WHAT IS... a Frame?

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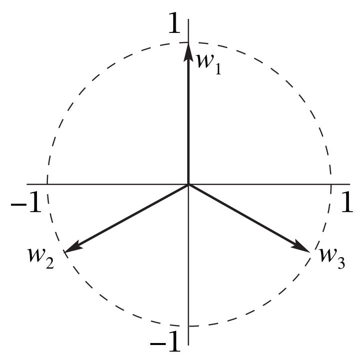

Figure 1. The *Mercedes frame* $\{w_1, w_2, w_3\}$.

A frame is a set of vectors in a Hilbert space that provides robust, basis-like representations even though the frame may be "redundant" or "overcomplete". In finite dimensions a frame is simply a spanning set, but this statement belies both the many practical applications of frames and the deep mathematical problems that remain unsolved. In infinite dimensions there are many shades of gray to the meanings of "spanning" and "independence", and some of the most important frames are overcomplete even though every finite subset is linearly independent. Though we do not have space to discuss them, applications drive much of the interest in frames. A short and incomplete list of areas in which frames play an important role includes analog-to-digital conversion and Sigma-Delta quantization, compressed sensing, phaseless reconstruction, reactive sensing, transmission with erasures (e.g., over the Internet), and classification and analysis of large data sets such as those obtained using LIDAR (a remote sensing method) or HSI (hyperspectral imagery).

Frames were introduced by Duffin and Schaeffer in a 1952 paper in the *Transactions of the AMS*. In that article (which is a model of clarity and well worth reading today), they declare a set of vectors $\mathcal{F} = \{f_n\}_{n\in J}$, J a countable index set, to be a *frame* for a Hilbert space H if there exist constants $A, B > 0$ such that

$$A\,\|f\|^2 \le \sum_{n\in J} |\langle f, f_n\rangle|^2 \le B\,\|f\|^2, \quad f \in H.$$

Sadly, Duffin and Schaeffer both passed away before anyone thought to ask why they called such a system a "frame". Is it because $A\,\|f\|^2$ and $B\,\|f\|^2$ "frame" the sum between them? We will never know. In any case, a frame is *tight* if we can take $A = B$, and it is *Parseval* if we can take $A = B = 1$.

Every orthonormal basis is a Parseval frame, but a Parseval frame need not be orthogonal or a basis. The *Mercedes frame* $\{w_1, w_2, w_3\}$, pictured in Figure 1, is a simple example of a tight frame (with $A = B = 3/2$). Rescaling, $\{u_1, u_2, u_3\}$, $u_i = cw_i$, $c = (2/3)^{1/2}$ is a Parseval frame for $\mathbb{R}^2$, and hence every vector $v \in \mathbb{R}^2$ satisfies

$$v = (v\cdot u_1)\,u_1 + (v\cdot u_2)\,u_2 + (v\cdot u_3)\,u_3.$$

The coefficients in this linear combination are not unique, because $\{u_1, u_2, u_3\}$ is dependent, yet this is actually an advantage in many situations. However, even in finite dimensions we usually require much larger sets of vectors that form a frame, often for a very high-dimensional space. Do there exist unit vectors $v_1, \dots, v_{97} \in \mathbb{R}^{43}$ that form a tight frame for $\mathbb{R}^{43}$? This is a problem about equidistributing

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DOI: http://dx.doi.org/10.1090/noti1011

points on the sphere, but not with respect to the usual notions of distribution. A collection of unit vectors that forms a tight frame for $\mathbb{R}^d$ or $\mathbb{C}^d$ is called a *finite unit norm tight frame*, or *FUNTF*. Benedetto and Fickus characterized FUNTFs in terms of the minima of a certain potential energy function on the sphere. An active area of research is to construct finite uniform norm tight frames that are *equiangular* or as close to equiangular as possible. Such frames would be important in applications such as signal processing, radar, and communications engineering.

If $\mathcal{F} = \{f_n\}_{n\in J}$ is a frame for a Hilbert space H, then $Sf = \sum_{n\in J} \langle f, f_n\rangle f_n$ is a continuous linear bijection of H onto itself. The *canonical dual frame* $\tilde{\mathcal{F}} = \{\tilde{f}_n\}_{n\in J}$, $\tilde{f}_n = S^{-1}f_n$, satisfies

$$f = \sum_{n\in J} \langle f, \tilde{f}_n\rangle f_n, \quad f \in H. \tag{1}$$

If the frame is tight, then $\tilde{f}_n = \frac{1}{A} f_n$. In general, the *frame coefficients* $\langle f, \tilde{f}_n\rangle$ need not be the only scalars c_n that satisfy $f = \sum c_n f_n$, but the *frame representation* in equation (1) enjoys useful "stability" properties. For example, the series converges *unconditionally*, i.e., regardless of the ordering of the index set J, and among all choices of c_n for a given f, the sequence of frame coefficients has minimal ℓ^2-norm (but the sequence that has minimal ℓ^1-norm is often sought for sparsity reasons). The representations in (1) are unique for every f if and only if $\mathcal{F}$ is a *Riesz basis* (the image of an orthonormal basis under a continuous linear bijection $A\colon H \to H$). No proper subset of a Riesz basis can be a frame, yet if a frame is not a Riesz basis, then there exist proper subsets that are frames.

