}) \lor (\exists f : P . \exists f' : N \cup P . Q_{f-f'})
\]

\( \equiv (Q_{-x} \equiv Q_x) \land (\exists f : P . \exists f' : P . Q_{f-f'}) \lor (\exists f : P . \exists f' : N . Q_{f-f'})
\]

\( \equiv (\text{Dom. split} \ \exists) \land (\exists f : P . \exists f' : P . Q_{f-f'}) \lor (\exists f : P . \exists f' : N . Q_{f-f'}) \tag{B}
\]

\( \equiv (\text{Simplify 2nd term}) — \text{Priority for what seems the more interesting term}
\]

\[ \exists f : P . \exists f' : N . \exists \ell : \mathbb{Z}^2 . x = \ell s \]

\( \equiv \) (Arithmetic \(< \)) \exists f : P . \exists f' : N . \exists \ell : \mathbb{Z}^2 . f - f' = s \land f - f' \in [2L, 2U]

\( \equiv \) (Loc: \( L \leq f \leq U \land -U \leq f' \leq -L \Rightarrow 2L \leq f - f' \leq 2U
\]

\( \equiv \) (Leibniz) \exists f : P . \exists f' : N . \exists \ell : \mathbb{Z}^2 . f - f' = s \land s \in [2L, 2U]

\( \equiv \) (Eliminate \( f' \))

\( \equiv \) (Swap \( \exists \)) \exists \ell : \mathbb{Z}^2 . \exists f : P . \exists f' : N . f - f' = s \land f \in [2L, 2U]

\( \equiv \) (Distrib. \land \exists) \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U] \land \exists f : P . \exists f' : N . f - f' = s

\( \equiv \) (Arithmetic) \exists f : P . \exists f' : N . f' = f - s

\( \equiv \) (Eliminate \( f \)) \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U]

\( \equiv \) (Loc: \( \forall \ v : [2L, 2U] . \exists f : [L, U] . f - v \in [-U, -L]. \text{Proof: witness } f := v/2
\]

\( \equiv \) (Simplify 2nd term) \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U]

\( \equiv \) (Similarly 1st term) (\( \exists \ell : \mathbb{Z}^2 . \ell s \in [-B, B] \) \lor (\( \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U] \))

\( \equiv \) (Prune negative \( \ell \))

\( \equiv \) (Symmetry) (\( \exists \ell : \mathbb{Z}^2 . \ell s \in [0, B] \) \lor (\( \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U] \))

\( \equiv \) (Absorb 1st term) \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U]

\( \equiv \) (Loc: (\( \exists \ell : \mathbb{Z}^2 . \ell s \in [0, B] \) \Rightarrow (\( \exists \ell : \mathbb{Z}^2 . \ell s \in [2L, 2U] \))

\( \quad \Rightarrow (\text{Eliminate } \ell) \quad s \leq B
\]

\( \quad \Rightarrow (\text{Loc: } (\exists \ell : \mathbb{Z}^2 . \ell s \leq B) \Rightarrow s \leq B \) (proven:

\( \Rightarrow (\text{Weaken: } B \leq 2B) \leq 2B
\]

\( \equiv \) (Euclid IDT) \( 0 \leq 2U - (2U + s)s < s \land s \leq 2B
\]

\( \equiv \) (Arithm. \(< \) ) \( 2L < (2U + s)s \leq 2U \land s \leq 2B
\]

\( \Rightarrow (2U + s \in \mathbb{N}_0) \exists \ell : \mathbb{N}_0 . \ell s \in [2L, 2U]
\]

\( \equiv \) (Def. [ ], arithm.) \exists \ell : \mathbb{N}_0 . 2L/\ell \leq s \leq 2U/\ell \tag{C}

Figure 15. Bandpass sampling: (b’) stepwise simplification to obtain a practical formula.


On the Measurement of Intangibles. A Principal Eigenvector Approach to Relative Measurement Derived from Paired Comparisons

Thomas L. Saaty

Introduction

Nearly all of us have been brought up to believe that clear-headed logical thinking is our only sure way to face and solve problems. But experience suggests that logical thinking is not natural to us. Indeed, we have to practice, and for a long time, before we can do it well. Since complex problems usually have many related factors, traditional logical thinking leads to sequences of ideas so tangled that the best solution cannot be easily discerned.

For a very long time people believed and argued strongly that it is impossible to express the intensity of human feelings with numbers. The epitome of such a belief was expressed by A. F. MacKay who writes [12] that pursuing the cardinal approaches is like chasing what cannot be caught. It was also expressed by Davis and Hersh [5]: “If you are more of a human being, you will be aware there are such things as emotions, beliefs, attitudes, dreams, intentions, jealousy, envy, yearning, regret, longing, anger, compassion and many others. These things—the inner world of human life—can never be mathematized.” In their book [11]

Lawrence LeShan and Henry Margenau write: “We cannot as we have indicated before, quantify the observables in the domain of consciousness. There are no rules of correspondence possible that would enable us to quantify our feelings. We can make statements of the relative intensity of feelings, but we cannot go beyond this. I can say: I feel angrier at him today than I did yesterday. We cannot, however, make meaningful statements such as, I feel three and one half times angrier than I did yesterday. The physicists’ schema, so faithfully emulated by generations of psychologists, epistemologists and aestheticians, is probably blocking their progress, defeating possible insights by its prejudicial force. The schema is not false—it is perfectly reasonable—but it is bootless for the study of mental phenomena.”

The Nobel Laureate Henri Bergson [4] writes: “But even the opponents of psychophysics do not see any harm in speaking of one sensation as being more intense than another, of one effort as being greater than another, and in thus setting up differences of quantity between purely internal states. Common sense, moreover, has not the slightest hesitation in giving its verdict on this point; people say they are more or less warm, or more or less sad, and this distinction of more and less, even when it is carried over to the region of subjective facts and unextended objects, surprises nobody.”

Thomas L. Saaty holds the Chair of Distinguished University Professor at the University of Pittsburgh, with appointments in several departments. His email address is saaty@katz.pitt.edu.

DOI: http://dx.doi.org/10.1090/noti944
If we were to ask what in practical terms measurement means, one would most likely propose a scale that is applied to measure objects: a set of numbers, a set of objects and a mapping from the objects to the numbers. Then we can agree that appropriate judgment must be used to interpret the scale readings and use them in practice. So judgment is also essential. But there is another way to think of a scale.

Henri Lebesgue, who was concerned with questions of measure theory and measurement, wrote [10]:

“It would seem that the principle of economy would always require that we evaluate ratios directly and not as ratios of measurements. However, in practice, all lengths are measured in meters, all angles in degrees, etc.; that is we employ auxiliary units and,
Table 0. Fundamental scale of absolute numbers.

<table>
<thead>
<tr>
<th>Intensity of Importance</th>
<th>Definition</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Equal Importance</td>
<td>Two activities contribute equally to the objective</td>
</tr>
<tr>
<td>2</td>
<td>Weak or slight</td>
<td>Experience and judgment slightly favor one activity over another</td>
</tr>
<tr>
<td>3</td>
<td>Moderate importance</td>
<td>Experience and judgment strongly favor one activity over another</td>
</tr>
<tr>
<td>4</td>
<td>Moderate plus</td>
<td>An activity is favored very strongly over another; its dominance demonstrated in practice</td>
</tr>
<tr>
<td>5</td>
<td>Strong importance</td>
<td>The evidence favoring one activity over another is of the highest possible order of affirmation</td>
</tr>
<tr>
<td>6</td>
<td>Strong plus</td>
<td>A better alternative way to assigning the small decimals is to compare two close activities with other widely contrasting ones, favoring the larger one as needed a little value over the smaller one using the 1-9 values.</td>
</tr>
<tr>
<td>7</td>
<td>Very strong or demonstrated importance</td>
<td>A logical assumption</td>
</tr>
<tr>
<td>8</td>
<td>Very, very strong</td>
<td>When it is desired to use such numbers in physical applications. Alternatively, often one estimates the ratios of such magnitudes by using judgment</td>
</tr>
<tr>
<td>9</td>
<td>Extreme importance</td>
<td>When activities are very close a decimal is added to 1 to show their difference as appropriate</td>
</tr>
<tr>
<td>1.1-1.9</td>
<td>Reciprocals of above nonzero numbers assigned to i when compared with activity j, then j has the reciprocal value when compared with i</td>
<td></td>
</tr>
<tr>
<td>Measurements from ratio scales</td>
<td>When we deal with intangible factors, which by definition have no scales of measurement, we can compare them in pairs. Making comparisons is a talent we all have. Not only can we indicate the preferred object, but we can also discriminate among intensities of preference. The philosopher, Arthur Schopenhauer [25], said, “Every truth is the reference of a judgment to something outside it, and intrinsic truth is a contradiction.”</td>
<td></td>
</tr>
</tbody>
</table>

Lebesgue did not go far enough in examining why we have to compare. When we deal with intangible factors, which by definition have no scales of measurement, we can compare them in pairs. Making comparisons is a talent we all have. Not
Table 1. The family's pairwise comparison matrix for the criteria.

<table>
<thead>
<tr>
<th>Size</th>
<th>Trans.</th>
<th>Nbrhd.</th>
<th>Age</th>
<th>Yard</th>
<th>Modern Cond.</th>
<th>Finance</th>
<th>Normalized Priority Vector w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>1/3</td>
<td>1/4</td>
</tr>
<tr>
<td>Trans.</td>
<td>1/5</td>
<td>1</td>
<td>1/3</td>
<td>5</td>
<td>3</td>
<td>1/5</td>
<td>1/7</td>
</tr>
<tr>
<td>Nbrhd.</td>
<td>1/3</td>
<td>3</td>
<td>1</td>
<td>6</td>
<td>3</td>
<td>1/2</td>
<td>1/5</td>
</tr>
<tr>
<td>Age</td>
<td>1/7</td>
<td>1/5</td>
<td>1/6</td>
<td>1</td>
<td>1/3</td>
<td>1/4</td>
<td>1/7</td>
</tr>
<tr>
<td>Yard</td>
<td>1/6</td>
<td>1/3</td>
<td>1/3</td>
<td>3</td>
<td>1</td>
<td>1/2</td>
<td>1/5</td>
</tr>
<tr>
<td>Modern</td>
<td>1/6</td>
<td>1/3</td>
<td>1/4</td>
<td>4</td>
<td>2</td>
<td>1/5</td>
<td>1/6</td>
</tr>
<tr>
<td>Cond.</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>7</td>
<td>5</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>Finance</td>
<td>4</td>
<td>7</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

$\lambda_{max} = 8.811$, Consistency Ratio (C.R.) = .083

Table 2. Pairwise comparison matrices for the alternative houses.

<table>
<thead>
<tr>
<th>Size of House A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>Yard Space A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>Modern Facilities A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>General Condition A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>Financing A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>5</td>
<td>.743</td>
<td>1.000</td>
<td>A</td>
<td>1</td>
<td>6</td>
<td>.691</td>
<td>1.000</td>
<td>A</td>
<td>1</td>
<td>9</td>
<td>.770</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1/5</td>
<td>1</td>
<td>.194</td>
<td>0.261</td>
<td>B</td>
<td>1/6</td>
<td>1</td>
<td>.091</td>
<td>0.132</td>
<td>B</td>
<td>1/9</td>
<td>1</td>
<td>.068</td>
<td>0.088</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1/9</td>
<td>1/4</td>
<td>.063</td>
<td>0.085</td>
<td>C</td>
<td>1/4</td>
<td>3</td>
<td>.218</td>
<td>0.315</td>
<td>C</td>
<td>1/6</td>
<td>3</td>
<td>.162</td>
<td>0.210</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Transportation A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>Neighborhood A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>Age of House A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
<th>Financing A</th>
<th>B</th>
<th>C</th>
<th>Distributive Priorities</th>
<th>Idealized Priorities</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>9</td>
<td>.717</td>
<td>1.000</td>
<td>A</td>
<td>1</td>
<td>2</td>
<td>.200</td>
<td>0.500</td>
<td>A</td>
<td>1</td>
<td>1</td>
<td>.072</td>
<td>0.111</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1/9</td>
<td>1</td>
<td>.066</td>
<td>0.092</td>
<td>B</td>
<td>2</td>
<td>1</td>
<td>.400</td>
<td>1.000</td>
<td>B</td>
<td>7</td>
<td>1</td>
<td>.650</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1/4</td>
<td>1</td>
<td>.217</td>
<td>0.303</td>
<td>C</td>
<td>5</td>
<td>1/3</td>
<td>.278</td>
<td>0.430</td>
<td>C</td>
<td>5</td>
<td>1/3</td>
<td>.278</td>
<td>0.430</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Distributive and ideal synthesis.
by making comparisons both consciously and subconsciously. It is a way of measurement that took place long before Nicole Oresme, Pierre de Fermat, and finally René Descartes more rigorously introduced general coordinate systems for physical measurements and assumed that they were extensible from zero to infinity by using an arbitrarily chosen unit applied uniformly over the entire range of measurement. Taking ratios removes the arbitrariness of the unit and creates relative absolute scales, invariant under the identity transformation. The example below is more staid than dealing with evanescent phenomena like political decisions, but it serves to illustrate the ideas whose mathematical foundations have been developed in a separate paper and in books.

Choosing the Best House

Consider the following (hypothetical) example: a family wishing to purchase a house identifies eight criteria that are important to them. The problem is to select one of three candidate houses. The first step is to structure the problem into a hierarchy (see Figure 1). On the first (top) level is the overall goal of Satisfaction with House. On the second level are the eight criteria that contribute to the goal, and on the third (bottom) level are the three candidate houses that are to be evaluated by considering the criteria on the second level.

The criteria important to the family are:

1. Size of House: Storage space; size of rooms; number of rooms; total area of house.
2. Transportation: Convenience and proximity of bus service.
4. Age of House: How long ago the house was built.
5. Yard Space: Front, back, and side space and space shared with neighbors.
6. Modern Facilities: Dishwasher, garbage disposal, air conditioning, alarm system, and other such items.
7. General Condition: Extent to which repairs are needed; condition of walls, carpet, drapes, wiring; cleanliness.
8. Financing: Availability of assumable mortgage, seller financing or bank financing.

The next step is to make comparative judgments. The family assesses the relative importance of all possible pairs of criteria with respect to the overall goal, Satisfaction with House, coming to a consensus judgment on each pair; and their judgments are arranged into a matrix. The questions to ask when comparing two criteria are, which is more important and how much more important is it with respect to satisfaction with a house?

The matrix of pairwise comparison judgments on the criteria given by the home-buyers in this case is shown in Table 1. The judgments are entered using the fundamental scale of the Analytic Hierarchy Process (AHP) Table 0: a criterion compared with itself is always assigned the value 1 so the main diagonal entries of the pairwise comparison matrix are all 1. We are permitted to interpolate values between the integers, if desired. Reciprocal values are automatically entered in the transpose position, so the family must make a total of twenty-eight pairwise judgments.

We have assumed that an element with weight zero is eliminated from comparison because zero can be applied to the whole universe of factors not included in the discussion.

The foregoing integer-valued fundamental scale of response used in making paired comparison judgments can be derived from the logarithmic response function of Weber Fechner in psychophysics as follows. For a given value of the stimulus, the magnitude of response remains the same until the value of the stimulus is increased sufficiently large in proportion to the value of the stimulus, thus preserving the proportionality of relative increase in stimulus for it to be detectable for a new response. This suggests the idea of just noticeable differences (jnd), well known in psychology. Thus, starting with a stimulus \( s_0 \), successive magnitudes of the new stimuli take the form [2]

\[
\begin{align*}
    s_1 &= s_0 + \Delta s_0 = s_0 + \frac{\Delta s_0}{s_0} s_0 = s_0 (1 + \alpha), \\
    s_2 &= s_1 + \Delta s_1 = s_1 (1 + \alpha) = s_0 (1 + \alpha)^2 = s_0 \alpha^2, \\
    &\vdots \\
    s_n &= s_{n-1} \alpha = s_0 \alpha^n \quad (n = 0, 1, 2, \ldots),
\end{align*}
\]

We consider the responses to these stimuli to be measured on a ratio scale \( (b = 0) \). A typical response has the form \( M_i = a \log \alpha^i \), \( i = 1, \ldots, n \), or one after another they have the form

\[ M_1 = a \log \alpha, \quad M_2 = 2a \log \alpha, \ldots, M_n = na \log \alpha. \]

We take the ratios \( M_i / M_1, \quad i = 1, \ldots, n \), of these responses in which the first is the smallest and serves as the unit of comparison, thus obtaining the integer values \( 1, 2, \ldots, n \) of the fundamental scale of the AHP. It appears that numbers are intrinsic to our ability to make comparisons and that they were not an invention by our primitive ancestors. We must be grateful to them for the discovery of the symbolism. In a less mathematical vein, we note that we are able to distinguish ordinarily between high, medium, and low at one level and for each of them in a second level below that also to distinguish between high, medium, and
low, giving us nine different categories. We assign the value one to (low, low), which is the smallest, and the value nine to (high, high), which is the highest, thus covering the spectrum of possibilities between two levels and giving the value nine for the top of the paired comparisons scale as compared with the lowest value on the scale. The scale consists of absolute numbers, which unlike the familiar numbers that belong to a ratio scale that are invariant under a similarity transformation (multiplication by a positive number) are invariant under the identity transformation.

In his book *The Number Sense: How the Mind Creates Mathematics* [6], the mathematician and cognitive neuropsychologist Stanislas Dehaene writes more than twenty-five years after we derived this scale that “Introspection suggests that we can mentally represent the meaning of numbers 1 through 9 with actual acuity. Indeed, these symbols seem equivalent to us. They all seem equally easy to work with, and we feel that we can add or compare any two digits in a small and fixed amount of time like a computer. In summary, the invention of numerical symbols should have freed us from the fuzziness of the quantitative representation of numbers.”

It can be shown that when comparisons involve greater contrast than 9, the elements can be aggregated into clusters, with a pivot element from one cluster to an adjacent one, again applying the same kind of comparisons within the next cluster using the scale 1–9. One then divides by the weight of the pivot element in the second cluster and multiplies by its weight from the first cluster, and now the priorities in the two clusters are comparable and can be put together and so on. This type of clustering is called homogeneous clustering [22]. In addition, as we shall see later, for stability of the priorities with respect to small changes in judgment, each cluster must not contain more than a few elements: about seven [20].

In the AHP model, the vector of priorities for the criteria is obtained by computing the principal eigenvector, the classical Perron vector, of the pairwise comparison matrix. The Perron vector is a necessary condition for obtaining the priorities when the judgments are inconsistent. As we shall see below, consistency and near consistency are important concepts in our considerations. Because the pairwise comparison matrix has positive entries, Perron’s theorem ensures that there is a unique positive vector (denoted by \( w \)) whose entries sum to one that is an eigenvector of the pairwise comparison matrix, associated with an eigenvalue (denoted by \( \lambda_{\text{max}} \)) of strictly largest modulus. That eigenvalue, the Perron eigenvalue, is positive and algebraically simple (multiplicity one as a root of the characteristic equation) [14]. Consistency of the family’s set of judgments is measured by the consistency ratio (C.R.), which we explain later.

Table 1 shows that size dominates transportation strongly since a 5 appears in the (size, transportation) position. In the (finance, size) position we have a 4, which means that finance is between moderately and strongly more important than size. The priority vector shows that financing is the most important criterion to the family as the entry of \( w \) corresponding to finance has the largest value, 0.345.

Consistency, which was alluded to previously, is an elaboration of the common sense view expressed in this statement: if you prefer spring to summer by 2, summer to winter by 3, and spring to winter by 6, then those three judgments are consistent.

The family’s next task is to compare the houses in pairs with respect to how much better (more dominant) one is than the other in satisfying each of the eight criteria. There are eight 3-by-3 matrices of judgments since there are eight criteria and three houses are to be compared for each criterion. The matrices in Table 2 contain the judgments of the family. In order to facilitate understanding of the judgments, we give a brief description of the houses.

In Table 1 an element on the left of the matrix is compared for dominance over another at the top.

**House A:** This house is the largest. It is located in a good neighborhood with little traffic and low taxes. Its yard space is larger than that of either house B or C. However, its general condition is not very good, and it needs cleaning and painting. It would have to be bank financed at a high interest rate.

**House B:** This house is a little smaller than house A and is not close to a bus route. The neighborhood feels insecure because of traffic conditions. The yard space is fairly small, and the house lacks basic modern facilities. On the other hand, its general condition is very good, and it has an assumable mortgage with a rather low interest rate.

**House C:** House C is very small and has few modern facilities. The neighborhood has high taxes but is in good condition and seems secure. Its yard is bigger than that of house B but smaller than house A’s spacious surroundings. The general condition of the house is good, and it has a pretty carpet and drapes. The financing is better than for house A but poorer than for house B.

In Table 2 both ordinary (distributive) and idealized priority vectors of the three houses are given for each of the criteria. The idealized priority vector is obtained by dividing each element of the distributive priority vector by its largest element. The composite priority vector for the houses is obtained by multiplying each priority vector by the priority of the corresponding criterion, adding
across all the criteria for each house and then normalizing. When we use the (ordinary) distributive priority vectors, this method of synthesis is known as the distributive mode and yields $A = .345$, $B = .369$, and $C = .285$. Thus house B is preferred to houses A and C in the ratios: $.369/.346$ and $.369/.285$, respectively.

In Table 2 again an element on the left of each matrix is compared for dominance over another at the top. If the top element is dominant, a fraction is entered and its inverse, a whole number, is entered in the reciprocal position.

When we use the idealized priority vector, the synthesis is called the ideal mode. This yields $A = .315$, $B = .383$, $C = .302$ and B is again the most preferred house. The two ways of synthesizing are shown in Table 3. They need not yield the same ranking. In general, the ideal mode is used when rating the alternatives one at a time (see later) with respect to the criteria and when the criteria priorities are independent of the alternatives. The ideal mode is used to force rank preservation when new houses are added to the collection by only comparing them with respect to the ideal and not the other alternatives. But always preserving rank is not always desirable [16].

The Pairwise Comparison Matrix

In comparing pairs of criteria with respect to the goal, one estimates which of the two criteria is more important and how much more. The result of these comparisons is arranged in a positive matrix $A = [a_{ij}]$ whose entries satisfy the reciprocal property $a_{ji} = 1/a_{ij}$.

We start with a positive reciprocal matrix such as Table 1. An $n$-by-$n$ table, $n(n - 1)/2$ judgments must be made, which is why the house-buying family had to make $(8 \times 7)/2 = 28$ judgments. These judgments are made independently, but they are not really “independent”. If the family feels that financing is twice as important as size and that size is twice as important as age, for consistency of judgments we should expect them to feel that financing is four times as important as age. The mathematical expression of our expectation is the set of identities

$$a_{ij} = a_{ik}/a_{jk} \quad \text{for all } i, j, k = 1, \ldots, n$$

among the entries of a consistent pairwise comparison matrix $A = [a_{ij}]$. Of course, real-world pairwise comparison matrices are very unlikely to be consistent, and we address the consequences of that reality next [16], [19].

Why the Principal Eigenvector?

Suppose a positive square matrix $A = [a_{ij}]$ is consistent. Then $A$ must have unit diagonal entries, since $a_{ii} = a_{kk}/a_{ik}$, for all $i, k = 1, \ldots, n$. Moreover, $A$ must be reciprocal since $a_{ij}a_{ji} = 1$ means that $a_{ij} = 1/a_{ji}$. Such a matrix has a very simple structure since $a_{ik} = a_{ii}a_{ki} = a_{i1}/a_{1k}$ for all $i, k$. Thus the entries in the first column of $A$ determine all other entries! For convenience, write $a_i \equiv a_{ii}$, so that $A = [a_{ij}] = [a_i/a_j]$. If we define the two positive $n$-by-1 vectors $x \equiv [a_i]$ and $y \equiv [a_j]^{-1}$, then it is clear that $A = xy^T$ has rank one. Thus, the positive matrix $A$ has one nonzero eigenvalue and $n - 1$ zero eigenvalues. It is easy to check that $Ax = \sum_{j=1}^n (a_i/a_j) a_j = na_i$, so the nonzero eigenvalue of $A$ (its Perron eigenvalue) is $n$, and an associated positive right eigenvector is $x \equiv [a_i]$. If we set $c = a_i + \cdots + a_n$, the Perron vector of $A$ (its unique positive eigenvector whose entries sum to one) may be written as $w = x/c = [a_i/c] = [w_i]$. The Perron vector determines all the entries of $A : A = [a_{ij}] = [a_i/a_j] = [(a_i/c)/(a_j/c)] = [w_i/w_j]$. When a matrix is consistent, right and left eigenvectors have reciprocal corresponding entries.

We know that an $n$-by-$n$ positive consistent matrix $A = [a_{ij}]$ has a unique positive eigenvector $w \equiv [w_i]$ (its Perron vector) whose entries sum to one and whose corresponding eigenvalue (its Perron eigenvalue) is $n$. Moreover, the ratios of the entries of $w$ are precisely the entries of $A : a_{ij} = w_i/w_j$. If we think of $A$ as a matrix of (perfectly) consistent pairwise comparisons for $n$ given elements, then the $n$ values $w_i$ are a natural set of priorities that underlie the set of pairwise judgments: $a_{ij} = w_i/w_j$. There are several ways to prove that the principal eigenvector is necessary when the judgments are inconsistent [18], [21].

The foregoing discussion is intended to motivate the central and critical choice of the Perron vector as the means to extract a vector of priorities from a given pairwise comparison matrix in the AHP model. If humans made perfectly consistent judgments all the time, the model would be perfect. But they do not, so we must now face the question of assessing the deviation from consistency of an actual pairwise comparison matrix and the consequences of inconsistency for the quality of decisions made according to the AHP model. In passing, we observe that if humans were always perfectly consistent, they would not be able to learn new things that modify or change the relations among what they knew before and they would be like robots. But there is a level of tolerable inconsistency that we must allow beyond which the judgments would appear to be uninformed, random, or arbitrary.

When we compare things, unlike assigning them numbers independently of one another, their priorities always depend on what other things they are compared with. Were we to assume that
the universe is interdependent (the subject of the Analytic Network Process (ANP) [17, 18]) with a complicated field of influences in every aspect, our traditional way of assigning numbers from scales of measurement would not be “the natural way” to determine their importance. More and more we are finding with this relative way of thinking that the world is different (has different rank orders) than we think we understand it to be today. Another useful observation is that comparisons are necessary for comparing criteria to derive their priorities because there are no scales for their measurement and also because their importance varies from decision to decision. In the ANP the criteria are compared with respect to each alternative and the alternatives are compared with respect to each criterion to derive their interdependent priorities.

Regrettably, the three laws of thought (identity, excluded middle, and contradiction), known to Plato and Aristotle and even to Leibniz, that are strict requirements we all adhere to in language, logic, science, and mathematics have precluded comparisons, our biological heritage. Without comparisons nothing can be known in and of itself without also knowing other things with which it is compared, including knowing it at an earlier time, so we can ensure it is the one we have in mind (the law of identity), thus recognizing it. Arthur Schopenhauer, who was not equipped to develop a mathematical theory to use comparisons, listed the laws of thought by adding a fourth one in his On the Fourfold Root of the Principle of Sufficient Reason. (1) a subject is equal to the sum of its predicates, or \( a = \sum \lambda \); (2) no predicate can be simultaneously attributed and denied to a subject, or \( a \neq \overline{a} \); (3) of every two contradictorily opposite predicates one must belong to every subject; and (4) truth is the reference of a judgment to something outside itself as its sufficient reason or ground.

We are all familiar with the arbitrarily imposed axiom in logic and mathematics that if A dominates B and B dominates C, then A must dominate C. But in the real world team A beats team B, B beats C but C beats A, contradicting theory. It appears that theory needs to be changed to accommodate reality.

**When is a Positive Reciprocal Matrix Consistent?**

Let \( A = [a_{ij}] \) be an \( n \)-by-\( n \) positive reciprocal matrix, so all \( a_{ij} = 1 \) and \( a_{ij} = 1/a_{ji} \) for all \( i, j = 1, \ldots, n \). Let \( w = [w_i] \) be the Perron vector of \( A \), let \( D = \text{diag}(w_1, \ldots, w_n) \) be the \( n \)-by-\( n \) diagonal matrix whose main diagonal entries are the entries of \( w \), and set \( E = D^{-1}AD = [a_{ij}w_j/w_i] = [\varepsilon_{ij}] \). Then \( E \) is similar to \( A \) and is a positive reciprocal matrix since \( \varepsilon_{ij} = a_{ij}w_i/w_j = (a_{ij}w_j/w_i)^{-1} = 1/\varepsilon_{ij} \). Moreover, all the row sums of \( E \) are equal to the Perron eigenvalue of \( A \):

\[
\sum_{j=1}^{n} \varepsilon_{ij} = \sum_{j} a_{ij}w_j/w_i = [Aw]_i/w_i = \lambda_{\text{max}} w_i/w_i = \lambda_{\text{max}}.
\]

The computation

\[
n\lambda_{\text{max}} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \varepsilon_{ij} \right) = \sum_{i=1}^{n} \varepsilon_{ii} + \sum_{i,j=1}^{n} (\varepsilon_{ij} + \varepsilon_{ji})
\]

is consistent. As our measure of deviation of \( A \) from consistency, we choose the **consistency index**

\[
\mu = \frac{\lambda_{\text{max}} - n}{n-1}.
\]

We have seen that \( \mu \geq 0 \) and \( \mu = 0 \) if and only if \( A \) is consistent. These two desirable properties explain the term “\( n \)” in the numerator of \( \mu \); what about the term “\( n - 1 \)” in the denominator? Since trace \( A = n \) is the sum of all the eigenvalues of \( A \), if we denote the eigenvalues of \( A \) that are different from \( \lambda_{\text{max}} \) by \( \lambda_2, \ldots, \lambda_{n+1} \), we see that \( n = \lambda_{\text{max}} + \sum_{i=2}^{n} \lambda_i \), so

\[
n - \lambda_{\text{max}} = \sum_{i=2}^{n} \lambda_i \quad \text{and} \quad \mu = -\frac{1}{n-1} \sum_{i=2}^{n} \lambda_i \quad \text{is the negative average of the non-Perron eigenvalues of} \quad A.
\]

It is an easy, but instructive, computation to show that \( \lambda_{\text{max}} = 2 \) for every 2-by-2 positive reciprocal matrix:

\[
\begin{pmatrix}
1 & \alpha \\
\alpha^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
1 + \alpha & 1 \\
(1 + \alpha)\alpha^{-1} & 1
\end{pmatrix}
= 2
\begin{pmatrix}
1 + \alpha \\
(1 + \alpha)\alpha^{-1}
\end{pmatrix}.
\]

Thus, every 2-by-2 positive reciprocal matrix is consistent.

Not every 3-by-3 positive reciprocal matrix is consistent, but in this case we are fortunate to have again explicit formulas for the Perron eigenvalue and eigenvector. For

\[
A = \begin{bmatrix}
1 & a & b \\
1/a & 1 & c \\
1/b & 1/c & 1
\end{bmatrix},
\]

we have \( \lambda_{\text{max}} = 1 + d + d^{-1} \), \( d = (ac/b)^{1/3} \), and

\[
w_1 = bd/(1 + bd + c), \quad w_2 = c/d/(1 + bd + c), \quad w_3 = 1/(1 + bd + c).
\]
Figure 2. Plot of random inconsistency.

Figure 3. Plot of first differences in random inconsistency—7 is critical.

<table>
<thead>
<tr>
<th>Order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>R.I.</td>
<td>0.00</td>
<td>0.52</td>
<td>0.89</td>
<td>1.11</td>
<td>1.25</td>
<td>1.35</td>
<td>1.40</td>
<td>1.45</td>
<td>1.49</td>
<td>1.52</td>
<td>1.54</td>
<td>1.56</td>
<td>1.58</td>
<td>1.59</td>
<td></td>
</tr>
<tr>
<td>First Order Differences</td>
<td>0.52</td>
<td>0.37</td>
<td>0.22</td>
<td>0.14</td>
<td>0.10</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Random index.

Note that \( \lambda_{\text{max}} = 3 \) when \( d = 1 \) or \( c = b/a \), which is true if and only if \( A \) is consistent.

In order to get some feel for what the consistency index might be telling us about a positive \( n \)-by-\( n \) reciprocal matrix \( A \), consider the following simulation: choose the entries of \( A \) above the main diagonal at random from the seventeen values \( \{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\} \). Then fill in the entries of \( A \) below the diagonal by taking reciprocals. Put ones down the main diagonal and compute the consistency index. Do this 50,000 times and take the average, which we call the random index. Table 4 shows the values obtained from one set of such simulations for matrices of size \( 1, 2, \ldots, 10 \).

Figure 2 is a plot of the first two rows of Table 4. It shows the asymptotic nature of random inconsistency.

The third row of Table 2 gives the differences between successive numbers in the second row. Figure 3 is a plot of these differences and shows the importance of the number seven as a cutoff point beyond which the differences are less than 0.10, where we are not sufficiently sensitive to make accurate changes in judgment on several elements simultaneously.

Since it would be pointless to try to discern any priority ranking from a set of random comparison judgments, we should probably be uncomfortable about proceeding unless the consistency index of a pairwise comparison matrix is very much smaller than the corresponding random index value in Table 4. The consistency ratio (C.R.) of a pairwise comparison matrix is the ratio of its consistency index (C. I.) to the corresponding random index value in Table 4.

As a rule of thumb, we do not recommend proceeding if the consistency ratio is more than about 0.10 for \( n > 4 \). For \( n = 3 \) and \( 4 \) we recommend that the C.R. be less than 0.05 and 0.09, respectively. Thus in general when asked, we require that C.R. not exceed 0.10 by much. How do we explain this outcome in general?

The notion of order of magnitude is essential in any mathematical consideration of changes in measurement. When one has a numerical value say between 1 and 10 for some measurement and one wants to determine whether a change in this value is significant or not, one reasons as follows: a change of a whole integer value is critical because it changes the magnitude and identity of the original number significantly. If the change or perturbation in value is of the order of a percent or less, it would be so small (by two orders of magnitude) and would be considered negligible. However, if this perturbation is a decimal (one order of magnitude smaller), we are likely to pay attention to modify the original value by this decimal without losing the
significance and identity of the original number as we first understood it to be. Thus in synthesizing near-consistent judgment values, changes that are too large can cause dramatic change in our understanding, and values that are too small cause no change in our understanding. We are left with only values of one order of magnitude smaller that we can deal with incrementally to change our understanding. It follows that our allowable consistency ratio should be not more than about .10. The requirement of 10% cannot be made smaller, such as 1% or 0.1%, without trivializing the impact of inconsistency. But inconsistency itself is important because without it new knowledge that changes preference cannot be admitted. Assuming that all knowledge should be consistent contradicts experience, which requires continued revision of understanding.

If the C.R. is larger than desired, we do three things: (1) find the most inconsistent judgment in the matrix; (2) determine the range of values to which that judgment can be changed corresponding to which the inconsistency would be improved; (3) ask the family to consider, if they can, changing their judgment to a plausible value in that range. If they are unwilling, we try with the second most inconsistent judgment, and so on. If no judgment is changed, the decision is postponed until better understanding of the criteria is obtained. In our house example the family initially made a judgment of 6 for the \( a_{37} \) entry in Table 1 and the consistency index of the set of judgments was C.I. = (9.669 - 8)/7 = 0.238. But C.R. = .238/1.40 = 0.17 is larger than the recommended value of 0.10. If we are going to ask the family to reconsider, and perhaps change, some of their pairwise comparisons, where should we start?

Three methods are plausible for this purpose. All require theoretical investigation of convergence and efficiency. The first uses an explicit formula for the partial derivatives of the Perron eigenvalue with respect to the matrix entries.

For a given positive reciprocal matrix \( A = [a_{ij}] \) and a given pair of distinct indices \( k > l \), define \( A(t) = [a_{ij}(t)] \) by \( a_{kl}(t) = a_{kl} + t, a_{lk}(t) = (a_{lk} + t)^{-1} \), and \( a_{ij}(t) = a_{ij} \) for all \( i \neq k, j \neq l \), so \( A(0) = A \). We use a linear function of \( t \) because multiplying by \( t \) when \( t \) is zero or close to zero can make the reciprocal very large, and thus we want \( t \) to be bounded away from zero. Also, we don’t want \( t \) to be very large because the judgments would be too widespread, violating the requirement of homogeneity. Thus our assumption on the functional relationship is reasonable. Let \( \lambda_{\text{max}}(t) \) denote the Perron eigenvalue of \( A(t) \) for all \( t \) in a neighborhood of \( t = 0 \) that is small enough to ensure that all entries of the reciprocal matrix \( A(t) \) are positive there. Finally, let \( \nu = [\nu_i] \) be the

<table>
<thead>
<tr>
<th></th>
<th>Size</th>
<th>Trans</th>
<th>Nbrhd</th>
<th>Age</th>
<th>Yard</th>
<th>Modern</th>
<th>Cond</th>
<th>Finance</th>
<th>( w )</th>
<th>( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>1/3</td>
<td>1/4</td>
<td>.173</td>
<td>.047</td>
</tr>
<tr>
<td>Trans.</td>
<td>1/5</td>
<td>1</td>
<td>1/3</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1/5</td>
<td>1/7</td>
<td>.054</td>
<td>.117</td>
</tr>
<tr>
<td>Nbrhd.</td>
<td>1/3</td>
<td>3</td>
<td>1</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>1/5</td>
<td>.188</td>
<td>.052</td>
</tr>
<tr>
<td>Age</td>
<td>1/7</td>
<td>1/5</td>
<td>1/6</td>
<td>1</td>
<td>1/3</td>
<td>1/4</td>
<td>1/7</td>
<td>1/8</td>
<td>.018</td>
<td>.349</td>
</tr>
<tr>
<td>Yard</td>
<td>1/6</td>
<td>1/3</td>
<td>1/3</td>
<td>3</td>
<td>1</td>
<td>1/2</td>
<td>1/5</td>
<td>1/6</td>
<td>.031</td>
<td>.190</td>
</tr>
<tr>
<td>Modern</td>
<td>1/6</td>
<td>1/3</td>
<td>1/4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1/5</td>
<td>1/6</td>
<td>.036</td>
<td>.166</td>
</tr>
<tr>
<td>Cond.</td>
<td>3</td>
<td>5</td>
<td>1/6</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>1/2</td>
<td>.167</td>
<td>.059</td>
</tr>
<tr>
<td>Finance</td>
<td>4</td>
<td>7</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>.333</td>
<td>.020</td>
</tr>
</tbody>
</table>

\( \lambda_{\text{max}} = 9.669, \text{ C.R.} = 0.17 \)

Table 5. A family’s housebuying pairwise comparison matrix for the criteria.

Table 6. Partial derivatives for the house example.
Table 7. $\epsilon_{ij} = a_{ij} w_j / w_i$.