Why do we need frames that are not orthonormal bases or Riesz bases? The *Classical Sampling Theorem* (also known as the *Shannon* or *Nyquist-Shannon Sampling Theorem*) is a cornerstone of information theory and signal processing. The Sampling Theorem is equivalent to the statement that the sequence $\mathcal{E}_b = \{e^{2\pi ibnx}\}_{n\in\mathbb{Z}}$ is a tight frame for $L^2[0,1]$ for each $0 < b \le 1$. Taking $b = 1$, we obtain an orthonormal basis. However, if $b < 1$, then $\mathcal{E}_b$ is not a Riesz basis for $L^2[0,1]$, and hence frame coefficients are not unique (even so, every finite subset of $\mathcal{E}_b$ is linearly independent). If $b = 1/N$, then $\mathcal{E}_{1/N}$ is a union of N orthonormal bases, but in general $\mathcal{E}_b$ cannot be written as a union of orthonormal bases. The Sampling Theorem underlies the encoding of bandlimited signals in digital form: we must have $b \le 1$ in order to have a hope of reconstructing the original signal from its encoding. Taking $b < 1$ corresponds to "oversampling" the signal or to using a frame that is not a Riesz basis. The "8 times oversampling" note that appears on the labels of compact discs is closely related. Oversampling aids in both noise reduction and error correction.

Many seemingly simple questions about frames lead to deep mathematical problems. For example, it is natural to ask if we can explicitly characterize the meaning of redundancy, especially for infinite frames. In general, a frame cannot be written as a union of orthonormal sequences, but can every frame $\mathcal{F} = \{f_n\}_{n\in J}$ be written as the union of finitely many nonredundant subsequences $\mathcal{E}_1, \dots, \mathcal{E}_N$? Here, a subsequence is *nonredundant* if it is a Riesz basis not for the entire space H but for the closure of its linear span. We call such a set a *Riesz sequence*. (In finite dimensions, this would simply be a linearly independent set.) Excluding the trivial case $\|f_n\| \to 0$ suggests the following conjecture.

The Feichtinger Conjecture. *If $\mathcal{F} = \{f_n\}_{n\in J}$ is a frame for a Hilbert space H and* $\inf \|f_n\| > 0$, *then there exist finitely many Riesz sequences $\mathcal{E}_1, \dots, \mathcal{E}_N$ whose union is $\mathcal{F}$.*

Casazza and Tremain have shown that the Feichtinger Conjecture is equivalent to the following *Kadison-Singer*, or *Paving*, Conjecture, which has been called the deepest open problem in operator theory today. (They also demonstrated that the Paving and Feichtinger Conjectures are equivalent to a number of other open problems from mathematics and engineering.)

Paving Conjecture. *For each $\varepsilon > 0$ there exists an integer $M > 0$ such that for every integer $n > 0$ and every $n \times n$ matrix S that has zero diagonal, there exists a partition $\{\sigma_1, \dots, \sigma_M\}$ of $\{1, \dots, n\}$ such that,*

$$\|P_{\sigma_j} S P_{\sigma_j}\| \le \varepsilon\, \|S\|, \quad j = 1, \dots, M,$$

where P_I denotes the orthogonal projection onto $\mathrm{span}\{e_i\}_{i\in I}$ *and $\|\cdot\|$ denotes operator norm.*

Duffin and Schaeffer were specifically interested in frames of the form $\mathcal{E} = \{e^{2\pi i\lambda_n x}\}_{n\in\mathbb{N}}$ for $L^2[0,1]$, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is an arbitrary countable subset of $\mathbb{R}$ or $\mathbb{C}$. Such *nonharmonic Fourier frames* yield "nonuniform" sampling theorems for bandlimited signals. Although frame theory was largely overlooked for many years after Duffin and Schaeffer, nonuniform sampling is today a major topic, both for bandlimited and nonbandlimited signals; e.g., it arises in magnetic resonance imaging (MRI).