<table>
<thead>
<tr>
<th>Size</th>
<th>Trans.</th>
<th>Nbrhd.</th>
<th>Age</th>
<th>Yard</th>
<th>Modern</th>
<th>Cond.</th>
<th>Finance</th>
<th>$w$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1.7779</td>
<td>1.756208</td>
<td>0.774933</td>
<td>1.163989</td>
<td>1.418734</td>
<td>0.425449</td>
<td>0.494088</td>
<td>0.174</td>
</tr>
<tr>
<td>Trans.</td>
<td>0.562461</td>
<td>1</td>
<td>0.548777</td>
<td>1.556678</td>
<td>1.637464</td>
<td>1.994957</td>
<td>0.717895</td>
<td>0.794016</td>
<td>0.062</td>
</tr>
<tr>
<td>Nbrhd.</td>
<td>0.569408</td>
<td>1.822233</td>
<td>2</td>
<td>1.134652</td>
<td>0.994177</td>
<td>1.615679</td>
<td>0.675211</td>
<td>0.102</td>
<td></td>
</tr>
<tr>
<td>Age</td>
<td>1.290434</td>
<td>0.642394</td>
<td>0.881328</td>
<td>1</td>
<td>0.584131</td>
<td>0.533978</td>
<td>1.64704</td>
<td>2.23156</td>
<td>0.019</td>
</tr>
<tr>
<td>Yard</td>
<td>0.859115</td>
<td>0.610968</td>
<td>1.005857</td>
<td>1.711945</td>
<td>1</td>
<td>0.609428</td>
<td>1</td>
<td>3.158533</td>
<td>1.697195</td>
</tr>
<tr>
<td>Modern</td>
<td>0.704854</td>
<td>0.501264</td>
<td>0.618935</td>
<td>1.872735</td>
<td>1.640883</td>
<td>1</td>
<td>1</td>
<td>1.079564</td>
<td>1.39304</td>
</tr>
<tr>
<td>Cond.</td>
<td>2.35046</td>
<td>1.392962</td>
<td>0</td>
<td>0.60715</td>
<td>0.759975</td>
<td>0.9263</td>
<td>2</td>
<td>0.774223</td>
<td>0.223</td>
</tr>
<tr>
<td>Finance</td>
<td>2.02393</td>
<td>1.259421</td>
<td>1.481018</td>
<td>0.448117</td>
<td>0.588958</td>
<td>0.717855</td>
<td>1.291617</td>
<td>1</td>
<td>0.345</td>
</tr>
</tbody>
</table>

Table 8.

<table>
<thead>
<tr>
<th>Size</th>
<th>Trans.</th>
<th>Nbrhd.</th>
<th>Age</th>
<th>Yard</th>
<th>Modern</th>
<th>Cond.</th>
<th>Finance</th>
<th>$w$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>1/3</td>
<td>1/4</td>
<td>.175</td>
</tr>
<tr>
<td>Trans.</td>
<td>1/5</td>
<td>1</td>
<td>1/3</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>3/5</td>
<td>1/7</td>
<td>.062</td>
</tr>
<tr>
<td>Nbrhd.</td>
<td>1/3</td>
<td>1</td>
<td>1/3</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>1/2</td>
<td>1/5</td>
<td>.103</td>
</tr>
<tr>
<td>Age</td>
<td>1/7</td>
<td>1/5</td>
<td>1/6</td>
<td>1</td>
<td>1/3</td>
<td>1/4</td>
<td>1/7</td>
<td>1/8</td>
<td>.019</td>
</tr>
<tr>
<td>Yard</td>
<td>1/6</td>
<td>1/3</td>
<td>1/3</td>
<td>3/5</td>
<td>1</td>
<td>1/2</td>
<td>1/5</td>
<td>1/5</td>
<td>.034</td>
</tr>
<tr>
<td>Modern</td>
<td>1/6</td>
<td>1/3</td>
<td>1/4</td>
<td>1/6</td>
<td>2</td>
<td>1</td>
<td>1/5</td>
<td>1/6</td>
<td>.041</td>
</tr>
<tr>
<td>Cond.</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>1/2</td>
<td>.221</td>
</tr>
<tr>
<td>Finance</td>
<td>4</td>
<td>7</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>.345</td>
</tr>
</tbody>
</table>

$\lambda_{\text{max}} = 8.811$, C.R. = 0.083

Table 9. Modified matrix in the $a_{37}$ and $a_{73}$ positions.

Thus, to identify an entry of $A$ whose adjustment within the class of reciprocal matrices would result in the largest rate of change in $\lambda_{\text{max}}$, we should examine the $n(n - 1)/2$ values $\{v_i w_j - a_{ij}^2 v_i w_j\}, i > j$, and select (any) one of largest absolute value. This is the method proposed for positive reciprocal matrices by Harker [8]. Ergu et al. [7] propose another method for dealing with consistency in the ANP. Here is how Harker’s method is applied to our house example with the initial judgments in Table 1 replaced by $a_{37} = 6, a_{73} = 1/6$ to make it more inconsistent.

Table 6 gives the array of partial derivatives for the matrix of criteria in Table 1.

The $(4, 8)$ entry in Table 5 (in bold print and underlined) is largest in absolute value. Thus, the family could be asked to reconsider their judgment of 1/8 for age vs. finance which indicates

unique positive eigenvector of the positive matrix $A^T$ that is normalized so that $v^T w = 1$. Then a classical perturbation formula [4, Theorem 6.3.12] tells us that

$$
\frac{d\lambda_{\text{max}}(t)}{dt} \bigg|_{t=0} = \frac{v^T A'(0) w}{v^T w} = v^T A'(0) w = \sum_{k\neq l} v_k w_l - \frac{1}{a_{ij}^2} v_i w_j.
$$

We conclude that

$$
\frac{\partial \lambda_{\text{max}}}{\partial a_{ij}} = v_i w_j - a_{ij}^2 v_j w_i \quad \text{for all } i, j = 1, \ldots, n.
$$

Because we are operating within the set of positive reciprocal matrices,

$$
\frac{\partial \lambda_{\text{max}}}{\partial a_{ij}} = -\frac{1}{a_{ij}^2} \frac{\partial \lambda_{\text{max}}}{\partial a_{ji}} \quad \text{for all } i \text{ and } j.
$$

202 Notices of the AMS Volume 60, Number 2
that finance is very strongly to extremely more important than age. One needs to know how much to change a judgment to improve consistency, and we show that next. One can then repeat this process with the goal of bringing the C.R. within the desired range. If the indicated judgments cannot be changed fully according to one’s understanding, they can be changed partially. Failing the attainment of a consistency level with justifiable judgments, one needs to learn more before proceeding with the decision. Actually, the values used in the original example were $a_{37} = 1/2$, $a_{73} = 2$, derived in the simpler approach described next.

Two other methods, presented here in order of increasing observed efficiency in practice, are conceptually different. They are based on the fact that

$$n\lambda_{\text{max}} - n = \sum_{i,j=1 \atop i \neq j}^n (\varepsilon_{ij} + \varepsilon_{ij}^{-1}).$$

This suggests that we examine the judgment for which $\varepsilon_{ij}$ is farthest from the number 1, that is, an entry $a_{ij}$ for which $a_{ij}w_j/w_i$ is the largest, and see if this entry can reasonably be made smaller. We hope that such a change of $a_{ij}$ also results in a new comparison matrix that has a smaller Perron eigenvalue. To demonstrate how improving judgments works, take the house example matrix in Table 1. To identify an entry ripe for consideration, construct the matrix $\varepsilon_{ij} = a_{ij}w_j/w_i$ (Table 7). The largest value in Table 7 is 5.32156, which focuses attention on $a_{37} = 6$.

How does one determine the most consistent entry for the (3,7) position? Harker [8] has shown that when we compute the new eigenvector $w$ after changing the (3,7) entry, we want the new (3,7) entry to be $w_j/w_i$ and the new (7,3) entry to be $w_3/w_7$. On replacing $a_{37}$ by $w_j/w_i$ and $a_{73}$ by $w_3/w_7$ and multiplying by the vector $w$, one obtains the same product as one would by replacing $a_{37}$ and $a_{73}$ with zeros and the two corresponding diagonal entries with two (see Table 8).

We take the Perron vector of the latter matrix to be our $w$ and use the now-known values of $w_3/w_7$ and $w_7/w_3$ to replace $a_{37}$ and $a_{73}$ in the original matrix. The family is now invited to change their judgment towards this new value of $a_{37}$ as much as they can. Here the value was $a_{37} = 0.102/0.223 = 1/2.18$, approximated by $1/2$ from the AHP fundamental scale, and we hypothetically changed it to $1/2$ to illustrate the procedure (see Table 9). If the family does not wish to change the original value of $a_{37}$, one considers the second most inconsistent judgment and repeats the process.

One by one, each reciprocal pair $a_{ij}$ and $a_{ji}$ in the matrix is replaced by zero and the corresponding diagonal entries $a_{ii}$ and $a_{jj}$ are replaced by 2. The principal eigenvalue $\lambda_{\text{max}}$ is then computed. The entry with the largest resulting $\lambda_{\text{max}}$ is identified for change as described above. This method is in use in the ANP software program SuperDecisions [26]. The SuperDecisions software is used in teaching the subject. Here is the link to the webpage from which the SuperDecisions software can be downloaded, and it is free to educators and researchers: www.superdecisions.com. Incidentally, the name of the software is borrowed from its use of a matrix whose entries are matrices, the supermatrix, and is not an attempt to sound like something extraordinary.

Alternatives in a decision may be compared in pairs or if there are many, they can be rated one at a time by assigning them numbers from appropriate priority scales developed for each criterion such as (high, medium, low), (excellent, outstanding, very good, good, poor and very poor) that are then compared in pairs and their priorities derived, with each scale divided by the largest derived eigenvector component, making the largest value equal to one and the rest proportionately smaller. Comparisons yield a more accurate ranking of alternatives than rating them one at a time because rating involves memory of an ideal that is likely to vary among different people. If the alternatives vary widely, then the scales developed must reflect different orders of magnitude that are appropriately linked together [22].

The Normalized Priority Vector is Unique
To choose the best alternative in a decision, the priorities must be unique. There is more to the concept of priority. When $A = [w_j/w_i]$ is consistent, $A^k = n^{k-1}A$. This says how much a criterion represented by a row of $A$ dominates other criteria through chains of $k$ arcs, uniquely determined by the single arc chains represented by the rows of $A$ itself. But this is not true when $A$ is inconsistent.

Criterion $i$ is said to dominate criterion $j$ in one step if the sum of the entries in row $i$ of $A$ is greater than the sum of the entries in row $j$. It is convenient to use the vector $e = (1, \ldots, 1)^T$ to express this dominance: criterion $i$ dominates criterion $j$ in one step if $(Ae)_i > (Ae)_j$. A criterion can dominate another criterion in more than one step by dominating other criteria that in turn dominate the second criterion. Two-step dominance is identified by squaring the matrix and summing its rows, three-step dominance by cubing it, and so on. Thus, criterion $i$ dominates criterion $j$ in $k$ steps if $(A^k e)_i > (A^k e)_j$. Criterion $i$ is said simply to dominate criterion $j$ if entry $i$ of the vector

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^m A^k e/e^T A^k e$$

February 2013
Notices of the AMS
The eigenvector $w_1$ is stable when the following hold:

1. The perturbation $\Delta A$ is small as observing the consistency index would ensure.
2. $\lambda_j$ is well separated from $\lambda_1$; when $A$ is consistent, $\lambda_1 = n$, $\lambda_j = 0$.
3. The product of left and right eigenvectors is not too large, which is the case for a consistent (and near-consistent) matrix if the elements are homogenous (compared here on the relative dominance scale of the absolute values 1–9) with respect to the criterion of comparison.
4. The number of their entries is small (hence perhaps why inconsistency becomes problematic as to which element causes it the most for $n > 7$, [13]).

The conclusion is that $n$ must be small, and one must compare homogeneous elements, which is in harmony with the axioms of the AHP [18].

**Synthesis of Individual Judgments into a Representative Group Judgment**

Kenneth Arrow's Impossibility Theorem, for which he received the Nobel Prize in 1972, stated that it was not possible to find a representative group judgment from the judgments of individuals using ordinal preferences. However, if one allows cardinal preferences and uses the geometric mean to combine individual judgments as we do in the AHP, it is possible. In 1983 we proved, in a paper coauthored with Janos Aczel, that the unique way to combine reciprocal individual judgments into a corresponding reciprocal group judgment is by using their geometric mean [1].

Arrow proved in his impossibility theorem, using ordinal preferences (either A is preferred to B or it is not) that there does not exist a social welfare function that satisfies all four conditions listed in Figure 4, at once. We showed in April 2011 [24] in a journal of which Arrow is an editor that with cardinal intensities of preference and the geometric mean to combine the individual judgments into a representative group judgment, a social welfare function exists that satisfies these four conditions. Thus we have a possibility theorem.

**Validation and Diverse Uses**

How do we test for the validity of the process? One of the things we can do is to get judgments from many people, even those who may not be experts in decision making but who are experts in what they do. Should the answer always match the data available? What if the data themselves are incorrect? What if we don’t know enough to create a very complete structure for a decision? These questions have been examined in the literature, but
at best one needs to apply the process in a number of decisions to develop confidence in its reliability. We have provided three examples to illustrate its accuracy when used with systematic knowledge and understanding, of both the decisions to which it is applied and of its limitations that revolve around the adequacy of the structure used to represent the decision, and the experience necessary to develop sound and accurate judgments used in making comparisons.

Relative Consumption of Drinks
Table 10 shows how an audience of about thirty people, using consensus to arrive at each judgment, provided judgments to estimate the dominance of the consumption of drinks in the U.S. (which drink is consumed more in the U.S. and how much more than another drink?). The derived vector of relative consumption and the actual vector, obtained by normalizing the consumption given in official statistical data sources, are at the bottom of the table.

Optics Example
Four identical chairs were placed on a line from a light source at the distances of nine, fifteen, twenty-one, and twenty-eight yards. The purpose was to see if one could stand by the light and look at the chair and compare their relative brightness in pairs, fill in the judgment matrix, and obtain a relationship between the chairs and their distance from the light source. This experiment was repeated twice with different judges whose judgment matrices are shown in Table 11.

The judges of the first matrix were the author’s young children, ages five and seven at that time, who gave their judgments qualitatively. The judge of the second matrix was the author’s wife, also a mathematician and not present during the children’s judgment process. In Table 12 we give the principal eigenvectors, eigenvalues, consistency indices, and consistency ratios of the two matrices.

First and second trial eigenvectors of Table 12 have been compared with the last column of Table 13 calculated from the inverse square law of optics. How close are the eigenvectors to the actual result for physics? We use a compatibility index that we developed for that purpose. We take the Hadamard product of a matrix of ratios of the entries of one vector with the transpose of a second matrix of the other vector. If the two vectors are identical, each entry of the Hadamard product would be equal to one and the sum of all resulting entries would be equal to $n^2$. Otherwise, one divides the resulting sum by $n^2$ and ensures that the ratio is about 1.01. It is interesting and important to observe in this example that the numerical judgments have captured a natural law. It would seem that they might do the same in other areas of perception or thought, like the one on estimating chess championship outcomes that we show in the next example, and, more generally, in continuous versions of these ideas.

Table 10. Relative consumption of drinks.

<table>
<thead>
<tr>
<th>Drink Consumption in the U.S.</th>
<th>Coffee</th>
<th>Wine</th>
<th>Tea</th>
<th>Beer</th>
<th>Sodas</th>
<th>Milk</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coffee</td>
<td>1</td>
<td>9</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>Wine</td>
<td>1/9</td>
<td>1</td>
<td>1/3</td>
<td>1/9</td>
<td>1/9</td>
<td>1/9</td>
<td>1/9</td>
</tr>
<tr>
<td>Tea</td>
<td>1/5</td>
<td>2</td>
<td>1</td>
<td>1/3</td>
<td>1/4</td>
<td>1/3</td>
<td>1/9</td>
</tr>
<tr>
<td>Beer</td>
<td>1/2</td>
<td>9</td>
<td>3</td>
<td>1</td>
<td>1/2</td>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>Sodas</td>
<td>1</td>
<td>9</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1/2</td>
</tr>
<tr>
<td>Milk</td>
<td>1</td>
<td>9</td>
<td>3</td>
<td>1</td>
<td>1/2</td>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>Water</td>
<td>2</td>
<td>9</td>
<td>9</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

The derived scale based on the judgments in the matrix is:

- Coffee 0.177
- Wine 0.019
- Tea 0.042
- Beer 0.116
- Sodas 0.190
- Milk 0.327
- Water 0.327

with a consistency ratio of .022.

The actual consumption (from statistical sources) is:

- Coffee 0.180
- Wine 0.106
- Tea 0.040
- Beer 0.120
- Sodas 0.180
- Milk 0.140
- Water 0.330

Table 11. Pairwise comparisons of the four chairs.

| Relative brightness eigenvector (1st Trial) | Relative brightness eigenvector (2nd Trial) |
|----------------|
| C1 | C2 | C3 | C4 |
| C1 | 1  | 5  | 6  | 7  |
| C2 | 1/5| 1  | 4  | 6  |
| C3 | 1/6| 1/4| 1  | 4  |
| C4 | 1/7| 1/6| 1/4| 1  |
| C1 | 1  | 4  | 6  | 7  |
| C2 | 1/4| 1  | 3  | 4  |
| C3 | 1/6| 1/3| 1  | 2  |
| C4 | 1/7| 1/4| 1/2| 1  |

Table 12. Principal eigenvectors and corresponding measures.

- $\lambda_{\text{max}} = 4.39$, C.I. = 0.13, C.R. = 0.14
- $\lambda_{\text{max}} = 4.10$, C.I. = 0.03, C.R. = 0.03

February 2013 Notices of the AMS 205
World Chess Championship Outcome Validation: The Karpov-Korchnoi Match

The following criteria (Table 14) and hierarchy (Figure 5) were used to predict the outcome of world chess championship matches using judgments of ten grandmasters in the former Soviet Union and the United States who responded to questionnaires mailed to them. The predicted outcomes that included the number of games played, drawn, and won by each player either was exactly as they turned out later or adequately close to predict the winner. The outcome of this exercise was officially notarized before the match took place. The notarized statement was mailed to the editor of the Journal of Behavioral Sciences along with the paper later published in May 1980. The prediction was that Karpov would win by six to five games over Korchnoi, which he did.

The AHP, the name of the decision process described above, has been used in various settings to make decisions.

This approach to prioritization provides the opportunity to help focus attention on the important issues in the world and allocate resources to them accordingly.

- Since its early development, the AHP has been used to predict correctly, a few months before the elections, the next candidate to be elected for president. The factors involved varied from election to election depending on the domestic and international circumstances prevailing at the time.
- In 1986 the Institute of Strategic Studies in Pretoria, a government-backed organization, used the AHP to analyze the conflict in South Africa and recommended actions ranging from the release of Nelson Mandela to the removal of apartheid and the granting of full citizenship and equal rights to the black majority. All of these recommended actions were quickly implemented by the white government.
- A company used it in 1987 to choose the best type of platform to build to drill for oil in the North Atlantic. A platform costs around three billion dollars to build, but the demolition cost was an even more significant factor in the decision.
- Xerox Corporation has used the AHP to allocate close to a billion dollars to its research projects.
- IBM used the process in 1991 in designing its successful mid-range AS 400 computer. IBM won the prestigious Malcolm Baldrige award for Excellence for that effort. The book about the AS 400 project has a chapter devoted to how AHP was used in benchmarking.
- The AHP has been used since 1992 in student admissions and prior to that in military personnel promotions and in hiring decisions.
- The process was applied to the U.S. versus China conflict in the intellectual property rights battle of 1995 over Chinese individuals copying music, video, and software tapes and CD’s. An AHP analysis involving three hierarchies for benefits, costs, and risks showed that it was much better for the U.S. not to sanction China. Shortly after the study was completed, the U.S. awarded China most-favored nation trading status and did not sanction it.
- In sports it was used in 1995 to predict which football team would go to the Super Bowl and win (correct outcome, Dallas won over my hometown, Pittsburgh). The AHP was applied in baseball to analyze which Padres players should be retained.
- British Airways used it in 1998 to choose the entertainment system vendor for its entire fleet of airplanes.
- The Ford Motor Company used the AHP in 1999 to establish priorities for criteria that improve customer satisfaction.
Table 14. Definitions of chess factors.

<table>
<thead>
<tr>
<th>T (1)</th>
<th>Calculation (Q): The ability of a player to evaluate different alternatives or strategies in light of prevailing situations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B (2)</td>
<td>Ego (E): The image a player has of himself as to his general abilities and qualification and his desire to win.</td>
</tr>
<tr>
<td>T (3)</td>
<td>Experience (EX): A composite of the versatility of opponents faced before, the strength of the tournaments participated in, and the time of exposure to a rich variety of chess players.</td>
</tr>
<tr>
<td>B (4)</td>
<td>Gamesmanship (G): The capability of a player to influence his opponent's game by destroying his concentration and self-confidence.</td>
</tr>
<tr>
<td>T (5)</td>
<td>Good Health (GH): Physical and mental strength to withstand pressure and provide endurance.</td>
</tr>
<tr>
<td>B (6)</td>
<td>Good Nerves and Will to Win (GN): The attitude of steadfastness that ensures a player's health perspective while the going gets tough. He keeps in mind that the situation involves two people and that if he holds out the tide may go in his favor.</td>
</tr>
<tr>
<td>T (7)</td>
<td>Imagination (IW): Ability to perceive and improvise good tactics and strategies.</td>
</tr>
<tr>
<td>T (8)</td>
<td>Intuition (IN): Ability to guess the opponent's intentions.</td>
</tr>
<tr>
<td>T (9)</td>
<td>Game Aggressiveness (GA): The ability to exploit the opponent's weaknesses and mistakes to one's advantage; occasionally referred to as &quot;killer instinct.&quot;</td>
</tr>
<tr>
<td>T (10)</td>
<td>Long Range Planning (LRP): The ability of a player to foresee the outcome of a certain move, set up desired situations that are more favorable, and work to alter the outcome.</td>
</tr>
<tr>
<td>T (11)</td>
<td>Memory M: Ability to remember previous games.</td>
</tr>
<tr>
<td>B (12)</td>
<td>Personality (P): Manners and emotional strength, and their effects on the opponent in playing the game and on the player in keeping his wits.</td>
</tr>
<tr>
<td>T (13)</td>
<td>Preparation (PR): Study and review of previous games and ideas.</td>
</tr>
<tr>
<td>T (14)</td>
<td>Quickness (Q): The ability of a player to see clearly the heart of a complex problem.</td>
</tr>
<tr>
<td>T (15)</td>
<td>Relative Youth (RY): The vigor, aggressiveness, and daring to try new ideas and situations, a quality usually attributed to young age.</td>
</tr>
<tr>
<td>T (16)</td>
<td>Seconds (S): The ability of other experts to help one to analyze strategies between games.</td>
</tr>
<tr>
<td>B (17)</td>
<td>Stamina (ST): Physical and psychological ability of a player to endure fatigue and pressure.</td>
</tr>
<tr>
<td>T (18)</td>
<td>Technique M: Ability to use and respond to different openings, improvise middle game tactics, and steer the game to a familiar ground to one's advantage.</td>
</tr>
</tbody>
</table>

- In 2001 it was used to determine the best site to relocate the earthquake-devastated Turkish city of Adapazari.
- A comprehensive analysis as to whether the Unites States should develop an anti-nuclear missile (estimated in the 1990s to cost sixty billion dollars and strongly opposed by scientists as technically infeasible) was presented to the National Defense University (NDU) in February 2002. In December of that year President Bush decided to go for it. The U.S. actually developed prototypes and tested them in stages successfully.
- An application by Professor Wiktor Adamus of Krakow University convinced the prime minister of Poland in 2007 not to adopt the Euro for currency until many years later.
- An AHP application, known to the military at the Pentagon, showed that occupying or bombing Iran in terms of benefits, opportunities, costs, and risks is not the best option for security in the Middle East.
- The AHP was used to assist the Green Bay Packers to hire the best players, perhaps partly the reason why they won the Super Bowl football championship in 2011 by beating the Pittsburgh Steelers. Other teams, including hockey and baseball, are also using it.
- In 1991, 2001, and 2009, AHP was used in three studies by economists to determine the turn-around dates of the U.S. economy and the strength of recovery. These studies were uncannily accurate.
- The latest application made in August 2011 was to the Israeli-Palestinian conflict.
The possibility of group choice: Pairwise comparisons were developed by J. H. Wilkinson [27]. We hope that we can have another opportunity to show the significance of the measurements obtained by using expert knowledge and judgment in that field.

This work on the AHP was developed independently of the Theory of Perron although I refer to him abundantly. Consistent matrices automatically satisfy Perron’s conditions, lead to his results, and generalize to acceptably inconsistent matrices through perturbation arguments some of which were developed by J. H. Wilkinson [27]. We hope that we can have another opportunity to show the reader how the ANP works and how the discrete mathematics of comparisons has been generalized to the continuous case involving Fredholm’s equation whose solution produces results associated with neural firing and synthesis.

References

Book Review

Transcending Tradition: Jewish Mathematicians in German-Speaking Academic Culture
Reviewed by Marjorie Senechal

Between 2006 and 2008, an exhibition entitled "Jewish Mathematicians in German-Speaking Academic Culture" was developed in Germany and shown in major German cities. Visitors and colleagues urged the organizers to take it abroad. To make it accessible to an international public, the exhibit has been redesigned, with English wall texts, English translations of primary documents, a supertitle "Transcending Tradition", and a handsome catalogue, reviewed here.

The international exhibition (with texts also in Hebrew and Arabic) opened in Israel in 2011 at the Museum of the Jewish Diaspora at Tel Aviv University. "The string quartet that performed one night last week at Beth Hatefusoth Museum of the Jewish Diaspora played a work by the German-Jewish composer Felix Mendelssohn," said the newspaper Haaretz, in its coverage of the opening.1

The violin whose sounds filled the hall that night once belonged to Georg Pick, a well-known Jewish mathematician from Vienna. He is best known for his eponymous formula, which concerns the connection between number theory and geometry, and he was said to have played this instrument along with his friend Albert Einstein, who also played the violin. On July 13, 1942, Pick was deported to the Theresienstadt concentration camp; two weeks later he died there, at the age of 82. Dr. Ruti Ungar, an Israeli historian who lives in Germany, brought this violin especially for the opening of the new exhibition she has curated. Ungar's grandfather, Herbert Ungar, was a friend of Pick's. A few days before Pick was sent to his death, he gave Ungar the violin. Herbert Ungar survived the Holocaust and hid the instrument until the war ended.

After Tel Aviv, the exhibition came to the United States, first to the John Crerar Library at the University of Chicago (October–December 2012). If you missed it there, you'll have another chance to see it, at the new Museum of Mathematics in New York City (date to be announced).

This is a review of the exhibition catalogue, not the exhibition itself. The catalogue stands on its own (though one would wish for a more complete index), with contributions from seven historians of mathematics in Germany. These essays are essential to our understanding of the role of Jewish mathematicians in German-speaking academic culture and the tragic end of that community. The text is enriched with illustrations, photographs, maps, tables, and primary documents. Some of these documents were not open to the public until the end of the last century and have changed the received picture.

A word about definitions. Who is a mathematician? Who is a Jew? These are sticky questions; some read-

Marjorie Senechal is Louise Wolff Kahn Professor Emerita in Mathematics and History of Science and Technology at Smith College. Her email address is senechal@smith.edu.

DOI: http://dx.doi.org/10.1090/noti
The Remak family exemplifies the fate of these families, says Vogt, "from expulsion to advance-
ment and acceptance, but then again to dismissal, persecution and genocide, all in less than one hundred years."

That wider public has probably never heard of Richard Courant, Max Dehn, William Feller, Felix Hausdorff, Adolf Hurwitz, Carl Gustav Jacob Jacobi, Leopold Kronecker, Edmund Landau, Hermann Minkowski, Emmy Noether, Issai Schur, Olga Taussky, and André Weil, to name a few of the 111 mathematicians featured in this book. For those of us to whom mathematics would be unimaginable without them, the catalogue is instructive too.

In the century between acceptance and persecution, the percentage of Jews in the professional German-speaking mathematical community rose from near zero to a third, despite still-pervasive anti-Semitism (discussed by Vogt in Chapter 7) in academia as well as in social, political, and cultural life. What were their names, where did they work, and what work did they do? The next five chapters address these questions. "People", by Birgit Bergmann, lists names and places, many familiar, some iconic, and some surprising. Her useful maps show their geographical spread: Aachen, Berlin, Bonn, Breslau, Darmsstadt, Dresden, Erlangen, Frankfurt, Freiburg, Giessen, Göttingen, Greifswald, Halle, Hamburg, Heidelberg, Karlsruhe, Kiel, Köln, Königsberg, Leipzig, Marburg, Munich, Münster, Strassburg, Tübingen, and Würzburg. Jewish mathematicians also worked in the German-speaking universities of Prague, Vienna, and Zürich. The next chapter comprises four richly detailed essays on Jewish mathematicians at the universities of Berlin, Göttingen (with special emphasis on the activities of Otto Blumenthal, Richard Courant, Emmy Noether, and Paul Bernays), Bonn, and Frankfurt. The first three of these universities are ancient and legendary, and their mathematical giants are well known to all of us.

Frankfurt University was different. Founded in 1914, it welcomed women students from the start and was forbidden, by statute, to consider religion in making appointments. Arthur Schoenflies, previously at Göttingen, was Frankfurt’s first full professor in mathematics. Perhaps not coincidentally, his masterpiece, *Krystallsysteme und Kristallstruktur* (1891), enumerating the 230 three-dimensional crystallographic groups, was newly recognized as a key to determining crystal structures (x-ray diffraction was discovered in 1912). Schoenflies died in 1928. His five children were Nazi victims. The catalogue also celebrates Frankfurt’s seminar in the history of mathematics, organized largely by Max Dehn. (Today we would describe it as a “great books” or “great works” seminar.) André Weil attended as a young student. “A text would be chosen and read in the original, with an effort

---

to follow closely not only the superficial lines but also the thrust of the underlying ideas.” In his autobiography (quoted in the catalogue), Weil describes Dehn as “a humanistic mathematician who saw mathematics as one chapter—certainly not the least important—in the history of human thought.” Ironically and tragically, one of the leading historians of mathematics in Germany, Kurt Vogel, wrote an anti-Semitic article in 1939, evidently to gain political support for this new academic field. The first page of his “Mathematics and Jewry” is shown (without English translation) on page 209 of the catalogue.

Chapter 4, compiled by Walter Purkert, “displays a selection of classic monographs, influential textbooks, and in some cases the collected works and papers of German-Jewish scholars. The list of authors and works spans a wide variety of mathematical fields and their applications,” including but not limited to abelian groups, integral operators, Fourier integrals, set theory, elliptic functions, aerodynamics, hydrodynamics, geometry of numbers, game theory, and probability theory. Next, in “Professional Commitment”, Moritz Epple and Volker R. Remmert describe the important role of the Springer publishing house in the Weimar era and Ferdinand Springer’s close cooperation with Richard Courant and other mathematicians, especially at Göttingen. Here too are periodicals, from Crelle’s Journal to the Mathematische Zeitschrift and the Mathematische Annalen, that counted Jewish mathematicians among their authors and/or editors. This essay includes Otto Blumenthal’s letter to Hilbert in 1933, offering to resign as managing editor of the Annalen for the journal’s sake. “I consider it my obligation to resign from my position, should you think that my ancestry or my uncertain situation as a dismissed university professor or anything else related to my person could damage the reputation or the efficacy of the Annalen,” he wrote, though “giving up this work will cause me pain.” Also, in “Mathematics in Culture” Birgit Bergmann describes the leading roles of Jewish mathematicians in popularization and dissemination.

The chapters just described have a twofold purpose. First, they show beyond question that Jewish mathematicians made fundamental contributions to all fields of twentieth-century mathematics and to all activities of the profession. Second, in Purkert’s words, they “completely disprove anti-Semitic stereotypes that had claimed the existence of a typical form of ‘Jewish mathematics’, remote from geometrical intuition or from applications.” Sadly, this nonsense did not vanish with the Nazis.3

3For a modern elaboration see, for example, Steven Gimbel, Einstein’s Jewish Science: Physics at the Intersection of Politics and Religion, The Johns Hopkins University Press, 2012.

In Chapter 8, “Dismissal and Exile”, Vogt lays bare the role of the Deutsche Mathematiker-Vereinigung (DMV) in the sordid 1930s. Twenty-two of its twenty-nine pages are primary documents (some also in English translation) and related photographs. The list of documents needs no comment:

* A letter (September 7, 1933) from the Prussian “Minister for science, art, and education of the people”, dismissing Reinhold Baer from his position at the University of Halle-Wittenberg.
* The questionnaire that Baer had had to fill out, including his religion (“evang.—bis 1920 mosaisch”) and his parents’ and grandparents’ religions (all of them “mosaisch”).
* A 1936 article by Emil Julius Gumbel, who had been dismissed from Heidelberg University for his political activities in 1932. “These Heidelberg professors, just like their colleagues all over the Reich, showed no fortitude,” he wrote. “Not a word of protest was uttered against the removal from office of so many scholars of outstanding merit… the idea of the university, the idealism, the intellectual forces…all of this evaporated when it came down to their pension rights.”
* A letter dated March 28, 1939, from DMV board member Emanuel Sperner to his colleagues Wilhelm Süss, Conrad Müller, and Helmut Hasse, dividing the “non-Aryan and foreigners” still in the DMV into three categories: Jews in Germany, Jews outside of Germany, and Foreigners.
* A letter from Süss to Friedrich Hartogs, four months later: “Dear Professor, You can no longer be a member of the German Mathematical Society. I therefore advise you to declare your resignation from our association. Otherwise we shall announce the termination of your membership at the next opportunity.” Signed, “Our sincerest respect, The President, SÜSS.” Süss sent this letter to other Jewish mathematicians as well.
* The list of sixty-seven mathematicians in the “List of Displaced German Scholars” compiled by the Academic Assistance Council, London, 1939 (most but not all of these mathematicians were Jewish).
* A letter from Friedrich Wilhelm Levi to Süss, 1950, describing the difficulties of living in Bombay.
* A photograph of Felix Hausdorff.
* A letter from Courant to Hausdorff (1939) expressing regret that he has been unable to find
a placement abroad for Hausdorff (presumably because of his advanced age).


After the war, Süss founded the Mathematisches Forschungsinstitut at Oberwolfach and made it into the very symbol of the international mathematical community. Though he had been a party (and SS) member, he successfully recast himself as a sort of Oskar Schindler, a Nazi who had used his position as president of the DMV throughout the war to protect Jews from within. The truth remained hidden until late in the twentieth century.4

"More than 65 years after the end of the Nazi regime," Vogt concludes this chapter, "additional questions can and should be raised, also with regard to the history of the DMV." There were honorable men: a Berlin pharmacologist, Otto Krayser, rejected a professorship in Frankfurt because its previous occupant had been dismissed. "One question which remains open," notes Vogt, "is whether there were other scientists who rejected positions at German universities because the previous holders had been dismissed. Where, in what disciplines, at what universities were there other scientists who did not wish to benefit from the injustices committed, who declined a promotion at this price? Mathematicians and students of mathematics should raise such questions now, and not shirk away from similar issues."

Three Jewish mathematicians—Friedrich Wilhelm Levi, Hans Hamburger, and Reinhold Baer— returned to a German university after the war. Most did not. "In a country being responsible of the cruel murder of five million Jews I could not breathe," Abraham Fraenkel told Erich Kamke in 1947. The rector of the University of Kiel had the year before inquired as to "whether [Fraenkel] would be interested in returning to the position from which he had been dismissed in 1933." Fraenkel’s letter of refusal, which he sent in both Hebrew and English, is reprinted in the catalogue.

The DMV was reestablished in 1948, with Erich Kamke, not Süss, at its head. Kamke invited expelled mathematicians to rejoin. The catalogue does not tell us if any accepted. Max Dehn was one who refused. He was in touch with German colleagues again, he said, but the DMV was another matter. "I have lost the confidence that such an association would act differently in the future than it did in 1935. I fear it would, once again, not resist an unjust measure coming from outside. …I am not afraid that the DMV will once again expel Jews, but perhaps next time it will be so-called communists, anarchists or ‘colored people’.

But, gradually, Jewish mathematicians returned for visits and for conferences at Oberwolfach; "on a personal level there were feelings, as one would expect, but on a professional level, people wanted mathematics to come first."5 "Substantial further historical research will be necessary in order to chart the landscape of Jewish émigré mathematicians and their relations to post-war Germany,” says Volker Remmert in the catalogue’s concluding chapter. "Mathematicians and historians have only just taken the first steps, and much remains to be done."

In lieu of a concert with Pick’s violin, the John Crerar Library at the University of Chicago mounted an exhibition of its own holdings, curated by graduate student Miriam Bilsker, to complement “Transcending Tradition”. Subtitled “A Bridge from Germany to America”, this exhibition featured material “relating to Jewish mathematicians who fled to America, their friends, and those influenced by them.” These included letters, articles, and photographs from the papers of James Franck, a Nobel laureate physicist who “was close friends with many of the mathematicians highlighted in Transcending Tradition” and the papers of Emil Gumbel, mentioned above; these demonstrate, says the library’s website, the difficulties of a life in exile. “Finally, the exhibit connects the intellectual world of the University of Chicago to the German academic world. It examines the figure of Saunders Mac Lane, the public intellectual and head of the University of Chicago mathematics department who received his doctorate from Göttingen in the 1930s. It presents a look at a short-lived University of Chicago exchange program with the University of Frankfurt which was designed to contribute to the re-education of Germans and to the maintenance of world peace.”6

An auxiliary exhibit such as this could be mounted in many places in this country. “By the end of the war the total migration [of mathematicians reaching America from the German-language world] was somewhere between 120 and 150,” tallied Nathan Reingold. “The actions of the American mathematicians,” he said, “is a story of the influence of the ideology of the universality of science; an example of transcendence of a different sort. Of course, one could argue that the American mathematicians were a small minority of a much larger group of émigré scientists, but that is not the point. The point is that the American mathematicians were a significant minority of a larger group of scientists, and that their actions had an impact on the world.”

5M. Senechal, “Oberwolfach 1944–1964”, The Mathematical Intelligencer, Vol. 20, no. 4, 1998, quoting Prof. John Todd. I wrote this article shortly after the DMV files were deposited in the archives of Freiburg University and opened to scholars, but I was unable to study them. I would like to thank Matthias Kreck for warning me that the portrait of Süss-as-Schindler would have to be redrawn. [http://news.lib.uchicago.edu/blog/2012/10/04/transcending-tradition-a-bridge-from-germany-to-america/]

of the hazards of Depression conditions; of the reactions to the policies of Nazi Germany; of the influence of nationalistic and anti-Semitic feelings in the United States; and of the persistence of the image of the United States as a haven for the oppressed. It is a story of a real world far removed from the certainty and elegance of mathematics as a monument to human rationality.\(^7\)

American mathematicians and historians have not shied away from this story, but we can and I think should do more to highlight the experiences of émigré mathematicians in the smaller or out-of-the-way colleges and universities that welcomed them in those terrible times. Life in America was difficult for many émigrés. They faced resentful colleagues, covert and not-so-covert anti-Semitism, heavy teaching loads, and language difficulties. Reingold adds that many “were startled and troubled by the different methods and attitudes in teaching in American colleges. Very few realized, as one émigré later wrote, ‘It takes a long time for anyone not born or brought up in this country to realize...that...the primary aim of a college...is to educate members of a democratic society.’” But that is only part of a more nuanced picture.

Antoni Zygmund, a Polish refugee (not Jewish), taught at Mount Holyoke College from 1940 to 1946; the “Zygmund Collection” is on permanent display in the mathematics department seminar room. MacTutor tells us that Zygmund “later spoke of the peacefulness and security that Mount Holyoke had brought to his family after the distress of their war-time experiences.” Mount Holyoke awarded Zygmund an honorary degree in 1988; the honor was mutual.

The November 2010 issues of the Notices carried a review of Siegmund-Schultze’s excellent book *Mathematicians Fleeing from Nazi Germany*. The reviewer, Michèle Audin, remarks that “such distinguished mathematicians as Max Dehn and André Weil were condemned to positions at Black Mountain College and Lehigh University, respectively.”8 Weil and Lehigh can speak for themselves, and Weil did so in his autobiography. But what do we know about Dehn at Black Mountain? (The college closed in 1956, four years after his death.) Evidently, he was happy there. Black Mountain College, in North Carolina, was avant-garde, not backwater, and Dehn valued that. It was founded in 1933 as a small and fervent bastion of academic freedom, “a community of learning in which decisions would be made in a democratic fashion through consensus reached in open meetings involving faculty and students,... The college was owned as a corporation by the faculty as a whole, and there was no outside board to exercise control. There were no required courses.”9 The arts held center stage, and many faculty and students were or became famous: Joseph and Anni Albers, John Cage, Merce Cunningham, Buckminster Fuller, and Robert Rauschenberg are a few. Dehn joined the faculty in January 1945, the end of an odyssey from Nazi Germany to Scandinavia, Russia, and Japan, and several stops in the United States. Dehn and Black Mountain College were a good fit; “Ornaments and rhythms—for Dehn these two were an expression of ‘the mathematical ability in humans.’”10 If the great topologist who had solved Hilbert’s third problem and organized the Frankfurt seminar in the history of mathematics felt he’d been condemned to his position, his colleagues never knew it. Black Mountain College is “a wonderful place where I can be together with young people without any institutional impediments,” he told Albers. “There, I can use what little abilities I have to transmit to them what I think is leading most surely towards a happy life.” He is buried on the campus, today the site of arts festivals.

Most stories did not have such happy endings, but I will end this review on another upbeat note, to record a footnote for posterity. Fritz John (1910–1994), Jewish on his father’s side, left Germany in 1933 for England; in 1935 he was appointed assistant professor of mathematics at the University of Kentucky in Lexington. Back in the 1930s, the University of Kentucky was small and isolated but, except for two years of war-related work, John stayed there until 1946, when he moved permanently to New York University. Surely he was glad to rejoin his mentor, Courant. But he made a difference in Lexington; I don’t know if he ever knew it.

I grew up near Lexington and took piano lessons from a teacher in town named Helen Lipscomb. Helen was a polio victim, confined to a wheelchair; her brother, Bill, was a chemist at the University of Minnesota. I met Bill Lipscomb for the first time in 2009, two years before he died at the age of ninety-two. By then he’d taught at Harvard for forty years and earned a Nobel prize (1976) for his work on boranes. Unlike me, Bill had attended the University of Kentucky after a Lexington public high school; he’d had a music scholarship and studied chemistry on the side. “Why did you decide to become a chemist instead of a musician?” I asked him. “What changed your mind?” “A math class,” he told me. “A math class taught by a German named Fritz John.”

---


\(^{10}\)Transcending Tradition, p. 119. For “ornaments” read “ornamental patterns” and “crystallographic groups”.

---

**February 2013**

**Notices of the AMS**

213
David Mumford—Selected Papers, Volume II

Reviewed by Frans Oort

The volume under review reproduces all papers by David Mumford in algebraic geometry not already included in Volume I (see [1]) and mathematical correspondence between Grothendieck and Mumford, plus several letters from Grothendieck to other mathematicians. Hence all papers in algebraic geometry by David Mumford are now collected and available in these two volumes [1], [2]. Let me first say a few words about the mathematics of Mumford.

The Style of Mumford

From the Autobiography of David Mumford: “At Harvard a classmate said ‘Come with me to hear Professor Zariski’s first lecture, even though we won’t understand a word’ and Oscar Zariski bewitched me. When he spoke the words ‘algebraic variety’, there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible. Especially, I became obsessed with a kind of passion flower in this garden, the moduli space of Riemann. I was always trying to find new angles from which I could see them better.” (1996; see [5], p. 225.)

Frans Oort is emeritus professor in pure mathematics at the University of Utrecht, the Netherlands. His email address is f.oort@uu.nl.

DOI: http://dx.doi.org/10.1090/noti951

During the period starting around 1960 and ending in the 1980s, David Mumford amazed us with new approaches to old problems in algebraic geometry. He rekindled interest in classical geometric ideas using modern methods.

Meeting Mumford and attending lectures by him meant an encounter with sparkling new ideas. The same holds true for reading the twelve books and more than sixty papers Mumford wrote in the years 1959–1982, when he was active in algebraic geometry. These carry fundamentally new insights, such as:

- revival of old ideas and techniques, long forgotten and reactivated by Mumford in a new spirit;
- unexpected views, questions, and directions in mathematics; and
- a deep understanding of the material studied, developing a sure and precise grip on the essence of topics.

His publications give us back that beautiful geometric feeling that was getting more and more lost in the algebraization and the functorialization of geometry.

His style is unique and fascinating. In a period when writing mathematics was increasingly done in a way where every symbol had many indices, where trees of definitions and concepts were difficult to climb, Mumford found a mathematical language that is clear and leads you straight to the central idea without losing precision. In reading Mumford, you’d better have a piece of paper and pencil at hand, because many arguments have to be worked out in greater detail by the reader himself, only to discover at the end that the author is correct, that he must have thought through all details of the situation being considered. This alone makes reading Mumford’s papers fascinating and stimulating.

In many cases Mumford does not aim at the greatest generality. The basic idea, the immediate
intuitive approach to the problem studied is central. Often he brings new insight and a fresh approach to classical questions. His work opens windows and gives rise to new developments.

As John Tate describes this: “Mumford has carried forward, after Zariski, the project of making algebraic and rigorous the work of the Italian school ... Mumford’s main interest [is] the theory of varieties of moduli. This is a central topic in algebraic geometry having its origins in the theory of elliptic integrals. The development of the algebraic and global aspects of this subject in recent years is due mainly to Mumford, who attacked it with a brilliant combination of classical, almost computational, methods and Grothendieck’s new scheme-theoretic techniques.” (1974, see [4]).

It would be wonderful to document developments arising from and stimulated by his pioneering work. He is generous to many of us, in contact and in writing, producing beautiful ideas and results and leaving open roads to new thoughts. We can hardly underestimate the influence this has had on all of us.

From his rich source of ideas, he often awarded inspiration to other mathematicians, who then finished the details. I have myself experienced this twice, and I am still very grateful for those opportunities.

About Selected Papers, Volume I

For a review of the first volume, see [7] and [8]. In that book, just about half of the papers by Mumford in algebraic geometry were published. As those two reviews point out, there were several flaws. It was not clear why some papers were not reproduced, and the papers did not appear in chronological order. That volume contains five different lists of references. No two agree. There are painful mistakes, such as page numbers that are omitted or wrong; names that are misspelled; references that are unsystematically abbreviated, even within the same list; papers from the volume itself that are referred to by different titles. We missed important papers by Mumford, such as the paper with Deligne, “The irreducibility of the space of curves of given genus”, [69e], appearing now in Volume II (with 325 citations, one of the most influential papers in modern algebraic geometry). But we also missed papers that were hard to find.

About Selected Papers, Volume II: Papers and Notes

The present volume corrects these flaws. The book contains a precise bibliography of Mumford. This is very useful; we can really trust this list. Furthermore, this volume contains all papers not appearing in Volume I.

The editors have done a great job of writing notes about the papers. In addition to correcting misprints, the notes indicate new developments and comment on information not available at the time a particular paper was written. This additional material makes reading these articles even more interesting.

Just sit down with this volume and be overwhelmed by ideas from, say, fifty years ago, still new and inspiring now. Read through the paper [61a] “The topology of normal singularities ...” or [65d] “Picard groups of moduli problems”, just to mention two of the papers featured here, and you see the fresh look, the powerful approach, and the inspiration communicated to the reader. But also read [78d], “Fields Medals (IV): An instinct for the key idea”, where Mumford and Tate describe work by the 1978 Fields Medalist Pierre Deligne; we see the unique quality of work by Deligne, its place in the history of algebraic geometry, and interaction with mathematics by Grothendieck in just two pages. Such papers collected in this volume give insight into this field in the years 1960-1980.

The Correspondence

Mumford wrote in 2008: “For me, personally, Grothendieck’s letters were priceless and enabled me to understand many of his ideas in their raw form before they were generalized too far and embedded in the daunting machinery of his ‘Élements’.” See [2], p. 5.

In this volume we see the fruit of a fascinating labor: reproducing a correspondence. It is a unique source of information and inspiration. We can compare a paper with the comments by Grothendieck on a preliminary version. Moreover, the editors have provided more than 160 footnotes explaining ideas in those letters and providing recent developments and references. Reviewing, typesetting, and editing this material was a big task, and we can be very grateful to the editors for this valuable work. Together with the Grothendieck-Serre correspondence [6], this book gives a beautiful picture and stimulating ideas around the development of algebraic geometry in the years 1960–1985. We know that Grothendieck destroyed many personal papers; as a consequence we mainly have the letters of Grothendieck to Mumford, while the greater part of the other half of the correspondence is lacking. This makes guessing even more interesting, though we would have appreciated seeing the other half.

Let me say some words about the interaction between Grothendieck and Mumford, the deep respect on both sides, and the difference in their style of research, thinking, and writing.

First Interaction between Mumford and Grothendieck

I remember meeting Grothendieck in 1961 in a Paris street; both of us were going to the same lecture. Grothendieck mentioned a construction made by a young American mathematician. In a 1961 letter to Grothendieck, Mumford described

February 2013 Notices of the AMS
his proof of “the key theorem in a construction of the arithmetic scheme of moduli of curves of any genus.” Grothendieck was excited about this idea, apparently completely new to him. Later Mumford pinned down the notion of a “coarse moduli scheme”, necessary in case the obvious moduli functor is not representable by a variety (or by a scheme). (See [2], pp. 635–638, where we see this excitement of Grothendieck reflected in several letters to Mumford.) Grothendieck explained that for “higher levels” he could represent moduli functors, but for all levels he could not perform the necessary construction. Mumford announced this method and explained it in his paper [61c]. Grothendieck, although positive about the new ideas of the young Mumford, wrote in 1961: “It seems to me that, because of your lack of some technical background on schemata, some proofs are rather awkward and unnatural” ([2], p. 636).

To French mathematicians of that period a construction in algebraic geometry should be done by solving a “universal problem”, in their terminology, by “representing a functor”. Igusa constructed (Ann. Math. 72 (1960), 621–649, in terminology developed by Mumford later) the coarse moduli scheme of curves of genus 2 over \( \mathbb{Z} \). To the “functorial thinking” of Paris mathematicians in 1961 this was strange, and we taste this atmosphere in the description by Samuel of this construction by Igusa: “Signalons aussitôt que le travail d’Igusa ne résoud pas, pour les courbes de genre 2, le problème des modules’ tel qu’il a été posé par Grothendieck à diverses reprises dans ce Séminaire. (We note that the work by Igusa for curves of genus 2 does not solve the problem of ‘moduli of curves’ as proposed by Grothendieck several times in this seminar.)” See the first lines of [3]. This is the background of the difference between Paris and Harvard mathematics at that time. Grothendieck classified work by Igusa as “most discouraging to read”; see [2], p. 636. (I like that paper of Igusa; it was on my desk on many occasions.)

Mumford completed this construction of moduli spaces over any base scheme (curves of any genus, polarized abelian varieties); it was a starting point of new developments. A clear line led from classical invariant theory to ideas by Igusa for \( g = 2 \) to Mumford’s geometric invariant theory, thereby creating an aspect of research that complemented Grothendieck’s work on these topics.

The stimulating difference between these two giants, Grothendieck and Mumford, their insight, and the respect they had for each other are a source of rich ideas. What a privilege for us to feel stimulating aspects contained in Mumford’s papers reproduced in these volumes and to be able to enjoy the exchange of their ideas, surviving in (part of) their correspondence, which is now available thanks to this beautiful volume.

Some Remarks about Grothendieck

Alexandre (Alexander) Grothendieck was active in algebraic geometry in the period 1958–1970. His style was fundamental. Anything he considered would be done in the most general situation. If there is a general solution, we can be sure Grothendieck puts us on the right track. The revolution he started (in algebraic geometry) has been fruitful, although not many of us can perform on the same heights and at the same level of abstraction.

We now have (partial) access to a fascinating biography. Who Is Alexandre Grothendieck? is a projected 3-volume biography of Alexandre Grothendieck.


Volume I is complete and available in German and English (2009). Volume III is complete and available in German (2010). Volume II is in preparation.

A few years ago Grothendieck sought to block publication of his unpublished writings:

Declaration of intent of non-publication. I do not intend to publish or republish any work or text of which I am the author, in any form whatsoever, printed or electronic, whether in full or in excerpts, texts of personal nature, of scientific character, or otherwise, or letters addressed to anybody, and any translation of texts of which I am the author. Any edition or dissemination of such texts which have been made in the past without my consent, or which will be made in the future and as long as I live, is against my will expressly specified here and is unlawful in my eyes. As I learn of these, I will ask the person responsible for such pirated editions, or of any other publication containing without permission texts from my hand (beyond possible citations of a few lines each), to remove from commerce these books; and librarians holding such books to remove these books from those libraries.

If my intentions, clearly expressed here, should go unheeded, then the shame of it falls on those responsible for the illegal editions, and those responsible for the libraries concerned (as soon as they have been informed of my intention).


(Translated by Scott Morrison; see http://sbseminar.wordpress.com/2010/02/09/grothendiecks-letter/ For the original French...
In 1985 Mumford wrote to Grothendieck: “...the letters that you wrote me are by far the most important things which explained your ideas and insights. The letters are vivid and clear and unencumbered by the customary style of formal French publications...My proposal would be to approach someone with a broad knowledge of your theories...and give...permission...I feel sure that such a collection would be extremely useful to the younger generation.” See [2], p. 750. We can be glad that the editors of the present volume (clearly with “broad knowledge of your theories”) got the permission of J. Malgoire (who had power-of-attorney for Grothendieck) to publish (part of) the correspondence of Grothendieck contained in this volume; see [2], pp. v and xii.

I put this beautiful volume back on my shelf, and I am sure I will consult it again many times.

References

Notices: One controversial issue that arose during your presidency was the Engage to Excel report by PCAST [President’s Council of Advisors on Science and Technology]. The Notices had two articles about this report in the October 2012 issue. What is the current status?

Friedlander: PCAST operates within the Office of Science and Technology Policy. Once a report is published, PCAST shifts its focus to other topics. Its most recent report is entitled Propelling Innovation in Drug Discovery, Development, and Evaluation. The report Engage to Excel will remain in place, and it will continue to have an influence on those concerned with the education of students aiming for STEM [science, technology, engineering, and mathematics] careers. There may be some new federal funding that will support some of its recommendations, and there are influential scientists who press the point of view in this report.

The PCAST report—which, in my view, is extremely unfair, unbalanced, and uninformed about what the mathematical community is doing—none-theless could be seen as constructive and useful. It served as a wake-up call. In most colleges and universities, mathematicians are trying various ways of reaching out to students who have weak mathematics backgrounds or who are undermotivated and of reaching out to other disciplines that wish their students to have certain mathematical skills or information. We should be doing more, and we should have a repository of information about what we are doing, but I don’t think the solution is to be found in some special teaching technique that would suddenly make us all great teachers. We need a more fundamental assessment of the future of mathematics education. I would like to see the AMS and the mathematics community shine a brighter light on what we can do and how we publicize what we are doing in mathematics education.

My concern with the PCAST report philosophically was this. We have to worry about not just the need for one million new STEM graduates—not just the increased numbers—but the quality of the education we offer and the knowledge students acquire. That message gets lost if you emphasize so much that we need to increase the numbers. The solution is not to dumb down the mathematics presented or to remove mathematics from the curriculum. We need to include and embrace as many people as possible. If we don’t, the basis of science will shrink to nothing. At the same time, we can’t say that certain aspects of science are too difficult or should be left for only a few elite people.

Notices: Another issue that came up during your presidency is “gold open access” publishing. Can you describe what gold open access is and how it might affect the math community and the AMS?

Friedlander: This is a really big deal. It concerns the AMS considerably. “Gold open access” means the author pays the cost of publication. “Open access” means that, once published, the publication is freely available, usually on the Web. In the current model, libraries pay a great deal of money to subscribe to journals, and that’s how journals are supported. Gold open access turns this around and makes that support come from authors.

This movement is related to the National Institutes of Health stipulation that work coming out of its grants should be made freely available shortly after publication—six months, or something like that. Although the details are not clear, both Britain and Germany seem to be suggesting that if you get

DOI: http://dx.doi.org/10.1090/noti956
national government funds, you should publish articles in gold open access journals. That’s really dramatic.

Also wrapped up in this is the Elsevier boycott, which involved many prestigious mathematicians, including Timothy Gowers and Terence Tao. The boycotters asserted that Elsevier was making much too much money and was behaving badly, so we should avoid working with Elsevier and should find new ways of publishing. One recent development is the creation of two new journals at Cambridge University Press, called Sigma and Pi. They will be electronic only and open access, and for the first three years CUP will absorb all costs. After that the journals will be based on the author-pay model.

This model is extremely tricky for the mathematics community. One hope is that mathematicians will get grant funds to cover author costs, which might be on the order of US$2,000 per article. At the moment, almost no grants have such funds. Another hope is that universities will say, “We now have lower subscription costs for libraries, so we’ll pass the savings down to individual departments, which will pass them down to individual faculty, who can use the money to pay publishing costs.” Some of us who have observed how university finances work doubt that that’s going to happen. A third possibility is that people will just pay out of their own pockets. If you want to get a job, if you want to get tenure, if you want to get promoted, if you want to get recognized, then you need publications. And you will pay for them. That one would need to be able to bankroll one’s career would be extremely unfortunate. I get one or two ads a week asking me to publish in open access journals—they are sprouting up all over the place, and they are strictly money making. A lot of vanity presses are seeking to exploit scientists’ need to publish. So there are real concerns about the author-pay model.

Mathematics is in a very awkward position because we are a small science. Many of us work by ourselves or maybe with one or two other people. Many of us work without grants. In areas such as biology or medicine, the grants are large and have publication funds. The sad thing is, governments may make decisions based on the dominant role of the biological and medical sciences. That’s where most of the money and the numbers are. Mathematics will probably have to go along with the decision of governments about gold open access, even though what’s best for biology is not necessarily what’s best for mathematics. It really is a quandary for us. At the moment, we are scrambling to position ourselves to comply with whatever governments stipulate. The AMS has made strong statements that we would not base a decision about acceptance of a paper on the ability of the author to pay.

Notices: One issue within the AMS that came up during your term is the Fellows Program. What is your sense of the reaction within the math community?

Friedlander: Most people are very enthusiastic. There was a minority from the beginning who called it an elitist program and didn’t like recognizing people who, from their point of view, had already been recognized. I’ve heard a few disgruntled comments but also many enthusiastic comments. I already know from my own university that this is going to give local recognition to mathematics. So I think the response is going to be very positive. The immediate response to the process of naming fellows has been that some people realize that their membership has lapsed; they would have qualified to be in the initial seed pool to become fellows, but they are no longer members. Messages from such excellent mathematicians show the value of the Fellows Program in the eyes of many.

Notices: If such people reinstate their membership, can they become fellows?

Friedlander: Yes, but it is not automatic. To start the program, a large number of mathematicians will become inaugural fellows in 2012, typically because they gave an International Congress of Mathematicians talk or an ICIAM [International Congress of Industrial and Applied Mathematics] plenary talk or an AMS invited address. Also, they must have been members for the last two years. Beginning in 2013, there will be a Fellow Selection Committee consisting of current fellows, and the committee composition will change every year. They will use whatever criteria they see fit to choose new fellows. But it will not be the case that those people who were not members but would otherwise have qualified to be inaugural fellows would automatically be in the next year.

I think the Fellows Program is good for mathematics and good for mathematicians. One thing I am very concerned about is membership, and I think the program is going to bolster membership in the AMS. A few people are going to resign because they object to the Fellows Program, but many more will join because they want to become fellows. You can’t be a fellow without being a member. Also, they’ll feel some identification with the AMS because they are fellows.

Notices: Another new program is the AMS graduate student chapters. How do they work?

Friedlander: We’ve identified, I think, twelve chapters to start in 2012. It will soon probably be over one hundred. A group of students with a faculty leader will write to the AMS and say, “We’d like to be an AMS chapter.” The AMS will respond and say, “Great, welcome aboard, and here is US$500 that you can spend on pizzas or movies or inviting speakers.” We hope to keep the bureaucracy to a minimum and to make the chapters a focus for graduate students in departments. Also, it will be
One thing I have found very striking is that we have many graduate student members— they used to be called nominee members—and yet, when I would visit various universities, the students didn’t know they were members. We are hoping the student chapters will make students more aware of the AMS and begin to get involved with the AMS. We also hope the chapters can help to create a good atmosphere in departments for graduate students. This is not a new idea. Other mathematical organizations already have student chapters. The AMS at the present time is in a sufficiently sound financial situation that it can offer small amounts of money to further support student activities.

Another thing we are starting is activity groups. This is something SIAM [Society for Industrial and Applied Mathematics] has had for many years. In fact, SIAM members often identify themselves more with their activity group than with SIAM itself. There has been a lot of discussion within the AMS, and the concept for the AMS activity groups has been changing over time.

**Notices:** What is the current idea of how it might work?

**Friedlander:** The current idea is to make it completely virtual, perhaps a combination of social networking and MathOverflow, based on people’s areas of interest. MathOverflow can be a little intimidating, because it rates the quality of people’s postings. The AMS model would be more informal. Although it would not be feasible for all the activity groups to meet at sectional or national meetings, one or two groups could get together and decide to organize one or two special sessions at an AMS meeting or to get together for lunch or dinner. The idea is to make it simple in concept and simple to run. The groups would provide an organization in which people with similar mathematical interests can get together virtually.

**Notices:** Is there anything else that the AMS is doing that you would like to talk about?

**Friedlander:** We have established an AMS Committee on Women in Mathematics. I hope this committee will have some financial resources behind it and can make specific recommendations. There is already a Joint Committee on Women with maybe eight mathematical organizations participating in it. That committee is a good nexus, but it doesn’t really have the ear of the AMS. I hope the new committee will.

I also want to mention how good the AMS staff is, from the leaders, like Don McClure, all the way down. They are remarkably dedicated, and it’s an amazingly good organization. Ideas get carried out. What I’ve found is that I have to be very careful not to propose too many things, because there could be gridlock!

One thing I would like to see us do is to have more contact with and influence in state and federal government. Maybe mathematics is too small to have a lot of influence, but I think we can do more. There are many other things I could tell you, but this is more than enough!

**Notices:** One last question. During your term as president, you talked to many mathematicians about the state of mathematics and the math profession. What are your impressions of the biggest problems, and the biggest successes, in the field right now?

**Friedlander:** The successes in the field are fantastic theorems, important theorems, applications of mathematics—the scientific successes.

But there are lots of worries about the overall health of the mathematical community. I am particularly worried about mathematics education at the university level. There is a lot of skepticism out there, and not just from the PCAST report. We need to think about how we’re going to teach and reach out to people who need mathematics. Some of them are interested in mathematics, some will become mathematicians—and some of them are not interested in mathematics but still need mathematics. If the AMS can help in the process of calling attention to good practices for doing this and being a reservoir of information and materials, that would help a lot. Another aspect is MOOCs, massive open online courses. Who knows how that’s going to influence mathematics teaching? Maybe one professor who normally teaches computer science will instead teach 500,000 students first-year calculus through a MOOC. Students don’t learn that way, and we mathematicians have to say so. We have to be aware that education doesn’t stand still.

I also see more and more that fundamental research is under threat. The NSF [National Science Foundation] now has grants that are specifically not to support research but to support the translation of research into the marketplace. It used to be that the NSF was the one place in the U.S. federal government that supported fundamental research. It still does, but it’s broadening its mission. There is a view of some that mathematics is fine—provided it informs investigations of, for example, genetics. We have to make the case that mathematics itself, and not just applications, is very important. Maybe the need to make this case has been there for millennia; maybe now is not a special time. But I’m more aware of this need now than I had been.
Interview with Endre Szemerédi

Martin Raussen and Christian Skau

Endre Szemerédi is the recipient of the 2012 Abel Prize of the Norwegian Academy of Science and Letters. This interview was conducted by Martin Raussen and Christian Skau in Oslo in May 2012 in conjunction with the Abel Prize celebration. This article originally appeared in the September 2012 issue of the Newsletter of the European Mathematical Society and is reprinted here with permission of the EMS.

Raussen and Skau: Professor Szemerédi, first of all we would like to congratulate you as the tenth Abel Prize recipient! You will receive the prize tomorrow from His Majesty, the King of Norway.

You were born in Budapest, Hungary, in 1940 during the Second World War. We have heard that you did not start out studying mathematics; instead, you started in medical school and only later on shifted to mathematics. Were you nevertheless interested in mathematical problems as a child or teenager? Did you like to solve puzzles?

Szemerédi: I have always liked mathematics and it actually helped me to survive in a way: when I was in elementary school, I was very short and weak and the stronger guys would beat me up. So I had to find somebody to protect me. I was kind of lucky, since the strongest guy in the class did not understand anything about mathematics. He could never solve the homework exercises, let alone pass the exam. So I solved the homework exercises for him and I sat next to him at the exam. Of course, we cheated and he passed the exam. But he was an honest person and he always protected me afterwards from the other big guys so I was safe. Hence my early interest in mathematics was driven more by necessity and self-interest than by anything else. In elementary school I worked a lot with mathematics but only on that level, solving elementary school exercises.

In high school I was good at mathematics. However, I did not really work on specific problems and, if I remember correctly, I never took part in any competitions. In Hungary there are different kinds of competitions. There is also a monthly journal KöMaL, where you may send in solutions to problems that are posed. At the end of the year the editors will add up points you get for good solutions. I never took part in this, the main reason being that my father wanted me to be a physician. At the time, this was the most recognized profession, prestigiously and also financially. So I studied mainly biology and some physics, but I always liked mathematics. It was not hard for me to solve high school exercises and to pass the exams. I even helped others, sometimes in an illegal way, but I did not do more mathematics than that.

My education was not the usual education you get in Hungary if you want to be a mathematician. In Hungary we have two or three extremely good elite high schools. The best is Fazekas, in Budapest; they produce every year about five to ten mathematicians who, by the time they go to the university, know a lot. I was not among those. This is not a particular Hungarian invention; also in the United States there are special schools concentrating on one subject. I can name a lot of mathematicians that are now considered to be the best ones in Hungary. Most of them (90 percent) finished the school at Fazekas. In Szeged, which is a town with about 200,000 inhabitants, there are two specialist schools also producing some really good mathematicians. One of those mathematicians was a student of Bourgain at the Institute for Advanced Study in Princeton who just recently defended his thesis with a stunning result. But again, I was not among those highly educated high school students.

R&S: Is it correct that you started to study mathematics at age 22?

Szemerédi: Well, it depends on how you define "started". I dropped out of medical school after half a year. I realized that, for several reasons, it was not for me. Instead, I started to work at a machine-making factory, which actually was a very
good experience. I worked there slightly less than two years.

In high school my good friend Gábor Ellmann was by far the best mathematician. Perhaps it is not proper to say this in this kind of interview but he was tall. I was very short in high school, at least until I was seventeen. I am not tall now, but at the time I was really short and that actually has its disadvantages. I do not want to elaborate. So I admired him very much because of his mathematical ability and also because he was tall.

It was actually quite a coincidence that I met him in the center of the town. He was to date a girlfriend, but he was fifteen minutes late so she had left. He was standing there, and I ran into him and he asked me what I was doing. Gábor encouraged me to go to Eötvös University, and he also told me that our mathematics teacher at high school, Sándor Bende, agreed with his suggestion. As always, I took his advice; this was really the reason why I went to university. Looking back, I have tried to find some other reason but so far I have not been successful.

At that time in Hungary you studied mathematics and physics for two years, and then one could continue to study physics, mathematics, and pedagogy for three years in order to become a mathematics teacher. After the third year they would choose fifteen out of about two hundred students who would specialize in mathematics.

**Turán and Erdős**

*R&S:* We heard that Paul Turán was the first professor in mathematics that made a lasting impression on you.

*Szemerédi:* That’s true. In my second year he gave a full-year lecture on number theory, which included elementary number theory, a little bit of analytic number theory, and algebraic number theory. His lectures were perfect. Somehow he could speak to all different kinds of students, from the less good ones to the good ones. I was so impressed with these lectures that I decided I would like to be a mathematician. Up to that point I was not sure that I would choose this profession, so I consider Paul Turán to be the one who actually helped me to decide to become a mathematician.

He is still one of my icons. I have never worked with him; I have only listened to his lectures and sometimes I went to his seminars. I was not a number theorist, and he mainly worked in analytic number theory.

*R&S:* By the way, Turán visited the Institute for Advanced Study in Princeton in 1948, and he became a very good friend of the Norwegian mathematician Atle Selberg.

*Szemerédi:* Yes, that is known in Hungary among the circle of mathematicians.

*R&S:* May we ask what other professors at the university in Budapest were important for you. Which of them did you collaborate with later on?

*Szemerédi:* Before the Second World War, Hungarian mathematics was very closely connected to German mathematics. The Riesz brothers, as well as Haar and von Neumann and many others, actually went to Germany after they graduated from very good high schools in Hungary. Actually, my wife Anna’s father studied there almost at the same time as von Neumann and, I guess, the physicist Wigner. After having finished high school, he and also others, went to Germany. And after having finished university education in Germany, most of them went to the United States. I don’t know the exact story, but this is more or less the case. After the Second World War, we were somehow cut off from Germany. We then had more connections with Russian mathematics.

In the late 1950s, Paul Erdős, the leading mathematician in discrete mathematics and combinatorics—actually, even in probability theory he did very good and famous work—started to visit Hungary, where his mother lived. We met quite often. He was a specialist in combinatorics. At the time, combinatorics had the reputation that you didn’t have to know too much. You just had to sit down and meditate on a problem. Erdős was outstanding in posing good problems. Well, of course, as happens to most people, he sometimes posed questions which were not so interesting. But many of the problems he posed, after being solved, had repercussions in other parts of mathematics—also in continuous mathematics, in fact. In that sense Paul Erdős was the most influential mathematician for me, at least in my early mathematical career. We had quite a lot of joint papers.

*R&S:* Twenty-nine joint papers, according to Wikipedia...

*Szemerédi:* Maybe, I’m not sure. In the beginning I almost exclusively worked with Paul Erdős. He definitely had a lasting influence on my mathematical thinking and mathematical work.

*R&S:* Was it usually Erdős who posed the problems, or was there an interaction from the very start?

*Szemerédi:* It was not only with me, it was with everybody. It was usually he who came up with the problems and others would work on them. Probably for many he is considered to be the greatest mathematician in that sense. He posed the most important problems in discrete mathematics which actually affected many other areas in mathematics. Even if he didn’t foresee that solving a particular problem would have some effect on something else, he had a very good taste for problems. Not only the solution but actually the methods used to obtain the solution often survived the problem itself and were applied in many other areas of mathematics.
R&S: Random methods, for instance?

Szemerédi: Yes, he was instrumental in introducing and popularizing random methods. Actually, it is debatable who invented random methods. The Hungarian mathematician Szegő used the so-called random method—it was not a method yet—to solve a problem. It was not a deterministic solution. But then Paul Erdős had a method yet—to solve a problem. It was not a random method—it was not introducing and popularizing random methods. Actually, it is debatable who invented random methods. Specifically, he proved that by two-coloring the edges of a complete graph with $n$ vertices randomly, almost certainly there will not be more than $2 \log n$ vertices so that all the connecting edges are of the same color.

In the United States, where I usually teach undergraduate courses, I present that solution. The audience is quite diverse; many of them do not understand the solution. But the solution is actually simple, and the good students do understand it. We all know it is extremely important—not only the solution but the method. Then Erdős systematically started to use random methods. To that point they just provided a solution for a famous problem but then he started to apply random methods to many problems, even deterministic ones.

And, of course, his collaboration with Rényi on the random graph is a milestone in mathematics; it started almost everything in random graph theory.

R&S: And that happened around 1960?

Szemerédi: Yes. It was in the 1960s, and it is considered to be the most influential paper in random graph theory. Their way of thinking and their methods are presently of great help for many, many mathematicians who work on determining the properties of real-life, large-scale networks and to find random methods that yield a good model for real-life networks.

Moscow: Gelfond and Gelfand

R&S: You did your graduate work in Moscow in the period 1967–1970 with the eminent mathematician Israel Gelfand as your supervisor. He was not a specialist in combinatorics. Rumors would have it that you, in fact, intended to study with another Russian mathematician, Alexander Gelfond, who was a famous number theorist. How did this happen, and whom did you actually end up working with in Moscow?

Szemerédi: This can be taken, depending on how you look at it, as a joke or it can be taken seriously. As I have already told you, I was influenced by Paul Turán, who worked in analytic number theory. He was an analyst; his mathematics was much more concrete than what Gelfand and the group around him studied. At the time, this group consisted of Kazhdan, Margulis, Manin, Arnold, and others, and he had his famous Gelfand seminar every week which lasted for hours. It was very frightening sitting there and not understanding anything. My education was not within this area at all. I usually had worked with Erdős on elementary problems, mainly within graph theory and combinatorics; it was very hard for me!

I wanted to study with Gelfond, but by some unfortunate misspelling of the name I ended up with Gelfand. That is the truth.

R&S: But why couldn’t you swap when you realized that you had got it wrong?

Szemerédi: I will try to explain. I was a so-called candidate student. That meant that you were sent to Moscow—or to Warsaw, for that matter—for three years. It had already been decided who would be your supervisor, and the system was quite rigid, though not entirely. I’m pretty sure that if you put a lot of effort into it, you could change your supervisor, but it was not so easy. However, it was much worse if you decided after half a year that it was not the right option for you and to go home. It was quite a shameful thing to just give up. You had passed the exams in Hungary and kind of promised you were going to work hard for the next three years. I realized immediately that this was not for me, and Gelfand also realized it and advised me not to do mathematics anymore, telling me: “Just try to find another profession; there are plenty in the world where you may be successful.” I was twenty-seven years old at the time, and he had all these star students aged around twenty, and twenty-seven was considered old! But in a sense, I was lucky: I went to Moscow in the fall of 1967, and in the spring next year, there was a conference on number theory in Hungary—in Debrecen, not Budapest. I was assigned to Gelfond; it was customary that every guest had his own Hungarian guide. I had a special role too, because Gelfond was supposed to buy clothes and shoes, which were hard to get in Russia at the time, for his wife. So I was in the driver’s seat because I knew the shops pretty well.

R&S: You spoke Russian then?

Szemerédi: Well, my Russian was not that good. I don’t know if I should tell this in this interview, but I failed the Russian exam twice. Somehow I managed to pass the final exam and I was sent to Russia. My Russian was good enough for shopping but not good enough for having more complex conversations. I only had to ask Gelfond for the size of the shoes he wanted for his wife and then I had a conversation in Hungarian with the shopkeepers. I usually don’t have good taste, but because I had to rise to the occasion, so to say, I was very careful and thought about it a lot. Later Gelfond told me that his wife was very satisfied. He was very kind and said that he would arrange the switch of supervisors!
This happened in the spring of 1968, but unfortunately he died that summer of a heart attack, so I stayed with Gelfand for a little more than a year after that. I could have returned to Hungary, but I didn’t want that; when I first agreed to study there, I felt I had to stay. They (i.e., Gelfand and the people around him) were very understanding when they realized that I would never learn what I was supposed to. Actually, my exam consisted of two exercises about representation theory taken from Kirillov’s book, which they usually give to third-year students. I did it, but there was an error in my solution. My supervisor was Bernstein, as you know a great mathematician and a very nice guy, too. He found the error in the solution, but he said that it was the effort that I had put into it that was important rather than the result, and he let me pass the exam.

To become a candidate you had to write a dissertation, and Gelfand let me write one about combinatorics. This is what I did. So, in a way, I finished my study in Moscow rather successfully. I did not learn anything, but I got the paper showing that I had become a candidate.

At this time there was a hierarchy in Hungary: doctorate of the university, then candidate, doctorate of the academy, then corresponding member of the academy, and then member of the academy. I achieved becoming a candidate of mathematics.

R&S: You had to work entirely on your own in Moscow?

Szemerédi: Yes, since I worked in combinatorics.

R&S: Gelfond must have realized that you were a good student. Did he communicate this to Gelfand in any way?

Szemerédi: That I don’t know. I only know that Gelfand very soon realized my lack of mathematical education. But when Gelfand came to Hungary, he talked to Turán and Erdős and also to Hungarian number theorists attending that meeting, and they were telling him: “Here is this guy who has a very limited background in mathematics.” This may be the reason why Gelfand agreed to take me as his student. But unfortunately he died early.

Hungarian Mathematics

R&S: We would like to come back to Hungarian mathematics. Considering the Hungarian population is only about ten million people, the list of famous Hungarian mathematicians is very impressive. To mention just a few, there is János Bolyai in the nineteenth century, one of the fathers of non-Euclidean geometry. In the twentieth century there is a long list, starting with the Riesz brothers, Frigyes and Marcel; Lipót Fejér; Gábor Szegő, Alfréd Haar; Tibor Rado; John von Neumann, perhaps the most ingenious of them all; Paul Turán; Paul Erdős; Alfréd Rényi; Raoul Bott (who left the country early but then became famous in the United States).

Among those still alive, you have Peter Lax, who won the Abel Prize in 2005; Béla Bollobás, who is in Great Britain; László Lovász; and now you. It’s all very impressive. You have already mentioned some facts that may explain the success of Hungarian mathematics. Could you elaborate, please?

Szemerédi: We definitely have a good system to produce elite mathematicians, and we have always had that. At the turn of the century—we are talking about the nineteenth century and the beginning of the twentieth century—we had two or three absolutely outstanding schools, not only the so-called Fasori, where von Neumann and Wigner studied, but also others. We were able to produce a string of young mathematicians, some of whom later went abroad and became great mathematicians—or great physicists, for that matter. In that sense I think the educational system was extremely good. I don’t know whether the general education was that good, but definitely for mathematics and theoretical physics it was extremely good. We had at least five top schools that concentrated on these two subjects, and that is already good enough to produce some great mathematicians and physicists.

Back to the question of whether the Hungarians are really so good or not. Definitely in discrete mathematics there was a golden period. This was mainly because of the influence of Erdős. He always travelled around the world, but he also spent a lot of time in Hungary. Discrete mathematics was certainly the strongest group.

The situation has changed now. Many Hungarian students go abroad to study at Princeton, Harvard, Oxford, Cambridge, or Paris. Many of them stay abroad, but many of them come home and start to build schools. Now we cover a much broader spectrum of mathematics, such as algebraic geometry, differential geometry, low-dimensional topology, and other subjects. In spite of my being a mathematician working in discrete mathematics who doesn’t know practically anything about these subjects, I am very happy to see this development.

R&S: You mentioned the journal KöMaL, which has been influential in promoting mathematics in Hungary. You told us that you were not personally engaged, but this journal was very important for the development of Hungarian mathematics. Isn’t that true?

Szemerédi: You are absolutely right. This journal is meant for a wide audience. Every month the editors present problems, mainly from mathematics but also from physics. At least in my time, in the late 1950s, it was distributed to every high school, and a lot of the students worked on these problems. In spite of solving the problems regularly, then by the time you finished high school you would know almost as much as the students in the elite high schools. The editors added the points you got from each correct solution at the end of
the year, giving a bonus for elegant solutions. Of course, the winners were virtually always from one of these elite high schools.

But it was intended for a much wider audience and it helped a lot of students, not only mathematicians. In particular, it also helped engineers. People may not know this, but we have very good schools for different kinds of engineering, and a lot of engineering students-to-be actually solved these problems. They may not have been among the best, but it helped them to develop a kind of critical thinking. You don’t just make a statement, but you try to see connections and put them together to solve the problems. So by the time they went to engineering schools, which by itself required some knowledge of mathematics, they were already quite well educated in mathematics because of KöMaL.

KöMaL plays an absolutely important role and, I would like to emphasize, not only in mathematics but more generally in natural sciences. Perhaps even students in the humanities are now working on these problems. I am happy for that and I would advise them to continue to do so (of course, not to the full extent, because they have many other things to study).

Important Methods and Results

R&S: We would now like to ask you some questions about your main contributions to mathematics.

You have made some groundbreaking (we don’t think that this adjective is an exaggeration) discoveries in combinatorics, graph theory, and combinatorial number theory. But arguably you are most famous for what is now called the Szemerédi theorem, the proof of the Erdős-Turán conjecture from 1936.

Your proof is extremely complicated. The published proof is forty-seven pages long and has been called a masterpiece of combinatorial reasoning. Could you explain first of all what the theorem says, the history behind it, and why and when you got interested in it?