In 1986 Daubechies, Grossmann, and Meyer brought frames back into the limelight with their work on *Gabor frames* and *wavelet frames* for $L^2(\mathbb{R})$. A (lattice) Gabor frame is a frame of the form $\mathcal{G}(g,a,b) = \{e^{2\pi ibnx} g(x - ak)\}_{k,n\in\mathbb{Z}}$, where $g \in L^2(\mathbb{R})$ and a, $b > 0$ are fixed (of course, g, a, and b must be carefully chosen in order for $\mathcal{G}(g,a,b)$ to actually form a frame). Thus a Gabor frame is produced by applying a discrete set of translation and modulation operators to g, and as a consequence there are underlying connections to representation theory, the Heisenberg group, and the uncertainty principle. Indeed, the *Balian-Low Theorem* states that, if a Gabor frame is a Riesz basis for $L^2(\mathbb{R})$, then the Heisenberg product $\left(\int_{-\infty}^{\infty} |x\, g(x)|^2\, dx\right) \left(\int_{-\infty}^{\infty} |\xi\, \hat{g}(\xi)|^2\, d\xi\right)$ must be infinite. Consequently, Gabor frames that are Riesz bases have limited interest. On the other hand, Feichtinger and Gröchenig proved that, once we find a reasonable function g that generates a Gabor frame $\mathcal{G}(g,a,b)$ for $L^2(\mathbb{R})$, then this frame provides stable basis-like representations not merely for square-integrable functions but also for functions in the entire family of Banach spaces $M_w^{p,q}(\mathbb{R})$ known as the *modulation spaces*. Thus, from a simple Hilbert space frame criterion we obtain representations valid across a wide range of function spaces. Similar representations hold for "irregular" Gabor frames of the form $\{e^{2\pi i b_k x} g(x - a_k)\}_{k\in\mathbb{N}}$, though the proofs in this setting are far more difficult. Recent advances in this area have come from profound new versions of Wiener's Lemma in noncommutative Banach algebras.

A wavelet frame is generated by the actions of translation and dilation. Specifically, if $\psi \in L^2(\mathbb{R})$ and a, $b > 0$ are fixed and if $\mathcal{W}(\psi,a,b) = \{a^{n/2}\psi(a^n x - bk)\}_{k,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then we call it a wavelet frame. In contrast to Gabor frames, it is possible to find very nice functions ψ such that $\mathcal{W}(\psi,a,b)$ is a Riesz basis or even an orthonormal basis for $L^2(\mathbb{R})$. This discovery by Meyer, Mallat, and Daubechies was the beginning of the "wavelet revolution". A wavelet frame or orthonormal basis generated by a "reasonably nice" function ψ will provide frame expansions not only for $L^2(\mathbb{R})$ but also for an entire suite of Banach spaces, including Sobolev spaces, Besov spaces, and Triebel–Lizorkin spaces. Wavelet frames have important applications today, as do various hybrid systems and generalizations such as curvelets and shearlets, which are especially important for analysis in higher dimensions (consider image or video processing). Even larger redundant "dictionaries", often so overcomplete that they are not even frames, are the basis for the theory and application of compressed sensing and *sparse representations*.

We cannot resist mentioning one last open problem. It is not difficult to show that any finite subset of the nonharmonic systems $\{e^{2\pi i\lambda_n x}\}_{n\in\mathbb{N}}$ studied by Duffin and Schaeffer is linearly independent. For *lattice* Gabor systems $\mathcal{G}(g,a,b) = \{e^{2\pi i bnx} g(x - ak)\}_{k,n\in\mathbb{Z}}$, it is likewise known that every finite subset is linearly independent, even if the system is not a frame. However, the answer is not known for irregular Gabor systems. As of this writing, the validity of the following conjecture is open.

Linear Independence of Time-Frequency Translates Conjecture. *If $g \in L^2(\mathbb{R})$ is not the zero function and $\{(a_k, b_k)\}_{k=1}^N$ is a set of finitely many distinct points in $\mathbb{R}^2$, then*

$$\{e^{2\pi i b_k x} g(x - a_k)\}_{k=1}^N$$

is linearly independent.

This conjecture is also known as the *HRT Conjecture*. It is known to be true for many special cases but not in general. For example, it is true if $N = 1$, 2, or 3. It is not known for $N = 4$, even if we impose the further condition that g is infinitely differentiable. In fact, even the following is open.

HRT Subconjecture. *If g is infinitely differentiable and $0 < \int_{-\infty}^{\infty} |g(x)|^2\, dx < \infty$, then*

$$\{g(x),\ g(x-1),\ e^{2\pi i x} g(x),\ e^{2\pi i\sqrt{2}x} g(x - \sqrt{2})\}$$

is linearly independent.

Further Reading

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