Szemerédi: Yes, I will start in a minute to explain what it is, but I suspect that not too many people have read it. I will explain how I got to the problem, but first I want to tell how the whole story started. It started with the theorem of van der Waerden: you fix two numbers, say five and three. Then you consider the integers up to a very large number, from 1 to n, say. Then you partition this set into five classes, and there will always be a class containing a three-term arithmetic progression. That was a fundamental result of van der Waerden—of course, not only with three and five but with general parameters.

Later, Erdős and Turán meditated over this result. They thought that maybe the reason why there is an arithmetic progression is not the partition itself; if you partition into five classes, then one class contains at least one fifth of all the numbers. They made the conjecture that what really counts is that you have dense enough sets.

That was the Erdős-Turán conjecture: if your set is dense enough in the interval 1 to n—we are of course talking about integers—then it will contain a long arithmetic progression. Later Erdős formulated a very brave and much stronger conjecture: let’s consider an infinite sequence of positive integers, a₁ < a₂ < …, such that the sum of the inverses {1/ai} is divergent. Then the infinite sequence contains arbitrarily long arithmetic progressions. Of course, this would imply the absolutely fundamental result of Green and Tao about arbitrarily long arithmetic progressions within the primes because for the primes, we know that the sum of the inverses is divergent.

That was a very brave conjecture; it isn’t even solved for arithmetic progressions of length k=3. But now people have come very close to proving it: Tom Sanders proved that if we have a subset between 1 and n containing at least n over log n log log n elements, then the subset contains a three-term arithmetic progression. Unfortunately, we need a little bit more, but we are getting close to solving Erdős’s problem for k=3 in the near future, which will be a great achievement. If I’m not mistaken, Erdős offered US $3,000 for the solution of the general case a long time ago. If you consider inflation, that means quite a lot of money.

R&S: Erdős offered 1,000 USD for the problem you solved, and that’s the highest sum he ever paid, right?

Szemerédi: Erdős offered US$1,000 as well for a problem in graph theory which was solved by V. Rödl and P. Frankl. These are the two problems I know about.

R&S: Let’s get back to how you got interested in the problem.

Szemerédi: That was very close to the Gelfand/Gelfond story, at least in a sense. At least the message is the same: I overlooked facts. I tried to prove that if you have an arithmetic progression, then it cannot happen that the squares are dense inside of it; specifically, it cannot be that a positive fraction of the elements of this arithmetic progression are squares. I was about twenty-five years old at the time and at the end of my university studies. At that time I already worked with Erdős. I very proudly showed him my proof, because I thought it was my first real result. Then he pointed out two, well, not errors, but deficiencies in my proof.

Firstly, I had assumed that it was known that r₃(n) = o(n), i.e., that if you have a set of positive upper density, then it has to contain an arithmetic progression of length four or, for that matter, of any length. I assumed that this was a true statement. Then I used [the fact] that there are no four

¡r₄(n) denotes the proportion of elements between 1 and n that a subset must contain in order for it to contain an arithmetic progression of length k.
squares that form an arithmetic progression. However, Erdős told me that the first statement was not known; it was an open problem. The other one was already known to Euler, which was two hundred fifty years before my time. So I had assumed something that is not known and, on the other hand, I had proved something that had been proven two hundred fifty years ago!

The only way to try to correct something so embarrassing was to start working on the arithmetic progression problem. That was the time I started to work on $r_k(n)$ and, more generally, on $r_k(n)$. First I took a look at Klaus Roth’s proof from 1953 of $r_3(n)$ being less than $n$ divided by log log $n$. I came up with a very elementary proof for $r_3(n) = o(n)$ so that even high school students could understand it easily. That was the starting point. Later I proved also that $r_k(n) = o(n)$.

Erdős arranged for me to be invited to Nottingham to give a talk on that result. But my English was virtually nonexistent. Right now you can still judge that there is room for improvement in my English, but at the time it was almost nonexistent. I gave a series of lectures; Peter Elliot and Edward Wirsing, both extremely strong mathematicians, wrote a paper based almost entirely on my pictures on the blackboard. Perhaps they understood some easy words in English that I used. Anyway, they helped to write the paper for me. A similar thing happened when I solved $r_k(n) = o(n)$ for general $k$. Then my good friend Andras Hajnal helped me to write the paper. That is actually an underestimate. The truth is that he listened to my explanations and then wrote the paper. I am very grateful to Peter Elliot, Edward Wirsing, and to my good friend Andras for their invaluable help.

R&S: When did all this happen?

Szemerédi: It was in 1973. The paper appeared in Acta Arithmetica in 1975. There is a controversial issue—well, maybe controversial is too strong a word—about the proof. It is widely said that one of the main tools in the proof is the so-called regularity lemma, which is not true in my opinion.

Well, everybody forgets about the proofs they produced thirty years ago. But I reread my paper and I couldn’t find the regularity lemma. There occurs a lemma in the proof which is similar to the regularity lemma, so maybe that lemma, which is definitely not the regularity lemma, inspired me later to prove the regularity lemma.

The real story is that I heard Bollobás’s lectures from 1974 about strengthening the Erdős-Stone theorem. The Erdős-Stone theorem from the 1940s was also a breakthrough result, but I don’t want to explain it here. Then Bollobás and Erdős strengthened it. I listened to Bollobás’s lectures and tried to improve their result. Then it struck me that a kind of regularity lemma would come in handy, and this led me to prove the regularity lemma. I am very grateful to Vasek Chvatal, who helped me to write the regularity paper. Slightly later the two of us gave a tight bound for the Erdős-Stone theorem.

R&S: I’ve seen that people refer to it in your proof of the Erdős-Turán conjecture as a weakened form of the regularity lemma.

Szemerédi: Yes, weaker, but similar in ideology, so to speak.

Connections to Ergodic Theory

R&S: Your proof of the Szemerédi theorem is the beginning of a very exciting story. We have heard from a reliable source that Hillel Furstenberg at the Hebrew University in Jerusalem first learned about your result when somebody gave a colloquium talk there in December 1975 and mentioned your theorem. Following the talk, there was a discussion in which Furstenberg said that his weak mixing of all orders theorem, which he already knew, would prove the ergodic version of the Szemerédi theorem in the weak mixing case. Since the Kronecker (or compact) case is trivial, one should be able to interpolate between them so as to get the full ergodic version. It took a couple of months for him to work out the details, which became his famous multiple recurrence theorem in ergodic theory.

We find it amazing that the Szemerédi theorem and Furstenberg’s multiple recurrence theorem are equivalent, in the sense that one can deduce one theorem from the other. We guess it is not off the mark to say that Furstenberg’s proof gave a conceptual framework for your theorem. What are your comments?

Szemerédi: As opposed to me, Furstenberg is an educated mathematician. He is a great mathematician and he already had great results in ergodic theory; he knew a lot. He proved that a measure-preserving system has a multiple recurrence property; this is a far-reaching generalization of a classical result by Poincaré. Using his result, Furstenberg proved my result on the $k$-term arithmetic progressions. So that is the short story about it. But I have to admit that his method is much stronger because it could be generalized to
could have used the Selberg sieve instead. And compared them, and then he and Ben said that he read all the proofs of the Szemerédi theorem. Terry Tao pointed out that later, while generalizing their theorem, they did not have to use my theorem. I would like to point out that they will not come up with a proof of the polynomial version within the next ten years by using elementary methods.

But then very interesting things happened. Tim Gowers started the so-called Polymath Project: many people communicated with each other on the Internet and decided that they would try to give a combinatorial proof of the Hales-Jewett density theorem using only elementary methods. After two months, they came up with an elementary proof. The density Hales-Jewett theorem was considered to be by far the hardest result proved by Furstenberg and Katznelson, and its proof is very long. The elementary proof of the density Hales-Jewett theorem is about twenty-five pages long.

There is now a big discussion among mathematicians whether one can use this method to solve other problems. Joint papers are very good when a small group of mathematicians cooperate. But the Polymath Project is different; hundreds of people communicate. You may work on something your whole life; then a hundred people appear, and many of them are ingenious. They solve your problem, and you are slightly disappointed. Is this a good thing? There is a big discussion among mathematicians about this method. I am for it. I will soon turn seventy-two years old, so I believe I can evaluate it without any self-interest.

R&S: Still, all this started with your proof of the Erdős–Turán conjecture. You mentioned Green-Tao. An important ingredient in their proof of the existence of arithmetic progressions of arbitrary length within the primes is a Szemerédi-type argument involving so-called pseudo-primes, whatever that is. So the ramifications of your theorem have been impressive.

Szemerédi: In their abstract they say that the three main ingredients in their proof are the Goldstone-Yildirim result, which gives an estimate for the difference of consecutive primes; their transference principle; and my theorem on arithmetic progressions.

R&S: By the way, according to Green and Tao one could have used the Selberg sieve instead.

Szemerédi: You are right. However, in my opinion the main revolutionary new idea is their transference principle which enables us to go from a dense set to a sparse set. I would like to point out that, while generalizing their theorem, they did not have to use my theorem. Terry Tao said that he read all the proofs of the Szemerédi theorem and compared them, and then he and Ben Green meditated on it. They were probably more inspired by Furstenberg’s method, the ergodic method. That is at least my take on this thing, but I am not an expert on ergodic theory.

R&S: But Furstenberg’s theorem came after and was inspired by yours. So however you put it, it goes back to you.

Szemerédi: Yes, that is what they say.

R&S: We should mention that Tim Gowers also gave a proof of the Szemerédi theorem.

Szemerédi: He started with Roth’s method, which is an estimation of exponential sums. Roth proved in his paper that \( r_d(n) \) is less than \( n \) divided by \( \log \log n \). Tim Gowers’s fundamental work not only gave an absolutely strong bound for the size of a set \( A \) in the interval \( [1,n] \) not containing a \( k \)-term arithmetic progression, he also invented methods and concepts that later became extremely influential. He introduced a norm (actually, several norms) that is now called the Gowers norm. This norm controls the randomness of a set. If the Gowers norm is big, he proved that it is correlated with a higher-order phase function, which is a higher-order polynomial. Gowers, and independently Rödl, Naegle, Schacht, and Skokan, proved the hypergraph regularity lemma and the hypergraph counting lemma, which are main tools in additive combinatorics and in theoretical computer science.

R&S: We should mention that Gowers received the Fields Medal in 1998 and that Terence Tao got it in 2006. Also, Roth was a Fields Medal recipient back in 1958.

Random Graphs and the Regularity Lemma

R&S: Let’s get back to the so-called Szemerédi regularity theorem. You have to explain the notions of random graphs and extremal graphs, because they are involved in this result.

Szemerédi: How can we imagine a random graph? I will talk only about the simplest example. You have \( n \) points and the edges are just the pairs, so each edge connects two points. We say that the graph is complete if you include all the edges, but that is, of course, not an interesting object. In one model of the random graph, you just close your eyes and with probability \( \frac{1}{2} \) you choose an edge. Then you will eventually get a graph. That is what we call a random graph, and most of them have very nice properties.

You just name any configuration, such as four-cycles \( C_4 \), for instance, or the complete graph \( K_4 \), then the number of such configurations is as you would expect. A random graph has many beautiful properties and it satisfies almost everything. Extremal graph theory is about finding a configuration in a graph. If you know that your graph is a random graph, you can prove a lot of things.

The regularity lemma is about the following. If you have any graph—unfortunately we have to assume a dense graph, which means that you have a
lot of edges—then you can break the vertex set into a relatively small number of disjoint vertex sets, so that if you take almost any two of these vertex sets, between them the so-called bipartite graph will behave like a random graph. We can break our graph into not too many pieces, so we can work with these pieces and we can prove theorems in extremal graph theory.

We can also use it in property testing, which belongs to theoretical computer science and many other areas. I was surprised that they use it even in biology and neuroscience, but I suspect that they use it in an artificial way, that they could do without the regularity lemma. But I am not an expert on this, so I can’t say this for sure.

**R&S:** The regularity lemma really has some important applications in theoretical computer science?

**Szemerédi:** Yes, it has, mainly in property testing but also in constructing algorithms. Yes, it has many important applications. Not only the original regularity lemma, but, since this is thirty years ago, there have appeared modifications of the regularity lemma which are more adapted for these purposes. The regularity lemma is for me just a philosophy, not an actual theorem. Of course, the philosophy is almost everything. That is why I like to say that in every chaos there is an order. The regularity lemma just says that in every chaos there is a big order.

**R&S:** Do you agree that the Szemerédi theorem, i.e., the proof of the Erdős-Turán conjecture, is your greatest achievement?

**Szemerédi:** It would be hard to disagree because most of my colleagues would say so. However, perhaps I prefer another result of mine with Ajtai and Komlós. In connection with a question about Sidon sequences we discovered an innocent-looking lemma. Suppose we have a graph of \( n \) vertices in which a vertex is connected to at most \( d \) other vertices. By a classical theorem of Turán, we can always find at least \( n/d \) vertices such that no two of them are connected by an edge. What we proved was that under the assumption that the graph contains no triangle, a little more is true: one can find \( n/d \) times log \( d \) vertices with the above property.

I am going to describe the proof of the lemma very briefly. We choose \( n/2d \) vertices of our graph randomly. Then we omit all the neighbors of the points in the chosen sets. This is, of course, a deterministic step. Then in the remaining vertex set we again choose randomly \( n/2d \) vertices and again deterministically omit the neighbors of the chosen set. It can be proved that this procedure can be repeated log \( d \) times and in the chosen set the average degree is at most 2. So in the chosen sets we can find a set of size at least \( n/4d \) such that no two points are connected with an edge.

Because of the mixture of random steps and deterministic steps, we called this new technique the “semirandom method”.

Historically, the first serious instance of a result of extremal graph theory was the famous theorem of Ramsey and, in a quantitative form, of Erdős and Szekeres. This result has also played a special role in the development of the “random method”. Therefore it has always been a special challenge for combinatorialists to try to determine the asymptotic behavior of the Ramsey functions \( R(k,n) \) as \( n \) (or both \( k \) and \( n \)) tend to infinity. It can be easily deduced from our lemma that \( R(3,n) < c n^2 / \log n \), which solved a long-standing open problem of Erdős. Surprisingly, about ten years later, Kim proved that the order of magnitude of our bound was the best possible. His proof is based on a brilliant extension of the “semirandom method”.

The “semirandom method” has found many other applications. For instance, together with Komlós and Pintz, I used the same technique to disprove a famous geometric conjecture of Heilbronn. The conjecture dates back to the 1940s. The setting is as follows: you have \( n \) points in the unit square and you consider the triangles defined by these points. Then the conjecture says that you can always choose a triangle of area smaller than a constant over \( n^2 \). That was the Heilbronn conjecture. For the bound \( 1/n \), this is trivial; then Klaus Roth improved this to 1 over \( n \log \log n \). Later Wolfgang Schmidt improved it further to 1 over \( n \log n \). Roth, in a very brilliant and surprising way, used analysis to prove that we can find a triangle of area less than \( 1/n^{1+\alpha} \), where \( \alpha \) is a constant.

We then proved, using the semirandom method, that it is possible to put down \( n \) points such that the smallest area of a triangle is at least \( \log n \) over \( n^2 \), disproving the Heilbronn conjecture. Roth told us that he gave a series of talks about this proof.

**Further Research Areas**

**R&S:** It is clear from just checking the literature and talking with people familiar with graph theory and combinatorics, as well as additive number theory, that you, sometimes with coauthors, have obtained results that have been groundbreaking and have set the stage for some very important developments. Apart from the Szemerédi theorem and the regularity lemma that we have already talked about, here is a short list of important results that you and your coauthors have obtained:

(i) The Szemerédi-Trotter theorem in the paper “Extremal problems in discrete geometry” from 1983.

\[ R(k,n) \] denotes the least positive integer \( N \) such that for any (red/blue)-coloring of the complete graph \( K_N \) on \( N \) vertices, there exists either an entirely red complete subgraph on \( k \) vertices or an entirely blue subgraph on \( n \) vertices.

(iii) The results obtained by AKS, which is the acronym for Miklós Ajtai, János Komlós, and Endre Szemerédi. The “sorting algorithm” is among the highlights.

Could you fill in some details, please?

**Szemerédi:** (i) Euclid’s system of axioms states some of the basic facts about incidences between points and lines in the plane. In the 1940s, Paul Erdős started asking slightly more complicated questions about incidences that even Euclid would have understood. How many incidences can occur among \( m \) points and \( n \) lines, where an “incidence” means that a line passes through a point? My theorem with Trotter confirmed Erdős’s rather surprising conjecture: the maximal number of incidences is much smaller in the real plane than in the projective one, much smaller than what we could deduce by simple combinatorial considerations.

(ii) Together with Paul Erdős, we discovered an interesting phenomenon and made the first nontrivial step in exploring it. We noticed, roughly speaking, that a set of numbers may have nice additive properties or nice multiplicative properties but not both at the same time. This has meanwhile been generalized to finite fields and other structures by Bourgain, Katz, Tao, and others. Their results had far-reaching consequences in seemingly unrelated fields of mathematics.

(iii) We want to sort \( n \) numbers, that is, to put them in increasing order by using comparisons of pairs of elements. Our algorithm is nonadaptive: the next comparison never depends on the outcome of the previous ones. Moreover, the algorithm can efficiently run simultaneously on \( cn \) processors such that every number is processed by only one of them at a time. Somewhat surprisingly, our algorithm does not require more comparisons than the best possible adaptive nonparallel algorithm. It is well known that any sorting algorithm needs at least \( n \log n \) comparisons.

**R&S:** What are, in your opinion, the most interesting and important open problems in combinatorics and graph theory?

**Szemerédi:** I admit that I may be somewhat conservative in taste. The problem that I would like to see solved is the very first problem of extremal graph theory: to determine the asymptotic behavior of the Ramsey functions.

### Combinatorics Compared to Other Areas of Mathematics

**R&S:** It is said, tongue in cheek, that a typical combinatorialist is a bright mathematician with an aversion to learning or embracing abstract mathematics. Does this description fit you?

**Szemerédi:** I am not sure. In combinatorics we want to solve a concrete problem, and by solving a problem we try to invent new methods. And it goes on and on. Sometimes we actually borrow from so-called well-established mathematics. People in other areas of mathematics often work in ways that are different from how we work in combinatorics.

Let’s exaggerate somewhat: they have big theories and they find sometimes a problem for the theory. In combinatorics, it is usually the other way around. We start with problems that actually are both relevant and necessary; that is, the combinatorics itself requires the solution of the problems, the problems are not randomly chosen. You then have to find methods that you apply to solve the problems and sometimes you might create a theory. But you start out by having a problem; you do not start by having a theory and then finding a problem to which you can apply the theory. Of course, that happens from time to time, but it is not the major trend.

Now, in the computer era, it is unquestionable that combinatorics is extremely important. If you want to run programs efficiently, you have to invent algorithms in advance, and these are basically combinatorial in nature. This is perhaps the reason why combinatorics today is a little bit elevated, so to say, and that mathematicians from other fields start to realize this and pay attention. If you look at the big results, many of them have big theories which I don’t understand, but at the very root there is often some combinatorial idea. This discussion is a little bit artificial. It’s true that combinatorics was a second-rated branch of mathematics thirty years ago but hopefully not any longer.

**R&S:** Do you agree with Bollobás, who in an interview from 2007 said the following: “The trouble with the combinatorial problems is that they do not fit into the existing mathematical theories. We much more prefer to get help from ‘mainstream’ mathematics rather than to use ‘combinatorial’ methods only; but this help is rarely forthcoming. However, I am happy to say that the landscape is changing.”

**Szemerédi:** I might agree with that.

**R&S:** Gowers wrote a paper about the two cultures within mathematics. There are problem solvers and there are theory builders. His argument is that we need both. He says that the organizing principles of combinatorics are less explicit than in core mathematics. The important ideas in combinatorial mathematics do not usually appear in the form of precisely stated theorems but more often as general principles of wide applicability.

**Szemerédi:** I guess that Tim Gowers is right. But there is interplay between the two disciplines. As Bollobás said, we borrow from the other branches of mathematics if we can when we solve concrete discrete problems, and vice versa. I once sat in class when a beautiful result in analytic number theory was presented. I understood only a part of it. The mathematician who gave the talk came to the bottleneck of the whole argument. I realized...
that it was a combinatorial statement and if you gave it to a combinatorialist, he would probably have solved it. Of course, one would have needed the whole machinery to prove the result in question, but at the root it actually boiled down to a combinatorial argument. A real interplay!

R&S: There is one question that we have asked almost all Abel Prize recipients; it concerns the development of important new concepts and ideas. If you recollect, would key ideas turn up when you were working hard at your desk on a problem, or did they show up in more relaxed situations? Is there any pattern?

Szemerédi: Actually, both! Sometimes you work hard on a problem for half a year and nothing comes out. Then suddenly you see the solution, and you are surprised and slightly ashamed that you hadn’t noticed these trivial things which actually finish the whole proof and which you did not discover for a long time. But usually you work hard and step-by-step you get closer to the solution. I guess that this is the case in every science. Sometimes the solution comes out of the blue, but sometimes several people are working together and find the solution.

I have to tell you that my success ratio is actually very bad. If I counted how many problems I have worked on and in how many problems I have been successful, the ratio would be very bad.

R&S: Well, in all fairness, this calculation should take into consideration how many problems you have tried to solve.

Szemerédi: Right, that is a nice remark.

R&S: You have been characterized by your colleagues—and this is meant as a huge compliment—as having an “irregular mind”. Specifically, you have been described as having a brain that is wired differently than most mathematicians’ [brains]. Many admire your unique way of thinking, your extraordinary vision. Could you try to explain to us how you go about attacking problems? Is there a particular method or pattern?

Szemerédi: I don’t particularly like the characterization of having an “irregular mind”. I don’t feel that my brain is wired differently and I think that most neurologists would agree with me. However, I believe that having unusual ideas can often be useful in mathematical research. It would be nice to say that I have a good general approach of attacking mathematical problems. But the truth is that after many years of research I still do not have any idea what the right approach is.

Mathematics and Computer Science

R&S: We have already talked about connections between discrete mathematics and computer science; you are in fact a professor in computer science at Rutgers University in the United States. Looking back, we notice that for some important mathematical theorems, like the solution of the four-color problem, for instance, computer power has been indispensable. Do you think that this is a trend? Will we see more results of this sort?

Szemerédi: Yes, there is a trend. Not only for this but also for other types of problems as well, where computers are used extensively. This trend will continue, even though I am not a computer expert. I am in the computer science department, but fortunately nobody asked me whether I could answer email, which I cannot! They just hired me because so-called theoretical computer science was highly regarded in the late 1980s. Nowadays, it does not enjoy the same prestige, though the problems are very important, the P versus NP problem, for instance. We would like to understand computation and how fast it is; this is absolutely essential mathematics, and not only for discrete mathematics. These problems lie at the heart of mathematics, at least in my opinion.

R&S: May we come back to the P versus NP problem, which asks whether every problem whose solution can be verified quickly by a computer can also be solved quickly by a computer. Have you worked on it yourself?

Szemerédi: I am working on two problems in computer science. The first one is the following: assume we compute an $n$-variable Boolean function with a circuit. For most of the $n$-variable Boolean functions the circuit size is not polynomial. But to the best of my knowledge, we do not know a particular function which cannot be computed with a Boolean circuit of linear size and depth $\log n$. I have no real idea how to solve this problem.

The second one is the minimum weight spanning tree problem. Again, so far I am unsuccessful.

I have decided that now I will, while keeping up with combinatorics, learn more about analytic number theory. I have in mind two or three problems, which I am not going to tell you. It is not the Riemann hypothesis, that I can tell.

R&S: The P versus NP conjecture is on the Clay list of problems, the prize money for a solution being one million USD, so it has a lot of recognition.

Szemerédi: Many people believe that the P versus NP problem is the most important one in current mathematics, regardless of the Riemann hypothesis and the other big problems. We should understand computation. What is in our power? If we can check easily that something is true, can we easily find a solution? Most probably not! Almost everybody will bet that P is not equal to NP, but not too much has been proved.

Soccer

R&S: You have described yourself as a sports fanatic.

Szemerédi: Yes, at least I was. I wanted to be a soccer player, but I had no success.

R&S: We have to stop you there. In 1953, when you were thirteen years old, Hungary had a
fantastic soccer team; they were called The Mighty Magyars. They were the first team outside the British Isles that beat England at Wembley and even by the impressive score of 6 to 3. At the return match in Budapest in 1954 they beat England 7 to 1, a total humiliation for the English team. Some of these players on the Hungarian team are well known in the annals of soccer, names like Puskás, Hidegkuti, Czibor, Bozsik, and Kocsis.

Szemerédi: Yes. These five were world-class players.

R&S: We have heard that the Hungarian team, before the game in Budapest, lived at the same place as you did. Bozsik watched you play soccer, and he said that you had real talent. Is this a true story?

Szemerédi: Yes, that is true, except that they did not live at the same place. My mother died early; this is why we three brothers lived at a boarding school. That school was very close to the hotel where the Hungarian team lived. They came sometimes to our soccer field to relax and watch our games, and one time we had a very important game against the team that was our strongest competitor. You know, boarding schools were competing like everyone else.

I was a midfielder like Bozsik. I was small and did not have the speed, but I understood the Hungarian team’s strategy. They revolutionized the soccer game, foreshadowing what was later called “Total Football”. They did not pass the ball to the nearest guy, but rather they aimed the ball to create space and openings, often behind the other team’s defense. That was a completely different strategy than the standard one, and therefore they were extremely effective.

I studied this and I understood their strategy and tried to imitate it. Bozsik saw this and he understood what I was trying to do.

R&S: You must have been very proud.

Szemerédi: Yes, indeed I was very proud. He was nice, and his praise is still something which I value very much.

R&S: Were you very disappointed with the World Cup later that year? As you very well know, the heavily favored Hungarian team first beat West Germany 8 to 3 in the preliminary round, but then they lost 2 to 3 in the final to West Germany.

Szemerédi: Yes. It was very unfortunate. Puskás was injured, so he was not at his best, but we had some other problems, too. I was very, very sad and for months I practically did not speak to anybody. I was a real soccer fan. Much later, in 1995, a friend of mine was the ambassador for Hungary in Cairo, and I visited him. Hidegkuti came often to the embassy because he was the coach for the Egyptian team. I tried to make him explain to me what happened in 1954, but I got no answer.

By the way, to my big surprise I quite often guess correct results. Several journalists came to me in Hungary for an interview after it was announced that I would receive the Abel Prize. The last question from one of them was about the impending European Cup quarter-final match between Barcelona and Milan. I said that up to now I have answered your questions without hesitation, but now I need three minutes. I reasoned that the defense of Barcelona was not so good (their defender Puyol is a bit old), but their midfield and attack are good, so: 3 to 1 to Barcelona. On the day the game was played, the paper appeared with my—as it turned out—correct prediction. I was very proud of this, and people on these blogs wrote that I could be very rich if I would enter the odds-prediction business!

R&S: We can at least tell you that you are by far the most sports-interested person we have met so far in these Abel interviews!

On behalf of the Norwegian, Danish, and European Mathematical Societies, and on behalf of the two of us, thank you very much for this most interesting interview.

Szemerédi: Thank you very much. I am very happy for the opportunity of talking to you.
Lloyd S. Shapley, professor emeritus of economics and mathematics at the University of California, Los Angeles, and Alvin E. Roth, George Gund Professor of Economics and Business Administration at Harvard University and Harvard Business School, have been awarded the 2012 Nobel Prize in Economics “for the theory of stable allocations and the practice of market design.” Though they worked independently of each other, the combination of Shapley’s basic theory and Roth’s empirical investigations, experiments, and practical design has generated a flourishing field of research and improved the performance of many markets.

On the Work of Roth and Shapley

The Notices asked Robert Aumann of the Hebrew University of Jerusalem to comment on the work of Alvin Roth and Lloyd Shapley. Aumann supplied the following description of their work.

“Matching markets” are the means by which we choose the many things in life that also choose us—schools, universities, jobs, life partners, and so on. “Market design” is about designing these markets efficiently, in much the same sense that the price mechanism is efficient in ordinary markets. This is a highly nontrivial task. The fundamental theory of market design was laid out by Lloyd Shapley, Herbert Scarf, and the late David Gale in the sixties and seventies of the last century. It has been applied in practice, with resounding success, largely through the efforts of one individual: Alvin Roth.

In 1962 Shapley and Gale published a paper entitled “College admissions and the stability of marriage”. There they developed an algorithm for matching—e.g., of men to women—that is “stable” in a very natural sense: No man and woman who are not matched to each other prefer each other to the partners with whom they are matched. In the early 1980s, Roth made a discovery with far-reaching consequences: In the first half of the twentieth century, the U.S. market for assigning medical students to hospitals (for the purpose of doing their residencies) had been in continuous turmoil. In the early 1950s, the American Hospital Association (AHA) introduced a computer algorithm for making these assignments, whereupon the previously chaotic market immediately became orderly. And Roth showed that the AHA algorithm was precisely the same as that of Gale and Shapley, though neither the AHA nor the inventors of their algorithm were familiar with the relevant theory.

This discovery opened the door to the application of the Gale-Shapley algorithm to a wide variety of important matching markets: in addition to the medical programs, perhaps most importantly to very large-scale programs of assigning students to schools, programs that are in actual use in New York and Boston and are under development in Denver and New Orleans. It also led to a large literature that modifies the algorithm to take account of new conditions that were not previously envisaged (e.g., when two medical students are married to each other and require residencies in the same hospital or at least in the same town). In addition to his applied work, Roth is a leader in these later theoretical developments.

Another very important development in market design is based on a paper entitled “On cores and indivisibility”, published by Shapley and Scarf in 1974. This paper lays out the theory of discrete exchange markets: markets where large indivisible items—like houses—are exchanged without any money changing hands. While initially the assumption of exchange without money appeared to be of largely theoretical interest, Roth brilliantly used the Shapley-Scarf results to design practical methods for facilitating donations of kidneys from potential donors to recipients, where the recipient

DOI: http://dx.doi.org/10.1090/noti946
intended by the original donor is unable to receive the kidney because of medical incompatibility.

Market design enables highly practical applications of sophisticated and deep theoretical results. Unlike much of economic theory, this really works in the real world. In my opinion, the work of Shapley and Roth in market design is the most outstanding success story in game theory and economics, more so even than auction theory, which has enjoyed great success and for which William Vickrey was rightfully awarded the 1996 Bank of Sweden Prize in Economic Sciences in Honor of Alfred Nobel.

In addition to the work just described, which laid the basis for market design, Shapley has been the top game theorist in the world for some sixty years—ever since the early fifties of the last century. He has been at the forefront of almost every major development in game theory. Perhaps his most visible impact has been on the “cooperative” or coalitional side of the theory, but also on the “noncooperative” or strategic side, his work has been of major importance. Here we mention only some of the outstanding items, starting with the coalitional side.

1. Development of the core as an independent solution concept for coalitional games in the early fifties. This led to several developments of major importance, as outlined below (items 2 and 3).

2. The principle of equivalence between the core of a large market and its competitive outcomes, as developed by Shapley and Shubik, Debreu, Scarf, and subsequently by many scores of others. For the first time, this work provided a firm theoretical foundation for the notion of competition—the “invisible hand” of Adam Smith. It is undoubtedly the most fundamental and significant application ever of game theory to economic theory.

3. The development of the notion of games with a continuum of players—“oceanic games”—by Shapley and Fields Medal winner John Milnor in the early sixties. The model of oceanic games turned out to have an enormous impact, including most prominently on the equivalence principle (item 2 above), but also on many other areas, including oligopoly, cost allocation, and even the routing of traffic.

4. Development of the “Shapley value”—an a priori evaluation of the “expected payoff” or “strength” of a player in a coalitional game. On this concept, I wrote in 1985 as follows: “The Shapley value is perhaps the most ‘successful’ of all cooperative solution concepts. It has a very broad spectrum—it almost always exists, and it applies to political as well as economic and ‘mixed’ contexts. It very often gives results with significant conceptual content...which is often of independent interest, not connected with the idea of value in any obvious or transparent manner.... A very important point is that the value is mathematically tractable. It lends itself to the application of mathematical methods from probability, measure theory, functional analysis and other areas. As a result, a very considerable body of theory has been built around the value; this theory may well be mathematically the richest and deepest in game theory. Of course this is intellectually pleasing, but that is not where its importance lies. Rather, its importance lies in the fact that this theory enables us to deal with the applications, to attack fairly complex models in a systematic manner and to solve them” (from Frontiers of Economics, edited by K. Arrow and S. Honkapohja, Basil Blackwell, Oxford, 1985).

These words were written more than thirty years ago. Since then, the theory and applications of the Shapley value have continued to flourish, and what was written then is all the more true today.

The work presaging market design and items 1–4 above are only some of the highlights of Shapley’s contributions to coalitional game theory. There are many others.

We pass now to two of Shapley’s most important contributions to the strategic (“noncooperative”) theory.

5. Stochastic games, introduced by Shapley in the early 1950s to model the dynamics of ongoing game situations. Starting in the mid-1970s, this subject began to attract more and more attention from some of the top mathematical game theorists in the world, such as J.-F. Mertens, A. Neyman, E. Kohlberg, T. Bewley, and others. In the last twenty years, it has really taken off, with additional contributions by the above-named workers, in addition to extremely deep work by the French school, notably N. Vieille, J. Renault, and others, and by R. Simon. The mathematical depth and beauty of this work matches that of the work on the Shapley value (item 4 above), though it is of course something completely different. The theory also has important conceptual applications.

6. There are sometimes simple examples that are very important. One such example is the famous “Prisoner’s dilemma”, which has had a justifiably enormous impact on game theory and social science in general. Not as well known to the general public, but of equal importance, and well known to the cognoscenti, is “Shapley’s game”:

\[
\begin{bmatrix}
0,0 & 4,5 & 5,4 \\
5,4 & 0,0 & 4,5 \\
4,5 & 5,4 & 0,0 \\
\end{bmatrix}
\]

This example has played a major role in the noncooperative theory. Originally advanced by Shapley to show that the important process known as “fictitious play” does not converge for nonzerosum games, it has since then taken on a life of its own; the first thing one does when working on a
noncooperative theory is to see how it applies to Shapley’s game.

The above represents only the tip of the iceberg. Shapley’s work in game theory—both applied and mathematical—is truly astounding in scope, in depth, in beauty, and in importance. On each of these counts, Shapley has done more than all the previous game theory Nobelists, even when taken together.

Alvin Roth: Biographical Sketch
Alvin E. Roth was born in 1951 and received his Ph.D. in operations research from Stanford University in 1974. He has held positions at the University of Illinois (1974–1982) and the University of Pittsburgh (1982–1998). Since 1998 he has been George Gund Professor of Economics and Business Administration at Harvard University and the Harvard Business School. He is currently a visiting professor at Stanford University, where he will become a full faculty member in 2013. He has held visiting professorships at the Technion in Israel (1986), at the Hebrew University of Jerusalem (1995), and at the University of Tel Aviv (1995). He has been a research associate at the National Bureau of Economic Research since 1998. His honors include fellowships from the Guggenheim Foundation (1983–1984) and the Alfred P. Sloan Foundation (1984–1986); the Lanchester Prize of the Operations Research Society of America (1991); and (with Itai Ashlagi) the NKR Terasaki Medical Innovation Award of the American Transplant Congress (2012). He is also a fellow of the American Academy of Arts and Sciences.

Lloyd Shapley: Biographical Sketch
Lloyd S. Shapley was born in 1923 in Cambridge, Massachusetts. He received his Ph.D. from Princeton University in 1953. He has held positions at Princeton (1952–1954) and at the RAND Corporation (1948–1949, 1954–1981). He joined the faculty of the University of California, Los Angeles, in 1981 and is currently professor emeritus at UCLA. He has held visiting appointments at the California Institute of Technology (1955–1956), the Indian Statistical Institute (1978–1979), the Hebrew University of Jerusalem (1979–1980), and Catholic University of Louvain (1982). He was awarded the John von Neumann Theory Prize in 1981. He is a fellow of the Econometric Society, the American Academy of Arts and Sciences, and the National Academy of Sciences, and is a distinguished fellow of the American Economic Association.

—Elaine Kehoe

---

### Upcoming Programs and Events

| SEMESTER-LONG PROGRAMS: |  
| (each with 3-4 associated workshops) |  
| **Low-dimensional Topology, Geometry, and Dynamics** | Fall 2013: September 9 – December 6  
| **Network Science and Graph Algorithms** | Spring 2014: February 3 – May 9  

| TOPICAL WORKSHOP: |  
| Issues in Solving the Boltzmann Equation for Aerospace Applications | June 3 – 7, 2013  

| SUMMER PROGRAMS: |  
| Summer@ICERM (Undergraduate Summer Research) |  
| Geometry and Dynamics | June 17 – August 9, 2013  
| IdeaLab (for Postdoctoral Researchers) | Topic 1: Tipping Points in Climate Systems  

| SPECIAL EVENTS: |  
| Scratching the Surface in Dynamic Visual Effects | Public lecture featuring Robert Bridson, UBC | March 11, 2013  
| Research Experiences for Undergraduate Faculty (REUF) | Co-sponsored with American Institute of Mathematics (AIM) | July 22–26, 2013  
| Modern Mathematics Workshop (during SACNAS) | October 3 – 6, 2013 in San Antonio, TX  

**Participation:** ICERM welcomes applications for long- and short-term visitors. Support for local expenses may be provided. Decisions about online applications are typically made 1–3 months before a program, as space and funding permit. ICERM encourages women and members of underrepresented minorities to apply.

To learn more about ICERM’s programs, organizers, confirmed program participants, and to submit an application, please visit our website: [http://icerm.brown.edu](http://icerm.brown.edu)

**About ICERM:** The Institute for Computational and Experimental Research in Mathematics is a National Science Foundation Mathematics Institute at Brown University in Providence, RI. Its mission is to broaden the relationship between mathematics and computation.
Utilization of Technology for Mathematical Talks. An Alarming Situation

V. V. Peller

I believe all mathematicians will agree that over the last ten years the percentage of mathematical talks given by means of a computer presentation has jumped and is still rising. I find this situation alarming. I have the strong opinion that this tendency leads to a degradation of the quality of talks and discredits the idea of mathematical talks.

Unfortunately, it is not rare for the organizers of mathematical conferences to write to the participants to encourage them to give a computer presentation or, even worse, they inform speakers that there will be no blackboards at all. On many occasions I have asked the organizers why they do not provide blackboards. They always responded that the conference would be held in a hotel (or another place) that has no blackboards. I do not find this explanation satisfactory. I strongly believe that if they want to organize a conference, it is their responsibility to provide blackboards. No excuse is acceptable. Incidentally, twenty years ago most conferences were held at universities, while presently it is more common to organize conferences at hotels.

Once I was invited to give a seminar talk and I was informed that there would be no blackboards and that I should prepare a computer presentation. I said that was absolutely impossible. Moreover, I said that it is as if you invite a pianist to give a performance, but unfortunately you have no piano and you encourage the pianist to record the notes onto a flash drive for their computer to be able to reproduce the notes. After my words, a blackboard was found.

Is a Mathematician a Performer or a Composer?

Once when giving a talk at a conference, I mentioned the above comparison with inviting a pianist and not having a piano. One mathematician disagreed with my comparison; he said that a mathematician is a composer, not a performer. I strongly disagree with his opinion. When a mathematician proves theorems, he is a composer. However, when he gives mathematical talks, he is a performer. Mathematicians should not underestimate the importance of being a performer.

What is Wrong with Computer Presentations?

Most speakers who give a computer presentation completely lose control over the speed of the presentation. They switch the screen very frequently, and they completely forget that the audience needs time to digest the information and to comprehend the material. As a result, it is very difficult (or even impossible) to follow the speaker.

Once I attended a plenary one-hour lecture. During the lecture the speaker stated all his results for the last three years and all the results of his students for the last three years. The speaker switched the screen over a hundred times. Needless to say, it was absolutely impossible to grasp the picture. Usually, when the speaker loses the audience, he does not realize it. He watches the screen and does not care about the audience.

Very often the speaker takes the PDF file of his paper designed for publication and displays it on the screen. This is a complete disaster: the speaker understands that it is impossible for the audience to read all the paper, he wants to skip certain slides, and he moves the material back and forth trying to find a page that he wants to concentrate on. The effect is disastrous. The only thing the audience can get from such a lecture is blinking screens.

Sometimes the speaker displays material with detailed text. At the same time, he talks and gives

V. V. Peller is professor of mathematics at Michigan State University. His email address is peller@math.msu.edu. The author is partially supported by NSF grant DMS 1001844.

DOI: http://dx.doi.org/10.1090/noti940
a verbal explanation of the material but his words do not coincide with the text on the screen. Apparently, he believes that people can read the text on the screen and listen to another text at the same time. As for me, I am not able to do two such things simultaneously.

Even if the speaker is considerate of the audience, goes at a reasonable speed, and thoroughly prepares the slides, there are still certain things that create problems for the audience. For example, when the speaker switches the screen, everything on the previous screen is gone. However, it is often important for people in the audience to look at a formula on the previous screen again. Sometimes speakers combine the computer and the blackboard and write important things on the blackboard. This can improve the presentation. However, this leads to the following problem: to read the screen, the lights are turned off; when the speaker writes on the blackboard, the lights are turned on. This can be very distracting.

Another problem is that when the speaker shows the new screen, everything on the screen appears at the same time. This is awkward. It is much easier to comprehend the material if the contents of the screen appear gradually, at a speed at which it can be digested. It is true that sometimes speakers using computers renew the screen in small portions. This can slightly improve the quality of the presentation. Nevertheless, it is still impossible to display a complicated formula gradually. However, for the audience, it is much easier to digest the material if a formula appears on the screen gradually.

I strongly believe that at least to a certain extent the talk should be an improvisation. The speaker should not lose the audience; he has to have feedback from the audience. The feedback should help the speaker adjust the speed of the presentation and decide whether he can include everything in the talk that was planned or whether to omit certain parts of the talk (or, something very, very rare, to add some material). As Paul Halmos said in [H], “The faces in the audience can be revealing and helpful: they can indicate the need to slow down, to speed up, to explain something, to omit something.” It is very difficult to achieve this if you give a computer presentation. Once the file is prepared, the speaker cannot alter it during the talk.

Let me also mention some technical problems. From time to time (usually such things happen at least once at each conference) the speaker brings a flash drive, inserts it into the computer, and the computer for some (mysterious) reason does not want to work with his file. Sometimes it takes ten to thirty minutes to fix the problem. Another common technical problem is pop-ups that appear on the screen. The speaker desperately attempts to delete them, often asking for assistance.

I do not want to say that all computer presentations are bad. There are exceptions. There are speakers who can minimize the negative features of computer presentations. However, if they were to use their skills to give a talk at the blackboard, the result would be better. I certainly do not want to say that all blackboard talks are good. Not at all! There are speakers who give blackboard talks very poorly. However, I am 100 percent sure that if those speakers were to give computer presentations, the result would be even worse.

**Why Do People Want to Give Computer Presentations?**

I think that, for many mathematicians, the idea of giving a computer presentation is attractive because they believe that it is easier for them to prepare a written text and then follow that text during the talk. Some speakers feel more confident if they prepare a computer talk, and they believe that they have already completed the job. The remaining part is easy: just follow the file prepared on the computer. This is certainly a wrong perception. The speaker must think about what is best for the audience, not for himself.

Also, many mathematicians give the same talk on several different occasions. In this case, they only need to prepare the talk once.

Several mathematicians told me that they believe that computer talks have an advantage because, they say, the fact that the speaker prepared a computer talk implies that he prepared himself for the talk. I strongly disagree with this opinion. On the contrary, it is often the case that the speaker, having prepared a computer talk, believes that he is all set, which is often wrong. Even if the file is prepared, the most important and difficult task is to speak in front of the audience.

I have heard from various mathematicians that for plenary lectures they use the blackboard, while in the case of short communications they have to give computer presentations because otherwise they have no chance of covering all the material. Again, this is a wrong perception. They can certainly cover more material in a computer presentation, but this does not mean that people in the audience will be able to digest the material. It is unnecessary to cover everything related to the subject of your talk. You have to select the material that can be understood by the audience and get the audience interested in what you are doing. People who become interested in your work can learn the details later. To attract people to your research, it is not a good idea to tell them everything you can in twenty minutes. I am afraid that in this way you will instead repel the audience from your work. In this respect I would like to quote Paul Halmos [H]: “If someone told you, in half an hour, the meaning of each ideogram on a page of Chinese, could you then read and enjoy..."
the poem on that page in the next half hour?” The speaker should not try to give too much detail. Sometimes, instead of writing complicated formulae or giving sophisticated definitions, it is more appropriate to describe verbally what one is doing.

Sometimes speakers try to give detailed proofs of certain results and believe that they do not have enough time for the proof if they give a blackboard talk. The same objection applies. It is not appropriate to give too many details during the talk.

Is the Overhead Projector a Better Alternative?

My opinion is that overhead projector talks are slightly better than computer presentations. Usually speakers who use overhead projecting don’t violate the speed limit as badly as certain speakers who give computer presentations. It is much less common among users of overhead projecting to display their papers designed for publication. However, with some speakers the speed is still excessive; when the screen is changed, everything on that screen is gone forever and the audience has no chance to review the contents of that screen. When the lecture room has two overhead projectors, the speaker can keep some important parts of his talk on one screen and change the contents of the other screen. This can slightly improve the quality of the presentation.

However, my opinion is that there are still many disadvantages of overhead projecting. Let me again quote Paul Halmos: “Do not, ever, greet an audience with a carefully prepared blackboard (or overhead projector sheets) crammed with formulas, definitions, and theorems. (An occasionally advisable exception to this rule has to do with pictures—if a picture, or two pictures, would help your exposition but would take too long to draw as you talk, at least with the care it deserves, the audience will forgive you for drawing it before the talk begins.) The audience can take pleasure in seeing the visual presentation grow before its eyes—the growth is part of your lecture, or should be.” I entirely agree with his opinion.

Is It Good When Students Teach Their Professors?

I heard from several colleagues that they never prepare computer presentations. However, occasionally when they had joint results with their students, the students gave computer talks and then sent the computer files to their professors. Later the professors themselves gave computer talks using the files of their students. This is ridiculous! Instead of teaching students how to give talks, professors are taught by their students.

Recently I attended a computer presentation at a conference. After the talk a student of the speaker, being in the audience, said that the speaker gave a computer presentation for the first time in his life: he had given only blackboard talks before. The student was very proud of his professor and proposed to congratulate him on this occasion. I do not think that such congratulations were a good idea.

It is common for young people to have a desire to master a modern technology. Sometimes this is good (in particular, it is certainly good to be able to prepare a mathematical paper with the help of a computer). However, this is not always the case. For example, young children should not use calculators to perform arithmetical operations, while college students should not use sophisticated calculators to differentiate elementary functions.

Our Old Friend the Blackboard

I strongly believe that the blackboard is the best medium for mathematical talks. It has many advantages over computer presentations or overhead projecting. On the other hand, I do not know any disadvantage (I speak from the point of view of people in the audience; the speaker can say that for him it is more convenient to prepare a computer presentation). Whatever can be performed with the help of a computer or overhead projector can also be performed much better on the blackboard.

First of all, the speed of a blackboard presentation is limited by the speed of writing on the board. I do not want to say that it is impossible to speak too fast. It is, however not to the same extent as many speakers do with computer presentations.

Next, the speaker does not have to erase the material as frequently as computer speakers switch the screen. It is certainly much better if the lecture room has several big blackboards. In this case the speaker can erase the contents of a board very rarely, and in the case of short communications there is no need to erase anything from the boards. Even if the speaker has to use an eraser, he can reserve one of the blackboards for the material that will not be erased during the talk.

What else? I have already said that the audience can digest the material much better if the material appears visually and gradually. It is much more difficult to comprehend the material if a large portion of it is suddenly displayed.

Finally, many computer (or overhead projector) speakers always look at the screen. The speaker at the blackboard does not look at the screen. He can watch the audience instead. As a result, he can have better contact with the audience, can have feedback from the audience, and can be sure that he has not lost the audience. As I have already said above, having good contact with the audience allows the speaker to adjust the speed of presentation and the amount of material to be presented. All of this is very important.

Again, let me repeat that not all blackboard talks are good talks. The speaker has to know how to use the advantages of the blackboard to be able to give a good talk. Nevertheless, I have no doubt that
the percentage of good blackboard talks is much higher than the percentage of good computer (or overhead projector) talks.

**What Can Be Done to Rescue Conferences?**

This is a tough question. My opinion is that only the blackboard should be used for mathematical talks. I know that some mathematicians will call me an extremist. However, even if the mathematical community is not ready to get rid of computer presentations, there are several steps that can be taken.

First of all, I think mathematicians who use computers to give talks should ask themselves why they do it. Perhaps they have no reason other than that it is more convenient for them to prepare a computer presentation. If this is the only reason, they should think about the audience rather than their own convenience. If they do it because otherwise they will not be able to cover all the material, they should understand that by using a computer they can increase the speed of the presentation, but they cannot increase the speed of comprehension.

I think it would be a good idea for Ph.D. advisors to teach their students how to give talks and to explain to them that if they learn how to give good talks at the blackboard, they will produce a much better impression on the audience, which will help their career.

Finally, I believe that the mathematical community ought to agree on a strict rule: *organizers of conferences must provide several blackboards to each lecture room.* Also, the organizers of conferences should by no means encourage participants to give computer presentations.

**Acknowledgment.** I would like to thank Nicholas Young for reading the manuscript and for making helpful suggestions.

**References**


---

**Is It Too Late to Learn Mathematics?**

**Minority and Nontraditional Students Show There Is a Second Chance**

*Estela A. Gavosto*

The typical comment when someone discovers that you are a mathematician is either “I was horrible in math” (the most common one) or “I wish I had taken more mathematics.” For many it may not be too late.

All students start learning mathematics at an early age and those who typically succeed in mathematics-intensive careers complete their formal math education while they are young too. Mathematics is the most commonly accelerated science subject during the K–12 years and the students who are talented in mathematics advance quickly in an intellectually nurturing environment. A common belief is that those who have mathematical talent will “display” it early in their lives and they will be properly tracked. International studies [10], however, show that in other countries a larger percentage of students than in the United States is identified as mathematically talented. That is, there is potentially a larger segment of the U.S. population with mathematical
the percentage of good blackboard talks is much higher than the percentage of good computer (or overhead projector) talks.

**What Can Be Done to Rescue Conferences?**

This is a tough question. My opinion is that only the blackboard should be used for mathematical talks. I know that some mathematicians will call me an extremist. However, even if the mathematical community is not ready to get rid of computer presentations, there are several steps that can be taken.

First of all, I think mathematicians who use computers to give talks should ask themselves why they do it. Perhaps they have no reason other than that it is more convenient for them to prepare a computer presentation. If this is the only reason, they should think about the audience rather than their own convenience. If they do it because otherwise they will not be able to cover all the material, they should understand that by using a computer they can increase the speed of the presentation, but they cannot increase the speed of comprehension.

I think it would be a good idea for Ph.D. advisors to teach their students how to give talks and to explain to them that if they learn how to give good talks at the blackboard, they will produce a much better impression on the audience, which will help their career.

Finally, I believe that the mathematical community ought to agree on a strict rule: organizers of conferences must provide several blackboards to each lecture room. Also, the organizers of conferences should by no means encourage participants to give computer presentations.

**Acknowledgment.** I would like to thank Nicholas Young for reading the manuscript and for making helpful suggestions.

**References**


---

**Is It Too Late to Learn Mathematics?**

Minority and Nontraditional Students Show There Is a Second Chance

*Estela A. Gavosto*

The typical comment when someone discovers that you are a mathematician is either “I was horrible in math” (the most common one) or “I wish I had taken more mathematics.” For many it may not be too late.

All students start learning mathematics at an early age and those who typically succeed in mathematics-intensive careers complete their formal math education while they are young too. Mathematics is the most commonly accelerated science subject during the K–12 years and the students who are talented in mathematics advance quickly in an intellectually nurturing environment. A common belief is that those who have mathematical talent will “display” it early in their lives and they will be properly tracked. International studies [10], however, show that in other countries a larger percentage of students than in the United States is identified as mathematically talented. That is, there is potentially a larger segment of the U.S. population with mathematical
trend than the one that we currently identify and train.

America’s prosperity in the twenty-first century is often linked to the capability of maintaining a leadership role worldwide in science and technology (see [4] and [12]). This matter brings to the forefront the need for improving K–12 science and mathematics education. In this article we will address the issue of students who at some point showed mathematical talent but either left mathematics or never pursued a career in mathematics-intensive majors. Should they resume their math studies later in their lives? We believe they should, and demographics and the math pipeline show that we need them to do so. Moreover, we will describe one approach that we have been developing to address this challenge.

Not Enough STEM Students Overall, Especially from Underrepresented Groups

According to the National Science Board’s Science and Engineering Indicators 2012 ([8]), the percentage of science and engineering degrees (excluding social and behavioral sciences) among the degrees in all fields granted to U.S. citizens and permanent residents did not change during the decade 2000–2009. During this period, the percentage of bachelor’s degrees in all fields earned by American Indian/Alaska Native, African American, and Hispanic students (URMs) increased slightly from 17 percent to 19 percent, while their percentage of science and engineering degrees remained steady at approximately 15 percent. However, these percentages could be misleading and show that, despite efforts to increase their involvement, the participation of underrepresented groups in STEM disciplines is not growing fast enough. This data does not reflect the changes in demographics, since the percentage of these underrepresented groups among the population in the eighteen- to twenty-four-year-old group has increased to 33 percent (2008 statistics, [9]). In particular, the data corresponding to Hispanics could be easily misinterpreted. The number of degrees awarded to Hispanics has increased substantially but their representation in science and engineering has not changed. The work in [6] states that underrepresented minorities, or URMs, are only two-thirds as likely as whites to earn bachelor’s degrees in STEM within six years, despite similar interest when they enter college. In mathematics the situation is even worse. The percentage of degrees awarded to URMs among those who have completed bachelor’s degrees in the subject has decreased from 14 percent in 2000 to 12 percent in 2009. In addition, Bressoud [1] points to the recent alarming decrease in the number of bachelor’s degrees in mathematics that are being awarded to African American students.

The analysis of this data is complex and poses many questions and challenges. We should also look at the numbers in mathematics not in isolation from other disciplines, but rather in the context of all the programs and opportunities that exist for underrepresented minorities in the sciences and engineering. Examples of these opportunities are the programs of the Division of Training, Workforce Development, and Diversity from the National Institute of General Medical Sciences [2]. These programs promote research training and the development of a strong and diverse biomedical research workforce. In general, the mathematical community does not have as many resources as other disciplines to recruit URMs. It is well known that other science and engineering programs welcome students with mathematical talent. These students often receive degrees in areas where they are recruited early on in their careers with offers of undergraduate research experiences and other resources. Many other socio-economic factors play a role, too, and steer students to other fields. In fact, during the period 1989–2008, the greatest increase in science and engineering bachelor’s degrees earned by underrepresented minorities has been in the fields of social, computer, and medical sciences; see [9].

Tapping into Nontraditional Students

Demographics show that current attempts to modify our pipeline are not producing changes as fast as they are needed. As we improve the curriculum paths for the future, we must also find alternative new models and strategies that provide reentry points for the wide portion of the population for whom the mathematics pipeline has not functioned.

The most common definitions of nontraditional students (see Kim [5]) include considerations of age, background (such as socio-economic status, ethnicity, first generation, and employment status), and at-risk characteristics. The nontraditional population forms the largest subset of students in the nation according to a recent report to the U.S. Congress and Secretary of Education by the Advisory Committee on Student Financial Assistance [11]. In particular, this report recommends a federal initiative to generate projects examining how to educate older adults in the sciences, engineering, technology innovation, and applied mathematics. Another report from the National Academy of Sciences [12] advocates that the federal government should give tax credits to employers who help their employees pursue continuing education in science and technology. The students included in these reports have a wide range of educational needs. A disproportionately high number of underrepresented minorities could be labeled as nontraditional students according to [5]. They often first enroll in public two-year
students can be mapped into this theory's learning principles: engaging prior understanding, providing learning opportunities for factual knowledge and conceptual understanding, and stimulating a metacognitive approach.

The design of Transitions in Mathematics is based on the above learning principles and it is taught in a studio format. Instead of starting over and taking sequences of courses with content that the students partially know, our course assists the students to fill the main "holes" in their background and teaches them how to gain the necessary knowledge to tackle new mathematical challenges in the future. The course starts with a math assessment, surveys, and interviews with each of the students. Based on the collected information, an individualized project grounded in the student's prior mathematical knowledge and life experiences is designed. The project outcomes are in tune with the future needs of the student. Parallel to the project, a concept map is constructed representing the connections between all the topics that the student will need to learn to complete the project. The students do worksheets in these topics to master the necessary skills and concepts. They are also introduced to a range of mathematical software and their proficient use is encouraged. The class meets at a media room or at a computer lab. The students work on their projects or worksheets individually and we foster collaboration among them. There is no formal lecture and the instructor and a graduate student assistant interact with the students giving feedback on their work. In every class period, the students give oral presentations on their current work and receive comments from the rest of the class on their conceptual understanding. The training in career development tools (library/databases search skills, public speaking, etc.) also increases long-term learning autonomy. The projects completed by the students cover a wide range of subjects. For example, they have included both applied and theoretical topics such as numerical solutions of diffusion equations for drug delivery, wavelet-based characterization of vertebral trabecular bone structure, and approximation theory in Lipschitz spaces.

Teaching the course has provided us with a collection of projects and a complex and interesting educational landscape. We are noticing patterns of topics and paths and eventually we could develop course material to be shared with others. Collaborating with faculty mentors in other departments is essential in developing meaningful applications. We have observed that traditional placement measures, like previous courses and their grades, do not dictate as much as in other settings what the students end up doing. Motivation is the most important factor in determining the students' progress, so having clear individualized goals in terms of a research project or more advanced courses is

---

1IMSD at Kansas University is funded by National Institutes of Health grant #R25GM62232 from the National Institute of General Medical Science.
key to the mathematical growth of the students. Overall, the students are very appreciative of the opportunity that Transitions in Mathematics gives them. Often they expressed that they have never learned so much mathematics in a course before. The ultimate goal of the initiative is to prepare students to complete a Ph.D. in a biomedical research area (which includes mathematics) and we track all the students after they have taken our course. Up to this point, two-thirds of the students who have completed our course and graduated have been admitted to graduate programs.

Transitions in Mathematics provides a rewarding and engaging experience where we challenge the students and ourselves to learn mathematics and its applications in alternative ways. Different backgrounds and experiences are equalized when we have to learn something that we do not know. As the median age of the population in the U.S. and in many regions of the world increases, older students, striving to learn the mathematics that they did not learn in their first college experience, inspire us to reach for new frontiers in areas of mathematics outside our comfort zone. Certainly, there is a second chance to learn mathematics at a higher level.

Acknowledgment. The author would like to thank the reviewers of this article for valuable comments and suggestions.

References
Giving a Talk

Bryna Kra

No one likes to sit through a bad talk, but unfortunately everyone does it much too frequently. No one sets out to give a bad talk, but probably all of us have done so. Paul Halmos [1] wrote a beautiful article on how to give a talk, and his advice remains apt today (and some of it is repeated here). But new technology, varying audiences, and different venues force modifications in how we give talks. This is subjective advice on how to give a good talk and, especially, how to avoid giving a bad talk. Of course, not all of it applies to everyone or to every situation, but my hope is that anyone reading this article is provoked into thinking a bit more when preparing a talk.

What Kind of Talk?

Think about the purpose of the talk: Is it a seminar? Colloquium? Conference? Job talk? What is the target audience: is it a survey talk? Is it aimed at graduate students? Is it meant for experts? The answers affect all aspects of the talk, starting with the format: an “old fashioned” chalk talk? Projecting via a document camera or overhead? A computer presentation?

There are no right or wrong answers, but before making a decision, think about the audience and the venue. For a survey talk with many references, a computer presentation might be appropriate. For a large conference, boards might not be available. For a small seminar, one can include details of a proof that can only be done on a board (or perhaps more accurately, should only be done on a board). Find out if the room is appropriate for using the board, particularly if it is a large room. When using technology, find out if the screen is easily visible, if it can be combined with use of a board, and if a computer is provided. These decisions depend on the culture of the venue for the talk, but with proper planning, any presentation method available can be used to give an excellent talk geared to the appropriate target audience.

Plan and Scope

The first piece of advice is obvious, but still needs stating: plan the talk. The talk should be a cohesive narrative, telling a story from the beginning to the end. Overall, plan to keep it simple. The whole audience should follow at least half of the talk, and most of the audience should be able to follow most of the talk. Do not just address the experts; save any comments meant for them until the last few minutes of the talk.

How does one go about planning? Perhaps the best advice is a negative piece of advice: do not try to cover too much! This especially applies to computer talks—just because it is easy to flash many theorems and explanations on a screen does not mean that anyone can follow. Instead, pick a focus. Do not set out to tell the audience every theorem in a subject or every theorem from a paper, but take the time to delve into a particular aspect.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.

Start with a hook: a general problem or subject likely to be of interest to the audience. By providing background, history, and placing the subject in a greater context, one can pique the audience’s interest before delving into any details. The exact type of motivation depends, as always, on the purpose of the talk and the target audience. For example, a colloquium on patterns in the primes might start with the Twin Prime Conjecture and one on the wave equation might start with a picture or graphic, whereas for a talk in a number theory or PDE seminar, one should assume familiarity with these topics.
and even experts do not always remember all the definitions and basic theorems. Of course, one can go too far by including too many basics for the particular audience: for a talk in a dynamics seminar, there is no need to define the word *ergodic*, but one should in a colloquium.

A good rule of thumb (other than for expository talks) is that at least half the talk should be devoted to the speaker’s own work. One should get to one’s own work as early as is practical—the earlier the better. If left to the last five minutes, the audience does not have time to absorb the material and pose questions.

Everyone likes coming out of a talk having learned something, and the speaker should try to include some sort of proof, even in a general audience talk. It might be a very simple fact or a much simplified toy version of the real problem. If the talk is a seminar, it might be an outline of steps in a long proof or the proof of a crucial, technical lemma. If the statement or proof can be given in a simple case, do it. The audience does not need to hear the details of the most general setting in which the theorem can be proven and can instead be directed to the paper(s) concerning it. By avoiding the most general possible statement, much heavy (and incomprehensible) notation is also avoided. More important than the details is the motivation of the subject, an overview of the ideas, and the intuition behind them.

Talks are meant to convey something that is not in the literature and not just to be a repetition of the introduction of a recent article. A useful exercise is to think back to how the result was proven: what were false steps? What were examples used to test the ideas? Explaining such steps can be instructive components of a talk.

For a seminar or typical conference talk, one theorem or a few related theorems are usually enough material. If the theorems covered naturally split into two different papers, then this is probably material for two separate talks. This does not mean that any result from another article ought to be omitted, but rather it should be used to bolster the main focus of the talk. Surveys are different: they should go slowly and have a goal but likely include many more (interrelated) topics.

The last step of preparing a talk, particularly for inexperienced speakers, is practice. Practice the talk out loud and time it. Frequently, people are surprised to find that what they thought would be easy to fit into twenty minutes really takes forty-five minutes, or vice versa. Even experienced speakers benefit by saying all the words before the actual talk, even if each iteration is somewhat different. I still write every word that I plan to say, though I rarely refer to these extensive notes during the talk—just the exercise of thinking through the words improves the presentation.

Citations and Thanks
Make it clear which results are yours and which are in the literature. Give citations for any results used or previous results on the same topic; there is no need to give complete bibliographic listings, just the author(s) and year of publication suffice.

Always thank the host or hosts for the invitation. Remember to cite any coauthors and anyone else who helped with the work.

Computer Talks
The biggest change over the past ten years has been the proliferation of computer talks, leading to a host of self-created problems. The most basic problem is unreadable slides: the computer is not set up correctly for projection, the type is too small, or a slide is too crowded.

The first of these problems is easily corrected with a few minutes of planning. Check in advance that the file works in full screen mode and know how to put the computer into this mode. If at all possible, try the file on the computer to be used during the talk. Before the talk begins, make sure to close any other programs running in the background, especially anything that might interrupt the presentation with a pop-up.

The other two issues need to be addressed during the preparation of the talk. Simply put, computer talks do not work if the slides are too crowded. The type size has to be large, and it should be readable at the back of the room. This means that there is plenty of space between the lines and a maximum of about ten lines per slide (fewer is even better). Leave out as many words as possible and do not write full paragraphs. Beamer [2], a commonly used \LaTeX{} class for creating slides, as with most other presentation programs, comes with preset margins and spacings. They are designed so that one cannot put too much on a single slide, and there is no reason to override the settings.

There is also no reason to put every word on the slide and then do nothing more than read it aloud—if that is the plan, just post the slides on a website and forget about giving the talk. It is better to have less on the slides and then use many words to explain what it all means. You should be able to state everything on a slide in less than a minute, but spend at least two to three minutes per slide fleshing out the explanations. Many excellent speakers spend five to six minutes per slide.

A pointer or the mouse is useful for highlighting what the audience should focus on, but do not overdo it. Use it to highlight a particular object for a few seconds and then let the audience continue to look (or not) themselves. A bouncing laser pointer is a sure way to make the audience dizzy.

Some speakers like to reveal the lines of a presentation one line at a time. But this means that the last line is visible for only a few seconds before it...
is replaced by a new page. It is often better to show the whole slide at once and let the audience see what is coming. To draw the audience’s attention to a particular statement, use the pointer, highlight the particular statement in another color (easy to do, for example, in Beamer), or use a lighter color to show the lower portions of the slide (again, easy in Beamer).

Slides are great for repeating something that is needed more than once. But that does not mean scrolling back to an earlier slide out of order. Instead, include another slide that repeats the result, instantly reminding the audience exactly what is needed—this is one of the benefits of a computer presentation.

Background graphics are almost always distracting. To get the audience to focus on the math, leave out the background of a beautiful sunset that obscures the equations and theorems. Some colors do not project well and should be avoided; particularly, many light colors do not work on rear projection screens. When in doubt, try the colors in advance on a real screen, not just on a small computer screen.

On the other hand, when all the talks look the same, the audience begins to glaze over. Judicious use of color on slides can break up the monotony without being distracting. Pictures and graphics can enhance a talk, and when appropriate, animation is an excellent tool for explaining an idea or a proof. The animations should be embedded in the file and tested in advance—no one enjoys watching a speaker search through hundreds of files on a computer.

Short Talks
Many conferences include short (twenty minutes or less) talks. Although one might think that these are easier to prepare, in some sense they are even harder. In twenty minutes, there is no time to present all the history, give a complete proof, or enter into the details of a construction. The goal is to convey a single result or several very closely related results, and this requires stripping out all other elements from the talk. Usually six to seven slides is more than enough for twenty minutes. Think of a twenty-minute talk as an extended movie trailer for the results: get the audience interested in finding out more.

Most advice for short presentations is the same as that for longer ones, just compressed: the results should be placed in context, important earlier results should be cited, and there should be a hint as to the ingredients that enter the proof. Heavy notation and long statements should be avoided, even if this means that the general statement is omitted or the statements are less precise (so long as the imprecisions are made clear to the audience).

Language Issues
For nonnative English speakers, giving a talk in English can often be a significant hurdle, but there are several ways to make it easier. The first is to practice the talk, even multiple times, with or without an audience. It may be easier to use a computer for the presentation and use the slides as a memory tool for difficult words or phrases. If it is more appropriate to give a chalk talk, a handout can help by summarizing key points. Always remember to speak in full sentences and try to explain everything as much as possible.

Pet Peeves
This is a personal list of egregious errors that I have witnessed. Having said that, I am sure that I have committed each of these violations at some point, but these errors can be easily avoided:

• Do not number your theorems. To refer back to a theorem, give it a memorable name, like the “Comparison Theorem” or the “Main Estimate”. No one can remember what Theorem 2 was, much less Theorem 8!
• Do not use the same letter for two different purposes, even if this seems entirely natural. For example, if there is a map that is $C^2$ and a constant $c$, use $k$ for the constant. Instead of using both a capital and lowercase version of a letter, such as $\phi$ and $\Phi$, use different letters unless they are very closely linked. Otherwise, it is too hard to distinguish the two while talking, leading to awkward statements like “little $\phi$” and “capital $\Phi$”.
• Do not introduce new definitions or a new topic in the last five minutes (the exception being a short talk). Instead, use this time to sum up main points, describe future directions, or address the experts.

• Do not reveal your own insecurities with comments such as “my result is trivial” or “I would extend the result to case X, but I don’t understand X” or “I am sure everyone in the audience knows more about this.” Never belittle your own work, as there are many other people who would be happy to do this for you.

Finishing Up
Do not rush the talk, but do not go over. It is better to finish with a minute or two left rather than keep impatient people in their seats for extra time. Be prepared for questions at the end and leave time for the audience to ask them.

Keep in mind that you, the speaker, know more about your work than the audience does. So relax, good luck, and enjoy yourself.

References
Codá Marques Awarded Ramanujan and TWAS Prizes

FERNANDO CODÁ MARQUES of the Instituto Nacional de Matemática Pura e Aplicada (IMPA) in Rio de Janeiro, Brazil, has been named the recipient of two major prizes for 2012: the Ramanujan Prize of the International Centre for Theoretical Physics (ICTP), the Niels Henrik Abel Memorial Fund, and the International Mathematical Union (IMU); and the TWAS Prize, awarded by the Academy of Sciences for the Developing World (TWAS).

The announcement from the ICTP/IMU for the Ramanujan Prize reads in part: “The prize is in recognition of his several outstanding contributions to differential geometry. Together with his coauthors, Fernando Codá Marques has solved long-standing open problems and obtained important results, including results on the Yamabe problem, the complete solution of Schoen’s conjecture, counterexamples to the rigidity conjecture of Min-Oo, connectivity of the space of positive curvature metrics on an orientable 3-manifold, and most recently, a proof of the Willmore conjecture.” The Ramanujan Prize is awarded annually to a researcher from a developing country who is younger than forty-five years of age on December 31 of the year of the award and who has conducted outstanding research in a developing country. Researchers working in any branch of the mathematical sciences are eligible. The prize carries a cash award of US$15,000, and the winner is invited to deliver a lecture at ICTP.

Codá Marques was honored with the TWAS Prize in Mathematics “for his fundamental contributions to the field of differential geometry, particularly for his work on variational problems in conformal geometry and applications of the theory of Ricci flow.” The TWAS Prizes honor individual scientists who have been working and living in a developing country for at least ten years. It carries a cash award of US$15,000, and the 2012 prizewinners will give lectures at TWAS’s general meeting in Argentina in 2013. Codá Marques was honored with the TWAS Prize in Mathematics “for his fundamental contributions to the field of differential geometry, particularly for his work on variational problems in conformal geometry and applications of the theory of Ricci flow.” The TWAS Prizes honor individual scientists who have been working and living in a developing country for at least ten years. It carries a cash award of US$15,000, and the 2012 prizewinners will give lectures at TWAS’s general meeting in Argentina in 2013.

Codá Marques received his Ph.D. from Cornell University in 2003. He has been affiliated with IMPA since then and has also held visiting positions at Stanford University, Princeton University, and the Institute for Advanced Study. He received a FAPERJ Young Scientist Fellowship in 2008 from the State of Rio de Janeiro, Brazil, and was an invited speaker at the International Congress of Mathematicians in Hyderabad, India, in 2010. He has also been awarded the 2012 UMALCA Prize of the Mathematical Union of Latin America and the Caribbean.

—Elaine Kehoe

CMS G. de B. Robinson Award Announced

TEODOR BANICA of the University of Toulouse, SERBAN BELINSCHI of Queens University, MIREILLE CAPITAINÉ of the Centre National de la Recherche Scientifique (CNRS), and BENOÎT COLLINS of the University of Ottawa have been awarded the 2012 G. de B. Robinson Award of the Canadian Mathematical Society (CMS) for their paper titled “Free Bessel laws”, published in the Canadian Journal of Mathematics 63 (2011), 3–37 http://dx.doi.org/10.4153/CJM-2010-060-6. The award is given in recognition of outstanding contributions to the Canadian Journal of Mathematics or the Canadian Mathematical Bulletin.

—From a CMS announcement

Przybyłowicz Awarded 2012 Information-Based Complexity Prize

PAWEL PRZYBYLOWICZ of AGH University of Science and Technology, Krakow, Poland, has been award the 2012 Information-Based Complexity (IBC) Young Researcher Award. The award consists of US$1,000 and a plaque. The prize is given annually for significant contributions to information-based complexity by a young researcher who has not reached his or her thirty-fifth birthday by September 30 the year of the award.

—Joseph Traub, Columbia University

Vinet Awarded CAP-CRM Prize

LUC VINET of the University of Montreal has been awarded the 2012 CAP-CRM Prize in Theoretical and Mathematical Physics by the Canadian Association of Physicists (CAP) and the Centre de Recherches Mathématiques (CRM). He was honored for “his outstanding and continued contributions to mathematical physics, mainly based on the study of symmetries, algebraic structures, and special functions.” The prize is intended to recognize research excellence in the fields of theoretical and mathematical physics.

—From a CAP-CRM announcement
Alexakis Awarded Aisenstadt Prize

SPYROS ALEXAKIS of the University of Toronto has been awarded the 2013 André-Aisenstadt Prize of the Centre de Recherches Mathématiques (CRM) for his work in analysis and mathematical physics. According to the prize citation, “his main contribution is a solution to a conjecture of Deser and Schwimmer regarding the structure of global conformal invariants. ... Secondly, together with Klainerman and Ionescu, he made important progress in the understanding of the Kerr solutions to Einstein’s equations. Finally, jointly with Mazzeo, he obtained deep results concerning minimal surfaces with bounded Willmore energy.” Alexakis received his Ph.D. from Princeton University in 2005 and has held Clay Research and Sloan fellowships. The prize recognizes outstanding research achievement by a young Canadian mathematician in pure or applied mathematics and consists of a cash award and a medal.

―From a CRM announcement

Athreya and Goswami Awarded 2012 Bhatnagar Prize

SIVA RAMACHANDRAN ATHREYA and DEBASHISH GOSWAMI of the Indian Statistical Institute won the Shanti Swarup Bhatnagar Prize for Science and Technology in the mathematical sciences. The prize is awarded by the Council of Scientific Research and Industrial Development to recognize outstanding Indian work in science and technology. Shanti Swarup Bhatnagar was the founding director of the Council. It is the highest award for science in India. The prize carries a cash award of 500,000 rupees (approximately US$9,000).

―Council of Scientific Research and Industrial Development, India

CAREER Awards Presented

The Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) has honored thirty-one young mathematicians in fiscal year 2012 with Faculty Early Career Development (CAREER) awards. The NSF established the awards to support promising scientists, mathematicians, and engineers who are committed to the integration of research and education. The grants provide funding of at least US$400,000 over a five-year period. The 2012 CAREER grant awardees and the titles of their grant projects follow.

ANDREW BLUMBERG, University of Texas, Austin, Algebraic K-Theory, Trace Methods, and Noncommutative Geometry; BENJAMIN BRUBAKER, University of Minnesota, Twin Cities, Multiple Dirichlet Series, Automorphic Forms, and Combinatorial Representation Theory; THOMAS CHEN, University of Texas, Austin, Dynamics of Complex Quantum Systems, Scaling Limits, and Renormalization; GUANG CHENG, Purdue University, Bootstrap M-Estimation in Seminonparametric Models; DANIJELA DAMJANOVIC, William Marsh Rice University, Smooth Group Actions: Persistence and Prevalence of Chaotic Behavior; YINGYING FAN, University of Southern California, High-Dimensional Variable Selection in Nonlinear Models and Classification with Correlated Data; JOSE FIGUEROA-LOPEZ, Purdue University, Bridging High-Frequency Data Analysis and Continuous-Time Features of Levy Models; TEENA GERHARDT, Michigan State University, Equivariant Homotopy and Algebraic K-Theory; JULIA GRIGSBY, Boston College, Connections between Algebraic and Geometric Invariants in Low-Dimensional Topology; MATTHEW HEDDEN, Michigan State University, Floer Homology and Low-Dimensional Topology; DANIEL KRASHEN, University of Georgia Research Foundation, The Arithmetic of Fields and the Complexity of Algebraic Structures; ANTON LEVKIN, Georgia Institute of Technology, Algorithms and Software for Computational Algebraic Geometry; PEIJUN LI, Purdue University, Direct and Inverse Scattering Problems for Wave Propagation in Complex and Random Environments; YEHUA LI, University of Georgia Research Foundation, New Topics in Functional Data Analysis; ROBERT LIPSHTITZ, Columbia University, Floer-Theoretic Approaches to Low-Dimensional Topology; LAURA MILLER, University of North Carolina, Chapel Hill, Mathematical Modeling and Experiments of Neuro-mechanical Pumping; ANDREW NEITZKE, University of Texas, Austin, Geometric Applications of Gauge Theory; IRINA NENCIU, University of Illinois, Chicago, Long-Time Asymptotics of Completely Integrable Systems with Connections to Random Matrices and Partial Differential Equations; DRAGOS OPREA, University of California, San Diego, Stable Sheaves, Stable Quotients, Stable Pairs; GRIGORIS PAOURIS, Texas A&M University, Geometry of Measures in High Dimensions; SAM PAYNE, Yale University, Tropical and Nonarchimedean Analytic Methods in Algebraic Geometry; ALESSANDRO RINALDO, Carnegie-Mellon University, Statistical Inference for Topological and Geometric Data Analysis; SEBASTIEN ROCH, University of Wisconsin, Madison, Phylogenomics: New Computational Methods through Stochastic Modeling and Analysis; SVETLANA ROUDENKO, George Washington University, Nonlinear Phenomena in Evolution PDE; BODHISATTVA SEN, Columbia University, Nonparametric Methods in Multiple Dimensions: Shape Restrictions, Bootstrap, and Beyond; RAMON VAN HANDEL, Princeton University, Conditional Theory of Large-Scale Stochastic Systems; JOHN VOIGHT, University of Vermont and State Agricultural College, Explicit Methods in Arithmetic Geometry; HUIXIA WANG, North Carolina State University, A New and Pragmatic Framework for Modeling and Predicting Conditional Quantiles in Data-Sparse Regions; BENJAMIN WEBSTER, Northeastern University, Representation Theory of Symplectic Singularities; YIFENG YU, University of California, Irvine, Analysis of G-Equations in the Modeling of Turbulent Flame Speed and Comparison with Other Math Models; HONG-KUN ZHANG, University of Massachusetts, Amherst, The Nature of SRB Measures for Nonequilibrium Hyperbolic Systems.

―Elaine Kehoe

246 Notices of the AMS Volume 60, Number 2
Professors of the Year Chosen

Four national winners and thirty-one state winners have been honored with U.S. Professor of the Year awards. Among the state winners are three mathematics professors: MIKE PINTER of Belmont University, Tennessee; JOHN HAMMAN of Montgomery College, Maryland; and STEPHEN DEBACKER of the University of Michigan, Ann Arbor. The U.S. Professors of the Year awards program celebrates outstanding instructors across the country. Sponsored by the Council for Advancement and Support of Education (CASE) and the Carnegie Foundation for the Advancement of Teaching, it is the only national program to recognize excellence in undergraduate education.

—From a CASE announcement

Rhodes Scholarships Announced

A senior mathematics student is among the thirty-two outstanding students named American Rhodes Scholars for 2013. EVAN R. SZABLOWSKI of Bakersfield, California, is a senior at the United States Military Academy, where he majors in mathematics. He has also studied at Al-Akhwawyn University in Morocco and has worked on projects encouraging entrepreneurship in Ethiopia and on emerging markets in the Czech Republic. He is a triathlete, conducts a West Point choir, and was a member of the first American team ever to win the Sandhurst military competition. At Oxford he plans to do the M.Sc. in mathematical modeling and scientific computing. The Rhodes Scholars were chosen from a pool of more than eight hundred students nominated by their colleges and universities.

—From a Rhodes Trust announcement

Mathematics Opportunities

Summer Program for Women Undergraduates

The 2013 Summer Program for Women in Mathematics (SPWM) will take place at George Washington University in Washington, D.C., from June 29 to August 3, 2013. This is a five-week intensive program for mathematically talented undergraduate women who are completing their junior year and may be contemplating graduate study in mathematical sciences. The goals of this program are to communicate an enthusiasm for mathematics, to develop research skills, to cultivate mathematical self-confidence and independence, and to promote success in graduate school. A number of seminars will be offered, led by active research mathematicians with the assistance of graduate students. The seminars will be organized to enable the students to obtain a deep understanding of basic concepts in several areas of mathematics, to learn how to do independent work, and to gain experience in expressing mathematical ideas orally and in writing. There will be panel discussions on graduate schools, career opportunities, and the job market. Weekly field trips will be organized to facilities of mathematical interest around the Washington area.

Applicants must be U.S. citizens or permanent residents studying at a U.S. university or college who are completing their junior year or the equivalent and have mathematical experience beyond the typical first courses in calculus and linear algebra. Sixteen women will be selected. Each will receive a travel allowance, campus room and board, and a stipend of US$1,750. The deadline for applications is March 1, 2013. Early applications are encouraged. Applications are accepted only by mail. For further information, please contact the director, Murli M. Gupta, email: mmg@gwu.edu; telephone: 202-994-4857; or visit the program’s website at http://www.gwu.edu/~spwm/. Application material is available on the website.

—From an SPWM announcement

News from the CRM

The Centre de Recerca Matemàtica (CRM) in Barcelona, Spain, will hold the International School and Research Workshop on Complex Systems April 8–13, 2013, organized by Álvaro Corral and Tomás Alarcón. The overall theme is the dynamics of complex systems, with an emphasis on emergent phenomena such as spatio-temporal patterns. The course focuses on the introduction and application of the basic quantitative concepts and tools for studying complexity via lectures and problem-solving classes. The deadline for registration is March 18, 2013. For more information see the website http://www.crm.cat/en/Activities/Pages/ActivityDescriptions/International-School-and-Research-Workshop-on-Complex-systems.aspx.

—From a CRM announcement
April 2013 Mathematics Awareness Month: Mathematics of Sustainability

Humanity continually faces the challenge of how to balance human needs against the world’s resources while operating within the constraints imposed by the laws of nature. Mathematics helps us better understand these complex issues and is used by mathematicians and practitioners in a wide range of fields to seek creative solutions for a sustainable way of life. Society and individuals will need to make difficult choices; mathematics provides us with tools to make informed decisions. See the poster, theme essays, and a sample press release that can be adapted for public awareness activities at the Mathematics Awareness Month website at [www.mathaware.org](http://www.mathaware.org).

Mathematics Awareness Month is sponsored each year by the Joint Policy Board for Mathematics (American Mathematical Society, American Statistical Society, Mathematical Association of America, and Society for Industrial and Applied Mathematics) to recognize the importance of mathematics through written materials and an accompanying poster that highlight mathematical developments and applications in a particular area.

—AMS announcement

EMS Code of Ethical Practice

In 2010 the European Mathematical Society (EMS) established an Ethics Committee, the first task of which was to draft a Code of Practice. This task was accomplished in April 2012. After discussion by the EMS Council, the final version of the document was approved by the EMS Executive Committee at the end of October 2012.

“The Code [of Practice] emphasizes ethical aspects of publication, dissemination, and assessment of mathematics,” the document states. “The European Mathematical Society considers the successful open and transparent publication and dissemination of mathematical research to be of the greatest importance for the future of our subject. Unethical behavior in publication and dissemination contaminates and jeopardizes the integrity and expansion of mathematics, and could have serious consequences for individuals....

“The Code will be revised within three years, in the light of experience with cases analyzed, and after consideration of comments received.

“The Ethics Committee is willing to consider cases involving allegations of unethical behavior in the publishing of mathematics. The practices that the Committee intends to follow are laid down in the section ‘Procedures’.”


—Allyn Jackson

Correction

The email address for Joseph Buckley, author of “Ernst Snapper (1913–2011)” in the January 2013 issue of the Notices was incorrect as published. The correct email address is joseph.buckley@wmich.edu.

—Sandy Frost
Math in Moscow Scholarships Awarded

The AMS has made awards to five mathematics students to attend the Math in Moscow program in the spring of 2013. Following are the names of the undergraduate students and their institutions: Matthew Chao, Rutgers University; Thao Do, Stony Brook University; Dave Lingenbrink, Harvey Mudd College; Cameron Marcott, University of Minnesota, College of Continuing Education; John Zanazzi, Northern Arizona University.

Math in Moscow is a program of the Independent University of Moscow that offers foreign students (undergraduate or graduate students specializing in mathematics and/or computer science) the opportunity to spend a semester in Moscow studying mathematics. All instruction is given in English. The fifteen-week program is similar to the Research Experiences for Undergraduates programs that are held each summer across the United States. The AMS awards several scholarships for U.S. students to attend the Math in Moscow program. The scholarships are made possible through a grant from the National Science Foundation. For more information about Math in Moscow, consult http://www.mccme.ru/mathinmoscow and the article “Bringing Eastern European Mathematical Traditions to North American students”, Notices, November 2003, pages 1250–4.

—Elaine Kehoe

AMS at the 2012 AMATYC Annual Conference

The 38th Annual Conference of the American Mathematical Association of Two-Year Colleges (AMATYC) was held in Jacksonville, Florida, in November 2012. The AMS hosted an exhibit at the conference, which included keynote speakers, posters, sessions, networking opportunities, committee meetings, and delegate assembly. See http://www.ams.org/meetings/amatyc2012-mtg.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers
paoffice@ams.org

AMS Email Support for Frequently Asked Questions

A number of email addresses have been established for contacting the AMS staff regarding frequently asked questions. The following is a list of those addresses together with a description of the types of inquiries that should be made through each address.

- abs-coord@ams.org for questions regarding a particular abstract or abstracts questions in general.
- acquisitions@ams.org to contact the AMS Acquisitions Department.
- ams@ams.org to contact the Society’s headquarters in Providence, Rhode Island.
- amsdc@ams.org to contact the Society’s office in Washington, D.C.
- amsfellows@ams.org to inquire about the Fellows of the AMS.
- amsmem@ams.org to request information about membership in the AMS and about dues payments or to ask any general membership questions; may also be used to submit address changes.
- ams-simons@ams.org for information about the AMS Simons Travel Grants Program.
- ams-survey@ams.org for information or questions about the Annual Survey of the Mathematical Sciences or to request reprints of survey reports.
- bookstore@ams.org for inquiries related to the online AMS Bookstore.
- classads@ams.org to submit classified advertising for the Notices.
- cust-serv@ams.org for general information about AMS products (including electronic products), to send address changes, place credit card orders for AMS products, to

From the AMS Public Awareness Office

2013 Calendar of Mathematical Imagery

To request a complimentary copy of the 2013 calendar featuring selected images from Mathematical Imagery (http://www.ams.org/mathimagery), please send email to the paoffice@ams.org with the subject line “2013 calendar-notice”. Please limit your order to three copies so that others may also have the opportunity to receive a copy.
correspond regarding a balance due shown on a monthly statement, or conduct any general correspondence with the Society’s Sales and Member Services Department.

development@ams.org for information about giving to the AMS, including the Epsilon Fund.

eims-info@ams.org to request information about Employment Information in the Mathematical Sciences (EIMS). For ad rates and to submit ads go to http://eims.ams.org.

emps-info@ams.org for information regarding AMS employment and career services.

eprod-support@ams.org for technical questions regarding AMS electronic products and services.

gradprg-ad@ams.org to inquire about a listing or ad in the Find Graduate Programs online service.

mathcal@ams.org to send information to be included in the “Mathematics Calendar” section of the Notices.

mathjobs@ams.org for questions about the online job application service MathJobs.org.

mathprograms@ams.org for questions about the online program application service Mathprograms.org.

mathrev@ams.org to submit reviews to Mathematical Reviews and to send correspondence related to reviews or other editorial questions.

meet@ams.org to request general information about Society meetings and conferences.

mmsb@ams.org for information or questions about registration and housing for the Joint Mathematics Meetings (Mathematics Meetings Service Bureau).

msn-support@ams.org for technical questions regarding MathSciNet.

notices@ams.org to send correspondence to the managing editor of the Notices, including items for the news columns. The editor (notices@math.wustl.edu) is the person to whom to send articles and letters. Requests for permission to reprint from the Notices should be sent to reprint-permission@ams.org (see below).

notices@ams.org to request general information about Society meetings and conferences.

notices-ads@ams.org to submit electronically paid display ads for the Notices.

notices-booklist@ams.org to submit suggestions for books to be included in the “Book List” in the Notices.

notices-letters@ams.org to submit letters and opinion pieces to the Notices.

notices-whatis@ams.org to comment on or send suggestions for topics for the “WHAT IS…?” column in the Notices.

nsagrants@ams.org for information about the NSA-AMS Mathematical Sciences Program.

paoffice@ams.org to contact the AMS Public Awareness Office.

president@ams.org to contact the president of the American Mathematical Society.

prof-serv@ams.org to send correspondence about AMS professional programs and services.

publications@ams.org to send correspondence to the AMS Publication Division.

pub-submit@ams.org to submit accepted electronic manuscripts to AMS publications (other than Abstracts). See http://www.ams.org/submit-book-journal to electronically submit accepted manuscripts to the AMS book and journal programs.

reprint-permission@ams.org to request permission to reprint material from Society publications.

sales@ams.org to inquire about reselling or distributing AMS publications or to send correspondence to the AMS Sales and Member Services Department.

secretary@ams.org to contact the secretary of the Society.

student-serv@ams.org for questions about AMS programs and services for students.

tech-support@ams.org to contact the Society’s typesetting Technical Support Group.

textbooks@ams.org to request examination copies or inquire about using AMS publications as course texts.

webmaster@ams.org for general information or for assistance in accessing and using the AMS website.

Deaths of AMS Members

SHREERAM SHANKAR ABHYANKAR, professor, Purdue University, died on November 2, 2012. Born on July 22, 1930, he was a member of the Society for 58 years.

HENRY FRANDSEN, of Knoxville, Tennessee, died on September 6, 2012. Born on May 21, 1933, he was a member of the Society for 51 years.

PHILIP G. KIRMSER, of Manhattan, Kansas, died on July 26, 2012. Born on December 17, 1919, he was a member of the Society for 63 years.

CHRISTOPH J. NEUGEBAUER, of Mechanicsville, Virginia, died on August 27, 2012. Born on April 21, 1927, he was a member of the Society for 59 years.
Reference and Book List

The Reference section of the Notices is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.

Contacting the Notices
The preferred method for contacting the Notices is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people’s mathematics research.

The managing editor is the person to whom to send requests for permissions, as well as all other inquiries, go to the managing editor. The electronic-mail addresses are notices@math.wustl.edu in the case of the editor and smf@ams.org in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Information for Notices Authors
The Notices welcomes unsolicited articles for consideration for publication, as well as proposals for such articles. The following provides general guidelines for writing Notices articles and preparing them for submission. Contact information for Notices editors and staff may be found on the Notices website, http://www.ams.org/notices.

Upcoming Deadlines
January 13, 2013: Applications for Jefferson Science Fellows Program. See email jsf@nas.edu, telephone 202-334-2643, or see the website http://sites.nationalacademies.org/PGA/Jefferson/PGA_046612.


January 31, 2013: Entries for AWM Essay Contest. Contact the contest organizer, Heather Lewis, at hlewis5@naz.edu.

February 1, 2013: Applications for February review for National Academies Research Associateship Programs. See the website http://sites.nationalacademies.org/PGA/RAP/PGA_050491 or contact Research Associateship Programs,

Where to Find It
A brief index to information that appears in this and previous issues of the Notices.

AMS Bylaws—January 2012, p. 73
AMS Email Addresses—February 2013, p. 249
AMS Ethical Guidelines—June/July 2006, p. 701
AMS Officers 2010 and 2011 Updates—May 2012, p. 708
AMS Officers and Committee Members—October 2012, p. 1290
Contact Information for Mathematical Institutes—August 2012, p. 979
Conference Board of the Mathematical Sciences—September 2012, p. 1128
IMU Executive Committee—December 2011, p. 1606
Information for Notices Authors—June/July 2012, p. 851
Mathematics Research Institutes Contact Information—August 2012, p. 979
National Science Board—January 2013, p. 109
NRC Board on Mathematical Sciences and Their Applications—March 2012, p. 444
NRC Mathematical Sciences Education Board—April 2011, p. 619
NSF Mathematical and Physical Sciences Advisory Committee—February 2013, p. 252
Program Officers for Federal Funding Agencies—October 2012, p. 1284 (DoD, DoE); December 2012, p. 1585 (NSF Mathematics Education)
Program Officers for NSF Division of Mathematical Sciences—November 2012, p. 1469
Reference and Book List


March 31, 2013: Applications for IPAM graduate summer school on computer vision. See www.ipam.ucla.edu.

April 1, 2013: Letters of intent for proposals for one-semester programs at the Bernoulli Center (CIB). See the website http://cib.epfl.ch/.

April 15, 2013: Applications for fall 2013 semester of Math in Moscow. See http://www.mccme.ru/mathinmoscow, or write to: Math in Moscow, P.O. Box 524, Wynnewood, PA 19096; fax: +7095-291-65-01; e-mail: mim@mccme.ru. Information and application forms for the AMS scholarships are available on the AMS website at http://www.ams.org/programs/travel-grants/mimMoscow, or by writing to: Math in Moscow Program, Membership and Programs Department, American Mathematical Society, 201 Charles Street, Providence RI 02904-2294; email student-serv@ams.org.

May 1, 2013: Applications for May review for National Academies Research Associateship Programs. See the website http://sites.nationalacademies.org/PGA/RAP/PGA_050491 or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email rap@nas.edu.


MPS Advisory Committee

Following are the names and affiliations of the members of the Advisory Committee for Mathematical and Physical Sciences (MPS) of the National Science Foundation. The date of the expiration of each member’s term is given after his or her name. The website for the MPS directorate may be found at www.nsf.gov/home/mps/. The postal address is Directorate for the Mathematical and Physical Sciences, National Science Foundation, 4201 Wilson Boulevard, Arlington, VA 22230.
Reference and Book List

James Berger (chair) (09/14)
Department of Statistical Science
Duke University

Daniela Bortolotto (09/14)
Department of Physics
Purdue University

Emery N. Brown (09/14)
Massachusetts Institute of Technology

R. Paul Butler (09/13)
Department of Terrestrial Magnetism
Carnegie Institution of Washington

Emily A. Carter (09/15)
Department of Mechanical and Aerospace Engineering
Princeton University

Eric A. Cornell (09/13)
JILA University of Colorado

George W. Crabtree (09/15)
Materials Science Division Argonne National Laboratory

Juan J. de Pablo (09/15)
Department of Chemical and Biological Engineering
University of Wisconsin-Madison

Francis J. DiSalvo Jr. (09/14)
Department of Chemistry
Cornell University

Bruce Elmegreen (09/14)
IBM Watson Research Center

Barbara J. Finlayson-Pitts (09/14)
Department of Chemistry
University of California, Irvine

Irene Fonseca (09/14)
Department of Mathematical Sciences
Carnegie Mellon University

Naomi J. Halas (09/13)
ECE Department
Rice University

Elizabeth Lada (09/14)
Department of Astronomy
University of Florida

Dennis L. Matthews (09/13)
College of Engineering and School of Medicine
University of California Davis

Juan C. Meza (09/15)
University of California Merced

Michael L. Norman (09/13)
Department of Physics
University of California San Diego

Eugenia Paulus (09/13)
Department of Chemistry
North Hennepin Community College

Elsa Reichmanis (09/14)
School of Chemical and Biomolecular Engineering
Georgia Institute of Technology

Esther S. Takeuchi (09/13)
State University of New York, Buffalo

Geoffrey West (09/14)
Santa Fe Institute

Book List

The Book List highlights recent books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. Suggestions for books to include on the list may be sent to notices-booklist@ams.org.

*Added to “Book List” since the list’s last appearance.


- Flatland, by Edwin A. Abbott, with notes and commentary by William F. Lindgren and Thomas F. Banchoff.
Reference and Book List


Number-Crunching: Taming Unruly Computational Problems from Mathematical Physics to Science Fiction, by Paul J. Nahin. Princeton


2012 Election Results

In the elections of 2012 the Society elected a vice president, a trustee, five members at large of the Council, three members of the Nominating Committee, and two members of the Editorial Boards Committee.

**Vice President**
Elected as the new vice president is Christoph Thiele from the University of California, Los Angeles. Term is three years (1 February 2013—31 January 2016).

**Trustee**
Elected as trustee is Karen Vogtmann from Cornell University. Term is five years (1 February 2013—31 January 2018).

**Members at Large of the Council**
Elected as new members at large of the Council are:
- Jesús A. De Loera from the University of California, Davis
- Allan Greenleaf from the University of Rochester
- Nataša Pavlović from the University of Texas at Austin
- Amber L. Puha from California State University, San Marcos
- Kenneth A. Ribet from the University of California, Berkeley
Terms are three years (1 February 2013—31 January 2016).

**Nominating Committee**
Elected as new members of the Nominating Committee are:
- Craig Huneke from the University of Virginia
- Ken Ono from Emory University
- Amie Wilkinson from the University of Chicago
Terms are three years (1 January 2013—31 December 2015).

**Editorial Boards Committee**
Elected as new members of the Editorial Boards Committee are:
- Walter Craig from McMaster University
- Walter D. Neumann from Columbia University
Terms are three years (1 February 2013—31 January 2016).
Vice President or Member at Large

One position of vice president and member of the Council ex officio for a term of three years is to be filled in the election of 2013. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations. The Council of 23 January 1979 stated the intent of the Council of nominating all persons on whose behalf there were valid petitions.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and operational considerations, which are described below.

Editorial Boards Committee

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The President will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and operational considerations, described below, should be followed.

Nominating Committee

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The President will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and operational considerations, described below, should be followed.

Rules and Procedures

Use separate copies of the form for each candidate for vice president, member at large, or member of the Nominating and Editorial Boards Committees.

1. To be considered, petitions must be addressed to Robert J. Daverman, Secretary, American Mathematical Society, Department of Mathematics, 1403 Circle Drive, University of Tennessee, 1534 Cumberland Avenue, Knoxville, TN 37996-1320, USA, and must arrive by 22 February 2013.

2. The name of the candidate must be given as it appears in the Combined Membership List (www.ams.org/cml). If the name does not appear in the list, as in the case of a new member or by error, it must be as it appears in the mailing lists, for example on the mailing label of the Notices. If the name does not identify the candidate uniquely, append the member code, which may be obtained from the candidate’s mailing label or by the candidate contacting the AMS headquarters in Providence (amsmem@ams.org).

3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.

4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.

5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.

6. The signature may be in the style chosen by the signer. However, the printed name and address will be checked against the Combined Membership List and the mailing lists. No attempt will be made to match variants of names with the form of name in the CML. A name neither in the CML nor on the mailing lists is not that of a member. (Example: The name Carla D. Savage is that of a member. The name C. Savage appears not to be.)

7. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.
Nomination Petition
for 2013 Election

The undersigned members of the American Mathematical Society propose the name of

as a candidate for the position of (check one):

☐ Vice President
☐ Member at Large of the Council
☐ Member of the Nominating Committee
☐ Member of the Editorial Boards Committee

of the American Mathematical Society for a term beginning 1 February, 2014

Return petitions by 22 February 2013 to:
Carla D. Savage, AMS Secretary, Computer Science Dept., Box 8206, North Carolina State University, Raleigh, NC 27695-8206 USA

Name and address (printed or typed)

Signature

Signature

Signature

Signature

Signature

Signature
Call for Nominations

Leroy P. Steele Prizes

The selection committee for these prizes requests nominations for consideration for the 2014 awards. Further information about the prizes can be found in the January 2012 Notices, pp. 79–100 (also available at http://ams.org/profession/prizes-awards/ams-prizes/steele-prize).

Three Leroy P. Steele Prizes are awarded each year in the following categories: (1) the Steele Prize for Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through Ph.D. students; (2) the Steele Prize for Mathematical Exposition: for a book or substantial survey or expository-research paper; and (3) the Steele Prize for Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. In 2014 the prize for Seminal Contribution to Research will be awarded for a paper in analysis.

Nomination with supporting information should be submitted to ams.org/profession/prizes-awards/nominations. Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included. Those who prefer to submit by regular mail may send nominations to the secretary, Carla Savage, Box 8206, Computer Science Department, North Carolina State University, Raleigh, NC 27695-8206. Those nominations will be forwarded by the secretary to the prize selection committee.

Deadline for nominations is March 31, 2013.
Mathematics Calendar

Please submit conference information for the Mathematics Calendar through the Mathematics Calendar submission form at [http://www.ams.org/cgi-bin/mathcal-submit.pl](http://www.ams.org/cgi-bin/mathcal-submit.pl). The most comprehensive and up-to-date Mathematics Calendar information is available on the AMS website at [http://www.ams.org/mathcal/](http://www.ams.org/mathcal/).

**January 2013**

* 23–25 **School on Quantum Ergodicity and Harmonic Analysis (Part Two)**, Göttingen University, Germany.
  **Description:** The school intends to give an introduction to quantum ergodicity. Its second part will deal with applications of quantum ergodicity to harmonic analysis, and consist of four lecture series by Martin Olbrich, Anke Pohl, Pablo Ramacher, and Lior Silberman.
  **Financial support:** Can be provided.
  **Registration:** For registration and further information please visit the website of the school. The first part of the school took place in Marburg in September 2012.
  **Information:** [http://www.mathematik.uni-marburg.de/~ramacher/QE2.html](http://www.mathematik.uni-marburg.de/~ramacher/QE2.html).

  **Description:** This workshop aims to bring together European researchers interested in the applications of ultrafilters in combinatorics of numbers, and related topics. The talks will present recent results in this area and provide an updated overview of the subject.
  **Information:** [http://crm.sns.it/event/268/](http://crm.sns.it/event/268/).

* 25–26 **Numerical Methods for PDEs: On the Occasion of Raytcho Lazarov’s 70th Birthday**, Texas A&M University, College Station, Texas.
  **Description:** The purpose of this workshop is to gather friends, colleagues, and collaborators of Raytcho Lazarov on the occasion of his 70th birthday. This workshop is made possible by the support of Texas A&M University’s Department of Mathematics, College of Science, Institute for Applied Mathematics and Computational Science, and Institute for Scientific Computation.
  **Attendance:** At the workshop is free-of-charge but there is a fee for attending the Friday celebration dinner. We ask all of the participants to register online. The deadline to register for the dinner is January 11, 2013. However, there is a limited number of dinner reservations available, so early registration is strongly encouraged.
  **Information:** Will be provided at [http://isc.tamu.edu/events/Lazarov70/](http://isc.tamu.edu/events/Lazarov70/) and will be updated frequently. Please join us as we celebrate our honored colleague.

**February 2013**

* 4–8 **Advances in Teichmüller theory**, Erwin Schrödinger Institute, Vienna, Austria.
  **Description:** The conference is part of a 3-month program (January 27–April 21, 2013) on Teichmüller theory, at the Erwin Schrödinger Institute of Vienna. The program will include all aspects of Teichmüller theory.

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.

An announcement will be published in the Notices if it contains a call for papers and specifies theplace, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.

In general, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the Notices in care of the American Mathematical Society in Providence or electronically to notices@ams.org or mathcal@ams.org.

In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the Notices prior to the meeting in question. To achieve this, listings should be received in Providence eight months prior to the scheduled date of the meeting. The complete listing of the Mathematics Calendar will be published only in the September issue of the Notices. The March, June, July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.

The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: [http://www.ams.org/](http://www.ams.org/).
Organizers: L. Funar, email: Louis.Funar@ujf-grenoble.fr; A. Papadopoulos, email: papadolp@math.unistra.fr and R. C. Penner, email: rpenner@caltech.edu. For participation please contact the organizers.


March 2013

* 25–29 Teichmüller theory: A quantization and relations with physics, Erwin Schrödinger Institute, Vienna, Austria.

**Description:** The conference is part of a 3-month program (January 27–April 21, 2013) on Teichmüller theory, at the Erwin Schrödinger Institute of Vienna. The program will include all aspects of Teichmüller theory. This conference will stress the quantization aspects and the relation with physics. A short course in the form of a series of lectures by Rinat Kashaev will be given during the conference.

**Organizers:** L. Funar, email: Louis.Funar@ujf-grenoble.fr; A. Papadopoulos, email: papadolp@math.unistra.fr and R. C. Penner, email: rpenner@caltech.edu. For participation please contact the organizers

**Information:** [http://modgroup.math.cnrs.fr/teichmuller.html](http://modgroup.math.cnrs.fr/teichmuller.html).


**Main Speakers:** Alice Guionnet (MIT), Michel Ledoux (Toulouse), Elchanan Mossel (Berkeley), Assaf Naor (NYU).

**Junior Speakers:** Marcel Nutz (Columbia), Percy Wong (Princeton).

**Registration:** Is free and will be open until March 13, 2013. The registration form and information pertaining to the conference can be found on the Probability Day website [http://orfe.princeton.edu/conferences/cpl3/](http://orfe.princeton.edu/conferences/cpl3/).

April 2013

* 8–13 International School and Research Workshop on Complex Systems, Centre de Recerca Matemàtica, Bellaterra, Barcelona.

**Organizers:** Álvaro Corral and Tomás Alarcón (CRM).


* 12–14 Graduate Research Conference in Algebra and Representation Theory, Kansas State University, Manhattan, Kansas.

**Description:** Graduate students working in algebra and representation theory are invited to participate in the 3rd Graduate Research Conference, hosted by Department of Mathematics at Kansas State University. The conference will feature invited lectures given by experts in algebra and representation theory and 20-min / 40-min talks by graduate students.


**Description:** The Great Lakes Geometry Conference is a well established conference series held annually in the Great Lakes region, rotating among different universities.

**Aim:** The aim of this conference series is to bring together distinguished speakers in a variety of areas in geometry, topology and mathematical physics.

**Talks:** The 2013 edition of the Great Lakes Geometry Conference will have geometric analysis as its main theme and will feature eight talks by experts in the field.

**Funding:** Partial travel support is available for participants (funds are pending from the NSF). Priority for funding will be given to graduate students and recent Ph.D.s. Women mathematicians and members of other underrepresented groups are especially encouraged to apply. Priority for funding will be given to those who register and request support before February 6, 2013.

**Information:** [http://www.math.northwestern.edu/greatlakes2013](http://www.math.northwestern.edu/greatlakes2013).


**Description:** The main aim of the conference is to promote, encourage, and bring together researchers in the fields of Optimization and Variational Analysis, Mathematical Systems Theory, Ordinary and Partial Differential Equations, Geometric Nonlinear Control and Applications, Fractional Calculus and Applications, and Calculus on Time Scales and Applications. The conference will consist of invited plenary talks and contributed paper presentations. It will be a mathematically enriching and socially exciting event.

**Information:** [http://sites.google.com/site/cvim2013](http://sites.google.com/site/cvim2013).

* 29–May 10 J-holomorphic Curves in Symplectic Geometry, Topology and Dynamics, Centre de Recherches Mathématiques, Montréal, Canada.

**Description:** With a focus on J-holomorphic curves and their applications to mirror symmetry and symplectic dynamics, this workshop will bring together most of the current top researchers in symplectic topology. It will consist of 34 research talks and two mini courses. Additionally, Helmut Hofer (IAS) will give an Andre-Aisenstadt series of lectures on a topic correlated with the workshop.

**Information:** [http://www.crm.umontreal.ca/2013/Curves13/index_e.php](http://www.crm.umontreal.ca/2013/Curves13/index_e.php).

May 2013

* 9–11 The International Conference on Technological Advances in Electrical, Electronics and Computer Engineering (TAECE2013), Mevlana University, Konya, Turkey.

**Description:** The conference aims to enable researchers build connections between different digital applications. All the accepted papers will be submitted to IEEE for potential inclusion to IEEE Xplore and EI Compendex.

**Information:** [http://sdiwc.net/conferences/2013/taece2013/](http://sdiwc.net/conferences/2013/taece2013/).

* 22–24 Algebra and Topology: A conference celebrating Lionel Schwartz’s 60th birthday, University of Nantes, France.

**Talk:** (First talk starting at 2pm on Wednesday May 22, 2013).

**Invited speakers:** Andrew Baker (Glasgow), Natalia Castellana (U. A. Barcelona), Nguyen Dang Ho Hai (Hue), Nick Kuhn (Virginia), Jean Lannes (Paris 7), Antoine Touzé (Paris 13), Sarah Whitehouse (Sheffield). Participants wishing to contribute a talk should send a title and an abstract.

**Deadline:** Deadline for talk submission: March 31, 2013.

**Scientific committee:** Haynes Miller (MIT), Bob Oliver (Paris 13), Jean-Claude Thomas (Angers).

**Organizing committee:** Aurélien Djament (Nantes), Vincent Franjou (Nantes), Geoffrey Powell (Angers), Christine Vespa (Strasbourg).

**Contact:** LS60@univ-nantes.fr.


* 30 “The first International Western Balkan Conference of Mathematical Sciences”, Elbasan University, Elbasan, Albania.

**Description:** The conference is an initiative to promote the results of mathematicians and other workers in this field. The conference has a wide range of mathematical directions to allow the young researcher to know the last developments of sciences in other countries and to be in contact with colleagues of the region. The aim is to create a tradition for the periodic meetings of this kind.

**Information:** [http://www.iwbcms.org](http://www.iwbcms.org).
Mathematics Calendar

June 2013

*3–14] Moduli Spaces and their Invariants in Mathematical Physics,
Centre de Recherches Mathématiques, Montréal, Canada.
Description: Apart from their intrinsic geometric interest, moduli spaces are a natural focus for the interplay with a wide range of ideas from physics, topology, and number theory. This workshop will aim to highlight links between the great variety of topics, loosely organized around four themes. During the first week, we will work primarily on symplectic and Poisson geometry, and derived categories and their moduli. The second week will concentrate on algebraic structures, and on matrix models and integrable systems.

Topics: The general topics and the special sessions proposed for the conference include but are not limited to: chaos and nonlinear dynamics, stochastic chaos, chemical chaos, data analysis and chaos, hydrodynamics, turbulence and plasmas, optics and chaos, chaotic oscillations and circuits, chaos in climate dynamics, geophysical flows, biology and chaos, neurophysiology and chaos, hamiltonian systems, chaos in astronomy and astrophysics, chaos and solitons, micro- and nano- electro-mechanical systems, neural networks and chaos, ecology and economy.
The publications of the conference include: The Book of Abstracts in electronic and in paper form, electronic proceedings in CD and in the web in a permanent website, publication in the journal of “Chaotic Modeling and Simulation”. Please see and download the papers of 2011 and 2012 issues at: http://www.cmsim.org/journal_issues.html.
Information: For more information and abstract/paper submission and special session proposals http://www.cmsim.org or send email to the conference secretariat at: secretariat@cmsim.org.

Description: The goal of the conference is to create a multidisciplinary round table for an open discussion on numerical modeling by using traditional and emerging computational paradigms. New technological challenges and fundamental ideas from theoretical computer science, linguistics, logic, set theory, and philosophy will meet requirements and new fresh applications from physics, chemistry, biology, and economy. Papers discussing new computational paradigms, relations with foundations of mathematics, and their impact on natural sciences are particularly solicited. A special attention will be also dedicated to numerical optimization and different issues related to theory and practice of the usage of infinitesimals and infinitesimals in numerical computations. Together with regular presentations at the conference there will be the Summer School offering tutorials and discussion sessions covering the topics of the conference. A special issue of a prestigious international journal will be published.

Description: The EASIAM conference gathers researchers in applied mathematics with respect to science, engineering and technology. The scope of the EASIAM conference covers all aspects of applied mathematics to science, industry, engineering and technology in the East and South East Asia. EASIAM is a section of the Society for Industrial and Applied Mathematics (SIAM). It promotes basic research and education in mathematics, which supports science and technology. Meanwhile, The Conference on Industrial and Applied Mathematics (CIAM) is a regular event held by the Industrial and Financial Mathematics Research Group, the Faculty of Mathematics and Natural Sciences, ITB.
Information: http://www.math.itb.ac.id/~easiam2013/.

*24–28] EACA’S Second International School on Computer Algebra and Applications, Faculty of Sciences, University of Valladolid, Spain.
Description: The school is an activity of the spanish network Red-EACA. It will consist of three courses (lectures and tutorials). There will be grants covering local expenses for young researchers. Candidates will be selected under the supervision of the school’s Scientific Committee. Young participants will also have the opportunity to give a short contributed talk on their ongoing work. The school also welcomes experienced researchers interested in these topics.
Lecturers: Aldo Conca (Univ. Genova, Italy), Irena Swanson (Reed College, USA), Volkmar Welker (Philippss-Univ. Marburg, Germany).
Organizers: Anna Bigatti (Univ. Genova, Italy), Philippe Gimenez (Univ. Valladolid, Spain), Eduardo Sáenz-de-Cabezón (Univ. La Rioja, Spain).

Description: The conference will be scheduled in plenary and keynote lectures followed by special and contributed sessions. The accepts of the conference will be on mathematical physics, solitons and transport processes, numerical methods, scientific computing, continuum mechanics, applied analysis, applied physics, biomathematics, complemented by some specific topics in contributed special sessions and symposia.
Confirmed speakers: V. Gerdjikov (INRNE, Bulgaria), L. C. Christov (Princeton University, USA), R. Ivanov (Dublin Institute of Technology, Ireland), A. Seyranian (MGU, Russia), L. Castro (University of Aveiro, Portugal), J. Gwinner (Universitaet der Bundeswehr, Muenchen, Germany), M. Neytcheva (Uppsala University, Sweden), E. Kansa (University of California, Davis, USA), B. Alexeev (Lomonosov University of Fine Chemical Technologies, Moscow, Russia), Z. Zlatev, Aarhus University, Denmark), etc.

July 2013

*1–5] 7th International Summer School on Geometry, Mechanics and Control (ICMAT School), La Cristalera, Miraflores de la Sierra, Madrid, Spain.
Description: The school is oriented to young researchers, Ph.D. and postdoctoral students in Mathematics, Physics and Engineering, in particular those interested in focusing their research on geometric control and its applications to mechanical and electrical systems, and optimal control. It is intended to present an up-to-date view of some fundamental issues in these topics and bring to the participants’
attention some open problems, in particular problems related to applications.

Courses: This year the courses will be delivered by: M. J. Gotay, Pacific Institute for the Mathematical Sciences, Canada. Momentum Maps, Constraints and Classical Field Theories; P. S. Krishnaprasad, University of Maryland, USA. Geometry of Collectives; Control, Dynamics and Reconstruction; D. Peralta Salas, Instituto de Ciencias Matematicas (ISC-CIC-UAM-UC3M-UCM), Spain. Topological structures in steady fluid flows.


* 3–6 International Conference on Anatolian Communications in Nonlinear Analysis (ANCNA 2013), Abant Izzet Baysal University, Bolu, Turkey.

Description: The aim of this meeting is bring together leading experts and researchers in nonlinear analysis and to assess new developments, ideas, and methods in this important and dynamic field. Additional emphasis will be put on applications in related areas, as well as other sciences, such as the natural sciences, economics, computers, and engineering.

Information: http://ancna.net/.


Description: There will be four main courses of six hours each, some seminars, communications, and posters. Courses taught by: Jacques Fejöz (Univ. Paris-Dauphine), Enrique Pujals (IMPA, Rio de Janeiro), Sandro Salsa (Politecnico di Milano), Luis Silvestre (University of Chicago).

Supporter: By the FME, UPC, SCM, RSME, SEMA.

Scientific Committee: Massimiliano Berti (Univ. Federico II), Rafael de la Llave (Georgia Tech), Jean-Michel Roquejoffre (Univ. Paul Sabatier, Toulouse), Alfonso Sorrentino (Univ. of Cambridge), Marco Antonio Teixeira (Univ. Estatal de Campinas), Juan Luis Vázquez (Univ. Autónoma de Madrid).

Information: http://www.mai.upc.edu/reerca/ seminaris-recerca/jisd2013/jisd2013

* 21–27 Applied Topology - Bedlewo 2013, Bedlewo, near Poznan, Poland.

Aim: Of the conference is to present various aspects of applied topology from classical to the recent research results.

Main topics: Theory of TC (robot motion planning) and its various modifications (as well as the Lusternik-Schnirelman category), topology of configuration spaces (and applications, including applications to combinatorics), stochastic algebraic topology (random complexes, random manifolds, applications), Morse theory in the context of applications, discrete Morse theory, some elements of computational topology, applications of topology beyond mathematics, e.g. to magnetohydrodynamics, population biology, medicine, engineering and other sciences.

Information: http://bcc.impan.pl/13AppTop/.

August 2013

* 12–16 2nd Strathmore International Mathematics Conference (SIMC-2013), Strathmore University, Nairobi, Kenya.


Description: The conference will be organized in plenary and parallel sessions of oral presentations. The Conference paper presentation sessions will be preceded by a number of 2-day schools (August 12 and 13) on broad research topics of contemporary interest and relevance.

Objective: The objective of this conference is to provide mathematicians researchers from around the world, with particular focus on Africa, to discuss latest developments and share their research results in contemporary areas of mathematics research and applications. This conference also intends to promote and explore new collaborations and further existing partnerships among the international mathematics community, and in particular, young researchers in Eastern Africa. Research areas: The conference is broad-based and will cover all branches of mathematics and interdisciplinary research.

Information: http://www.strathmore.edu/mathsc2/.

* 25–27 The 7th Global Conference on Power Control and Optimization (PCO’2013), Prague, Czech Republic.

Description: Past conferences were held in Chiang Mai (2008), Bali (2009), Gold Coast (2010), Kuching (2010), Dubai (2011) and Las Vegas (2012).

Scope: Of the conference is contemporary and original research and educational development in the area of mechanical, electrical, communication, sustainable energy, controllers, robotics, wireless sensors, bio-medicine computing, nano-science, management, environment, business, continuous and hybrid optimization. Prospective authors from universities or other educational institutes and industry are invited to submit abstract and/or full paper by email before the deadline. All papers submitted before the deadline will be peer reviewed by independent specialists. The conference proceedings will be published in the PCO CD (ISBN: 978-983-44483-56). Selected reviewed and registered papers will be published online by AIP proceeding. Some other selected papers will be requested to submit an extended version to Elsevier, Springer, InScience, GTJO, Taylor & Francis, PEP and others for special issue journal publication. These will be reviewed by the journal editorial boards. Proposals for holding tutorials, workshops, special sessions, exhibition are invited from academia and industrial bodies, and should be addressed to the conference secretariat. The conference organizing committee is currently looking for financial sponsors from industry, academia, and professional bodies.

Organizers: Nader Barsoum (Australia), Jeffrey Webb (UK), Rabi Habash (Iraq).

Local organizers: Ivan Zelinka (Czech Republic), Roman Senkerik (Czech Republic), P. Vasant (Malaysia).

Participating University: Faculty of Electrical Engineering and Computer Science V.B-TUO, Czech Republic.

Information: http://www.pcoglobal.com/Prague.htm; email: pcoglobal@gmail.com.


Aim: The aim of this conference is to contribute to the development of mathematical sciences and its applications and to bring together the mathematics community, interdisciplinary researchers, educators, mathematicians, statisticians and engineers from all over the world.

Topics: Of interest for submission include, but are not limited to applied mathematics, analysis and its applications, algebra and its applications, topology, geometry and its applications, logic and its applications, probability, statistics and their applications, mathematical physics, mathematics education. We invite you to submit an abstract and participate in this conference. The abstracts will be published in the conference proceeding and selected peer review articles will be published in the international journals (including SCI or MathSciNet, Mathematics Review, etc.) Information: http://www.iecmsa.org.
September 2013
1-6 Kangro-100, Methods of Analysis and Algebra, International Conference dedicated to the Centennial of Professor Gunnar Kangro, University of Tartu, Tartu, Estonia.
Description: Professor Gunnar Kangro (1913-1975), member of the Estonian Academy of Sciences, was the most famous Estonian mathematician of his time. He was a world-class professional in his main research area — summability theory. His excellent courses and textbooks in algebra and analysis advocated the use of new theories developed in the first half of the twentieth century, and led the transition of Estonian mathematics to modern basis. Having also supervised 23 Cand. Sci. Theses, he is fully considered the founder of contemporary Estonian mathematical school.
Topics: Modern methods of analysis and algebra. The working program of the conference consists of plenary lectures in the mornings and parallel sessions in afternoons.
Proceedings: Peer-reviewed contributions will be published in a special issue of the open access journal “Acta et Commentationes Universitatis Tartuensis de Mathematica”; http://math.ut.ee/acta.
Social events: Include excursions introducing Estonia http://www.visitestonia.com and an accompanying persons’ program.
Information: http://kangro100.ut.ee.
8-14 Combinatorial Methods in Topology and Algebra, Il Palazzo, Cortona, Italy.
Description: CoMeTA is an opportunity of meeting and sharing ideas among emerging researchers, enhanced by the interaction with a small group of experts. We plan to organize talks, poster sessions, and thematic discussions. The proceedings of the conference will be published in the "Springer INdAM series". We particularly welcome young researchers and female researchers.
Venue: "Il Palazzo", a beautiful Renaissance palace, whose frescoed halls have hosted several important scientific events. It is located in Cortona, one of the most well-preserved medieval cities in Tuscany, on a hilltop with free-ranging views of the surrounding countryside.
Organizers: Bruno Benedetti (KTH), Emanuele Delucchi (Bremen), Luca Moci (INdAM Fellow). Funding: INdAM (Istituto Nazionale di Alta Matematica).
October 2013
28-November 9 Lévy Processes and Self-similarity 2013, Tunis, Tunisia.
Description: A CIMPA school will be coupled with a conference under the name “Lévy Processes and Self-similarity 2013”. The conference is a follow up to those organized in Clermont-Ferrand (2002), Toulouse (2005), Angers (2009) and Le Touquet-Paris-Plage (2011).
Organizers: Lévy and self-similar processes will be studied under many aspects through a series of 5 lectures: Philippe Biane, Free probability and free Lévy processes; Lo Chaumont, An introduction to self-similar processes; Sonia Fourati, Complex analysis and exit problem for Lévy processes; Jean Jacod, Lévy Processes and statistics; René Schilling, Probabilistic and analytic aspects of subordination. The conference Lévy Processes and Self-similarity will be held during the second week.
November 2013
13-16 International Conference on Fractals and Wavelets, Rajagiri School of Engineering & Technology Kakkanad Cochin, Kerala, India.
Conference focus on following areas: Fractals, self similarity, holomorphic dynamics, wavelets, image processing, signal processing.
Information: http://rajagiritech.ac.in.
19-21 Gulf International Conference on Applied Mathematics (GICAM13), Mubarak Al-Abdullah Al-Jaber Area, Kuwait.
Description: This is a conference on applied mathematics organized and hosted by the Department of Mathematics & Natural Sciences at the Gulf University for Science & Technology. The objective is to bring together applied mathematicians and other researchers using mathematics as a problem solving tool. Some of the major areas of interest are mathematical biology, fluid mechanics, mathematics of finance & economics, numerical analysis and computational science.
Information: http://conferences.gust.edu.kw.

The following new announcements will not be repeated until the criteria in the next to the last paragraph at the bottom of the first page of this section are met.

July 2014
14-18 The 30th International Colloquium on Group Theoretical Methods in Physics, Ghent University, Ghent, Belgium.
Description: The ICGTPM series is traditionally dedicated to the application of symmetry and group theoretical methods in physics, chemistry and mathematics, and to the development of mathematical tools and theories for progress in group theory and symmetries. Over the years, it has further broadened and diversified due to the successful application of group theoretical, geometric and algebraic methods in life sciences and other areas. The conference has an interdisciplinary character. It aims at bringing together experts and young researchers from different fields encouraging cross-disciplinary interactions.
August 2014
Description: This 2-day workshop will showcase the contributions of female mathematicians to the three main themes of the associated MSRI program: Shimura varieties, p-adic automorphic forms, periods and L-functions. It will bring together women who are working in these areas in all stages of their careers, featuring lectures by both established leaders and emerging researchers. In addition, there will be a poster session open to all participants and an informal panel discussion on career issues.
Information: http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9806.
Math in the Media

www.ams.org/mathmedia
coverage of today’s applications of mathematics and other math news

“Is That Painting Real? Ask a Mathematician”
Christian Science Monitor

“Journeys to the Distant Fields of Prime”
The New York Times

“He’s Too Good at Math”
Slate Magazine

“Sensor Sensibility”
Science News

“Into the Fold”
Smithsonian

“Fast Routing in Road Networks with Transit Nodes”
Science

“Puzzle Me This”
Chronicle of Higher Education

“Added Dimensions to Grain Growth”
Nature

“Students Learn to Rhythmic Beat of Rap”
NCTimes.com

“The Science of Steadying a Wobbly Table”
National Public Radio

“Fish Virus Spreads in Great Lakes”
National Public Radio

“The Geometry of Music”
Time

“The Monty Hall Problem”
abcnews.com

“The Prosecutor’s Fallacy”
The New York Times

“By the NUMB3RS”
The Baltimore Sun

See the current Math in the Media and explore the archive at www.ams.org/mathmedia
New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to http://www.ams.org/bookstore-email.

Algebra and Algebraic Geometry

Hopf Algebras and Tensor Categories

Nicolás Andruskiewitsch, Universidad Nacional de Córdoba, Argentina, and Juan Cuadra and Blas Torrecillas, Universidad de Almería, Spain, Editors

This volume contains the proceedings of the Conference on Hopf Algebras and Tensor Categories, held July 4–8, 2011, at the University of Almería, Spain. The articles in this volume cover a wide variety of topics related to the theory of Hopf algebras and its connections to other areas of mathematics. In particular, this volume contains a survey covering aspects of the classification of fusion categories using Morita equivalence methods, a long comprehensive introduction to Hopf algebras in the category of species, and a summary of the status to date of the classification of Hopf algebras of dimensions up to 100. Among other topics discussed in this volume are a study of normalized class sum and generalized character table for semisimple Hopf algebras, a contribution to the classification program of finite dimensional pointed Hopf algebras, relations to the conjecture of De Concini, Kac, and Procesi on representations of quantum groups at roots of unity, a categorical approach to the Drinfeld double of a braided Hopf algebra via Hopf monads, an overview of Hom-Hopf algebras, and several discussions on the crossed product construction in different settings.


Contemporary Mathematics, Volume 585


A Study of Singularities on Rational Curves Via Syzygies

David Cox, Amherst College, MA, Andrew R. Kustin, University of South Carolina, Columbia, SC, Claudia Polini, University of Notre Dame, IN, and Bernd Ulrich, Purdue University, West Lafayette, IN

Contents: Introduction, terminology, and preliminary results; The general lemma; The triple lemma; The BiProj Lemma; Singularities of multiplicity equal to degree divided by two; The space of true triples of forms of degree \( d \): the base point free locus, the birational locus, and the generic Hilbert-Burch matrix; Decomposition of the space of true triples; The Jacobian matrix and the ramification locus; The conductor and the branches of a rational plane curve; Rational plane quartics: A stratification and the correspondence between the Hilbert-Burch matrices and the configuration of singularities; Bibliography.

Memoirs of the American Mathematical Society, Volume 222, Number 1045

Character Identities in the Twisted Endoscopy of Real Reductive Groups

Paul Mezo, Carleton University, Ottawa, ON, Canada

Contents: Introduction; Notation; The foundations of real twisted endoscopy; The local Langlands correspondence; Tempered essentially square-integrable representations; Spectral transfer for essentially square-integrable representations; Spectral transfer for limits of discrete series; Appendix A. Parabolic descent for geometric transfer factors; Bibliography; Index.

Memoirs of the American Mathematical Society, Volume 222, Number 1042

Applications

A Mutation-Selection Model with Recombination for General Genotypes

Steven N. Evans, David Steinsaltz, and Kenneth W. Wachter, University of California, Berkeley, CA

Contents: Introduction; Definition, existence, and uniqueness of the dynamical system; Equilibria; Mutation, selection, and recombination in discrete time; Shattering and the formulation of the convergence result; Convergence with complete Poissonization; Supporting lemmas for the main convergence result; Convergence of the discrete generation system; Appendix A. Results cited in the text; Bibliography; Index; Glossary of notation.

Memoirs of the American Mathematical Society, Volume 222, Number 1044

Recent Advances in Scientific Computing and Applications

Jichun Li and Hongtao Yang, University of Nevada, Las Vegas, NV, and Eric Machorro, National Security Technologies, LLC, Las Vegas, NV, Editors

This volume contains the proceedings of the Eighth International Conference on Scientific Computing and Applications, held April 1–4, 2012, at the University of Nevada, Las Vegas.

The papers in this volume cover topics such as finite element methods, multiscale methods, finite difference methods, spectral methods, collocation methods, adaptive methods, parallel computing, linear solvers, applications to fluid flow, nano-optics, biofilms, finance, magnetohydrodynamics flow, electromagnetic waves, the fluid-structure interaction problem, and stochastic PDEs.

This book will serve as an excellent reference for graduate students and researchers interested in scientific computing and its applications.

Differential Equations

Pseudo-Differential Operators with Discontinuous Symbols: Widom's Conjecture

A. V. Sobolev, University College London, United Kingdom

Contents: Introduction; Main result; Estimates for PDO’s with smooth symbols; Trace-class estimates for operators with non-smooth symbols; Further trace-class estimates for operators with non-smooth symbols; A Hilbert-Schmidt class estimate; Localisation; Model problem in dimension one; Partitions of unity, and a reduction to the flat boundary; Asymptotics of the trace (9.1); Proof of Theorem 2.9; Closing the asymptotics; Proof of Theorems 2.3 and 2.4; Appendix 1: A lemma by H. Widom; Appendix 2: Change of variables; Appendix 3: A trace-class formula; Appendix 4: Invariance with respect to the affine change of variables; Bibliography.

Memoirs of the American Mathematical Society, Volume 222, Number 1043


Mathematics Subject Classification: 47G30; 35S05, 47B10, 47B35,
In the interest of fostering a greater awareness and appreciation of mathematics and its connections to other disciplines and everyday life, MSRI and the AMS are publishing books in the Mathematical Circles Library series as a service to young people, their parents and teachers, and the mathematics profession.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

Contents: Problems; Answers; Hints; Solutions; Thematic index.

MSRI Mathematical Circles Library, Volume 1


Mathematics Subject Classification: 97-01, 97A20, 97D50; 00A07, 00A09, 97A80, 97U30, 97U40, AMS members US$28, List US$35, Order code MCL/12

Number Theory

Diophantine Methods, Lattices, and Arithmetic Theory of Quadratic Forms

Wai Kiu Chan, Wesleyan University, Middletown, CT, Lenny Fukshansky, Claremont McKenna College, CA, Rainer Schulze-Pillot, Universität des Saarlandes, Saarbrücken, Germany, and Jeffrey D. Vaaler, University of Texas at Austin, TX, Editors

This volume contains the proceedings of the International Workshop on Diophantine Methods, Lattices, and Arithmetic Theory of Quadratic Forms, held November 13–18, 2011, at the Banff International Research Station, Banff, Alberta, Canada.

The articles in this volume cover the arithmetic theory of quadratic forms and lattices, as well as the effective Diophantine analysis with height functions. Diophantine methods with the use of heights are usually based on geometry of numbers and ideas from lattice theory. The target of these methods often lies in the realm of quadratic forms theory. There are a variety of prominent research directions that lie at the intersection of these areas, a few of them presented in this volume:

- Representation problems for quadratic forms and lattices over global fields and rings, including counting representations of bounded height.
- Small zeros (with respect to height) of individual linear, quadratic, and cubic forms, originating in the work of Cassels and Siegel, and related Diophantine problems with the use of heights.
- Hermite’s constant, geometry of numbers, explicit reduction theory of definite and indefinite quadratic forms, and various generalizations.
- Extremal lattice theory and spherical designs.


Contemporary Mathematics, Volume 587


Mathematics Subject Classification: 11Exx, 11Hxx, 11G50, 11D09, 00A08, 00A09, 97A80, 97U30, 97U40, AMS members US$77.60, List US$97, Order code CONM/587

New AMS-Distributed Publications

Algebra and Algebraic Geometry

Galois–Teichmüller Theory and Arithmetic Geometry

Hiroaki Nakamura, Okayama University, Japan, Florian Pop, University of Pennsylvania, Philadelphia, PA, Leila Schneps, University of Paris VI, France, and Akio Tamagawa, Kyoto University, Japan, Editors

Since the 1980s, Grothendieck’s “Esquisse d’un Programme” has triggered tremendous developments in number theory and arithmetic geometry, extending from the studies of anabelian geometry and related Galois representations to those of polylogarithms and multiple zeta values, motives, rational points on arithmetic varieties, and effectiveness questions in arithmetic geometry.

This volume contains twenty-four articles based on talks presented at two international meetings that focused on the above themes. The meetings were held in Kyoto in October 2010. The volume
includes both survey articles and research papers that provide useful information about this area of investigation.

This item will also be of interest to those working in number theory. Published for the Mathematical Society of Japan by Kinokuniya, Tokyo, and distributed worldwide, except in Japan, by the AMS.

Contents: A. Auel, Remarks on the Milnor conjecture over schemes; F. C. S. Brown, On the decomposition of motivic multiple zeta values; S. Carr and L. Schneps, Combinatorics of the double shuffle Lie algebra; P. Cartier, On the double zeta values; S. Corry, Harmonic Galois theory for finite graphs; P. Débes and F. C. Legrand, Twisted covers and specializations; H. Furusho, Geometric interpretation of double shuffle relation for multiple L-values; K. Hashimoto and H. Tsunogai, Noether’s problem for transitive permutation groups of degree 6; Y. Ihara, Comparison of some quotients of fundamental groups of algebraic curves over p-adic fields; N. Imai, Dimensions of moduli spaces of finite flat models; P. Lochak, Results and conjectures in profinite Teichmüller theory; I. Marin, Galois actions on complex braid groups; A. Obus, The (local) lifting problem for curves; G. Quick, Some remarks on profinite completion of spaces; C. Rasmussen, An abelian surface with constrained 3-power torsion; M. Saito, Fake liftings of Galois covers between smooth curves; A. Schmidt, Motivic aspects of anabelian geometry; J. Stix, On cuspidal sections of algebraic fundamental groups; H. Tokunaga, A note on quadratic residue curves on rational ruled surfaces; K. Wickelgren, n-nilpotent obstructions to \( \pi_1 \) sections of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and Massey products; Z. Wojtkowiak, Lie algebras of Galois representations on fundamental groups; G. Yamashita, p-adic multiple zeta values, p-adic multiple L-values, and motivic Galois groups; Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: Inertia groups and profinite Dehn twists; H. Nakamura, Some congruence properties of Eisenstein invariants associated to elliptic curves.

Advanced Studies in Pure Mathematics, Volume 63

Mathematics Subject Classification: 14D10; 11M32, 14G32, 14H30, 32G15, AMS members US$118.40, List US$148, Order code ASPM/63

Geometry and Topology

Introduction à la Théorie de Jauge

Andrei Teleman, Aix-Marseille University, France

The fundamental idea of mathematical gauge theory is to study the moduli spaces of solutions of certain systems of partial differential equations on a differentiable manifold and to obtain information about this manifold (for instance, information on its diffeomorphism type) using them. This idea brought the first spectacular results in 4-dimensional differential topology:

- The ability to show that the intersection form of a compact, oriented, differentiable 4-manifold is standard over \( \mathbb{Z} \) whenever it is (positively or negatively) defined.

The goal of these lecture notes is to give a solid introduction to mathematical gauge theory and to explain in detail some of its important applications in 4-dimensional differential topology, e.g., the Donaldson theorem concerning the intersection form of differentiable 4-manifolds and the Van de Ven conjecture concerning the differential topological classification of complex surfaces.

This book deals essentially with Seiberg-Witten theory, which is easily accessible to students, but also contains elements of Donaldson theory: the gauge group of a principal fiber-bundle, Yang-Mills equations, ASD-equations, and examples of moduli spaces of Yang-Mills equations.

These lecture notes are fully accessible to students who have attended lectures on differentiable geometry and algebraic topology and have a basic background in modern analysis (Sobolev spaces, distributions, and differential operators).

This item will also be of interest to those working in analysis.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Introduction; Théorie de Hodge sur les variétés compactes; Connexions linéaires et courbure; Fibrés principaux et connexions sur les fibrés principaux; Connexions de Yang-Mills et connexions auto-antidécalées; Structures Spin et Spinc, opérateurs de Dirac, la formule de Weitzenböck; Espaces de modules de monopoles de Seiberg-Witten. Le théorème de Donaldson sur la forme d’intersection d’une 4-variété; Les invariants de Seiberg-Witten; Monopoles sur les surfaces kählériennes; Exemples et applications; Appendices; Bibliographie.

Cours Spécialisés—Collection SMF, Number 18

Mathematics Subject Classification: 57R57, 32Q55, 32G13, Individual member US$81, List US$90, Order code COSP/18

Number Theory

Automorphic Representations and L-functions

D. Prasad, C. S. Rajan, A. Sankaranarayanan, and J. Sengupta, Tata Institute of Fundamental Research, Mumbai, India, Editors

This volume contains the proceedings of the International Colloquium on Representations and L-functions organized by the Tata Institute of Fundamental Research in January 2012, one of a series of colloquia that began in 1956. It covers a wide spectrum of mathematics, including classifications of representations, arithmeticity, the average size of the 2-Selmer group of Jacobians for certain hyperelliptic curves, certain Kuznetsov formula for symmetry types of families of L-functions, sub-convexity bounds in the level aspect, Linnik’s ergodic
method, beyond endoscopy, and harmonic analysis for the relative trace formula.

This volume contains refereed articles by leading experts in these fields and includes original results as well as expository materials in these areas.

A publication of the Tata Institute of Fundamental Research. Distributed worldwide except in India, Bangladesh, Bhutan, Maldavis, Nepal, Pakistan, and Sri Lanka.


Tata Institute of Fundamental Research


Mathematics Subject Classification: 11F11, 11F67, 11F70, 11F72, 11K70, 11M41, 11R34, 14G05; 06B15, 11F41, 11F46, 11F80, 11S40, 14G10, 37A45, AMS members US$48, List US$60, Order code TIFR/18

About the Cover

Displaced mathematicians

This month’s cover was suggested by Marjorie Senechal’s review of Transcending tradition: Jewish Mathematicians in German-Speaking Academic Culture in this issue. It displays the life-lines of the more familiar Jewish mathematicians of German background whose lives were seriously affected by the Nazi regime (not at all a complete list of those affected). The break in each line marks, roughly, at what point their lives were transformed, almost always for the worse, by either emigration or death.

Details of these lives are of course far more complicated than a simple graph can possibly illustrate. Fritz Noether (Emmy’s brother) emigrated to the Soviet Union in 1938, but was executed after the German invasion as an enemy alien; Robert Remak escaped to Holland in 1939 but was caught by the Nazis when that country was invaded and then shipped off to be murdered in Auschwitz. Many of those who eventually found a stable life went through a long and undoubtedly painful period of anxiety in several countries.

One thing that the graph does show is that most of these mathematicians decided to leave early. It also shows well, however, one of the sadder facts of that sad time. Remak seems to have been the youngest of the mathematicians who suffered the worst fate. It was older mathematicians—Ludwig Berwald (deported to squalor and death in Poland); Paul Epstein, Friedrich Hartogs, and Felix Hausdorff (all suicides), Otto Blumenthal and Georg Pick (died in Theresienstadt)—who found it impossible to take up a new life through emigration and therefore suffered from discrimination against their age as well as against their family origin.

The biographies of these people on the MacTutor website are generally rather good, and the book Mathematicians Fleeing from Nazi Germany by Reinhard Siegmund-Schultze (reviewed in the Notices, November 2010) is a thorough account of the effect of the Nazi take-over on German mathematics. In particular, three appendices in Schultze’s book list all German mathematicians persecuted by the Nazis. The book under review in this issue captures well the cultural and academic world that passed with World War II.

We thank Moritz Epple and Michael Korey for their help.

—Bill Casselman
Graphics editor
(notices-covers@ams.org)
The University of Maine seeks a faculty member to serve as Chairperson of the Department of Liberal Arts and Sciences and is responsible for the general conduct of departmental affairs and has authority for all decisions concerning such affairs. The chair leads the department on a daily basis with duties including, but not limited to, working collaboratively with faculty and students from multiple disciplines, attracting and retaining top-tier faculty, enhancing existing undergraduate and graduate degree programs, advancing the status of the department in terms of research and graduate education, supporting and promoting the service mission of the department, and budgeting and allocating departmental financial resources and determining allocation of resources among competing interests.

Candidates must have academic leadership experience, excellent interpersonal and organizational skills, as well as knowledge of effective strategies for working with diverse faculty, staff, and students. The successful candidate is also required to have outstanding written and oral communication skills, a leadership style that is both visionary and collaborative, familiarity with national trends in both undergraduate and graduate education in mathematics, and a record of innovation and excellence in mathematics pedagogy. A distinguished record of scholarly achievement in the mathematical sciences, administrative experience, and a record of excellence in teaching are also required.

This position offers a competitive salary and additional compensation for chair responsibilities.

The University of Maine is the primary graduate institution in the State of Maine. The department offers BA and MA degrees. Further information about the department is available at: www.math.umaine.edu

The University of Maine is just 60 miles from the beautiful Bar Harbor area and Acadia National Park. It is approximately the same distance from skiing and remote hiking areas. Numerous cultural activities, excellent public schools, and a reasonable cost of living make the greater Bangor area a pleasant place to live.

To apply, submit a cover letter, curriculum vitae, and three or more letters of reference at least one of which addresses your potential as a mathematics department chair. The cover letter should address your background; your research, teaching, and administrative accomplishments; and your leadership vision and goals as Chair of the Department of Mathematics and Statistics at the University of Maine.

You may submit your application to: http://www.mathjobs.org/jobs/umaine, send a single pdf to: chairsearch@math.umaine.edu, or mail to: Chair Search Department of Mathematics & Statistics 5752 Neville Hall University of Maine.

**UNIVERSITY OF MAINE**

Department of Mathematics & Statistics

The University of Maine seeks a faculty member to serve as Chairperson of the Department of Mathematics and Statistics, effective July 1, 2013. Having met the department evaluation criteria for both the respective academic rank and the granting of tenure, the successful applicant will be hired as a tenured faculty member in the department and be appointed to a five-year term as chair with reappointment as chairperson beyond the two-year probationary period contingent upon satisfactory performance. Preference will be given to candidates currently at the rank of full professor but outstanding candidates holding the rank of associate professor will also be considered.

The chair reports to the Dean of the College of Liberal Arts and Sciences and is responsible for the general conduct of departmental affairs and has authority for all decisions concerning such affairs. The chair leads the department on a daily basis with duties including, but not limited to, working collaboratively with faculty and students from multiple disciplines, attracting and retaining top-tier faculty, enhancing existing undergraduate and graduate degree programs, advancing the status of the department in terms of research and graduate education, supporting and promoting the service mission of the department, and budgeting and allocating departmental financial resources and determining allocation of resources among competing interests.

Candidates must have academic leadership experience, excellent interpersonal and organizational skills, as well as knowledge of effective strategies for working with diverse faculty, staff, and students. The successful candidate is also required to have outstanding written and oral communication skills, a leadership style that is both visionary and collaborative, familiarity with national trends in both undergraduate and graduate education in mathematics, and a record of innovation and excellence in mathematics pedagogy. A distinguished record of scholarly achievement in the mathematical sciences, administrative experience, and a record of excellence in teaching are also required.

This position offers a competitive salary and additional compensation for chair responsibilities.

The University of Maine is the primary graduate institution in the State of Maine. The department offers BA and MA degrees. Further information about the department is available at: www.math.umaine.edu

The University of Maine is just 60 miles from the beautiful Bar Harbor area and Acadia National Park. It is approximately the same distance from skiing and remote hiking areas. Numerous cultural activities, excellent public schools, and a reasonable cost of living make the greater Bangor area a pleasant place to live.

To apply, submit a cover letter, curriculum vitae, and three or more letters of reference at least one of which addresses your potential as a mathematics department chair. The cover letter should address your background; your research, teaching, and administrative accomplishments; and your leadership vision and goals as Chair of the Department of Mathematics and Statistics at the University of Maine.

You may submit your application to: http://www.mathjobs.org/jobs/umaine, send a single pdf to: chairsearch@math.umaine.edu, or mail to: Chair Search Department of Mathematics & Statistics 5752 Neville Hall University of Maine.

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

The 2013 rate is $3.50 per word with a minimum two-line headline. No discounts for multiple ads or the same ad in consecutive issues. For an additional $10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the “Positions Available” classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

**Upcoming deadlines** for classified advertising are as follows: March 2013 issue—February 28, 2013; June/July 2013 issue—April 26, 2013; August 2013 issue—May 29, 2013; September 2013 issue—July 1, 2013. **U.S. laws prohibit** discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available” advertisements from institutions outside the U.S. cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to U.S. laws. Details and specific wording may be found on page 1373 (vol. 44).

**Submissions** of wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the U.S. and Canada or 401-455-4084 worldwide for further information.

**Submission:** Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or email to classifieds@ams.org. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 02940. Advertisers will be billed upon publication.
University of Maine
Orono, ME 04469-5752
Incomplete applications cannot be considered. Appropriate background checks will be required. General correspondence about this position should be sent to chairsearch@math.umaine.edu. Review of applications will begin December 10, 2012, and continue until the position is filled. UMaine is committed to diversity in our workforce.

On January 1, 2011, UMaine became a tobacco-free campus. Information regarding UMaine’s tobacco-free policy is online at http://umaine.edu/tobaccofree/

The University of Maine is an Equal Employment Opportunity/Affirmative Action Employer.

MISSOURI

MISSOURI UNIVERSITY
Department of Mathematics and Statistics

Missouri University of Science & Technology invites applications for the position of Chair of the Department of Mathematics and Statistics, with the assumption of the position in the Fall of 2013. For details about the position and the application process please click the academic position link on the website http://hrad1.mst.edu/hr/employment/. Information about the department can be found at http://math.mst.edu/.

Missouri S&T is an AA/EEO institution. Women, minorities, and persons with disabilities are encouraged to apply. Missouri S&T participates in E-Verify. For more information on E-Verify, please contact DHS at 1-888-464-4218.

PENNSYLVANIA

THE PENNSYLVANIA STATE UNIVERSITY
Department of Mathematics

Penn State is embarking on a transformative cluster hiring initiative in cyber-science, computation, and data-enabled science and engineering, to lead through cyber-enabled innovation in interdisciplinary research. This cross-college endeavor will coordinate multiple faculty appointments to develop new functional capabilities centered on data, models, and simulations for deeper insights into the critical problems in Science and Engineering. We seek outstanding faculty who can work across disciplines and in a team to advance algorithms, software, and theory to exploit “Big Data” and “Big Simulations”, for scientific discovery and engineering design in a variety of disciplines.

The Department of Mathematics is seeking candidates who have a strong research record or have demonstrated potential with expertise in mathematical modeling, analysis, and simulations for data and computation-intensive problems arising from all areas in physical, engineering, life, and social sciences. Candidates with experiences in large-scale simulations using parallel multicore architectures and heterogeneous computer systems are especially encouraged to apply. Applicants are required to have a Ph.D. in mathematics (or closely related discipline) and have a strong teaching record. The successful candidate is expected to actively collaborate with colleagues across the campus to engage in Penn State’s Cyberscience initiative, and to pursue external funding for interdisciplinary research with a strong mathematical component.

For information about the Department of Mathematics, please visit http://www.math.psu.edu and for information about Penn State Cyberscience Cluster Hire in Computation and Data-Enabled Science and Engineering, please visit http://www.ics.psu.edu/hire.html

The appointment is expected to commence in the fall semester of 2013. Appointments are possible at all levels from Assistant and Associate Professors through Full Professors and Endowed Chairs. Online application via http://www.mathjobs.org is strongly preferred. Applications completed by February 15, 2013, will be guaranteed full consideration. Required application materials include: Online application, Statement of professional interests, Curriculum vitae, Names and addresses of four references emailed to: rse15@psu.edu or mailed to: Search Committee, Department of Mathematics, The Pennsylvania State University, 107 McAllister Building, University Park, PA 16802.

Employment will require successful completion of background check(s) in accordance with university policies. Penn State is committed to Affirmative Action, Equal Opportunity, and the diversity of its workforce.

SOUTH CAROLINA

CLEMSON UNIVERSITY
Department of Mathematical Sciences
Department Chair

Applications and nominations are invited for the position of Chair of the Department of Mathematical Sciences at Clemson University. Qualifications include a rank of Full Professor, or equivalent, and proven leadership experience. Administrative experience is highly desirable and an earned doctorate or equivalent is required. Candidates for Chair must demonstrate both a dynamic vision for supporting, directing, and enhancing the multiple missions of the department and an appreciation for the department’s broad and diverse mathematical sciences disciplines.

The Mathematical Sciences Department at Clemson has successfully integrated the areas of algebra and discrete mathematics; analysis; bioinformatics; computational mathematics; operations research; mathematical statistics and probability; and applied statistics into balanced educational programs at both the undergraduate and graduate levels. It offers B.A., B.S., M.S., and Ph.D. degree programs. The department has achieved national recognition in a number of areas from pure and applied research to program and classroom innovation. The department houses 46 tenured/tenure-track faculty members, 28 lecturers, 12 visiting faculty, 8 full-time staff members, as well as 116 graduate and 231 undergraduate students. It is the largest unit within the College of Engineering and Sciences, and it contributes to the university with a significant service course load. Further information regarding the department and its programs can be found at its website: http://www.math.clemson.edu

Applications must be electronically filed at www.mathjobs.org by uploading a cover letter, vita (including names, telephone numbers, and e-mail addresses of three references), and statements on teaching and research; in addition, three reference letters should be submitted to: http://www.mathjobs.org. Applications received by February 1, 2013, will receive full consideration, but later applications may be considered until the position is filled. The position will be available August 15, 2013. Salary will be commensurate with credentials and experience.

Clemson University is an Affirmative Action/Equal Opportunity Employer and does not discriminate against any individual or group of individuals on the basis of age, color, disability, gender, national origin, race, religion, sexual orientation, veteran status, or genetic information.
General Information Regarding Meetings & Conferences of the AMS

Speakers and Organizers: The Council has decreed that no paper, whether invited or contributed, may be listed in the program of a meeting of the Society unless an abstract of the paper has been received in Providence prior to the deadline.

Special Sessions: The number of Special Sessions at an Annual Meeting is limited. Special Sessions at annual meetings are held under the supervision of the Program Committee for National Meetings and, for sectional meetings, under the supervision of each Section Program Committee. They are administered by the associate secretary in charge of that meeting with staff assistance from the Meetings and Conferences Department in Providence. (See the list of associate secretaries on page 288 of this issue.) Each person selected to give an Invited Address is also invited to generate a Special Session, either by personally organizing one or by having it organized by others. Proposals to organize a Special Session are sometimes solicited either by a program committee or by the associate secretary. Other proposals should be submitted to the associate secretary in charge of that meeting (who is an ex officio member of the program committee) at the address listed on page 288. These proposals must be in the hands of the associate secretary at least seven months (for sectional meetings) or nine months (for national meetings) prior to the meeting at which the Special Session is to be held in order that the committee may consider all the proposals for Special Sessions simultaneously. Special Sessions must be announced in the Notices in a timely fashion so that any Society member who so wishes may submit an abstract for consideration for presentation in the Special Session. Talks in Special Sessions are usually limited to twenty minutes; however, organizers who wish to allocate more time to individual speakers may do so within certain limits. A great many of the papers presented in Special Sessions at meetings of the Society are invited papers, but any member of the Society who wishes to do so may submit an abstract for consideration for presentation in a Special Session, provided it is submitted to the AMS prior to the special early deadline for consideration. Contributors should know that there is a limit to the size of a single Special Session, so sometimes all places are filled by invitation. An author may speak by invitation in more than one Special Session at the same meeting. Papers submitted for consideration for inclusion in Special Sessions but not accepted will receive consideration for a contributed paper session, unless specific instructions to the contrary are given.

The Society reserves the right of first refusal for the publication of proceedings of any Special Session. If published by the AMS, these proceedings appear in the book series Contemporary Mathematics. For more detailed information on organizing a Special Session, see [http://www.ams.org/meetings/specialsessionmanual.html](http://www.ams.org/meetings/specialsessionmanual.html).

Contributed Papers: The Society also accepts abstracts for ten-minute contributed papers. These abstracts will be grouped by related Mathematical Reviews subject classifications into sessions to the extent possible. The title and author of each paper accepted and the time of presentation will be listed in the program of the meeting. Although an individual may present only one ten-minute contributed paper at a meeting, any combination of joint authorship may be accepted, provided no individual speaks more than once.

Other Sessions: In accordance with policy established by the AMS Committee on Meetings and Conferences, mathematicians interested in organizing a session (for either an annual or a sectional meeting) on employment opportunities inside or outside academia for young mathematicians should contact the associate secretary for the meeting with a proposal by the stated deadline. Also, potential organizers for poster sessions on a topic of choice should contact the associate secretary before the deadline.

Abstracts: Abstracts for all papers must be received by the meeting coordinator in Providence by the stated deadline. Unfortunately, late papers cannot be accommodated.

Submission Procedures: Visit the Meetings and Conferences homepage on the Web at [http://www.ams.org/meetings](http://www.ams.org/meetings) and select “Submit an abstract”.

Site Selection for Sectional Meetings
Sectional meeting sites are recommended by the associate secretary for the section and approved by the Secretariat. Recommendations are usually made eighteen to twenty-four months in advance. Host departments supply local information, ten to fifteen rooms with overhead projectors and a laptop projector for contributed paper sessions and Special Sessions, an auditorium with twin overhead projectors and a laptop projector for Invited Addresses, space for registration activities and an AMS book exhibit, and registration clerks. The Society partially reimburses for the rental of facilities and equipment and for staffing the registration desk. Most host departments volunteer; to do so, or for more information, contact the associate secretary for the section.
Oxford, Mississippi  
University of Mississippi  
March 1–3, 2013  
Friday – Sunday  
Meeting #1087  
Southeastern Section  
Associate secretary: Robert J. Daverman  
Announcement issue of Notices: December 2012  
Program first available on AMS website: December 13, 2012  
Program issue of electronic Notices: March 2013  
Issue of Abstracts: Volume 34, Issue 2  

Deadlines  
For organizers: Expired  
For consideration of contributed papers in Special Sessions: Expired  
For abstracts: Expired  

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.  

Invited Addresses  
Patricia Hersh, North Carolina State University, An interplay of combinatorics with topology.  
Daniel Krashen, University of Georgia, Topology, arithmetic, and the structure of algebraic groups.  
Washington Mio, Florida State University, Taming shapes and understanding their variation.  
Slawomir Solecki, University of Illinois at Urbana-Champaign, An abstract approach to Ramsey theory with applications.  

Special Sessions  
Algebraic Combinatorics, Patricia Hersh, North Carolina State University, and Dennis Stanton, University of Minnesota.  
Approximation Theory and Orthogonal Polynomials, David Benko, University of South Alabama, Erwin Mina-Diaz, University of Mississippi, and Edward Saff, Vanderbilt University.  
Banach Spaces and Operators on Them, Qingying Bu and Gerard Buskes, University of Mississippi, and William B. Johnson, Texas A&M University.  
Commutative Algebra, Sean Sather-Wagstaff, North Dakota State University, and Sandra M. Spiroff, University of Mississippi.  
Connections between Matroids, Graphs, and Geometry, Stan Dziobiak, Talmage James Reid, and Haidong Wu, University of Mississippi.  
Dynamical Systems, Alexander Grigo, University of Oklahoma, and Saša Kočić, University of Mississippi.  
Fractal Geometry and Ergodic Theory, Manav Das, University of Louisville, and Mrinal Kanti Roychowdhury, University of Texas-Pan American.  
Graph Theory, Laura Sheppardson and Bing Wei, University of Mississippi, and Hefui Wu, McGill University.  
Modern Methods in Analytic Number Theory, Nathan Jones and Micah B. Milinovich, University of Mississippi, and Frank Thorne, University of South Carolina.  
Set Theory and Its Applications, Christian Rosendal, University of Illinois at Chicago, and Slawomir Solecki, University of Illinois at Urbana-Champaign.
Chestnut Hill, Massachusetts

Boston College

April 6–7, 2013
Saturday – Sunday

Meeting #1088

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of Notices: January 2013

Program first available on AMS website: February 21, 2013

Program issue of electronic Notices: April 2013

Issue of Abstracts: Volume 34, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: February 12, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Roman Bezrukavnikov, Massachusetts Institute of Technology, Canonical bases and geometry.

Marston Conder, University of Auckland, Discrete objects with maximum possible symmetry.

Alice Guionnet, École Normale Supérieure de Lyon, Title to be announced.

Yanir Rubinstein, University of Maryland, Geometry: (very) local meets global.

Special Sessions

Algebraic and Geometric Structures of 3-manifolds (Code: SS 3A), Ian Biringer, Tao Li, and Robert Meyerhoff, Boston College.

Algorithmic Problems of Group Theory and Applications to Information Security (Code: SS 14A), Delaram Kahrobaei, City University of New York Graduate Center and New York College of Technology, and Vladimir Shpilrain, City College of New York and City University of New York Graduate Center.

Arithmetic Dynamics and Galois Theory (Code: SS 6A), John Cullinan, Bard College, and Farshid Hajir and Siman Wong, University of Massachusetts, Amherst.

Combinatorics and Classical Integrability (Code: SS 16A), Amanda Redlich and Shabnam Beheshti, Rutgers University.

Commuting Matrices and the Hilbert Scheme (Code: SS 11A), Anthony Iarrobino, Northeastern University, and Leila Khatami, Union College.


Counting and Equidistribution on Symmetric Spaces (Code: SS 5A), Dubi Kelmer, Boston College, and Alex Kontorovich, Yale University.

Discrete Geometry of Polytopes (Code: SS 10A), Barry Monson, University of New Brunswick, and Egon Schulte, Northeastern University.


History and Philosophy of Mathematics (Code: SS 7A), James T. Tattersall, Providence College, and V. Frederick Rickey, United States Military Academy.

Homological Invariants in Low-dimensional Topology. (Code: SS 1A), John Baldwin, Joshua Greene, and Eli Grigsby, Boston College.

Homology and Cohomology of Arithmetic Groups (Code: SS 13A), Avner Ash, Boston College, Darrin Doud, Brigham Young University, and David Pollack, Wesleyan University.

Hopf Algebras and their Applications (Code: SS 9A), Timothy Kohl, Boston University, and Robert Underwood, Auburn University Montgomery.

Moduli Spaces in Algebraic Geometry (Code: SS 4A), Dawei Chen and Maksym Fedorchuk, Boston College, and Joe Harris and Yu-Jong Tzeng, Harvard University.

Real and Complex Dynamics of Difference Equations with Applications (Code: SS 12A), Ann Brett, Johnson and Wales University, and M. R. S. Kulenovic, University of Rhode Island.

Recursion and Definability (Code: SS 15A), Rachel Epstein, Harvard University, Karen Lange, Wellesley College, and Russell Miller, Queens College and City University of New York Graduate Center.

Research by Undergraduates and Students in Post-Baccalaureate Programs (Code: SS 8A), Chi-Keung Cheung, Boston College, David Damiano, College of the Holy Cross, Steven J. Miller, Williams College, and Suzanne L. Weekes, Worcester Polytechnic Institute.

Topology and Generalized Cohomologies in Modern Condensed Matter Physics (Code: SS 17A), Claudio Chamon and Robert Kotiuga, Boston University.

Boulder, Colorado

University of Colorado Boulder

April 13–14, 2013
Saturday – Sunday

Meeting #1089

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of Notices: January 2013

Program first available on AMS website: February 28, 2013

Program issue of electronic Notices: April 2013

Issue of Abstracts: Volume 34, Issue 2
Deadlines
For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: February 19, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional1.html.

Invited Addresses
Gunnar Carlsson, Stanford University, Title to be announced.
Jesus A. De Loera, University of California, Davis, Title to be announced.
Brendan Hassett, Rice University, Title to be announced.
Raphael Rouquier, University of California Los Angeles, Title to be announced.

Special Sessions
Advances in Mathematical Biology (Code: SS 16A), Liming Wang, California State University, Los Angeles, and Jiangguo Liu, Colorado State University.
Algebraic Geometry (Code: SS 14A), Sebastian Casalaina-Martin, University of Colorado, Renzo Cavalieri, Colorado State University, Brendan Hassett, Rice University, and Jonathan Wise, University of Colorado.
Algebras, Lattices and Varieties (Code: SS 5A), Keith A. Kearnes and Ágnes Szendrei, University of Colorado, Boulder.
Analysis of Dynamics of the Incompressible Fluids (Code: SS 18A), Mimi Dai and Congming Li, University of Colorado, Boulder.
Arithmetic Statistics and Big Monodromy (Code: SS 23A), Jeff Achter, Colorado State University, and Chris Hall, University of Wyoming.
Associative Rings and Their Modules (Code: SS 1A), Greg Oman and Zak Mesyan, University of Colorado, Colorado Springs.
Cluster Algebras and Related Combinatorics (Code: SS 6A), Gregg Musiker, University of Minnesota, Kyungyong Lee, Wayne State University, and Li Li, Oakland University.
Combinatorial Avenues in Representation Theory (Code: SS 21A), Richard Green, University of Colorado Boulder, Anne Shepler, University of North Texas, and Nathaniel Thiem, University of Colorado Boulder.
Combinatorial and Computational Commutative Algebra and Algebraic Geometry (Code: SS 7A), Hirotachi Abo, University of Idaho, Zach Teitler, Boise State University, and Alexander Woo, University of Idaho.
Diophantine Approximation on Manifolds and Fractals: Dynamics, Measure Theory and Schmidt Games. (Code: SS 15A), Wolfgang Schmidt, University of Colorado at Boulder, and Lior Fishman, University of North Texas.
Dynamical Systems: Thermodynamic Formalism and Connections with Geometry (Code: SS 10A), Keith Burns, Northwestern University, and Dan Thompson, The Ohio State University.

Dynamics and Arithmetic Geometry (Code: SS 2A), Sunion Ih, University of Colorado at Boulder, and Thomas J. Tucker, University of Rochester.
Elliptic Systems and Their Applications (Code: SS 20A), Wenxiong Chen, Yeshiva University, and Congming Li, University of Colorado at Boulder.
Extremal Graph Theory (Code: SS 3A), Michael Ferrara, University of Colorado Denver, Stephen Hartke, University of Nebraska-Lincoln, and Michael Jacobson, University of Colorado Denver.
Foundations of Computational Mathematics (Code: SS 13A), Susan Margulies, Pennsylvania State University, and Jesus De Loera, University of California, Davis.
Harmonic Analysis of Frames, Wavelets, and Tilings (Code: SS 12A), Veronika Furst, Fort Lewis College, Keri Kornelson, University of Oklahoma, and Eric Weber, Iowa State University.
Noncommutative Geometry and Geometric Analysis (Code: SS 22A), Carla Farsi and Alexander Gorokhovsky, University of Colorado, Boulder.
Number Theory with a Focus on Diophantine Equations and Recurrence Sequences (Code: SS 8A), Patrick Ingram, Colorado State University, and Katherine E. Stange, University of Colorado, Boulder.
Set Theory and Boolean Algebras (Code: SS 17A), Natasha Dobrinen, University of Denver, and Don Monk, University of Colorado, Boulder.
Singular Spaces in Geometry, Topology, and Algebra (Code: SS 11A), Greg Friedman, Texas Christian University, and Laurentiu Maxim, University of Wisconsin, Madison.
Themes in Applied Mathematics: From Data Analysis through Fluid Flows and Biology to Topology (Code: SS 4A), Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico, and Enfitek, Inc.

Ames, Iowa
Iowa State University
April 27–28, 2013
Saturday – Sunday
Meeting #1090
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: February 2013
Program first available on AMS website: March 14, 2013
Program issue of electronic Notices: April 2013
Meetings & Conferences

Issue of Abstracts: Volume 34, Issue 2

Deadlines
For organizers: Expired
For consideration of contributed papers in Special Sessions: January 18, 2013
For abstracts: March 5, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Kevin Costello, Northwestern University, Title to be announced.
Marianne Csornyei, University of Chicago, Title to be announced.
Vladimir Markovic, California Institute of Technology, Title to be announced.
Eitan Tadmor, University of Maryland, Title to be announced.

Special Sessions
Algebraic and Geometric Combinatorics (Code: SS 4A), Sung Y. Song, Iowa State University, and Paul Terwilliger, University of Wisconsin-Madison.
Analysis, Dynamics and Geometry In and Around Teichmüller Spaces (Code: SS 17A), Alistair Fletcher, Northern Illinois University, Vladimir Markovic, California Institute of Technology, and Dragomir Saric, Queens College CUNY.
Commutative Algebra and its Environments (Code: SS 6A), Olgur Celikbas and Greg Piepmeyer, University of Missouri, Columbia.
Commutative Ring Theory (Code: SS 8A), Michael Axtell, University of St. Thomas, and Joe Stickles, Millikin University.
Computability and Complexity in Discrete and Continuous Worlds (Code: SS 11A), Jack Lutz and Tim McNicholl, Iowa State University.
Computational Advances on Special Functions and Tropical Geometry (Code: SS 14A), Lubjana Beshaj, Oakland University, and Emma Previato, Boston University.
Control Theory and Qualitative Analysis of Partial Differential Equations (Code: SS 16A), George Avalos, University of Nebraska-Lincoln, and Scott Hansen, Iowa State University.
Discrete Methods and Models in Mathematical Biology (Code: SS 18A), Dora Matache, University of Nebraska-Omaha, and Stephen J. Willson, Iowa State University.
Extremal Combinatorics (Code: SS 7A), Steve Butler and Ryan Martin, Iowa State University.
Generalizations of Nonnegative Matrices and Their Sign Patterns (Code: SS 3A), Minerva Catral, Xavier University, Shaun Fallat, University of Regina, and Pauline van den Driessche, University of Victoria.
Geometric Elliptic and Parabolic Partial Differential Equations (Code: SS 13A), Brett Kotschwar, Arizona State University, and Xuan Hien Nguyen, Iowa State University.
Graphs, Hypergraphs and Counting (Code: SS 26A), Eva Czabarka and Laszlo Szekely, University of South Carolina.
Kinetic and Hydrodynamic PDE-based Descriptions of Multi-scale Phenomena (Code: SS 25A), James Evans and Hailiang Liu, Iowa State University, and Eitan Tadmor, University of Maryland.
Logic and Algebraic Logic (Code: SS 9A), Jeremy Alm, Illinois College, and Andrew Ylvisaker, Iowa State University.
Multi-Dimensional Dynamical Systems (Code: SS 15A), Jayadev Athreya, University of Illinois, Urbana-Champaign, Jonathan Chaika, University of Chicago, and Joseph Rosenblatt, University of Illinois at Urbana-Champaign.
Numerical Analysis and Scientific Computing (Code: SS 20A), Hailiang Liu, Songting Luo, James Rossmanith, and Jue Yan, Iowa State University.
Numerical Methods for Geometric Partial Differential Equations (Code: SS 22A), Gerard Awanou, University of Illinois at Chicago, and Nicolae Tarfulea, Purdue University.
Operator Algebras and Topological Dynamics (Code: SS 1A), Benton L. Duncan, North Dakota State University, and Justin R. Peters, Iowa State University.
Partial Differential Equations (Code: SS 12A), Gary Lieberman and Paul Sacks, Iowa State University, and Mahamadi Warma, University of Puerto Rico at Rio Piedras.
Probabilistic and Multiscale Modeling Approaches in Cell and Systems Biology (Code: SS 19A), Jasmine Foo, University of Minnesota, and Anastasios Matzavinos, Iowa State University.
Quasigroups, Loops, and Nonassociative Division Algebras (Code: SS 21A), C. E. Ealy, Jr. and Annette Paul, Western Michigan University, Benjamin Phillips, University of Michigan Dearborn, J. D. Phillips, Northern Michigan University, and Petr Vojtechovsky, University of Denver.
Ring Theory and Noncommutative Algebra (Code: SS 24A), Victor Camillo, University of Iowa, and Miodrag C. Iovanov, University of Bucharest and University of Iowa.
Stochastic Processes with Applications to Physics and Control (Code: SS 10A), Jim Evans and Arka Ghosh, Iowa State University, Jon Peterson, Purdue University, and Alexander Roitershtein, Iowa State University.
Topology of 3-Manifolds (Code: SS 23A), Marion Campisi and Alexander Zupan, University of Texas at Austin.
Zero Forcing, Maximum Nullity/Minimum Rank, and Colin de Verdiere Graph Parameters (Code: SS 2A), Leslie Hogben, Iowa State University and American Institute of Mathematics, and Bryan Shader, University of Wyoming.

Accommodations
Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include hotel tax. Participants must state that they are with the American Mathematical Society (AMS) Meeting at Iowa State University to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. Hotels have varying

278 Notices of the AMS Volume 60, Number 2
cancellation and early checkout penalties; be sure to ask for details.

Gateway Hotel & Conference Center, 2100 Green Hills Drive, Ames, IA 50014, 515-292-8600; www.gatewayames.com/. Rates are US$109 per night for single/double occupancy. Please note that hotel tax in Ames is 12%. Amenities include free wireless Internet in rooms; heated indoor swimming pool; sauna and whirlpool; fitness facility featuring treadmills, weight machines, and ellipticals; Business Center with work station; and The IowaStater Restaurant on site with room service available. Free parking is available. The hotel is located within walking distance of Iowa State University in Ames, Iowa, and is on the local campus shuttle route. Cancellation and early check-out policies vary; be sure to check when you make your reservation. Checkin time is 4:00 p.m.; checkout time: 12:00 p.m. The deadline for reservations at this rate is March 27, 2013.

Best Western Plus University Park Inn & Suites, 2500 University Blvd., Ames, IA 50010, 515-296-2500; bestwesterniowa.com/hotels/best-western-plus-university-park-inn-and-suites. Rates are US$109 per night for doubles and US$104 per night for singles. Please note that hotel tax in Ames is 12%. Amenities include guestrooms equipped with a refrigerator, cable satellite television, and free high-speed Internet access. Guests will wake each morning to a complimentary full hot breakfast. This pet-friendly Iowa hotel also features an indoor heated swimming pool and hot tub, fitness center, 24-hour business center, and meeting facilities. This property is located within a mile from the campus and is on the local campus shuttle route. Cancellation and early check-out policies vary; be sure to check when you make your reservation. Checkin time is 3:00 p.m. and checkout time is 12:00 p.m. The deadline for reservations at this rate is March 27, 2013.

Holiday Inn Ames Conference Center, 2609 University Blvd., Ames, IA 50010, 515-268-8808; www.holidayinn.com/hotels/us/en/ames/amwia/hoteldetail. Rates are US$129 per night for single/double occupancy. Please note that hotel tax in Ames is 12%. Amenities include free high-speed Internet access, indoor heated pool and children’s wading pool open 8:00 a.m.-10:00 p.m., business center, and health and fitness center. All rooms have a work desk with an ergonomic chair or work area with two side chairs, microwave, and refrigerator. This property is located approximately 2 mile from the campus by car and is on the local campus shuttle route. Cancellation and early check-out policies vary; be sure to check when you make your reservation. The deadline for reservations at this rate is March 27, 2013.

AmeriInn & Suites, 2507 SE16th Street, Ames, IA 50010, 515-233-1005; www.americinn.com/hotels/IA/Ames. Rates are US$72 per night for single and double occupancy. Please note that hotel tax in Ames is 12%. Amenities include indoor pool; free hot, home-style AmeriInn Perk breakfast; and free hotel-wide high-speed Internet. This property is located approximately 5 miles from the campus. Cancellation and early check-out policies vary; be sure to check when you make your reservation. Checkin time is 3:00 p.m. and checkout time is 11:00 a.m. The deadline for reservations at this rate is March 27, 2013.

Country Inn & Suites, 2605 SE 16th Street, Ames, IA 50010, 515-233-3935; www.countryinns.com/ames-hotel-ia-50010/iaames. Rates are US$99 per night for single and double occupancy. Amenities include indoor pool, on-site parking, high-speed wireless Internet access, a Read It & Return® lending library, Business Center, and a complimentary daily breakfast. This is a 100% smoke free property. This property is located approximately 4 miles from the campus. Cancellation and early check-out policies vary; be sure to check when you make your reservation. Checkin time is 2:00 p.m.; Checkout time is 12:00 p.m. The deadline for reservations at this rate is March 27, 2013.

Microtel Inn & Suites, 2216 SE16the Street, Ames, IA 50010, 515-233-4444; www.microtelames.com/. Rates are US$65 per night for doubles and US$55 for singles. Amenities include free continental breakfast, free Wi-Fi Internet access, fitness center, and free parking. 24-hour front desk service and guest laundry facilities. Some hotel rooms feature 32-inch flat-screen TVs, microwaves, refrigerators, and sofa beds. This property is located approximately 5 miles from the campus. Cancellation and early check-out policies vary; be sure to check when you make your reservation. Checkin time is 2:00 p.m.; checkout time is 12:00 p.m. The deadline for reservations at this rate is March 27, 2013.

Food Services

On Campus: Union Drive Marketplace is located on the west side of campus in the Union Drive neighborhood and is open Saturday and Sunday 10:00 a.m. to 2:00 p.m. for lunch and 4:00 p.m. to 7:00 p.m. for dinner. This all-you-can-eat dining center is US$10.50 per person (US$7.25 for under 10) and offers the following dining options: Country Cuisine, Wok Your Way, Oregano’s, Backyard Grill, Farmer’s Market, Fresh Choice Deli, and Sweet Temptations. The Memorial Union Food Court is open Saturday from 11:00 a.m.-2:00 p.m. Options include: Cy’s & Fries, Burrito Works, Panda Express (also open Sunday 4:00 p.m. to 8:00 p.m.), Subway, Sunset Strips, and World Bistro.

Off Campus: There are many choices for dining convenient to campus and hotels. The Cafe, 2616 Northridge Pkwy, 515-292-0100; offering good food, good drink, and good company, and utilizing some of the most rewarding ingredients life has to offer by buying directly from the people who grow them. Open Friday and Saturday 7:00 a.m.-10:00 p.m. and Sunday 7:00 a.m.-9:00 p.m.

Fuji Japanese Steakhouse, 1614 S Kellogg Ave. #101, 515-232-8383; offering the personalized experience of having your choice of succulent seafood, tender chicken, juicy steaks, and fresh vegetables grilled to delicious perfection in an extraordinary theater that will feed the senses and entertain the appetite. Open Friday and Saturday for lunch from 11:00 a.m.-2:00 p.m. and dinner from 2:00 p.m.-10 p.m., and Sunday from 11:00 a.m.-9:00 p.m.

Lucullan’s Italian Grill, 400 Main Street, 515-232-8484; for 33 years we have been feeding the friends and families of
Events and ISU. We have a rich history of incredible authen-
tic Italian food, rich sauces, aged steaks, and baked breads. 
Friday and Saturday for dinner from 4:30 p.m.-10:00 p.m. 
and Sunday from 5:00 p.m.-9:00 p.m.
Olde Main Brewing Company, 316 Main Street, 515-
232-0553; is located in downtown Ames’ quaint Main 
Street district and offers handcrafted foods, brews, and
atmosphere which will make you feel right at home. The
restaurant is open daily for dinner from 5:00 p.m.-10:00 
p.m., Saturday lunch 11:00 a.m.-3:00 p.m., and Sunday 
brunch 10:30 a.m.-2:00 p.m. The Pub is open daily from 
11:00 a.m.-2:00 a.m.
Hickory Park, 1404 South Duff Avenue, 515-232-8940;
established in 1970, and since its inception has become an
Ames tradition! Hickory Park’s reputation has flourished 
from an expansive menu filled with a variety of hickory 
smoked meats served in generous portions. There is cer-
tainly something for everyone with over 100 menu sele-
tions. Open Friday and Saturday 10:30 a.m.-10:00 p.m. and 
Sunday 10:30 a.m.-9:00 p.m.
Please visit www.visitames.com/shop_and_dine/
dining.aspx, for more casual and fine dining options 
in Ames.

Registration and Meeting Information
Registration and the AMS book exhibit will be located 
in the Room 305 of Carver Hall. Special Sessions and all In-
vited Addresses will be held in Carver Hall and the Erdős 
Memorial Lecture will be held in Room 2055 of Hoover 
Hall. Please refer to the campus map www.math.iastate.
edu/Events/2013AMS/getting.html for specific locations. 
The registration desk will be open on Saturday, 
April 27, 7:30 a.m.-4:00 p.m. and Sunday, April 28, 8:00 
am.-12:00 p.m. Fees are US$53 for AMS members, US$74 
for nonmembers; and US$5 for students, unemployed 
mathematicians, and emeritus members. Fees are payable 
on site via cash, check, or credit card; advance registration 
is not available.

Other Activities
Book Sales: Stop by the on-site AMS bookstore and review 
the newest titles from the AMS, enjoy up to 25% off all AMS 
publications, or take home an AMS t-shirt! Complimentary 
coffee will be served courtesy of AMS Membership 
Services.
AMS Editorial Activity: An acquisitions editor from the 
AMS book program will be present to speak with prospective 
authors. If you have a book project that you would like 
to discuss with the AMS, please stop by the book exhibit.
Friday Poster Session: The Department of Mathematics 
will be hosting a Graduate Student Poster Session Friday, 
April 26, 2013 6:30 p.m.-9:00 p.m. in the Sun Room and 
South Ballroom of the Memorial Union. Refreshments will 
be served. Easels, backing boards, and clips provided.
Submission deadline is March 31, 2013. Address inquiries 
to Luis Garcia at lagarcia@iastate.edu. For more in-
formation please visit http://www.math.iastate.edu/
Events/2013AMS/posterinfo.html.

Local Information and Maps
This meeting will take place on the Ames Campus of Iowa 
State University. A campus map for all of Iowa State 
University can be found at www.math.iastate.edu/
Events/2013AMS/getting.html Information about the 
Iowa State University Department of Mathematics may 
be found at http://www.math.iastate.edu/. Please 
watch the website available at www.ams.org/meetings/
sectional/sectional.html for additional information 
on this meeting. Please visit the Iowa State University 
website at www.iastate.edu/ for additional information 
on the campus.

Parking
Visitor parking is available in the garages on campus. The 
following link directs you to a map which shows where 
the Parking Ramp (hourly fees apply) and Parking Lot 50B 
(free parking) are in relation to Carver Hall: http://www.
math.iastate.edu/Events/2013AMS/mapMeeting.pdf. For 
information on hourly and daily rates for the Parking 
Ramp please visit: http://www.mu.iastate.edu/index.
cfm?nodeID=21695&audienceID=1.

Travel
Iowa State University is located in Ames, Iowa. The Des 
Moines International Airport is 40 miles south of Ames.
Shuttle: The University has organized a shuttle from 
the airport to Ames for US$55 roundtrip/person. Please visit 
the link to reserve your shuttle: https://www.visitames.
By Car: From I-35: Take exit 111B, merging onto U.S.- 
30W toward Ames/Iowa State. On U.S.-30W, take exit 146, 
merging onto University Boulevard. Continue on University 
Boulevard 1.4 miles to Lincoln Way. Turn left (west) onto 
Lincoln Way and continue .6 miles to the Union parking 
ramp entrance on your right.

Car Rental
Hertz is the official car rental company for the meeting.
To make a reservation accessing our special meeting rates 
online go to www.hertz.com, click on the box “I have a 
discount”, and type in our convention number (CV): 04N30003. You can also call Hertz directly at 800-654-
2240 (U.S. and Canada) or 405-749-4434 (other countries).
At the time this announcement was prepared, rates were 
US$24.00 to US$74.00 per day on the weekend. At the time 
of your reservation, the meeting rates will be automatically 
compared to other Hertz rates and you will be quoted the 
best comparable rate available.

Local Transportation
CyRide: The CyRide bus system serves the Iowa State 
campus and city of Ames. ISU students ride free. Regular 
fare for others is US$1, with reduced fares for seniors, 
K-12 students, and others.

Weather
The average high temperature for April is approxi-
mately 60 degrees Fahrenheit and the average low is ap-
proximately 40 degrees Fahrenheit. Rain can be common in for this time of year. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

**Information for International Participants**

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at sites.nationalacademies.org/pga/biso/visas/ and travel.state.gov/visa/visa_1750.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to mk1@ams.org.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

- Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  - family ties in home country or country of legal permanent residence
  - property ownership
  - bank accounts
  - employment contract or statement from employer stating that the position will continue when the employee returns;

- Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;

- Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

- Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

- If travel plans will depend on early approval of the visa application, specify this at the time of the application;

- Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

**Alba Iulia, Romania**

**June 27–30, 2013**

*Thursday – Sunday*

**Meeting #1091**

First Joint International Meeting of the AMS and the Romanian Mathematical Society, in partnership with the

“Simion Stoilow” Institute of Mathematics of the Romanian Academy.

Associate secretary: Steven H. Weintraub

Announcement issue of Notices: January 2013

Program first available on AMS website: Not applicable

Program issue of electronic Notices: Not applicable

Issue of Abstracts: Not applicable

**Deadlines**

For organizers: To be announced

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

**Invited Addresses**

Viorel Barbu, Universitatea Cuza, *Title to be announced.*

Sergiu Klainerman, Princeton University, *Title to be announced.*

George Lusztig, Massachusetts Institute of Technology, *Title to be announced.*

Stefan Papadima, Institute of Mathematics of the Romanian Academy of Sciences, *Title to be announced.*

Dan Timotin, Institute of Mathematics of the Romanian Academy of Sciences, *Title to be announced.*

Srinivasa Varadhan, New York University, *Title to be announced.*

**Special Sessions**

**Algebraic Geometry**, Marian Aprodu, Institute of Mathematics of the Romanian Academy, Mihnea Mustata, University of Michigan, Ann Arbor, and Mihnea Popa, University of Illinois, Chicago.


**Calculus of Variations and Partial Differential Equations**, Marian Borcea, Loyola University, Chicago, Liviu Ignat, Institute of Mathematics of the Romanian Academy, Mihai Mihailescu, University of Craiova, and Daniel Onofrei, University of Houston.

**Commutative Algebra**, Florian Enescu, Georgia State University, and Cristodor Ionescu, Institute of Mathematics of the Romanian Academy.

**Discrete Mathematics and Theoretical Computer Science**, Sebastian Cioaba, University of Delaware, Gabriel Istrate, Universitatea de Vest, Timisoara, Ioan Tomescu, University of Bucharest, and Marius Zimand, Towson University.


Harmonic Analysis and Applications, Ciprian Demeter, Indiana University, Bloomington, and Camil Muscalu, Cornell University.

Hopf Algebras, Coalgebras, and their Categories of Representations, Miodrag C. Iovanov, University of Bucharest and University of Iowa, Susan Montgomery, University of Southern California, and Siu-Hung Ng, Iowa State University.

Local and Nonlocal Models in Wave Propagation and Diffusion, Anca V. Ion, Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, and Petronela Radu, University of Nebraska, Lincoln.


Nonlinear Evolution Equations, Daniel Tataru, University of California, Berkeley, and Monica Visan, University of California, Los Angeles.

Operator Algebra and Noncommutative Geometry, Marius Dadarlat, Purdue University, and Florin Radulescu, Institute of Mathematics of the Romanian Academy and University of Rome Tor Vergata.

Operator Theory and Function Spaces, Aurelian Gheondea, Institute of Mathematics of the Romanian Academy and Bilkent University, Mihai Putinar, University of California, Santa Barbara, and Dan Timotin, Institute of Mathematics of the Romanian Academy.

Probability and its Relation to Other Fields of Mathematics, Krzysztof Burdzy, University of Washington, and Mihai N. Pascu, Transilvania University of Brașov.

Random Matrices and Free Probability, Ioana Dumitriu, University of Washington, and Ionel Popescu, Georgia Institute of Technology and Institute of Mathematics of the Romanian Academy.

Several Complex Variables, Complex Geometry and Dynamics, Dan Coman, Syracuse University, and Cezar Joiţa, Institute of Mathematics of the Romanian Academy.

Topics in Geometry and Algebraic Topology, Stefan Papadima, Institute of Mathematics of the Romanian Academy, and Alexandru I. Suciu, Northeastern University.

Louisville, Kentucky

University of Louisville

October 5–6, 2013
Saturday – Sunday

Meeting #1092
Southeastern Section
Associate secretary: Robert J. Daverman
Announcement issue of Notices: June 2013
Program first available on AMS website: August 22, 2013
Program issue of electronic Notices: October 2013
Issue of Abstracts: Volume 33, Issue 3

Deadlines
For organizers: March 5, 2013
For consideration of contributed papers in Special Sessions: June 18, 2013
For abstracts: August 13, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Michael Hill, University of Virginia, Title to be announced.

Suzanne Lenhart, University of Tennessee, Title to be announced.

Ralph McKenzie, Vanderbilt University, Title to be announced.

Victor Moll, Tulane University, Title to be announced.

Special Sessions
Extremal Graph Theory (Code: SS 1A), Jozsef Balogh, University of Illinois at Urbana-Champaign, and Louis DeBiasio and Tao Jiang, Miami University, Oxford, OH.
Set Theory and Its Applications (Code: SS 1A), Paul Larson, Miami University, Justin Moore, Cornell University, and Grigor Sargsyan, Rutgers University.

Philadelphia, Pennsylvania

Temple University

October 12–13, 2013
Saturday – Sunday

Meeting #1093
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: June 2013
Program first available on AMS website: To be announced
Program issue of electronic Notices: October 2013
Issue of Abstracts: Volume 33, Issue 3
Meetings & Conferences

Deadlines
For organizers: March 12, 2013
For consideration of contributed papers in Special Sessions: June 25, 2013
For abstracts: August 20, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Patrick Brosnan, University of Maryland, Title to be announced.
Xiaojung Huang, Rutgers University, Title to be announced.
Barry Mazur, Harvard University, Title to be announced (Erdo˝s Memorial Lecture).
Robert Strain, University of Pennsylvania, Title to be announced.

Special Sessions
Contact and Symplectic Topology (Code: SS 5A), Joshua M. Sabloff, Haverford College, and Lisa Traynor, Bryn Mawr College.
History of Mathematics in America (Code: SS 4A), Thomas L. Bartlow, Villanova University, Paul R. Wolfson, West Chester University, and David E. Zitarelli, Temple University.
Recent Advances in Harmonic Analysis and Partial Differential Equations (Code: SS 1A), Cristian Gutiérrez and Irina Mitrea, Temple University.
Recent Developments in Noncommutative Algebra (Code: SS 6A), Edward Letzter and Martin Lorenz, Temple University.
Several Complex Variables and CR Geometry (Code: SS 7A), Andrew Raich, University of Arkansas, and Yuan Zhang, Indiana University-Purdue University Fort Wayne.

St. Louis, Missouri
Washington University
October 18–20, 2013
Friday – Sunday
Meeting #1094
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: August 2013
Program first available on AMS website: September 5, 2013
Program issue of electronic Notices: October 2013
Issue of Abstracts: Volume 33, Issue 4

Deadlines
For organizers: April 2, 2013
For consideration of contributed papers in Special Sessions: July 15, 2013
For abstracts: September 10, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Ronny Hadani, University of Texas at Austin, Title to be announced.
Effie Kalfagianni, Michigan State University, Title to be announced.
Jon Kleinberg, Cornell University, Title to be announced.
Vladimir Sverak, University of Minnesota, Title to be announced.

Special Sessions
Algebraic and Combinatorial Invariants of Knots (Code: SS 1A), Heather Dye, McKendree University, Allison Hennrich, Seattle University, and Louis Kauffman, University of Illinois.
Computability Across Mathematics (Code: SS 2A), Wesley Calvert, Southern Illinois University, and Johanna Franklin, University of Connecticut.
Geometric Topology in Low Dimensions (Code: SS 4A), William H. Kazez, University of Georgia, and Rachel Roberts, Washington University in St. Louis.
Interactions between Geometric and Harmonic Analysis (Code: SS 3A), Leonid Kovalev, Syracuse University, and Jeremy Tyson, University of Illinois, Urbana-Champaign.
Noncommutative Rings and Modules (Code: SS 5A), Greg Marks and Ashish Srivastava, St. Louis University.

Riverside, California
University of California Riverside
November 2–3, 2013
Saturday – Sunday
Meeting #1095
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: August 2013
Program first available on AMS website: September 19, 2013
Program issue of electronic Notices: November 2013
Issue of Abstracts: Volume 33, Issue 4

Deadlines
For organizers: March 20, 2013
For consideration of contributed papers in Special Sessions: July 2, 2013
For abstracts: August 27, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Ronny Hadani, University of Texas at Austin, Title to be announced.
Effie Kalfagianni, Michigan State University, Title to be announced.
Jon Kleinberg, Cornell University, Title to be announced.
Vladimir Sverak, University of Minnesota, Title to be announced.
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
- Michael Christ, University of California, Berkeley, Title to be announced.
- Mark Gross, University of California, San Diego, Title to be announced.
- Matilde Marcolli, California Institute of Technology, Title to be announced.
- Paul Vojta, California Institute of Technology, Title to be announced.

Special Sessions
- The Mathematics of Planet Earth (Code: SS 1A), John Baez, University of California, Riverside.

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 15–18, 2014
Wednesday - Saturday

Meeting #1096
Joint Mathematics Meetings, including the 120th Annual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Georgia Benkart
Announcement issue of Notices: October 2013
Program first available on AMS website: November 1, 2013
Program issue of electronic Notices: January 2014
Issue of Abstracts: Volume 35, Issue 1

Deadlines
For organizers: April 1, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Albuquerque, New Mexico

University of New Mexico

April 5–6, 2014
Saturday - Sunday

Meeting #1099
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: April 2014
Issue of Abstracts: To be announced

Deadlines
For organizers: September 5, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: February 11, 2014
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions
The Inverse Problem and Other Mathematical Methods Applied in Physics and Related Sciences (Code: SS 1A), Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico and Enfitek Inc.

Lubbock, Texas
Texas Tech University
April 11–13, 2014
Friday – Sunday
Meeting #2000
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: September 18, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions
Topology and Physics (Code: SS 1A), Razvan Gelca and Alastair Hamilton, Texas Tech University.

Eau Claire, Wisconsin
University of Wisconsin-Eau Claire
September 20–21, 2014
Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: February 20, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: August 5, 2014

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

Special Sessions
Mirror Symmetry and Representation Theory, David Kazhdan, Hebrew University, and Roman Bezrukavnikov, Massachusetts Institute of Technology.
Nonlinear Analysis and Optimization, Boris Mordukhovich, Wayne State University, and Simeon Reich and Alexander Zaslavski, The Technion-Israel Institute of Technology.
Qualitative and Analytic Theory of ODE’s, Yosef Yomdin, Weizmann Institute.

Tel Aviv, Israel
Bar-Ilan University, Ramat-Gan and Tel-Aviv University, Ramat-Aviv
June 16–19, 2014
Monday – Thursday
The 2nd Joint International Meeting between the AMS and the Israel Mathematical Union.
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Francisco, California
San Francisco State University
October 25–26, 2014
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: October 2014
Issue of Abstracts: To be announced

Deadlines
For organizers: March 25, 2014
For consideration of contributed papers in Special Sessions: To be announced
Meetings & Conferences

For abstracts: September 3, 2014

San Antonio, Texas

Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio

January 10–13, 2015
Saturday – Tuesday
Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Steven H. Weintraub
Announcement issue of Notices: October 2014
Program first available on AMS website: To be announced
Program issue of electronic Notices: January 2015
Issue of Abstracts: Volume 36, Issue 1

Deadlines
For organizers: April 1, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Chicago, Illinois

Loyola University Chicago

October 3–4, 2015
Saturday – Sunday
Central Section

Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: October 2015
Issue of Abstracts: To be announced

Deadlines
For organizers: March 10, 2015
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Las Vegas, Nevada

University of Nevada, Las Vegas

April 18–19, 2015
Saturday – Sunday
Western Section

Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Program issue of electronic Notices: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: September 18, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 6–9, 2016
Wednesday – Saturday
Joint Mathematics Meetings, including the 122nd Annual Meeting of the AMS, 99th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus
Announcement issue of Notices: October 2015
Program first available on AMS website: To be announced
Program issue of electronic Notices: January 2016
Issue of Abstracts: Volume 37, Issue 1

Deadlines
For organizers: April 1, 2015
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Porto, Portugal

University of Porto

June 11–14, 2015
Thursday – Sunday
First Joint International Meeting between the AMS and the Sociedade Portuguesa de Matemática.
Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017
Wednesday – Saturday
Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Georgia Benkart
Announcement issue of Notices: October 2016
Program first available on AMS website: To be announced
Program issue of electronic Notices: January 2017
Issue of Abstracts: Volume 38, Issue 1

Deadlines
For organizers: April 1, 2016
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018
Wednesday – Saturday
Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Matthew Miller
Announcement issue of Notices: October 2017
Program first available on AMS website: To be announced
Program issue of electronic Notices: To be announced
Issue of Abstracts: To be announced

Electronic Mathematical Reviews® (eMR) Sections

You can now stay informed of new research published in your subject area by subscribing to the electronic Mathematical Reviews® (eMR) Section of your choice. Subject areas are assigned using MSC 2010.

eMR Sections Offer:
• Improved accessibility and portability, as eMR Section issues will be posted monthly and available as downloadable PDFs
• Email notifications when new issues are posted
• Perpetual access
• Discounts to MR Reviewers and AMS individual members
• Low-cost option for researchers and faculty whose institutions don’t currently subscribe to the full database, MathSciNet

For ordering information, please visit:
www.ams.org/bookstore/emrsections
Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Robert J. Daverman, Department of Mathematics, University of Tennessee, Knoxville, TN 37996-6900, e-mail: daverman@math.utk.edu; telephone: 865-974-6900.

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated. Up-to-date meeting and conference information can be found at www.ams.org/meetings/.

Meetings:

2013

March 1–3

April 6–7

April 13–14

April 27–28

June 27–30

October 5–6

October 12–13

October 18–20

November 2–3

Oxford, Mississippi

Chestnut Hill, Massachusetts

Boulder, Colorado

Ames, Iowa

Alba Iulia, Romania

Louisville, Kentucky

Philadelphia, Pennsylvania

St. Louis, Missouri

Riverside, California

p. 275

p. 276

p. 276

p. 277

p. 281

p. 282

p. 282

p. 283

p. 283

2014

January 15–18

March 21–23

March 29–30

April 5–6

April 11–13

June 16–19

September 20–21

October 25–26

Baltimore, Maryland

Knoxville, Tennessee

Baltimore, Maryland

Albuquerque, New Mexico

Lubbock, Texas

Tel Aviv, Israel

Eau Claire, Wisconsin

San Francisco, California

Annual Meeting

Annual Meeting

Annual Meeting

Annual Meeting

p. 284

p. 284

p. 284

p. 284

p. 285

p. 285

p. 285

p. 285

2015

January 10–13

April 18–19

June 11–14

October 3–4

San Antonio, Texas

Las Vegas, Nevada

Porto, Portugal

Chicago, Illinois

p. 286

p. 286

p. 286

p. 286

2016

January 6–9

Seattle, Washington

Annual Meeting

p. 287

2017

January 4–17

Atlanta, Georgia

Annual Meeting

p. 287

2018

January 10–13

San Diego, California

Annual Meeting

p. 287

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 111 in the January 2012 issue of the Notices for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of \LaTeX is necessary to submit an electronic form, although those who use \LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in \LaTeX. Visit http://www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences in Cooperation with the AMS: (see http://www.ams.org/meetings/ for the most up-to-date information on these conferences.)

July 22–26, 2013: Samuel Eilenberg Centenary Conference (E100), Warsaw, Poland.
The following publications examine the different roles that mathematics can play in addressing questions related to Planet Earth.

**Portraits of the Earth: A Mathematician Looks at Maps**

Timothy G. Feeman, Villanova University, PA

"I became hooked on this book ... (It) is interesting, entertaining, mathematical, and, so it seems to me, a labor of love ... I recommend this for yourselves, for your bookshelves, and for your students."

—Robert W. Vallin, MAA Online

Maps are exciting, visual tools that we encounter on a daily basis. This book explores the mathematical ideas involved in creating and analyzing maps, and is the first modern book to present the famous problem of mapping the earth in a style that is highly readable and mathematically accessible to most students. Through the visual context of maps and mapmaking, students will see how contemporary mathematics can help them to understand and explain the world.


**Introduction to PDEs and Waves for the Atmosphere and Ocean**

Andrew Majda, Courant Institute of Mathematical Sciences, New York University, NY

"The author presents rigorous mathematical theory ... and offers deep insights ... The contribution of these notes to the modern literature is very valuable and unique."

—Mathematical Reviews

Written by a leading specialist in the area of atmosphere/ocean science (AOS), this book aims to introduce mathematicians to this fascinating and important topic and, conversely, to develop a mathematical viewpoint on basic topics in AOS of interest to the disciplinary AOS community, ranging from graduate students to researchers.


For more AMS resources on mathematics and the environment, visit: ams.org/samplings/mpe-2013

**Mathematical Methods for Analysis of a Complex Disease**

Frank C. Hoppensteadt, Courant Institute of Mathematical Sciences, New York University, NY

Complex diseases involve most aspects of population biology, including genetics, demographics, epidemiology, and ecology. Mathematical methods have been used effectively in all of these areas. The aim of this book is to provide sufficient background in such mathematical and computational methods to enable the reader to better understand complex systems in biology, medicine, and the life sciences.

Titles in this series are co-published with the Courant Institute of Mathematical Sciences at New York University.


**Mathematical Methods in Immunology**

Jerome K. Percus, Courant Institute of Mathematical Sciences and Department of Physics, New York University, NY

The complexity of the mammalian adaptive immune system calls for its encapsulation by mathematical models, and this book aims at the associated description and analysis. In the process, it introduces tools that should be in the armory of any current or aspiring applied mathematician, in the context of, arguably, the most effective system nature has devised to protect an organism from its manifold invisible enemies.

Titles in this series are co-published with the Courant Institute of Mathematical Sciences at New York University.


**Mathematical Modelling: A Case Studies Approach**

Reinhard Illner, C. Sean Bohun, Samantha McCollum, and Thea van Roode, University of Victoria, BC, Canada

Mathematical modelling is a subject without boundaries. It is the means by which mathematics becomes useful to virtually any subject, and has been and continues to be a driving force for the development of mathematics itself. This book explains, in the form of case studies, the process of modelling real situations to obtain mathematical problems that can be analyzed, thus solving the original problem.


**Modelling in Healthcare**

The Complex Systems Modelling Group (CSMG), The IRMACS Center, Simon Fraser University, Burnaby, BC, Canada

"How many patients will require admission to my hospital in two days? How widespread will influenza be in my community in two weeks? These and similar questions are the province of Modelling in Healthcare. This new volume ... uses plain language, sophisticated mathematics and vivid examples to guide and instruct... [T]he content and logic are readily understandable by modelers, administrators and clinicians alike. This volume will surely serve as their common and thus preferred reference for modeling in healthcare for many years."

—Timothy G. Buchman, Ph.D., M.D., FACS, FCCM